4

DIFFERENTIATION

SUMMARY OF CONCEPTS

DERIVATIVE OF A FUNCTION

Let $y = f(x)$ be a function defined on the interval $[a, b]$. Let for a small increment δx in x, the corresponding increment in the value of y be δy. Then

$$y = f(x) \text{ and } y + \delta y = f(x + \delta x)$$

On subtraction, we get

$$\delta y = f(x + \delta x) - f(x)$$

or
$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Taking limit on both sides when $\delta x \to 0$ we have,

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

if this limit exists, is called the *derivative* or *differential coefficient* of y with respect to x and is written as $\frac{dy}{dx}$ or $f'(x)$. Thus

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

Derivative at a Point

The value of $f'(x)$ obtained by putting $x = a$, is called the derivative of $f(x)$ at $x = a$ and it is denoted by $f'(a)$ or $\left\{\frac{dy}{dx}\right\}_{x=a}$.

Standard Derivatives

The following formulae can be applied directly for finding the derivative of a function:

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
7. $\frac{d}{dx}(e^x) = e^x$
8. $\frac{d}{dx}(a^x) = a^x \log_e a,\ a > 1$
9. $\frac{d}{dx}(\log_e x) = \frac{1}{x},\ x > 0$
10. $\frac{d}{dx}(x^n) = nx^{n-1}$
11. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}},\ -1 < x < 1$
12. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}},\ -1 < x < 1$
13. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2},\ -\infty < x < \infty$
14. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}},\ |x| > 1$
15. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}},\ |x| > 1$
16. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2},\ -\infty < x < \infty$
17. $\frac{d}{dx}(|x|) = \frac{x}{|x|}$ or $\frac{|x|}{x},\ x \neq 0.$

RULES FOR DIFFERENTIATION

1. The derivative of a constant function is zero, i.e.
$$\frac{d}{dx}(c) = 0.$$
2. The derivative of constant times a function is constant times the derivative of the function, i.e.
$$\frac{d}{dx}\{c \cdot f(x)\} = c\frac{d}{dx}\{f(x)\}.$$
3. The derivative of the sum or difference of two function is the sum or difference of their derivatives, i.e.
$$\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}.$$

5. If the function $f(x) = \dfrac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$ is continuous at each point of its domain, then the value of $f(0)$ is

(a) 2 (b) $\dfrac{2}{3}$

(c) $-\dfrac{1}{3}$ (d) $\dfrac{2}{3}$

[Based on PET (Raj.) 2000]

6. If $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then, at $x = 0$, the function $f(x)$ is:

(a) Differentiable (b) Not differentiable

(c) Continuous but not differentiable

(d) None of these **[Based on PET (Raj.) 1998]**

7. If $f(x) = |x - 3|$, then $f'(3)$ is:

(a) -1 (b) 1

(c) 0 (d) Does not exist

[Based on PET (Raj.) 1997, 1998]

8. The function $f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & x \neq 0 \\ \dfrac{k}{2}, & x = 0, \end{cases}$

is continuous at $x = 0$, then $k =$

(a) 3 (b) 6

(c) 9 (d) 12

[Based on EAMCET 2000]

9. The value of the constants a, b and c for which the function

$$f(x) = \begin{cases} (1+ax)^{1/x} & , x < 0 \\ b & , x = 0 \\ \dfrac{(x+c)^{1/3} - 1}{(x+1)^{1/2} - 1}, & x > 0 \end{cases}$$

may be continuous at $x = 0$, are

(a) $a = \log\dfrac{2}{3}$, $b = \dfrac{-2}{3}$, $c = 1$

(b) $a = \log\dfrac{2}{3}$, $b = \dfrac{2}{3}$, $c = -1$

(c) $a = \log\dfrac{2}{3}$, $b = \dfrac{2}{3}$, $c = 1$

(d) None of these **[Based on Roorkee 1999]**

10. If $f(x) = \begin{cases} e^{-1/x^2}\sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0 & , x = 0 \end{cases}$ then graph of the function $f(x)$

(a) is broken at the point $x = 0$

(b) is continuous at the point $x = 0$

(c) has a definite tangent at the point $x = 0$

(d) does not have a definite tangent at the point $x = 0$.

11. The function $f(x) = a[x+1] + b[x-1]$, where $[x]$ is the greatest integer function is continuous at $x = 1$, if:

(a) $a + b = 0$ (b) $a = b$

(c) $2a - b = 0$ (d) None of these

[Based on UPSEAT 2001]

Answers

1. (d)	**2.** (c)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (c)	**7.** (d)
8. (b)	**9.** (c)	**10.** (b)	**11.** (a)			

Product Rule of Differentiation The derivative of the product of two functions

$$\frac{d}{dx}\{f(x)\cdot g(x)\}=f(x)\cdot\frac{d}{dx}\{g(x)\}+g(x)\cdot\frac{d}{dx}\{f(x)\}$$

= (first function) × (derivative of second function) + (second function) × (derivative of first function)

Quotient Rule of Differentiation The derivative of the quotient of two functions

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\}=\frac{g(x)\cdot\frac{d}{dx}\{f(x)\}-f(x)\cdot\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$$

$$=\frac{(\text{denom.}\times\text{derivative of num.})-(\text{num.}\times\text{derivative of denom.})}{(\text{denominator})^2}$$

Derivative of a Function of a Function (Chain Rule) If y is a differentiable function of t and t is a differentiable function of x *i.e.* $y = f(t)$ and $t = g(x)$, then

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

Similarly, if $y = f(u)$, where $u = g(v)$ and $v = h(x)$, then,

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

DERIVATIVE OF PARAMETRIC FUNCTIONS

Sometimes x and y are separately given as functions of a single variable t (called a parameter) i.e. $x = f(t)$ and $y = g(t)$. In this case,

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{f'(t)}{g'(t)}.$$

and $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$=\frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\Big/\frac{dx}{dt}.$$

DERIVATIVE OF IMPLICIT FUNCTIONS

The derivative of an implicit function, given by the relation $f(x, y) = 0$ in which y is not expressible explicitly in terms of x, can be found by the following steps:

Step 1. Differentiate each term of the equation $f(x, y) = 0$ w.r.t. x, keeping in mind that $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$; $\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$ and so on.

Step 2. Collect the terms containing $\frac{dy}{dx}$ on one side and the terms not involving $\frac{dy}{dx}$ on the other side.

Step 3. Divide by coefficient of $\frac{dy}{dx}$ to get $\frac{dy}{dx}$ as a function of x or y or both.

Shorter Method for Finding the Derivative of an Implicit Function

Step 1. Take all the terms of the function to be differentiated to the left hand side and put left hand side equal to $\phi(x, y)$.

Step 2.

$$\frac{dy}{dx}=-\frac{\text{derivatie of }\phi(x,y)\text{ w.r.t. }x\text{ treating }y\text{ as constant}}{\text{derivatie of }\phi(x,y)\text{ w.r.t. }y\text{ treating }x\text{ as constant}}.$$

DIFFERENTIATION OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

If $y = f(x)$ and $z = g(x)$, then in order to find the derivative of $f(x)$ w.r.t. $g(x)$, we use the formula

$$\frac{dy}{dz}=\frac{dy/dx}{dz/dx}=\frac{f'(x)}{g'(x)}.$$

LOGARITHMIC DIFFERENTIATION

The process of taking logarithms before differentiation is called logarithmic differentiation. When the function to be differentiated involves a function in its power or when the function is the product or quotient of a number of functions, we first take log on both sides and then differentiate each logarithmic term separately.

Properties of Logarithms

(i) $\log_e(mn)=\log_e m+\log_e n$ (ii) $\log_e\left(\frac{m}{n}\right)=\log_e m-\log_e n$

(iii) $\log_e(m)^n=n\log_e m$ (iv) $\log_e e=1$

(v) $\log_n m=\frac{\log_e m}{\log_e n}$ (vi) $\log_n m\cdot\log_m n=1.$

Shorter Methods of Finding the Derivative of a Logarithmic Function

If $y = [f(x)]^{g(x)}$, then to find $\frac{dy}{dx}$, in addition to the method discussed above, we can also apply any of the following two methods:

Method 1

Step 1. Express $y=[f(x)]^{g(x)}=e^{g(x)\log f(x)}$ $[\because a^x=e^{x\log a}]$

Step 2. Differentiate w.r.t. x to obtain $\frac{dy}{dx}$.

Method 2

Step 1. Evaluate

A = Differential coefficient of y treating $f(x)$ as constant.

Step 2. Evaluate

B = Differential coefficient of y treating $g(x)$ as constant.

Step 3. $\frac{dy}{dx}$ = A + B.