Notations: In the following, $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers.

1. Find all pairs (x, y) with x, y real, satisfying the equations:

$$\sin\left(\frac{x+y}{2}\right) = 0, \quad |x| + |y| = 1.$$

2. Suppose that PQ and RS are two chords of a circle intersecting at a point O. It is given that $PO = 3$ cm and $SO = 4$ cm. Moreover, the area of the triangle POR is 7 cm^2. Find the area of the triangle QOS.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}$ and for all $t \geq 0$,

$$f(x) = f(e^t x).$$

Show that f is a constant function.

4. Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function such that for all $x \in (0, \infty)$,

$$f(2x) = f(x).$$

Show that the function g defined by the equation

$$g(x) = \int_x^{2x} f(t)\frac{dt}{t} \text{ for } x > 0$$

is a constant function.

P.T.O.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that its derivative f' is a continuous function. Moreover, assume that for all $x \in \mathbb{R}$,

$$0 \leq \left|f'(x)\right| \leq \frac{1}{2}.$$

Define a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ by:

$$a_1 = 1,$$

$$a_{n+1} = f(a_n) \text{ for all } n \in \mathbb{N}.$$

Prove that there exists a positive real number M such that for all $n \in \mathbb{N}$,

$$|a_n| \leq M.$$

6. Let $a \geq b \geq c > 0$ be real numbers such that for all $n \in \mathbb{N}$, there exist triangles of side lengths $a^n,\ b^n, c^n$. Prove that the triangles are isosceles.

7. Let $a, b, c \in \mathbb{N}$ be such that

$$a^2 + b^2 = c^2 \text{ and } c - b = 1.$$

Prove that

(i) a is odd,
(ii) b is divisible by 4,
(iii) $a^b + b^a$ is divisible by c.

8. Let $n \geq 3$. Let $A = ((a_{ij}))_{1\leq i,j\leq n}$ be an $n \times n$ matrix such that $a_{ij} \in \{1, -1\}$ for all $1 \leq i, j \leq n$. Suppose that

$$a_{k1} = 1 \text{ for all } 1 \leq k \leq n \text{ and}$$

$$\sum_{k=1}^{n} a_{ki}a_{kj} = 0 \text{ for all } i \neq j.$$

Show that n is a multiple of 4.