Notation

$\mathbb{Z}$ = the set of integers
$\mathbb{N} = \{n \in \mathbb{Z} : n \geq 1\}$
$\mathbb{R}$ = the set of real numbers
$\mathbb{Q}$ = the set of rational numbers
$\mathbb{C}$ = the set of complex numbers

(1) Let H be a Hilbert space and S be a subspace of H. Let $x \in H$ and $\|x\| = 1$. Prove that

$$\inf_{z \in S^{\perp}} \|x - z\| = \sup\{|\langle x, y\rangle| : y \in S, \|y\| \leq 1\}.$$

(2) Let A be an $n \times n$ complex matrix, and suppose that

$$A^n \neq 0.$$

Prove that

$$A^k \neq 0,$$

for all $k \in \mathbb{N}$.

(3) Find two non-singular matrices B and C such that

$$BC + CB = 0.$$

(4) Let $l^1 = \{\{\alpha_n\}_{n\geq 1} : \sum_{n\geq 1} |\alpha_n| < \infty\}$ and $l^2 = \{\{\alpha_n\}_{n\geq 1} : \sum_{n\geq 1} |\alpha_n|^2 < \infty\}$ be equipped with the usual norms. Let $T : l^1 \to l^2$ be defined by

$$T(\{\alpha_n\}_{n\geq 1}) = \{\alpha_n\}_{n\geq 1}.$$

Show that T is a continuous operator which is not a compact operator.

(5) Let R be a commutative ring with 1 and P be a prime ideal of R. Consider the polynomial ring $R[x]$ and let $P[x]$ be the ideal of $R[x]$ consisting of polynomials whose coefficients all belong to P. Show that the ideal

$$P[x] + \langle x\rangle := \{f(x) + xg(x) : f(x) \in P[x],\, g(x) \in R[x]\},$$

is a prime ideal of $R[x]$.

(6) Fix $n \in \mathbb{N}$. Count the number of functions $h : \{1, 2, 3, \ldots, 2n\} \to \{1, -1\}$ such that

$$\sum_{j=1}^{2n} h(j) > 0.$$

(7) Let $q, q' \in \mathbb{N}$ and suppose that q' divides q. Let $U(m)$ denote the multiplicative group of residue classes coprime to m, that is

$$U(m) = \left(\mathbb{Z}/m\mathbb{Z}\right)^*.$$

Let $\pi : U(q) \to U(q')$ be such that if $a \in U(q)$, $\pi(a)$ is the unique element in $U(q')$ such that

$$a \equiv \pi(a) \pmod{q'}.$$

Show that π is onto.

(8) Let G be a group of order 12. Prove that G has a normal subgroup of order 3 or 4.

(9) Define $\phi : \mathbb{N} \to \mathbb{N}$ by $\phi(m)$ equals the number of elements in

$$\{k : 1 \leq k \leq m, \text{ g.c.d}(k, m) = 1\}.$$

Let $n \in \mathbb{N}$, $n \geq 2$. Show that $\phi(2^n - 1)$ is divisible by n.

(10) Prove that $(\mathbb{Q}, +)$ and $(\mathbb{Q} \times \mathbb{Q}, +)$ are not isomorphic as groups.