Sample Questions 2019
TESTCODE : PHB

There are three parts. Part I is compulsory for all candidates and carries a credit of 30% of the total. Besides, each candidate has to choose only one of the Parts II & III and answer from that part as per instructions. The credit for each of these parts is 70% of the total.

Part I

Answer all questions

1. For any real x, let $f(x) = min\{\sqrt{x}, x^2\}$. Compute $\int_0^3 f(x)\, dx$.

[5]

2. There are three balls labeled 1, 2, 3 and three boxes also labeled 1, 2, 3. Balls are placed at random into the boxes. Let X be the random variable that denotes the number of empty boxes. Find $E(X)$, the expectation of X.

[5]

3. Let $A = (a_{ij})$ be a 17×17 matrix with entries a_{ij} defined as follows.

$$a_{ij} = \begin{cases} +1 & \text{if } i > j \\ 0 & \text{if } i = j \\ -1 & \text{if } i < j \end{cases}$$

Is A invertible? Justify your answer.

[5]

4. Calculate the change in entropy if x grams of water at temperature $T_1\,^\circ$C is added to y grams of water at temperature $T_2\,^\circ$C ($T_1 > T_2$).

[5]

5. A pendulum is suspended in a lift. When the lift is stationary, the period of oscillation of the pendulum is T_0. Determine the period of oscillation of the pendulum when the lift begins to accelerate downwards with an acceleration $\frac{3}{4}g$, where g is the acceleration due to gravity.

[5]

6. Consider a single electron atom with orbital angular momentum $L = \sqrt{2}\hbar$. What are the possible values of a measurement of L_z, the z-component of L?

[5]

Part II

Physics

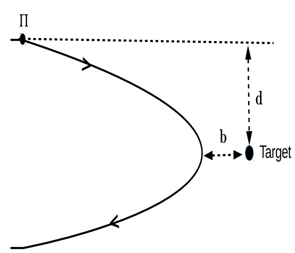
Answer any five questions

1. (a) A particle of mass m is sliding on a smooth surface along a path that satisfies $r^2 = az$ where a is a constant. Using cylindrical coordinates r, ϕ, z where $x = r\ cos\phi, y = r\ sin\phi, z = z$:

   i. Set up the Lagrangian and find the equations of motion.

   ii. Identify the cyclic coordinate.

   (b) In a non-relativistic system, a charged pion (with charge $+q$ for π^+ or $-q$ for π^-) has kinetic energy T and it is moving towards a massive target nucleus with charge Q. The pion is considered to hit the nucleus if its distance is b from the nucleus (see figure). The collision cross-section is given by $\Sigma = \pi d^2$, where d is the impact parameter.



   Show that the cross-sections, Σ^+ and Σ^- for π^+ and π^- respectively are given by

$$\Sigma^+ = \frac{\pi b^2(T - V)}{T}, \qquad \Sigma^- = \frac{\pi b^2(T + V)}{T}$$

   where $V = \frac{qQ}{b}$ is the Coulomb potential.

   • Impact parameter d is defined as the length of the perpendicular drawn from the target (nucleus) to the line of motion that the pion would have taken if there was no interaction.

[(5+1)+8]

2. (a) A particle of mass m is placed in a finite spherical well of radius a with the following potential:

$$V(r) = \begin{cases} -V_0 \ , & \text{if } \ r \leq a \\ \\ 0 \ , & \text{if } \ r > a \end{cases}$$

i. Solving the radial equation with $\ell = 0$ find the ground state wave function.

ii. Show that there is no bound state if

$$V_0 < \frac{\pi^2 \hbar^2}{8ma^2}$$

(b) The Hamiltonian for a spin-$\frac{1}{2}$ particle of mass m with charge $+e$ in an external magnetic field $\vec{B}$ is

$$H = -\frac{ge}{2mc} \, \vec{s}.\vec{B}$$

where the symbols have their usual meaning.

i. Derive the expression for $\frac{d\vec{s}}{dt}$.

ii. Assuming $\vec{B} = B\hat{y}$, find $s_z(t)$ in terms of the given quantities.

[(4+3)+(4+3)]

3. (a) Two synchronized clocks A and B are at rest in an inertial reference frame. The distance between them is L. Another clock X is moving with a velocity $\frac{3}{5}c$ along the line joining A and B, c being the velocity of light in vacuum. Both the clocks A and X read zero when X passes A. When X reaches the mid point of the line joining A and B, what are the readings of clocks A and B with respect to the inertial frame attached with X?

(b) Consider the following state of a quantum harmonic oscillator

$$|\psi\rangle = c_0|0\rangle + c_k|k\rangle$$

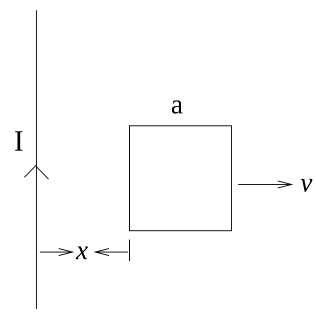
where $|0\rangle$ and $|k\rangle$ are the energy eigenstates. The non-zero real coefficients c_0 and c_k satisfy $c_0^2 + c_k^2 = 1$. Find the allowed values of k for which

$$\langle\psi|\frac{1}{2}m\omega^2\hat{x}^2|\psi\rangle = \langle\psi|\frac{\hat{p}^2}{2m}|\psi\rangle$$

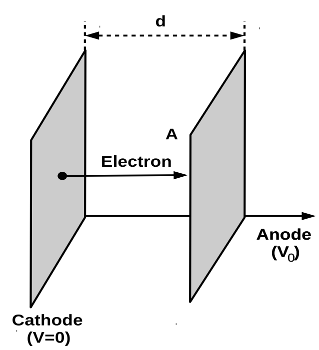
where the symbols have their usual meaning.

[6+8]

4. (a) A square frame with side a and a long wire carrying a current I are located in the same plane as shown in the figure below. The frame translates to the right with a constant velocity v. Find the e.m.f. induced in the frame as a function of x.



(b) Suppose there are two parallel plate electrodes of area A, at voltages 0 and V_0 respectively, separated by a distance d (see the figure below). The dimensions of the plates are much larger than the separation between them. With an unlimited supply of electrons at rest to the lower potential electrode (placed at $x = 0$), a steady current I flows between the plates.



i. Write the Poisson's equation for the region between the plates.
ii. What is the speed of the electrons at point x, where the potential is $V(x)$?
iii. Show that $\frac{d^2V}{dx^2} = \beta V^{-1/2}$ and find the constant β.

[6+(1+2+5)]

5. (a) A system with two degrees of freedom is described by the Hamiltonian

$$H = q_1p_1 - q_2p_2 - aq_1^2 + bq_2^2$$

where q_i, p_j obey canonical Poisson brackets and a, b are numerical constants. Show that

$$F_1 = \frac{p_1 - aq_1}{q_2} \quad , \qquad F_2 = q_1q_2$$

are constants of motion.

(b) A K-meson of rest energy 494 MeV decays into a muon of rest energy 106 MeV and a neutrino of zero rest energy. Find the kinetic energies of muon and neutrino in the rest frame of K-meson in which K-meson decays at rest.

(c) State the conservation laws that are violated for the following processes:

i. $\nu_\mu + n \longrightarrow e^- + p$

ii. $p + \bar{p} \longrightarrow \Lambda^0 + \Lambda^0$

[(3+3)+6+(1+1)]

6. (a) Suppose the density of states of a free electron gas in three dimensions gets increased by eight times.

i. Explain on physical grounds whether the Fermi temperature of the system increases or decreases.

ii. Find out the factor by which it increases or decreases.

(b) Consider a system of N non-interacting spins each with a magnetic moment of magnitude μ. The system is placed in an external uniform magnetic field $\vec{B}$.

i. Write down the Hamiltonian of the system

ii. Calculate the magnetization per spin at temperature T.

[(2+4)+(3+5)]

7. Consider a one-dimensional tight-binding periodic lattice with lattice constant a, on-site energy ϵ and nearest-neighbor hopping strength t.

i. Determine the energy dispersion (viz, E-k) relation.

ii. Find the wave vector k in terms of lattice constant and total number of lattice sites for an N-site lattice under periodic and finite boundary conditions.

iii. Calculate the energy band width for the periodic case.

[6+(3+3)+2]

8. (a) Consider a free scalar field.

i. Derive an expression for the Hamiltonian in terms of creation and annihilation operators.

ii. What is the energy of the vacuum?

(b) Consider a field theory where a Dirac electron ψ interacts with a charged scalar ϕ and a neutral η.

i. Give examples of simplest possible interaction terms in both cases, maintaining gauge invariance.

ii. Draw Feynman diagrams for the processes $\psi^- \ \phi^+ \ \longrightarrow \ \psi^- \ \phi^+$ and $\psi^- \ \eta \longrightarrow \psi^- \ \eta$.

[(4+3)+(4+3)]

Part III

Mathematics

Answer <u>any five</u> questions

1. (a) Synchronized clocks A and B are at rest in an inertial reference frame. Clock C is moving with velocity $(3/5)c$ along the line joining A and B, c being the velocity of light in vacuum. When C passes A, both the clocks A and C read $t = 0$. Answer the following questions.

   i. What time does C read when it reaches B?

   ii. How far apart are A and B in the inertial frame in which clock C is at rest?

   iii. In C's frame, when A passes C, what time does B read?

   (b) A solid circular cylinder of radius a rotating about its axis is placed gently with its axis horizontal on a rough plane, whose inclination to the horizon is α. Initially, the friction acts up the plane and the coefficient of friction is μ. Show that the cylinder will move upwards if $\mu > \tan\alpha$. Also show the time that elapses before rolling commence is

$$\frac{a\,\omega}{g(3\mu\cos\alpha - \sin\alpha)},$$

   where ω is the initial angular velocity of the cylinder.

[(2+2+2)+8]

2. (a) Evaluate $\displaystyle\int_C \frac{e^{2z}}{(z+1)^4}dz$ where C stands for the circle $|z| = 3$.

   (b) Use Fourier transform to solve the equation

$$\frac{\partial u(x,t)}{\partial t} = \kappa\frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = f(x), \quad |u(x,t)| < M,$$

   where κ, M are constants, $t > 0$ and $-\infty < x < \infty$.

[7+7]

3. (a) Show that

$$(1-x^2)P_n'(x) = xP_{n-1}(x) - nxP_n(x)$$

   where $P_n(x)$ denotes nth order Legendre Polynomial and the symbol $'$ indicates derivative with respect to x.

(b) Show that the general solution of the equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ can be written as $\phi(x, y) = f_1(x + iy) + f_2(x - iy)$ where $f_i(x, y)$ $(i = 1, 2)$ are twice differentiable arbitrary functions.

[8+6]

4. Let G_{25}^* denote the set of all integers between 1 and 25 which are coprime to 25. Define a binary operation $\odot$ on G_{25}^* as follows.

For $a, b \in G_{25}^*$, $a \odot b = c$ if $c \in G_{25}^*$ and $ab \equiv c \bmod 25$.

(a) Show that $(G_{25}^*, \odot)$ is a group.

(b) Hence, or otherwise, show that

$$13^{20} \equiv 1 \bmod 25.$$

[8+6]

5. (a) For $x \geq 0$, define $f(x) = \int_0^x e^{-t^2} dt$.

Show that for $x > 0$, $\quad f(x) > x\, e^{-x^2}$.

(b) Show that for any real or complex square matrix A, there is an α such that $\alpha I + A$ is non-singular, where I denotes the identity matrix.

[7+7]

6. (a) Find the values of r at which bifurcation occur and classify them as saddle-node, transcritical, pitchfork bifurcation of $\frac{dx}{dt} = rx - \frac{x}{1+x^2}$. Sketch the bifurcation diagram of equilibrium points x^* vs.r.

(b) Obtain the equation of motion for a particle falling vertically under the influence of gravity when frictional forces obtainable from a dissipation function $\frac{1}{2}kv^2$ are present. Integrate the equation to obtain the velocity as a function of time and show that the maximum possible velocity for fall from rest is $v = \frac{mg}{k}$.

[8+6]

7. Suppose f be a Borel-measurable function and μ be a σ -finite measure.

(a) Let $\int_{\mathbb{R}} f d\mu$ exists and let $\{B_n\}$ be a sequence of disjoint sets. Show that $\sum_{n=1}^{\infty} \int_{B_n} f d\mu$ is either an absolutely convergent series or it diverges to $+\infty$ (or $-\infty$).

(b) Let f be defined on reals and μ be the Lebesgue measure. If f is Riemann integrable then show that the set of continuity points of f is Borel measurable and its complement has measure zero.

[7+7]

8. (a) Let $\{X_t\}$ be a continuous time stochastic process. Write down the exact conditions under which $\{X_t\}$ can be said to be a standard Brownian motion.

(b) Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion. Define a process $\{X_t\}_{t\geq 0}$ as follows:

$$X_t = \begin{cases} tB_{\frac{1}{t}} & \text{for } t \neq 0 \\ 0 & \text{for } t = 0. \end{cases}$$

Show that $\{X_t\}$ is a standard Brownian motion.

(c) Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion. Define a process $\{M_t\}_{t\geq 0}$ as $M_t = B_t^2 - t$. Show that $\{M_t\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}$, where $\mathcal{F}_t = \sigma\{B_s : 0 \leq s \leq t\}$.

[2+7+5]