Sample Question 2019
TEST CODE: QEA

1. Answer the following questions.

   (a) Let $f : [0,1] \to [0,1]$ be a twice differentiable convex function. Define
   $$g(x) = [f(x)]^2 \ \forall\ x \in [0,1].$$
   Is g convex? Justify your answer. [4 marks]

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as
   $$f(x) = \frac{|x|}{2x} \ \forall\ x \in \mathbb{R} \setminus \{0\}.$$
   Can $f(0)$ be defined in a way such that f is continuous at 0? Justify your answer. [4 marks]

2. Let $f : [0,1] \to [0,1]$ be a strictly increasing function with $f(0) = 0$ and $f(1) = 1$. Define a function $g : [0,1] \to [0,1]$ such that $g(f(x)) = x$ for all $x \in [0,1]$.

   (a) Prove that $g'(y) = \frac{1}{f'(g(y))}$. [4 marks]

   (b) Is g strictly increasing? [2 marks]

   (c) If $f(x) = x^2$ for all $x \in [0,1]$, then find $g'(\frac{1}{2})$. [2 marks]

3. Three courses, A, B, and C are offered to a class of 100 students. The number of students in each course is: A is taken by 50 students, B by 80 students, and C by 40 students. Out of all the students, 20 have taken all the three courses. Every student has taken at least one course.

   (a) How many students have taken exactly two courses? [4 marks]

   (b) How many students have taken exactly one course? [4 marks]

4. Suppose A is any 2×2 matrix with a_{ij} denoting the entry of i-th row and j-th column. Suppose $a_{11}a_{22} = a_{12}a_{21}$.

   (a) Prove that one of the eigenvalues of A is zero. [4 marks]

   (b) Suppose A is the following matrix. Find all its eigenvalues. [4 marks]
   $$A = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$$

5. Consider the 3×3 matrix X shown below, where x, α are real numbers.

$$X = \begin{pmatrix} x & \alpha & 1 \\ x & x & 1 \\ 6 & 0 & 1 \end{pmatrix}$$

   (a) Show that if $x = \alpha$ then X does not have full rank. [2 marks]
   (b) Show that if $x = 6$ then X does not have full rank. [2 marks]
   (c) Show that if X does not have full rank, then either $x = 6$ or $x = \alpha$. [4 marks]

6. Answer the following questions.

   (a) Consider the following optimization problem:

$$\max_{x \in [0,\beta]} x(1-x),$$

   where $\beta \in [0,1]$. Let x^* be an optimal solution of the above optimization problem. For what values of β will we have $x^* = \beta$? [6 marks]

   (b) A firm is producing two products a and b. The market price (per unit) of a and b are respectively 3 and 2. The firm has resources to produce only 10 units of a and b together. Also, the quantity of a produced cannot exceed double the quantity of b produced. What is the revenue-maximizing production plan (i.e., how many units of a and b) of the firm? [6 marks]

7. Consider two random variables: (1) rain level (r), which can be either HIGH or LOW and (2) school attendance (s), which can be either 0 or 1. The joint probability distribution of (r, s) is as follows:

$$\text{Prob (HIGH,0)} = 0.2, \text{Prob (LOW,0)} = 0.3,$$

$$\text{Prob (HIGH,1)} = 0.35, \text{Prob (LOW,1)} = 0.15.$$

   (a) Find the marginal distribution of rain level. [4 marks]
   (b) Find the conditional probability of HIGH rain given that attendance is 0. [4 marks]
   (c) Are the two random variables independent? Justify your answer. [4 marks]

8. A slip of paper is given to person A, who marks it with either (+) or (−). The probability of her writing (+) is $\frac{1}{3}$. Then, the slip is passed sequentially to B, C, and D. Each of them either changes the sign on the slip with probability $\frac{2}{3}$ or leaves it as it is with probability $\frac{1}{3}$.

(a) Compute the probability that the final sign is (+) if A wrote (+). [3 marks]

(b) Compute the probability that the final sign is (+) if A wrote (−). [3 marks]

(c) Compute the probability that A wrote (+) if the final sign is (+). [6 marks]

9. There are n houses on a street numbered $h_1, \ldots, h_n$. Each house can either be painted BLUE or RED.

   (a) How many ways can the houses $h_1, \ldots, h_n$ be painted? [2 marks]

   (b) Suppose $n \geq 4$ and the houses are situated on n points on a circle. There is an additional constraint on painting the houses: exactly two houses need to be painted BLUE and they cannot be next to each other. How many ways can the houses $h_1, \ldots, h_n$ be painted under this new constraint? [5 marks]

   (c) How will your answer to the previous question change if the houses are located on n points on a line. [5 marks]

10. There are n biased coins $C_1, C_2, \ldots, C_n$. They are tossed sequentially - coin C_1 is tossed first, followed by C_2, ..., finally C_n. The probability of heads in coin C_k for each $k \in \{1, \ldots, n\}$ is $\frac{1}{2k+1}$. For every k, let P_k be the probability that we have odd number of heads after coins $C_1, \ldots, C_k$ are tossed.

    (a) Find P_1, P_2, P_3. [3 marks]

    (b) For every $k \in \{1, \ldots, n\}$, write P_k as a function of P_{k-1}. [5 marks]

    (c) What is the value of P_n? [4 marks]