Sample PQB 2019

PART I (STATISTICS / MATHEMATICS STREAM)

ATTENTION: Answer a total of **SIX** questions taking **at least TWO** from each Group **- S1 & S2.**

GROUP S1: Statistics

1. **(a)** Assume that the variables $X_1, X_2, \cdots, X_{2n}$ all have same variance σ^2, and correlation coefficient between X_i and X_j is given by ρ for $i \neq j$ and $i, j = 1,2, \cdots, 2n$. Prove that the correlation coefficient between Y and Z is

$$\frac{n\rho}{1 + (n-1)\rho},$$

where $Y = \sum_{i=1}^{n} X_i$ and $Z = \sum_{i=n+1}^{2n} X_i$.

(b) If the random variables X_1, X_2 and X_3 are such that

$$X_3 = aX_1 + bX_2,$$

where $a, b \neq 0$; then show that

(i) the partial correlation coefficients are numerically equal to unity;

(ii) $\rho_{13\cdot 2}$ has the same sign as a, $\rho_{23\cdot 1}$ has the same sign as b and $\rho_{12\cdot 3}$ has the opposite sign of a/b.

[10 + (5+5) = 20]

2. **(a)** Let $\bar{x}$ be the arithmetic mean of three variates x_1, x_2 and x_3. Show that

$$\frac{1}{2}(x_1 - x_2)^2 + \frac{1}{3 \cdot 2}(x_1 + x_2 - 2x_3)^2 = \sum_{i=1}^{3}(x_i - \bar{x})^2.$$

(b) Suppose, in a city 40% of the households have no motorized vehicle, 30% have one, 15% have two, 10% have three and 5% have four or more vehicles.

(i) Draw a graph to represent the distribution function of the number of vehicles per household.

(ii) Find the population median.

(iii) You have been asked to report a measure of location for the number of vehicles owned by households in the city. Do you think population median would represent the reality better than the population mean? Explain briefly.

(iv) Suppose you have surveyed 500 households in the city. Let n_i be the number of vehicles owned by the i-th household, $i = 1, 2, \cdots, 500$. Which diagram may be used to present this data? Explain briefly.

(v) A random sample of 10 households revealed the following values of the number of motorized vehicles owned by them: 4, 2, 1, 0, 0, 0, 0, 0, 0, 1. Draw a graph of the empirical distribution function.

[8 + (3+1+3+3+2) = 20]

3. (a) Suppose there are 200 students in a particular class. To study the age of the students, a random sample of size 20 with replacement is drawn and their ages are noted as $x_1, x_2, \cdots, x_{20}$.

It is given that $\sum_{i=1}^{20} x_i = 280$ and $\sum_{i=1}^{20} x_i^2 = 4000$. On the basis of these information, get a 90% confidence interval for the average age of the students in the class. [Given that $t_{.05,\,19} = 1.729$, $t_{.05,\,20} = 1.725$, and $z_{.05} = 1.645$.]

(b) Consider a process producing balls. The chance of each ball being defective is P (with $P \leq 0.02$). These balls are packed in N boxes of 100 each. The value of N is known to you. Let X be the number of boxes that do not contain any defective balls. Get an unbiased estimate of e^{-100P} on the basis of X. Also, give the standard error of the estimator.

[10 + 10 = 20]

4. A car tyre manufacturing company intends to test four different types of rubber improver for their effects on the wear of car tyres. Four test cars (four-wheeler) are available for the experiment. Each improver can be tested on one tyre only. The position of a tyre in a car may influence the wear. No interaction is expected between different sources of variation. The cars will be driven under normal conditions for six months, after which the wear of each tyre on each car will be measured. These values will then be suitably compared for selecting the best rubber improver amongst the four.

(i) Identify, with proper justification, a suitable design for the experiment and name it.

(ii) Give the layout of the design.

(iii) Identify the treatments.

(iv) How will you randomize your design?

(v) Write down the skeleton analysis of variance table, showing different sources of variation, degrees of freedom and the computational formulae for respective sum of squares.

[**Note:** If the identification of the design is not logical then no credit will be given for rest of the answers.]

[4 + 2 + 1 + 4 + 9 = 20]

5. **(a)** Consider a box containing M smart phones, where M is unknown to you. Each phone is labelled 1 through M. In order to estimate M, you are allowed to draw a random sample (without replacement) of three phones, and the observed labels are 18, 5 and 27. Write down the likelihood function and provide maximum likelihood estimate of M.

(b) Ajay and Bhaskar are playing a game to see whether they are equal in ability. The game is such that it cannot result in a draw. It was decided that the game will be played 7 times and if either person wins 6 times or more, they would accept that their abilities differ.

(i) Formulate the hypotheses H_0 and H_1.

(ii) Are H_0 and H_1 simple or composite? Explain briefly.

(iii) Write down any one point in the sample space.

(iv) List the points in the sample space that represent the critical region.

(v) Find the level of significance.

(vi) What is the equation of the power function?

(vii) Is the test unbiased? Explain briefly.

(viii) What assumptions did you make?

[5 + (1+2+1+1+2+3+2+3) = 20]

GROUP S2: Probability

6. **(a)** A five-digit number is formed using the digits 0, 1, 2, 3, 4 without repetition. Find the probability that it will be divisible by 4.

(b) A multiple-choice-test has 30 questions. Each question has four possible answers, exactly one of which is correct. A student knows the correct answer to 16 questions, and guesses at random for the remaining ones.

- **(i)** Obtain the probability distribution of the number of total questions this student gets right.
- **(ii)** Use **(i)** to derive mean and variance of the number of correct answers this student gets.

(c) In ten independent throws of a biased die, the probability that an even number will appear five times is twice the probability that an even number will appear four times. Find the probability that an even number will not appear in ten independent throws.

[7 + (5+3) + 5 = 20]

7. **(a)** Choose a point at random in the unit disk around origin. What is its expected distance from the origin?

(b) At a measuring station for air pollutants, the amount of ozone and carbon particles are recorded at noon every day. Let X be the concentration of carbon particles (μg/cc), and Y the concentration of ozone (ppm). Suppose that (X, Y) has a bivariate normal distribution, where X has mean 10.7 and variance 32.0, and Y has mean 0.1 and variance 0.02. The correlation coefficient between X and Y is 0.80. The ozone is considered unhealthy if it exceeds 0.35.
Suppose that the instrument used to measure ozone fails, so that we can measure only the carbon level. If this turns out to be 20.7 μg/cc, what is the

- **(i)** predicted ozone level, and
- **(ii)** probability that the ozone level is unhealthy.

[You may leave the answer in terms of standard normal distribution function $\Phi(\cdot)$.]

[10 + (4+6) = 20]

8. **(a)** There are $k + 1$ machines in a shop, all engaged in the mass production of an item. The ith machine produces defectives with a probability of i/k, $i = 0, 1, 2, \ldots, k$. A machine is randomly selected and then the items produced are repeatedly sampled. If first n products are all defectives then show that the conditional probability that $(n + 1)$st sampled product will also be defective is approximately equal to $(n+1)/(n+2)$ when k is large.

[**Hint:** Use the integral approximation

$$\frac{1}{k}\sum_{j=0}^{k}\left(\frac{j}{k}\right)^{n} \approx \int_{0}^{1} x^{n}\, dx = \frac{1}{n+1}$$

for large k.]

(b) Suppose that the number of telephone calls received by an operator between 11 a.m. to 12 noon follows Poisson distribution with rate λ=2 per hour. Find the probability that he will receive a total of one call in next three days during that time-period.

[15 + 5 = 20]

9. **(a)** Let $X_1, X_2, \cdots, X_n, n \geq 3$, be independent random variables with continuous uniform distribution over [0, 1].

(i) Find $P(X_1 < X_2 < X_3)$.

(ii) Define $Y = X_1 X_2 \cdots X_n$. Find $V(Y)$.

(b) Suppose that (X_1, X_2, X_3) has trinomial distribution with probability vector $(p_1, p_2, p_3), p_1 + p_2 + p_3 = 1, p_i > 0,\ i = 1, 2, 3$; and given that $X_1 + X_2 + X_3 = 20$. Find expectation and variance of X_2 given $X_3 = 8$.

[(5+5) + 10 = 20]

10. **(a)** Trucks arrive at a particular toll booth on the highway according to Poisson process with rate one per minute.

Assume that midnight 12 o'clock as the start-time of the process. Compute the

(i) probability that the 10th truck arrives two or more minutes after the arrival of 9th one,

(ii) probability that the 10th truck arrives after $00{:}20$ a.m.,

(iii) probability that there are two arrivals during $[01{:}01, 01{:}04]$ a.m. and three arrivals in the interval $[01{:}03, 01{:}05]$ a.m.

(b) Consider a system having two possible states: 0 or 1. Let X_n be the state of the system at time n for $n = 0, 1, 2, \cdots$ and

$$\mathrm{P}[X_{n+1} = 1 | X_n = 0] = \tfrac{1}{2}, \qquad \mathrm{P}[X_{n+1} = 0 | X_n = 1] = \tfrac{1}{3}.$$

(i) Write down the transition probability matrix.

(ii) Let $\mathrm{P}[X_0 = 1] = \alpha$ for $0 < \alpha < 1$. Find the value of $\mathrm{P}[X_3 = 0]$.

(iii) Assuming $\mathrm{P}[X_0 = 1] = \alpha$ for $0 < \alpha < 1$, find $\lim_{n\to\infty} \mathrm{P}[X_n = 1]$.

[(3+3+6) + (1+2+5) =20]

PART II (ENGINEERING STREAM)

ATTENTION: Answer a total of **SIX** questions taking **at least TWO** from each Group **- E1 & E2.**

GROUP E1: Mathematics

1. **(a)** How many numbers from 1 to 1000 are not divisible by 2, 3 and 5?

(b) Let

$$f(x) = \frac{4^x}{4^x + 2}.$$

Find

$$\sum_{r=1}^{2001} f\left(\frac{r}{2002}\right).$$

[8 + 12 =20]

2. **(a)** Find the greatest value of $a^2b^3c^4$ subject to $a + b + c = 18$, and $a, b, c > 0$.

(b) For what values of p, the following series converges:

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}.$$

[10 + 10 = 20]

3. **(a)** Show that the area lying in the region $x \geq 4$ and between the circle $x^2 + y^2 = 32$, the x-axis and the tangent drawn at the point (4,4) on the circle is $4(4 - \pi)$ square units.

(b) Consider the triangle ABC, where $\angle A = 30^0$ and $\angle B = 45^0$. Prove that $AB(AB - CA) = BC^2$.

[10 + 10 = 20]

4. **(a)** Find the equation of the curve which is such that the portion of the x-axis cut-off between origin and the tangent at any point is proportional to the ordinate of that point.

(b) For real a, b, c, x, y and z, show that

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}.$$

[8 + 12 = 20]

GROUP E2: Engineering & Technology

Engineering Mechanics and Thermodynamics

5. **(a)** With reference to the coordinate axes x and y, locate the centroid of the area of the cross-section shown in Figure 1.

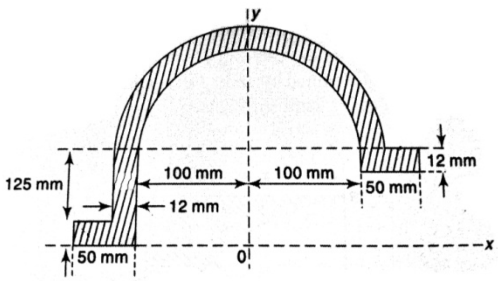

Figure 1

(b) A horizontal beam AB is hinged to a vertical wall at A and supported at its mid-point C by a tie rod CD as shown in

Figure 2. Find the tension in the tie rod and the reaction at A due to a vertical load applied at B.

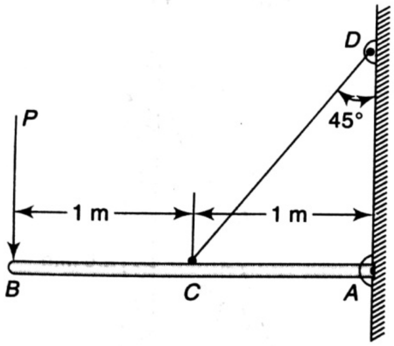

Figure 2

[10+10=20]

6. **(a)** A certain quantity of energy Q is transferred from a body at constant temperature T_1 to another body at constant temperature T_2 ($T_1 > T_2$). Show that the loss in the available energy is $T_0\, \Delta S_{Uni}$, where T_0 is the ambient temperature and ΔS_{Uni} is the total change in the entropy of the universe.

(b) An engineer claims to have developed a refrigerator that has a COP of 14 and that maintains the cold temperature at 1^0C while operating in a kitchen where the temperature is 25^0C. Is the claim true or false?

[10+10=20]

7. **(a)** The specific heat at constant volume for a particular substance is given by the relation

$$C_v = a + bT + cT^2$$

where a, b and c are constants. If it is desired to change the temperature of the substance from T_1 to T_2 at constant volume, then show that the relation to calculate the heat interaction is

$$q = a(T_2 - T_1) + \frac{b}{2}(T_2^2 - T_1^2) + \frac{c}{3}(T_2^3 - T_1^3)\,.$$

(b) A hammer weighing m_1 kg is used to drive horizontally a nail weighing m_2 kg into a timber post. The striking velocity of the hammer is 10 m/s and the nail is driven 25 mm at each blow. Find the resistance offered by the timber to the nail and the energy lost in driving it 75 mm inside the post.

[10+(5+5)=20]

Electrical and Electronics Engineering

8. **(a)** The open-circuit (OC) and short-circuit (SC) test data for a single phase 1 kVA, 100 V / 200 V, 50 Hz transformer are given below:

OC test from the low voltage side	100 V	0.5 A	30 W
SC test from the high voltage side	10 V	5 A	40 W

Calculate the following:

(i) magnetizing current;
(ii) core-loss current;
(iii) magnetizing reactance;
(iv) short-circuit impedance.

(b) A four-pole, wave-wound separately excited DC generator is being driven by a turbine at 1500 rpm speed. Compute

(i) the induced voltage in the armature, if the flux per pole is 12 mWb and total number of armature conductor is 500,
(ii) the line current and the terminal voltage when an electric heater of rating 250 V, 1250 W is connected across the armature. Consider the armature circuit resistance as 2 Ω, and
(iii) the line current and the terminal voltage when a 12.5 Ω resistor is connected across the heater.

[10+(4+4+2)=20]

9. **(a)** An *npn* transistor shown in Figure 3 below, is used in common-emitter amplifier mode with $\beta = 49$, $V_{CC} = 10$ V, and

$R_L = 2$ kΩ. If a 100 kΩ resistor R_B is connected between the collector and the base of the transistor, then calculate

(i) the quiescent collector current, and

(ii) the collector to emitter voltage drop between points A and B. Assume base to emitter voltage drop is 0.7 V.

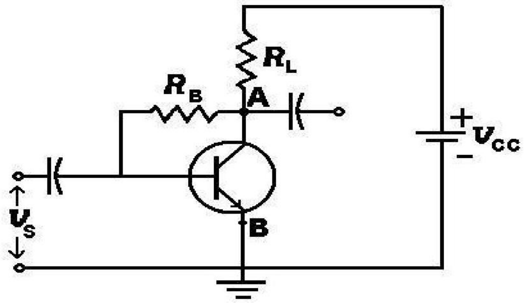

Figure 3

(b) For the circuit, given below in Figure 4, find the output voltage v_O for $v_S = 0.5$ V and $v_S = -1.0$ V. Assume that the diode D_2 is ideal, $R_1 = 2$ kΩ, $R_2 = 8$ kΩ, $R = 2$ kΩ, $R_3 = 8$ kΩ and $v_{S2} = 0.5$ V.

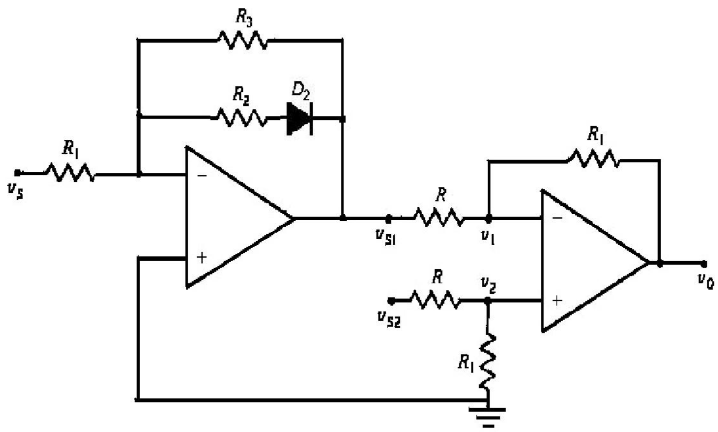

Figure 4

[(4+4)+(6+6)=20]

Engineering Drawing

10. **(a)** Sketch three views of the object shown in Figure 5.

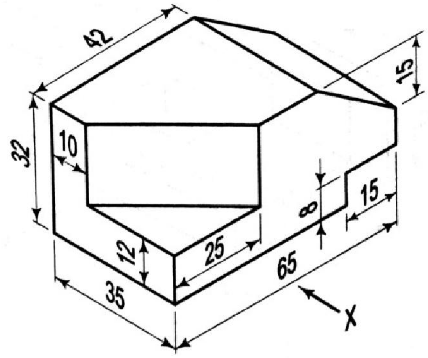

Figure 5

(b) Two views of an object are shown in Figure 6. Sketch the isometric view of the object.

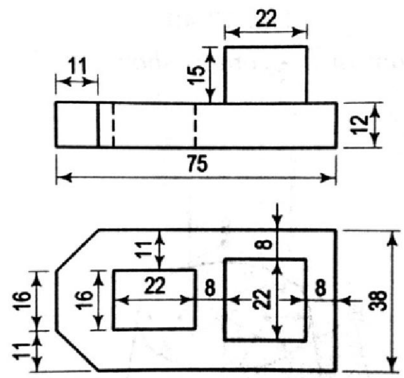

Figure 6

[10+10=20]