JRF Mathematics Examination
RM II

Solve any six questions.

1. Let $n \geq 3$ be a natural number. Prove that the three cycle $(1, 2, 3)$ is not a cube of any element in the symmetric group S_n.

2. Prove that any group of order 35 is cyclic.

3. (a) Let F be a field of odd characteristic and let K be a field extension of F of degree 2. Prove that there exists an $a \in F$ such that $K \cong F(\sqrt{a})$.

   (b) Give an example of a degree 2 extension K of a field F of characteristic two which is not obtained by attaching a square root of an element of F.

4. Let $r \leq n$ be natural numbers and let $\{v_1, \ldots, v_r\}$ and $\{w_1, \ldots, w_r\}$ be two linearly independent subsets of $\mathbb{R}^n$ such that

$$\langle v_i, v_j \rangle = \langle w_i, w_j \rangle \ \ \forall\ 1 \leq i, j \leq r,$$

   where $\langle\ ,\ \rangle$ denotes the standard inner product on $\mathbb{R}^n$. Prove that there exists an orthogonal operator T on $\mathbb{R}^n$ such that $T(v_i) = w_i$ for all $1 \leq i \leq r$.

5. Show that the space $C[0, 1]$ of real-valued continuous functions on the unit interval $[0, 1]$ with the sup norm

$$||f|| = \sup_{x \in [0,1]} |f(x)|$$

is not a Hilbert space with respect to any inner product.

6. Let $\mathcal{H}$ be a Hilbert space with complete orthonormal basis $\{e_n | n \in \mathbb{N}\}$. Let R be the shift operator on $\mathcal{H}$ defined by $R(e_n) = e_{n+1}$, and extended by linearity and continuity. Show that there is no $x \in \mathcal{H}$, such that $\text{Span}\{R^{2n}(x) | n \geq 0\}$ is dense in $\mathcal{H}$.

7. Let $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$. Find the eigenvalues of A and identify the set

$$E = \{a \in \mathbb{R} : \lim_{n\to\infty} a^n A^n \text{ exists and is different from zero}\}.$$

Note that convergence of a sequence of matrices is taken entrywise.

8. Let T be a bounded operator on a normed linear space X such that $T^2 = T$. Compute the inverse of $\lambda I - T$, for any complex number $\lambda \neq 0, 1$.

9. Prove that for any natural number n, there exist n consecutive integers each of which is divisible by a perfect square greater than one.

10. Let $\{a_1, \dots, a_{n^2+1}\}$ be a permutation of the set $\{1, 2, \dots, n^2+1\}$. Prove that the sequence $\{a_i\}$ contains a monotone subsequence of length $n+1$.

11. Let $p > 3$ be a prime number and $\mathbb{F}_p$ denote the finite field of order p. Prove that the polynomial $X^2 + X + 1$ is reducible in $\mathbb{F}_p[X]$ if and only if $p \equiv 1 (\text{mod } 3)$.