- $\mathbb{C}$ denotes the set of complex numbers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{Q}$ denotes the set of rational numbers.
- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{N}$ denotes the set of positive integers.

Q 1. Let G be a group of order n, H a subgroup of G of order m, $k = \frac{n}{m}$ and S_k the symmetric group on k symbols.

(a) Show that there is a nontrivial group homomorphism $\phi : G \to S_k$.

(b) Assuming $\frac{k!}{2} < n$, show that G has a nontrivial proper normal subgroup.

Q 2. Let G be the multiplicative group of complex numbers of modulus 1 and G_n (n a positive integer) the subgroup consisting of the n-th roots of unity. For positive integers m and n, show that G/G_m and G/G_n are isomorphic groups.

Q 3. Let $A = \mathbb{Q}[X]/(X^3 - 1)$.

(a) Prove that A is a direct product of two integral domains.

(b) Is the ring A isomorphic to $\mathbb{Q}[X]/(X^3+1)$? Justify your answer.

Q 4. Let X be an $n \times n$ complex matrix of rank 1 and I the $n \times n$ identity matrix. Show that

$$\det(I + X) = 1 + \mathrm{tr}(X),$$

where $\mathrm{tr}(X)$ denotes the trace of X and $\det(X)$ denotes the determinant of X.

Q 5. Let A and X be invertible complex matrices such that $XAX^{-1} = A^2$. Prove that there exists a natural number m such that each eigenvalue of A is an m-th root of unity.

Q 6. For $A = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, we define a sequence of vectors $\vec{v}_1 = \vec{v}, \vec{v}_{n+1} = A\vec{v}_n$ for $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} \vec{v}_n$ exists and is equal to $\begin{pmatrix} \frac{a+b}{2} \\ \frac{a+b}{2} \end{pmatrix}$.

Q 7. Let p_k be the k-th prime number. Show that there are infinitely many k such that

$$p_{k+1} - p_k > 2.$$

Q 8. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$ and P_n the orthogonal projection onto $\text{span}\{e_1, e_2, \dots, e_n\}, n \geq 1$. Prove that for all bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ and $h \in \mathcal{H}$, $P_nTP_nh \to Th$ as $n \to \infty$.

Q 9. Let S be a linear subspace of $C([0,1])$ which is closed in $L^2([0,1])$.
(a) Show that S is closed in $(C([0,1]), \|\cdot\|_\infty)$.
(b) Show that there exists $M > 0$ such that for all $f \in S$,

$$\|f\|_2 \leq \|f\|_\infty \leq M\|f\|_2.$$

Q 10. Let $\ell^p(\mathbb{Z}) = \left\{\{x_n\}_{n\in\mathbb{Z}} : x_n \in \mathbb{C} \text{ and } \lim_{N\to\infty} \sum_{n=-N}^{N} |x_n|^p < \infty\right\}$ for $p \in [1, \infty)$. Let $\{x_n\}_{n\in\mathbb{Z}}$ and $\{y_n\}_{n\in\mathbb{Z}}$ be any two elements of $\ell^1(\mathbb{Z})$.

(a) Prove that $\lim_{N\to\infty} \sum_{m=-N}^{N} x_{n-m}y_m$ exists for every $n \in \mathbb{Z}$.

(b) If $z_n = \lim_{N\to\infty} \sum_{m=-N}^{N} x_{n-m}y_m$, then prove that $\{z_n\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$.

(c) Conclude that $\{z_n\}_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ for all $p \in (1, \infty)$.