

$\mathbb{N} = \{1, 2, \dots\}$ .

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ .

$\mathbb{Q}$  = the set of rational numbers.

$\mathbb{R}$  = the set of real numbers.

$\mathbb{R}^n$  = the  $n$ -dimensional real space with the Euclidean topology.

$\mathbb{C}$  = the set of complex numbers.

$\mathbb{C}^n$  = the  $n$ -dimensional complex space with the Euclidean topology.

$M_n(\mathbb{R}), M_n(\mathbb{C})$  = the vector space of  $n \times n$  real or complex matrices, respectively.

$f', f''$  = the first and second derivatives of the function  $f$ , respectively.

$f^{(n)}$  = the  $n$ th. derivative of the function  $f$ .

$\int_C$  stands for the line integral over the curve  $C$ .

$I_n$  = the  $n \times n$  identity matrix.

$A^{-1}$  = the inverse of an invertible matrix  $A$ .

$S_n$  = the permutation group on  $n$  symbols.

$\hat{i} = (1, 0, 0), \hat{j} = (0, 1, 0)$  and  $\hat{k} = (0, 0, 1)$ .

$\ln x$  = the natural logarithm of  $x$  (to the base  $e$ ).

$|X|$  = the number of elements in a finite set  $X$ .

$\mathbb{Z}_n$  = the additive group of integers modulo  $n$ .

$\arctan(x)$  denotes the unique  $\theta \in (-\pi/2, \pi/2)$  such that  $\tan \theta = x$ .

All vector spaces are over the real or complex field, unless otherwise stated.

**SECTION – A**  
**MULTIPLE CHOICE QUESTIONS (MCQ)**

**Q. 1 – Q. 10 carry one mark each.**

Q. 1 Let  $0 < \alpha < 1$  be a real number. The number of differentiable functions  $y : [0, 1] \rightarrow [0, \infty)$ , having continuous derivative on  $[0, 1]$  and satisfying

$$\begin{aligned} y'(t) &= (y(t))^\alpha, \quad t \in [0, 1], \\ y(0) &= 0, \end{aligned}$$

is

- (A) exactly one. (B) exactly two.  
(C) finite but more than two. (D) infinite.

Q. 2 Let  $P : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that  $P(x) > 0$  for all  $x \in \mathbb{R}$ . Let  $y$  be a twice differentiable function on  $\mathbb{R}$  satisfying  $y''(x) + P(x)y'(x) - y(x) = 0$  for all  $x \in \mathbb{R}$ . Suppose that there exist two real numbers  $a, b$  ( $a < b$ ) such that  $y(a) = y(b) = 0$ . Then

- (A)  $y(x) = 0$  for all  $x \in [a, b]$ . (B)  $y(x) > 0$  for all  $x \in (a, b)$ .  
(C)  $y(x) < 0$  for all  $x \in (a, b)$ . (D)  $y(x)$  changes sign on  $(a, b)$ .

Q. 3 Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function satisfying  $f(x) = f(x + 1)$  for all  $x \in \mathbb{R}$ . Then

- (A)  $f$  is not necessarily bounded above.  
(B) there exists a unique  $x_0 \in \mathbb{R}$  such that  $f(x_0 + \pi) = f(x_0)$ .  
(C) there is no  $x_0 \in \mathbb{R}$  such that  $f(x_0 + \pi) = f(x_0)$ .  
(D) there exist infinitely many  $x_0 \in \mathbb{R}$  such that  $f(x_0 + \pi) = f(x_0)$ .



Q. 4 Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that for all  $x \in \mathbb{R}$ ,

$$\int_0^1 f(xt) dt = 0. \quad (*)$$

Then

- (A)  $f$  must be identically 0 on the whole of  $\mathbb{R}$ .
- (B) there is an  $f$  satisfying  $(*)$  that is identically 0 on  $(0, 1)$  but not identically 0 on the whole of  $\mathbb{R}$ .
- (C) there is an  $f$  satisfying  $(*)$  that takes both positive and negative values.
- (D) there is an  $f$  satisfying  $(*)$  that is 0 at infinitely many points, but is not identically zero.

Q. 5 Let  $p$  and  $t$  be positive real numbers. Let  $D_t$  be the closed disc of radius  $t$  centered at  $(0, 0)$ , i.e.,  $D_t = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq t^2\}$ . Define

$$I(p, t) = \iint_{D_t} \frac{dxdy}{(p^2 + x^2 + y^2)^p}.$$

Then  $\lim_{t \rightarrow \infty} I(p, t)$  is finite

- (A) only if  $p > 1$ .
- (B) only if  $p = 1$ .
- (C) only if  $p < 1$ .
- (D) for no value of  $p$ .

Q. 6 How many elements of the group  $\mathbb{Z}_{50}$  have order 10?

- (A) 10
- (B) 4
- (C) 5
- (D) 8



Q. 7 For every  $n \in \mathbb{N}$ , let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  be a function. From the given choices, pick the statement that is the negation of

“For every  $x \in \mathbb{R}$  and for every real number  $\epsilon > 0$ , there exists an integer  $N > 0$  such that  $\sum_{i=1}^p |f_{N+i}(x)| < \epsilon$  for every integer  $p > 0$ .”

- (A) For every  $x \in \mathbb{R}$  and for every real number  $\epsilon > 0$ , there does not exist any integer  $N > 0$  such that  $\sum_{i=1}^p |f_{N+i}(x)| < \epsilon$  for every integer  $p > 0$ .
- (B) For every  $x \in \mathbb{R}$  and for every real number  $\epsilon > 0$ , there exists an integer  $N > 0$  such that  $\sum_{i=1}^p |f_{N+i}(x)| \geq \epsilon$  for some integer  $p > 0$ .
- (C) There exists  $x \in \mathbb{R}$  and there exists a real number  $\epsilon > 0$  such that for every integer  $N > 0$ , there exists an integer  $p > 0$  for which the inequality  $\sum_{i=1}^p |f_{N+i}(x)| \geq \epsilon$  holds.
- (D) There exists  $x \in \mathbb{R}$  and there exists a real number  $\epsilon > 0$  such that for every integer  $N > 0$  and for every integer  $p > 0$  the inequality  $\sum_{i=1}^p |f_{N+i}(x)| \geq \epsilon$  holds.

Q. 8 Which one of the following subsets of  $\mathbb{R}$  has a non-empty interior?

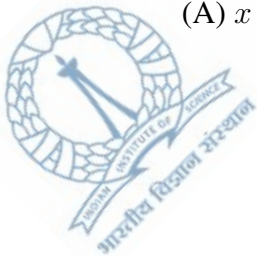
- (A) The set of all irrational numbers in  $\mathbb{R}$ .
- (B) The set  $\{a \in \mathbb{R} : \sin(a) = 1\}$ .
- (C) The set  $\{b \in \mathbb{R} : x^2 + bx + 1 = 0 \text{ has distinct roots}\}$ .
- (D) The set of all rational numbers in  $\mathbb{R}$ .

Q. 9 For an integer  $k \geq 0$ , let  $P_k$  denote the vector space of all real polynomials in one variable of degree less than or equal to  $k$ . Define a linear transformation  $T : P_2 \rightarrow P_3$  by

$$Tf(x) = f''(x) + xf(x).$$

Which one of the following polynomials is not in the range of  $T$ ?

- (A)  $x + x^2$                       (B)  $x^2 + x^3 + 2$                       (C)  $x + x^3 + 2$                       (D)  $x + 1$



Q. 10 Let  $n > 1$  be an integer. Consider the following two statements for an arbitrary  $n \times n$  matrix  $A$  with complex entries.

- I. If  $A^k = I_n$  for some integer  $k \geq 1$ , then all the eigenvalues of  $A$  are  $k^{\text{th}}$  roots of unity.  
II. If, for some integer  $k \geq 1$ , all the eigenvalues of  $A$  are  $k^{\text{th}}$  roots of unity, then  $A^k = I_n$ .

Then

(A) both I and II are TRUE.

(B) I is TRUE but II is FALSE.

(C) I is FALSE but II is TRUE.

(D) neither I nor II is TRUE.



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**Q. 11 – Q. 30 carry two marks each.**

Q. 11 Let  $M_n(\mathbb{R})$  be the real vector space of all  $n \times n$  matrices with real entries,  $n \geq 2$ . Let  $A \in M_n(\mathbb{R})$ . Consider the subspace  $W$  of  $M_n(\mathbb{R})$  spanned by  $\{I_n, A, A^2, \dots\}$ . Then the dimension of  $W$  over  $\mathbb{R}$  is necessarily

- (A)  $\infty$ .                      (B)  $n^2$ .                      (C)  $n$ .                      (D) at most  $n$ .

Q. 12 Let  $y$  be the solution of

$$(1+x)y''(x) + y'(x) - \frac{1}{1+x}y(x) = 0, \quad x \in (-1, \infty),$$

$$y(0) = 1, \quad y'(0) = 0.$$

Then

- (A)  $y$  is bounded on  $(0, \infty)$ .                      (B)  $y$  is bounded on  $(-1, 0]$ .  
 (C)  $y(x) \geq 2$  on  $(-1, \infty)$ .                      (D)  $y$  attains its minimum at  $x = 0$ .

Q. 13 Consider the surface  $S = \{(x, y, xy) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\}$ . Let  $\vec{F} = y\hat{i} + x\hat{j} + \hat{k}$ . If  $\hat{n}$  is the continuous unit normal field to the surface  $S$  with positive  $z$ -component, then

$$\iint_S \vec{F} \cdot \hat{n} \, dS$$

equals

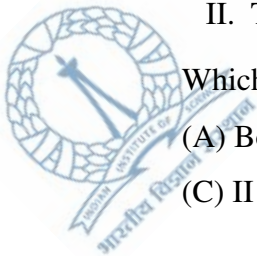
- (A)  $\frac{\pi}{4}$ .                      (B)  $\frac{\pi}{2}$ .                      (C)  $\pi$ .                      (D)  $2\pi$ .

Q. 14 Consider the following statements.

- I. The group  $(\mathbb{Q}, +)$  has no proper subgroup of finite index.  
 II. The group  $(\mathbb{C} \setminus \{0\}, \cdot)$  has no proper subgroup of finite index.

Which one of the following statements is true?

- (A) Both I and II are TRUE.                      (B) I is TRUE but II is FALSE.  
 (C) II is TRUE but I is FALSE.                      (D) Neither I nor II is TRUE.



Q. 15 Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a bijective map such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^2} < +\infty.$$

The number of such bijective maps is

- (A) exactly one. (B) zero.  
(C) finite but more than one. (D) infinite.

Q. 16 Define

$$S = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right).$$

Then

- (A)  $S = 1/2$ . (B)  $S = 1/4$ . (C)  $S = 1$ . (D)  $S = 3/4$ .

Q. 17 Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an infinitely differentiable function such that for all  $a, b \in \mathbb{R}$  with  $a < b$ ,

$$\frac{f(b) - f(a)}{b - a} = f'\left(\frac{a + b}{2}\right).$$

Then

- (A)  $f$  must be a polynomial of degree less than or equal to 2.  
(B)  $f$  must be a polynomial of degree greater than 2.  
(C)  $f$  is not a polynomial.  
(D)  $f$  must be a linear polynomial.



Q. 18 Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{0\}, \\ 1 - \frac{1}{p} & \text{if } x = \frac{n}{p}, n \in \mathbb{Z} \setminus \{0\}, p \in \mathbb{N} \text{ and } \gcd(n, p) = 1. \end{cases}$$

Then

- (A) all  $x \in \mathbb{Q} \setminus \{0\}$  are strict local minima for  $f$ .
- (B)  $f$  is continuous at all  $x \in \mathbb{Q}$ .
- (C)  $f$  is not continuous at all  $x \in \mathbb{R} \setminus \mathbb{Q}$ .
- (D)  $f$  is not continuous at  $x = 0$ .

Q. 19 Consider the family of curves  $x^2 - y^2 = ky$  with parameter  $k \in \mathbb{R}$ . The equation of the orthogonal trajectory to this family passing through  $(1, 1)$  is given by

- (A)  $x^3 + 3xy^2 = 4$ .
- (B)  $x^2 + 2xy = 3$ .
- (C)  $y^2 + 2x^2y = 3$ .
- (D)  $x^3 + 2xy^2 = 3$ .

Q. 20 Which one of the following statements is true?

- (A) Exactly half of the elements in any even order subgroup of  $S_5$  must be even permutations.
- (B) Any abelian subgroup of  $S_5$  is trivial.
- (C) There exists a cyclic subgroup of  $S_5$  of order 6.
- (D) There exists a normal subgroup of  $S_5$  of index 7.

Q. 21 Let  $f : [0, 1] \rightarrow [0, \infty)$  be a continuous function such that

$$(f(t))^2 < 1 + 2 \int_0^t f(s) ds, \text{ for all } t \in [0, 1].$$

Then

- (A)  $f(t) < 1 + t$  for all  $t \in [0, 1]$ .
- (B)  $f(t) > 1 + t$  for all  $t \in [0, 1]$ .
- (C)  $f(t) = 1 + t$  for all  $t \in [0, 1]$ .
- (D)  $f(t) < 1 + \frac{t}{2}$  for all  $t \in [0, 1]$ .





Q. 22 Let  $A$  be an  $n \times n$  invertible matrix and  $C$  be an  $n \times n$  nilpotent matrix. If  $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$  is a  $2n \times 2n$  matrix (each  $X_{ij}$  being  $n \times n$ ) that commutes with the  $2n \times 2n$  matrix  $B = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$ , then

- (A)  $X_{11}$  and  $X_{22}$  are necessarily zero matrices.
- (B)  $X_{12}$  and  $X_{21}$  are necessarily zero matrices.
- (C)  $X_{11}$  and  $X_{21}$  are necessarily zero matrices.
- (D)  $X_{12}$  and  $X_{22}$  are necessarily zero matrices.

Q. 23 Let  $D \subseteq \mathbb{R}^2$  be defined by  $D = \mathbb{R}^2 \setminus \{(x, 0) : x \in \mathbb{R}\}$ . Consider the function  $f : D \rightarrow \mathbb{R}$  defined by

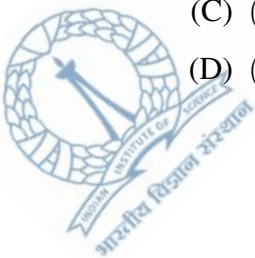
$$f(x, y) = x \sin \frac{1}{y}.$$

Then

- (A)  $f$  is a discontinuous function on  $D$ .
- (B)  $f$  is a continuous function on  $D$  and cannot be extended continuously to any point outside  $D$ .
- (C)  $f$  is a continuous function on  $D$  and can be extended continuously to  $D \cup \{(0, 0)\}$ .
- (D)  $f$  is a continuous function on  $D$  and can be extended continuously to the whole of  $\mathbb{R}^2$ .

Q. 24 Which one of the following statements is true?

- (A)  $(\mathbb{Z}, +)$  is isomorphic to  $(\mathbb{R}, +)$ .
- (B)  $(\mathbb{Z}, +)$  is isomorphic to  $(\mathbb{Q}, +)$ .
- (C)  $(\mathbb{Q}/\mathbb{Z}, +)$  is isomorphic to  $(\mathbb{Q}/2\mathbb{Z}, +)$ .
- (D)  $(\mathbb{Q}/\mathbb{Z}, +)$  is isomorphic to  $(\mathbb{Q}, +)$ .



Q. 25 Let  $y$  be a twice differentiable function on  $\mathbb{R}$  satisfying

$$y''(x) = 2 + e^{-|x|}, \quad x \in \mathbb{R},$$

$$y(0) = -1, \quad y'(0) = 0.$$

Then

- (A)  $y = 0$  has exactly one root.
- (B)  $y = 0$  has exactly two roots.
- (C)  $y = 0$  has more than two roots.
- (D) there exists an  $x_0 \in \mathbb{R}$  such that  $y(x_0) \geq y(x)$  for all  $x \in \mathbb{R}$ .

Q. 26 Let  $f : [0, 1] \rightarrow [0, 1]$  be a non-constant continuous function such that  $f \circ f = f$ . Define

$$E_f = \{x \in [0, 1] : f(x) = x\}.$$

Then

- (A)  $E_f$  is neither open nor closed.
- (B)  $E_f$  is an interval.
- (C)  $E_f$  is empty.
- (D)  $E_f$  need not be an interval.

Q. 27 Let  $g$  be an element of  $S_7$  such that  $g$  commutes with the element  $(2, 6, 4, 3)$ . The number of such  $g$  is

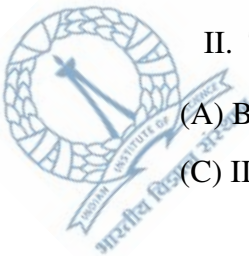
- (A) 6.
- (B) 4.
- (C) 24.
- (D) 48.

Q. 28 Let  $G$  be a finite abelian group of odd order. Consider the following two statements:

I. The map  $f : G \rightarrow G$  defined by  $f(g) = g^2$  is a group isomorphism.

II. The product  $\prod_{g \in G} g = e$ .

- (A) Both I and II are TRUE.
- (B) I is TRUE but II is FALSE.
- (C) II is TRUE but I is FALSE.
- (D) Neither I nor II is TRUE.



Q. 29 Let  $n \geq 2$  be an integer. Let  $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be the linear transformation defined by

$$A(z_1, z_2, \dots, z_n) = (z_n, z_1, z_2, \dots, z_{n-1}).$$

Which one of the following statements is true for every  $n \geq 2$ ?

- (A)  $A$  is nilpotent. (B) All eigenvalues of  $A$  are of modulus 1.  
(C) Every eigenvalue of  $A$  is either 0 or 1. (D)  $A$  is singular.

Q. 30 Consider the two series

$$\text{I. } \sum_{n=1}^{\infty} \frac{1}{n^{1+(1/n)}} \quad \text{and} \quad \text{II. } \sum_{n=1}^{\infty} \frac{1}{n^{2-n^{1/n}}}$$

Which one of the following holds?

- (A) Both I and II converge. (B) Both I and II diverge.  
(C) I converges and II diverges. (D) I diverges and II converges.



**SECTION – B**  
**MULTIPLE SELECT QUESTIONS (MSQ)**

**Q. 31 – Q. 40 carry two marks each.**

Q. 31 Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function with the property that for every  $y \in \mathbb{R}$ , the value of the expression

$$\sup_{x \in \mathbb{R}} [xy - f(x)]$$

is finite. Define  $g(y) = \sup_{x \in \mathbb{R}} [xy - f(x)]$  for  $y \in \mathbb{R}$ . Then

(A)  $g$  is even if  $f$  is even.

(B)  $f$  must satisfy  $\lim_{|x| \rightarrow \infty} \frac{f(x)}{|x|} = +\infty$ .

(C)  $g$  is odd if  $f$  is even.

(D)  $f$  must satisfy  $\lim_{|x| \rightarrow \infty} \frac{f(x)}{|x|} = -\infty$ .

Q. 32 Consider the equation

$$x^{2021} + x^{2020} + \dots + x - 1 = 0.$$

Then

(A) all real roots are positive.

(B) exactly one real root is positive.

(C) exactly one real root is negative.

(D) no real root is positive.

Q. 33 Let  $D = \mathbb{R}^2 \setminus \{(0, 0)\}$ . Consider the two functions  $u, v : D \rightarrow \mathbb{R}$  defined by

$$u(x, y) = x^2 - y^2 \text{ and } v(x, y) = xy.$$

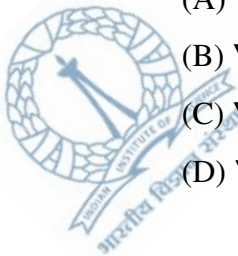
Consider the gradients  $\nabla u$  and  $\nabla v$  of the functions  $u$  and  $v$ , respectively. Then

(A)  $\nabla u$  and  $\nabla v$  are parallel at each point  $(x, y)$  of  $D$ .

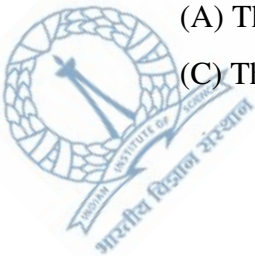
(B)  $\nabla u$  and  $\nabla v$  are perpendicular at each point  $(x, y)$  of  $D$ .

(C)  $\nabla u$  and  $\nabla v$  do not exist at some points  $(x, y)$  of  $D$ .

(D)  $\nabla u$  and  $\nabla v$  at each point  $(x, y)$  of  $D$  span  $\mathbb{R}^2$ .



- Q. 34 Consider the two functions  $f(x, y) = x + y$  and  $g(x, y) = xy - 16$  defined on  $\mathbb{R}^2$ . Then
- (A) the function  $f$  has no global extreme value subject to the condition  $g = 0$ .
  - (B) the function  $f$  attains global extreme values at  $(4, 4)$  and  $(-4, -4)$  subject to the condition  $g = 0$ .
  - (C) the function  $g$  has no global extreme value subject to the condition  $f = 0$ .
  - (D) the function  $g$  has a global extreme value at  $(0, 0)$  subject to the condition  $f = 0$ .
- Q. 35 Let  $f : (a, b) \rightarrow \mathbb{R}$  be a differentiable function on  $(a, b)$ . Which of the following statements is/are true?
- (A)  $f' > 0$  in  $(a, b)$  implies that  $f$  is increasing in  $(a, b)$ .
  - (B)  $f$  is increasing in  $(a, b)$  implies that  $f' > 0$  in  $(a, b)$ .
  - (C) If  $f'(x_0) > 0$  for some  $x_0 \in (a, b)$ , then there exists a  $\delta > 0$  such that  $f(x) > f(x_0)$  for all  $x \in (x_0, x_0 + \delta)$ .
  - (D) If  $f'(x_0) > 0$  for some  $x_0 \in (a, b)$ , then  $f$  is increasing in a neighbourhood of  $x_0$ .
- Q. 36 Let  $G$  be a finite group of order 28. Assume that  $G$  contains a subgroup of order 7. Which of the following statements is/are true?
- (A)  $G$  contains a unique subgroup of order 7.
  - (B)  $G$  contains a normal subgroup of order 7.
  - (C)  $G$  contains no normal subgroup of order 7.
  - (D)  $G$  contains at least two subgroups of order 7.
- Q. 37 Which of the following subsets of  $\mathbb{R}$  is/are connected?
- (A) The set  $\{x \in \mathbb{R} : x \text{ is irrational}\}$ .
  - (B) The set  $\{x \in \mathbb{R} : x^3 - 1 \geq 0\}$ .
  - (C) The set  $\{x \in \mathbb{R} : x^3 + x + 1 \geq 0\}$ .
  - (D) The set  $\{x \in \mathbb{R} : x^3 - 2x + 1 \geq 0\}$ .



Q. 38 Consider the four functions from  $\mathbb{R}$  to  $\mathbb{R}$ :

$$f_1(x) = x^4 + 3x^3 + 7x + 1, \quad f_2(x) = x^3 + 3x^2 + 4x, \quad f_3(x) = \arctan(x)$$

and

$$f_4(x) = \begin{cases} x & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Which of the following subsets of  $\mathbb{R}$  are open?

- (A) The range of  $f_1$ . (B) The range of  $f_2$ .  
 (C) The range of  $f_3$ . (D) The range of  $f_4$ .

Q. 39 Let  $V$  be a finite dimensional vector space and  $T : V \rightarrow V$  be a linear transformation. Let  $\mathcal{R}(T)$  denote the range of  $T$  and  $\mathcal{N}(T)$  denote the null space  $\{v \in V : Tv = 0\}$  of  $T$ . If  $\text{rank}(T) = \text{rank}(T^2)$ , then which of the following is/are necessarily true?

- (A)  $\mathcal{N}(T) = \mathcal{N}(T^2)$ . (B)  $\mathcal{R}(T) = \mathcal{R}(T^2)$ .  
 (C)  $\mathcal{N}(T) \cap \mathcal{R}(T) = \{0\}$ . (D)  $\mathcal{N}(T) = \{0\}$ .

Q. 40 Let  $m > 1$  and  $n > 1$  be integers. Let  $A$  be an  $m \times n$  matrix such that for some  $m \times 1$  matrix  $b_1$ , the equation  $Ax = b_1$  has infinitely many solutions. Let  $b_2$  denote an  $m \times 1$  matrix different from  $b_1$ . Then  $Ax = b_2$  has

- (A) infinitely many solutions for some  $b_2$ . (B) a unique solution for some  $b_2$ .  
 (C) no solution for some  $b_2$ . (D) finitely many solutions for some  $b_2$ .



**SECTION – C**  
**NUMERICAL ANSWER TYPE (NAT)**

**Q. 41 – Q. 50 carry one mark each.**

Q. 41 The number of cycles of length 4 in  $S_6$  is \_\_\_\_\_.

Q. 42 The value of

$$\lim_{n \rightarrow \infty} \left( 3^n + 5^n + 7^n \right)^{\frac{1}{n}}$$

is \_\_\_\_\_.

Q. 43 Let  $B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$  and define  $u(x, y, z) = \sin((1 - x^2 - y^2 - z^2)^2)$  for  $(x, y, z) \in B$ . Then the value of

$$\iiint_B \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) dx dy dz$$

is \_\_\_\_\_.

Q. 44 Consider the subset  $S = \{(x, y) : x^2 + y^2 > 0\}$  of  $\mathbb{R}^2$ . Let

$$P(x, y) = \frac{y}{x^2 + y^2} \text{ and } Q(x, y) = -\frac{x}{x^2 + y^2}$$

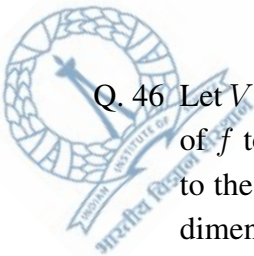
for  $(x, y) \in S$ . If  $C$  denotes the unit circle traversed in the counter-clockwise direction, then the value of

$$\frac{1}{\pi} \int_C (P dx + Q dy)$$

is \_\_\_\_\_.

Q. 45 Consider the set  $A = \{a \in \mathbb{R} : x^2 = a(a+1)(a+2) \text{ has a real root}\}$ . The number of connected components of  $A$  is \_\_\_\_\_.

Q. 46 Let  $V$  be the real vector space of all continuous functions  $f : [0, 2] \rightarrow \mathbb{R}$  such that the restriction of  $f$  to the interval  $[0, 1]$  is a polynomial of degree less than or equal to 2, the restriction of  $f$  to the interval  $[1, 2]$  is a polynomial of degree less than or equal to 3 and  $f(0) = 0$ . Then the dimension of  $V$  is equal to \_\_\_\_\_.



Q. 47 The number of group homomorphisms from the group  $\mathbb{Z}_4$  to the group  $S_3$  is \_\_\_\_\_.

Q. 48 Let  $y : \left(\frac{9}{10}, 3\right) \rightarrow \mathbb{R}$  be a differentiable function satisfying

$$(x - 2y) \frac{dy}{dx} + (2x + y) = 0, \quad x \in \left(\frac{9}{10}, 3\right), \quad \text{and } y(1) = 1.$$

Then  $y(2)$  equals \_\_\_\_\_.

Q. 49 Let  $\vec{F} = (y + 1)e^y \cos(x)\hat{i} + (y + 2)e^y \sin(x)\hat{j}$  be a vector field in  $\mathbb{R}^2$  and  $C$  be a continuously differentiable path with the starting point  $(0, 1)$  and the end point  $\left(\frac{\pi}{2}, 0\right)$ . Then

$$\int_C \vec{F} \cdot d\vec{r}$$

equals \_\_\_\_\_.

Q. 50 The value of

$$\frac{\pi}{2} \lim_{n \rightarrow \infty} \cos\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{8}\right) \cdots \cos\left(\frac{\pi}{2^{n+1}}\right)$$

is \_\_\_\_\_.





**Q. 51 – Q. 60 carry two marks each.**

Q. 51 The number of elements of order two in the group  $S_4$  is equal to \_\_\_\_\_.

Q. 52 The least possible value of  $k$ , accurate up to two decimal places, for which the following problem

$$\begin{aligned} y''(t) + 2y'(t) + ky(t) &= 0, t \in \mathbb{R}, \\ y(0) = 0, y(1) = 0, y(1/2) &= 1, \end{aligned}$$

has a solution is \_\_\_\_\_.

Q. 53 Consider those continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  that have the property that given any  $x \in \mathbb{R}$ ,

$$f(x) \in \mathbb{Q} \text{ if and only if } f(x+1) \in \mathbb{R} \setminus \mathbb{Q}.$$

The number of such functions is \_\_\_\_\_.

Q. 54 The largest positive number  $a$  such that

$$\int_0^5 f(x) dx + \int_0^3 f^{-1}(x) dx \geq a$$

for every strictly increasing surjective continuous function  $f : [0, \infty) \rightarrow [0, \infty)$  is \_\_\_\_\_.

Q. 55 Define the sequence

$$s_n = \begin{cases} \frac{1}{2^n} \sum_{j=0}^{n-2} 2^{2j} & \text{if } n > 0 \text{ is even,} \\ \frac{1}{2^n} \sum_{j=0}^{n-1} 2^{2j} & \text{if } n > 0 \text{ is odd.} \end{cases}$$

Define  $\sigma_m = \frac{1}{m} \sum_{n=1}^m s_n$ . The number of limit points of the sequence  $\{\sigma_m\}$  is \_\_\_\_\_.



Q. 56 The determinant of the matrix

$$\begin{pmatrix} 2021 & 2020 & 2020 & 2020 \\ 2021 & 2021 & 2020 & 2020 \\ 2021 & 2021 & 2021 & 2020 \\ 2021 & 2021 & 2021 & 2021 \end{pmatrix}$$

is \_\_\_\_\_.

Q. 57 The value of

$$\lim_{n \rightarrow \infty} \int_0^1 e^{x^2} \sin(nx) dx$$

is \_\_\_\_\_.

Q. 58 Let  $S$  be the surface defined by

$$\{(x, y, z) \in \mathbb{R}^3 : z = 1 - x^2 - y^2, z \geq 0\}.$$

Let  $\vec{F} = -y\hat{i} + (x-1)\hat{j} + z^2\hat{k}$  and  $\hat{n}$  be the continuous unit normal field to the surface  $S$  with positive  $z$ -component. Then the value of

$$\frac{1}{\pi} \iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$$

is \_\_\_\_\_.

Q. 59 Let  $A = \begin{pmatrix} 2 & -1 & 3 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{pmatrix}$ . Then the largest eigenvalue of  $A$  is \_\_\_\_\_.

Q. 60 Let  $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ . Consider the linear map  $T_A$  from the real vector space  $M_4(\mathbb{R})$  to itself defined by  $T_A(X) = AX - XA$ , for all  $X \in M_4(\mathbb{R})$ . The dimension of the range of  $T_A$  is \_\_\_\_\_.

**END OF THE QUESTION PAPER**