JEE-Main-26-06-2022-Shift-2 (Memory Based)

MATHEMATICS

Question: Find $\cos^{-1}\left(\frac{3}{10}\cos\left(\tan^{-1}\frac{4}{3}\right)+\frac{2}{5}\sin\left(\tan^{-1}\frac{4}{3}\right)\right)$

Options:
(a) 0
(b) $\frac{\pi}{3}$
(c) $\frac{\pi}{6}$
(d) $\frac{\pi}{2}\cos\frac{\pi}{4}$

Answer: (b)

Solution:

$$\cos^{-1}\left(\frac{3}{10}\times\frac{3}{5}+\frac{2}{5}\times\frac{4}{5}\right)$$

$$\cos^{-1}\left(\frac{9}{50}+\frac{8}{25}\right)$$

$$\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$$

Question: If function $f(x)=x-1$ and $g(x)=\frac{x^2}{x^2+1}$ then fog is:

Options:
(a) One-one and onto
(b) One-one but not onto
(c) Onto but not one-one
(d) Neither one-one nor onto

Answer: (d)

Solution:

$$f(g(x))=g(x)=\frac{x^2}{x^2+1}-1$$

$$=\frac{-1}{x^2+1}$$

$f(g(x))$ is even function $\Rightarrow$ many-one

$f(g(x))$ is always negative $\Rightarrow$ into

$\Rightarrow f\left(g\left(x\right)\right)$ is neither one-one nor onto

Question: $\lim\limits_{x\to 0}\frac{\cos\left(\sin x\right)-\cos x}{x^4}=?$

Answer: $\frac{1}{6}$

Solution:

$$\lim_{x\to 0}\frac{\cos\left(\sin x\right)-\cos x}{x^4}$$

$$=\lim_{x\to 0}\frac{2\sin\left(\frac{x+\sin x}{2}\right)\sin\left(\frac{x-\sin x}{2}\right)}{x^4}$$

As $x\to 0\Rightarrow \sin x\to 0$

$$=\lim_{x\to 0}\left(\frac{x+\sin x}{2}\right)\left(\frac{x-\sin x}{2}\right)\frac{2\sin\left(\frac{x+\sin x}{2}\right)\sin\left(\frac{x-\sin x}{2}\right)}{x^4\left(\frac{x+\sin x}{2}\right)\left(\frac{x-\sin x}{2}\right)}$$

$$=\lim_{x\to 0}\frac{x^2-\sin^2 x}{2x^4}$$

It is of the form $\frac{0}{0}$, so applying L-Hospital's rule

$$=\lim_{x\to 0}\frac{2x-2\sin x\cos x}{8x^3}$$

$$=\lim_{x\to 0}\frac{2x-\sin 2x}{8x^3}$$

Again applying L-Hospital's rule

$$=\lim_{x\to 0}\frac{2-2\cos 2x}{24x^2}$$

Again applying L-Hospital's rule

$$=\lim_{x\to 0}\frac{4\sin 2x}{48x}$$

Again applying L-Hospital's rule

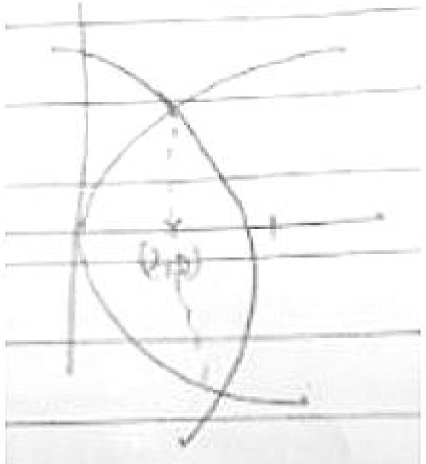
$$=\lim_{x\to 0}\frac{8\cos 2x}{48}$$

$$=\frac{1}{6}$$

Question: Find area bounded by $y^2=8x$ & $y^2=16\left(3-x\right)$

Answer: 16.00

Solution:



$y^2 = 8x;\ y^2 = 16(3-x)$

$8x = 16(3-x)$

$\Rightarrow x = 6-2x$

$\Rightarrow x = 2,\ y = \pm 4$

$$\text{Area } = 2\left[\int_0^4 \left(3-\frac{y^2}{16}\right)-\left(\frac{y^2}{8}\right)dy\right]$$

$$= 2\left[\int_0^4 3-\frac{3y^2}{16}dy\right]$$

$$= 6\left[4-\frac{4}{3}\right]$$

$$= 16$$

Question: If $g(x) = \int \frac{1}{x}\times\sqrt{\frac{1-x}{1+x}}\,dx,\ g(0)=1$ then $g(1) = ?$

Answer: $\frac{\pi}{3}+\ln\left|2-\sqrt{3}\right|$

Solution:

Given, $g(x) = \int \frac{1}{x}\times\sqrt{\frac{1-x}{1+x}}\,dx$

Put $x = \cos 2\theta$

$dx = -2\sin 2\theta\, d\theta$

$$\Rightarrow g(x) = \int \frac{-2\sin 2\theta}{\cos 2\theta}\tan\theta\, d\theta$$

$$= 2\int\left(\frac{1-2\sin^2\theta}{1-2\sin^2\theta}-\frac{1}{1-2\sin^2\theta}\right)d\theta$$

$$= 2\left[\int 1\,d\theta-\int\frac{\sec^2\theta}{\sec^2\theta-2\tan^2\theta}d\theta\right]$$

$$= 2\theta-2\int\frac{\sec^2\theta\, d\theta}{1-\tan^2\theta}$$

$$g(x) = 2\theta - 2\frac{1}{2}\ln\left|\frac{1+\tan\theta}{1-\tan\theta}\right| + C$$

$$g(x) = \cos^{-1}(x) - \ln\left|\frac{1+\sqrt{\frac{1-x}{1+x}}}{1-\sqrt{\frac{1-x}{1+x}}}\right| + C$$

Now, $g(1) = 0$

$\Rightarrow 0 = 0 - 0 + C$

$C = 0$

$$\therefore g\left(\frac{1}{2}\right) = \frac{\pi}{3} - \ln\left|\frac{1+\sqrt{\frac{\frac{1}{2}}{\frac{3}{2}}}}{1-\sqrt{\frac{\frac{1}{2}}{\frac{3}{2}}}}\right|$$

$$= \frac{\pi}{3} + \ln\left|2-\sqrt{3}\right|$$

Question: If ${}^{40}C_0 + {}^{41}C_1 + {}^{42}C_2 + ... + {}^{60}C_{20} = \frac{n}{m}\ {}^{60}C_{20}$, and m, n are coprime, then $n + m = ?$

Answer: 102.00

Solution:

${}^{41}C_0 + {}^{41}C_1 + {}^{42}C_2 + ... + {}^{60}C_{20}$

$\Rightarrow {}^{42}C_1 + {}^{42}C_2 + ... + {}^{60}C_{20}$

$\Rightarrow {}^{43}C_2 + {}^{44}C_3 + ... + {}^{60}C_{20}$

$\Rightarrow {}^{60}C_{19} + {}^{60}C_{20}$

$\Rightarrow {}^{61}C_{20} = \frac{m}{n}\ {}^{60}C_{20}$

$$\frac{61!}{20!41!} = \frac{m}{n}\frac{60!}{20!40!}$$

$$\frac{m}{n} = \frac{61}{41}$$

$\therefore m + n = 102$

Question: If $p + q = 3, p^4 + q^4 = 369$ then $\left(\frac{1}{p} + \frac{1}{q}\right)^{-2} = ?$

Answer: 4.00 or 64.00

Solution:

$p + q = 3 \Rightarrow p^2 + q^2 + 2pq = 9$

Also, $p^4 + q^4 = 369$

$$\left(p^2+q^2\right)^2-2p^2q^2=369$$

$$\left[\left(p+q\right)^2-2pq\right]^2-2\left(pq\right)^2=369$$

$$\left(16-2pq\right)^2-2\left(pq\right)^2=369$$

Let $pq=t$

$$81+4t^2-36t-2t^2=369$$

$$2t^2-36t-288=0$$

$$t^2-18t-144=0$$

$$t=\frac{18\pm\sqrt{324+576}}{2}$$

$$t=\frac{18+30}{2}$$

$$t=24,-6$$

$$\therefore\left(\frac{1}{p}+\frac{1}{q}\right)^{-2}=\left(\frac{p+q}{pq}\right)^{-2}=\left(\frac{pq}{3}\right)^2$$

$$=\left(\frac{24}{3}\right)^2 \text{ or } \left(-\frac{6}{3}\right)^2$$

$$=4 \text{ or } 64$$

Question: Side lengths of a cuboid are $2x, 5x$ & $4x$. There is closed hemisphere of radius r such that sum of surface area of cuboid & hemisphere is constant. Find ratio of x & r such that sum of volumes is maximum.

Answer: $\frac{45}{19}$

Solution:

Surface area $=76x^2+3\pi r^2=$ constant $=k$

$$V=40x^3+\frac{2}{3}\pi r^3=40x^3+\frac{2}{3}\pi\left(\frac{k-76x^2}{3\pi}\right)^{\frac{3}{2}}$$

$$\frac{dV}{dx}=0\Rightarrow 120x^2+\frac{2}{3}\pi\cdot\frac{3}{2}\left(\frac{k-76x^2}{3\pi}\right)^{\frac{1}{2}}\cdot\left(\frac{-152x}{3\pi}\right)=0$$

$$120x^2=\frac{152x}{3}\left(\frac{k-76x^2}{3\pi}\right)^{\frac{1}{2}}$$

$$\left(\frac{45x}{19}\right)^2=\left(\frac{k-76x^2}{3\pi}\right)^{\frac{1}{2}}$$

$$\left(\frac{45}{19}\right)^2\cdot x^2=r^2$$

$$\Rightarrow\frac{r}{x}=\frac{45}{19}$$

Question: A 3 digit number is randomly formed, find the probability that its common divisor with 36 is only 2.

Answer: $\frac{1}{6}$

Solution:

Total 3 digit numbers will be = 900

$n(s) = 900$

Now, need 3 digit number whose common divisor with 36 is only 2

Such numbers will be

$102, 106, 110, \ldots .998$

$\therefore$ 225 numbers

But in these $102, 114, 126, \ldots .990$

i.e., 75 numbers and to be deleted

$$\therefore \text{Probability } = \frac{225-75}{900}$$

$$= \frac{150}{900}$$

$$= \frac{1}{6}$$

Question: If l_1 is the tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ and l_2 is a straight lines passing through (0, 0) and perpendicular to l_1. If the locus of the point of intersection of l_1 and l_2 is $(x^2 + y^2)^2 = \alpha x^2 + \beta y^2$. Then, the values of $\alpha + \beta$ is ____.

Answer: 5.00

Solution:

Equation of tangent $l_1 : y = mx \pm \sqrt{9m^2 - 4}$ ....(1)

Then $l_2 : y = -\frac{1}{m}x$ ....(2)

For locus of point of intersection, eliminate m

$$\Rightarrow y = -\frac{x^2}{y} \pm \sqrt{9\frac{x^2}{y^2} - 4}$$

$$(x^2 + y^2)^2 = 9x^2 - 4y^2$$

$$\Rightarrow \alpha = 9, \beta = -4$$

$$\Rightarrow \alpha + \beta = 5$$

Question: $\int_0^{\sqrt{2}} \frac{(x^2 - 2)}{(x^2 + 2)\sqrt{x^4 + 4}} = ?$

Answer: $\frac{-\pi}{8}$

Solution:

$$I=\int_0^{\sqrt{2}}\frac{\left(x^2-2\right)}{\left(x^2+1\right)\sqrt{x^4+4}}$$

$$I=\int_0^{\sqrt{2}}\frac{\left(1-\frac{2}{x^2}\right)dx}{\left(x+\frac{2}{x}\right)\sqrt{x^2+\frac{4}{x^2}}}$$

$$I=\int_0^{\sqrt{2}}\frac{\left(1-\frac{2}{x^2}\right)dx}{\left(x+\frac{2}{x}\right)\sqrt{\left(x+\frac{2}{x}\right)^2-4}}$$

Put $x+\frac{2}{x}=t$

$$\left(1-\frac{2}{x^2}\right)dx=dt$$

$$I=\int_0^{2\sqrt{2}}\frac{dt}{t\sqrt{t^2-4}}=\frac{1}{2}\sec^{-1}\left(\frac{t}{2}\right)\Big]_{\infty}^{2\sqrt{2}}$$

$$I=\frac{1}{2}\left[\frac{\pi}{4}-\frac{\pi}{2}\right]=\frac{-\pi}{8}$$

Question: $A=\sum_{n=1}^{\infty}\frac{1}{\left(3+\left(-1\right)^n\right)^n}$, $B=\sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{\left(3+\left(-1\right)^n\right)^n}$. Find $\frac{A}{B}$.

Answer: $\frac{-11}{9}$

Solution:

$$A=\sum_{n=1}^{\infty}\frac{1}{\left[3+\left(-1\right)^n\right]^n}=\frac{1}{2}+\frac{1}{4^2}+\frac{1}{2^3}+\frac{1}{4^4}+\frac{1}{2^5}+\frac{1}{4^6}+....$$

$$=\frac{1}{2}\left[1+\frac{1}{2^2}+\frac{1}{2^4}+....\infty\right]+\frac{1}{4^2}\left[1+\frac{1}{4^2}+\frac{1}{4^6}+....\infty\right]$$

$$=\frac{1}{2}\left[\frac{1}{1-\frac{1}{4}}\right]+\frac{1}{16}\left[\frac{1}{1-\frac{1}{16}}\right]$$

$$=\frac{2}{3}+\frac{1}{15}=\frac{11}{15}$$

$$B=\sum_{n=1}^{\infty}\frac{(-1)^n}{\left[3+(-1)^n\right]^n}=\frac{-1}{2}+\frac{1}{4^2}-\frac{1}{2^3}-\frac{1}{4^4}+....\infty$$

$$=\frac{-2}{3}+\frac{1}{15}=\frac{-9}{15}$$

$$\therefore \frac{A}{B}=\frac{-11}{9}$$

Question: $16\sin 20\sin 40\sin 80=?$

Answer: $2\sqrt{3}$

Solution:

$$16\sin 20\cdot\sin(60-20)\cdot\sin(60+20)$$

$$=16\sin 20\cdot\left[\sin^2 60-\sin^2 20\right]$$

$$=16\sin 20\left[\frac{3}{4}-\sin^2 20\right]=4\left[3\sin 20-4\sin^3 20\right]$$

$$=4\sin(3\times 20)$$

$$=4\sin 60$$

$$=2\sqrt{3}$$

Question: Mean and standard deviation of 50 observations is 15 and 2. It was found that one observation was incorrect. Sum of correct and incorrect observation is 7. The mean of correct observation is 16. Find standard deviation of correct observation.

Answer: $\sqrt{43}$

Solution:

$n=50,\ \bar{x}=15,\ \sigma=2$

Let incorrect observation is x_1 and correct is x_1'

$$\therefore \bar{x}=\frac{x_1+x_2+x_3+...x_{50}}{50}=15$$

$$\Rightarrow \sum_{x=1}^{50}x_1^0=750$$

$$x_1+x_1'=70$$

Now, $\bar{x}'=\dfrac{x_1'+x_2+x_3+....x_{50}}{50}=16$

$$x_1'+\sum_{i=2}^{50}x_i=800$$

$$\therefore x_1'-x_1=50$$

$$\Rightarrow x_1'=60,\ x_1=10$$

Now, $\sigma^2=4=\dfrac{x_1^2+x_2^2+x_3^2+...x_{50}^2}{50}-225$

$\Rightarrow {x_2}^2 + {x_3}^2 + \ldots + {x_{50}}^2 = 11350$

$\therefore \left(\sigma^2\right)' = \dfrac{\left(x_1'\right)^2 + {x_2}^2 + {x_3}^2 + \ldots . {x_{50}}^2}{50} - (16)^2$

$= 6.56 = \sqrt{43}$

Question: If $z^2 + z + 1 = 0,\ x \in C$. Find $\left| \sum_{k=1}^{15} \left(z^k + \frac{1}{z^k} \right)^2 \right|$.

Answer: 30.00

Solution:

$z^2 + z + 1 = 0 \Rightarrow z = \omega$ (cube root of units)

$\left| \sum_{k=1}^{15} \left(\omega^k + \frac{1}{\omega^k} \right)^2 \right| = \left| \sum_{k=1}^{15} \left(\omega^k + \omega^{2k} \right)^2 \right|$

$= 5(2)^2 + 10(-1)^2$

$= 30$