03/7/17

RELATIONS AND FUNCTIONS

* Consider, $A = \{1,2,3\}$

Cartesian product on $A = A \times A$

$A \times A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,2)\}$

$n(A) = 3 \quad \therefore \; n(A \times A) = 3 \times 3 = 9$

* Relation $\subseteq A \times A$

Eg: $R_1 = \{(x,y) : x \leq y\}$

$= \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$

Types of Relations:

i] Reflexive relation : $\forall a \in A, (a,a) \in R$

ii] Symmetric relation : $\forall (a,b) \in R \Rightarrow (b,a) \in R$

iii] Transitive relation : $\forall (a,b),(b,c) \in R \Rightarrow (a,c) \in R$

iv] Equivalence relation : A relation which is all R, S, T.

Eg: *] $A \times A$ on set A

*] '$\parallel^k$' on set of straight lines

*] '=' on set of $\mathbb{R}$

*] $\cong$ on set of triangles

*] $\simeq$ on set of triangles.

EXERCISE 1.1

q] Show that each of the relation R in the set $A = \{x \in Z, 0 \leq x \leq 12\}$ given by (i) $R = \{(a,b) : |a-b|$ is a multiple of $4\}$

Sol] Gn: $R = \{(a,b) : |a-b|$ is a multiple of $4\}$

$= \{(a,b) : |a-b| = 4k\}$

Let $a,b,c \in A$

(i) Reflexive property : $\forall a \in A, (a,a) \in R$

$\forall a \in A, |a-a| = 0 = 4(0) \in A$ is true.

$\therefore R$ is reflexive $\longrightarrow$ ①

(ii) Symmetric property : $\forall (a,b) \in R \Rightarrow (b,a) \in R$

$\forall (a,b) \in R \Rightarrow |a-b| = 4k$

$\Rightarrow |-(a-b)| = 4k$

$\Rightarrow |b-a| = 4k \Rightarrow (b,a) \in R$

$\therefore$ R is symmetric $\longrightarrow$ ②

(iii) Transitive property : $\forall (a,b),(b,c) \in R \rightarrow (a,c) \in R$

$\forall (a,b),(b,c) \in R \Rightarrow |a-b| = 4k$ and $|b-c| = 4l$

$\Rightarrow |a-\cancel{b}+\cancel{b}-c| = 4(k+l)$

$\Rightarrow |a-c| = 4m$

$\Rightarrow$ R is transitive $\longrightarrow$ ③

①, ②, ③ $\Rightarrow$ R is an equivalence relation

Eg 5] $R = \{(a,b) : 2 \text{ divides } (a-b)\}$ on Z

$= \{(a,b) : a-b = 2k\}$

Let $a,b,c \in A$

(i) Reflexive property : $\forall a \in A, (a,a) \in R$

$\forall a \in A,\ a-b = 2k \in A$ is true.

$\Rightarrow$ R is reflexive $\longrightarrow$ ①

(ii) Symmetric property : $\forall (a,b) \in R \Rightarrow (b,a) \in R$

$\forall (a,b) \in R \Rightarrow (a-b) = 2k$

$\Rightarrow -(a-b) = -2k$

$\Rightarrow b-a = -2k \Rightarrow (b,a) \in R$

$\therefore$ R is symmetric $\longrightarrow$ ②

(iii) Transitive property : $\forall (a,b),(b,c) \in R \Rightarrow (a,c) \in R$

$\forall (a,b),(b,c) \in R \Rightarrow (a-b) = 2k$ and $(b-c) = 2l$

$\Rightarrow a-\cancel{b}+\cancel{b}-c = 2(k+l)$

$\Rightarrow a-c = 2m$

$\Rightarrow$ R is transitive $\longrightarrow$ ③

①, ②, ③ $\Rightarrow$ R is an equivalence relation.

9] ii] $R = \{(a,b) : a = b\}$ on $A = \{x \in Z, 0 \le x \le 12\}$

(i) Reflexive property :

$\forall a \in A$, $a = a$ is true

Hence, R is reflexive $\rightarrow$ ①

(ii) Symmetric property :

$\forall (a,b) \in R \Rightarrow a = b$

$\Rightarrow b = a$ is true and $\therefore (b,a) \in R$

$\therefore$ R is symmetric $\longrightarrow$ ②

(iii) Transitive property :

$\forall (a,b),(b,c) \in R \Rightarrow (a,c)$ should $\in R$

i.e, $a = b$ and $b = c$ in R

$\Rightarrow a = c$ is true

$\Rightarrow (a,c) \in R$

$\therefore$ R is transitive $\longrightarrow$ ③

①, ②, ③ $\Rightarrow$ R is an equivalence relation.

Eg 3] Let L be the set of all lines in a plane and R be the relation in L defined as $R = \{(l_1, l_2) : l_1 \perp^r l_2\}$. Show that R is symmetric but neither reflexive nor transitive.

Sol) Gn : $R = \{(l_1, l_2) : l_1 \perp^r l_2\}$ on L

Let $l_1, l_2, l_3 \in L$

(i) Reflexive : $\forall a \in L, (a,a) \in R$

$\forall a \in L$, $l_1 \perp l_1$ is not true

($\because$ a line can't be $\perp^r$ to itself)

$\therefore$ R is not reflexive

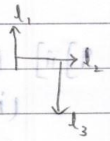
(ii) Symmetric : $\forall (a,b) \in R$, $(b,a) \in R$

$\forall (l_1, l_2) \in R$, $\Rightarrow l_1 \perp l_2$

$\Rightarrow l_2 \perp l_1$ is true on L

$\therefore$ R is symmetric

(iii) Transitive : $\forall (a,b),(b,c) \in R \Rightarrow (a,c) \in R$



$\forall (l_1,l_2),(l_2,l_3) \in R \Rightarrow l_1 \perp l_2$ and $l_2 \perp l_3$

$\not\Rightarrow l_1 \not\perp l_3$

$\Rightarrow R$ is not transitive

Hence given R is only symmetric, neither reflexive nor transitive.

14] Let L be set of all lines and R be the relation in L defined as $R = \{(L_1,L_2) : L_1 \parallel L_2\}$. Show that R is equivalence relation. Find the set of all lines related to the line $y = 2x+4$.

Sol] Gn: $R = \{(L_1,L_2) : L_1 \parallel L_2\}$

(i) Reflexive :

$\forall L_1 \in L, \quad L_1 \parallel L_1$ is true

$\therefore R$ is reflexive $\longrightarrow$ ①

(ii) Symmetric

$\forall (L_1,L_2) \in R, \quad L_1 \parallel L_2$

$\Rightarrow L_2 \parallel L_1$ is true on L

$\therefore R$ is symmetric $\longrightarrow$ ②

(iii) Transitive :

$\forall (L_1,L_2),(L_2,L_3) \in R \Rightarrow L_1 \parallel L_2$ and $L_2 \parallel L_3$

$\Rightarrow L_1 \parallel L_3$

$\Rightarrow R$ is transitive $\longrightarrow$ ③

①, ②, ③ $\Rightarrow R$ is an equivalence relation.

All the lines related to the line $y = 2x+4$ and $y = 2x+k$ where k is a real number.

12] Show that R defined in set A of all triangles as $R = \{(T_1,T_2) : T_1 \sim T_2\}$ is equivalence relation. Consider 3 right triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2, T_3 are related ?

Sol] Gn : $R = \{(T_1, T_2) : T_1 \sim T_2\}$

(i) Reflexive :

$\forall\ T_1 \in A$, $T_1 \sim T_1$ is true

$\therefore$ R is reflexive $\longrightarrow$ ①

(ii) Symmetric :

$\forall (T_1, T_2) \in R$, $T_1 \sim T_2$

$\Rightarrow T_2 \sim T_1$ is true

$\therefore$ R is symmetric $\longrightarrow$ ②

(iii) Transitive :

$\forall (T_1, T_2), (T_2, T_3) \in R$, $T_1 \sim T_2$ and $T_2 \sim T_3$

$\Rightarrow T_1 \sim T_3$ is true

$\Rightarrow$ R is transitive $\longrightarrow$ ③

①, ②, ③ $\Rightarrow$ R is an equivalence relation.

Since two Δ^{le}s are similar if corresponding sides are proportional.

$\therefore \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{1}{2}$

$\therefore$ T_1 and T_3 are related.

15] Let R be the relation in $A = \{1, 2, 3, 4\}$ given by $R = \{(1,2), (2,2), (1,1), (4,4), (1,3), (3,3), (3,2)\}$. Find whether R is R,S,T.

Sol] i) $\forall\ a \in A$, $(a,a) \in R$

$\therefore$ R is reflexive.

(ii) $(1,2) \in R$ but $(2,1) \notin R$,

$\therefore$ R is not symmetric

(iii) $\forall (a,b), (b,c) \in R$, $(a,c) \in R$

$\therefore$ R is transitive.

** 10] Give an eg of Relation which is :

i] S but neither R nor T :
on $A = \{1,2,3\}$, $R = \{(2,3),(3,2)\}$

ii] T but neither R nor S :
on $A = \{1,2,3\}$, $R = \{(1,2),(2,3),(1,3)\}$

iii] R and S but not T :
on $A = \{1,2,3\}$, $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$
$(1,2),(2,3) \in R$ but $(1,3) \notin R$
$\therefore$ R is not transitive.

iv] R and T but not S :
on $A = \{1,2,3\}$, $R = \{(1,1),(2,2),(3,3),(1,2)\}$

v] S and T but not R :
on $A = \{1,2,3\}$, $R = \{(1,2),(2,1),(1,1)\}$

1] Determine whether the following relations are R, S, T :

i] on $A = \{1,2,3,4 \cdots 14\}$ defined as
$R = \{(x,y) : 3x - y = 0\}$

Sol] (i) Reflexive property :
$1 \in A$, $3(1) - 1 \neq 0$
$\therefore$ R is not reflexive.

(ii) Symmetric :
$(1,3) \in R \not\Rightarrow (3,1) \in R$
$\therefore$ R is not symmetric

(iii) Transitive :
$(1,3),(3,9) \in R$, but $(1,9) \notin R$
$\because (1,9)$ does not satisfy $3x - y = 0$
$\therefore$ R is not Transitive.

NOTE : Polynomial functions are never R nor S nor T in general

ii] R in $\mathbb{N}$ defined by $R = \{(x,y) : y = x + 5, x < 4\}$

Sol] Being polynomial, R is neither R nor S nor T.

iii] Relation R in $A = \{1,2,3,4,5,6\}$; $R = \{(x,y) : y$ is divisible by $x\}$

(i) Reflexive :

x divides x, is true.

(ii) Symmetric :

x divides y but y does not divides x

Eg : 2 divides 4 but 4 does not divides 2

(iii) Transitive :

$x|y, y|z \Rightarrow x|z$ is true on N

Eg : $2|4, 4|8 \Rightarrow 2|8$

iv] On $\mathbb{Z}$, $R = \{(x,y) : x-y$ is an integer$\}$

(i) Reflexive :

$x - x = 0 \in \mathbb{Z}$ $\therefore R$ is reflexive.

(ii) Symmetric :

$x - y \in \mathbb{Z} \Rightarrow y - x \in \mathbb{Z}$ $\therefore R$ is symmetric

(iii) Transitive :

$x - y \in \mathbb{Z}, y - z \in \mathbb{Z} \Rightarrow x - z \in \mathbb{Z}$ $\therefore R$ is transitive.

v] Relation R on the set of human beings in a town given by

(a) $R = \{(x,y) : x$ & y work at the same place$\}$.

Reflexive ✓

Symmetric ✓

Transitive ✓

(c) $R = \{(x,y) : x$ is exactly 7 cm taller than $y\}$

Reflexive ✗

Symmetric ✗

Transitive ✗

(d) $R = \{(x,y) : x$ is wife of $y\}$

Reflexive ×

Symmetric ×

Transitive ×

2] Show that R in the set $\mathbb{R}$ of real no.s, defined as $R = \{(a,b) : a \le b^2\}$ is neither reflexive nor symmetric nor transitive.

Sol] (i) Reflexive :

$\forall (a,a) \in R$, $a \le a^2$ is false for real no.s

$\therefore$ R is not reflexive.

(ii) Symmetric :

$\forall (a,b) \in R$, $a \le b^2$

$\not\Rightarrow b \le a^2$

$\therefore$ R is not symmetric

(iii) Transitive :

$\forall (a,b),(b,c) \in R$, $a \le b^2$, $b \le c^2$

$\Rightarrow a \le c^4$ which is false

$\therefore$ R is not transitive.

4] Show that R in $\mathbb{R}$ defined as $R = \{(a,b) : a \le b\}$, is R and T but not S.

Sol] (i) Reflexive :

$\forall (a,a) \in R$, $a \le a$ is true

$\therefore$ R is reflexive.

(ii) Symmetric :

$\forall (a,b) \in R$, $a \le b$ $\quad 2 \le 3$ ✓

$\not\Rightarrow b \le a$ $\quad 3 \le 2$ ×

$\therefore$ R is not symmetric.

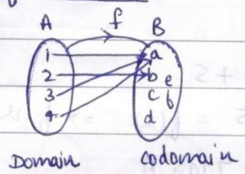
(iii) Transitive :

$\forall (a,b),(b,c) \in R$, $a \le b$, $b \le c$ $\quad 2 \le 3$, $3 \le 4$ ✓

$\Rightarrow a \le c$ is true $\quad \Rightarrow 2 \le 4$ ✓

$\therefore$ R is transitive.

Types of Functions:

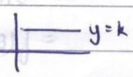


Range = {a, b}

Range ⊂ CD

Types: ① Identity $y = f(x) = x$

② Constant : $f(x) = k$

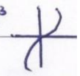


③ Polynomial : $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} \cdots + a_n$

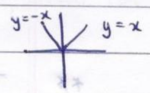
eg: $y = x^2$

$y^2 = x$

$y = x^3$

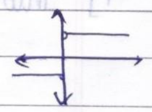


④ ~~modu~~ Modulus : $f(x) = |x| = \begin{cases} x ; x \geq 0 \\ -x ; x < 0 \end{cases}$

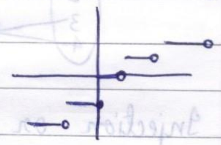


⑤ Rational : $\frac{f(x)}{g(x)}$; $g(x) \neq 0$

⑥ Signum : $f(x) = \frac{|x|}{x} = \begin{cases} 1 , x > 0 \\ 0 , x = 0 \\ -1 , x < 0 \end{cases}$



⑦ Greatest integer : $f(x) = [x]$



Odd and Even Function:

Odd fun^n : If $\boxed{f(-x) = -f(x)}$, then $f(x)$ is an odd fun^n

Eg : $f(x) = x^5 - 5x^3 + 7x$

$f(-x) = (-x)^5 - 5(-x)^3 + 7(-x) = -x^5 + 5x^3 - 7x$

$= -(x^5 - 5x^3 + 7x) = -f(x)$ ∴ f is odd

$f(\theta) = 2\sin 3\theta - \operatorname{cosec} 5\theta + \tan^3\theta$

$f(-\theta) = 2\sin(-3\theta) - \operatorname{cosec}(-5\theta) + (\tan(-\theta))^3$

$= -2\sin 3\theta + \operatorname{cosec} 5\theta - \tan^3\theta$

$= -f(\theta)$

⇒ f is an odd fun^n.

Even fun^n : $\boxed{f(-x) = f(x)}$, then $f(x)$ is an even fun^n.

Eg : $f(x) = 4x^4 - 3x^2 + 5$

$f(-x) = 4(-x)^4 - 3(-x)^2 + 5$

$= 4x^4 - 3x^2 + 5 = f(x) \Rightarrow f$ is even

$f(\theta) = \sin^2\theta - \cos\theta + 7\tan^2\theta$

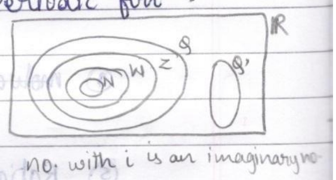
$f(-\theta) = [\sin(-\theta)]^2 - \cos(-\theta) + 7(\tan(-\theta))^2$

$= \sin^2\theta - \cos\theta + 7\tan^2\theta$

$= f(\theta)$ $\therefore f$ is even

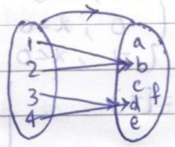
Periodic Fun^n : A fun^n whose value repeats for every equal no. of period or phase is called periodic fun^n.

Eg : Trignometric fun^n.



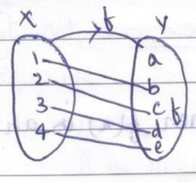
** Types of Functions :

1] Into fun^n : $\boxed{\text{Range} \subset CD}$ i.e., free elements present in CD



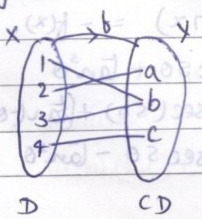
also Many-one fun^n

2] Injection or 1-1 function :



- Presence of one to one correspondence b/w D and CD.
- $\boxed{f(x_1) = f(x_2)}$
  $\Rightarrow x_1 = x_2$

3] Surjection or onto fun^n :



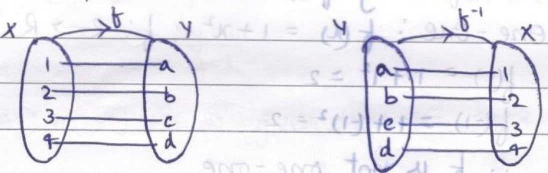
(no free elements in CD)

Range = CD

$\boxed{\forall y \in CD, \exists x \in D}$ $\therefore f$ is onto

4] Bijection or 1-1-onto fun^n:

A bijection is a fun^n which is both injection and surjection. i.e, it should be both 1-1 and onto.



$f : X \to Y$ $\qquad$ $f^{-1} : Y \to X$

EXERCISE 1.2

7] In each of cases, state whether the fun^n is 1-1, onto or bijective. Justify.

i) $f : R \to R$ defined by $f(x) = 3 - 4x$

Gn: $f(x) = 3 - 4x$; $f : R \to R$

(a) TPT: f is one-one:

choose $x_1, x_2 \in R$

Consider $f(x_1) = f(x_2)$

$$\cancel{3} - 4x_1 = \cancel{3} - 4x_2$$

$$-4x_1 = -4x_2$$

$$x_1 = x_2$$

$\because f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, $f(x)$ is 1-1 fun^n $\longrightarrow$ ①

(b) TPT: f is onto:

wkt, $y = f(x)$

$$y = 3 - 4x$$

$$x = \frac{3-y}{4}$$

$\therefore \forall y \in CD, \exists x \in D$, given by $x = \frac{3-y}{4}$

$\therefore f(x)$ is onto fun^n $\longrightarrow$ ②

① and ② ⇒ $f^{-1}(x)$ exists given by $\boxed{f^{-1}(x) = \frac{3-x}{4}}$

ii] $f: R \to R$ defined by $f(x) = 1 + x^2$

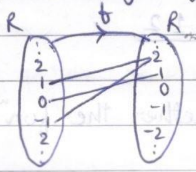
(a) One - One : $f(x) = 1 + x^2$; $f: R \to R$

$f(1) = 1 + 1^2 = 2$

$f(-1) = 1 + (-1)^2 = 2$

∴ f is not one-one

(b) Onto : All −ve real no.s in CD are free and do not form images for any element in D.



∴ f is not onto.

Hence given funⁿ is neither one-one nor onto.

10] Let $A = R - \{3\}$, $B = R - \{1\}$. Consider $f: A \to B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Is f one-one and onto. Justify.

(a) One - One :

Choose $(x_1), (x_2)$ from D

Consider $f(x_1) = f(x_2)$

$$\frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\cancel{x_1x_2} - 2x_2 - 3x_1 + \cancel{6} = \cancel{x_1x_2} - 2x_1 - 3x_2 + \cancel{6}$$

$$-2x_2 + 3x_2 = -2x_1 + 3x_1$$

$$x_2 = x_1$$

∵ $f(x_1) = f(x_2) \Rightarrow x_2 = x_1$ ∴ $f(x)$ is one one → ①

(b) onto :

wkt, $y = f(x)$

$$y = \frac{x-2}{x-3}$$

$$xy - 3y = x - 2$$
$$xy - x = 3y - 2$$
$$x(y-1) = 3y - 2$$
$$x = \frac{3y-2}{y-1}$$

$\forall y \in CD$, $\exists x \in D$ given by $x = \frac{3y-2}{y-1}$

$\therefore f$ is onto $\longrightarrow$ ②

① & ② $\Rightarrow f^{-1}(x)$ exists

given by $\boxed{f^{-1}(x) = \frac{3x-2}{x-1}}$

Eg 17) $f(x) = \frac{3x+4}{5x-7}$ $f: R - \{\frac{7}{5}\} \to R - \{\frac{3}{5}\}$. Show that f is one-one and onto. Write $f^{-1}(x)$.

(a) One-One :

Choose x_1, x_2 from D

Consider $f(x_1) = f(x_2)$

$$\frac{3x_1+4}{5x_1-7} = \frac{3x_2+4}{5x_2-7}$$
$$(3x_1+4)(5x_2-7) = (3x_2+4)(5x_1-7)$$
$$\cancel{15x_1x_2} - 21x_1 + 20x_2 - \cancel{28} = \cancel{15x_1x_2} - 21x_2 + 20x_1 - \cancel{28}$$
$$-21x_1 + 20x_2 = -21x_2 + 20x_1$$
$$x_1 = x_2$$

$\because f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $\therefore f(x)$ is one-one $\longrightarrow$ ①

(b) Onto :

wkt, $y = f(x)$

$$y = \frac{3x+4}{5x-7}$$
$$5xy - 7y = 3x + 4$$
$$5xy - 3x = 4 + 7y$$
$$x(5y-3) = 7y + 4$$
$$x = \frac{7y+4}{3y-3}$$

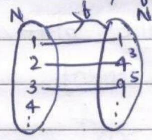
$\forall\, y \in CD,\ \exists\, x \in D$ given by $x = \dfrac{7y+4}{3y-5}$

$\therefore f$ is onto $\rightarrow$ ②

①, ② $\Rightarrow f^{-1}(x)$ exists given by $f^{-1}(x) = \dfrac{7x+4}{3x-5}$

2] Check injectivity and surjectivity of :

i] $f: N \rightarrow N\ ;\ f(x) = x^2$



Injective, ~~Surjective~~

f is one-one ~~by~~ but not onto

$\because$ Non-perfect squares in CD 3, 5 ... are not images to any element in D.

ii] $f: Z \rightarrow Z\ ;\ f(x) = x^2$ $\cancel{I}$ $\cancel{S}$ $\Big\}\ \because f(1) = 1,\ f(-1) = 1$

iii] $f: R \rightarrow R\ ;\ f(x) = x^2$ $\cancel{I}$ $\cancel{S}$

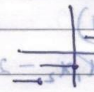
iv] $f: N \rightarrow N\ ;\ f(x) = x^3$ I $\cancel{S}$

v] $f: Z \rightarrow Z\ ;\ f(x) = x^3$ I $\cancel{S}$

3] (i) $f: R \rightarrow R\ ;\ f(x) = [x]$. PT its neither 1-1 nor onto.

(ii) Write all the points of discontinuity of $[x]$ on $[-3, 3]$.

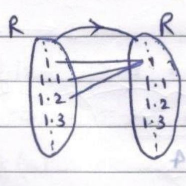
Sol] (ii)



$x \in \{-2, -1, 0, 1, 2\}$

(i) One-One : $\left.\begin{array}{l}[1] = 1 \\ [1.1] = 1 \\ [1.5] = 1\end{array}\right\} \Rightarrow [x]$ is not 1-1

Onto :



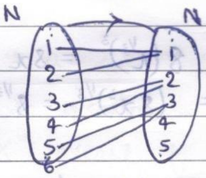
Nos like 1.2, 1.3 ... i.e all the rationals in CD do not form images for any element in D.

$\therefore [x]$ is not 1-1 and also not onto

9] $f: N \to N$; $f(n) = \begin{cases} \frac{n+1}{2} ; \text{ if } n \text{ is odd} \\ \frac{n}{2} ; \text{ if } n \text{ is even } \forall n \in N \end{cases}$

State whether f is bijective. Justify.

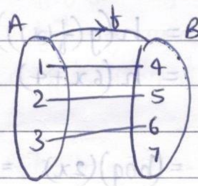
Sol]



From the mapping we observe that $f(n)$ is not 1-1 but is onto

$\therefore$ f is not bijective.

6] $A = \{1,2,3\}$, $B = \{4,5,6,7\}$. $f = \{(1,4),(2,5),(3,6)\}$ be a fun^n from A to B. Show that f is 1-1.



$\because$ there is one to one from D to CD,

f is 1-1.

Composition of Functions:

* Function inside a fun^n is called composite fun^n, like

gof, fog, fof, goh, gog, hogof (or) gf, fg, ff...

(or) f.g, g.f ...

Eg: $f(x) = x^2+1$, $g(x) = x-1$. find fog, gof, fof, gog.

(i) $fog(x) = f[g(x)] = f(x-1) = (x-1)^2+1$

$= x^2 - 2x+1+1 = x^2-2x+2$

(ii) $gof(x) = g[f(x)] = g(x^2+1) = (x^2+1)-1 = x^2$

(iii) $fof(x) = f[f(x)] = f(x^2+1) = (x^2+1)^2+1 = x^4+2+2x^2$

(iv) $gog(x) = g[g(x)] = g(x-1) = x-1-1 = x-2$

EXERCISE 1.3

**3] Find fog and gof if

i] $f(x) = |x|$, $g(x) = |5x-2|$

(a) $fog(x) = f[g(x)] = f(|5x-2|) = ||5x-2|| = |5x-2|$

(b) $gof(x) = g[f(x)] = g(|x|) = |5|x|-2|$

ii] $f(x) = 8x^3$, $g(x) = x^{1/3}$

(a) $fog(x) = f[g(x)] = f(x^{1/3}) = 8(x^{1/3})^3 = 8x$

(b) $gof(x) = g[f(x)] = g(8x^3) = (8x^3)^{1/3} = 8^{1/3}x = \sqrt[3]{8}\,x = 2x$

Eg 26] $f: N \to N$, $g: N \to N$, $h: N \to R$ defined as $f(x) = 2x$, $g(y) = 3y+4$, $h(z) = \sin z$. Show that ~~(hog)of~~ $ho(gof) = (hog)of$

Sol] LHS $= ho(gof)(x) = h(gof(x)) = h(g(f(x)))$

$= h(g(2x)) = h(3(2x)+4) = h(6x+4)$

$= \cancel{h}\sin(6x+4) \longrightarrow (1)$

RHS $= ((hog)of)(x) = (hog)(f(x)) = (hog)(2x) = h(g(2x))$

$= h(3(2x)+4) = h(6x+4) = \sin(6x+4) \longrightarrow (2)$

(1), (2) $\Rightarrow$ LHS = RHS.

13] If $f: R \to R$, $f(x) = (3-x^3)^{\frac{1}{3}}$, then $fof(x)$ is

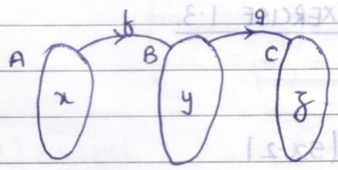
Sol] $f(f(x)) = f[(3-x^3)^{1/3}]$

$= [3-[(3-x^3)^{1/3}]^3]^{1/3}$

$= [3-(3-x^3)]^{1/3}$

$= (0+x^3)^{1/3} = x$

Let $f: A \to B$, $g: B \to C$ then the composition of f and g is gof and is defined as the fun^n from A to C, and hence $gof: A \to C$



Let identity element of A be x.

" " " B be y.

" " " C be z.

From mapping: $y = f(x)$ and $z = g(y)$

Consider $z = g(y) = g[f(x)] = gof(x)$

Eg: Consider $2 \times \frac{1}{2} = 1 \Rightarrow$ '$\times$' is operation

'1' is identity element under $\times^{n}$.

$\because a * a^{-1} = e$, e is identity element.

$\underset{\text{operation}}{*}$

$\Rightarrow$ $\times^{ve}$ inverse of $2 = \frac{1}{2}$

& $\times^{ve}$ inverse of $\frac{1}{2} = 2$

Illy, if $gof(x) = x$ and

$fog(y) = y$

Then, $f^{-1} = g$ and $g^{-1} = f$

Hence f is invertible

Eg 17] Show that if $f: R - \left\{\frac{7}{5}\right\}$ to $R - \left\{\frac{3}{5}\right\}$ is defined by $f(x) = \frac{3x+4}{5x-7}$. Show that f is invertible. Write $f^{-1}(x)$.

Sol) Gn: $f: R - \left\{\frac{7}{5}\right\} \longrightarrow R - \left\{\frac{3}{5}\right\}$

Given $y = f(x) = \frac{3x+4}{5x-7}$

$5xy - 7y = 3x + 4$

$5xy - 3x = 7y + 4$

$x(5y-3) = 7y + 4$

$\boxed{x = \frac{7y+4}{5y-3}} = g(y)$

Consider $gof(x) = g[f(x)] = g\left(\frac{3x+4}{5x-7}\right)$

$= g\left[\frac{7\left(\frac{3x+4}{5x-7}\right) + 4}{5\left(\frac{3x+4}{5x-7}\right) - 3}\right]$

$$= \frac{21x + \cancel{28} + 20x - \cancel{28}}{\cancel{15x} + 20 \cancel{+} \cancel{15x} + 21} = \frac{41x}{41} = x = I_{R-\{\frac{7}{5}\}}$$

Consider, $fog(y) = f[g(y)] = f\left(\frac{7y+4}{5y-3}\right)$

$$= \left[\frac{3\left(\frac{7y+4}{5y-3}\right)+4}{5\left(\frac{7y+4}{5y-3}\right)-7}\right] = \frac{21y + \cancel{12} + 20y - \cancel{12}}{\cancel{35y} + 20 - \cancel{35y} + 21}$$

$$= \frac{41y}{41} = y = I_{R-\{\frac{3}{5}\}}$$

$\Rightarrow f$ is invertible

$\therefore f^{-1}(x) = \frac{7x+4}{5x-3}$

4] If $f(x) = \frac{4x+3}{6x-4}$, $x \neq \frac{2}{3}$. ST $fof(x) = x$ and f is invertible. Write $f^{-1}(x)$

Sol] Gn: $f(x) = \frac{4x+3}{6x-4}$, $x \neq \frac{2}{3}$

$$LHS = fof(x) = f[f(x)] = f\left(\frac{4x+3}{6x-4}\right) = \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$$

$$= \frac{16x + \cancel{12} + 18x - \cancel{12}}{\cancel{24x} + 18 - \cancel{24x} + 16} = \frac{34x}{34} = x = RHS$$

wkt, $y = f(x) = \frac{4x+3}{6x-4}$

$$6xy - 4y = 4x + 3$$
$$6xy - 4x = 4y + 3$$
$$x(6y-4) = 4y+3$$
$$x = \frac{4y+3}{6y-4} = g(y)$$

Consider $gof(x) = g[f(x)] = g\left(\frac{4x+3}{6x-4}\right)$

$$= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} = \frac{16x+\cancel{12}+18x-\cancel{12}}{\cancel{24x}+18-\cancel{24x}+16}$$

$$= \frac{34x}{34} = x$$

Consider $fog(y) = f[g(y)] = f\left(\frac{4y+3}{6y-4}\right) = \frac{4\left(\frac{4y+3}{6y-4}\right)+3}{6\left(\frac{4y+3}{6y-4}\right)-4}$

$$= \frac{16y+\cancel{12}+18y-\cancel{12}}{\cancel{24y}+18-\cancel{24y}+16} = \frac{34y}{34} = y$$

$\Rightarrow f$ is invertible and $f^{-1}(x) = \frac{4x+3}{6x-4}$

We infer that

9] Consider $f: R \to [-5, \infty)$ given by $f(x) = 9x^2+6x-5$ ST f is invertible. Write $f^{-1}(x)$.

Sol] $y = f(x) = 9x^2+6x-5$

$y = [(3x)^2 + 2\cdot 3x\cdot 1 + 1^2] - 1^2 - 5$ Complete square method

$y = (3x+1)^2 - 6$

$\frac{\sqrt{y+6}-1}{3} = x = g(y)$

Consider, $gof(x) = g[f(x)] = g[(3x+1)^2-6]$

$$= \frac{\sqrt{(3x+1)^2-\cancel{6}+\cancel{6}}-1}{3} = \frac{3x+\cancel{1}-\cancel{1}}{3}$$

$$= \frac{3x}{3} = x = I_R$$

Consider, $fog(y) = f[g(y)] = f\left(\sqrt{\frac{y+6}{3}}-1\right)$

$$= \left[3\left(\sqrt{\frac{y+6}{3}}-1\right)+1\right]^2-6$$

$$= \left(\sqrt{y+6}-\cancel{1}+\cancel{1}\right)^2-6$$

$$= y+\cancel{6}-\cancel{6}$$

$$= y = I_{[-5,\infty)}$$

$\Rightarrow f$ is invertible and $f^{-1}(x) = \sqrt{\frac{x+6}{3}}-1$

Eg 25] Let $f: N \to R$ defined by $f(x) = 4x^2+12x+15$ ST $f: N \to S$ where S is range of f is invertible. Write $f^{-1}(x)$.

Sol] Gn: $f: N \to S$, $f(x) = 4x^2+12x+15$

wkt, $y = f(x) = 4x^2+12x+15$

$$y = (2x)^2+2\cdot 2x\cdot 3+3^2-3^2+15$$

$$y = (2x+3)^2+6 = f(x)$$

$$\frac{\sqrt{y-6}-3}{2} = x = g(y)$$

Consider $gof(x) = g[f(x)] = g[(2x+3)^2+6]$

$$= \frac{\sqrt{(2x+3)^2+\cancel{6}-\cancel{6}}-3}{2}$$

$$= \frac{2x+\cancel{3}-\cancel{3}}{2} = \frac{2x}{2} = x = I_N$$

Consider, $fog(y) = f[g(y)] = f\left(\frac{\sqrt{y-6}-3}{2}\right)$

$$= \left[\cancel{2}\left(\frac{\sqrt{y-6}-3}{\cancel{2}}\right)+3\right]^2+6$$

$$= y-\cancel{6}+\cancel{6} = y = I_S$$

$\Rightarrow f$ is invertible. and $f^{-1}(x) = \frac{\sqrt{x-6}-3}{2}$.

Eg 23] Let $f: N \to Y$, $f(x) = 4x+3$. ST f is invertible and find f^{-1}.

Sol] wkt $y = f(x) = 4x+3$

$$\boxed{x = \frac{y-3}{4} = g(y)}$$

Consider, $gof(x) = g[f(x)] = g(4x+3)$

$$= \frac{4x+\cancel{3}-\cancel{3}}{4} = \frac{4x}{4} = x = I_N$$

$$fog(y) = f[g(y)] = f\left(\frac{y-3}{4}\right) = \cancel{4}\left(\frac{y-3}{\cancel{4}}\right)+3 = y-3+3 = y = I_Y$$

$\Rightarrow f$ is invertible, $f^{-1}(x) = \frac{x-3}{4}$

6] f is a funⁿ from $[-1,1]$ to R given by $f(x) = \frac{x}{x+2}$. ST f is 1-1. Find inverse of f.

Sol] Gn: $f: [-1,1] \to R$

(a) One - One: Choose x_1, x_2 belonging to D.

Consider, $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1}{x_1+2} = \frac{x_2}{x_2+2}$$

$$\Rightarrow \cancel{x_1x_2} + 2x_1 = \cancel{x_1x_2} + 2x_2$$

$$\cancel{2}x_1 = \cancel{2}x_2$$

$$\Rightarrow x_1 = x_2$$

$\because f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, $f(x)$ is one - one.

(b) Inverse (onto):

wkt, $y = f(x)$

$$y = \frac{x}{x+2}$$

$$xy + 2y = x$$

$\cancel{xy - x = -2y}$ $2y = (x - xy)$

$$2y = x(1-y)$$

$$\boxed{x = \frac{2y}{1-y}} \quad \therefore f^{-1}(x) = \frac{2x}{1-x}$$

14] Let $f: R - \{-\frac{4}{3}\}$, $f(x) = \frac{4x}{3x+4}$. Inverse of f is given

by (a) $g(y) = \frac{3y}{3-4y}$ (b) $g(y) = \frac{4y}{4-3y}$ (c) $g(y) = \frac{4y}{3-4y}$

(d) $g(y) = \frac{3y}{4-3y}$

Sol] wkt, $y = f(x)$

$$y = \frac{4x}{3x+4}$$

$$3xy + 4y = 4x$$

$$4y = (4x - 3xy)$$

$$4y = x(4-3y)$$

$$x = \frac{4y}{4-3y} \quad (b)$$

8] Consider $f: R_+ \longrightarrow [4, \infty)$ given by $f(x) = x^2+4$. ST f is invertible and write $f^{-1}(x)$.

Sol] wkt, $y = f(x)$

$$y = x^2+4$$

$$\boxed{x = \sqrt{y-4}} = g(y)$$

Consider, $gof(x) = g(f(x)) = g(x^2+4) = \sqrt{(x^2+\cancel{4})-\cancel{4}} = x = I$

$fog(y) = f(g(y)) = f(\sqrt{y-4})$

$$= (\sqrt{y-4})^2+4$$

$$= y \cancel{-4} + \cancel{4} = y = I_{(4,\infty)}$$

$\Rightarrow f$ is invertible, $f^{-1}(x) = \sqrt{x-4}$

Binary Operations:

Eg: $N = \{1, 2, 3, 4 \ldots\}$

Let $2, 4 \in N$,

$* \to +^{m}$: $2+4 = 6 \in N$

$\Rightarrow$ '+' is b.o on N

$* \rightarrow -^m : 2-4 = -2 \notin N$

$\therefore$ '−' is not b.o on N

$* \rightarrow -^m$ on Z : $\forall a, b \in Z$, $a - b \in Z$

$\therefore$ − is bo on Z.

A binary operation $*$ on a set A is a function $* : A \times A \rightarrow A$ denoted by $a * b$.

EXERCISE 1.4

1] Determine whether or not $*$ given below is b.o or not. Justify

i] On Z^+; $a * b = a - b$

Sol] $\forall a, b \in Z^+$, $a * b = a - b \notin Z^+$

$\therefore *$ is not b.o on Z^+ $\because 2 - 4 = -2 \in Z^+$

ii] On Z^+; $a * b = ab$

Sol] $\forall a, b \in Z^+$, $a * b = ab \in Z^+$

$\therefore *$ is b.o on Z^+

iii] On R ; $a * b = ab^2$

Sol] $\forall a, b \in R$, $a * b = ab^2 \in R$

$\therefore *$ is b.o on R

iv] On Z^+; $a * b = |a - b|$

Sol] $\forall a, b \in Z^+$; $a * b = |a - b| \in R$

$\therefore *$ is b.o on Z^+

v] On Z^+; $a * b = a$

Sol] $\forall a, b \in Z^+$, $a * b = a \in Z^+$

$\therefore *$ is b.o on Z^+

Properties on B.O :

(a) Commutative ÷ $\forall a,b \in A$, $a*b = b*a$

(b) Associative ÷ $\forall a,b,c \in A$, $a*(b*c) = (a*b)*c$

(c) (e) Existence of Identity ÷ $\forall a, e \in A$, $a*e = e*a = a$

[To find id , use $a*e = a$]

(d) Existence of Inverse (a^{-1}) ÷ $\forall a, a^{-1} \in A$, $a*a^{-1} = a^{-1}*a = e$

[To find a^{-1} , use $a*a^{-1} = e$]

2] For each operation given below, determine whether * is

(a) bo (b) commutative (c) associative.

i] On Z , $a*b = a-b$

(a) $\forall a,b \in Z$, $a*b = a-b \in Z$

$\therefore$ * is b.o

(b) Commutative Law : $\forall a,b \in A$; $a*b = b*a$

$$\left.\begin{array}{l} LHS = a*b = a-b \\ RHS = b*a = b-a \end{array}\right\} \begin{array}{l} LHS \neq RHS \\ \therefore * \text{ is not commutative} \end{array}$$

(c) Associative : $\forall a,b,c \in A$, $a*(b*c) = (a*b)*c$

$$\left.\begin{array}{l} LHS = a*(b*c) = a*(b-c) = a-b+c \\ RHS = (a*b)*c = (a-b)*c = a-b-c \end{array}\right\} \begin{array}{l} LHS \neq RHS \\ \therefore * \text{ is not associative} \end{array}$$

ii] On Q , $a*b = ab+1$

(a) $\forall a,b \in Q$, $a*b = ab+1 \in Q$

$\therefore$ * is bo on Q

$$\text{(b)} \left.\begin{array}{l} LHS = a*b = ab+1 \\ RHS = b*a = ba+1 = ab+1 \end{array}\right\} \begin{array}{l} LHS = RHS \\ \therefore * \text{ is commutative.} \end{array}$$

$$\text{(c)} \left.\begin{array}{l} LHS = a*(b*c) = a*(bc+1) = abc+a+1 \\ RHS = (a*b)*c = (ab+1)*c = abc+c+1 \end{array}\right\} \begin{array}{l} LHS \neq RHS \\ * \text{ is not associati} \end{array}$$

iii] On Q , $a*b = \frac{ab}{2}$

(d) Existence of Id : $\forall a, e \in A$, $a * e = a$

Consider, $a * e = a$

$$\frac{ae}{4} = a$$

$$\boxed{e = 4}$$

(e) Existence of inverse : $\forall a, a^{-1} \in A$, $a * a^{-1} = e$

Consider, $a * a^{-1} = e$

$$\frac{aa^{-1}}{4} = e = 4$$

$$aa^{-1} = 16$$

$$\boxed{a^{-1} = \frac{16}{a}}$$

v] On Z^+, $a * b = 2^{ab}$

(a) $\forall a, b \in Z^+$, $a * b = 2^{ab} \in Z^+$

$\therefore$ $*$ is b.o on Z^+

(b) LHS $= a * b = 2^{ab}$

RHS $= b * a = 2^{ba} = 2^{ab}$

LHS = RHS $\therefore$ $*$ is commutative

(c) LHS $= a * (b * c) = a * 2^{bc} = 2^{2^{bc}a}$

RHS $= (a * b) * c = 2^{ab} * c = 2^{2^{ab}c}$

LHS $\neq$ RHS $\therefore$ $*$ is not associative.

vi] On Z^+, $a * b = a^b$

(a) $\forall a, b \in Z^+$, $a * b = a^b \in Z^+$

$\therefore$ $*$ is b.o on Z^+

(b) LHS $= a * b = a^b$

RHS $= b * a = b^a$

LHS $\neq$ RHS $\therefore$ $*$ is not commutative

(c) LHS $= a * (b * c) = a * (b^c) = a^{(b^c)}$

RHS $= (a * b) * c = (a^b) * c = (a^b)^c$

LHS $\neq$ RHS $\therefore$ $*$ is not associative

vii] On R, $a * b = \frac{a}{b+1}$

(a) $\forall a, b \in R$, $a * b = \frac{a}{b+1} \in R$

$\therefore *$ is b.o on R

(b) LHS $= a * b = \frac{a}{b+1}$

RHS $= b * a = \frac{b}{a+1}$

LHS $\neq$ RHS

$\therefore *$ is not commutative

(c) LHS $= a*(b*c) = a*\left(\frac{b}{c+1}\right) = \frac{a}{\frac{b}{c+1}+1}$

RHS $= (a*b)*c = \left(\frac{a}{b+1}\right)*c = \frac{\frac{a}{b+1}}{c+1} = \frac{a}{(b+1)(c+1)}$

LHS $\neq$ RHS

$\therefore *$ is not associative.

3] Consider the bo '$\wedge$' on the set $\{1,2,3,4,5\}$ defined by $a \wedge b = \min\{a,b\}$. Write the operation table.

$\wedge$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

HR

R_5

*] since The table is symmetric about pd

$\wedge$ is commutative i.e $a*b = b*a$

*] since $R_5 \sim HR$, 5 is identity element

4] Conside b.o $*$ on set $\{1,2,3,4,5\}$ given by

$*$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

(i) Compute $(2*3)*4$ and $2*(3*4)$

$= 1*4$ and $2*1$

$= 1$ and 1

$\Rightarrow (2*3)*4 = 2*(3*4)$

$*$ is associative.

(ii) Is $*$ commutative?

Yes, since the table is symmetric about pd.

(iii) Compute $(2*3)*(4*5)$

$= 1*1$

$= 1$

5] Let $*'$ be the b.o on $\{1, 2, 3, 4, 5\}$ defined by $a*'b$ = HCF of a and b. Construct the table.

$*'$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

6] Let $*$ be b.o on N given by $a*b$ = LCM of a and b.

Find (i) $5*7$

= LCM of 5 & 7 = 35

2	20, 16
2	10, 8
2	5, 4
2	5, 2
5	5, 1

(ii) $20*16$

= LCM of 20 & 16

= 80

(iii) Is $*$ commutative?

LCM of a & b = LCM b and a

$\therefore *$ is commutative

(iv) Is $*$ associative?

LHS $= a*(b*c)$

$= a*$(LCM of b & c)

= LCM of a, b, c

= LCM of a & b $*c$

= ~~LCM of~~ $(a*b)*c$

= RHS

$\therefore *$ is associative

7] Is $*$ defined on the set $\{1,2,3,4,5\}$ by $a*b =$ LCM of a & b, a b.o. Justify.

Sol] No, $*$ is not b.o on given set

$\because$ LCM of $3,4 = 12 \notin$ set.

9] Let $*$ be b.o on Q. Find which of the b.o are commutative, associative?

i] $a*b = a-b$

(a) Commutative:

$LHS = a*b = a-b \neq b-a = b*a = RHS$

$\because LHS \neq RHS$, $\therefore$ $*$ is not commutative

(b) Associative: $*$ is not associative

ii] $a*b = a^2+b^2$

(a) Commutative: $\forall a,b \in A$, $a*b = b*a$

$LHS = a*b = a^2+b^2 = b^2+a^2 = b*a = RHS$

$\therefore$ $*$ is commutative

(b) Associative: $\forall a,b,c \in A$, $(a*b)*c = a*(b*c)$

$LHS = a*(b*c) = a*(b^2+c^2) = a^2+(b^2+c^2)^2$

$RHS = (a*b)*c = (a^2+b^2)*c = (a^2+b^2)^2+c^2$

$LHS \neq RHS$, $\therefore$ $*$ is not associative

iii] $a*b = a+ab$

(a) Commutative:

$LHS = a*b = a+ab$

$RHS = b*a = b+ba$

$\because LHS \neq RHS$, $\therefore$ $*$ is not commutative.

(b) Associative

$LHS = a*(b*c) = a*(b+bc) = a+a(b+bc)$

$RHS = (a*b)*c = (a+ab)*c = a+ab+(a+ab)c$

$\because LHS \neq RHS$, $\therefore$ $*$ is not associative iv] $a * b = (a-b)^2$

(a) Commutative :

$LHS = a*b = (a-b)^2$
$RHS = b*a = (b-a)^2$
$\because LHS \neq RHS$
$*$ is not commutative

(b) Associative :

$LHS = a*(b*c) = a*(b-c)^2 = [a-(b-c)^2]^2$
$RHS = (a*b)*c = (a-b)^2 * c = [(a-b)^2 - c]^2$
$\because LHS \neq RHS$
$\therefore *$ is not associative

v] $a*b = ab^2$

(a) Commutative :

$LHS = a*b = ab^2$
$RHS = b*a = ba^2$
$\because LHS \neq RHS$
$\therefore *$ is not commutative

(b) Associative :

$LHS = a*(b*c) = a*(bc^2) = a(bc^2)^2$
$RHS = (a*b)*c = (ab^2)*c = ab^2c^2$
$\because LHS \neq RHS$
$\therefore *$ is not associative

11] Let $A = N \times N$ and $*$ be b.o defined by $(a,b)*(c,d) = (a+c, b+d)$. ST $*$ is commutative and associative. Find e if any.

Sol] Given: $(a,b)*(c,d) = (a+c, b+d)$ on A

(a) Commutative :

Let $(a,b), (c,d), (e,f)$ be 3 pairs $\in A$

$LHS = (a,b)*(c,d) = (a+c, b+d)$
$RHS = (c,d)*(a,b) = (c+a, d+b)$
$= (a+c, b+d)$

$LHS = RHS$, $\therefore *$ is commutative

(b) Associative :

$LHS = (a,b)*[(c,d)*(e,f)] = (a,b)*[c+e, d+f]$
$= (a+c+e, b+d+f)$

$RHS = [(a,b)*(c,d)]*(e,f) = (a+c, b+d)*(e,f)$
$= (a+c+e, b+d+f)$

$LHS = RHS$, $\therefore *$ is associative

(c) Existence of ~~inver~~ identity element : $\forall\, a, e \in A$, $a * e = a$

$(a,b) * (e,e) = (a,b)$

$(a+e, b+e) = (a,b)$

$a + e = a$ | $b + e = b$

$e = 0$ | $\boxed{e = 0 \notin N}$

$\therefore$ identity element does not exist for the given operation on the given set.

10] Find which of the operations in Q9 have identity.

i] $a * b = a - b$

$\forall\, a, e \in A$, $a * e = a$

Consider, $a * e = a$

$a - e = a$

$e = 0$

ii] $a * b = a^2 + b^2$

Consider, $a * e = a$

$a^2 + e^2 = a$

$e^2 = a - a^2$

$e = \sqrt{a - a^2}$

iii] $a * b = a + ab$

Consider, $a * e = a$

$a + ae = a$

$ae = 0$

$e = 0$

iv] $a * b = (a - b)^2$

$a * e = a$

$(a - e)^2 = a$

$a - e = \sqrt{a}$

$e = a - \sqrt{a}$

v] $a * b = ab^2$

$a * e = a$

$ae^2 = a$

$e^2 = 1$

$e = 1$