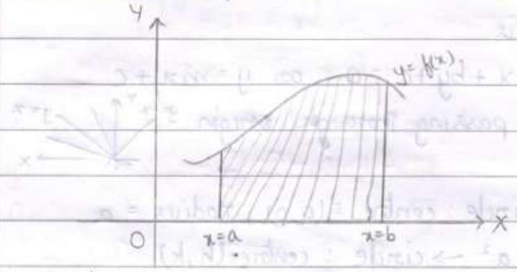
APPLICATION OF INTEGRALS

Area enclosed between the curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$.



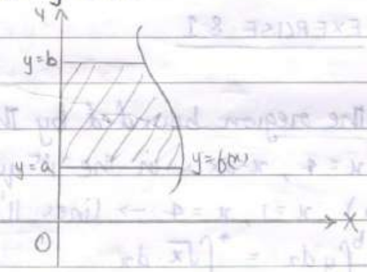
$$A = \int_a^b f(x)\,dx$$

$dx \rightarrow$ w.r.t 'x'

i.e, Area measured over x-axis

$x = a$, $x = b$ } lines $\parallel^{le}$ to y-axis

Area enclosed between the curve $y = f(x)$, the lines $y = a$, $y = b$ and y-axis



$$A = \int_a^b x \cdot dy$$

$dy \rightarrow$ w.r.t 'y' $\Rightarrow$ Area measured over y-axis

$x = f(y)$

$y = a$, $y = b$ } lines $\parallel^{le}$ to x-axis

General Lines and Curves:

1. $x = k \rightarrow$ line $\parallel^{le}$ to y-axis

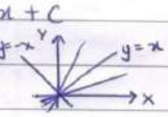
$x = 0 \rightarrow$ y-axis

$y = k \rightarrow$ line $\parallel^{le}$ to x-axis

$y = 0 \rightarrow x$-axis

General line: $ax + by + c = 0$ or $y = mx + c$

** $y = mx \rightarrow$ lines passing through origin

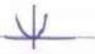


$x^2 + y^2 = a^2 \rightarrow$ circle: centre $\equiv (0,0)$, radius $= a$

$(x-h)^2 + (y-k)^2 = a^2 \rightarrow$ circle : centre$\equiv(h,k)$

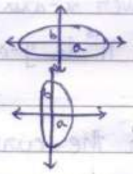
$y^2 = 4ax$ 

Parabola

$x^2 = 4ay$ 

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $b > a$

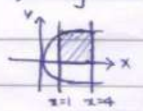


Ellipes $\quad y = \frac{b}{a}\sqrt{a^2 - x^2}$

EXERCISE 8.1

1] Find the area of the region bounded by the curve $y^2 = x$ and $x = 1$, $x = 4$, x-axis in the 1^{st} quadrant.

Gn: $y^2 = x$ (parabola) $\quad y = \sqrt{x}$ $\quad x = 1,\ x = 4 \rightarrow$ lines $\parallel^{le}$ to y-axis



$$\text{Area} = \int_a^b y\,dx = \int_1^4 \sqrt{x}\cdot dx$$

$$= \frac{2}{3}x\sqrt{x}\Big]_1^4$$

$$= \frac{2}{3}[8-1] = \frac{14}{3} \text{ sq.ut}$$

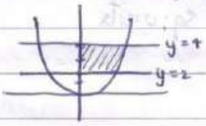
2] Find the area of the region bounded by $y^2 = 9x$, $x=2$, $x=4$, x-axis in 1^{st} quadrant.

Gn: $y^2 = 9x$, $y = \sqrt{9x} = 3\sqrt{x}$

$A = \int_a^b y\,dx = \int_2^4 3\sqrt{x}\,dx = \cancel{3}\cdot\frac{2}{\cancel{3}}x\sqrt{x}\Big]_2^4 = 2(8-2\sqrt{2})$ sq. units

3] Find the area of the region bounded by $x^2 = 4y$, $y=2$, $y=4$, y axis and 1^{st} quadrant.

Gn: $x^2 = 4y \Rightarrow x = 2\sqrt{y}$



$A = \int_a^b x\cdot dy = \int_2^4 2\sqrt{y}\,.dy$

$A = 2\cdot\frac{2}{3}y\sqrt{y}\Big]_2^4$

$= \frac{4}{3}\left[8-2\sqrt{2}\right]$ sq. units

Eg 1] Find the area enclosed by the circles

(a) $x^2+y^2=a^2$ $\quad \pi a^2$

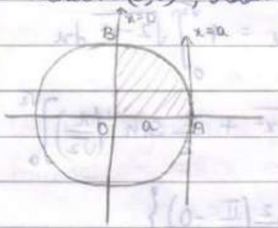
(b) $x^2+y^2=8$ $\quad 8\pi$ $\quad \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

(c) $x^2+y^2=25$ $\quad 25\pi$ $\quad x^2+y^2=a^2 \Rightarrow y=\sqrt{a^2-x^2}$

(d) $x^2+y^2=2$ $\quad 2\pi$

a] $x^2+y^2=a^2 \Rightarrow y=\sqrt{a^2-x^2}$

center $=(0,0)$, radius $=a$



From fig:

Area of circle $= 4\times$ Area of $\widehat{OAB}$ (sector)

$A = 4\int_0^a y\,dx = 4\int_0^a\sqrt{a^2-x^2}\,dx$

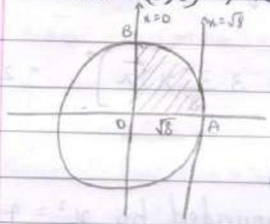
$A = 4\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$= 4\left\{0+\frac{a^2}{2}\left(\frac{\pi}{2}-0\right)\right\}$

$A = 4\cdot\frac{\pi a^2}{4} = \pi a^2$ sq. units

b) $x^2+y^2=8 \Rightarrow y=\sqrt{8-x^2}$

Center = (0,0), radius = $\sqrt{8}$



Area of circle = 4 × Area of $\widehat{OAB}$

$$A = 4\int_0^a y\cdot dx = 4\int_0^{\sqrt{8}} \sqrt{8-x^2}\, dx$$

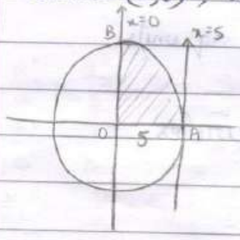
$$A = 4\left[\frac{x}{2}\sqrt{8-x^2}+\frac{8}{2}\sin^{-1}\left(\frac{x}{\sqrt{8}}\right)\right]_0^{\sqrt{8}}$$

$$= 4\left\{0-\frac{8}{2}\left(\frac{\pi}{2}-0\right)\right\}$$

$$= \cancel{4}\cdot\frac{8\pi}{\cancel{4}} = 8\pi \text{ sq. units}$$

c) $x^2+y^2=25 \Rightarrow y=\sqrt{25-x^2}$

center = (0,0), radius = 5



$$A = 4\int_0^a y\cdot dx = 4\int_0^5 \sqrt{25-x^2}\, dx$$

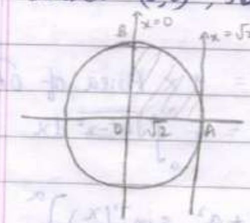
$$A = 4\left[\frac{x}{2}\sqrt{25-x^2}+\frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_0^5$$

$$= 4\left\{0-\frac{25}{2}\left(\frac{\pi}{2}-0\right)\right\}$$

$$= \cancel{4}\cdot\frac{25\pi}{\cancel{4}} = 25\pi \text{ sq. units}$$

d) $x^2+y^2=2 \Rightarrow y=\sqrt{2-x^2}$

center = (0,0), radius = $\sqrt{2}$



$$A = 4\int_0^a y\cdot dx = 4\int_0^{\sqrt{2}} \sqrt{2-x^2}\, dx$$

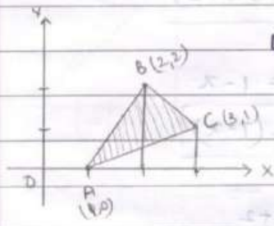
$$A = 4\left[\frac{x}{2}\sqrt{2-x^2}+\frac{2}{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\sqrt{2}}$$

$$= 4\left\{0-\frac{2}{2}\left(\frac{\pi}{2}-0\right)\right\}$$

$$= \cancel{4}\cdot\frac{2\pi}{\cancel{4}} = 2\pi \text{ sq. units}$$

** Eg 9] Find the area of the region bounded by the Δ^{le} whose vertices are (1,0), (2,2), (3,1).

Sol] Gn: $A=(1,0)$, $B=(2,2)$, $C=(3,1)$



Eq. of sides: $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$

$\overline{AB}$: $\dfrac{y-0}{2-0} = \dfrac{x-1}{2-1} \Rightarrow \boxed{y = 2(x-1)}$

$\overline{BC}$: $\dfrac{y-2}{1-2} = \dfrac{x-2}{3-2} \Rightarrow y-2 = 2-x$

$\boxed{y = 4-x}$

$\overline{AC}$: $\dfrac{y-0}{1-0} = \dfrac{x-1}{3-1} \Rightarrow \boxed{y = \frac{1}{2}(x-1)}$

Area of $\Delta ABC = \int_1^2 \text{Ar under } \overline{AB} + \int_2^3 \text{Ar under } \overline{BC} - \int_1^3 \text{Ar under } \overline{AC}\, dx$

$A = \int_1^2 2(x-1)\,dx + \int_2^3 (4-x)\,dx - \int_1^3 \frac{1}{2}(x-1)\,dx$

$= \left[2\frac{x^2}{2} - 2x\right]_1^2 + \left[4x - \frac{x^2}{2}\right]_2^3 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^3$

$= 3 - 2(1) + 4(1) - \frac{1}{2}(5) - \frac{1}{4}(8) + \frac{1}{2}(2)$

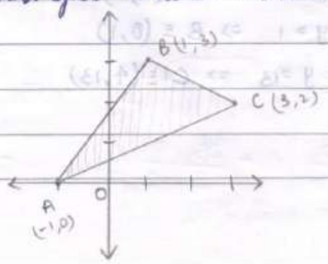
$= 3 - \cancel{2} + \cancel{4} - \cancel{2} + 1 - \frac{5}{2} = 4 - \frac{5}{2} = \boxed{\frac{3}{2} \text{ sq. units}}$

Verification: Ar of $\Delta = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = \frac{1}{2}\left[1 + 1(-4)\right] = \frac{3}{2}$

EXERCISE 8.2

4] Using integration find the area of region bounded by the triangles whose vertices are (-1,0), (1,3), (3,2).

Sol]



Gn: $A=(-1,0)$, $B=(1,3)$, $C=(3,2)$

Eq. of sides : $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$\overline{AB}$: $\frac{y-0}{3-0} = \frac{x+1}{1+1} \Rightarrow \boxed{y = \frac{1}{2}(3x+3)}$

$\overline{BC}$: $\frac{y-3}{2-3} = \frac{x-1}{3-1} \Rightarrow 2y-6 = 1-x$

$\boxed{y = \frac{1}{2}(7-x)}$

$\overline{AC}$: $\frac{y-0}{2-0} = \frac{x+1}{3+1} \Rightarrow 4y = 2x+2$

$\boxed{y = \frac{1}{2}(x+1)}$

Ar of ΔABC $= \int_{-1}^{1} \frac{3}{2}(x+1)\,dx + \int_{1}^{3} \frac{1}{2}(7-x)\,dx - \int_{-1}^{3} \frac{1}{2}(x+1)\,dx$

$A = \frac{3}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1} + \frac{1}{2}\left[7x-\frac{x^2}{2}\right]_{1}^{3} - \frac{1}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{3}$

$= \frac{3}{2}(2) + \frac{1}{2}(10) - \frac{1}{2}(8)$

$= 3+5-4 = \boxed{4 \text{ sq units}}$

3] Find the area of the region bounded by the curves $y = x^2+2$, $y = x$, $x = 0$ and $x = 3$

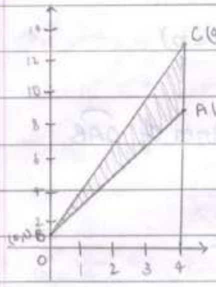
5] Find the area of the triangular region whose sides have the equations $y = 2x+1$, $y = 3x+1$, $x = 4$.

Sol] Gn: $\overline{AB}: y = 2x+1$, $\overline{BC}: y = 3x+1$, $\overline{AC}: x = 4$ line ∥ to y-axis

pt. A : $\overline{AB}$ & $\overline{AC}$: $y = 2(4)+1 = 9 \Rightarrow A \equiv (4,9)$

pt. B : $\overline{AB}$ & $\overline{BC}$: $0 = x \Rightarrow y = 1 \Rightarrow B \equiv (0,1)$

pt. C : $\overline{AC}$ & $\overline{BC}$: $y = 3(4)+1 \Rightarrow y = 13 \Rightarrow C \equiv (4,13)$



Area of $\Delta ABC = \int_0^4 \text{Ar under } \overline{BC} - \int_0^4 \text{Ar under } \overline{BA}$

$$A = \int_0^4 (3x+1)\,dx - \int_0^4 (2x+1)\,dx$$

$$= \int_0^4 (3x+1) - (2x+1)\,dx$$

$$= \int_0^4 x \cdot dx = \left.\frac{x^2}{2}\right]_0^4 = \frac{16}{2} = \boxed{8 \text{ sq. units}}$$

EXERCISE 8·1

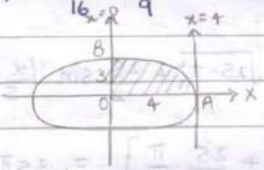
4] Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ $(a > b)$

Sol] Area, $A = \pi ab = \pi(4)(3) = 12\pi$ sq. units

5] Find the area of region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ $(a < b)$

Sol] $A = \pi ab = \pi(2)(3) = 6\pi$ sq. units

4] Gn: $\frac{x^2}{16} + \frac{y^2}{9} = 1$; $y = \frac{3}{4}\sqrt{16-x^2}$



From fig,

Area of ellipse $= 4 \times$ Area of $\widehat{OAB}$

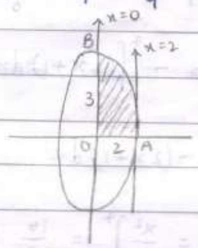
$$A = 4\int_a^b y\,dx$$

$$A = 4\int_0^4 \frac{3}{4}\sqrt{16-x^2}\,dx$$

$$= 4 \cdot \frac{3}{4}\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4$$

$$= 3\left[8 \cdot \frac{\pi}{2}\right] = 12\pi \text{ sq. units}$$

5] Gn: $\frac{x^2}{4}+\frac{y^2}{9}=1$; $y=\frac{3}{2}\sqrt{4-x^2}$ $(a<b)$



Area of ellipse = 4 x Area of $\widehat{OAB}$

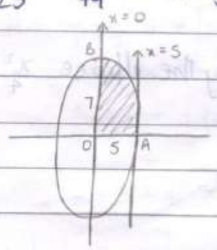
$A = 4\int_a^b y\,dx$

$A = 4\int_0^2 \frac{3}{2}\sqrt{4-x^2}\,dx$

$A = \overset{2}{\cancel{4}}\cdot\frac{3}{\cancel{2}}\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$

$A = 6\left[0+\frac{\cancel{4}}{\cancel{2}}\cdot\frac{\pi}{\cancel{2}}\right] = 6\pi$ sq. units

*] $\frac{x^2}{25}+\frac{y^2}{49}=1$; $y=\frac{7}{5}\sqrt{25-x^2}$ $(a<b)$



Area of ellipse = 4 x Area of $\widehat{OAB}$

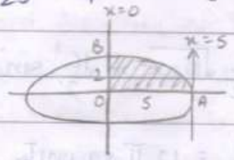
$A = 4\int_a^b y\,dx$

$A = 4\int_0^5 \frac{7}{5}\sqrt{25-x^2}\,dx$

$A = 4\cdot\frac{7}{5}\left[\frac{x}{2}\sqrt{25-x^2}+\frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_0^5$

$= \cancel{4}\cdot\frac{7}{\cancel{5}}\left[0+\frac{\overset{5}{\cancel{25}}}{\cancel{2}}\cdot\frac{\pi}{\cancel{2}}\right] = 35\pi$ sq. units

*] $\frac{x^2}{25}+\frac{y^2}{4}=1$; $y=\frac{2}{5}\sqrt{25-x^2}$ $(a>b)$



Area of ellipse $= 4\int_a^b y\cdot dx$

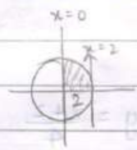
$A = 4\int_0^5 \frac{2}{5}\sqrt{25-x^2}\,dx$

$= 4\cdot\frac{2}{5}\left[\frac{x}{2}\sqrt{25-x^2}+\frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_0^5$

$= \cancel{4}\cdot\frac{2}{\cancel{5}}\left[0+\frac{\overset{5}{\cancel{25}}}{\cancel{2}}\cdot\frac{\pi}{\cancel{2}}\right] = 10\pi$ sq. units

12] Area lying in the first quadrant and bounded by the circle $x^2+y^2=4$ and the lines $x=0$ and $x=2$ is

a) π
b) $\frac{\pi}{2}$
c) $\frac{\pi}{3}$
d) $\frac{\pi}{4}$



$x^2+y^2=4$; center $(0,0)$; radius $=2$

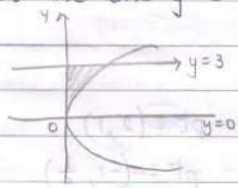
Area of req. region $= \frac{1}{4}$ Area of circle

$A = \frac{1}{4}\pi(4)$

$A = \pi$ sq. units

13] Area of the region bounded by the curve $y^2=4x$, y-axis and the line $y=3$ is

Sol]



$y^2=4x \rightarrow$ parabola

$y=3$, line $\parallel^{le}$ to x-axis

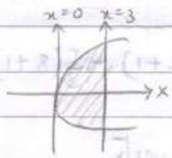
$x = \frac{y^2}{4}$

Limits : $y=0$, $y=3$

$A = \int_0^3 x \cdot dy = \int_0^3 \frac{y^2}{4} dy = \frac{1}{4} \cdot \left.\frac{y^3}{3}\right]_0^3 = \frac{1}{4} \cdot \frac{1}{3} \cdot 3(9) = \frac{9}{4}$ sq. unit

11] Find the area of the region bounded by the curve $y^2=4x$, line $x=3$

Sol]



$y^2=4x \rightarrow$ parabola

$y=\sqrt{4x} = 2\sqrt{x}$

$x=3$, line $\parallel^{le}$ to y-axis

Limits : $x=0$, $x=3$

$A = 2\int_0^3 y \cdot dx = 2\int_0^3 2\sqrt{x}\, dx = 2 \cdot 2 \cdot \frac{2}{3} \left. x\sqrt{x} \right]_0^3$

$A = 2 \cdot \frac{4}{3} \cdot 3\sqrt{3} = 8\sqrt{3}$ sq. units

10] Find the area bounded by the curve $x^2 = 4y$ and line $x = 4y - 2$

Sol] $x^2 = 4y \rightarrow ①$ $\quad x = 4y - 2 \rightarrow ②$

$(4y-2)^2 = 4y$ $\quad y = \frac{x^2}{4}$

$16y^2 + 4 - 16y - 4y = 0$ $\quad ② \Rightarrow y = \frac{x+2}{4}$

$16y^2 - 20y + 4 = 0$

$4(4y^2 - 5y + 1) = 0$

$4y^2 - 4y - y + 1 = 0$

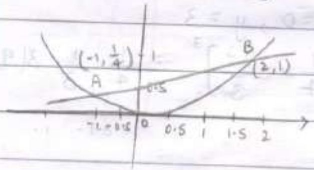
$4y(y-1) - 1(y-1) = 0$

$(4y-1)(y-1) = 0$

$y = 1, \frac{1}{4}$

$y = 1 \Rightarrow x = 4(1) - 2 \Rightarrow x = 2$ $\quad pt \equiv (2,1)$

$y = \frac{1}{4} \Rightarrow x = 4\left(\frac{1}{4}\right) - 2 \Rightarrow x = -1$ $\quad pt \equiv \left(-1, \frac{1}{4}\right)$



Required area $= \widehat{OAB}$ = Area under line – Area under parabola

$$A = \int_{-1}^{2} \left(\frac{x+2}{4}\right) dx - \int_{-1}^{2} \frac{x^2}{4}\, dx$$

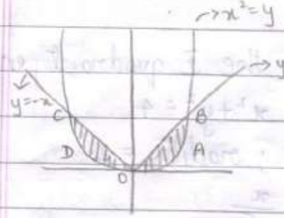
$$A = \int_{-1}^{2} \frac{1}{4}(x + 2 - x^2)\, dx$$

$$A = \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{1}{4}\left[\frac{3}{2} + 2(2+1) - \frac{1}{3}(8+1)\right]$$

$$A = \frac{1}{4}\left(\frac{3}{2} + 6 - 3\right) = \frac{1}{4}\left(\frac{9}{2}\right) = \frac{9}{8} \text{ sq. units}$$

9] Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$

Sol] $x^2 = y \rightarrow ①$ $\quad ; \; y = |x| = \begin{cases} x \; ; \; x \geq 0 \\ -x \; ; \; x < 0 \end{cases}$



Req. area = $\widehat{OABCD}$ = 2 × Area of $\widehat{OAB}$

$A = \int_a^b y\,dx$

$= 2\int_0^1 (\text{Ar under line} - \text{Ar. under parabola})\,dx$

$= 2\int_0^1 (x - x^2)\,dx$

$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{2}{6}$ sq. units $= \frac{1}{3}$ sq. unit

pt. of intersection:

$y = x,\ y = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

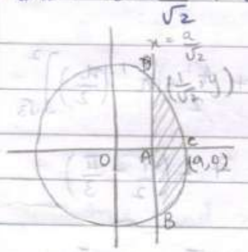
$x = 0, 1$

pt ≡ (0,0), (1,1)

7] Find the area of the smaller part of the circle $x^2+y^2=a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$

Sol] $x^2+y^2=a^2 \longrightarrow ①$; center ≡ (0,0) ; radius = a

line : $x = \frac{a}{\sqrt{2}}$ – line $\parallel^{le}$ to y-axis



$① \Rightarrow y = \sqrt{a^2-x^2}$

Req. area = $\widehat{ABCD}$ = 2 × Area $\widehat{ACD}$

$A = 2\int_a^b y\,dx$

$= 2\int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2-x^2}\,dx$

$= 2\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{a/\sqrt{2}}^{a}$

$A = 2\left[\frac{1}{2}\left(0 - \frac{a}{\sqrt{2}}\frac{a}{\sqrt{2}}\right) + \frac{a^2}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$

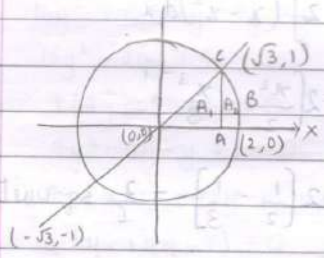
$= 2\left[\frac{-a^2}{4} + \frac{a^2}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right] = -\frac{a^2}{2} + a^2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$

$A = \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$ sq. units

6] Find the area of the region in the I quadrant enclosed by x-axis, $x = \sqrt{3}y$ and circle $x^2+y^2=4$.

Sol] $x^2+y^2=4 \rightarrow (1)$; centre $(0,0)$; radius $= 2$

line : $x = \sqrt{3}y \rightarrow (2)$ $\quad (2) \Rightarrow y = \frac{x}{\sqrt{3}}$



$x^2 + \frac{x^2}{3} = 4 \Rightarrow \frac{3x^2+x^2}{3} = 4$

$\Rightarrow 4x^2 = 12$

$\Rightarrow x = \pm\sqrt{3}$

$(2) \Rightarrow y = \frac{\sqrt{3}}{\sqrt{3}}$

$y = \pm 1$

pts $(\sqrt{3},1), (-\sqrt{3},-1)$

$(1) \Rightarrow y = \sqrt{4-x^2}$

Req. Area $= \widehat{OBC} = \Delta OAC +$ Area $\widehat{ABC}$

$A = \int_a^b y\,dx = \int_0^{\sqrt{3}}$ Area under line $+ \int_{\sqrt{3}}^{2}$ Area under circle

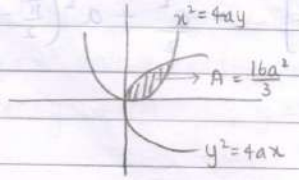
$A = \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\,dx + \int_{\sqrt{3}}^{2} \sqrt{4-x^2}\,dx$

$= \left.\frac{x^2}{2\sqrt{3}}\right]_0^{\sqrt{3}} + \left.\frac{x}{2}\sqrt{4-x^2} + \frac{\cancel{4}^2}{\cancel{2}}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$

$= \frac{1}{2\sqrt{3}}(3) + \frac{1}{2}(0-\sqrt{3}) + 2\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + 2 \cdot \frac{\pi}{\cancel{6}3} = \frac{\pi}{3}$ sq. units

** Area enclosed between 2 parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16a^2}{3}$ sq. units



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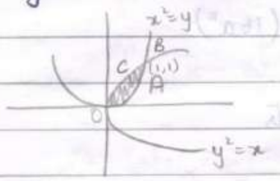
*** Eg6] Find the area of the region bounded by the parabola $y = x^2$ and $y^2 = x$

Sol] $y = x^2 \longrightarrow ①$

$y^2 = x \longrightarrow ②$



pt. of intersection

$x^2 = y$

$x^4 = y^2$

$x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0 \Rightarrow x = 0, x = 1$

① $\Rightarrow y = 0, 1$

pt. $(0,0), (1,1)$

Req. area $= \widehat{OABC} = \int_0^1 \text{Ar under } \underset{②}{x} - \int_0^1 \text{Ar. under } \underset{①}{x^2}\, dx$

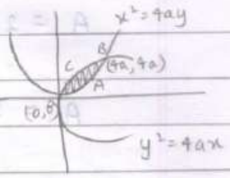
$A = \int_0^1 \sqrt{x} - x^2\, dx = \frac{2}{3} x\sqrt{x} - \frac{x^3}{3}\Big]_0^1$

$A = \frac{2}{3}(1) - \frac{1}{3} = \frac{1}{3}$ sq. units

Shortcut : $4a = 1 \Rightarrow a = \frac{1}{4}$ $A = \frac{16a^2}{3} = \frac{16}{3}\left(\frac{1}{16}\right) = \frac{1}{3}$ sq. units

*] $y^2 = 4ax$ and $x^2 = 4ay \rightarrow ①$

$\Rightarrow y = 2\sqrt{ax}$ $\Rightarrow x = 2\sqrt{ay}$



① $\Rightarrow x^2 = 4ay$

$x^4 = 4^2a^2y^2$

$x^4 = 4^2a^2(4ax)$

$x(x^3 - 4^3a^3) = 0$

$x = 0, 4a$

① $\Rightarrow y = 0, 4a$

pts are $(0,0), (4a, 4a)$

$$\text{Req. area} = \widehat{OABC} = \int_0^{4a} \text{Ar under ①} - \text{Ar under ② } dx$$

$$A = \int_0^{4a}\left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4a}\right)dx = 2\sqrt{a}\cdot\frac{2}{3}x\sqrt{x} - \frac{1}{4a}\frac{x^3}{3}\Big]_0^{4a}$$

$$A = \frac{4\sqrt{a}}{3}a(2\sqrt{a}) - \frac{1}{4a(3)}(4a)(16a^2)$$

$$= \frac{16a^2}{3}(2-1) = \frac{16a^2}{3} \text{ sq. units}$$

*] $y^2 = 8x \rightarrow ①$ $x^2 = 8y \rightarrow ②$

$y = \sqrt{8x}$ $\frac{x^2}{8} = y$

$$x^4 = 8^2y^2 = 8^2(8x)$$

$$x^4 - 8^3x = 0$$

$$x(x^3 - 8^3) = 0$$

$$x = 0, 2$$

$$y = 0, 2$$

pts are $(0,0), (2,2)$

$$\text{Req. area} = OABC = \int_0^2 \text{Ar under ①} - \text{Ar under ② } dx$$

$$A = \int_0^2\left(\sqrt{8}\sqrt{x} - \frac{x^2}{8}\right)dx = \sqrt{8}\,\frac{2}{3}x\sqrt{x} - \frac{1}{8}\frac{x^3}{3}\Big]_0^2$$

$$A = 2\sqrt{2}\cdot\frac{2}{3}(2-0)\sqrt{2} - \frac{1}{8}\left(\frac{8-0}{3}\right)$$

$$A = \frac{16}{3} - \frac{1}{3} = \frac{15}{3} = 5 \text{ sq. units}$$

*] $y^2 = 16x \rightarrow ①$, $x^2 = 16y \rightarrow ②$

$$x^4 = 16^2y^2$$

$$x^4 = 16^2(16x)$$

$$x(x^3 - 16) = 0$$

$$x = 0, 16$$

$$\Rightarrow y = 0, 16$$

pts are $(0,0), (16,16)$

Req area = OABC = $\int_a^b y\,dx$

① $\Rightarrow y = 4\sqrt{x}$, ② $\Rightarrow y = \frac{x^2}{16}$

Ar = $\int_0^{16}$ Ar under ① − Ar under ② dx

$= \int_0^{16} 4\sqrt{x} - \frac{x^2}{16}\,dx = \left[4 \cdot \frac{2}{3} x\sqrt{x} - \frac{x^3}{16 \cdot 3}\right]_0^{16}$

$= \frac{8}{3}(16\sqrt{16}) - \frac{16^3 - 0}{3 \cdot 16} = \frac{8}{3}(16 \times 4) - \frac{16^2 \cdot \cancel{16}}{3 \cdot \cancel{16}}$

$= \frac{8}{3}(64) - \frac{16^2}{3} = \frac{16}{3}(32 - 16) = \frac{256}{3}$ sq. units

Verification: $y^2 = 16x$; $x^2 = 16y$

$4a = 16 \Rightarrow a = 4$

$A = \frac{16a^2}{3} = \frac{16(16)}{3} = \frac{256}{3}$

EXERCISE 8·2

Eg 10] Find the area of region enclosed b/w 2 circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$.

Sol] $x^2 + y^2 = 4 \rightarrow$ ①
center $\equiv (0,0)$
radius = 2
$y = \sqrt{4 - x^2}$

$(x-2)^2 + y^2 = 4 \rightarrow$ ②
$C \equiv (2,0)$, $r = 2$
$y = \sqrt{4 - (x-2)^2}$

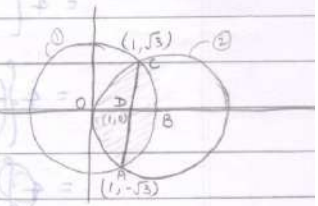
$\Rightarrow \cancel{4} - x^2 = \cancel{4} - (x-2)^2$

$x^2 = (x-2)^2$

$x^2 = x^2 + 4 - 4x$

$4x = 4$

$\boxed{x = 1}$



① $\Rightarrow y = \pm\sqrt{3}$

∴ pts are $(1, \sqrt{3})$, $(1, -\sqrt{3})$

Req. area $= \widehat{OABC} = 4 \times \widehat{BDC}$

$$A = 4\int_a^b y\,dx = 4\int_1^2 \sqrt{4-x^2}\,dx$$

$$A = 4\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2$$

$$= 4\left\{\frac{1}{2}(0-\sqrt{3}) + 2\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\right\}$$

$$= -2\sqrt{3} + 8\frac{\pi}{3} = 2\left(\frac{4\pi}{3} - \sqrt{3}\right) \text{ sq. units}$$

2] Find the area bounded by the curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.

Sol] $x^2+y^2 = 1 \rightarrow ①$ | $(x-1)^2 + y^2 = 1$

$C \equiv (0,0)$, $r = 1$ | $C \equiv (1,0)$, $r = 1$

$y = \sqrt{1-x^2}$ | $y = \sqrt{1-(x-1)^2}$

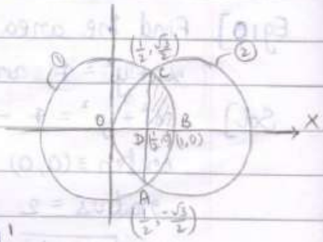
$$\Rightarrow \cancel{1} - x^2 = \cancel{1} - (x-1)^2$$

$$x^2 = x^2 + 1 - 2x$$

$$2x = 1$$

$$\boxed{x = \frac{1}{2}} \qquad ① \Rightarrow y = \pm\frac{\sqrt{3}}{2}$$

pts are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$



Req. Area $= \widehat{OABC} = 4\ \widehat{BDC}$

$$A = 4\int_a^b y\,dx = 4\int_{\frac{1}{2}}^1 \sqrt{1-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_{\frac{1}{2}}^1$$

$$= 4\left\{\left(-\frac{1}{4}\cdot\frac{\sqrt{3}}{2}\right) + \left(\frac{\pi}{2} - \frac{\pi}{6}\right)\frac{1}{2}\right\}$$

$$= -\frac{\sqrt{3}}{2} + \frac{2\pi}{3}$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \text{ sq. units}$$