JEE-Main-28-07-2022-Shift-1 (Memory Based)

MATHEMATICS

Question: Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ & $B_0 = A^{49} + 2A^{98}$. If $B_n = adj\left(B_{n-1}\right) \forall n \geq 1$ then $\left|B_4\right|$

Options:
(a) 3^{28}
(b) 3^{30}
(c) 3^{32}
(d) 3^{36}

Answer: (c)

Solution:

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore B_0 = I + 2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

As $B_n = adj\left(B_{n-1}\right)$

$$\therefore B_4 = adj\left(B_3\right) = adj\left(adj\, B_2\right) = adj\left(adj\left(adj\, B_1\right)\right)$$

$$= adj\left(adj\left(adj\left(adj\, B_0\right)\right)\right)$$

$$\therefore \left|B_4\right| = \left|B_0\right|^{2^4} = \left|B_0\right|^{16}$$

$$\left|B_0\right| = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3(-3)$$

$$\therefore \left|B_0\right|^{16} = 9^{16} = 3^{32}$$

Question: Considering the principal value of ITF, the sum of all solutions of $\cos^{-1} x - 2\sin^{-1} x = \cos^{-1} 2x$ is

Options:
(a) 0
(b) 1
(c) $\frac{1}{2}$
(d) $-\frac{1}{2}$

Answer: (a)
Solution:

$$\cos^{-1} x - 2\sin^{-1} x = \cos^{-1} 2x$$
$$\cos^{-1} x - \cos^{-1} 2x = 2\sin^{-1} x$$
$$\cos\left(\cos^{-1} x - \cos^{-1} 2x\right) = \cos\left(2\sin^{-1} x\right)$$
$$x(2x) + \sqrt{1-x^2}\sqrt{1-4x^2} = 1-2x^2$$
$$4x^2 - 1 = \sqrt{1-x^2}\sqrt{1-4x^2}$$
$$\left(4x^2-1\right)^2 = \left(1-4x^2-x^2+4x^4\right)$$
$$16x^4 - 8x^2 + 1 = 1 - 5x^2 + 4x^4$$
$$\Rightarrow 12x^4 - 3x^2 = 0$$
$$\Rightarrow x^2\left(4x^2-1\right) = 0$$
$$\Rightarrow x = 0, \frac{1}{2}, -\frac{1}{2}$$

Sum = 0

Question: If minimum value of $f(x) = \frac{5x^2}{2} + \frac{\alpha}{x^5}, x > 0$ is 14 then $\alpha = ?$

Options:
(a) 32
(b) 64
(c) 128
(d) 256
Answer: (c)
Solution:

$$f(x) = \frac{5x^2}{2} + \frac{\alpha}{x^5}$$
$$f'(x) = 5x - \frac{5\alpha}{x^6} = 0$$
$$\Rightarrow x^7 = \alpha$$
$$\Rightarrow x = \alpha^{\frac{1}{7}}$$
$$f\left(\alpha^{\frac{1}{7}}\right) = 14$$
$$\frac{5\alpha^{\frac{2}{7}}}{2} + \frac{\alpha}{\alpha^{\frac{5}{7}}} = 14$$
$$\frac{7\alpha^{\frac{2}{7}}}{2} = 14$$
$$\alpha^{\frac{2}{7}} = 4$$

$\alpha = 4^{\frac{7}{2}} = 2^7 = 128$

Question: $x\,dy = \left(\sqrt{x^2+y^2}+y\right)dx$ curve passes through $(1,0)$. Find $y(2) = ?$

Answer: $\frac{3}{2}$

Solution:

$x\,dy = \left(\sqrt{x^2+y^2}+y\right)dx$

$\frac{dy}{dx} = \sqrt{1+\frac{y^2}{x^2}}+\frac{y}{x}$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$x\frac{dv}{dx} = \sqrt{1+v^2}$

$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

$= \ln\left|v+\sqrt{1+v^2}\right| = \ln|x| + \ln c$

$\Rightarrow v+\sqrt{1+v^2} = cx$

$\Rightarrow \frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}} = cx$

$x = 1, y = 0$

$\Rightarrow 0+1 = c$

$\Rightarrow c = 1$

$\Rightarrow \frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}} = x$

$\Rightarrow \frac{y}{2}+\sqrt{1+\frac{y^2}{4}} = 2$

$\Rightarrow \sqrt{1+\frac{y^2}{4}} = 2-\frac{y}{2}$

$\Rightarrow 1+\frac{y^2}{4} = 4+\frac{y^2}{4}-2y$

$\Rightarrow 2y = 3$

$\Rightarrow y = \frac{3}{2}$

Question: Find remainder when $7^{2022}+3^{2022}$ is divided by 5

Answer: 3.00

Solution:

Given, $7^{2022}+3^{2022}$

$7^{2022}+3^{2022}=(49)^{1011}+9^{1011}$

$=(50-1)^{1011}+(10-1)^{1011}$

$=(5k-1)+(5\lambda-1)$

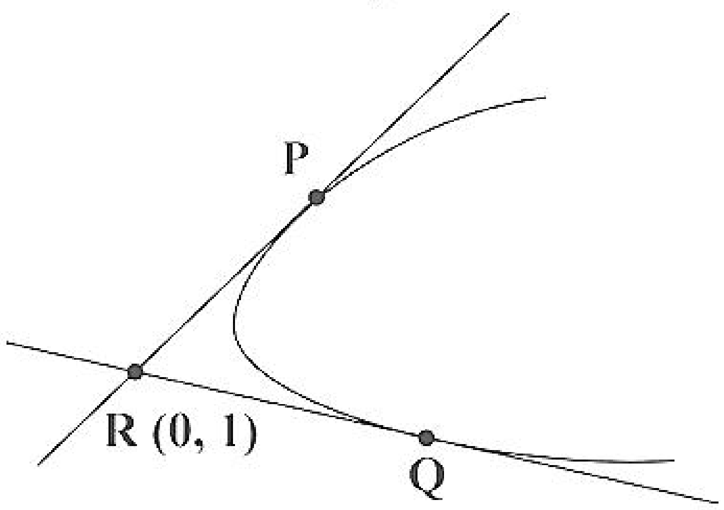
Remainder $=5-2=3$

Question: $y^2=2x+3$. For $A(1,0)$, two tangents are drawn which meets parabola at P & Q. Find orthocentre of APQ.

Answer: (2, -1)

Solution:

$y^2=2x-3 \qquad ....(1)$

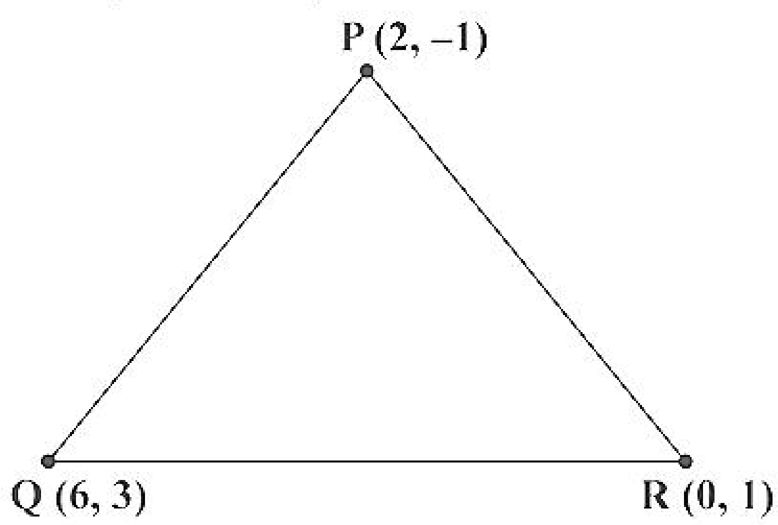


$PQ=T=0$

$y=x-3 \qquad ....(2)$

Solving (1) & (2) we get

$(2,-1)$ and $(6,3)$



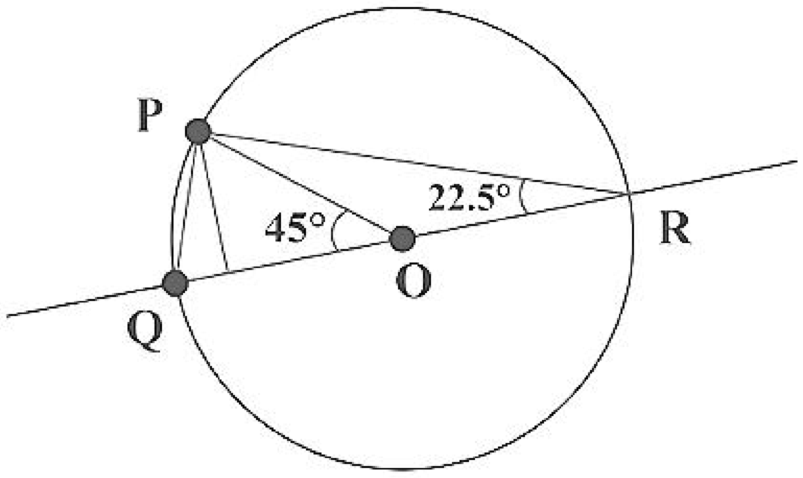
$m_{QR}=-1,\ m_{PQ}=1$

Orthocentre $\equiv(6,3)$

Question: A line passing through centre “O” of circle with radius 4 units, cuts the circle at Q & R. P is a point on the circle such that OP makes $45°$ with QR. Find area of ΔPQR.

Answer: $\frac{16}{\sqrt{2}}$

Solution:



$QR = 8$

$PR = QR\cos 22.5° = 8\cos 22.5°$

$PQ = QR\sin 22.5° = 8\sin 22.5°$

Area $= \frac{1}{2} PR \cdot PQ$

$= \frac{1}{2} 8\cos 22.5° \cdot 8\sin 22.5°$

$= 16\sin 22.5°$

$= \frac{16}{\sqrt{2}}$

Question: A 6-8 digit password is to be made using A, B, C, D, E, 1, 2, 3, 4, 5, with repetition allowed. If number of such passwords containing atleast one digit is $x \times 5^6$ then $x = ?$

Answer: 7073.00

Solution:

6 digit _ _ _ _ _ _ $: 10^6 - 5^6$

7 digit _ _ _ _ _ _ _ $: 10^7 - 5^7$

8 digit _ _ _ _ _ _ _ _ $: 10^8 - 5^8$

Total: $10^6 - 5^6 + 10^7 - 5^7 + 10^8 - 5^8$

$= 7073\left(5^6\right)$

$\therefore x = 7073$

Question: Out of total candidate 60% were female, 40% were male. 60% of total passed. Female passed was twice of males passed. A candidate who passes was chosen. Find probability that it was a female.

Answer: $\frac{2}{3}$

Solution:

Given,

Total = 100

Female = 60
Male = 40
Passed = 60
Female passed = 40, Male passes = 20

Probability $= \frac{40}{60} = \frac{2}{3}$

Question: If $f(x) = \int_0^x e^{x-t} f'(t)\,dt + e^x(x^2 - x + 1)$, then find $f(x)|_{\min}$.

Answer: ()

Solution:

$$f(x) = \int_0^x e^{x-t} f'(t)\,dt + e^x(x^2 - x + 1)$$

$$f(x) = e^x \int_0^x e^{-t} f'(t)\,dt + e^x(x^2 - x + 1)$$

$$f'(x) = e^x\left[\int_0^x e^{-t} f'(t)\,dt + e^{-x} f'(x)\right] + e^x(x^2 - x + 1 + 2x - 1)$$

$$f'(x) = e^x\left[\int_0^x e^{-t} f'(x)\,dt\right] + f'(x) + e^x(x^2 + x)$$

$$\Rightarrow 0 = e^x\left[\int_0^x e^{-t} f'(t)\,dt + x^2 + x\right]$$

$$\Rightarrow 0 = \int_0^x e^{-t} f'(x)\,dt + x^2 + x$$

$$\Rightarrow f'(x) = -e^{-x}(2x+1)$$

Put $f'(x) = 0 \Rightarrow x = \frac{-1}{2}$

$$f'(x) = -e^x(2x + 2 - 1)$$

$$f'(x) = -e^x(2x + 2) + e^x$$

$$\Rightarrow f(x) = -e^x(2x) + e^x + C$$

$$f(0) = 1$$

$$\Rightarrow C = 0$$

$$f(x) = -e^x(2x) + e^x$$

$$f\left(\frac{-1}{2}\right) = -e^{\frac{-1}{2}}\left(2\left(\frac{-1}{2}\right)\right) + e^{\frac{-1}{2}} = 0$$

Question: Find principal range of $\cos^{-1}\left(\frac{x^2-4x+2}{x^2+3}\right)$

Answer: ()

Solution:

$$y=\frac{x^2-4x+2}{x^2+3}$$

$$\Rightarrow x^2y+3y=x^2-4x+2$$

$$\Rightarrow x^2(y-1)+4x+3y-2=0$$

$$D\geq 0$$

$$16-4(y-1)(3y-2)\geq 0$$

$$4-(3y^2-2y-3y+2)\geq 0$$

$$\Rightarrow -3y^2+5y+2\geq 0$$

$$\Rightarrow 3y^2-5y-2\leq 0$$

$$\Rightarrow y\in\left[\frac{-1}{3},2\right]$$

$$\text{Range}\in\left[0,\cos^{-1}\left(\frac{-1}{3}\right)\right]$$

Question: Eccentricity of $x^2-y^2=1$ is reciprocal of eccentricity of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. $y=\left(\frac{\sqrt{5}}{2}\right)x+k$ is common tangent. Find $4(a^2+b^2)$.

Answer: $\frac{6}{7}$

Solution:

$$x^2-y^2=1$$

$$e=\sqrt{2}$$

$$b^2=a^2\left(1-\frac{1}{2}\right)$$

$$b^2=\frac{a^2}{2}$$

$$y=\frac{\sqrt{5}}{2}x+R$$

$$R^2=\frac{1\times 5}{4}-1=\frac{1}{4}$$

$$R^2=a^2m^2+b^2$$

$$\frac{1}{4}=\frac{a^2 5}{4}+b^2$$

$1 = 5a^2 + 4b^2$

$1 = 5a^2 + 2a^2$

$a^2 = \frac{1}{7}, b^2 = \frac{1}{14}$

$4\left(a^2 + b^2\right) = 4\left(\frac{1}{7} + \frac{1}{14}\right)$

$= 4\left(\frac{3}{14}\right)$

$= \frac{6}{7}$

Question: If $a_{n+2} = \frac{2}{a_{n+1}} + a_n, a_1 = 1, a_2 = 2$ & $\left(\frac{a_1 + \frac{1}{a_2}}{a_3}\right)\left(\frac{a_2 + \frac{1}{a_3}}{a_4}\right) \times \ldots \times \left(\frac{a_{30} + \frac{1}{a_{31}}}{a_{32}}\right) = 2^{\alpha} \times {}^{61}C_{31}$

then $\alpha = ?$

Answer: ()

Solution:

$\because a_{n+2} \cdot a_{n+1} - a_{n+1} \cdot a_n = 2$ where $a_1 = 1, a_2 = 2$ and $a_3 = 2$

Let $T_r = a_{r+1} \cdot a_r$

So, T_r is A.P. with common difference 2 and first term 2.

Clearly $T_r = 2r$

Now $\prod_{i=1}^{30}\left(\frac{a_i + \frac{1}{a_{i+1}}}{a_{i+2}}\right) = \prod_{i=1}^{30}\left(\frac{T_r + 1}{T_{r+1}}\right)$

$= \prod_{i=1}^{30}\left(\frac{2r+1}{2r+2}\right) = \frac{3 \cdot 5 \cdot 7 \ldots. 61}{4 \cdot 6 \cdot 8 \ldots 62} = \frac{62!}{2\left(4 \cdot 6 \cdot 8 \ldots. 62\right)^2}$

$= \frac{62!}{2^{61} \cdot (31!)^2}$

$= 2^{-61} \cdot {}^{62}C_{31}$

$= 2^{-60} \cdot {}^{61}C_{30}$

$2^{-60} \cdot {}^{61}C_{31}$

Question: Let $z_1 \in C, \left|z_1 - 3\right| = \frac{1}{2}, z_2 \in C, \left|z_2 + \left|z_2 - 1\right|\right| = \left|z_2 - \left|z_2 + 1\right|\right|$, then least value of $\left|z_1 - z_2\right|$ is:

Answer: $\frac{3}{2}$

Solution:

$\left|z_2+|z_2-1|\right|^2=\left|z_2-|z_2+1|\right|^2$

$\Rightarrow\left(z_2+|z_2-1|\right)\left(\overline{z}_2+|z_2-1|\right)=\left(z_2-|z_2+1|\right)\left(\overline{z}_2-|z_2+1|\right)$

$\Rightarrow z_2\left(|z_2-1|+|z_2+1|\right)+\overline{z}_2\left(|z_2-1|+|z_2+1|\right)=|z_2+1|^2-|z_2-1|^2$

$\Rightarrow\left(z_2+\overline{z}_2\right)\left(|z_2+1|+|z_2-1|\right)=2\left(z_2+\overline{z}_2\right)$

Either $z_2+\overline{z}_2=0$ or $|z_2+1|+|z_2-1|=2$

So, z_2 lies on imaginary axis or on real axis with in $[-1,1]$. Also $|z_1-3|=\frac{1}{2}$, lies on the circle having centre 3 and radius $\frac{1}{2}$.

Clearly $|z_1-z_2|_{\min}=\frac{3}{2}$