INTEGRALS

$\frac{d}{dx}(\log x) = \frac{1}{x}$

i.e derivative of $\log x$ w.r.t $x = \frac{1}{x}$

$\therefore$ antiderivative of $\frac{1}{x} = \log x \Rightarrow \int \frac{1}{x} dx = \log x$

$\therefore$ Integration is reverse process of defrentiation and hence called antiderivative.

$\frac{d}{dx}(\log x) = \frac{1}{x} + 0$

$\frac{d}{dx}(\log x + 1) = \frac{1}{x} \Rightarrow \int \frac{1}{x} dx = \log x + 1$

$\frac{d}{dx}(\log x + 5) = \frac{1}{x} \Rightarrow \int \frac{1}{x} dx = \log x + 5$

$\Rightarrow$ a constant always exists for an antiderivative etc

$\therefore \int \frac{1}{x} \cdot dx = \log x + c$

i.e, $\boxed{\int f(x) \cdot dx = F(x) + C}$

constant of integration

integral value

Symbol of integration elongated 'S' which represents summation

any function of x

w.r.t x

There are two types of integrals:

i) Indefinite ii) Definite

Definite : $\int_a^b f(x)\, dx$

b upper limit

a lower limit

Integration is used to find the areas under lines and curves. Entire area is divided into infinitesimally small regions, area of each region is found and added to get the entire area.

i.e, the small areas are integrated.

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\cdot dx$
x^n	$\frac{x^{n+1}}{n+1}+C$	k	$kx+C$
$\frac{1}{x}$	$\log x + C$	$\sqrt{x}$	$\frac{2}{3}x^{\frac{3}{2}}+C$
$\frac{1}{x^2}$	$-\frac{1}{x}+C$	$\frac{1}{\sqrt{x}}$	$2\sqrt{x}+C$
e^x	e^x+C	a^x	$\frac{a^x}{\log a}+C$
$\sin x$	$-\cos x+C$	$\cos x$	$\sin x+C$
$\tan x$	$\log(\sec x)+C$	$\cot x$	$\log(\sin x)+C$
$\sec x$	$\log(\sec x+\tan x)+C$	$\operatorname{cosec} x$	$\log(\operatorname{cosec} x-\cot x)+C$
$\sec^2 x$	$\tan x+C$	$\operatorname{cosec}^2 x$	$-\cot x+C$
$\sec x\cdot\tan x$	$\sec x+C$	$\operatorname{cosec} x\cdot\cot x$	$-\operatorname{cosec} x+C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x+C$	$\frac{1}{1+x^2}$	$\tan^{-1}x+C$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1}x+C$		

1] $\int x\cdot dx = \frac{x^2}{2}+C$

2] $\int x^7 dx = \frac{x^8}{8}+C$

3] $\int x^{100} dx = \frac{x^{101}}{101}+C$

4] $\int x^5 dx = \frac{x^6}{6}+C$

5] $\int \frac{1}{x} dx = \log x + C$

6] $\int \frac{1}{x^7} dx = \int x^{-7} dx = \frac{x^{-6}}{-6}+C$

7] $\int \frac{1}{x^5} dx = \int x^{-5} dx = \frac{x^{-4}}{-4}+C$

8] $\int \frac{1}{x^{77}} dx = \frac{x^{-76}}{-76}+C$

9] $\int \frac{1}{x^{100}} dx = \frac{x^{-99}}{-99}+C$

10] $\int \sqrt[3]{x^2}\, dx = \int x^{\frac{2}{3}} dx = \frac{x^{\frac{5}{3}}}{5/3}+C$

11] $\int \frac{1}{\sqrt[3]{x^2}} \, dx = \int x^{-2/3} \, dx = \frac{3x^{1/3}}{1} + C$

12] $\int \sqrt[7]{x^5} \cdot dx = \int x^{5/7} \cdot dx = \frac{7x^{12/7}}{12} + C$

13] $\int \frac{1}{\sqrt[7]{x^5}} \cdot dx = \int x^{-5/7} \cdot dx = \frac{7x^{2/7}}{2} + C$

14] $\int \frac{1}{\sqrt{x^5}} \cdot dx = \int x^{-5/2} \, dx = \frac{2x^{-3/2}}{-3} + C$

15] $\int \sqrt[3]{x^2} \cdot dx = \int x^{4/3} \cdot dx = \frac{3x^{5/3}}{5} + C$

16] $\int \frac{x^2 + 2x}{x} \, dx = \int x + 2 \cdot dx = \frac{x^2}{2} + 2x + C$

17] $\int \frac{3x^2 + 2x^5 - 7x}{x^2} \, dx = \int 3 + 2x^3 - \frac{7}{x} \, dx = 3x + \frac{2x^4}{4} - 7\log x + C$

18] $\int 5x^3 - 7x^2 + 5x - 5 \cdot dx = \frac{5x^4}{4} - \frac{7x^3}{3} + \frac{5x^2}{2} - 5x + C$

19] $\int \left(x + \frac{1}{x}\right)^2 \cdot dx = \int x^2 + \frac{1}{x^2} + 2 \cdot dx = \frac{x^3}{3} - \frac{1}{x} + 2x + C$

20] $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \cdot dx = \int x + \frac{1}{x} - 2 \cdot dx = \frac{x^2}{2} + \log x - 2x + C$

21] $\int (3x - 5)^2 \cdot dx = \int 9x^2 - 30x + 25 \cdot dx = \frac{9x^3}{3} + \left(-\frac{30x^2}{2}\right) + 25x + C$

22] $\int \frac{2x - 1}{\sqrt{x}} \cdot dx = \int 2\sqrt{x} - \frac{1}{\sqrt{x}} \cdot dx = 2\left(\frac{2}{3}\right)x^{3/2} - 2\sqrt{x} + C$

23] $\int \frac{(3 - x)^2}{\sqrt{x}} \cdot dx = \int \frac{9 + x^2 - 6x}{\sqrt{x}} \, dx = \int \frac{9}{\sqrt{x}} + x^{3/2} - 6\sqrt{x} \cdot dx$

$= 9(2\sqrt{x}) + \frac{x^{5/2} \cdot 2}{5} - 6\left(\frac{2}{3}\right)x^{3/2} + C$

24] $\int x(1 - \sqrt{x}) \, dx = \int x - x^{3/2} \cdot dx = \frac{x^2}{2} - \frac{x^{5/2} \cdot 2}{5} + C$

25) $\int \sqrt{x}\,(x^2-5x+1)\cdot dx = \int x^{5/2} - 5x^{3/2} + \sqrt{x}\, dx = \frac{2}{7}x^{7/2} - 5\cdot\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2}$

26) $\int 3\sqrt{x}\cdot(3x^2-5x+7)\cdot dx = \int 3\left(3x^{5/2} - 5x^{3/2} + 7\sqrt{x}\right) dx$

$= 9\cdot\frac{2}{7}x^{7/2} - 15\left(\frac{2}{5}\right)x^{5/2} + 21\left(\frac{2}{3}\right)x^{3/2} + C$

27) $\int (1-x)^3\cdot dx = a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$

$= \int 1 - 3x + 3x^2 - x^3 \cdot dx = x - \frac{3x^2}{2} + x^3 - \frac{x^4}{4} + C$

28) $\int 3\sin x + \tan x - 4\operatorname{cosec} x \cdot dx = -3\cos x + \log(\sec x) - 4\log(\operatorname{cosec} x - \cot x)$

29) $\int 3\cos x - \cot x + 5\sec x \cdot dx = 3\sin x - \log(\sin x) + 5\log(\sec x + \tan x) + C$

NOTE: * $\int \frac{\cdots}{1 \pm \sin/\cos}$ => rationalise

* $\int \frac{\sin^2 x}{\cos x} = \int \frac{\sin x}{\cos x}\cdot \sin x$

* $\int \frac{\sin x}{\cos^2 x} = \int \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \int \sec x \cdot \tan x$

* $\int \frac{\cos x}{\sin^2 x} = \int \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} = \int \operatorname{cosec} x \cdot \tan x$

* $\frac{1}{\sqrt{a^2-x^2}} \,/\, \sqrt{a^2-x^2}$, put $x = a\sin\theta \,/\, a\cos\theta$

* $\frac{1}{a+x^2} \,/\, \sqrt{a+x^2} \,/\, \frac{1}{\sqrt{a^2+x^2}}$, put $x = a\tan\theta$

* $\frac{1}{\sqrt{x^2-a}} \,/\, \sqrt{x^2-a^2} \,/\, \frac{1}{x^2-a^2}$, put $x = a\sec\theta \,/\, a\operatorname{cosec}\theta$

* $\int \sin^2 \rightarrow \frac{1-\cos 2x}{2}$

* $\int \cos^2 \rightarrow \frac{1+\cos 2x}{2}$

* $\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$

* $\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$

* $\sin^4 = (\sin^2)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1+\cos^2 2x - 2\cos 2x}{4}$

$= \frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right]$

$\int \sin^4 = \int \frac{1}{4}\left[\frac{3}{2} + \frac{\cos 4x}{2} - 2\cos 2x\right]$

* $\cos^4 = (\cos^2)^2 = \left(\frac{1+\cos 2x}{2}\right)^2 + \cdots$

* $\tan^2 x = \sec^2 x - 1$
* $\cot^2 x = \operatorname{cosec}^2 x - 1$
* $\tan^3 = \tan^2 \cdot \tan = (\sec^2 x - 1)\tan x = \tan x \cdot \sec^2 x - \tan x \ldots$
* $\cot^3 = \cot^2 \cdot \cot = (\operatorname{cosec}^2 x - 1)\cot x = \cot x \cdot \operatorname{cosec}^2 x - \cot x$

<u>Integration by Substitution</u>:

~~$\int \sin 2x \, dx = - \cos$~~ ~~$2x = t$~~

* $\int \sin 2x \cdot dx$ $\quad 2x = t$

$= \int \sin t \cdot \frac{dt}{2}$ $\quad 2 \cdot dx = dt$, $dx = \frac{dt}{2}$

$= -\frac{1}{2}\cos t + C$

$= -\frac{1}{2}\cos 2x + C$

* $\int e^{3x} \cdot dx = \frac{e^{3x}}{3} + C$

* $\int \tan 2x \cdot dx = \frac{\log(\sec 2x)}{2} + C$

Eg: $\int (2x+1)(3x^2+3x+7)^5 \cdot dx$

$3x^2+3x+7=t$

$(6x+3)dx = dt$

$3(2x+1)dx = dt$

$(2x+1)dx = \frac{dt}{3}$

$I = \frac{1}{3}\int t^5 dt$

$= \frac{1}{3}\frac{t^6}{6} + C$

$= \frac{1}{18}(3x^2+3x+7)^6 + C$

Eg: $\int \frac{(\log x)^6}{x} \cdot dx$

$\log x = t$

$\frac{1}{x} \cdot dx = dt$

$\int t^6 \cdot dt$

$= \frac{t^7}{7} + C = \frac{(\log x)^7}{7} + C$

Eg: $\int (\tan^2 x - 3\tan x + 5)\sec^2 x \cdot dx$

$\tan x = t$

$\sec^2 x \cdot dx = dt$

$= \int (t^2 - 3t + 5) \cdot dt$

$= \frac{t^3}{3} - \frac{3t^2}{2} + 5t + C$

$= \frac{\tan^3 x}{3} - \frac{3}{2}\tan^2 x + 5\tan x + C$

* $\int [f(x)]^n \cdot f'(x) \cdot dx = \frac{[f(x)]^{n+1}}{n+1} + C$

* $\int \frac{f'(x) \cdot dx}{f(x)} = \log[f(x)] + C$

EXERCISE 7.1

Eg 1] i] $\int \cos 2x \cdot dx = I$

$I = \frac{\sin 2x}{2} + C$

Eg 2] ii] $I = \int (3x^2 + 4x^3) dx = \cancel{3}\frac{x^3}{\cancel{3}} + \cancel{4}\frac{x^4}{\cancel{4}} + C = x^3 + x^4 + C$

iii] $I = \int \frac{1}{x} \cdot dx = \log x + C$

Eg 2] i] $I = \int \frac{x^3-1}{x^2}\, dx = \int \left(x - \frac{1}{x^2}\right) dx = \frac{x^2}{2} + \frac{1}{x} + C$

iii] $I = \int \left(x^{3/2} + 2e^x - \frac{1}{x}\right) dx = \frac{2}{5}x^{5/2} + 2e^x - \log x + C$

Eg 3] i] $I = \int (\sin x + \cos x)\, dx = -\cos x + \sin x + C$

ii] ** $I = \int \operatorname{cosec} x \cdot (\operatorname{cosec} x + \cot x)\, dx = \int (\operatorname{cosec}^2 x + \operatorname{cosec} x \cdot \cot x)\, dx$
$= -\cot x - \operatorname{cosec} x + C$

18] ** $I = \int \sec x\,(\sec x + \tan x)\, dx = \int (\sec^2 x + \sec x \cdot \tan x)\, dx$
$= \tan x + \sec x + C$

iii] ** $I = \int \frac{1-\sin x}{\cos^2 x}\, dx = \int \frac{1}{\cos^2 x} - \frac{\sin x}{\cos x \cdot \cos x}\, dx$
$= \int (\sec^2 x - \sec x \cdot \tan x)\, dx = \tan x - \sec x + C$

** $I = \int \frac{1-\cos x}{\sin^2 x}\, dx = \int (\operatorname{cosec}^2 x - \operatorname{cosec} x \cdot \cot x)\, dx$
$= -\cot x + \operatorname{cosec} x + C$

1] $\int \sin 2x = \frac{-\cos 2x}{2} + C$

2] $I = \int \cos 3x = \frac{\sin 3x}{3} + C$

3] $I = \int e^{2x} \cdot dx = \frac{e^{2x}}{2} + C$

4] $I = \int (ax+b)^2 = \int (a^2x^2 + b^2 + 2abx)\, dx$
$= \frac{a^2 x^3}{3} + b^2 x + \frac{2abx^2}{2} + C$

5] $I = \int \sin 2x - 4e^{3x} = -\frac{\cos 2x}{2} - \frac{4e^{3x}}{3} + C$

6] $I = \int (4e^{3x}+1)\,dx = \frac{4e^{3x}}{3} + x + C$

7] $I = \int x^2\left(1 - \frac{1}{x^2}\right) dx = \int (x^2 - 1)\,dx = \frac{x^3}{3} - x + C$

8] $I = \int (ax^2 + bx + c)\,dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$

9] $I = \int (2x^2 + e^x)\,dx = \frac{2x^3}{3} + e^x + C$

10] $I = \int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx = \int \left(x + \frac{1}{x} - 2\right) dx = \frac{x^2}{2} + \log x - 2x + C$

11] $I = \int \frac{x^3 + 5x^2 - 4}{x^2}\,dx = \int \left(x + 5 - \frac{4}{x^2}\right) dx$

$= \frac{x^2}{2} + 5x + 4\frac{1}{x} + C$

12] $I = \int \frac{(x^3 + 3x + 4)}{\sqrt{x}}\,dx = \int \left(x^{\frac{5}{2}} + 3\sqrt{x} + \frac{4}{\sqrt{x}}\right) dx$

$= \frac{2}{7}x^{7/2} + 3\left(\frac{2}{3}\right)x^{3/2} + 4(2\sqrt{x}) + C$

13] ** $I = \int \frac{(x^3 - x^2 + x - 1)}{x-1}.dx = \int \frac{x^2(x-1) + 1(x-1)}{(x-1)}dx$

$= \int (x^2 + 1)\,dx = \frac{x^3}{3} + x + C$

14] $I = \int (1-x)\sqrt{x}\,.dx = \int (\sqrt{x} - x^{3/2})\,dx = \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C$

15] $I = \int \sqrt{x}\,(3x^2 + 2x + 3)\,dx = \int (3x^{5/2} + 2x^{3/2} + 3\sqrt{x})\,dx$

$= 3.\frac{2}{7}x^{7/2} + 2.\frac{2}{5}x^{5/2} + 3.\frac{2}{3}x^{3/2} + C$

16] $I = \int (2x - 3\cos x + e^x)\,dx = x^2 - 3\sin x + e^x + C$

17] $I = \int (2x^2 - 3\sin x + 5\sqrt{x})\, dx = \frac{2x^3}{3} + 3\cos x + 5.\frac{2}{3}x^{3/2} + C$

18] $I = \int \sec x(\sec x + \tan x)\, dx = \int(\sec^2 x + \sec x \tan x)\, dx$
$= \tan x + \sec x + C$

19] $I = \int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\, dx = \int \frac{\sin^2 x}{\cos^2 x}\, dx = \int \tan^2 x \cdot dx$
$= \int (\sec^2 x - 1)\, dx = \tan x - x + C$

20] $I = \int \frac{(2 - 3\sin x)}{\cos^2 x}\, dx = \int \frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x}\, dx$
$= \int (2\sec^2 x - 3\sec x \cdot \tan x)\, dx = 2\tan x - 3\sec x + C$

21] $I = \int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \frac{2}{3}x^{3/2} + 2\sqrt{x} + C$

22] If $\frac{d}{dx}[f(x)] = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$, then $f(x) = ?$

Gn : $\frac{d}{dx}[f(x)] = 4x^3 - \frac{3}{x^4}$

Integrating on both sides,

$\int \frac{d}{dx}[f(x)]\, dx = \int \left(4x^3 - \frac{3}{x^4}\right) dx$

$f(x) = \frac{4x^4}{4} - \frac{3x^{-3}}{-3} + C = x^4 + x^{-3} + C$

$f(2) = 16 + \frac{1}{8} + C$

$0 = \frac{129}{8} + C$

$\Rightarrow C = -\frac{129}{8}$

$\therefore f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$

Eg 4] Find the antiderivative F of f defined by $f(x) = 4x^3 - 6$ where $F(0) = 3$.

Gn : $f(x) = 4x^3 - 6$

Integrating on both the sides.

$\int f(x) \cdot dx = \int (4x^3 - 6)\, dx$

$F(x) = \frac{4x^4}{4} - 6x + C \longrightarrow (1)$

$F(0) = 0 - 0 + C$

$\boxed{3 = C}$

$\therefore (1) \Rightarrow F(x) = x^4 - 6x + 3$

EXERCISE 7.2

Eg 5] i] $I = \int \sin mx \cdot dx = -\frac{\cos mx}{m} + C$

ii] $I = \int 2x \sin(x^2+1)\, dx$ $\qquad x^2 + 1 = t$, $2x \cdot dx = dt$

$= \int \sin t \cdot dt$

$= -\cos t + C$

$= -\cos(x^2+1) + C$

iii]** $I = \int \frac{\tan^4\sqrt{x} \cdot \sec^2\sqrt{x}}{\sqrt{x}}\, dx$ $\qquad \tan\sqrt{x} = t$, $\frac{\sec^2\sqrt{x}}{2\sqrt{x}} \cdot dx = dt$, $\frac{\sec^2\sqrt{x}}{\sqrt{x}} \cdot dx = 2dt$

$= \int 2t^4 \cdot dt$

$= 2\frac{t^5}{5} + C$

$= \frac{2}{5}\tan^5\sqrt{x} + C$

iv] $I = \int \frac{\sin(\tan^{-1}x)}{(1+x^2)} dx$ $\tan^{-1}x = t$

$\frac{1}{1+x^2} dx = dt$

$= \int \sin t \cdot dt$

$= -\cos t + C$

$= -\cos(\tan^{-1}x) + C$

** Eg 6] i] $I = \int \sin^3 x \cdot \cos^2 x \cdot dx$

$= \int \sin^2 x \cdot \cos^2 x \cdot \sin x \cdot dx$ $\cos x = t$

$= \int (1-\cos^2 x) \cos^2 x \sin x \, dx$ $-\sin x \cdot dx = dt$

$= \int (1-t^2) t^2 (-) dt$ $\sin x \cdot dx = -dt$

$= -\int (t^2 - t^4) dt$

$= -\left[\frac{t^3}{3} - \frac{t^5}{5}\right] + C$

$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

ii] $I = \int \frac{\sin x}{\sin(x+a)} dx$ $x + a = t$

$dx = dt$

$I = \int \frac{\sin(t-a)}{\sin t} dt$ $x = t - a$

$= \int \frac{\sin t \cos a - \cos t \sin a}{\sin t} dt$

$= \int (\cos a - \sin a \cdot \cot t) dt$

$= \cos a\, t - \sin a \cdot \log(\sin t) + C$

$= \cos a\,(x+a) - \sin a \cdot \log(\sin(x+a)) + C$

iii] $I = \int \frac{dx}{1+\tan x} = \int \frac{dx}{1+\frac{\sin x}{\cos x}} = \int \frac{\cos x \cdot dx}{\cos x + \sin x}$

$= \frac{1}{2} \int \frac{2\cos x \cdot dx}{\cos x + \sin x} = \frac{1}{2} \int \frac{(\cos x + \cos x)\, dx}{\cos x + \sin x}$

$$I = \frac{1}{2}\int \frac{\cos x + \sin x - \sin x + \cos x}{\cos x + \sin x} \cdot dx$$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\cos x + \sin x} \cdot dx$$

$$= \frac{1}{2}\int 1 + \frac{\cos x - \sin x}{\cos x + \sin x} \cdot dx$$

$$= \frac{1}{2}\left[x + \log(\cos x + \sin x)\right] + C$$

32) $I = \int \frac{dx}{1+\cot x} = \int \frac{dx}{1 + \frac{\cos x}{\sin x}} = \int \frac{\sin x \; dx}{\sin x + \cos x} \cdot \frac{2}{2}$

$$= \frac{1}{2}\int \frac{2\sin x \cdot dx}{\sin x + \cos x} = \frac{1}{2}\int \frac{\sin x + \sin x + \cos x - \cos x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) - (\cos x - \sin x)}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int 1 - \frac{(\cos x - \sin x)}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\left[x - \log(\cos x + \sin x)\right] + C$$

1) $I = \int \frac{2x}{1+x^2} dx = \log|1+x^2| + C$

2) $I = \int \frac{(\log x)^2}{x} \cdot dx$ $\qquad \log x = t$

$\qquad \frac{1}{x} dx = dt$

$$= \int t^2 \, dt$$

$$= \frac{t^3}{3} + C = \frac{(\log x)^3}{3} + C$$

** 3) $I = \int \frac{1}{x + x\log x} dx = \int \frac{1}{x(1+\log x)} dx$

$$= \log|1+\log x| + C$$

4] $I = \int \sin x \cdot \sin(\cos x)\, dx$ $\cos x = t$

$= -\int \sin t \cdot dt$ $-\sin x \cdot dx = dt$

$= -(-\cos t) + c$

$= \cos t + c$

$= \cos(\cos x) + c$

5] $I = \int \sin(ax+b) \cdot \cos(ax+b) \cdot dx$ $\cos(ax+b) = t$

$-\sin(ax+b) \cdot a \cdot dx = dt$

$= -\int \frac{1}{a} t \cdot dt$ $\sin(ax+b) \cdot dx = -\frac{1}{a} dt$

$= -\frac{1}{a} \cdot \frac{t^2}{2} + c$

$= -\frac{1}{2a} \cdot \cos^2(ax+b) + c$

(or)

$I = \int \sin(ax+b) \cdot \cos(ax+b) \cdot dx$ $\sin x \cdot \cos x = \frac{\sin 2x}{2}$

$= \int \frac{\sin 2(ax+b)}{2} dx$

$= \frac{1}{2}\left[\frac{-\cos(2ax+2b)}{2a}\right] + c$

$= \frac{-\cos(2ax+2b)}{4a} + c$

6] $I = \int \sqrt{ax+b} \cdot dx$

$= \frac{2}{3} \frac{(ax+b)^{3/2}}{a} + c$

** 7] $I = \int x \cdot \sqrt{x+2}\, dx$ $x+2 = t$

$dx = dt$

$= \int (t-2)\sqrt{t} \cdot dt$ $x = t-2$

$= \int t^{3/2} - 2\sqrt{t} \cdot dt$

$= \frac{2}{5} t^{5/2} - 2 \cdot \frac{2}{3} t^{3/2} + c$

$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C$

8) $I = \int x\sqrt{1+2x^2}\cdot dx$ $\qquad 1+2x^2 = t$

$4x \cdot dx = dt$

$= \frac{1}{4}\int \sqrt{t}\cdot dt$ $\qquad x\cdot dx = \frac{1}{4}\cdot dt$

$= \frac{1}{\cancel{4}_2}\cdot\frac{\cancel{2}}{3}\cdot t^{3/2} + C$

$= \frac{1}{6}(1+2x^2)^{3/2} + C$

9) $I = \int (4x+2)\sqrt{x^2+x+1}\, dx$ $\qquad x^2+x+1 = t$

$= \int 2dt\sqrt{t}$ $\qquad (2x+1)\,dx = dt$

$(4x+2)\,dx = 2dt$

$= 2\cdot\frac{2}{3}t^{3/2} + C$

$= \frac{4}{3}(x^2+x+1)^{3/2} + C$

10) $I = \int \frac{1}{x-\sqrt{x}}\,dx = \int \frac{1}{\sqrt{x}(\sqrt{x}-1)}\,dx$ $\qquad \sqrt{x}-1 = t$

$\frac{1}{2\sqrt{x}}\,dx = dt$

$= \int \frac{2\,dt}{t} = 2\log t + C$ $\qquad \frac{1}{\sqrt{x}}\cdot dx = 2dt$

$= 2\log(\sqrt{x}-1) + C$

** 11)(a) $I = \int \frac{x}{\sqrt{x+4}}\cdot dx = \int \frac{(x+4)-4}{\sqrt{x+4}}\cdot dx$

$= \int\left(\sqrt{x+4} - \frac{4}{\sqrt{x+4}}\right) dx$

$= \frac{2}{3}(x+4)^{3/2} - 4(2\sqrt{x+4}) + C$

(b) $I = \int \frac{x}{\sqrt{x+5}}\,dx = \int \frac{(x+5)-5}{\sqrt{x+5}}\,dx$

$= \int\left(\sqrt{x+5} - \frac{5}{\sqrt{x+5}}\right) dx = \frac{2}{3}(x+5)^{3/2} - 5(2\sqrt{x+5}) + C$

(c) $I = \int \frac{x}{x+5}\, dx = \int \frac{(x+5)-5}{(x+5)}\, dx$

$= \int \left(1 - \frac{5}{x+5}\right) dx = x - 5\log(x+5) + C$

12] $I = \int (x^3-1)^{1/3} \cdot x^5 \cdot dx$ $\quad x^3 - 1 = t$

$3x^2 \cdot dx = dt$

$= \int (x^3-1)^{1/3} \cdot x^2 \cdot x^3 \cdot dx$ $\quad x^2 \cdot dx = \frac{1}{3} dt$

$= \int t^{1/3} \cdot (t+1) \cdot \frac{1}{3} \cdot dt$ $\quad x^3 = t+1$

$= \frac{1}{3}\int t^{4/3} + t^{1/3} \cdot dt$

$= \frac{1}{3}\left(\frac{3}{7} t^{7/3} + \frac{3}{4} \cdot t^{4/3}\right) + C$

$= \frac{1}{7}(x^3-1)^{7/3} + \frac{1}{4}(x^3-1)^{4/3} + C$

13] $I = \int \frac{x^2 \cdot dx}{(2+3x^3)^3} \cdot dx$ $\quad 2 + 3x^3 = t$

$9x^2 \cdot dx = dt$

$= \frac{1}{9}\int \frac{dt}{t^3}$ $\quad x^2 \cdot dx = \frac{1}{9} dt$

$= \frac{1}{9} \cdot \frac{t^{-2}}{-2} + C$

$= -\frac{1}{18} \cdot \frac{1}{(2+3x^3)^2} + C$

14] $I = \int \frac{1}{x(\log x)^m} \cdot dx$ $\quad \log x = t$

$\frac{1}{x} \cdot dx = dt$

$= \int \frac{dt}{t^m}$

$= \frac{t^{-m+1}}{-m+1} + C$

$= \frac{1}{(-m+1)} (\log x)^{1-m} + C$

15] $I = \int \frac{x \cdot dx}{9 - 4x^2}$ $\quad 9 - 4x^2 = t$

$-8x \cdot dx = dt$

$x \cdot dx = -\frac{1}{8} dt$

$= \frac{-1}{8}\int \frac{dt}{t}$

$= \frac{-1}{8} \log t + c$

$= -\frac{1}{8} \log(9 - 4x^2) + c$

16] $I = \int e^{2x+3} \, dx = \frac{e^{2x+3}}{2} + c$

17] $I = \int \frac{x}{(e^{x^2})} \cdot dx$ $\quad x^2 = t$

$2x \cdot dx = dt$

$x dx = \frac{dt}{2}$

$= \frac{1}{2}\int \frac{dt}{e^t}$

$= \frac{1}{2}\int e^{-t} \cdot dt$

$= \frac{1}{2} \frac{e^{-t}}{-1} + c = -\frac{1}{2} e^{-x^2} + c$

18] $I = \int \frac{e^{\tan^{-1} x}}{1 + x^2} \cdot dx$ $\quad \tan^{-1} x = t$

$\frac{1}{1+x^2} \cdot dx = dt$

$I = \int e^t \cdot dt = e^t + c$

$= e^{\tan^{-1} x} + c$

** 19] $I = \int \frac{e^{2x} - 1}{e^{2x} + 1} \cdot dx$ $\quad \div$ num & den by e^x

$= \int \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}}$

$= \log \left| e^x + \frac{1}{e^x} \right| + c$

20] $I = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \, dx$

$= \frac{1}{2} \int \frac{2(e^{2x} - e^{-2x})}{e^{2x} + e^{-2x}} \, dx$

$= \frac{1}{2} \log |e^{2x} + e^{-2x}| + c$

21] $I = \int \tan^2(2x-3) \, dx = \int \sec^2(2x-3) - 1 \cdot dx$

$= \frac{\tan(2x-3)}{2} - x + C$

22] $I = \int \sec^2(7-4x) \cdot dx = \frac{\tan(7-4x)}{-4} + C$

23] $I = \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} \, dx$ $\qquad \sin^{-1} x = t$

$\frac{1}{\sqrt{1-x^2}} \cdot dx = dt$

$= \int t \cdot dt$

$= \frac{t^2}{2} + C$

$= \frac{(\sin^{-1} x)^2}{2} + C$

24] $I = \int \frac{(2\cos x - 3\sin x)}{6\cos x + 4\sin x} \, dx = \frac{1}{2} \int \frac{2\cos x - 3\sin x}{3\cos x + 2\sin x} \, dx$

$= \frac{1}{2} \log |3\cos x + 2\sin x| + C$

25] $I = \int \frac{1}{\cos^2 x (1 - \tan x)^2} \, dx = \int \frac{\sec^2 x \cdot dx}{(1-\tan x)^2}$ $\qquad 1 - \tan x = t$

$-\sec^2 x \cdot dx = dt$

$= -\int \frac{dt}{t^2} = \frac{1}{t} + C = \frac{1}{1 - \tan x} + C$

26] a] $I = \int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$ $\quad \sqrt{x} = t$

$\frac{1}{2\sqrt{x}} dx = dt$

$\frac{1}{\sqrt{x}} \cdot dx = 2dt$

$= \int \cos t \cdot 2dt$

$= \sin t \cdot 2 + c$

$= 2\sin\sqrt{x} + c$

(b) $I = \int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$ $\quad \sqrt{x} = t$

$\frac{1}{2\sqrt{x}} dx = dt$

$\frac{1}{\sqrt{x}} \cdot dx = 2dt$

$= \int \sin t \cdot 2dt$

$= -\cos t \cdot 2 + c$

$= -2\cos\sqrt{x} + c$

27] $I = \int \sqrt{\sin 2x} \cdot \cos 2x \cdot dx$ $\quad \sin 2x = t$

$2\cos 2x \cdot dx = dt$

$\cos 2x \cdot dx = \frac{dt}{2}$

$= \frac{1}{2}\int \sqrt{t}\, dt$

$= \frac{1}{2} \cdot \frac{2}{3} t^{3/2} + c$

$= \frac{1}{3}(\sin 2x)^{3/2} + c$

28] $I = \int \frac{\cos x}{\sqrt{1+\sin x}} \cdot dx$ $\quad 1 + \sin x = t$

$\cos x \cdot dx = dt$

$= \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + c$

$= 2\sqrt{1+\sin x} + c$

29] $I = \int \cot x \cdot \log(\sin x) \cdot dx$ $\quad \log(\sin x) = t$

$\frac{\cos x}{\sin x} dx = dt$

$\cot x \cdot dx = dt$

$= \int t \cdot dt$

$= \frac{t^2}{2} + c$

$= \frac{(\log(\sin x))^2}{2} + c$

30] $I = \int \frac{\sin x}{(1+\cos x)} dx = -\int \frac{(-\sin x)\, dx}{1+\cos x}$

$= -\log|1+\cos x| + c$

$= \log\left(\frac{1}{1+\cos x}\right) + c$

31] $I = \int \frac{\sin x \cdot dx}{(1+\cos x)^2}$ $\quad 1+\cos x = t$

$-\sin x \cdot dx = dt$

$= \int \frac{-dt}{t^2}$ $\quad \sin x \cdot dx = -dt$

$= \frac{1}{t} + c = \frac{1}{1+\cos x} + c$

33] $I = \int \frac{1}{1-\tan x} dx = \int \frac{\cos x \cdot dx}{\cos x - \sin x} = \frac{1}{2}\int \frac{2\cos x \, dx}{\cos x - \sin x}$

$= \frac{1}{2}\int \frac{(\cos x - \sin x) - (-\cos x - \sin x)}{\cos x - \sin x} dx$

$= \frac{1}{2}\int 1 - \frac{(-\cos x - \sin x)}{\cos x - \sin x} dx$

$= \frac{1}{2}\left[x - \log|\cos x - \sin x|\right] + c$

** 34] $I = \int \frac{\sqrt{\tan x}\, dx}{\sin x \cdot \cos x}$ Multiply & ÷ denominator by $\cos x$.

$= \int \frac{\sqrt{\tan x} \cdot dx}{\left(\frac{\sin x}{\cos x}\right)\cos x \cdot \cos x} = \int \frac{\sqrt{\tan x} \cdot \sec^2 x \cdot dx}{\tan x}$

$= \int \frac{\sec^2 x \cdot dx}{\sqrt{\tan x}}$ $\quad \tan x = t$

$\sec^2 x \cdot dx = dt$

$= \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C$

$= 2\sqrt{\tan x} + C$

35) $I = \int \frac{(1+\log x)^2}{x} dx$ $\quad 1+\log x = t$

$\frac{1}{x} dx = dt$

$= \int t^2 \, dt$

$= \frac{t^3}{3} + C = \frac{(1+\log x)^3}{3} + C$

36) $I = \int \frac{(x+1)(x+\log x)^2}{x} dx$

$= \int \frac{(x+1)}{x} (x+\log x)^2 dx = \int \left(1+\frac{1}{x}\right)(x+\log x)^2 \cdot dx$

$= \int t^2 \cdot dt$ $\quad x + \log x = t$

$\left(1+\frac{1}{x}\right) dx = dt$

$= \frac{t^3}{3} + C = \frac{1}{3}(x+\log x)^3 + C$

37) $I = \int x^3 \frac{\sin(\tan^{-1} x^4)}{1+x^8} dx$ $\quad \tan^{-1} x^4 = t$

$\frac{4x^3}{1+x^8} dx = dt$

$= \frac{1}{4}\int \sin t \cdot dt$

$= -\frac{1}{4}\cos t + C$

$= -\frac{1}{4}\cos(\tan^{-1} x^4) + C$

**38) $I = \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx = \log|x^{10} + 10^x| + C$

39) $I = \int \frac{dx}{\sin^2 x \cdot \cos^2 x} = \int \frac{\cos^2 x + \sin^2 x}{\sin^2 x \cdot \cos^2 x} dx$

$= \int \operatorname{cosec}^2 x + \sec^2 x \cdot dx$

$= \tan x - \cot x + C$

EXERCISE 7.3

1] $I = \int \sin^2(2x+5)\cdot dx = \int \frac{1-\cos(4x+10)}{2}\,dx$ $\qquad \sin^2 x = \frac{1-\cos 2x}{2}$

$= \frac{1}{2}\left[x - \frac{\sin(4x+10)}{4}\right] + C$

2] (a) $I = \int \sin 3x \cdot \cos 4x \cdot dx$ $\qquad SA \cdot CB = \frac{1}{2}[S(A+B) + S(A-B)]$

$= \frac{1}{2}\int(\sin 7x - \sin x)\,dx$ $\qquad CA\ SB = \frac{1}{2}[S(A+B) - S(A-B)]$

$\qquad CA \cdot CB = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$= \frac{1}{2}\left[\frac{-\cos 7x}{7} + \cos x\right] + C$ $\qquad SA\ SB = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

(b) $I = \int \cos 3x \cdot \sin 5x \cdot dx = \frac{1}{2}\int \sin 8x + \sin 2x \cdot dx$

$= \frac{1}{2}\left[\frac{-\cos 8x}{8} - \frac{\cos 2x}{2}\right] + C$

(c) $I = \int \cos 3x \cdot \cos 5x \cdot dx = \frac{1}{2}\int \cos 8x + \cos 2x \cdot dx$

$= \frac{1}{2}\left[\frac{\sin 8x}{8} + \frac{\sin 2x}{2}\right] + C$

(d) $I = \int \sin 6x \cdot \sin x \cdot dx = -\frac{1}{2}\int \cos 7x - \cos 5x \cdot dx$

$= -\frac{1}{2}\left[\frac{\sin 7x}{7} - \frac{\sin 5x}{5}\right] + C$

3] $I = \int \cos 2x \cdot \cos 4x \cdot \cos 6x \cdot dx$

$= \frac{1}{2}\int[\cos 8x + \cos 4x]\cos 4x \cdot dx$

$= \frac{1}{2}\int(\cos 8x \cdot \cos 4x + \cos^2 4x)\cdot dx$ $\qquad \cos^2 x = \frac{1+\cos 2x}{2}$

$= \frac{1}{2}\int\left[\frac{1}{2}(\cos 12x + \cos 4x) + \frac{(1+\cos 8x)}{2}\right]dx$

$= \frac{1}{4}\left[\frac{\sin 12x}{12} + \frac{\sin 4x}{4} + x + \frac{\sin 8x}{8}\right] + C$

4) $I = \int \sin^3(2x+1)\, dx$ $\quad \sin^3 x = \frac{3\sin x - \sin 3x}{4}$

$= \frac{1}{4}\int 3\sin(2x+1) - \sin(6x+3)\, dx$

$= \frac{1}{4}\left[-\frac{3\cos(2x+1)}{2} + \frac{\cos(6x+3)}{6}\right] + C$

5) $I = \int \sin^3 x \cdot \cos^3 x\, dx = \int (\sin x \cdot \cos x)^3\, dx = \int \left(\frac{\sin 2x}{2}\right)^3 dx$

$= \frac{1}{8}\int \frac{3\sin 2x - \sin 6x}{4}\, dx = \frac{1}{32}\left[-\frac{3\cos 2x}{2} + \frac{\cos 6x}{6}\right] + C$

6) $I = \int \sin x \cdot \sin 2x \cdot \sin 3x\, dx$

$= -\frac{1}{2}\int (\cos 4x - \cos 2x)\sin 2x\, dx$

$= -\frac{1}{2}\int (\cos 4x \cdot \sin 2x - \cos 2x \cdot \sin 2x)\, dx$

$= -\frac{1}{2}\int \left[\frac{1}{2}(\sin 6x - \sin 2x) - \frac{\sin 4x}{2}\right] dx$

$= -\frac{1}{4}\left[\frac{-\cos 6x}{6} + \frac{\cos 2x}{2} - \frac{\cancel{\sin}\cos 4x}{4}\right] + C$

7) $I = \int \sin 4x \cdot \cancel{\cos 8x}\, \sin 8x\, dx$

$I = -\frac{1}{2}\int (\cos 12x - \cos 4x)\, dx$

$= -\frac{1}{2}\left[\frac{\sin 12x}{12} - \frac{\sin 4x}{4}\right] + C$

8) $I = \int \frac{1-\cos x}{1+\cos x}\, dx = \int \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} = \int \tan^2\frac{x}{2}\, dx$

$= \int \left(\sec^2\frac{x}{2} - 1\right) dx = \frac{2\tan\frac{x}{2}}{1} - x + C$

9) $I = \int \frac{\cos x\, dx}{1+\cos x} = \int \frac{\cos x}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x}\, dx$

$$I = \int \frac{\cos x - \cos^2 x}{\sin^2 x}\, dx = \int (\operatorname{cosec} x \cdot \cot x - \cot^2 x)\, dx$$

$$= \int \operatorname{cosec} x \cdot \cot x - (\operatorname{cosec}^2 x - 1)\, dx$$

$$= -\operatorname{cosec} x + \cot x + x + C$$

12] $$I = \int \frac{\sin^2 x}{1+\cos x}\, dx = \int \frac{\sin^2 x}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x}\, dx$$

$$= \int \frac{\sin^2 x (1-\cos x)}{\sin^2 x} = \int 1 - \cos x = x - \sin x + C$$

10] $$I = \int \sin^4 x \cdot dx = \int (\sin^2 x)^2 \cdot dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1 + \cos^2 2x - 2\cos 2x)\, dx$$

$$= \frac{1}{4}\int \left[1 + \left(\frac{\cos 4x + 1}{2}\right) - 2\cos 2x\right] dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + \frac{\cos 4x}{2} - 2\cos 2x\right) dx$$

$$= \frac{1}{4}\left[\frac{3x}{2} + \frac{\sin 4x}{8} - \frac{\cancel{2}\sin 2x}{\cancel{2}}\right] + C$$

11] $$I = \int \cos^4 2x \cdot dx = \int (\cos^2 2x)^2 \cdot dx = \int \left(\frac{1+\cos 4x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1 + \cos^2 4x + 2\cos 4x)\, dx = \frac{1}{4}\int 1 + \left(\frac{\cos 8x + 1}{2}\right) + 2\cos 4x \cdot dx$$

$$= \frac{1}{4}\int \frac{3}{2} + \frac{\cos 8x}{2} + 2\cos 4x \cdot dx$$

$$= \frac{1}{4}\left[\frac{3x}{2} + \frac{\sin 8x}{16} + \frac{\cancel{2}\sin 4x}{\cancel{4}\,2}\right] + C$$

13) $I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha} dx = \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha} dx$

$= \int \frac{2(\cos^2 x - \cos^2\alpha) \cancel{-1} \cancel{+1}}{\cos x - \cos\alpha} dx$

$= 2\int \frac{(\cos x + \cos\alpha)\cancel{(\cos x - \cos\alpha)}}{\cancel{(\cos x - \cos\alpha)}} dx$

$= 2[\sin x + \cos\alpha \cdot x] + C$

14) $I = \int \frac{\cos x - \sin x}{1 + \sin 2x} \cdot dx = \int \frac{\cos x - \sin x}{\cos^2 x + \sin^2 x + 2\sin x\cos x} \cdot dx$

$= \int \frac{\cos x - \sin x \cdot dx}{(\cos x + \sin x)^2}$ $\qquad \cos x + \sin x = t$

$(-\sin x + \cos x)\, dx = dt$

$= \int \frac{dt}{t^2} = -\frac{1}{t} + C$

$= \frac{-1}{\cos x + \sin x} + C$

15) $I = \int \tan^3 2x \cdot \sec 2x \cdot dx = \int \tan^2 2x \cdot \tan 2x \cdot \sec 2x \cdot dx$

$= \int (\sec^2 2x - 1) \tan 2x \cdot \sec 2x \cdot dx$

$= \int \sec^2 2x \cdot \sec 2x \cdot \tan 2x \cdot dx - \int \sec 2x \cdot \tan 2x \cdot dx$

$\sec 2x = t$

$2\sec 2x \cdot \tan 2x \cdot dx = dt$

$I = \frac{1}{2}\int t^2\, dt - \frac{\sec 2x}{2} + C$

$= \frac{1}{2}\frac{t^3}{3} - \frac{\sec 2x}{2} + C$

$= \frac{1}{2}\left[\frac{\sec^3 2x}{3} - \frac{\sec 2x}{2}\right] + C$

16] $I = \int \tan^4 x \cdot dx = \int (\tan^2 x)(\tan^2 x)\, dx$

$= \int (\sec^2 x - 1)\tan^2 x\, dx = \int \sec^2 x \tan^2 x - \tan^2 x\, dx$

$= \int \sec^2 x \tan^2 x \cdot dx - \int \tan^2 x \cdot dx$

$\tan x = t$

$\sec^2 x\, dx = dt$

$= \int t^2 \cdot dt - \int (\sec^2 x - 1)\, dx$

$= \frac{t^3}{3} - \tan x + x + C$

$= \frac{\tan^3 x}{3} - \tan x + x + C$

17] $I = \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \int \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}\, dx$

$= \int \sec x \cdot \tan x + \operatorname{cosec} x \cdot \cot x\, dx$

$= \sec x - \operatorname{cosec} x + C$

18] $I = \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\, dx = \int \frac{(1 - 2\sin^2 x) + 2\sin^2 x}{\cos^2 x}\, dx$

$= \int \frac{1}{\cos^2 x} \cdot dx = \int \sec^2 x\, dx = \tan x + c$

19] $I = \int \frac{1}{\sin x \cdot \cos^3 x} \cdot dx$ Multiply and divide denominator by $\cos x$

$= \int \frac{dx}{\frac{\sin x}{\cos x} \cdot \cos x \cdot \cos^3 x} = \int \frac{\sec^2 x \cdot \sec^2 x \cdot dx}{\tan x}$

$= \int \frac{(1 + \tan^2 x)}{\tan x} \cdot \sec^2 x \cdot dx$ $\tan x = t$

$\sec^2 x \cdot dx = dt$

$$I = \int \left(\frac{1+t^2}{t}\right) dt = \int \left(\frac{1}{t} + t\right) dt = \log t + \frac{t^2}{2} + c$$

$$= \log(\tan x) + \frac{\tan^2 x}{2} + c$$

20) $I = \int \frac{\cos 2x \cdot dx}{(\cos x + \sin x)^2} = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} \cdot dx$

$$= \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2}\, dx$$

$$= \log|\cos x + \sin x| + c$$

21) $I = \int \sin^{-1}(\cos x) \cdot dx = \int \sin^{-1}\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx$

$$= \int \frac{\pi}{2} - x \, dx = \frac{\pi x}{2} - \frac{x^2}{2} + c$$

** 22) $I = \int \frac{dx}{\cos(x-a)\cos(x-b)}$ Multiply and divide by $\sin(b-a)$.

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(b-a)\, dx}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(b-a)} \int \frac{\sin(b-a+x-x)\, dx}{\cos(x-a)\cdot\cos(x-b)}$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin[(x-a)-(x-b)]\, dx}{\cos(x-a)\cos(x-b)}$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(x-a)\cdot\cos(x-b) - \cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\sin(b-a)} \int \tan(x-a) - \tan(x-b)\, dx$$

$$= \frac{1}{\sin(b-a)} \left[\log|\sec(x-a)| - \log|\sec(x-b)|\right] + c$$

$$= \frac{1}{\sin(b-a)} \log\left|\frac{\sec(x-a)}{\sec(x-b)}\right| + c$$

23) $I = \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cdot \cos^2 x} dx = \int \sec^2 x - \operatorname{cosec}^2 x \, dx$

$= \tan x + \cot x + C$

24) $I = \int \frac{e^x(1+x)\,dx}{\cos^2(xe^x)}$ $\quad xe^x = t$

$(xe^x + e^x)\,dx = dt$

$= \int \frac{dt}{\cos^2 t} = \int \sec^2 t \, dt$ $\quad e^x(x+1)\,dx = dt$

$= \tan t + C = \tan(xe^x) + C$

Special Integrals:

I

$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$ $\quad x = a\tan\theta$

$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$ $\quad$ partial fractions or adjustments

$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$

II

$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$ $\quad x = a\sin\theta$

$\int \frac{1}{\sqrt{a^2+x^2}} dx = \log\left|x + \sqrt{a^2+x^2}\right| + C$ $\quad x = a\tan\theta$

$\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x + \sqrt{x^2-a^2}\right| + C$ $\quad x = a\sec\theta$

III

$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

$\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2+x^2}\right| + C$

$\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2-a^2}\right| + C$ $\quad$ by parts

PROOFS:

1] $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

LHS $= I = \int \frac{dx}{a^2+x^2}$ $\quad x = a\tan\theta$; $\theta = \tan^{-1}\frac{x}{a}$

$dx = a\sec^2\theta\cdot d\theta$

$= \int \frac{a\sec^2\theta\,d\theta}{a^2 + a^2\tan^2\theta}$

$= \int \frac{\cancel{a}\sec^2\theta\cdot d\theta}{a^{\cancel{2}}(1+\tan^2\theta)} = \frac{1}{a}\int \frac{\sec^2\theta\cdot d\theta}{\sec^2\theta}$

$= \frac{1}{a}\int 1\cdot d\theta$

$= \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$ = RHS

$\therefore \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

2] $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

LHS $\Rightarrow \frac{1}{a^2-x^2} = \frac{1}{\underset{\text{linear factor}}{(a+x)(a-x)}} = \frac{A}{a+x} + \frac{B}{a-x}$

$1 = A(a-x) + B(a+x)$

$x = -a$: $1 = A(2a) \Rightarrow \boxed{A = \frac{1}{2a}}$

$x = a$: $1 = B(2a) \Rightarrow \boxed{B = \frac{1}{2a}}$

$I = \int \frac{dx}{a^2-x^2} = \int \frac{1}{2a(a+x)}\,dx + \int \frac{1}{2a(a-x)}\,dx$

$= \frac{1}{2a}\left[\log|a+x| + \frac{\log|a-x|}{-1}\right] + C$

$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

3] $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$

$1 = A(x+a) + B(x-a)$

put $x = a$: $1 = A(2a) \Rightarrow A = \frac{1}{2a}$

put $x = -a$: $1 = B(-2a) \Rightarrow B = \frac{-1}{2a}$

$\therefore I = \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int \frac{1}{x-a}\,dx - \frac{1}{2a}\int \frac{dx}{x+a}\,dx$

$= \frac{1}{2a}\left[\log|x-a| - \log|x+a|\right] + C$

$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

4] $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

$I = \int \frac{1}{\sqrt{a^2-x^2}}\,dx$ $x = a\sin\theta$; $\theta = \sin^{-1}\frac{x}{a}$

$dx = a\cos\theta \cdot d\theta$

$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$

$= \int \frac{a\cos\theta\, d\theta}{a\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta}{\cancel{\cos\theta}}\,d\theta = \int 1 \cdot d\theta$

$= \theta + C$

$= \sin^{-1}\frac{x}{a} + C$

5] $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left|x+\sqrt{a^2+x^2}\right| + C$

$I = \int \frac{dx}{\sqrt{a^2+x^2}}$ $x = a\tan\theta$

$\tan\theta = \frac{x}{a}$

$dx = a\sec^2\theta \cdot d\theta$

$$I = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2+a^2\tan^2\theta}}$$

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{x^2}{a^2}}$$

$$= \int \frac{\cancel{a}\sec^2\theta \cdot d\theta}{\cancel{a}\sqrt{1+\tan^2\theta}}$$

$$\sec\theta = \frac{\sqrt{a^2+x^2}}{a}$$

$$= \int \frac{\sec^{\cancel{2}}\theta \cdot d\theta}{\cancel{\sec\theta}} = \int \sec\theta \cdot d\theta$$

$$= \log|\sec\theta + \tan\theta| + C$$

$$= \log\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + C$$

$$= \log\left|\frac{x+\sqrt{a^2+x^2}}{a}\right| + C$$

$$= \log|x+\sqrt{a^2+x^2}| \underbrace{-\log a + C}_{C}$$

$$\therefore \int \frac{1}{\sqrt{a^2+x^2}}\, dx = \log|x+\sqrt{a^2+x^2}| + C$$

6] $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \log|x+\sqrt{x^2-a^2}| + C$

$$I = \int \frac{1}{\sqrt{x^2-a^2}}\, dx$$

$$x = a\sec\theta$$

$$\sec\theta = \frac{x}{a}$$

$$= \int \frac{a\sec\theta \cdot \tan\theta\, d\theta}{\sqrt{a^2\cdot\sec^2\theta - a^2}}$$

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\frac{x^2}{a^2}-1}$$

$$= \int \frac{\cancel{a}\sec\theta \tan\theta\, d\theta}{\cancel{a}\sqrt{\sec^2\theta - 1}}$$

$$= \int \frac{\sec\theta \cancel{\tan\theta}\, d\theta}{\cancel{\tan\theta}}$$

$$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$$

$$= \int \sec\theta \cdot d\theta$$

$$dx = a\sec\theta\cdot\tan\theta\cdot d\theta$$

$$= \log|\sec\theta + \tan\theta| + C$$

$$I = \log\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

$$= \log\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C$$

$$= \log\left|x + \sqrt{x^2-a^2}\right| \underbrace{- \log a + C}_{C}$$

$$= \log\left|x + \sqrt{x^2-a^2}\right| + C$$

1] Integration by parts: I L A T E → exponential

ITF, Log, Algebra, trigonometry

$$\int \mathrm{I} \cdot \mathrm{II} \cdot dx = \mathrm{I} \int \mathrm{II} - \int \left[\int \mathrm{II} \cdot \frac{d}{dx}(\mathrm{I})\right] dx$$

Eg 1] $\int x \cdot e^x \, dx = x \cdot e^x - \int e^x \cdot 1 \, dx = xe^x - e^x + C$

Eg 2] $\int x \cdot \cos x \cdot dx = x \sin x - \int \sin x \cdot 1 \, dx = x \sin x + \cos x + C$

Eg 3] $\int x \cdot \sec^2 x \cdot dx = x \cdot \tan x - \int \tan x \cdot 1 \, dx = x \tan x - \log|\sec x| + C$

Eg 4] $\int x^2 \cdot e^{2x} \cdot dx = x^2 \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot 2x \cdot dx$

$$= x^2 \cdot \frac{e^{2x}}{2} - \left[x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot 1 \cdot dx\right] + C$$

$$= x^2 \cdot \frac{e^{2x}}{2} - x\frac{e^{2x}}{2} + \frac{1}{2} \cdot \frac{e^{2x}}{2} + C$$

7] $\int \sqrt{a^2-x^2} \, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

$$I = \int \overset{\mathrm{I}}{\sqrt{a^2-x^2}} \cdot \overset{\mathrm{II}}{1} \, dx$$

$$= \sqrt{a^2-x^2} \cdot x - \int x \cdot \frac{(-2x)}{2\sqrt{a^2-x^2}} \cdot dx$$

$$I = x\sqrt{a^2-x^2} - \left[\int \frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\right]dx$$

$$I = x\sqrt{a^2-x^2} - \underbrace{\int \sqrt{a^2-x^2}\,dx}_{I} + \int \frac{a^2}{\sqrt{a^2-x^2}}\,dx$$

$$I + I = x\sqrt{a^2-x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$2I = x\sqrt{a^2-x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

8) $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2+x^2}\right| + C$

$$I = \int \sqrt{a^2+x^2}\cdot 1\,dx = \sqrt{a^2+x^2}\cdot x - \int x \cdot \frac{(2x)}{2\sqrt{a^2+x^2}}\,dx$$

$$I = x\sqrt{a^2+x^2} - \left[\int \frac{(a^2+x^2)-a^2}{\sqrt{a^2+x^2}}\right]dx$$

$$I = x\sqrt{a^2+x^2} - \underbrace{\int \sqrt{a^2+x^2}\,dx}_{I} + a^2\int \frac{1\,dx}{\sqrt{a^2+x^2}}$$

$$2I = x\sqrt{a^2+x^2} + a^2 \log\left|x + \sqrt{a^2+x^2}\right| + C$$

$$I = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2+x^2}\right| + C$$

9) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2-a^2}\right| + C$

$$I = \int \sqrt{x^2-a^2}\cdot 1\,dx = \sqrt{x^2-a^2}\cdot x - \int x\cdot \frac{(2x)}{2\sqrt{x^2-a^2}}\,dx$$

$$I = x\sqrt{x^2-a^2} - \left[\int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}\right]dx$$

$$I = x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

$$2I = x\sqrt{x^2-a^2} - a^2\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

Completing the square:

$$ax^2+bx+c$$

$$= a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$$

$$= a\left\{\left(x+\frac{1}{2}\cdot\frac{b}{a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right\} = a\left\{\left(x+\frac{b}{2a}\right)^2+(\quad)^2\right\}$$

Eg: $1-x-2x^2$

$$= -2\left\{x^2+\frac{1}{2}x-\frac{1}{2}\right\} = -2\left\{\left(x+\frac{1}{4}\right)^2-\frac{1}{2}-\frac{1}{16}\right\}$$

$$= -2\left\{\left(x+\frac{1}{4}\right)^2-\frac{9}{16}\right\} = -2\left\{\left(x+\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2\right\}$$

$$= 2\left\{\left(\frac{3}{4}\right)^2-\left(x+\frac{1}{4}\right)^2\right\}$$

Eg] x^2+2x+2

$$= (x+1)^2+2-1$$

$$= (x+1)^2+1^2$$

Eg] $1+2x+3x^2$

$$= 3\left\{x^2+\frac{2}{3}x+\frac{1}{3}\right\}$$

$$= 3\left\{\left(x+\frac{1}{3}\right)^2+\frac{1}{3}-\frac{1}{9}\right\}$$

$$= 3\left\{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right\}$$

$$= 3\left\{\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right\}$$

Eg] $(x-5)(x-4) = x^2-9x+20$

$$= \left(x-\frac{9}{2}\right)^2+20-\frac{81}{4}$$

$$= \left(x-\frac{9}{2}\right)^2-\frac{1}{4}$$

$$= \left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

EXERCISE 7.4

Eg 9 i] $x^2-6x+13$

$= (x-3)^2+13-9$

$= (x-3)^2+2^2$

(a) $\int \frac{dx}{x^2-6x+13}$

$I = \int \frac{dx}{(x-3)^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+c$

(b) $I = \int \frac{dx}{\sqrt{x^2-6x+13}} = \int \frac{dx}{\sqrt{(x-3)^2+2^2}} = \log\left|(x-3)+\sqrt{(x-3)^2+2^2}\right|+c$

(c) $I = \int \sqrt{x^2-6x+13}\,dx = \int \sqrt{(x-3)^2+2^2}$

$= \frac{x-3}{2}\sqrt{(x-3)^2+2} + \frac{4}{2}\log\left|(x-3)+\sqrt{(x-3)^2+2^2}\right|+c$

$= \frac{x-3}{2}\sqrt{x^2-6x+13} + 2\log\left|(x-3)+\sqrt{x^2-6x+13}\right|+c$

Eg 9 ii] $3x^2-13x-10$

$= 3\left\{x^2+\frac{13}{3}x-\frac{10}{3}\right\} = 3\left\{\left(x+\frac{13}{6}\right)^2-\frac{10}{3}-\frac{169}{36}\right\}$

$= 3\left\{\left(x+\frac{13}{6}\right)^2-\frac{289}{36}\right\} = 3\left\{\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2\right\}$

(a) $\int \frac{dx}{3x^2-13x-10} = \frac{1}{3}\int \frac{dx}{\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2}$

$= \frac{1}{3}\cdot\frac{1}{2\left(\frac{17}{6}\right)}\log\left|\frac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right|+c$

$= \frac{1}{17}\log\left|\frac{6x-4}{6x+30}\right|+c$

$= \frac{1}{17}\log\left|\frac{3x-2}{3x+15}\right|+c$

(b) $\int \frac{dx}{\sqrt{3x^2+13x-10}} = \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{(x+\frac{13}{6})^2 - (\frac{17}{6})^2}}$

$= \frac{1}{\sqrt{3}} \log\left|\left(x+\frac{13}{6}\right) + \sqrt{\left(x+\frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2}\right| + c$

Eg 8] i] $\int \frac{dx}{x^2-16} = \int \frac{dx}{x^2-4^2} = \frac{1}{2(4)} \log\left|\frac{x-4}{x+4}\right| + C$

ii] $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$

$= \sin^{-1}\left(\frac{x-1}{1}\right) + C$

$-x^2+2x$
$-[x^2-2x]$
$-[(x-1)^2-1]$
$1^2-(x-1)^2$

Eg 9] i] $\int \frac{dx}{x^2-6x+13} = \int \frac{dx}{(x-3)^2+2^2}$

$= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + C$

$x^2-6x+13$
$(x-3)^2+13-9$
$(x-3)^2+4$

ii] $\int \frac{dx}{3x^2+13x-10}$

$3x^2+13x-10$
$3\left[x^2+\frac{13}{3}x-\frac{10}{3}\right]$
$3\left\{\left(x+\frac{13}{6}\right)^2-\frac{10}{3}-\frac{169}{36}\right\}$
$=3\left\{\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2\right\}$

$I = \frac{1}{3}\int \frac{dx}{(x+\frac{13}{6})^2-(\frac{17}{6})^2}$

$= \frac{1}{3}\cdot\frac{6}{(2)(17)} \log\left|\frac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right| + C$

$= \frac{1}{17}\log\left|\frac{6x-4}{6x+30}\right| + C = \frac{1}{17}\log\left|\frac{3x-2}{3x+15}\right| + C$

iii] $I = \int \frac{dx}{\sqrt{5x^2-2x}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{(x-\frac{1}{5})^2-(\frac{1}{5})^2}}$

$= \frac{1}{\sqrt{5}}\log\left|\left(x-\frac{1}{5}\right)+\sqrt{\left(x-\frac{1}{5}\right)^2-\left(\frac{1}{5}\right)^2}\right| + C$

$5x^2-2x$
$5\left(x^2-\frac{2}{5}x\right)$
$5\left\{\left(x-\frac{1}{5}\right)^2-\frac{1}{25}\right\}$
$5\left\{\left(x-\frac{1}{5}\right)^2-\left(\frac{1}{5}\right)^2\right\}$

1] $I = \int \frac{3x^2}{x^6+1} dx = \int \frac{3x^2}{(x^3)^2+1^2} dx$ $x^3 = t$, $3x^2 \, dx = dt$

$= \int \frac{dt}{t^2+1} = \frac{1}{1} \tan^{-1}\left(\frac{t}{1}\right) + c$

$= \tan^{-1}(x^3) + c$

2] $I = \int \frac{1}{\sqrt{1+4x^2}} . dx = \int \frac{dx}{\sqrt{1+(2x)^2}}$ $2x = t$, $2 \cdot dx = dt$, $dx = \frac{1}{2} dt$

$= \frac{1}{2} \int \frac{dt}{\sqrt{1+t^2}}$

$= \frac{1}{2} \log \left| t + \sqrt{1+t^2} \right| + c$

$= \frac{1}{2} \log \left| 2x + \sqrt{1+4x^2} \right| + c$

(or)

(a) $I = \int \frac{1}{\sqrt{1+4x^2}} \cdot dx = \int \frac{dx}{\sqrt{1+(2x)^2}} = \frac{\log \left| 2x + \sqrt{1+4x^2} \right|}{2} + c$

(b) $I = \int \frac{dx}{\sqrt{1-9x^2}} = \int \frac{dx}{\sqrt{1-(3x)^2}} = \frac{\sin^{-1}(3x)}{3} + c$

(c) $I = \int \frac{dx}{\sqrt{9x^2-1}} = \int \frac{dx}{\sqrt{(3x)^2-1}} = \frac{\log \left| 3x + \sqrt{9x^2-1} \right|}{3} + c$

3] $I = \int \frac{1}{\sqrt{(2-x)^2+1}} = \frac{\log \left| 2-x + \sqrt{(2-x)^2+1} \right|}{-1} + c$

4] $I = \int \frac{1}{\sqrt{9-25x^2}} dx = \int \frac{1}{\sqrt{3^2-(5x)^2}} . dx = \frac{1}{5} \sin^{-1}\left(\frac{5x}{3}\right) + c$

5] $I = \int \frac{3x}{1+2x^4} . dx = \int \frac{3x \cdot dx}{1^2 + (\sqrt{2}x^2)^2}$ $\sqrt{2}x^2 = t$, $\sqrt{2} \cdot 2x \cdot dx = dt$, $x dx = \frac{1}{2\sqrt{2}} dt$

$= \int \frac{3}{2\sqrt{2}} . \frac{dt}{(1+t^2)}$

$$I = \frac{3}{2\sqrt{2}} \cdot \tan^{-1}(t) + c \quad = \frac{3}{2\sqrt{2}} \cdot \tan^{-1}\left(\sqrt{2}x^2\right) + c$$

6] $I = \int \frac{x^2}{1-x^6} dx = \int \frac{x^2}{1-(x^3)^2} dx$ $\qquad x^3 = t$

$3x^2 \cdot dx = dt$

$= \int \frac{1}{3} \cdot \frac{dt}{1-t^2} = \frac{1}{3} \cdot \frac{1}{2(1)} \log\left|\frac{1+t}{1-t}\right| + c$ $\qquad x^2 \cdot dx = \frac{dt}{3}$

$= \frac{1}{6} \log\left|\frac{1+x^3}{1-x^3}\right| + c$

** 7] $I = \int \frac{(x-1)\,dx}{\sqrt{x^2-1}} = \int \frac{x \cdot dx}{\sqrt{x^2-1}} - \int \frac{1 \cdot dx}{\sqrt{x^2-1}}$

$x^2 - 1 = t$

$2x \cdot dx = dt$

$x \cdot dx = \frac{1}{2} dt$

$I = \frac{1}{2}\int \frac{dt}{\sqrt{t}} - \log\left|x + \sqrt{x^2-1}\right| + c$

$= \frac{1}{2} 2\sqrt{x^2-1} - \log\left|x + \sqrt{x^2-1}\right| + c$

8] $I = \int \frac{x^2}{\sqrt{x^6+a^6}} dx = \int \frac{x^2 \cdot dx}{\sqrt{(x^3)^2 + (a^3)^2}}$ $\qquad x^3 = t$

$3x^2 \cdot dx = dt$

$= \int \frac{1}{3} \cdot \frac{dt}{\sqrt{t^2 + (a^3)^2}} = \frac{1}{3} \int \frac{dt}{\sqrt{t^2 + (a^3)^2}}$ $\qquad x^2 \cdot dx = \frac{1}{3} dt$

$= \frac{1}{3} \log\left|t + \sqrt{t^2 + (a^3)^2}\right| + c$

$= \frac{1}{3} \cdot \log\left|x^3 + \sqrt{x^6 + a^6}\right| + c$

** 9) $I = \int \frac{\sec^2 x \cdot dx}{\sqrt{\tan^2 x + 4}}$ $\qquad \tan x = t$

$\sec^2 x \cdot dx = dt$

$$I = \int \frac{dt}{\sqrt{t^2+4}} = \log\left|t + \sqrt{t^2+4}\right| + C$$

$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$$

10] $I = \int \frac{1 \cdot dx}{\sqrt{x^2+2x+2}}$

$$= \int \frac{dx}{\sqrt{(x+1)^2+1^2}}$$

$$= \log\left|(x+1) + \sqrt{x^2+2x+2}\right| + C$$

x^2+2x+2

$(x+1)^2 + 2 - 1$

$(x+1)^2 + 1^2$

11] $I = \int \frac{1 \cdot dx}{9x^2+6x+5} = \frac{1}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$

$$= \frac{1}{9} \cdot \frac{3}{2} \tan^{-1}\left[\frac{x+\frac{1}{3}}{\frac{2}{3}}\right] + C$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{3x+1}{2}\right) + C$$

$9x^2+6x+5$

$9\left(x^2 + \frac{2}{3}x + \frac{5}{9}\right)$

$= 9\left\{\left(x+\frac{1}{3}\right)^2 + \frac{5}{9} - \frac{1}{9}\right\}$

$= 9\left\{\left(x+\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right\}$

12] $I = \int \frac{dx}{\sqrt{7-6x-x^2}} = \int \frac{dx}{\sqrt{4^2-(x+3)^2}}$

$$= \sin^{-1}\left(\frac{x+3}{4}\right) + C$$

$-x^2-6x+7$

$= -(x^2+6x-7)$

$= -\left\{(x+3)^2 - 7 - 9\right\}$

$= -\left\{(x+3)^2 - (4)^2\right\}$

$= \left\{4^2 - (x+3)^2\right\}$

13] $I = \int \frac{1 \cdot dx}{\sqrt{(x-1)(x-2)}} = \int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$

$$= \log\left|\left(x-\frac{3}{2}\right) + \sqrt{(x-1)(x-2)}\right| + C$$

$(x-1)(x-2)$

x^2-3x+2

$\left(x-\frac{3}{2}\right)^2 + 2 - \frac{9}{4}$

$\left(x-\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2$

14] $I = \int \frac{dx}{\sqrt{8+3x-x^2}}$

$$I=\int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}$$

$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

$$-(x^2-3x-8)$$
$$-\left[\left(x-\frac{3}{2}\right)^2-8-\frac{9}{4}\right]$$
$$-\left[\left(x-\frac{3}{2}\right)^2-\left(\frac{\sqrt{41}}{2}\right)^2\right]$$
$$=\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2$$

15] $$I=\int\frac{dx}{\sqrt{(x-a)(x-b)}}=\int\frac{dx}{\sqrt{x^2-(a+b)x+ab}}$$

$$=\int\frac{dx}{\sqrt{\left[x-\frac{(a+b)}{2}\right]^2-\left(\frac{a-b}{2}\right)^2}}$$

$$=\log\left|\left(x-\frac{a+b}{2}\right)+\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}\right|+C$$

$$x^2-(a+b)x+ab$$
$$\left[x-\frac{(a+b)}{2}\right]^2+ab-\frac{(a+b)^2}{4}$$
$$\left[x-\frac{(a+b)}{2}\right]^2+\frac{4ab-(a+b)^2}{4}$$
$$\left[x-\frac{(a+b)}{2}\right]^2-\left(\frac{a-b}{2}\right)^2$$

24] $$I=\int\frac{dx}{x^2+2x+2}=\int\frac{dx}{(x+1)^2+1^2}$$

$$=\tan^{-1}(x+1)+C$$

$$x^2+2x+2$$
$$(x+1)^2+2-1$$
$$(x+1)^2+1^2$$

25] $$I=\int\frac{dx}{\sqrt{9x-4x^2}}=\int\frac{1}{2}\cdot\frac{dx}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

$$9x-4x^2$$
$$-4\left[x^2-\frac{9}{4}x\right]$$
$$-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]$$

EXERCISE 7.7

Eg 23] $\int \sqrt{x^2+2x+5}\,dx$

x^2+2x+5

$(x+1)^2+5-1$

$(x+1)^2+2^2$

$I = \int \sqrt{(x+1)^2+2^2}\,dx$

$= \frac{(x+1)}{2}\sqrt{(x+1)^2+2^2} + \frac{2^2}{2}\log\left|(x+1)+\sqrt{(x+1)^2+2^2}\right| + C$

$= \frac{(x+1)}{2}\sqrt{x^2+2x+5} + 2\log\left|(x+1)+\sqrt{x^2+2x+5}\right| + C$

** Eg 24] $\int \sqrt{3-2x-x^2}\,dx$

$-(x^2+2x-3)$

$-\left[(x+1)^2-3-1\right]$

$= 2^2-(x+1)^2$

$I = \int \sqrt{2^2-(x+1)^2}\,dx$

$= \frac{(x+1)}{2}\sqrt{2^2-(x+1)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x+1}{2}\right) + C$

$= \frac{(x+1)}{2}\sqrt{3-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + C$

1] $\int \sqrt{4-x^2}$

$I = \int \sqrt{2^2-x^2} = \frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) + C$

2] $\int \sqrt{1-4x^2}\,dx$

$= \int \sqrt{1-(2x)^2}\,dx$

$\cancel{1-4x^2}$

$\cancel{-4\left(\frac{1}{4}+x^2\right)}$

$I = \frac{1}{2}\left[\frac{2x}{2}\sqrt{1-4x^2} + \frac{1}{2}\sin^{-1}\left(\frac{2x}{1}\right)\right] + C$

$1-4x^2$

$-4\left(x^2-\frac{1}{4}\right)$

$I = \frac{1}{2}\left[x\sqrt{1-4x^2} + \frac{1}{2}\sin^{-1}(2x)\right] + C$

$4\left[\left(\frac{1}{2}\right)^2-x^2\right]$

(or)

$I = 2\int \sqrt{\left(\frac{1}{2}\right)^2-x^2}\,dx = 2\left[\frac{x}{2}\frac{\sqrt{1-4x^2}}{\sqrt{4}} + \frac{1}{8}\sin^{-1}\frac{x}{1/2}\right]$

$I = \frac{x}{2}\sqrt{1-4x^2} + \frac{1}{4}\sin^{-1}2x$

3] $\int \sqrt{x^2+4x+6}\, dx = \int \sqrt{(x+2)^2+(\sqrt{2})^2}\, dx$

$= \frac{(x+2)}{2}\sqrt{x^2+4x+6} + \frac{2}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right| + C$

4] $\int \sqrt{x^2+4x+1} = \int \sqrt{(x+2)^2-(\sqrt{3})^2}\, dx$

$= \frac{(x+2)}{2}\sqrt{x^2+4x+1} - \frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right| + C$

5] $\int \sqrt{1-4x-x^2} = \int \sqrt{(\sqrt{5})^2-(x+2)^2}\, dx$

$= \frac{(x+2)}{2}\sqrt{1-4x-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C$

$1-4x-x^2$
$-(x^2+4x-1)$
$-[(x+2)^2-1-4]$
$-[(x+2)^2-5]$
$=(\sqrt{5})^2-(x+2)^2$

6] $\int \sqrt{x^2+4x-5} = \int \sqrt{(x+2)^2-3^2}$

$= \frac{(x+2)}{2}\sqrt{x^2+4x-5} - \frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right| + C$

x^2+4x-5
$(x+2)^2-5-4$
$(x+2)^2-3^2$

7] $I = \int \sqrt{1+3x-x^2}\, dx$

$= \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\, dx$

$-(x^2-3x-1)$
$-\left[\left(x-\frac{3}{2}\right)^2-1-\frac{9}{4}\right]$
$\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2$

$= \frac{\left(x-\frac{3}{2}\right)}{2}\sqrt{1+3x-x^2} + \frac{13}{8}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right) + C$

$= \frac{(2x-3)}{4}\sqrt{1+3x-x^2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + C$

8] $I = \int \sqrt{x^2+3x}\, dx = \int \sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}\, dx$

$= \frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{x^2+3x} - \frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right| + C$

$= \frac{(2x+3)}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x}\right| + C$

9] $I = \int \sqrt{1+\frac{x^2}{9}}\, dx = \frac{1}{3}\int \sqrt{3^2+x^2}\, dx$

$= \frac{1}{3}\left[\frac{x}{2}\sqrt{9+x^2} + \frac{9}{2}\log\left|x+\sqrt{9+x^2}\right|\right] + C$

10] $I = \int \sqrt{1+x^2}\, dx$

$= \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|x+\sqrt{1+x^2}\right| + C$

11] $I = \int \sqrt{x^2-8x+7}\, dx$

x^2-8x+7

$(x-4)^2+7-16$

$(x-4)^2-(3)^2$

$= \int \sqrt{(x-4)^2-(3)^2}\, dx$

$= \frac{(x-4)}{2}\sqrt{x^2-8x-7} - \frac{9}{2}\log\left|(x-4)+\sqrt{x^2-8x-7}\right| + C$

<u>Integrals of the form</u> : $\int \frac{(px+q)\,dx}{\sqrt{ax^2+bx+c}}$ / $\int \frac{(px+q)\,dx}{ax^2+bx+c}$

→ Numerator $= A\frac{d}{dx}(de) + B$

→ Compare coefficients of x and constants to get values of A, B.

→ Replace numerator, split and integrate.

→ $I = A\int \frac{\frac{d}{dx}(de)}{de}\,dx + B\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$

$= A\log|de| + B$ (complete the eq)

Eg 10] i] $I = \int \frac{(x+2)\,dx}{2x^2+6x+5}$

$Nr = A\frac{d}{dx}(de) + B$

$(x+2) = A(4x+6) + B$

coefficients of x : $1 = 4A \Rightarrow A = \frac{1}{4}$

constants : $2 = 6A + B$

$2 - \frac{6}{4} = B \Rightarrow B = \frac{1}{2}$

$$I = \int \frac{A(4x+6)+B}{2x^2+6x+5}\,dx = \int \frac{\frac{1}{4}(4x+6)+\frac{1}{2}}{2x^2+6x+5}\,dx$$

$$= \frac{1}{4}\int \frac{(4x+6)\,dx}{2x^2+6x+5} + \frac{1}{2}\int \frac{dx}{2x^2+6x+5}\,dx$$

$2x^2+6x+5 = t$

$(4x+6)\cdot dx = dt$

$2\left(x^2+3x+\frac{5}{2}\right)$

$2\left[\left(x+\frac{3}{2}\right)^2+\frac{5}{2}-\frac{9}{4}\right]$

$2\left[\left(x+\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2\right]$

$$I = \frac{1}{4}\int \frac{dt}{t} + \frac{1}{2}\int \frac{dx}{(2)\left[\left(x+\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2\right]}$$

$$= \frac{1}{4}\log\left|2x^2+6x+5\right| + \frac{1}{4_2}\cdot\frac{2}{1}\tan^{-1}\left(\frac{x+\frac{3}{2}}{\frac{1}{2}}\right)+c$$

$$= \frac{1}{4}\log\left|2x^2+6x+5\right| + \frac{1}{2}\tan^{-1}(2x+3)+c$$

ii] $I = \int \frac{x+3}{\sqrt{5-4x+x^2}}\,dx$

Num $= A\,\frac{d}{dx}(de) + B$

$x+3 = A(2x-4)+B$

coeff of x : $1 = 2A \Rightarrow A = \frac{1}{2}$

constants : $3 = -\overset{2}{\cancel{4}}\left(\frac{1}{2}\right)+B \Rightarrow B = 5$

$$I = \int \frac{\frac{1}{2}(2x-4)+5}{\sqrt{5-4x+x^2}}\,dx = \frac{1}{2}\int \frac{(2x-4)}{\sqrt{5-4x+x^2}}\,dx + 5\int \frac{dx}{\sqrt{5-4x+x^2}}$$

$5-4x+x^2=t$

$(2x-4)\,dx=dt$

x^2-4x+5

$(x-2)^2+5-4$

$(x-2)^2+1^2$

$$I=\frac{1}{2}\int\frac{dt}{\sqrt{t}}+5\int\frac{dx}{\sqrt{(x-2)^2+1^2}}$$

$$=\frac{1}{2}\cdot 2\sqrt{t}+5\left\{\log\left|(x-2)+\sqrt{x^2-4x+5}\right|+C\right.$$

$$=\sqrt{x^2-4x+5}+5\log\left|(x-2)+\sqrt{x^2-4x+5}\right|+C$$

EXERCISE 7.4

Special 16] $I=\int\frac{(4x+1)\,dx}{\sqrt{2x^2+x-3}}$

$2x^2+x-3=t$

$(4x+1)dx=dt$

$$I=\int\frac{dt}{\sqrt{t}}=2\sqrt{t}+C$$

$$I=2\sqrt{2x^2+x-3}+C$$

17] $I=\int\frac{(x+2)}{\sqrt{x^2-1}}\,dx$

Numerator $=A\frac{d}{dx}(de)+B$

$x+2=A(2x)+B$

coeff of x : $1=2A \Rightarrow A=\frac{1}{2}$

constants : $2=B$

$$I=\int\frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}\,dx=\frac{1}{2}\int\frac{(2x)\,dx}{\sqrt{x^2-1}}+2\int\frac{dx}{\sqrt{x^2-1}}$$

$x^2 - 1 = t$

$(2x)dx = t$

$x^2 - 1$

$$I = \frac{1}{2}\int\left(\frac{dt}{\sqrt{t}}\right) + 2\int \frac{dx}{\sqrt{(x^2) - (1)^2}}$$

$$= \frac{1}{2}(2\sqrt{t}) + 2\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2 - 1}\right| + C$$

$$= \sqrt{x^2-1} + 2\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2-1}\right| + C$$

18] $\int \frac{5x-2}{1+2x+3x^2}dx$

~~$3x^2+2x+1 = t$~~

~~$(6x+2)dx = dt$~~

Num $= A\frac{d}{dx}(dc) + B$

$5x-2 = A(6x+2) + B$

coefficients of x : $5 = 6A \Rightarrow A = \frac{5}{6}$

constants : $-2 = 2A + B$

$-2 - 2\left(\frac{5}{6}\right) = B$

$B = \frac{-11}{3}$

$$I = \int \frac{\frac{5}{6}(6x+2) - \frac{11}{3}}{1+2x+3x^2}dx = \frac{5}{6}\int\frac{6x+2}{1+2x+3x^2}dx - \frac{11}{3}\int\frac{dx}{1+2x+3x^2}$$

$1 + 2x + 3x^2 = t$

$(6x+2)dx = dt$

$3x^2 + 2x + 1$

$3\left(x^2 + \frac{2}{3}x + \frac{1}{3}\right)$

$3\left[\left(x+\frac{1}{3}\right)^2 + \frac{1}{3} - \frac{1}{9}\right]$

$3\left[\left(x+\frac{1}{3}\right)^2 + \frac{2}{9}\right]$

$= 3\left[\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]$

$$I = \frac{5}{6}\int\frac{dt}{t} - \frac{11}{3(3)}\int\frac{dx}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2}$$

$$= \frac{5}{6}\log|t| - \frac{11}{9}\cdot\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right) + C$$

$$= \frac{5}{6}\log|3x^2+2x+1| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

19] $\displaystyle\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}\,dx = \int \frac{(6x+7)\,dx}{\sqrt{x^2-9x+20}}$

Num $= A\frac{d}{dx}(de) + B$

$6x+7 = A(2x-9) + B$

coeff of x : $6 = 2A \Rightarrow A = 3$

const^n : $7 = -27 + B \Rightarrow B = 34$

$I = \displaystyle\int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}\,dx = 3\int \frac{(2x-9)dx}{\sqrt{x^2-9x+20}} + 34\int \frac{dx}{\sqrt{x^2-9x+20}}$

$x^2-9x+20 = t$

$(2x-9)\,dx = dt$

$\left(x-\frac{9}{2}\right)^2 + 20 - \frac{81}{4}$

$\left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$

$I = 3\displaystyle\int \frac{dt}{\sqrt{t}} + 34\int \frac{dx}{\sqrt{(x-\frac{9}{2})^2-(\frac{1}{2})^2}}$

$= 3(2\sqrt{t}) + 34\log\left|\left(x-\frac{9}{2}\right) + \sqrt{\left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + c$

$= 6\sqrt{x^2-9x+20} + 34\log\left|\left(\frac{2x-9}{2}\right) + \sqrt{x^2-9x+20}\right| + C$

22] $\displaystyle\int \frac{(x+3)}{x^2-2x-5}\,dx$

Num $= A\frac{d}{dx}(de) + B$

$x+3 = A(2x-2) + B$

coeff of x : $1 = 2A \Rightarrow A = \frac{1}{2}$

constants : $3 = -2A + B \Rightarrow 3 + 1 = B = 4$

$I = \displaystyle\int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}\,dx = \frac{1}{2}\int \frac{(2x-2)\,dx}{x^2-2x-5} + 4\int \frac{dx}{x^2-2x-5}$

$x^2-2x-5 = t$

$(2x-2)\,dx = dt$

x^2-2x-5

$(x-1)^2 - 5 - 1$

$(x-1)^2 - (\sqrt{6})^2$

$$I = \frac{1}{2}\int \frac{dt}{t} + 4\int \frac{dx}{(x-1)^2 - (\sqrt{6})^2}$$

$$= \frac{1}{2}\log(t) + 4 \cdot \frac{1}{2\sqrt{6}} \log\left|\frac{(x-1)-\sqrt{6}}{(x-1)+\sqrt{6}}\right| + C$$

$$= \frac{1}{2}\log|x^2-2x-5| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

20) $\displaystyle\int \frac{(x+2)\,dx}{\sqrt{4x-x^2}}$

Num $= A\frac{d}{dx}(de) + B$

$x+2 = A(4-2x) + B$

coeffs of x : $1 = -2A \Rightarrow A = -\frac{1}{2}$

constant : $2 = 4A + B \Rightarrow 2+2 = B = 4$

$$I = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}\,dx = -\frac{1}{2}\int \frac{(4-2x)}{\sqrt{4x-x^2}}\,dx + 4\int \frac{dx}{\sqrt{4x-x^2}}$$

$4x - x^2 = t$

$(4-2x)\,dx = dt$

$-(x^2-4x)$

$-[(x-2)^2-4]$

$2^2-(x-2)^2$

$$I = \frac{-1}{2}\int \frac{dt}{\sqrt{t}} + 4\int \frac{dx}{\sqrt{2^2-(x-2)^2}}$$

$$= -\frac{1}{2}(2\sqrt{t}) + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

$$= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

21) $\displaystyle\int \frac{(x+2)}{\sqrt{x^2+2x+3}}\,dx$

Num $= A\frac{d}{dx}(de) + B$

$x+2 = A(2x+2) + B$

coeffs of x : $1 = 2A \Rightarrow A = \frac{1}{2}$

const^n : $2 = 2A + B \Rightarrow 2 - 1 = B = 1$

$$I = \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{(2x+2)\,dx}{\sqrt{x^2+2x+3}} + \int \frac{dx}{\sqrt{x^2+2x+3}}$$

$x^2+2x+3 = t$

$(2x+2)dx = dt$

x^2+2x+3
$= (x+1)^2+3-1$
$= (x+1)^2+(\sqrt{2})^2$

$$I = \frac{1}{2}\int \frac{dt}{\sqrt{t}} + \int \frac{dx}{\sqrt{(x+1)^2+(\sqrt{2})^2}}$$

$$= \frac{1}{2}(2\sqrt{t}) + \log\left|(x+1)+\sqrt{(x+1)^2+(\sqrt{2})^2}\right| + C$$

$$= \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

23] $\displaystyle\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx$

$5x+3 = A(2x+4)+B$

coeff of x : $5 = 2A \Rightarrow A = \frac{5}{2}$

constⁿ : $3 = 4A+B \Rightarrow 3-10 = B = -7$

$$I = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\int \frac{(2x+4)\,dx}{\sqrt{x^2+4x+10}} - 7\int \frac{dx}{\sqrt{x^2+4x+10}}$$

$x^2+4x+10 = t$

$(2x+4)dx = dt$

$x^2+4x+10$
$(x+2)^2+10-4$
$(x+2)^2+6 = (x+2)^2+(\sqrt{6})^2$

$$I = \frac{5}{2}\int \frac{dt}{\sqrt{t}} - 7\int \frac{dx}{\sqrt{(x+2)^2+(\sqrt{6})^2}}$$

$$= \frac{5}{2}(2\sqrt{t}) - 7\log\left|(x+2)+\sqrt{(x+2)^2+(\sqrt{6})^2}\right| + C$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

EXERCISE 7.6

ILATE

1] $I = \int \underset{I}{x} \cdot \underset{II}{\sin x} \cdot dx$

$= x(-\cos x) - \int (-\cos x) \cdot 1 \, dx$

$= -x\cos x + \sin x + C$

2] $I = \int x \cdot \sin 3x \cdot dx$

$= x\left(-\frac{\cos 3x}{3}\right) - \int \left(-\frac{\cos 3x}{3}\right) \cdot 1 \, dx$

$= -\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + C$

3] $I = \int x^2 \cdot e^x \cdot dx$

$= x^2 \cdot e^x - \int e^x \cdot 2x \cdot dx$

$= x^2 \cdot e^x - 2\left[x \cdot e^x - \int e^x \cdot 1 \, dx\right]$

$= x^2 \cdot e^x - 2xe^x + 2e^x + C$

$= e^x\left[x^2 - 2x + 2\right] + C$

(or)

By Bernoulli's rule :

$I = \int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C$

* $I = \int x^3 e^{2x} \, dx$

$= x^3 \cdot \frac{e^{2x}}{2} - 3x^2 \cdot \frac{e^{2x}}{4} + \cancel{6}^{3} x \cdot \frac{e^{2x}}{\cancel{8}_{4}} - \cancel{6}^{3} \frac{e^{2x}}{\cancel{16}_{8}} + C$

$= \frac{e^{2x}}{2}\left[x^3 - \frac{3x^2}{2} + \frac{3x}{2} - \frac{3}{4}\right] + C$

4] $I = \int x \cdot \log x \, dx$

NOTE : If log or IT function is present, the same is the 1^{st} funcn.

$I = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx$

$= \frac{x^2 \log x}{2} - \frac{1}{2} \cdot \frac{x^2}{2} + C$

5] $I = \int x \cdot \log 2x \cdot dx = \log 2x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{2}{2x} \cdot dx$

$= \frac{x^2}{2} \log 2x - \frac{1}{2} \cdot \frac{x^2}{2} + C$

6] $I = \int x^2 \cdot \log x \cdot dx = \log x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$

$= \frac{x^3}{3} \log x - \frac{1}{3} \frac{x^3}{3} + C$

7] $I = \int x \cdot \sin^{-1} x \, dx = \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} dx$

$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \left[\int \frac{(1-x^2)-1}{\sqrt{1-x^2}} \right] dx$

$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1-x^2} \, dx - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}}$

$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1}(x) \right] - \frac{1}{2} \sin^{-1} x + C$

$= \frac{x^2}{2} \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + C$

8] $I = \int x \cdot \tan^{-1} x \, dx = \tan^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{(1+x^2)} dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \left[\int \frac{(x^2+1)-1}{1+x^2} \right] dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int 1 \cdot dx + \frac{1}{2} \int \frac{1}{1+x^2} \cdot dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C$

9] $I = \int x \cdot \cos^{-1} x \, dx = \cos^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \left(\frac{-1}{\sqrt{1-x^2}} \right) dx$

$= \frac{x^2}{2} \cos^{-1} x - \frac{1}{2} \left[\int \frac{(1-x^2)-1}{\sqrt{1-x^2}} \right] dx$

$$= \frac{x^2}{2}\cos^{-1}x - \frac{1}{2}\int\sqrt{1-x^2}\,dx + \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2}{2}\cos^{-1}x - \frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right] + \frac{1}{2}\sin^{-1}x + C$$

$$= \frac{x^2}{2}\cos^{-1}x - \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1}x + C$$

3m ** 13] $I = \int \tan^{-1}x \cdot 1\,dx = \tan^{-1}x \cdot x - \int \frac{x}{1+x^2}\,dx$

$$= \tan^{-1}x \cdot x - \frac{1}{2}\int\frac{2x}{1+x^2}\,dx \quad \text{(adjusting)}$$

$$= x\tan^{-1}x - \frac{1}{2}\log|1+x^2| + C$$

14] $I = \int x(\log x)^2\,dx = (\log x)^2 \cdot \frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{2\log x}{x}\,dx$

$$= (\log x)^2 \cdot \frac{x^2}{2} - \int x \cdot \log x \cdot dx$$

$$= (\log x)^2 \cdot \frac{x^2}{2} - \left[\log x \cdot \frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}\,dx\right]$$

$$= (\log x)^2 \cdot \frac{x^2}{2} - \frac{x^2}{2}\log x + \frac{1}{2}\int x \cdot dx$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + c$$

15] $I = \int (x^2+1)\log x \cdot dx = \log x\left(\frac{x^3}{3}+x\right) - \int\left(\frac{x^3}{3}+x\right)\frac{1}{x}\cdot dx$

$$= \left(\frac{x^3}{3}+x\right)\log x - \int\left(\frac{x^2}{3}+1\right)dx$$

$$= \left(\frac{x^3}{3}+x\right)\log x - \left[\frac{x^3}{9}+x\right] + C$$

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21) (b) NOTE : (a) $\int e^{ax}\cdot \sin bx \cdot dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right] + c$

(b) $\int e^{ax}\cos bx \cdot dx = \dfrac{e^{x}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + c$

21] $I = \int e^{2x}\cdot \sin x \cdot dx = \dfrac{e^{2x}}{4+1}\left[2\sin x - \cos x\right] + C$

$$I = \int e^{2x}\cdot \sin x\, dx = -e^{2x}\cos x + \int \cos x \cdot 2e^{2x}\cdot dx$$
$$= -e^{2x}\cos x + 2\left[e^{2x}\cdot \sin x - \int \sin x \cdot 2e^{2x}\cdot dx\right]$$
$$= -e^{2x}\cos x + 2\left[e^{2x}\sin x - 2\int \sin x \cdot e^{2x}\cdot dx\right]$$
$$= -e^{2x}\cos x + 2e^{2x}\sin x - 4\underbrace{\int \sin x \cdot e^{2x}\, dx}_{I} + C$$

$$5I = -e^{2x}\cos x + 2e^{2x}\sin x + c$$
$$I = \frac{e^{2x}}{5}\left[2\sin x - \cos x\right] + c$$

$$I = \int e^{x}\sin x\, dx = -e^{x}\cos x + \int \cos x \cdot e^{x}\cdot dx$$
$$= -e^{x}\cos x + \left[e^{x}\sin x - \int \sin x \cdot e^{x}\cdot dx\right] + c$$
$$= -e^{x}\cos x + e^{x}\sin x - I$$
$$2I = -e^{x}\cos x + e^{x}\sin x + c$$
$$I = \frac{e^{x}}{2}\left[-\cos x + \sin x\right] + c$$

$$\cancel{I = \int e^{2x}\cos 3x \cdot dx = e^{2x}\cdot\frac{\sin 3x}{3} - \int \frac{\sin 3x}{3}\cdot 2e^{2x}\, dx}$$
$$\cancel{= e^{2x}\frac{\sin 3x}{3} - 2\left[e^{2x}\frac{(-\cos 3x)}{9}\right.}$$

$\dfrac{e^{2x}}{13}\left[2\cos 3x + 3\sin 3x\right] +$

$$I = \int e^{2x}\cdot \cos 3x \cdot dx = e^{2x}\cdot \frac{\sin 3x}{3} - \int \frac{\sin 3x}{3}\cdot 2e^{2x}\cdot dx$$
$$= e^{2x}\cdot\frac{\sin 3x}{3} - \frac{2}{3}\left[e^{2x}\frac{(-\cos 3x)}{3} + \int \frac{\cos 3x}{3}\cdot 2e^{2x}\cdot dx\right]$$
$$= e^{2x}\frac{\sin 3x}{3} + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}\int I + C$$
$$\frac{13}{9}I = \frac{e^{2x}}{3}\left[\sin 3x + \frac{2}{3}\cos 3x\right] + C$$

$$I = \int e^{3x} \cdot \sin 4x \cdot dx = \frac{e^{3x}}{25}\left[3\sin 4x - 4\cos 4x\right] + C$$

$$= -e^{3x}\frac{\cos 4x}{4} + \int \frac{\cos 4x}{4} \cdot 3e^{3x} \cdot dx$$

$$= -e^{3x}\frac{\cos 4x}{4} + \frac{3}{4}\left[e^{3x} \cdot \frac{\sin 4x}{4} - \int \frac{\sin 4x}{4} \cdot 3 \cdot e^{3x} dx\right]$$

$$= -e^{3x}\frac{\cos 4x}{4} + \frac{3}{16}e^{3x}\sin 4x - \frac{9}{16}\underbrace{\int e^{3x} \cdot \sin 4x \cdot dx}_{I}$$

$$I + \frac{9}{16}I = e^{3x}\left[\frac{3}{16}\sin 4x - \frac{1}{4}\cos 4x\right] + c$$

$$\frac{25I}{16} = \frac{e^{3x}}{16}\left[3\sin 4x - 4\cos 4x\right] + C$$

$$I = \frac{e^{3x}}{25}\left[3\sin 4x - 4\cos 4x\right] + C$$

Property under By parts :

$$\int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + C$$

Proof : LHS $= I = \int e^x\left[f(x) + f'(x)\right]dx$

$$I = \int e^x f(x)\, dx + \int e^x f'(x)\, dx$$

$$= \cancel{\int e^x f(x) dx} + e^x f(x) - \cancel{\int f(x) \cdot e^x dx} + C$$

$$= e^x f(x) + C = \text{RHS}$$

16] $I = \int e^x(\underset{f}{\sin x} + \underset{f'}{\cos x})\, dx = e^x \sin x + C$

24] $I = \int e^x \sec x(1 + \tan x)\, dx = \int e^x\left[\underset{f}{\sec x} + \underset{f'}{\sec x \cdot \tan x}\right] \cdot dx$

$= e^x \sec x + C$

Eg 22] (a) $I = \int e^x \left[\tan^{-1}x + \frac{1}{1+x^2}\right] dx = e^x \tan^{-1}x + C$

$a^2+b^2 = (a+b)^2 - 2ab$

(b)* $I = \int \frac{(x^2+1)e^x}{(x+1)^2}\, dx = \int e^x \left[\frac{(x+1)^2 - 2x}{(x+1)^2}\right] dx$

$= \int e^x \left[1 - 2\frac{x}{(x+1)^2}\right] dx$

$= \int e^x\, dx - 2\int e^x \cdot \frac{x}{(x+1)^2}\, dx$

$= e^x - 2\left[\int e^x \cdot \frac{(x+1)-1}{(x+1)^2}\right] dx$

$= e^x - 2\left[\int e^x \left(\underbrace{\frac{1}{x+1}}_{f} - \underbrace{\frac{1}{(x+1)^2}}_{f'}\right)\right] dx$

$= e^x - 2\frac{e^x}{x+1} + C$

17*) $I = \int \frac{x \cdot e^x}{(1+x)^2}\, dx$

$= \int e^x \left(\frac{(x+1)-1}{(1+x)^2}\right) dx = \int e^x \left(\underbrace{\frac{1}{x+1}}_{f} - \underbrace{\frac{1}{(x+1)^2}}_{f'}\right) dx$

$= \frac{e^x}{x+1} + C$

18] $I = \int e^x \cdot \frac{(1+\sin x)}{1+\cos x}\, dx$

$= \int e^x \left[\frac{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right] dx = \int e^x \left[\underbrace{\frac{1}{2}\sec^2\frac{x}{2}}_{f'} + \underbrace{\tan\frac{x}{2}}_{f}\right] dx$

$= e^x \tan\frac{x}{2} + C$

19] $I = \int e^x \cdot \frac{(x-1)}{x^2}\, dx = \int e^x \left(\underbrace{\frac{1}{x}}_{f} - \underbrace{\frac{1}{x^2}}_{f'}\right) dx$

$= \frac{e^x}{x} + C$

20) $I = \int \frac{(x-3)e^x}{(x-1)^3}dx = \int \frac{(x-1-2)e^x}{(x-1)^3}dx$

$= \int e^x \left(\underset{f}{\frac{1}{(x-1)^2}} - \underset{f'}{\frac{2}{(x-1)^3}}\right)dx$

$= \frac{e^x}{(x-1)^2} + c$

22) $I = \int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$ $x = \tan\theta$ $\theta = \tan^{-1}x$

$dx = \sec^2\theta \cdot d\theta$

$= \int \sin^{-1}(\sin 2\theta)\sec^2\theta \cdot d\theta$

$= \int 2\theta \cdot \sec^2\theta \cdot d\theta = 2\left[\theta\tan\theta - \int \tan\theta \cdot d\theta\right]$

$= 2\theta\tan\theta - 2\log(\sec\theta) + c$

$= 2x\tan^{-1}x - 2\log\left[\sec(\tan^{-1}x)\right] + c$

Integration by Partial Fractions - Rules:

* If the denominator has linear factors, take one constant for every factor.

$\frac{2x-3}{x(x^2-1)} = \frac{2x-3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

* For every repeated linear factor of the form $(ax+b)^n$ take 'n' no. of constants each time increasing the power.

$\frac{2x-3}{\underset{LF}{x}\underset{RLF}{(x+1)^2}} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

* $\frac{2x-3}{(3x+1)^3}$: Single repeated linear factor in denominator then put it as y

$\therefore 3x+1 = y \Rightarrow x = \frac{y-1}{3}$

$\frac{2(y-1)-9}{3y^3} = \frac{2y-11}{3y^3}$

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* If the denominator has quadratic polynomial as ax^2+bx+c or ax^2+c, use 2 constants simultaneously as $Ax+B$.

Eg: $\frac{2x-3}{x(x+1)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{(Dx+E)}{x^2+1}$
LF RLF QF

NOTE: If in $\frac{f(x)}{g(x)}$, degree of numerator $\geq$ degree of denominator, then it is improper fraction (IF).

Divide and reduce IF as $IF = Q_{uotient} + \frac{R_{emainder}}{Divisor}$

Eg: $\frac{9}{2}$ .... nu > de ⇒ IF

$D \leftarrow 2\overline{)9}(4 \rightarrow Q$
$\frac{8}{1} \rightarrow R$

$\frac{9}{2} = 4 + \frac{1}{2}$

EXERCISE 7.5

1] $\int \frac{x}{(x+1)(x+2)}dx$

° of nu = 1
° of de = 2
° of nu < ° of de ⇒ PF

$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

$x = -1$: $A = -1$

$x = -2$: $B = 2$

$I = \int \left(\frac{-1}{x+1} + \frac{2}{x+2}\right)dx = -\log(x+1) + 2\log(x+2) + c$

$= \log\left[\frac{(x+2)^2}{(x+1)}\right] + c$

2] $\int \frac{1}{x^2-9}dx$

$\frac{1}{x^2-9} = \frac{1}{(x+3)(x-3)} = \frac{A}{x-3} + \frac{B}{x+3}$

$I = \int \frac{1}{6}\frac{1}{(x-3)}dx + -\frac{1}{6}\int\frac{1}{x+3}dx$

$$I = \frac{1}{6}\left[\log(x-3) - \log(x+3)\right] + C$$

$$= \frac{1}{6}\log\left|\frac{x-3}{x+3}\right| + C$$

3] $\int \frac{3x-1 \, dx}{(x-1)(x-2)(x-3)}$

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$\cancel{x=1 : 2 =}\ 3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

$x=1 : 2 = 2A \Rightarrow A = 1$

$x=2 : 5 = -B \Rightarrow B = -5$

$x=3 : 8 = 2C \Rightarrow C = 4$

$$I = \int\frac{dx}{x-1} + -5\int\frac{dx}{x-2} + 4\int\frac{dx}{x-3}$$

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$$

4] $\int \frac{x \, dx}{(x-1)(x-2)(x-3)}$

$$\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

put $x=1 : 1 = 2A \Rightarrow A = \frac{1}{2}$

put $x=2 : 2 = -B \Rightarrow B = -2$

put $x=3 : 3 = 2C \Rightarrow C = \frac{3}{2}$

$$I = \frac{1}{2}\int\frac{dx}{x-1} - 2\int\frac{dx}{x-2} + \frac{3}{2}\int\frac{dx}{x-3}$$

$$I = \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$$

5] $I = \int \frac{2x \cdot dx}{x^2+3x+2} = \int \frac{2x \cdot dx}{(x+1)(x+2)}$

x^2+3x+2
$x^2+x+2x+2$
$x(x+1)+2(x+1)$
$(x+2)(x+1)$

$$\frac{2x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$2x = A(x+2) + B(x+1)$$

put $x=-1$: $-2 = A \Rightarrow A = -2$

put $x=-2$: $-4 = -B \Rightarrow B = 4$

$$I = \int \frac{-2\,dx}{x+1} + \int \frac{4\,dx}{x+2}$$

$$= -2\int \frac{dx}{x+1} + 4\int \frac{dx}{x+2}$$

$$= -2\log|x+1| + 4\log|x+2| + C$$

$$= 2\left[2\log|x+2| - \log|x+1|\right] + C = 2\log\left|\frac{(x+2)^2}{(x+1)}\right| + C$$

6] $I = \int \frac{1-x^2}{x(1-2x)}dx$

$°$ of $nu = °$ of $de \Rightarrow$ IF
$Q + \frac{R}{D}$

$-2x^2+x)\,-x^2+1\,(\frac{1}{2} \to Q$
D
$\quad -x^2+\frac{x}{2}$
$\quad \ominus \quad \ominus$
$\quad -\frac{x}{2}+1 = \left(\frac{2-x}{2}\right) - R$

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left[\frac{2-x}{x(1-2x)}\right] \longrightarrow ①$$

$$\frac{2-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x}$$

$x = 0: \quad A = 2 \quad \left(\frac{2}{1}\right)$

$x = \frac{1}{2}: \quad B = 3 \quad \left(\frac{3}{2} \cdot \frac{2}{1}\right)$

① $\Rightarrow \dfrac{1-x^2}{x(1-2x)} = \dfrac{1}{2} + \dfrac{1}{2}\left[\dfrac{2}{x} + \dfrac{3}{1-2x}\right]$

$\therefore I = \displaystyle\int \frac{1-x^2}{x(1-2x)} = \frac{1}{2}\int\left(1 + \frac{2}{x} + \frac{3}{1-2x}\right)dx$

$= \dfrac{1}{2}\left[x + 2\log x + \dfrac{3\log(1-2x)}{2}\right] + C$

8] $I = \displaystyle\int \frac{x}{(x-1)^2(x+2)}\,dx$

$\dfrac{x}{(x+2)(x-1)^2} = \dfrac{A}{x+2} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

$x = A(x-1)^2 + B(x-1)(x+2) + C(x+2)$

$x = -2: \; -2 = A(9) \Rightarrow A = \frac{-2}{9}$

$x = 1: \; 1 = C(3) \Rightarrow C = \frac{1}{3}$

coeff of x^2: $0 = A + B \Rightarrow B = \frac{2}{9}$

$I = \displaystyle\int \frac{-2}{9}\cdot\frac{1}{x+2}\,dx + \frac{2}{9}\int\frac{1}{x-1}\,dx + \frac{1}{3}\int\frac{1}{(x-1)^2}\,dx$

$= \frac{-2}{9}\log|x+2| + \frac{2}{9}\log|x-1| + \frac{1}{3}\left(\frac{-1}{x-1}\right) + C$

$= \dfrac{2}{9}\log\left|\dfrac{x-1}{x+2}\right| - \dfrac{1}{3(x-1)} + C$

1] $I = \displaystyle\int \frac{x}{(x^2+1)(x-1)}\,dx$

$\dfrac{x}{(x^2+1)(x-1)} = \dfrac{A}{x-1} + \dfrac{(Bx+C)}{x^2+1}$

QF LF

$x = A(x^2+1) + (Bx+C)(x-1)$

$x = 1: \; 1 = A(2) \Rightarrow A = \frac{1}{2}$

coeff of x^2: $0 = A + B \Rightarrow B = \frac{-1}{2}$

coeff of x : $1 = C - B \Rightarrow C = \frac{1}{2}$

$$I = \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{-x+1}{x^2+1}dx$$

$$= \frac{1}{2}\log|x-1| + \frac{1}{2}\int \frac{1}{x^2+1}dx - \frac{1}{2}\int \frac{x}{x^2+1}dx$$

$x^2+1 = t$

$2x\,dx = dt$

$x\,dx = \frac{1}{2}dt$

$$= \frac{1}{2}\log|x-1| + \frac{1}{2}\tan^{-1}x - \frac{1}{2}\cdot\frac{1}{2}\int \frac{dt}{t}$$

$$= \frac{1}{2}\left[\log(x-1) + \tan^{-1}x - \frac{1}{2}\log t\right] + C$$

$$= \frac{1}{2}\left[\tan^{-1}x + \log\left|\frac{x-1}{\sqrt{x^2+1}}\right|\right] + C$$

9] $\int \frac{(3x+5)dx}{x^3-x^2-x+1} = \int \frac{(3x+5)dx}{x^2(x-1)-1(x-1)}$

$$\frac{3x+5}{(x^2-1)(x-1)} = \frac{3x+5}{(x-1)(x+1)(x-1)} = \frac{3x+5}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$3x+5 = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$$

$x = 1$: $8 = 2C \Rightarrow C = 4$

$x = -1$: $2 = 4A \Rightarrow A = \frac{1}{2}$

coeff of x^2: $0 = A + B \Rightarrow B = -\frac{1}{2}$

$$I = \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x-1} + 4\int \frac{dx}{(x-1)^2}$$

$$= \frac{1}{2}\left[\log|x+1| - \log(x-1)\right] + 4\frac{(-1)}{x-1} + C$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C$$

10] $\int \frac{2x-3}{(x^2-1)(2x+3)} dx$

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{2x+3}$$

$$2x-3 = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x+1)(x-1)$$

put $x=-1$ $\quad -5 = -2A \Rightarrow A = \frac{5}{2}$

put $x=1$ $\quad -1 = 10B \Rightarrow B = \frac{-1}{10}$

put $x=\frac{-3}{2}$ $\quad -6 = \frac{5}{4}C \Rightarrow C = \frac{-24}{5}$

$$I = \frac{5}{2}\int \frac{dx}{x+1} - \frac{1}{10}\int \frac{dx}{x-1} - \frac{24}{5}\int \frac{dx}{2x+3}$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5}\,\frac{\log|2x+3|}{2}$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3|$$

11] $I = \int \frac{5x}{(x+1)(x^2-4)} dx$

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-2}$$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2)$$

put $x=-1$: $\quad -5 = -3A \Rightarrow A = \frac{5}{3}$

$x=-2$: $\quad -10 = 4B \Rightarrow B = \frac{-5}{2}$

$x=2$: $\quad 10 = 12C \Rightarrow C = \frac{5}{6}$

$$I = \frac{5}{3}\int \frac{dx}{x+1} - \frac{5}{2}\int \frac{dx}{x+2} + \frac{5}{6}\int \frac{dx}{x-2}$$

$$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$$

$^\circ$ of nu $>$ $^\circ$ of de $\Rightarrow$ IF

** 12] $\int \frac{(x^3+x+1)\,dx}{x^2-1}$

$x^2-1 \,)\, x^3+x+1 \,(\, x \rightarrow Q$
D
$\quad \ominus x^3 \oplus x$
$\quad 2x+1 \rightarrow R$

$\frac{x^3+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1} \longrightarrow (1)$

$\frac{2x+1}{(x-1)(x+1)} = \frac{3}{2(x-1)} + \frac{-1}{-2(x+1)}$

(1) $\Rightarrow I = \int x\,dx + \frac{3}{2}\int \frac{1\,dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1}$

$= \frac{x^2}{2} + \frac{3}{2}\log|x-1| + \frac{1}{2}\log|x+1| + C$

13] $\int \frac{2\cdot dx}{(1-x)(1+x^2)}$

$\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{(Bx+C)}{1+x^2}$

$2 = A(1+x^2) + (Bx+C)(1-x)$

put $x=1$: $2 = A(2) \Rightarrow A = 1$

coeff of x^2 : $0 = A - B \Rightarrow B = 1$

coeff of x : $0 = B - C \Rightarrow C = 1$

$I = \int \left(\frac{1}{1-x} + \frac{x+1}{1+x^2}\right) dx = \int \frac{1}{1-x} + \frac{x}{1+x^2} + \frac{1}{1+x^2}\, dx$

$= \int \frac{1}{1-x} + \frac{2x}{2(1+x^2)} + \frac{1}{1+x^2}\, dx$

$= \frac{\log|1-x|}{-1} + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + c$

14] $I = \int \frac{3x-1}{(x+2)^2}\,dx$

$\frac{3x-1}{(x+2)^2} = \frac{3(y-2)-1}{y^2} = \frac{3y-7}{y^2}$ put $x+2=y$

$x = y-2$

$= \frac{3}{y} - \frac{7}{y^2}$

$I = 3\int \frac{1}{y}\,dy - 7\int \frac{1}{y^2}\,dy$

$= 3\log y + \frac{7}{y} + C$

$= 3\log|x+2| + 7\left(\frac{1}{x+2}\right) + C$

15] $\int \frac{1}{x^4-1}\,dx = \int \frac{1\,dx}{(x^2-1)(x^2+1)} = \int \frac{dx}{(x-1)(x+1)(x^2+1)}$

$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{(Cx+D)}{x^2+1}$

$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$

put $x=1$: $1 = 4A \Rightarrow A = \frac{1}{4}$

$x=-1$: $1 = -4B \Rightarrow B = -\frac{1}{4}$

coeff of x^3: $0 = A+B+C \Rightarrow C = 0$

coeffs of x^2: $0 = D$

$I = \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1}$

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***16] $\int \frac{dx}{x(x^n+1)}$ ÷ & x by x^{n-1}

$I = \int \frac{x^{n-1}\,dx}{x^n(x^n+1)}$ $\qquad x^n = t$

$n x^{n-1}\,dx = dt$

$= \frac{1}{n}\int \frac{dt}{t(t+1)}$ $\qquad x^{n-1}\,dx = \frac{1}{n}dt$

$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} = \frac{1}{t} + \frac{-1}{t+1}$

$I = \frac{1}{n}\int\left(\frac{1}{t} - \frac{1}{t+1}\right)dt = \frac{1}{n}\left[\log t \mp \log|t+1|\right] + c$

$= \frac{1}{n}\log\left|\frac{t}{t+1}\right| + c$

$$\boxed{\int \frac{dx}{x(x^n+1)} = \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C}$$

*] $\int \frac{dx}{x(x^7+1)} = \frac{1}{7}\log\left|\frac{x^7}{x^7+1}\right| + c$ ÷ & x by x^6

$I = \int \frac{x^6\,dx}{x^7(x^7+1)} = \frac{1}{7}\int \frac{dt}{t(t+1)}$ $\qquad x^7 = t$

$7x^6\,dx = dt$

$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} = \frac{1}{t} - \frac{1}{t+1}$ $\qquad x^6\,dx = \frac{1}{7}dt$

$I = \frac{1}{7}\int \frac{1}{t} - \frac{1}{t+1}\,dt = \frac{1}{7}\left[\log t + \log|t+1|\right] + c$

$= \frac{1}{7}\log\left|\frac{t}{t+1}\right| + c$

$$\boxed{\int \frac{dx}{x(x^7+1)} = \frac{1}{7}\log\left|\frac{x^7}{x^7+1}\right| + c}$$

17] $\int \frac{\cos x\,dx}{(1-\sin x)(2-\sin x)}$ $\qquad \sin x = t$

$\cos x \cdot dx = dt$

$I = \int \frac{dt}{(1-t)(2-t)}$

$\frac{1}{(1-t)(2-t)} = \frac{A}{1-t} + \frac{B}{2-t} = \frac{1}{1-t} - \frac{1}{2-t}$

$I = \int dt \; \frac{1}{1-t} - \frac{1}{2-t} = \frac{\log|1-t|}{-1} - \frac{\log|2-t|}{-1}$

$= \log\left|\frac{2-t}{1-t}\right| + C = \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$

18] $I = \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx$

$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$ In partial fractions, $x^2 = t$

$\Rightarrow \frac{(t+1)(t+2)}{(t+3)(t+4)} = \frac{t^2+3t+2}{t^2+7t+12}$ (Improper fraction)

$t^2+7t+12 \,)\, t^2+3t+2 \,(\, 1$

$\quad t^2+7t+12$

$\quad -4t-10 = -2(2t+5)$

$\frac{t^2+3t+2}{t^2+7t+12} = 1 - \frac{2(2t+5)}{(t+3)(t+4)} \longrightarrow (1)$

$\frac{2t+5}{(t+3)(t+4)} = \frac{A}{t+3} + \frac{B}{t+4} = \frac{-1}{t+3} + \frac{3}{t+4}$

$\therefore \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx = \int 1 - 2\left[\frac{-1}{x^2+(\sqrt{3})^2} + \frac{3}{x^2+4}\right] dx$

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$= x + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{6^3}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$

** 21] $I = \int \frac{dx}{e^x - 1} = \int \frac{dt}{t(t-1)}$

$e^x = t$

$e^x dx = dt$

$\int \frac{-1}{t} + \frac{1}{t-1} \, dt$

$dx = \frac{dt}{e^x} = \frac{dt}{t}$

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$= \log|t-1| - \log|t| + c$

$= \log\left|\frac{t-1}{t}\right| + c = \log\left|\frac{e^x-1}{e^x}\right| + c$

19] $\int \frac{2x\,dx}{(x^2+1)(x^2+3)}$

$\frac{2x}{(x^2+1)(x^2+3)} = \frac{(Ax+B)}{x^2+1} + \frac{(Cx+D)}{x^2+3}$

$2x = (Ax+B)(x^2+3) + (Cx+D)(x^2+1)$

put $x^2 = t$

$2x\,dx = dt$

$\int \frac{dt}{(t+1)(t+3)}$ Now do partial fractions.

Bernoullis rule :

$$\int uv\,dx = u\int v - u_1\iint v + u_2\iiint v \cdots$$

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Eg 20] $\int \frac{x(\sin^{-1}x)}{\sqrt{1-x^2}}dx$ $\quad \sin^{-1}x = t \;;\; x = \sin t$

$\frac{1}{\sqrt{1-x^2}}dx = dt$

$I = \int t\sin t\, dt$ Bernoulli's rule $\Rightarrow \cos t = \sqrt{1-x^2}$

$= -t\cos t + \sin t + C$

$= -(\sin^{-1}x)\sqrt{1-x^2} + x + C$

Definite Integrals :

Fundamental theorem of integral calculus :

Theorem : Let f be a continuous funⁿ defined on $[a, b]$, and F be its anti derivative, then

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

EXERCISE 7.9

1] $I = \int_{-1}^{1}(x+1)dx = \left[\frac{x^2}{2} + x\right]_{-1}^{1} = \frac{1}{2}(1-1) + [1-(-1)] = 2$

** im 2] $I = \int_2^3 \frac{1}{x}dx = \log x\Big]_2^3 = \log 3 - \log 2 = \log\frac{3}{2}$

* $I = \int_1^2 \frac{1}{x}dx = \log x\Big]_1^2 = \log 2 - \cancel{\log 1}^{0} = \log 2$

3] $I = \int_1^2 (4x^3 - 5x^2 + 6x + 9)dx = \frac{\cancel{4}x^4}{\cancel{4}} - \frac{5x^3}{3} + \frac{6x^2}{2} + 9x\Big]_1^2$

$= (16-1) - \frac{5}{3}(8-1) + 3(4-1) + 9(2-1)$

$= 15 - \frac{35}{3} + 9 + 9 = 33 - \frac{35}{3} = \frac{64}{3}$

4] $\int_0^{\frac{\pi}{4}} \sin 2x\,dx = -\frac{\cos 2x}{2}\Big]_0^{\frac{\pi}{4}} = -\frac{1}{2}(0-1) = \frac{1}{2}$

5] $\int_0^{\frac{\pi}{2}} \cos 2x \cdot dx = \frac{\sin 2x}{2}\Big]_0^{\frac{\pi}{2}} = \frac{1}{2}[0-0] = 0$

6] $\int_4^5 e^x \cdot dx = e^x\Big]_4^5 = e^5 - e^4$

7] $\int_0^{\frac{\pi}{4}} \tan x\, dx = \log(\sec x)\Big]_0^{\frac{\pi}{4}} = \log\sqrt{2} - \log 1 = \log\sqrt{2} = \frac{1}{2}\log 2$

8] $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \cdot dx = \log(\operatorname{cosec} x - \cot x)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \log(\sqrt{2}-1) - \log(2-\sqrt{3}) = \log\left|\frac{\sqrt{2}-1}{2-\sqrt{3}}\right|$

9] $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Big]_0^1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

10] $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x\Big]_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

11] $\int_2^3 \frac{dx}{x^2-1} = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\Big]_2^3 = \frac{1}{2}\left[\log\left(\frac{2}{4}\right) - \log\left(\frac{1}{3}\right)\right]$

$= \frac{1}{2}\log\left|\frac{1}{2}\cdot\frac{3}{1}\right| = \frac{1}{2}\log\left(\frac{3}{2}\right)$

$= \log\sqrt{\frac{3}{2}}$

12] $\int_0^{\frac{\pi}{2}} \cos^2 x \cdot dx = \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$

$I = \frac{1}{2}\left\{\left(\frac{\pi}{2} - 0\right) + \frac{1}{2}(0-0)\right\} = \frac{\pi}{4}$

13] $I = \int_2^3 \frac{x \cdot dx}{x^2+1} = \frac{1}{2}\int_2^3 \frac{2x \cdot dx}{x^2+1} = \frac{1}{2}\log|x^2+1|\Big]_2^3$

$= \frac{1}{2}\{\log 10 - \log 5\} = \frac{1}{2}\log\left(\frac{10}{5}\right) = \frac{1}{2}\log 2 = \log\sqrt{2}$

If a substitution is made for a definite integral, its limits also change.

14] $\int_0^1 \frac{2x+3}{5x^2+1}dx = \int_0^1 \frac{2x\,dx}{5x^2+1} + \int_0^1 \frac{3\,dx}{5x^2+1}$

$5x^2+1 = t$

$10x\cdot dx = dt$

$x\cdot dx = \frac{dt}{10}$

$= 2\int_0^1 \frac{dt}{10(t)} + \frac{3}{5}\int_0^1 \frac{dx}{(x^2+\frac{1}{5})}$

$= \frac{2}{10}\int_0^1 \frac{dt}{t} + \frac{3}{5}\int_0^1 \frac{dx}{x^2+(\frac{1}{\sqrt{5}})^2}$

$= \frac{1}{5}\log|5x^2+1|\Big]_0^1 + \frac{3\sqrt{5}}{5}\left[\tan^{-1}(x\sqrt{5})\right]_0^1$

$= \frac{1}{5}\log 6 - \cancel{\log 1} + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$

15] $\int_0^1 x\cdot e^{x^2}dx$

$x^2 = t$

$2x\cdot dx = dt$

$x\cdot dx = \frac{dt}{2}$

put $x = 0,\ t = 0$

$x = 1,\ t = 1$

$I = \int_0^1 \frac{1}{2}dt\cdot e^t = \frac{1}{2}\left. e^t\right]_0^1$

$= \frac{1}{2}(e^1 - e^0) = \frac{1}{2}(e-1)$

16] $\int_1^2 \frac{5x^2\,dx}{x^2+4x+3} = 5\int_1^2 \frac{x^2\,dx}{x^2+4x+3}$

$^\circ$ of nr $= {}^\circ$ of de $\Rightarrow$ IF

$x^2+4x+3)\,x^2\,(1$

$\underline{x^2+4x+3}$

$-(4x+3)$

$\frac{x^2}{x^2+4x+3} = 1 - \frac{(4x+3)}{x^2+4x+3}$

$= 1 - \frac{(4x+3)}{(x+1)(x+3)}$

$= 1 - \left[\frac{A}{x+1} + \frac{B}{x+3}\right]$

$x = -1;\ \frac{-1}{2} = A$

$x = -3;\ B = \frac{-9}{-2} = \frac{9}{2}$

$= 1 - \left[\frac{-1}{2(x+1)} + \frac{9}{2(x+3)}\right]$

$I = 5\int_1^2 \left(1 + \frac{1}{2(x+1)} - \frac{9}{2(x+3)}\right)dx$

$= 5\left\{x + \frac{1}{2}\log(x+1) - \frac{9}{2}\log(x+3)\right\}_1^2$

$$= 5\left\{(2-1) + \frac{1}{2}(\log 3 - \log 2) - \frac{9}{2}(\log 5 - \log 4)\right\}$$

$$= 5\left\{1 + \log\sqrt{\frac{3}{2}} - \frac{9}{2}\log\left(\frac{5}{4}\right)\right\}$$

17) $I = \int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)\,dx$

$$= 2\tan x + \frac{x^4}{4} + 2x\Big]_0^{\frac{\pi}{4}} = 2\left(\tan\frac{\pi}{4} - \tan 0\right) + \left(\frac{\left(\frac{\pi}{4}\right)^4}{4} - \frac{0}{4}\right) + \left(2\frac{\pi}{4} - 0\right)$$

$$= 2 + \frac{\pi^4}{1024} + \frac{\pi}{2}$$

imp *** 18) $\int_0^{\pi} \sin^2\frac{x}{2} - \cos^2\frac{x}{2}\,dx = \int_0^{\pi} -\cos 2\left(\frac{x}{2}\right).dx = \int_0^{\pi} -\cos x$

$$= -\sin x\Big]_0^{\pi} = 0$$

Bernoulli's theorem

20) $\int_0^1 x\cdot e^x + \sin\left(\frac{\pi x}{4}\right)dx = x e^x - e^x\Big]_0^1 - \cos\left(\frac{\pi x}{4}\right)\cdot\frac{4}{\pi}\Big]_0^1$

$$= (e-0) - (e-1) - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= \cancel{e} - \cancel{e} + 1 - \frac{4}{\pi\sqrt{2}} + \frac{4}{\pi} \quad = 1 - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}} - 1\right)$$

19) $\int_0^2 \frac{6x+3}{x^2+4}\,dx$ split & integrate

$\int_{-5}^{5} |x+2|\,dx$

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21) $\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1}x\Big]_1^{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{4} = 60^\circ - 45^\circ = \frac{\pi}{12}$

22) $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = \int_0^{\frac{2}{3}} \frac{dx}{(2)^2+(3x)^2} = \frac{1}{(3)2} \tan^{-1}\left(\frac{3x}{2}\right)\Big]_0^{\frac{2}{3}}$

$= \frac{1}{6}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{24}$

EXERCISE 7.10

1) $\int_0^1 \frac{x}{x^2+1}\,dx$ $\quad x^2+1 = t$, $2x\cdot dx = dt$, $x\,dx = \frac{1}{2}dt$ $\quad x=0, t=1$; $x=1, t=2$

$I = \int_1^2 \frac{1}{2}\cdot\frac{dt}{t}$

$= \frac{1}{2}\log t\Big]_1^2 = \frac{1}{2}(\log 2 - \log 1) = \frac{1}{2}\log 2 = \log\sqrt{2}$

3) $\int_0^1 \sin^{-1}\frac{2x}{1+x^2}\,dx$ $\quad x = \tan\theta$, $\theta = \tan^{-1}x$, $dx = \sec^2\theta\,d\theta$ $\quad x=0, \theta=0$; $x=1, \theta=\frac{\pi}{4}$

$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\,d\theta$

$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta\,d\theta = 2\int_0^{\frac{\pi}{4}} \theta\sec^2\theta\,d\theta$

by parts

$I = 2\left[\theta\cdot\tan\theta - \int\tan\theta\,d\theta\right]_0^{\frac{\pi}{4}} = 2\left[\theta\tan\theta - \log(\sec\theta)\right]_0^{\frac{\pi}{4}}$

$= 2\left\{\left(\frac{\pi}{4} - 0\right) - \left(\log\sqrt{2} - \log 1\right)\right\} = 2\left[\frac{\pi}{4} - \log\sqrt{2}\right] = \frac{\pi}{2} - \log 2$

4] $I = \int_0^2 x\sqrt{x+2}\,dx$ $\quad x+2 = t$, $x = t-2$, $dx = dt$ $\quad \left| \begin{array}{l} x=0, t=2 \\ x=2, t=4 \end{array} \right.$

$I = \int_2^4 (t-2)\sqrt{t}\,dt$

$= \int_2^4 \left(t^{\frac{3}{2}} - 2\sqrt{t}\right) dt = \frac{2}{5}t^{\frac{5}{2}} - (2)\frac{2}{3}t^{\frac{3}{2}}\Big]_2^4$

$= \frac{2}{5}\left[t^2\sqrt{t}\right] - \frac{4}{3}\left[t\sqrt{t}\right]_2^4 = \frac{2}{5}(32-4\sqrt{2}) - \frac{4}{3}(8-2\sqrt{2})$

$= \left(\frac{64}{5} - \frac{32}{3}\right) - \left(\frac{8}{5} + \frac{8}{3}\right)\sqrt{2} = \frac{32}{15} - \frac{8}{15}(8\sqrt{2}) = \frac{32-64\sqrt{2}}{15}$

compulsory (6m) **

Properties of Definite Integrals

1] $P_0 : \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ (change of variable does not affect the value of integral)

Proof:

LHS $= \int_a^b f(x)\,dx = \int_a^b f(t)\cdot dt$ = RHS $\quad x = t$, $dx = dt$ $\quad \left| \begin{array}{l} x=a, t=a \\ x=b, t=b \end{array} \right.$

2] $P_1 : \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

LHS $= \int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a) = -[F(a) - F(b)]$

$= -\int_b^a f(x)\,dx$ = RHS

3] $P_2 : \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

LHS $= \int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$

$= F(b) - F(a) - F(c) + F(c)$

$= [F(b) - F(c)] + [F(c) - F(a)]$

$= \int_c^b f(x)\,dx + \int_a^c f(x)\,dx$ = RHS

4] $P_3 : \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Eg: $I = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot dx = \int_0^{\frac{\pi}{2}} \sin^2\left(\frac{\pi}{2}-x\right)dx = \int_0^{\frac{\pi}{2}} \cos^2 x\, dx \rightarrow (2)$

$\hookrightarrow (1)$

(1) + (2)

$2I = \int_0^{\frac{\pi}{2}}(\sin^2 x + \cos^2 x)\,dx = \int_0^{\frac{\pi}{2}} 1 \cdot dx = x\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

$\boxed{I = \frac{\pi}{4}}$ Illy $\int_0^{\frac{\pi}{2}} \cos^2 x \cdot dx = \frac{\pi}{4}$

Proof : LHS $= \int_a^b f(x)\,dx$ put $x = a+b-t$, $dx = -dt$ | $x = a,\ t = b$; $x = b,\ t = a$

$I = \int_b^a f(a+b-t)(-)\,dt$

$= -\int_b^a f(a+b-t)\,dt = \int_a^b f(a+b-t)\,dt$ (using P_1)

$= \int_a^b f(a+b-x)\,dx$ (using P_0) = RHS

5] $P_4 : \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Proof : LHS $= \int_0^a f(x)\,dx$ put $x = a-t$, $dx = -dt$ | $x = 0,\ t = a$; $x = a,\ t = 0$

$I = \int_a^0 f(a-t)(-)\,dt$

$= -\int_a^0 f(a-t)\,dt = \int_0^a f(a-t)\,dt$ (using P_1)

$= \int_0^a f(a-x)\,dx$ (using P_0) = RHS

6] P_5 : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

LHS $= I = \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

In 2^{nd} integral : $x = 2a-t$, $dx = -dt$ | $x = a, t = a$; $x = 2a, t = 0$

$I = \int_0^a f(x)\,dx + \int_a^0 f(2a-t)\,dt\,(-)$

$= \int_0^a f(x)\,dx - \int_a^0 (2a-t)\,dt = \int_0^a f(x)\,dx + \int_0^a f(2a-t)\,dt$ (using P_1)

$\therefore \int_0^{2a} f(x)\,dx + \int_0^a f(2a-x)\,dx$ (using P_0)

7] P_6 : $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

Proof : LHS $= I = \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

In 2^{nd} integral : $x = 2a-t$, $dx = -dt$ | $x = a, t = a$; $x = 2a, t = 0$

$I = \int_0^a f(x)\,dx + \int_a^0 f(2a-t)(-)\,dt$

$= \int_0^a f(x)\,dx - \int_a^0 f(2a-t)\,dt = \int_0^a f(x)\,dx + \int_0^a f(2a-t)\,dt$ (using P_1)

$\therefore \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$ [using P_0]

case (i) $f(2a-x) = f(x)$

$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$

case (ii) If $f(2a-x) = -f(x)$

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0$$

cases (i) and (ii) $\Rightarrow$

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

8) P_7 : $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x) & ; \text{if } f(x) \text{ is an even fun}^n \\ 0 & ; \text{if } f(x) \text{ is an odd fun}^n \end{cases}$

NOTE: Even fun^n : If $f(-x) = f(x)$

Odd fun^n : If $f(-x) = -f(x)$

LHS $= \int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$

In 1st integral : put $x = -t$, $dx = -dt$ | $x = -a, t = a$; $x = 0, t = 0$

$I = \int_a^0 f(-t)\,dt(-) + \int_0^a f(x)dx$

$\int f(x)dx = -\int_a^0 f(-t)dt + \int_0^a f(x)dx = \int_0^a f(-t)dt + \int_0^a f(x)dx$ (P_1)

$\int_{-a}^{a} f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx$ [using P_0]

case (i) : If $f(x)$ is an even fun^n, $f(-x) = f(x)$

$\Rightarrow \int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$

case (ii) : If $f(x)$ is an odd fun^n, $f(-x) = -f(x)$

$\int_{-a}^{a} f(x)dx = \int_0^a -f(x)dx + \int_0^a f(x)dx = 0$

EXERCISE 7.11

Evaluate the following integrals :

1] $\int_0^{\frac{\pi}{2}} \cos^2 x \cdot dx = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2}-x\right)dx = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot dx \rightarrow (2)$

$\hookrightarrow (1)$

(1) + (2)

$I + I = \int_0^{\frac{\pi}{2}} (\cos^2 x + \sin^2 x)\,dx$

$2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx = x\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

$I = \frac{\pi}{4}$

Eg 34] $I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x\, dx}{\sin^4 x + \cos^4 x} \longrightarrow (1)$

$I = \int_0^{\frac{\pi}{2}} \frac{\sin^4\left(\frac{\pi}{2}-x\right)dx}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)} = \int_0^{\frac{\pi}{2}} \frac{\cos^4 x \cdot dx}{\cos^4 x + \sin^4 x} \rightarrow (2)$

(1) + (2)

$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x + \cos^4 x}{\sin^4 x + \cos^4 x}\,dx = \int_0^{\frac{\pi}{2}} 1 \cdot dx = x\Big]_0^{\frac{\pi}{2}}$

$2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

2] $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}\, dx}{\sqrt{\sin x} + \sqrt{\cos x}} \longrightarrow (1)$

$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}\, dx}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)} + \sqrt{\cos\left(\frac{\pi}{2}-x\right)}} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}} \rightarrow (2)$

(1) + (2)

$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx = \int_0^{\frac{\pi}{2}} 1 \cdot dx = x\Big]_0^{\frac{\pi}{2}}$

$2I = \frac{\pi}{2}$

$I = \frac{\pi}{4}$

3) $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x \, dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} \longrightarrow (1)$

$I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2}\left(\frac{\pi}{2} - x\right) dx}{\sin^{3/2}\left(\frac{\pi}{2} - x\right) + \cos^{3/2}\left(\frac{\pi}{2} - x\right)} = \int_0^{\frac{\pi}{2}} \frac{\cos^{3/2} x \, dx}{\cos^{3/2} x + \sin^{3/2} x} \longrightarrow (2)$

(1) + (2)

$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2} x + \cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx = \int_0^{\frac{\pi}{2}} 1 \cdot dx = x \Big]_0^{\frac{\pi}{2}}$

$2I = \frac{\pi}{2}$

$\therefore I = \frac{\pi}{4}$

4) $I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x \, dx}{\sin^5 x + \cos^5 x} \longrightarrow (1)$

$I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2} - x\right) dx}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)} = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx \longrightarrow (2)$

(1) + (2)

$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx = \int_0^{\frac{\pi}{2}} 1 \cdot dx = x \Big]_0^{\frac{\pi}{2}}$

$2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

7) $I = \int_0^1 x(1-x)^n dx$

$= \int_0^1 (1-x)\left[1 - (1-x)\right]^n dx = \int_0^1 (1-x)\, x^n dx$

$= \int_0^1 x^n - x^{n+1} dx = \frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\Big]_0^1$

$= \frac{1}{n+1} - \frac{1}{n+2} = \frac{n+2-n-1}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}$

9] $I = \int_0^2 x\sqrt{2-x}\, dx$

$= \int_0^2 (2-x)\sqrt{2-(2-x)}\, dx \quad = \int_0^2 (2-x)\sqrt{x}\, dx$

$= \int_0^2 2\sqrt{x} - x^{3/2}\, dx \quad = 2\cdot\frac{2}{3}x^{3/2} - x^{5/2}\cdot\frac{2}{5}\Big]_0^2$

$= \frac{4}{3}x\sqrt{x} - \frac{2}{5}x^2\sqrt{x}\Big]_0^2 = \frac{4}{3}(2\sqrt{2}) - \frac{2}{5}(4\sqrt{2})$

$= 8\sqrt{2}\left(\frac{2}{15}\right) = \frac{16\sqrt{2}}{15}$

***17] $I = \int_0^a \frac{\sqrt{x}\cdot dx}{\sqrt{x}+\sqrt{a-x}} \longrightarrow (1)$

$I = \int_0^a \frac{\sqrt{a-x}\cdot dx}{\sqrt{a-x}+\sqrt{a-(a-x)}} = \int_0^a \frac{\sqrt{a-x}\cdot dx}{\sqrt{a-x}+\sqrt{x}} \longrightarrow (2)$

(1) + (2)

$2I = \int_0^a \frac{\sqrt{a-x}+\sqrt{x}}{\sqrt{a-x}+\sqrt{x}}\, dx = \int_0^a 1\cdot dx = x\Big]_0^a$

$2I = a \Rightarrow I = \frac{a}{2}$

*] $\int_2^3 \frac{\sqrt{x}\cdot dx}{\sqrt{x}+\sqrt{5-x}} \longrightarrow (1)$

$I = \int_2^3 \frac{\sqrt{5-x}\, dx}{\sqrt{5-x}+\sqrt{5-(5-x)}} = \int_2^3 \frac{\sqrt{5-x}\, dx}{\sqrt{5-x}+\sqrt{x}} \longrightarrow (2)$

(1) + (2)

$2I = \int_2^3 \frac{\sqrt{5-x}+\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\, dx = \int_2^3 1\, dx = x\Big]_2^3$

$2I = 3-2 = 1$

$I = \frac{1}{2}$

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15) $I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx \longrightarrow ①$

$I = \int_0^{\frac{\pi}{2}} \frac{\sin(\frac{\pi}{2} - x) - \cos(\frac{\pi}{2} - x)}{1 + \sin(\frac{\pi}{2} - x)\cos(\frac{\pi}{2} - x)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \cos x \cdot \sin x} dx \rightarrow ②$

① + ②

$2I = \int_0^{\frac{\pi}{2}} \frac{\cancel{\sin x} - \cancel{\cos x} + \cancel{\cos x} - \cancel{\sin x}}{1 + \sin x \cos x} dx = 0 \Rightarrow I = 0$

NOTE: $\boxed{\int_a^b \frac{\sqrt{x}\, dx}{\sqrt{x} + \sqrt{a+b-x}} = \frac{b-a}{2}}$

*** Eg 35] $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}} \longrightarrow ①$ $\qquad \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$

$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan(\frac{\pi}{2} - x)}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\cot x}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \frac{1}{\sqrt{\tan x}}}$

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x} \cdot dx}{\sqrt{\tan x} + 1} \longrightarrow ②$

① + ②

$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x} + 1}{\sqrt{\tan x} + 1} dx = \int_{\pi/6}^{\pi/3} 1 \cdot dx = x \Big]_{\pi/6}^{\pi/3}$

$2I = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \Rightarrow I = \frac{\pi}{12}$

*** 8] $I = \int_0^{\pi/4} \log(1 + \tan x) dx = \boxed{\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8} \log 2}$ CET ****

$I = \int_0^{\frac{\pi}{4}} \log(1 + \tan x) x$

$= \int_0^{\frac{\pi}{4}} \log\left[1 + \tan(\frac{\pi}{4} - x)\right] dx = \int_0^{\frac{\pi}{4}} \log\left[1 + \left(\frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\right)\right] dx$

$= \int_0^{\frac{\pi}{4}} \log\left[1 + \frac{1 - \tan x}{1 + \tan x}\right] dx = \int_0^{\frac{\pi}{4}} \log\left(\frac{1 + \cancel{\tan x} + 1 - \cancel{\tan x}}{1 + \tan x}\right) d$

$$I = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right)dx = \int_0^{\frac{\pi}{4}} \log 2 \cdot dx - \underbrace{\int_0^{\frac{\pi}{4}} \log(1+\tan x)dx}_{I}$$

$$2I = \log 2 \cdot \int_0^{\frac{\pi}{4}} 1 \cdot dx = \log 2 \left[x\right]_0^{\frac{\pi}{4}} = \log 2 \left(\frac{\pi}{4} - 0\right)$$

$$2I = \frac{\pi}{4}\log 2$$

$$I = \frac{\pi}{8}\log 2$$

If we put $x=-2$ then $(x+2)$ becomes zero
∴ we put $x=-2$ and split the integration

5] $I = \int_{-5}^{5} |x+2|\,dx = \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5}(x+2)dx$

$$= -\frac{x^2}{2} - 2x\Big]_{-5}^{-2} + \frac{x^2}{2} + 2x\Big]_{-2}^{5}$$

$$= -\frac{1}{2}(4-25) - 2(-2+5) + \frac{1}{2}(25-4) + 2(5+2)$$

$$= \frac{21}{2} + \frac{21}{2} - 6 + 14 = 21 + 8 = 29$$

6] $I = \int_2^8 |x-5|\,dx = \int_2^5 -(x-5)dx + \int_5^8 (x-5)\,dx$

$$= -\frac{x^2}{2} + 5x\Big]_2^5 + \frac{x^2}{2} - 5x\Big]_5^8$$

$$= \frac{-1}{2}(25-4) + \cancel{5(3)} + \frac{1}{2}(64-25) - \cancel{5(3)}$$

$$= \frac{-21}{2} + \frac{39}{2} = \frac{18}{2} = 9$$

18] $I = \int_0^4 |x-1|\,dx = \int_0^1 (1-x)\,dx + \int_1^4 (x-1)\,dx$

$$= x - \frac{x^2}{2}\Big]_0^1 + \frac{x^2}{2} - x\Big]_1^4$$

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$$= \frac{1}{2} + (8-4) - \left(-\frac{1}{2}\right) = \frac{1}{2} + \frac{9}{2} = \frac{10}{2} = 5$$

**11] $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cdot dx$ $P_7 : \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; f \text{ is even} \\ 0 & ; f \text{ is odd} \end{cases}$

$f(-x) = [\sin(-x)]^2 = \sin^2 x = f(x)$

$\Rightarrow f$ is even

$I = 2\int_0^{\frac{\pi}{2}} \sin^2 x \cdot dx = 2\int_0^{\frac{\pi}{2}} \sin^2\left(\frac{\pi}{2} - x\right) dx \rightarrow ①$

$= 2\int_0^{\frac{\pi}{2}} \cos^2 x \cdot dx \rightarrow ②$

① + ②

$2I = 2\int_0^{\frac{\pi}{2}} 1 \cdot dx = x\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

$I = \frac{\pi}{2}$

**13] $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \cdot dx$

$f(-x) = [\sin(-x)]^7 = -\sin^7 x = -f(x)$

$\Rightarrow f$ is an odd function.

$\Rightarrow I = 0$

***20] $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\underset{\text{odd}}{x^3} + \underset{\text{odd}}{x\cos x} + \underset{\text{odd}}{\tan^5 x} + 1)\,dx$

$I = 0 + 0 + 0 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot dx = x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Eg 33] $I = \int_{-1}^{1} (\underbrace{\sin^5 x \cdot \cos^4 x}_{\text{odd}})\,dx \quad \because f(-x) = -f(x)$

$I = 0$

14] $I = \int_0^{2\pi} \cos^5 x \cdot dx$ $\quad P_6 : \int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x) & ; f(2a-x) = f(x) \\ 0 & ; f(2a-x) = -f(x) \end{cases}$

$f(2\pi - x) = [\cos(2\pi - x)]^5 = \cos^5 x = f(x)$

$\Rightarrow I = 2\int_0^{\pi} \cos^5 x \cdot dx$

$f(\pi - x) = [\cos(\pi - x)]^5 = -\cos^5 x = -f(x)$ odd funⁿ

$I = 0$

Eg 32] $I = \int_0^{\pi} \frac{x \sin x \cdot dx}{1+\cos^2 x} = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)\, dx}{1+\cos^2(\pi - x)}$

$= \int_0^{\pi} \frac{(\pi - x)\sin x \cdot dx}{1+\cos^2 x} = \int_0^{\pi} \frac{\pi \sin x \cdot dx}{1+\cos^2 x} - \underbrace{\int_0^{\pi} \frac{x \cdot \sin x \cdot dx}{1+\cos^2 x}}_{I}$

$2I = \pi \int_0^{\pi} \frac{\sin x \cdot dx}{1+\cos^2 x}$ $\quad \cos x = t$, $-\sin x \cdot dx = dt$ $\quad \Big| \quad x = 0, t = 1$; $x = \pi, t = -1$

$2I = \pi \int_1^{-1} \frac{-dt}{1+t^2} = -\pi \left[\tan^{-1} t\right]_1^{-1}$

$2I = -\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = \pi \cdot \frac{\pi}{2}$

$I = \frac{\pi^2}{4}$

Eg 31] $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x \cdot dx$

$f(-x) = [\sin(-x)]^2 = \sin^2 x = f(x)$

$\Rightarrow f$ is even

$I = 2\int_0^{\frac{\pi}{4}} \sin^2 x \cdot dx = \cancel{2} \int_0^{\frac{\pi}{4}} \left(\frac{1 - \cos 2x}{\cancel{2}}\right) dx$

$= x - \frac{\sin 2x}{2}\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{2}(1 - 0)$

$I = \frac{\pi}{4} - \frac{1}{2}$

12] $I = \int_0^{\pi} \frac{x\,dx}{1+\sin x} = \int_0^{\pi} \frac{(\pi - x)\,dx}{1+\sin(\pi - x)} = \int_0^{\pi} \frac{(\pi - x)\,dx}{1+\sin x}$

$= \int_0^{\pi} \frac{\pi\,dx}{1+\sin x} - \int_0^{\pi} \frac{x\,dx}{1+\sin x}$

$2I = \pi \int_0^{\pi} \frac{1}{1+\sin x}\,dx$

$2I = \pi \int_0^{\pi} \left(\frac{1}{1+\sin x} \cdot \frac{1-\sin x}{1-\sin x}\right) dx = \pi \int_0^{\pi} \frac{(1-\sin x)\,dx}{\cos^2 x}$

$= \pi \int_0^{\pi} \sec^2 x - \sec x \cdot \tan x \cdot dx$

$= \pi \left[\tan x - \sec x\right]_0^{\pi} = \pi \left[(0-0) - (-1-1)\right]$

$2I = 2\pi$

$I = \pi$

** CET

Eg 36] $\boxed{I = \int_0^{\frac{\pi}{2}} \log(\sin x) = \int_0^{\frac{\pi}{2}} \log(\cos x) = -\frac{\pi}{2} \log 2 = \frac{\pi}{2} \log\left(\frac{1}{2}\right)}$

$I = \int_0^{\frac{\pi}{2}} \log(\sin x)\,dx \rightarrow ①$

$= \int_0^{\frac{\pi}{2}} \log\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx = \int_0^{\frac{\pi}{2}} \log(\cos x)\,dx \rightarrow ②$

① + ②

$2I = \int_0^{\frac{\pi}{2}} \log(\sin x) + \log(\cos x)\,dx = \int_0^{\frac{\pi}{2}} \log(\sin x \cdot \cos x)\,dx$

$2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right) dx = \underbrace{\int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx}_{I_1} - \int_0^{\frac{\pi}{2}} \log 2 \cdot dx \rightarrow ③$

$I_1 = \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx$

$2x = t$

$2 \cdot dx = dt$

$dx = \frac{1}{2} dt$

$x = 0,\ t = 0$

$x = \frac{\pi}{2},\ t = \pi$

$I_1 = \int_0^{\pi} \log(\sin t)\, dt \cdot \frac{1}{2}$

$f(\pi - t) = \log[\sin(\pi - t)] = \log(\sin t) = f(t)$

$\Rightarrow I_1 = 2\int_0^{\frac{\pi}{2}} \log(\sin t)\, dt \cdot \frac{1}{2}$

$I_1 = \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx = I \quad [\text{using } P_0]$

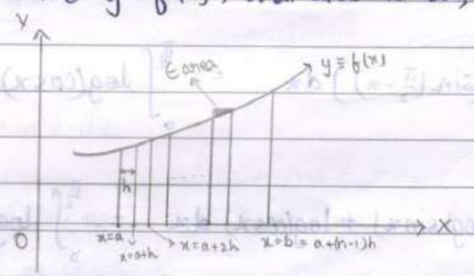
③ $\Rightarrow 2I = I - \log 2 \int_0^{\frac{\pi}{2}} 1 \cdot dx$

$I = -\log 2\, [x]_0^{\frac{\pi}{2}} = -\log 2 \cdot \left(\frac{\pi}{2}\right)$

$\therefore \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}$

Definite integrals as the limit of sum:

The definite integral $\int_a^b f(x)\, dx$ is the area bounded by the curve $y = f(x)$, ordinates $x = a$, $x = b$ and x-axis.



$$\int_a^b f(x)\, dx = \lim_{h \to 0} h\left\{f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)\right\}$$

where $h = \frac{b-a}{n} \Rightarrow \boxed{nh = b-a}$

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EXERCISE 7.8

Find the integral value as limit of sum

1] $\int_a^b x \cdot dx$; $f(x) = x$, $nh = b - a$

$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left\{f(a) + f(a+h) + f(a+2h) \cdots + f(a+(n-1)h)\right\}$

$\int_a^b x \cdot dx = \lim_{h\to 0} h\left\{a + a+h + a+2h + \cdots + a+(n-1)h\right\}$

$= \lim_{h\to 0} h\left\{[a+a+a \cdots + n\text{ times}] + h(1+2+3 \cdots + (n-1))\right\}$

AP : $a=1, d=1, n=n-1$

$S_n = \frac{n}{2}[2a+(n-1)d]$

$= \lim_{h\to 0} h\left\{na + h\frac{(n-1)}{2}(\cancel{2} + n - \cancel{1} - 1)\right\}$

$= \lim_{h\to 0}\left\{a(nh) + \left(\frac{nh-h}{2}\right)nh\right\}$

$= \lim_{h\to 0}\left\{a(b-a) + \left(\frac{b-a-h}{2}\right)(b-a)\right\}$

$= \cancel{\lim_{h\to 0}}\ \cancel{\{}ab - a^2 + \frac{(b-a)}{2}(b-a)$

$= \frac{2ab - 2a^2 + b^2 + a^2 - 2ab}{2} = \frac{b^2-a^2}{2}$

Verification

$\left.\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2}$

2] $I = \int_0^5 (x+1)\,dx$; $f(x) = x+1$, $nh = 5-0 = 5$ $\left.\frac{x^2}{2} + x\right]_0^5 = \frac{25}{2} + 5 = \frac{35}{2}$

$\int_0^5 (x+1)\,dx = \lim_{h\to 0} h\left\{f(0) + f(h) + f(2h) + \cdots + f[(n-1)h]\right\}$

$= \lim_{h\to 0} h\left\{1 + h+1 + 2h+1 + \cdots + (n-1)h+1\right\}$

$= \lim_{h\to 0} h\left\{(1+1+1+\cdots n\text{ times}) + h(1+2+3 \cdots + (n-1))\right\}$

AP : $a=1, d=1, n=n-1$

$S_n = \frac{n}{2}[2a+(n-1)d]$

$$= \lim_{h\to 0} h\left\{n + h\frac{(n-1)}{2}\left(\cancel{2} + (n-\cancel{1}) - \cancel{1}\right)\right\}$$

$$= \lim_{h\to 0} h\left\{n + \frac{h(n-1)\,n}{2}\right\} = \lim_{h\to 0}\left\{nh + nh\frac{(nh-h)}{2}\right\}$$

$$= \lim_{h\to 0}\left\{5 + 5\frac{(5-h)}{2}\right\} = 5 + \frac{25}{2} = \frac{35}{2}$$

** Eg26] $\int_0^2 e^x\,dx \qquad f(x) = e^x,\ nh = 2-0 = 2$

$$\int_0^2 e^x\,dx = \lim_{h\to 0} h\left\{f(0) + f(h) + f(2h) + \cdots + f[(n-1)h]\right\}$$

$$= \lim_{h\to 0} h\left\{1 + \underbrace{e^h + e^{2h} + \cdots + e^{(n-1)h}}_{GP:\ a=e^h,\ r=e^h,\ S_n = \frac{a(r^n-1)}{r-1}}\right\}$$

$$= \lim_{h\to 0} h\left\{1 + e^h\left(\frac{e^{(n-1)h}-1}{e^h-1}\right)\right\}$$

$$= \lim_{h\to 0}\left\{h + he^h\left[\frac{e^{nh-h}-1}{(e^h-1)}\right]\right\}$$

$$= \lim_{h\to 0}\left\{h + \frac{e^h(e^{2-h}-1)}{\left(\frac{e^h-1}{h}\right)}\right\} \qquad \lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = 1$$

$$= 0 + \frac{e^0(e^2-1)}{1} = \boxed{e^2-1}$$

5] $\int_{-1}^{1} e^x\,dx \ ;\ f(x) = e^x,\ nh = 1+1 = 2$

$$\int_{-1}^{1} e^x\,dx = \lim_{h\to 0} h\left\{f(-1) + f(-1+h) + f(-1+2h) + \cdots + f[-1+(n-1)h]\right\}$$

$$= \lim_{h\to 0} h\left\{e^{-1} + e^{-1+h} + e^{-1+2h} + \cdots + e^{-1+(n-1)h}\right\}$$

$$= \lim_{h\to 0} h\left\{e^{-1} + e^{-1}\cdot e^h + e^{-1}\cdot e^{2h} + \cdots + e^{-1}\cdot e^{(n-1)h}\right\}$$

$= \lim_{h \to 0} h \left\{ (e^{-1} + e^{-1} + \cdots + n \text{ times}) \right\}$

$= \lim_{h \to 0} h \left\{ e^{-1} \left[1 + e^{h} + e^{2h} + \cdots + e^{(n-1)h} \right] \right\}$

GP : $a = 1$, $r = e^{h}$, $n = n$. $S_n = a\left(\frac{r^n - 1}{r - 1}\right)$

$= \lim_{h \to 0} h \left\{ e^{-1} \cdot \frac{e^{nh} - 1}{e^{h} - 1} \right\} = \lim_{h \to 0} e^{-1} \left[\frac{e^{2} - 1}{\frac{e^{h} - 1}{h}} \right]$

$= \frac{e^{-1}(e^{2} - 1)}{1} = \frac{e^{2} - 1}{e}$

3] $\int_2^3 x^2 \, dx$; $f(x) = x^2$, $nh = 3 - 2 = 1$. $\left.\frac{x^3}{3}\right]_2^3 = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$

$\int_2^3 x^2 \, dx = \lim_{h \to 0} h \left\{ f(2) + f(2+h) + f(2+2h) + \cdots + f(2 + (n-1)h) \right\}$

$= \lim_{h \to 0} h \left\{ 4 + (2+h)^2 + (2+2h)^2 + \cdots + [2 + (n-1)h]^2 \right\}$

$= \lim_{h \to 0} h \left\{ 4 + 4 + 4h + h^2 + 4 + 8h + 4h^2 + \cdots + 4 + 4(n-1)h + (n-1)^2 h^2 \right\}$

$= \lim_{h \to 0} h \left\{ (4 + 4 + \cdots + n \text{ times}) + 4h[1 + 2 + 3 + \cdots (n-1)] + h^2 (1^2 + 2^2 + \cdots + (n-1)^2) \right\}$

AP: $\Sigma n = \frac{n(n+1)}{2}$, $n = n-1$ | $\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$

$= \lim_{h \to 0} h \left\{ 4n + 4h \frac{(n-1)n}{2} + h^2 \frac{(n-1)n[2(n-1)+1]}{6} \right\}$

$= \lim_{h \to 0} \left\{ 4nh + 2nh(nh - h) + \frac{nh(nh-h)(2nh-h)}{6} \right\}$

$= \lim_{h \to 0} \left\{ 4 + 2(1-h) + \frac{1(1-h)(2-h)}{6} \right\}$

$= 4 + 2(1) + \frac{1(1)(2)}{6}$

$= 6 + \frac{1}{3} = \frac{19}{3}$