TRIGONOMETRY

Trigonometry is made of three words “tri”, “gono”, “metry”. Where “tri” means “three”, “gono” means “side” and “metry” means measurement. So, trigonometry is study of measuring three side figure which is triangle.

Usually we use right angle triangle to solve problem based on $\theta \; \frac{Base}{Hypotenuse}$ trigonometry. Problem in trigonometry are usually based on trigonometric ratio.

Trigonometric Ratio

Trigonometric ratio are the ratio between two sides of a triangle. At particular angle the ratio between two sides will remain same irrespective to their length.

There are six Trigonometric Ratios which are as:

Sine: It is a ratio between a perpendicular and hypotenuse. It is represented as “sin” in all trigonometric identities.

$$\sin \theta \; \frac{Perpendicular}{Hypotenuse} =$$

Where θ represents the angle for which the ratio is derived.

Cosine: It is a ratio between a base and hypotenuse. It is represented as “cos” in all trigonometric identities.

$$\cos \theta = \frac{Base}{Hypotenuse}$$

Secant: It is a ratio between a hypotenuse and base. It is represented as “sec” in all trigonometric identities.

$$\sec \theta = \frac{Hypotenuse}{Base}$$

Cosecant: It is a ratio between a hypotenuse and perpendicular. It is represented as “cosec” in all trigonometric identities.

$$\text{cosec} \; \theta = \frac{Hypotenuse}{perpendicular}$$

Tangent: It is a ratio between a perpendicular and base. It is represented as “tan” in all trigonometric identities.

$$\tan \theta \; \frac{perpendicular}{Base} =$$

Cotangent: It is a ratio between a base and perpendicular. It is represented as “cot” in all trigonometric identities.

cot θ $\frac{Base}{Perpendicular}$ =

Angle: When two rays (initial and terminal) meet at a point after rotation in a plane then they are said to have described an angle. In other words we can say, the circular distance between two inclined lines is called angle.

Unit of Angle:

- Degree (°)
- Radian (^{c})

Relationship between degree and radian:

π rad = 180°

For below particular angles the value of trigonometric ratios are constant.

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$∞	N.D/
cot	N.D/ ∞	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2 ∞	N.D/
cosec	N.D/ ∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Signs of Trigonometric Ratio in quadrants:

1st quadrant: All positive

2nd quadrant: sin and cosec positive

3rd quadrant: tan and cot positive

4th quadrant: cos and sec positive

Relation between Trigonometric Ratios:

sin θ × θ cosec = 1

cos θ × θ sec = 1

tan θ × co θ = 1

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sec\theta}{\csc\theta}$

$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\csc\theta}{\sec\theta}$

Trigonometric Ratios of Allied Angles:

With θ

sin (- θ) = - θ sin

cos (- θ) = θ cos

tan (- θ) = - tan

cot (- θ) = - cot θ

sec (- θ) = sec θ

cosec (- θ) = - cosec θ

With (90° – θ)

sin (90° – θ) = cos θ

cos (90° – θ) = sin θ

tan (90° – θ) = θcot

cot (90° – θ) = tan θ

sec (90° – θ) = cosec θ

cosec (90° – θ) = sec θ

With (90° + θ)

sin (90° + θ) = cos θ

cos (90° + θ = – sin θ

tan (90° + θ) = – cot θ

cot (90° + θ) = – tan θ

sec (90° + θ) = – cosec θ

cosec (90° + θ) = – sec θ

With (180° – θ)

$\sin (180^\circ - \theta) = \sin \theta$

$\cos (180^\circ - \theta) = -\cos \theta$

$\tan (180^\circ - \theta) = -\tan \theta$

$\cot (180^\circ - \theta) = -\cot \theta$

$\sec (180^\circ - \theta) = -\sec \theta$

$\operatorname{cosec} (180^\circ - \theta = \operatorname{cosec} \theta$

With $(180^\circ + \theta)$

$\sin (180^\circ + \theta = -\sin \theta$

$\cos (180^\circ + \theta) = -\cos \theta$

$\tan (180^\circ + \theta) = \tan \theta$

$\cot (180^\circ + \theta) = \cot \theta$

$\sec (180^\circ + \theta) = -\sec \theta$

$\operatorname{cosec} (180^\circ + \theta = -\operatorname{cosec} \theta$

With $(270^\circ - \theta$

$\sin (270^\circ - \theta = -\cos \theta$

$\cos (270^\circ - \theta) = -\sin \theta$

$\tan (270^\circ - \theta = \cot \theta$

$\cot (270^\circ - \theta) = \tan \theta$

$\sec (270^\circ - \theta) = -\operatorname{cosec} \theta$

$\operatorname{cosec} (270^\circ - \theta) = -\sec \theta$

With $(270^\circ + \theta)$

$\sin (270^\circ + \theta = -\cos \theta$

$\cos (270^\circ + \theta = \sin \theta$

$\tan (270^\circ + \theta = -\cot \theta$

$\cot (270^\circ + \theta = -\tan \theta$

$\sec (270^\circ + \theta) = \operatorname{cosec} \theta$

$\operatorname{cosec} (270^\circ + \theta = -\sec \theta$

With (360° – θ)

sin (360° - θ) = – sin θ

cos (360° – θ) = cos θ

tan (360° – θ) = – tan θ

cot (360° – θ) = – cot θ

sec (360° – θ) = sec θ

cosec (360° – θ= – cosec θ

With (360° + θ)

sin (360° + θ) = sin θ

cos (360° + θ) = cos θ

tan (360° + θ) = tan θ

cot (360° + θ = cot θ

sec (360° + θ) = sec θ

cosec (360° + θ) = cosec θ

Some Useful Identities

1) $\sin^2\theta + \cos^2\theta = 1$
   It can also be expressed as
   $\sin^2\theta = 1 - \cos^2\theta$
   $\cos^2\theta = 1 - \sin^2\theta$

2) $\sec^2\theta - \tan^2\theta = 1$
   It can also be expressed as
   $\sec^2\theta = 1 + \tan^2\theta$
   $\sec^2\theta - 1 = \tan^2\theta$

3) $\text{cosec}^2\theta - \cot^2\theta = 1$
   It can also be expressed as
   $\text{cosec}^2\theta = 1 + \cot^2\theta$
   $\text{cosec}^2\theta - 1 = \cot^2\theta$

4) sin (A + B) = sin A cos B + cos A sin B

5) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

6) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

7) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

8) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

9) $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

10) $2 \sin A \sin B = \cos(A - B) - \sin(A + B)$

11) $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

12) $\sin^2 A - \sin^2 B = \sin(A + B) \sin(A - B)$

13) $\cos^2 A - \cos^2 B = \cos(A+B) \cos(A - B)$

14) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A . \tan B}$

15) $\tan(A - B) = \frac{\tan A + \tan B}{1 - \tan A . \tan B}$

16) $\cot(A + B) = \frac{\cot A . \cot B - 1}{\cot A + \cot B}$

17) $\cot(A - B) = \frac{\cot A . \cot B + 1}{\cot B - \cot A}$

18) $\sin 2A = 2 \sin A \cos A = \frac{2\tan A}{1+\tan^2 A}$

19) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - \sin^2 A \quad \frac{1-\tan^2 A}{1+\tan^2 A} \quad =$

20) $\tan 2A = \frac{2 \tan A}{1-\tan^2 A}$

21) $\sin 3A = 3 \sin A - 4 \sin^3 A$

22) $\cos 3A = 4 \cos^3 A - 3 \cos A$

23) $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}$

24) $\sin C + \sin D = 2 \sin \left(\frac{C+D}{2}\right) \quad \left(\frac{C-D}{2}\right)$. cos

25) $\sin C - \sin D = 2 \cos \left(\frac{C+D}{2}\right) \quad \left(\frac{C-D}{2}\right)$. sin

26) $\cos C + \cos D = 2 \cos \left(\frac{C+D}{2}\right) \quad \left(\frac{C-D}{2}\right)$. cos

27) $\cos C - \cos D = 2 \sin \left(\frac{C+D}{2}\right) \sin \left(\frac{D-C}{2}\right)$

28) If $4\theta < 60$

i. $\sin \theta . \sin 2\theta . \sin 4\theta = \frac{1}{4} \sin 3\theta$

ii. $\cos \theta . \cos 2\theta . \cos 4\theta = \frac{1}{4} \cos 3\theta$

iii. $\tan \theta . \tan 2\theta . \tan 4\theta = \tan 3\theta$

iv. $\cot \theta . \cot 2\theta . \cot 4\theta = \cot 3\theta$

29) For all value of θ

i. $\sin (60 - \theta) \sin \theta . \sin (60 + \theta) = \frac{1}{4} \sin 3\theta$

ii. $\cos (60 - \theta) \cos \theta . \cos (60 + \theta = \frac{1}{4} \cos 3\theta$

iii. $\tan (60 - \theta) \tan \theta . \tan (60 + \theta) = \tan 3\theta$

iv. $\cot (60 - \theta) \cot . \cot (60 + \theta) = \cot 3\theta$

30) If $A + B = 45^{\circ}$

i. $(1 + \tan A)(1 + \tan B) = 2$

ii. $(1 - \cot A)(1 - \cot B) = 2$

31) If $A + B = 90^{\circ}$

i. $\sin A = \cos B$

ii. $\operatorname{cosec} A = \sec B$

iii. $\tan A = \cot B$

32) If $A + B + C = 90^{\circ}$

i. $\tan A . \tan B + \tan B . \tan C + \tan C . \tan A = 1$

ii. $\cot A + \cot B + \cot C = \cot A . \cot B . \cot C$

33) If $A + B + C = 180^{\circ}$

i. $\tan A + \tan B + \tan C = \tan A . \tan B . \tan C$

ii. $\cot A . \cot B + \cot B . \cot C + \cot C . \cot A = 1$

iii. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A . \sin B . \sin C$

34) $\tan (45 + \theta) = \frac{1 + \tan \theta}{1 - \tan \theta} \quad \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \quad =$

35) tan (45 – θ) $\frac{1-\tan\theta}{1+\tan\theta}$ $\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$ =

Solved Examples:

1. If 12tan θ5, then find the trigonometric ratio.

Solution:

tan θ $\frac{5}{12}=\frac{perpendicular}{base}$

It means perpendicular is 5 and base will be 12. By using Pythagoras Theorem, we can easily find hypotenuse.

Hypotenuse^2 = perpendicular^2 + base^2

Hypotenuse = $\sqrt{5^2+12^2}$= 13

sin θ $\frac{Perpendicular}{Hypotenuse}$ = $\frac{5}{13}$

cos θ $\frac{Base}{Hypotenuse}$ $\frac{12}{13}$ =

sec θ $\frac{Hypotenuse}{Base}$ $\frac{13}{12}$ =

cosec θ $\frac{Hypotenuse}{perpendicular}$ $\frac{13}{5}$ =

cot θ $\frac{Base}{Perpendicular}$ $\frac{12}{5}$ =

$\frac{1+\sin\theta}{1-\sin\theta}$ $\frac{1+\frac{15}{17}}{1-\frac{15}{17}}$

2. If tan = θ, then find the value of $\frac{a}{b}$ $\frac{a\sin\theta+b\cos\theta}{a\sin\theta-b\cos\theta}$

Solution:

$\frac{a\sin\theta+b\cos\theta}{a\sin\theta-b\cos\theta}$

Divide both numerator and denominator by

$\frac{a\frac{\sin\theta}{\cos\theta}+b}{a\frac{\sin\theta}{\cos\theta}-b}$ θ

θ $\frac{\sin\theta}{\cos\theta}$ $\frac{a}{b}$ tan = =

$\frac{a\times\frac{a}{b}+b}{a\times\frac{a}{b}-b}=\frac{a^2+b^2}{a^2-b^2}$

OR

As both numerator and denominator have sin and cos, which have hypotenuse their denominator thus we can use a as θn and b as cc θ

Now,

$\frac{a\sin\theta+b\cos\theta}{a\sin\theta-b\cos\theta}=\frac{a\times a+b\times b}{a\times a-b\times b}=\frac{a^2+b^2}{a^2-b^2}$

$\theta \quad \frac{15}{8}$

3. sin 720° – cot 270° – sin 150° cos 120° is equal to –

Solution:

sin (2×360° + 0°) – cot (360° – 90°) – sin (180° – 30°). cos (90° + 30°)

sin 0° – cot 90° – sin 30°. sin 30°

$0 - 0 + \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

4. If tan (x + y) . tan (x – y) = 1, then find the value of tan x

Solution:

tan (x + y) = $\frac{1}{\tan(x-y)}$ = cot (x – y)

x + y + x – y = 90°

2x = 90°

x = 45°

tan 45° = 1

5. If cot 2A cot 3A = 1 , then find the value of sin $\frac{5A}{2}$. cos $\frac{5A}{2}$

Solution:

2A + 3A = 90°

A = 18°

Sin $\frac{5\times18}{2}$ cos $\frac{5\times18}{2}$

sin 45° × cos 45° = $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2}$

6. If $\frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta} = \frac{17}{7}$ then find the value of tan θ.

Solution:

Apply componendo and dividendo –

$\frac{\sin\theta + \cos\theta + \sin\theta - \cos\theta}{\sin\theta + \cos\theta - \sin\theta + \cos\theta} = \frac{17+7}{17-7}$

$\frac{2\sin\theta}{2\cos\theta}$ = tan θ = $\frac{24}{10} = \frac{12}{5}$

7. If $\tan\theta + \cot\theta = 2$, then find the value of $\tan^{100}\theta + \cot^{110}\theta$

Solution:

$\tan^{100}\theta + \cot^{110}\theta = 1 + 1 = 2$

8. If $\sin^2\theta + \sin\theta = 1$, then find the value of $\cos^4\theta + \cos^2\theta$.

Solution:

$\sin\theta = 1 - \sin^2\theta = \cos^2\theta$

$\cos^4\theta = \sin^2\theta$

$\cos^4\theta + \cos^2\theta = \sin^2\theta + \sin\theta = 1$

9. Solve: $\tan 1^\circ . \tan 2^\circ . \tan 3^\circ \ldots \tan 89^\circ$

Solution:

$\tan 1^\circ . \tan 2^\circ . \tan 3^\circ \ldots \tan 45^\circ \ldots \cot 3^\circ . \cot 2^\circ . \cot 1^\circ$

$\tan 1^\circ \cot 1^\circ \tan 2^\circ \cot 2^\circ \tan 3^\circ \cot 3^\circ \ldots \tan 45^\circ\ 1 \times 1 \times 1 \ldots . \times 1 = 1$

10. Solve $\cos 1^\circ . \cos 2^\circ . \cos 3^\circ \ldots\ldots \cos 179^\circ$.

Solution:

$\cos 1^\circ . \cos 2^\circ . \cos 3^\circ \ldots \cos 90^\circ \ldots \cos 179^\circ$

$\cos 90^\circ = 0$

$\cos 1^\circ . \cos 2^\circ . \cos 3^\circ \ldots 0 \ldots \cos 179^\circ = 0$