Notation

$\mathbb{Z}$ = the set of integers
$\mathbb{N} = \{n \in \mathbb{Z} : n \geq 1\}$
$\mathbb{R}$ = the set of real numbers
$\mathbb{Q}$ = the set of rational numbers
$\mathbb{C}$ = the set of complex numbers

(1) Let X be a compact topological space. Suppose that for any $x, y \in X$ with $x \neq y$, there exist open sets U_x and U_y containing x and y, respectively, such that

$$U_x \cup U_y = X \qquad \text{and} \qquad U_x \cap U_y = \emptyset.$$

Let $V \subseteq X$ be an open set. Let $x \in V$. Show that there exists a set U which is both open and closed and $x \in U \subseteq V$.

(2) Let $C[0,1]$ denote the set of all real-valued continuous functions on $[0,1]$. Consider the normed linear space

$$X = \{f \in C[0,1] : f(\frac{1}{2}) = 0\},$$

with the sup-norm, $\|f\| = \sup\{|f(t)| : t \in [0,1]\}$. Show that the set

$$P = \{f \in X : f \text{ is a polynomial }\}$$

is dense in X.

(3) Let $g : [0, \frac{1}{2}] \to \mathbb{R}$ be a continuous function. Define $g_n : [0, \frac{1}{2}] \to \mathbb{R}$ by $g_1 = g$ and

$$g_{n+1}(t) = \int_0^t g_n(s)\, ds,$$

for all $n \geq 1$. Show that

$$\lim_{n\to\infty} n! g_n(t) = 0,$$

for all $t \in [0, \frac{1}{2}]$.

(4) Let $\sum_{n\geq 1} a_n$ be an absolutely convergent series of complex numbers. Let

$$b_n = \begin{cases} a_n & \text{if } 1 \leq n < 100 \\ \frac{n+1}{n^2} a_n^2 & \text{if } n \geq 100. \end{cases}$$

Prove that $\sum_{n\geq 1} b_n$ is an absolutely convergent series.

(5) Let $f : [0,1] \times [0,1] \to [0, \infty)$ be a continuous function. Suppose that

$$\int_0^1 \Big(\int_0^1 f(x,y)\ dy \Big)\ dx = 0.$$

Prove that f is the identically zero function.

(6) Let m denote the Lebesgue measure on $[0,1]$. Give an example of a sequence of continuous functions $\{f_n\}_{n\geq 1} \subseteq L^1[0,1]$ such that

$$\sup_{t\in[0,1]} |f_n(t)| = 1,$$

for all n and

$$\int_0^1 |f_n| \, dm \to 0,$$

as $n \to \infty$.

(7) Let Γ denote the positively oriented circle of radius 2 with center at the origin. Let f be an analytic function on $\{z \in \mathbb{C} : |z| > 1\}$, and let

$$\lim_{z\to\infty} f(z) = 0.$$

Prove that

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{z-\zeta} \, d\zeta,$$

for all $z \in \mathbb{C}$ with $|z| > 2$.

(8) Prove that there is no sequence of complex polynomials that converges to $\frac{1}{z^2}$ uniformly on the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

(9) Consider the differential equation

$$\dot{x} = x(1-x) - \frac{1}{4},$$

where $\dot{x} = \frac{dx}{dt}$. For any solution $x(t)$, find the limit of $x(t)$ as $t \to \infty$.

(10) Consider the system

$$\dot{X} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} X$$

where $X(t) = \binom{x(t)}{y(t)}$, $\dot{X} := \frac{dX}{dt} = \binom{\dot{x}(t)}{\dot{y}(t)}$ and λ is a fixed real number. Show that if $\lambda < 0$ then $X(t) \to \binom{0}{0}$ and $X(t)$ is asymptotic to the line $y = 0$ in the xy-plane, as $t \to \infty$.