CSB
2019

GROUP A

Answer any three out of four questions.

A1. Consider a snail at the foot of a pole that is `h` metres high. The snail starts climbing up the pole at an initial speed of `c` metres / hour. After every `d` hours of climbing, the snail takes a nap for `z` hours. While it is asleep, it slips down the pole at `s` metres / hour. After waking up from the nap, it continues climbing, but its speed reduces by `p`% of the speed that it had just before it took this nap.

In your answer booklet, write the expressions / conditions corresponding to the blanks in the pseudo-code below, so that the code determines the total number of hours it takes the snail to reach the top of the pole. If the snail can never reach the top of the pole, the code should print an appropriate message.

Step 1. Set `remainingHeight` to `h`, `speed` to `c`, and `numHours` to `0`.

Step 2. If ____________________, print message:
"Snail reaches the top in ____________________ hours"
and STOP.

Step 3. Otherwise, increment `numHours` by ________________;
decrement `remainingHeight` by ____________________.

Step 4. If ____________________, print message:
"Snail will never be able to reach the top"
and STOP.

Step 5. Otherwise, set `speed` to ____________________,
and repeat from Step 2.

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[10]

A2. Suppose there are five pairs of shoes in a closet, from which four shoes are taken out at random. What is the probability that among these four shoes, there is at least one complete pair?

[10]

A3. Let n be a positive integer. Consider the set

$$S = \{-n, -n+1, -n+2, \ldots, -1, 0, 1, 2, \ldots, n-1, n\}.$$

Let $f : S \times S \to \{-1, 1\}$ be a function. If $f(-a, -b) = -f(a, b) \;\; \forall a, b \in S$, f is said to be **odd**. If $f(-a, -b) = f(a, b) \;\; \forall a, b \in S$, f is said to be **even**.

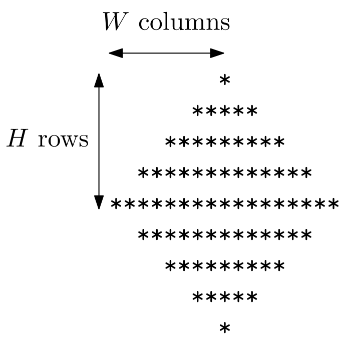
(a) What is the total number of different functions from $S \times S$ to $\{-1, 1\}$?

(b) How many of these are odd?

(c) How many of these are neither odd nor even?

[2+5+3=10]

A4. Write pseudo-code to print a diamond formed using the character * as shown in the figure below. Let H and W denote the number of rows, and the number of columns occupied by the upper half and the left half of the diamond, respectively. In the example shown below, $H = 5$ and $W = 9$. Your code should take H and W as input parameters. You may assume that $W - 1$ is an integral multiple of $H - 1$.



[10]

GROUP B

Select any one from the following five sections, and answer questions from that section.

I. COMPUTER SCIENCE

Answer any five out of eight questions.

C1. Consider the task of multiplying two integer matrices A and B, each of size 500×500. Each matrix can have at most 500 non-zero entries. Any row of A or B that contains at least one non-zero element, must have no less than 50 non-zero elements.

(a) Propose a space-efficient data structure for this task.

(b) Based on your proposed data structure, design an efficient algorithm to multiply A and B. The resulting matrix should be stored separately.

(c) What is the maximum number of integer multiplications needed to complete this task as per your implementation?

[4+6+4=14]

C2. Let $S = \{1, 2, \ldots, n\}$ and $f : S \to S$. If $R \subseteq S$, define $f(R) = \{f(x) \mid x \in R\}$.

(a) Suppose you are given the values of $f(1), f(2), \ldots, f(n)$, in order. Propose a data structure for storing these values so that, for a given x, the values of $f(x)$ and $f^{-1}(x) = \{y \mid f(y) = x\}$ can be determined efficiently.

(b) Suppose that we call a set $R \subseteq S$ "*special*" if $f(R) = R$. For $n = 3$, give an example of a function f and subsets $R_i \subseteq S$ such that R_i is special, and $|R_i| = i$ for $i = 1, 2, 3$.

(c) Using your data structure from (a), design an efficient algorithm to find a set $R \subseteq S$ that is *special* (as defined in (b)). You will get full credit only if your algorithm takes $O(n)$ time in the worst case.

[4+2+8=14]

C3. Consider a set A containing n positive integers $a_0, a_1, \ldots, a_{n-1}$ such that $a_i > \sum_{j=0}^{i-1} \big((j+1)a_j\big)$ for $i = 1, 2, \ldots, n-1$. You are also given a target positive integer T.

(a) Write an efficient algorithm to determine whether there is any subset of A, such that the sum of the elements in this subset is T.

(b) Analyse the worst case time complexity of your algorithm.

[8+6=14]

C4. Consider the sets X and Y given below:

$$\begin{aligned} X &= \{x_1^{(a)}, x_1^{(b)}, x_2^{(a)}, x_2^{(b)}, \ldots, x_k^{(a)}, x_k^{(b)}\} \\ Y &= \{y_1^{(a)}, y_1^{(b)}, y_1^{(c)}, y_2^{(a)}, y_2^{(b)}, y_2^{(c)}, \ldots, y_m^{(a)}, y_m^{(b)}, y_m^{(c)}\}. \end{aligned}$$

Construct an undirected graph $G = (X \cup Y, E)$ by including the following edges in E:

- for each $i = 1, 2, \ldots, k$, join $x_i^{(a)}$ and $x_i^{(b)}$ by an edge;
- for each $i = 1, 2, \ldots, m$, join $y_i^{(a)}$ and $y_i^{(b)}$ by an edge; also join $y_i^{(b)}$ and $y_i^{(c)}$ by an edge, and $y_i^{(c)}$ and $y_i^{(a)}$ by an edge;
- for each $y \in Y$, join y to exactly one element x of X.

(a) Show that a vertex cover of G cannot have less than $k + 2m$ elements.

[Recall that, for a graph $G = (V, E)$, a subset V' of V is said to be a *vertex cover*, if for every $e = (v_1, v_2) \in E$, either $v_1 \in V'$ or $v_2 \in V'$.]

(b) Suppose G' is obtained from G by deleting all edges of the form (x, y) where $x \in X$ and $y \in Y$. Compute the number of distinct vertex covers for G'.

[8+6=14]

C5. (a) Consider the following relations:

Student(*snum*: integer, *sname*: string, *major*: string, *level*: string, *age*: integer)
Class(*name*: string, *meets at*: string, *room*: string, *fid*: integer)
Enrolled(*snum*: integer, *cname*: string)
Faculty(*fid*: integer, *fname*: string, *deptid*: integer).

Write an SQL query to find the names of faculty members who teach in every room in which some class is taught.

(b) Let R be a relation with functional dependencies $\mathcal{F}$. For any subset of attributes $X \subset R$, the closure of X is defined as the set

$$X^+ = \{A \in R | X \in A \text{ holds with respect to } \mathcal{F}\}.$$

For two non-empty attribute sets Y and Z in R, prove or disprove each of the following statements:

(i) $(Y^+Z)^+ = (YZ)^+$;

(ii) $(YZ)^+ = Y^+Z^+$.

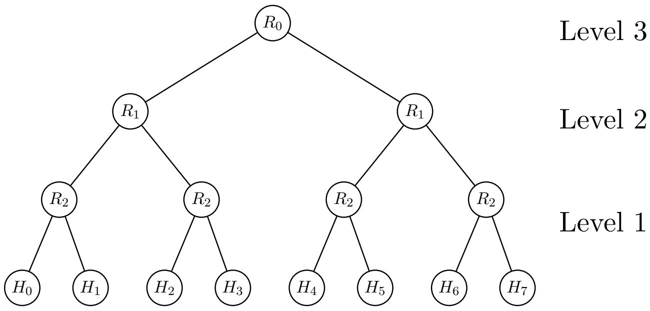
[6+(4+4)=14]

C6. (a) Let α be the percentage of a program code which can be executed simultaneously by n processors in a computer system. Assume that the remaining code must be executed sequentially by a single processor. Each processor has an execution rate of x MIPS, and all the processors are of equal capability.

(i) Derive an expression for the effective MIPS rate when the code is executed using n processors, in terms of n, α and x.

(ii) Given $n = 16$ and $x = 4$ MIPS, determine the value of α to yield a system performance of 40 MIPS.

(b) In a computer network, 2^k host computers $H_0, H_1, \ldots, H_{2^k-1}$ are connected via routers that are connected in the form of a binary tree. An example is shown in the figure below for $k = 3$. Each node labelled R_i corresponds to a router at level $3-i$ $(i = 1, 2, 3)$; the nodes at the leaf level correspond to the hosts.



All the lines are full-duplex, and routers are faster than the links. Also, all links at the same level are of the same capacity. Traffic is always routed via shortest path. Suppose the capacity of each link at level 1 is 1 MBps.

(i) Find the minimum capacities to be assigned to the links at upper layers i ($1 < i \leq k$), such that the router R_0 is never congested. Prove the correctness of your result.

(ii) With the link capacities found in (i), show that there always exist traffic patterns that may congest any router R_i other than R_0. Justify your answer.

[(4+2)+(4+4)=14]

C7. The byte-addressable logical address space in a lightweight computing platform consists of 128 segments, each of capacity 32 pages, each of which can hold 4096 bytes. The physical memory (also byte-addressable) consists of 2^{14} page frames, each 4096 bytes long.

(a) Formulate the logical and physical address formats.

(b) If segment table entries and page table entries both occupy 4 bytes each, compute the size in bytes of (i) the segment table and (ii) page tables for any process.

(c) Assume that no swap space is available. If the kernel takes up 12×2^{20} bytes, and the rest of the physical memory may be used to hold processes, compute the maximum number of processes that the kernel can handle concurrently in the worst case.

[(2+2)+(3+3)+4=14]

C8. Let $M = (Q, \Sigma, \delta, q_o, F)$ be a Deterministic Finite Automaton (DFA), and let $L = L(M)$ be the language accepted by M. Consider another DFA $M' = (Q, \Sigma, \delta, q_o, Q)$, and let $L' = L(M')$.

(a) Give an example of a language $L \subseteq \{0, 1\}^*$ for which $L(M') = \mathit{prefix}(L)$.

(b) Give an example of a language $L \subseteq \{0, 1\}^*$ for which $L(M') \neq \mathit{prefix}(L)$.

Justify your answer in each case.

Recall that for a given alphabet Σ, a string $\beta \in \Sigma^*$ is said to be a prefix of a string $\alpha \in \Sigma^*$ if $\alpha = \beta\gamma$ for some $\gamma \in \Sigma^*$. For example, the prefixes of 010 are $\varepsilon, 0, 01$ and 010. Given a language $L \subseteq \Sigma^*$, $\mathit{prefix}(L)$ is defined as

$$\mathit{prefix}(L) = \{x \in \Sigma^* \mid x \text{ is a prefix of some } y \in L\}.$$

[7+7=14]

II. ELECTRICAL AND ELECTRONICS ENGINEERING

Answer any five out of eight questions.

E1. We would like to design a counter that generates, in binary, the sequence of primes $2 \to 3 \to 5 \to 7 \to 11 \to 13 \to 2 \ldots$.

(a) Draw the state transition diagram of the circuit.

(b) How many Flip-Flops are required to implement this counter?

(c) Draw the circuit diagram using Flip-Flops and logic gates of your choice. Explain each step of your design methodology.

[2+2+(5+5)=14]

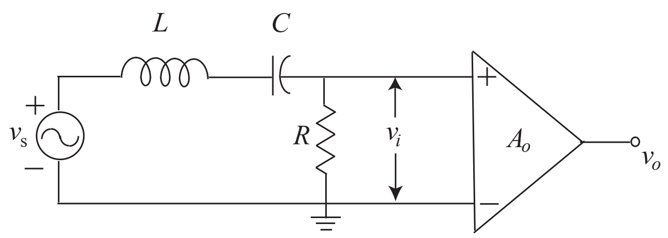
E2. Given two integers x, y, $0 \le x, y \le 7$, design a combinational logic circuit to compute the function

$$z = (-1)^x x + (-1)^y y.$$

Integers are represented in 2's complement form. Use only full adder blocks and exclusive-OR gates in your design.

[14]

E3. Consider the OP-AMP circuit shown below:



(a) Derive the transfer function

$$A_v(j\omega) = \frac{v_o}{v_s},$$

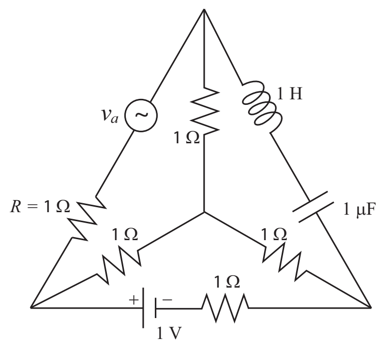
assuming that the amplifier provides a gain $A_0 = \frac{v_o}{v_i}$ which is positive and constant for all frequencies of operation. Show that the circuit acts as an active band-pass filter.

(b) Derive the expressions for the centre frequency ω_0, the quality factor Q, and the bandwidth.

(c) Given that the midband voltage gain $A_0 = 50$, a centre frequency $\omega_0 = 1000$ rad/sec, and $C = 0.2\ \mu F$, find the values of L and R to achieve a Q factor of 10.

[5+6+3=14]

E4. In the circuit shown below, for what value of frequency ω will the current through the inductance be in phase with the alternating voltage source $v_a = 2\cos(\omega t)$ V? If the frequency is indeed chosen this way, what will be the current through the resistor R, at time t? Both the voltage sources may be assumed to be ideal.



[2+12=14]

E5. (a) Let X be a discrete random variable and $X \in \{-3, -2, -1, 0, 1, 2, 3\}$. The probabilities of seven possible values of X are given in the table below:

Value	-3	-2	-1	0	1	2	3
Probability	1/4	1/8	1/16	1/8	1/16	1/8	1/4

Let us define a function $f(X)$ of the random variable X as follows:

$$f(X) = X^2 + 1.$$

Compute the following:

(i) $H(X)$ - entropy of the random variable X;

(ii) $H(f(X))$ - entropy of the function $f(X)$; and

(iii) $H(X, f(X))$ - joint entropy between the random variable X and the function $f(X)$.

(b) Prove that for any discrete random variable X and any function $f(X)$, $H(X) \geq H(f(X))$.

(c) Devise an optimum variable-length code for the set of symbols (discrete values) of $f(X)$ of Part (a), with strings of 0's and 1's based on Huffman coding.

[(1+1+2)+5+5=14]

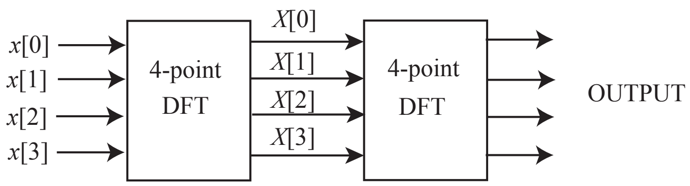
E6. (a) The N-point Discrete Fourier Transform (DFT) for a signal $x[n]$ is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp\left\{\frac{-j2\pi kn}{N}\right\}, \qquad k = 0, \ldots, N-1.$$

Compute the output if the signal, defined as

$$x[n] = \begin{cases} n+1 & \text{for } n = 0, 1, 2, 3 \\ 0 & \text{otherwise,} \end{cases}$$

passes through a cascade of two identical systems, each of which implements a 4-point DFT as shown below.



HINT: Explicit computation of the DFT may not be required.

(b) Consider a system with impulse response

$$h[n] = \begin{cases} 1 & \text{for } n = 0, 1, \ldots, 5 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Determine the locations of the poles and zeros of this system.

(ii) Can this system be used to implement a low-pass or a high-pass filter? Justify your answer.

[6+(4+4)=14]

E7. (a) A motor, operating at 220V, has 90% efficiency. If the armature resistance is 1.1Ω, what is the current in the armature when the motor

(i) is just switched on; and

(ii) running at full speed?

(b) A 100V source drives a DC motor, with stator resistance 2Ω, at 1200 rpm at no load. What will be the torque and current generated by the DC motor if it is fed by a 220V supply and the load is such that the motor speed is 1500 rpm? Assume constant field and no other loss.

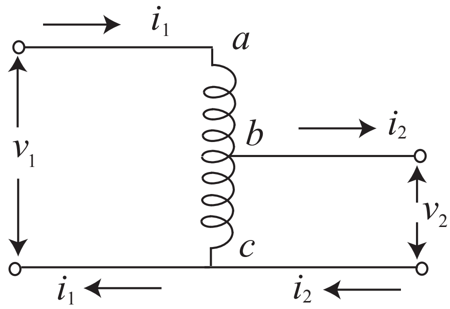
[(3+3)+(6+2)=14]

E8. (a) A 250kW generator provides emf to a purely resistive load through a step-down transformer which reduces 50kV transmission voltage to a lower voltage. The transmission line is a 2-cable supply with line resistance as 0.3Ω/cable. Calculate

(i) the reduction in voltage along the transmission line; and

(ii) the rate of dissipation of thermal energy from the transmission line.

(b) The circuit of an autotransformer is shown below.



The turns ratio of the autotransformer is 6. If i_{ab} is 3A, find i_{cb}.

[(3+3)+8=14]

III. MATHEMATICS

Answer any five out of ten questions.

Notation

$\mathbb{Z}$ = the set of integers

$\mathbb{N} = \{n \in \mathbb{Z} : n \geq 1\}$

$\mathbb{R}$ = the set of real numbers

$\mathbb{Q}$ = the set of rational numbers

$\mathbb{C}$ = the set of complex numbers

M1. Let A be a diagonalisable $m \times m$ matrix with entries from $\mathbb{C}$. Define $\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ assuming A^0 is the identity matrix. Prove that $\det(\exp(A)) = e^{\mathrm{Tr}(A)}$.

[14]

M2. Let $H = \{1 + 4k : k \in \mathbb{Z},\ k \geq 0\}$. An element $x \in H$ is called H-*prime* if $x \neq 1$ and x cannot be written as product of two strictly smaller elements of H.

(a) Show that $xy \in H$ for all $x, y \in H$.

(b) Prove that every $x \in H$ greater than 1, can be factored as a product of H-primes but unique factorisation does not hold.

[5+9=14]

M3. Find an ideal I in $A = \frac{\mathbb{Z}[X]}{(X^4 + X^2 + 1)}$ such that A/I is a finite field with 25 elements.

[14]

M4. (a) Suppose A and B are closed subsets of a topological space such that $A \cap B$ and $A \cup B$ are connected. Prove that A and B are connected.

(b) Demonstrate that the conclusion may not hold if the assumption of A and B being closed subsets, is dropped.

[7+7=14]

M5. Examine whether there is a polynomial $f(X) \in \mathbb{R}[X]$ such that $\dfrac{\mathbb{R}[X]}{(f(X))}$ is isomorphic **as rings** to the product ring $\mathbb{C} \times \mathbb{C}$.

[14]

M6. Let $f(x), g(x) \in \mathbb{Z}[X]$ with $f(x) = \sum_{j=0}^{n} a_j x^j$ and $b(x) = \sum_{j=0}^{n} b_j x^j$. For $m \in \mathbb{N}$, we say $f \equiv g \mod m$ if $a_j = b_j \mod m$ for $0 \le j \le n$. For an odd prime $p > 0$, let

$$f(x) = x^{p-1} - 1 \quad \text{and} \quad g(x) = (x-1)(x-2)\cdots(x-p+1).$$

Prove that

(a) the polynomial $f(x) - g(x)$ has degree $p - 2$, and

(b) $f \equiv g \mod p$.

[4+10=14]

M7. Prove that the following two groups are isomorphic:

- $\mathbb{Z}[X]$, the group of polynomials with integer coefficients under addition, and
- $\mathbb{Q}_{>0}$, the group of positive rational numbers under multiplication.

HINT: Fundamental theorem of Arithmetic

[14]

M8. Let $A = \dfrac{\mathbb{C}[X,Y]}{(X^2+Y^2-1)}$, and x, y denote the images of X, Y in A respectively. If $u = x + iy$, then prove that

(a) u is a unit in A, and

(b) $u - i$ generates a maximal ideal of A.

[4+10=14]

M9. Prove that there cannot be any topological space X such that $\mathbb{R}$ is homeomorphic to $X \times X$ with the product topology.

[14]

M10. Let A be a real symmetric $m \times m$ matrix with m distinct eigenvalues and $v_1, \ldots, v_m$ be the corresponding eigenvectors. Let C be an $m \times m$ matrix satisfying $\langle Cv_j \, , v_j \rangle = 0$ for $1 \leq j \leq m$. Prove that there exists an $m \times m$ matrix X such that $AX - XA = C$.

[14]

IV. PHYSICS

Answer any five out of eight questions.

P1. (a) The transformation between two sets of coordinates $\{q,\ p\}$ and $\{Q,\ P\}$ is given by

$$Q = \ln(1 + \sqrt{q}\cos p)$$

$$P = 2\ (1 + \sqrt{q}\cos p)\sqrt{q}\sin p.$$

(i) Using this transformation show that if $\{q,\ p\}$ are canonical, then $\{Q,\ P\}$ are also canonical variables.

(ii) Show that the generating function between the two sets of canonical variables is

$$F_3 = -(\exp Q - 1)^2 \tan p.$$

(b) A particle of mass M is restricted to move on the parabola $z = \frac{x^2}{a}$ in the Z-X plane under a constant gravitational acceleration g, where a is a constant.

(i) Write the Lagrangian in a suitable generalized coordinate and derive the equation of motion.

(ii) Find the equilibrium position for the particle.

(iii) Calculate the frequency of small oscillation about the equilibrium position.

(HINT: Approximate the Lagrangian for small displacement and velocity.)

[(3+3)+(5+1+2)=14]

P2. (a) A classical analogue of a linear triatomic molecule such as CO_2 is a central mass M connected with two masses (m each) on either side by two springs of equal spring constant k, in a straight line, where $M > m$.

(i) Using the principle of small oscillations, find out the possible frequencies of oscillation for such a molecule.

(ii) Show that there is one mode of oscillation where the central mass M remains stationary while the other two oscillate with identical amplitudes in opposite directions.

(iii) If one of the masses m receives an impulse P_0 at $t = 0$, find the motion of that mass as a function of time.

(b) As seen from an inertial frame S, an event occurs at point A on the x-axis. After 10^{-6} seconds another event occurs at point B, 600 meters apart, on the same axis.

(i) Show that there exists an inertial frame S' with respect to which two events appear simultaneous.

(ii) What are the magnitude and direction of the velocity of S' with respect to S?

[(4+2+2)+(2+4)=14]

P3. (a) A thin insulating rod, running from $z=-a$ to $z=+a$, carries the line charge $\lambda = k \sin\left(\frac{\pi z}{a}\right)$, where k is a constant. Find the leading term in the multipole expansion of the potential.

(b) A very long solenoid of radius a, with n turns per unit length, carries a current I_s. Coaxial with the solenoid, at radius $b >> a$, is a circular ring of metallic wire, with resistance R. When the current in the solenoid is gradually decreased, a current I_r is induced in the ring.

(i) Find I_r in terms of $\frac{dI_s}{dt}$.

(ii) Calculate the power delivered to the ring using Poynting vector.

[6+(3+5)=14]

P4. (a) Let the n-th energy eigenstate of a harmonic oscillator with mass m and angular frequency ω be given by $\psi_n(x)$.

(i) Obtain the expression for the probability of finding the particle outside the classical region in terms of $\psi_n(x)$.

(ii) Calculate the probability for the same when the harmonic oscillator is in the ground state given by

$$\psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{\alpha x^2}{2}\right], \text{ where } \alpha = \frac{m\omega}{\hbar}.$$

(b) Consider a particle of mass m in a one dimensional infinite square well of width a. The centre is perturbed by a delta-function bump $H' = \gamma\delta(x - \frac{a}{2})$, where γ is a constant.

(i) Find the first-order correction to the allowed energies.

(ii) Are the energies perturbed for even values of n? Justify your answer.

[(3+5)+(4+(1+1))=14]

P5. (a) (i) For the angular momentum operator $\hat{J}$, show how $\hat{J}_x$ and $\hat{J}_y$ transform under a rotation of finite angle θ about the z-axis.

(ii) Using these results, evaluate how $\hat{J}$ transforms under the rotation.

(b) The Hamiltonian of a system is $\hat{H} = \epsilon\vec{\sigma}.\vec{n}$, where ϵ is a constant, $\vec{n}$ is an arbitrary unit vector, and σ_x, σ_y, and σ_z are the Pauli spin matrices. Find the energy eigenvalues and normalized eigenvectors of $\hat{H}$.

[(5+3)+6=14]

P6. Assume a crystal with a paramagnetic impurity whose energy levels are $\pm\epsilon$ in a given magnetic field.

(a) Write the contribution of this impurity to the partition function of the crystal in the presence of the field.

(b) Find the contribution of the impurity to the specific heat.

(c) Compare the temperature dependence of this specific heat to that of the phonon specific heat (Debye theory) in the limit of very low temperatures, and give a physical interpretation to the difference thereof.

(d) Carry out a similar comparison, as in (c), for the limit of very high temperatures.

[2+4+4+4=14]

P7. (a) Consider a collection of N two-level systems in thermal equilibrium at a temperature T. Each system has only two states: a ground state of energy zero and an excited state of energy ϵ.

(i) Find an expression for the probability that a given system will be found in the excited state. Sketch the temperature dependence of this probability.

(ii) Calculate the entropy of the entire collection.

(b) You are given four identically looking objects:

- a permanent magnet
- a diamagnetic material
- a paramagnetic material
- an unmagnetized ferromagnetic material.

How can you identify each of the above objects without using any external electromagnetic field? Give physical explanations.

[(4+4)+6=14]

P8. (a) Consider a relativistic field theory of electrons (of charge e and mass m) interacting with electromagnetic field.

(i) Write down the action and show that it is invariant under local gauge transformation. Why is a photon mass term not allowed in this scheme?

(ii) Draw possible Feynman diagrams for electron-electron scattering to $O(e^4)$.

(iii) Consider momenta of incoming and outgoing electrons to be $\mathbf{p_1}, \mathbf{p_2}$ and $\mathbf{p_3}, \mathbf{p_4}$, respectively. How will you write down the amplitude corresponding to one of the above Feynman diagrams in momentum space? (Explicit numerical factors are not necessary.)

(b) Consider the following elementary particle reaction that takes place on a proton target at rest in the laboratory:

$$K^- + p \to \pi^0 + \Lambda^0.$$

Show that the energy of the incident K^- particle such that the Λ^0 can be produced at rest in the laboratory is given by

$$E_K = \frac{m_K^2 - m_\pi^2 + (m_\Lambda - m_p)^2}{2(m_\Lambda - m_p)},$$

where m_i is the rest mass of i-th particle.

[(4+2+2)+6=14]

V. STATISTICS

Answer any five out of eight questions.

S1. Consider a count variable X following a Poisson distribution with parameter $\theta > 0$, where zero count (i.e., $X = 0$) is not observable. We have n observations $X_1, \ldots, X_n$ from this distribution. Let $\overline{X}$ denote the sample mean.

(a) Derive the quantity for which $\overline{X}$ is an unbiased estimator.

(b) Suppose that the observed value of $\overline{X}$ is strictly greater than 1. Show that the likelihood function of θ has a unique maximizer.

[4+10=14]

S2. Let $\mathcal{P} = \{f_\theta : \theta \in \Theta\}$, where f_θ is a continuous probability density over the support $\mathbb{R}$ for each $\theta \in \Theta$. Suppose that, if X_1, X_2 are independent and identically distributed with density f_θ, then $X_1 + X_2$ is sufficient for θ.

(a) Fix $\theta_0 \in \Theta$ and define $s(x, \theta) = \log f_\theta(x) - \log f_{\theta_0}(x) - \log f_\theta(0) + \log f_{\theta_0}(0)$. Prove that

$$s(x_1 + x_2, \theta) = s(x_1, \theta) + s(x_2, \theta), \qquad \text{for all } \theta \in \Theta,\ x_1, x_2 \in \mathbb{R},$$

and hence show that $s(x, \theta) = xs(1, \theta)$ for all x and all θ.

(b) Using (a), or otherwise, prove that $\mathcal{P}$ must be an exponential family indexed by θ.

[(5+4)+5=14]

S3. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with a common density function $f(x, \theta) = e^{-(x-\theta)} I(x \geq \theta)$, where $\theta \in \mathbb{R}$.

(a) Find the maximum likelihood estimator $\widehat{\theta}_n$ of θ based on $X_1, \ldots, X_n$.

(b) Show that $\widehat{\theta}_n$ is consistent for θ.

(c) For a suitable normalizing factor k_n (to be specified by you), find a non-degenerate limiting distribution of $k_n(\widehat{\theta}_n - \theta)$.

[3+5+6=14]

S4. Consider the Gauss-Markov model, $Y = X\beta + \epsilon$, where $\epsilon \sim N_n(0, \sigma^2 I_n)$ and $X_{n\times p}$ has rank $r < p$. Suppose $T_{n\times(p-1)}$ is the matrix formed by the first $p-1$ columns of X and it also has rank r. Let B denote any generalized inverse of $T'T$. Prove that $\widehat{\beta} = \begin{pmatrix} BT'Y \\ 0 \end{pmatrix}$ minimizes $(Y - X\beta)'(Y - X\beta)$.

[14]

S5. Suppose $X_1, X_2, \ldots, X_n$ are independent with $X_i \sim N(i\theta, \tau^2)$ for $i = 1, \ldots, n$. Define

$$U = \frac{\sum_{i=1}^{n} iX_i}{\sum_{i=1}^{n} i^2}, \qquad V^2 = \frac{\sum_{i=1}^{n}(X_i - iU)^2}{n-1}.$$

Show that $\frac{1}{\tau^2}(n-1)V^2$ has a chi-square distribution with $(n-1)$ degrees of freedom.

[14]

S6. Suppose that $T_1, \ldots, T_n$ are lifetimes of n items started together which are independent and identically distributed having exponential distribution with mean $1/\lambda$. Also let $0 < \tau_1 < \tau_2$ are two prefixed time points when they are observed. At time τ_1 we remove each surviving item, if any, with probability $p \in (0, 1)$, and at time τ_2 we remove all the surviving items, if any, from the study. Instead of observing the T_is, we observe only the four counts as follows:

X_1 = the number of items failed before time τ_1,

X_2 = the number of items removed at time τ_1,

X_3 = the number of items failed between times τ_1 and τ_2,

X_4 = the number of items removed at time τ_2.

(a) Obtain the joint distribution of (X_1, X_2, X_3, X_4).

(b) Find a maximum likelihood estimate of p based on these four counts.

[9+5=14]

S7. A spider and a fly move between locations 1 and 2 at discrete times $1, 2, 3, \ldots$ according to Markov chains with respective transition matrices $\begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$ and $\begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}$. The spider starts from location 1 while the fly starts from location 2. Once they are at the same location, there is no further movement.

(a) Find the transition matrix of their joint movement over the following three states:

$$S_1 = \text{Spider is at location 1 but the fly is at location 2,}$$
$$S_2 = \text{Spider is at location 2 but the fly is at location 1,}$$
$$S_3 = \text{Both spider and fly are at the same location.}$$

(b) What is the expected time till the two meet at the same location?

[7+7=14]

S8. Let $X_1, \ldots, X_n$ be independent and identically distributed having the discrete uniform distribution on $\{1, 2, \ldots, \theta\}$, where $\theta \in \Theta = \{2, 3, 4, 5, \ldots\}$.

(a) Given $\theta_0 \in \Theta$ and $0 < \alpha < 1$, find a level-α likelihood ratio test for testing

$$H_0 : \theta \leq \theta_0 \quad \text{against} \quad H_1 : \theta > \theta_0.$$

(b) Show that the largest order statistic is not complete.

[6+8=14]