MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. If the term independent of x in the expansion of $\left(x^{\frac{2}{3}}+\frac{\alpha}{x^3}\right)^{22}$ is 7315, then $|\alpha|$ is

(1) 1

(2) 2

(3) 0

(4) 3

Answer (1)

Sol. $T_{r+1} = {}^{22}C_r\left(x^{\frac{2}{3}}\right)^{22-r}\left(\frac{\alpha}{x^3}\right)^r$

$\Rightarrow \frac{2(22-r)}{3} - 3r = 0$

$\Rightarrow 44 - 2r - 9r = 0 \Rightarrow r = 4$

$\therefore T_5 = {}^{22}C_4\alpha^4 = 7315$

$\alpha^4 = \frac{7315}{7315}$

$\Rightarrow |\alpha| = 1$

2. The value of $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x+\frac{\pi}{4}}{2-\cos 2x}dx$ equals

(1) $\frac{3\pi^2}{\sqrt{6}}$

(2) $\sqrt{3}\pi^2$

(3) $\frac{\pi^2}{6\sqrt{3}}$

(4) $\frac{6\pi^2}{\sqrt{3}}$

Answer (3)

Sol. $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x\,dx}{2-\cos 2x} + \frac{\pi}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{2-\cos 2x}$

$= 0 + \frac{\pi}{4}\cdot 2\int_0^{\frac{\pi}{4}} \frac{dx}{2-\cos 2x}$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \frac{\sec^2 x\,dx}{1+3\tan^2 x}$

Now, $\tan x = t$

$= \frac{\pi}{2}\int_0^1 \frac{dt}{1+3t^2}$

$= \frac{\pi}{2\sqrt{3}}\tan^{-1}\left(\sqrt{3}\,t\right)\Big|_0^1$

$= \frac{\pi}{2\sqrt{3}}\left(\frac{\pi}{3}\right) = \frac{\pi^2}{6\sqrt{3}}$

3. The area determined by $xy < 8$, $y < x^2$ and $y > 1$ is

(1) $4\ln 2 - \frac{14}{3}$

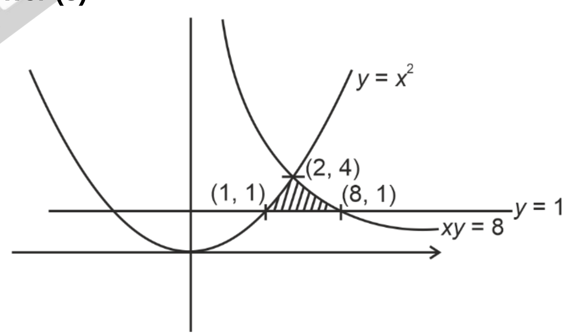
(2) $4\ln 2 + \frac{20}{3}$

(3) $8\ln 4 - \frac{14}{3}$

(4) $8\ln 4 - \frac{20}{3}$

Answer (3)

Sol.



$\therefore$ Area $= \int_1^2\left(x^2-1\right)dx + \int_2^8\left(\frac{8}{x}-1\right)dx$

$= \frac{x^3}{3} - x\Big|_1^2 + \left(8\ln x - x\right)\Big|_2^8$

$= \left(\frac{8}{3}-2\right) - \left(\frac{1}{3}-1\right) + (8\ln 8 - 8) - (8\ln 2 - 2)$

$= \frac{4}{3} + 8\ln 4 - 6 = 8\ln 4 - \frac{14}{3}$

4. If $f(x)+f\left(\frac{1}{1-x}\right)=1-x$, then $f(2)$ equals

(1) $\frac{1}{4}$ (2) $\frac{-5}{4}$

(3) $\frac{3}{4}$ (4) $-\frac{3}{4}$

Answer (2)

Sol. $f(x)+f\left(\frac{1}{1-x}\right)=1-x$ ...(i)

Put $x = 2$ in (i)

$f(2) + f(-1) = -1$ ...(ii)

Put $x = -1$ in (i)

$f(-1)+f\left(\frac{1}{2}\right)=2$ ...(iii)

Put $x=\frac{1}{2}$ in (i)

$f\left(\frac{1}{2}\right)+f(2)=\frac{1}{2}$ ...(iv)

From (iii) and (iv)

$2-f(-1)=\frac{1}{2}-f(2)$

$f(-1)=\frac{3}{2}+f(2)$ ...(v)

From (ii) and (v)

$-1-f(2)=\frac{3}{2}+f(2)$

$2f(2)=-1-\frac{3}{2}=\frac{-5}{2}$

$\boxed{f(2)=\frac{-5}{4}}$

5. If $f(x) = x^x$, $x > 0$ then $f''(2) + f'(2)$ is

(1) $10 + 12 \ln 2 + 4 (\ln 2)^2$

(2) $10 + 4 (\ln 2)^2$

(3) $10 + 12 \ln 2$

(4) $2^{\ln 2} + (\ln 2)^2$

Answer (1)

Sol. $y = x^x$

$\therefore$ $f'(2) = 4(1 + \ln 2)$

$y' = x^x (1 + \ln x)$

$y''=\frac{x^x}{x}+x^x(1+\ln x)^2$

$\Rightarrow$ $f''(2) = 2 + 4 (1 + \ln 2)^2$

$f''(2) + f'(2) = 4 + 4\ln 2 + 6 + 8 \ln 2 + 4(\ln 2)^2$

$= 10 + 12 \ln 2 + 4 (\ln 2)^2$

6. Which of the following is a tautology?

(1) $p \to (\sim p \wedge q)$ (2) $p \to (p \vee q)$

(3) $p \to (\sim p \vee q)$ (4) $p \to (\sim p \wedge \sim q)$

Answer (2)

Sol. For tautology of $p \to q$

(i) $p \to$ T and $q \to$ T OR

(ii) $p \to$ F and $q \to$ T or F

So, option (2) is true.

7. If the system of equations

$\alpha x + y + z = 1$

$x + \alpha y + z = 1$

$x + y + \alpha z = \beta$ has infinitely many solutions, then

(1) $\alpha = 1, \beta = 1$ (2) $\alpha = 1, \beta = -1$

(3) $\alpha = -1, \beta = -1$ (4) $\alpha = -1, \beta = 1$

Answer (1)

Sol. For infinite solutions

$\Delta = \Delta_x = \Delta_y = \Delta_z = 0$

$$\Rightarrow \begin{vmatrix}\alpha & 1 & 1\\ 1 & \alpha & 1\\ 1 & 1 & \alpha\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ 1 & \alpha & 1\\ \beta & 1 & \alpha\end{vmatrix}=\begin{vmatrix}\alpha & 1 & 1\\ 1 & 1 & 1\\ 1 & \beta & \alpha\end{vmatrix}=\begin{vmatrix}\alpha & 1 & 1\\ 1 & \alpha & 1\\ 1 & 1 & \beta\end{vmatrix}=0$$

$\therefore$ $\alpha = 1 = \beta$

Clearly $\alpha = \beta = 1$ makes all the above equations identical i.e., three co-incidence planes.

8. If $A=\frac{1}{2}\begin{bmatrix}1 & \sqrt{3}\\ -\sqrt{3} & 1\end{bmatrix}$, then which of the following is true?

(1) $A^{30}=A^{25}$ (2) $A^{30}+A^{25}+A=I$

(3) $A^{30}-A^{25}+A=I$ (4) $A^{30}=A^{25}+A$

Answer (3)

Sol. $\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$

$|A-\lambda I|=0$

$$\begin{vmatrix}\frac{1}{2}-\lambda & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}-\lambda\end{vmatrix}=0$$

$\lambda^2+\frac{1}{4}-\lambda+\frac{3}{4}=0$

$A^2 - A + I = 0 \Rightarrow A^3 - A^2 + A = 0$

$A^4 = (A-I)^2 = A^2 + I - 2A = A - I + I - 2A = -A$

$\boxed{A^4 = -A}$

$A^{30} - A^{25} + A = A^2(A^4)^7 - (A^4)^6 \cdot A + A$

$= -A^2A^7 - A^6 \cdot A + A$

$= -A^9 - A^7 + A$

$= -A^8 \cdot A - A^4 \cdot A^3 + A$

$= -A^3 - A + A$

$= -A^3$

$= A - A^2$

$= I$

9. 2 unbiased die are thrown independently. A is the event such that the number on the first die is greater than second die. B is the event such that number on the first die is even and number on the second die is odd. C is the event such that first die shows odd number & second die shows even number. Then,

(1) $n((A \cup B) \cap C) = 6$

(2) A and B are mutually exclusive events

(3) A and B are independent events

(4) $n(A) = 18,\ n(B) = 6,\ n(C) = 6$

Answer (1)

Sol. $n(S) = 36$

$$A = \begin{Bmatrix} (1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,3),(2,4),(2,5),(2,6) \\ (3,4),(3,5),(3,6) \\ (4,5),(4,6) \\ (5,6) \end{Bmatrix}$$

$n(A) = 15$

$$B = \begin{Bmatrix} (2,1),(2,3),(2,5) \\ (4,1),(4,3),(4,5) \\ (6,1),(6,3),(6,5) \end{Bmatrix}$$

$n(B) = 9$

$$C = \begin{Bmatrix} (1,2),(1,4),(1,6) \\ (3,2),(3,4),(3,6) \\ (5,2),(5,4),(5,6) \end{Bmatrix}$$

$n(C) = 9$

$P(A) = \frac{15}{36},\ P(B) = \frac{9}{36},\ P(C) = \frac{9}{36}$

$n(B \cap C) = 0$

$A \cap B = \{(2,3),(2,5),(4,5)\}$

$P(A \cap B) = \frac{3}{36} = \frac{1}{12}$

$P(A) \cdot P(B) = \frac{15}{36} \cdot \frac{6}{36}$

$= \frac{5}{72}$

$$((A \cup B) \cap C) = \begin{Bmatrix} (1,2),(1,4),(1,6) \\ (3,4),(3,6) \\ (5,6) \end{Bmatrix}$$

$n((A \cup B) \cap C) = 6$

10. If $\frac{dy}{dx} = \frac{x^2 + 3y^2}{3x^2 + y^2}$, $y(1) = 0$ Then

(1) $\frac{2x^2}{(x^2 - y^2)^2} = \ln|x - y| + \frac{2x}{x - y}$

(2) $\frac{2x}{(x^2 - y^2)^2} = \ln|x - y| + 1$

(3) $\frac{2x^2}{(x^2 - y^2)^2} = \ln|x - y| + \frac{y}{x - y}$

(4) $\frac{2x}{(x^2 - y^2)^2} = \ln|x - y| + \frac{y}{x - y}$

Answer (1)

Sol. $\frac{dy}{dx} = \frac{x^2 + 3y^2}{3x^2 + y^2}$

Let $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

So, $v + x\frac{dv}{dx} = \frac{1 + 3v^2}{3 + v^2}$

$x\frac{dv}{dx} = \frac{1 + 3v^2}{3 + v^2} - v = \frac{-v^3 + 3v^2 - 3v + 1}{v^2 + 3}$

$\frac{v^2 + 3}{-v^3 + 3v^2 - 3v + 1}dv = \frac{1}{x}dx$

$\Rightarrow \int \frac{v^2 + 3}{(1 - v)^3}dv = \frac{1}{x}dx$

$\Rightarrow \int \frac{1}{1 - v}dv - \int \frac{2dv}{(1 - v)^2} + \int \frac{4}{(1 - v)^3}dv = \int \frac{1}{x}dx$

$\Rightarrow \quad -\ln|-v| - \frac{2}{1-v} + \frac{2}{(1-v)^2} = \ln|x| + C$

$\because \quad y(1) = 0, \Rightarrow v(1) = 0$

$\Rightarrow \quad \boxed{C = 0}$

$\therefore \quad \frac{2}{\left(1-\frac{y}{x}\right)^2} = \ln\left|1-\frac{y}{x}\right| + \frac{2}{1-\frac{y}{x}} + \ln(x)$

$\Rightarrow \quad \frac{2x^2}{(x^2-y^2)^2} = \ln|x-y| + \frac{2x}{x-y}$

11. $\vec{a} = \hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{c} = 4\hat{i} + 5\hat{j} - \hat{k}$. If $\vec{r} \cdot \vec{b} = 0$ and $\vec{r} \times \vec{a} = \vec{b} \times \vec{c}$ then $\vec{r}$ is equal to

(1) $-12\hat{i} - 8\hat{j} + \hat{k}$ (2) $-12\hat{i} - \frac{23}{3}\hat{j} + \hat{k}$ (3) $12\hat{i} + \frac{23}{3}\hat{j} + \hat{k}$ (4) $12\hat{i} + 8\hat{j} + \hat{k}$

Answer (2)

Sol. $\vec{r}\ \vec{a} = \vec{b} \times \vec{c}$

$\vec{b} \times (\vec{r} \times \vec{a}) = \vec{b} \times (\vec{b} \times \vec{c})$

$\vec{b} \cdot \vec{a}\ \vec{r} - \vec{b} \cdot \vec{r}\ \vec{a} = \vec{b} \cdot \vec{c}\ \vec{b} - \vec{b} \cdot \vec{b}\ \vec{c}$

$6\vec{r} = -8(2\hat{i} - 3\hat{j} + \hat{k}) - 14(4\hat{i} + 5\hat{j} - \hat{k})$

$6\vec{r} = -72\hat{i} - 46\hat{j} + 6\hat{k}$

$\vec{r} = -12\hat{i} - \frac{23}{3}\hat{j} + \hat{k}$

12. If $2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x \in (0, 1)$ has

(1) 2 solutions for $x < \frac{1}{2}$

(2) 2 solutions for $x > \frac{1}{2}$

(3) One solution for $x < \frac{1}{2}$

(4) One solution for $x > \frac{1}{2}$

Answer (3)

Sol. Put $x = \tan\theta$ we get

$2\tan^{-1}\left(\tan\left(\frac{\pi}{4} - \theta\right)\right) = \cos^{-1}\cos 2\theta$

$2\left(\frac{\pi}{4} - \theta\right) = 2\theta$

$\Rightarrow \quad \theta = \frac{\pi}{8}$

$\therefore \quad x = \sqrt{2} - 1$ only one value $< \frac{1}{2}$

13.

14.

15.

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

21. Number of non-negative integral solutions of $x + y + z = 21$ if $x \geq 1, y \geq 3, z \geq 6$

Answer (78)

Sol. $\because \quad x + y + z = 21 \qquad [\because x \geq 1, y \geq 3, z \geq 6]$

then $x_1 + y_1 + z_1 = 11$

Now $x_1 \geq 0, y_1 \geq 0, z_1 \geq 0$

Total ${}^{11+3-1}C_{3-1}$ solution

${}^{13}C_2 = \frac{13!}{2!11!} = 6 \times 13 = 78$

22. Total 6 digit numbers using the digits 4, 5, 9 which are divisible by 6 are

Answer (81.00)

Sol. For this, 4 will be fixed on unit place digit

		Total number
Case-I	4's → 6 time	1
Case-II	4's → 4 time	$\frac{5!}{3!} = 20$
	5's → 1 time	
	9's → 1 time	
Case-III	4's → 3 time	$\frac{5!}{2!3!} = 10$
	5's → 3 time	
Case-IV	4's → 3 time	$\frac{5!}{2!3!} = 10$
	9's → 3 time	
Case-V	4's → 2 time	$\frac{5!}{2!2!} = 30$
	5's → 2 time	
	9's → 2 time	
Case-VI	4's → 1 time	$\frac{5!}{4!} = 5$
	5's → 1 time	
	9's → 4 time	
Case-VII	4's → 1 time	$\frac{5!}{4!} = 5$
	5's → 4 time	
	9's → 1 time	

Total numbers = 81

23. Let 3 A.P′s be

S_1 = 2, 5, 8, 11, ……..394

S_2 = 1, 3, 5, 7, ………397

and S_3 = 2, 7, 12, ……..397

then sum of common terms of these three A.P's is

Answer (2364)

Sol. S_2 has all odd numbers upto 397

∴ Common terms in S_1 and S_2 gives 5, 11, 17, 23, …….391 = S_4

Common terms S_3 and S_2 … 7, 17, 27, ……..397 = S_5

∴ Required terms will be common terms of S_4 and S_5

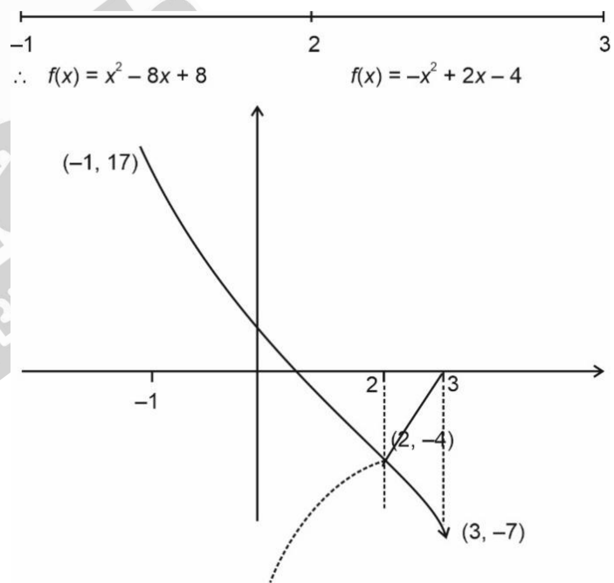
i.e., 17, 47, 77, 107, ……..377

$$\text{Sum} = \frac{12}{2}(17 + 377) = 2364$$

24. Let $f(x) = |(x-3)(x-2)| - 3x + 2$ for $x \in [1, 3]$. If m and M are absolute maximum and absolute minimum value of $f(x)$ then $|m| + |M|$ equals

Answer (24)

Sol.



∴ $m = -7$ and $M = 17$

∴ $|m| + |M| = 7 + 17 = 24$

25. Let $X_1, X_2, X_3 \ldots X_7$ is an A.P. such that $X_1 < X_2 < X_3 \ldots. < X_7$, $X_1 = 9$, $\sigma = 4$. The value of $\overline{X} + X_6$ is equal to _________.

Answer (34)

Sol. Let the series be $a - 3d$, $a - 2d$, $a - d$, a, $a + d$, $a + 2d$, $a + 3d$, $a - 3d = 9$

Now, if we shift the origin, the variance remains same and $\bar{x}$ became $\bar{x}-a$

$\therefore$ For $-3d, -2d, -d, 0, d, 2d, 3d$

$$16=\frac{2}{7}\left(9d^2+4d^2+d^2\right)-\left(\bar{x}-a\right)^2$$

$$\Rightarrow\ 16=\frac{2}{7}d^2(14)-(0)^2$$

$\Rightarrow$ $d = 2$

$a - 3d = 9$

$\Rightarrow$ $a = 15$

$\bar{x}=15$

$x_6 = a + 2d$

$=15 + 2(2) = 19$

$\bar{x}+x_6=15+19=34$

26.

27.

28.

29.

30.

❑ ❑ ❑