JEE-Main-27-06-2022-Shift-1 (Memory Based)

MATHEMATICS

Question: $x=\sum_{n=0}^{\infty} a^n, y=\sum_{n=0}^{\infty} b^n, z=\sum_{n=0}^{\infty} c^n, |a|,|b|,|c|<1$ & a,b,c are in A.P. then

Options:

(a) x, y, z are in AP

(b) x, y, z are in GP

(c) $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ are in AP

(d) $\frac{1}{x},\frac{1}{y},\frac{1}{z}=1-(a+b+c)$

Answer: (c)

Solution:

$x=\sum_{n=0}^{\infty} a^n=\frac{1}{1-a}$

$y=\frac{1}{1-b}, z=\frac{1}{1-c}$

$a,b,c \Rightarrow \text{AP}$

$1-a, 1-b, 1-c \Rightarrow \text{AP}$

$\Rightarrow \frac{1}{1-a},\frac{1}{1-b},\frac{1}{1-c} \Rightarrow \text{HP}$

$\Rightarrow x, y, z \Rightarrow \text{HP}$

$\Rightarrow \frac{1}{x},\frac{1}{y},\frac{1}{z} \Rightarrow AP$

Question: Find number of distinct real roots of $x^4-4x-1=0$

Answer: 2.00

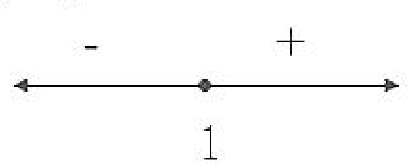
Solution:

Let $f(x)=x^4-4x-1$

$f'(x)=4x^3-4$

$\Rightarrow x^3-1=0$

$x=1$



Thus $f(x)$ will take only one minimum at $x=1$, thus, number of real roots will be '2'.

Question: $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=$

Answer: $\frac{-1}{2}$

Solution:

Given, $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$

$$\Rightarrow \frac{\sin 3\frac{\left(\frac{2\pi}{7}\right)}{2}}{\sin\left(\frac{\frac{2\pi}{7}}{2}\right)}\cos\left\{\frac{2\pi}{7}+\left(\frac{3-1}{2}\right)\frac{2\pi}{7}\right\}$$

$$\Rightarrow \frac{\sin\left(\frac{3\pi}{7}\right)}{\sin\frac{\pi}{7}}\cos\left\{\frac{4\pi}{7}\right\}$$

$$\Rightarrow \frac{-1}{2}$$

Question: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ satisfies $\left(4\sqrt{\frac{2}{5}},3\right)$ & $e=\frac{1}{4}$. Find $3a^2-b^2$.

Answer: 31.00

Solution:

Given, $e=\frac{1}{4}$

$$\Rightarrow b^2=a^2\left(1-e^2\right)$$

$$=a^2\left(1-\frac{1}{16}\right)$$

$$b^2=\frac{15}{16}a^2$$

Now, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ satisfy $\left(4\sqrt{\frac{2}{5}},3\right)$

$$\Rightarrow \frac{16\left(\frac{2}{5}\right)}{a^2}+\frac{9}{b^2}=1$$

$$\frac{32}{5a^2}+\frac{9\times 16}{15a^2}=1$$

$$\Rightarrow \frac{32+48}{5a^2}=1$$

$$\Rightarrow 5a^2=80$$

$$\Rightarrow a^2=16$$

$$\therefore b^2=\frac{15}{16}\times 16=15$$

Thus, $a^2+b^2=16+15=31$

Question: $\lim_{x\to 7}\frac{\left(18-[1-x]\right)}{\left([x]-3a\right)}$ exists, find a.

Answer: -6.00

Solution:

$$\frac{17-[-x]}{[x]-3a}$$

$$\text{LHL }=\frac{17-(-7)}{6-3a}=\frac{24}{6-3a}$$

$$\text{RHL }=\frac{17-(-8)}{7-3a}=\frac{25}{7-3a}$$

$$\text{LHL = RHL }\Rightarrow \frac{24}{6-3a}=\frac{25}{7-3a}$$

$$\Rightarrow 168-72=150-75a$$

$$\Rightarrow 18=-3a$$

$$\Rightarrow a=-6$$

Question: $f(x)=\frac{2e^{2x}}{e^{2x}+e}$, $f\left(\frac{1}{100}\right)+f\left(\frac{2}{100}\right)+...+f\left(\frac{99}{100}\right)=?$

Answer: 99.00

Solution:

$$f(x)=\frac{2e^{2x}}{e^{2x}+e}$$

$$f(1-x)=\frac{2e^{2-2x}}{e^{2-2x}+e}=\frac{\frac{2e^2}{2^{2x}}}{\frac{e^2}{e^{2x}}+e}$$

$$\Rightarrow \frac{2e^2}{e^2+e\cdot e^{2x}}=\frac{2e}{e+e^{2x}}$$

$$f(x)+f(1-x)=\frac{2e^{x}}{e^{2x}+e}+\frac{2e}{e+e^{2x}}=2$$

$\Rightarrow f\left(\frac{1}{100}\right)+f\left(\frac{91}{100}\right)=2$

$f\left(\frac{2}{100}\right)+f\left(\frac{98}{100}\right)=2$

$\vdots$

$f\left(\frac{49}{100}\right)+f\left(\frac{51}{100}\right)=2$

$f\left(\frac{50}{100}\right)=f\left(\frac{1}{2}\right)=\frac{2e}{e+e}=1$

$\therefore$ Required answer $=2\times 49+1$

Question: $\frac{1}{5}+\frac{2}{65}+\frac{3}{325}+\frac{4}{102}+....$ up to 10 terms?

Answer: $\frac{55}{221}$

Solution:

$$t_r=\frac{r}{4r^4+1}$$

$$\Rightarrow t_r=\frac{r}{4r^4+1+4r^2-4r^2}$$

$$\Rightarrow t_r=\frac{4r}{4\left(2r^2+1\right)^2-\left(2r\right)^2}$$

$$\Rightarrow t_r=\frac{1}{4}\times\frac{4r}{\left(2r^2+1-2r\right)\left(2r^2+1+2r\right)}$$

$$\Rightarrow t_r=\frac{1}{4}\left(\frac{1}{\left(2r^2+1-2r\right)}-\frac{1}{\left(2r^2+1+2r\right)}\right)$$

$$\Rightarrow t_r=\frac{1}{4}\left(\frac{1}{2r^2+1-2r}-\frac{1}{2\left(r+1\right)^2-2\left(r+1\right)+1}\right)$$

$$\Rightarrow \sum_{r=1}^{\infty}t_r=\frac{1}{4}\sum_{r=1}^{\infty}\left(\frac{1}{2r^2+1-2r}-\frac{1}{2\left(r+1\right)^2-2\left(r+1\right)+1}\right)$$

$$\Rightarrow \sum_{r=1}^{\infty}t_r=\left(\frac{1}{4}\right)\left[\left(\frac{1}{1}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{13}\right)+\left(\frac{1}{13}-\frac{1}{25}\right)+...\right]$$

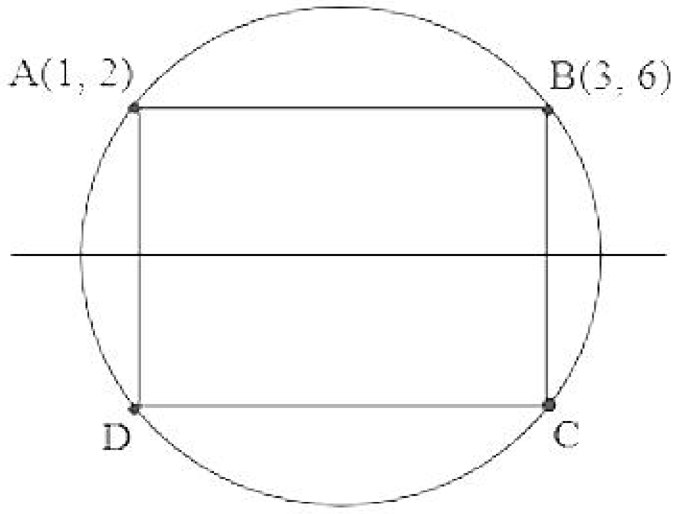
$$\Rightarrow \sum_{r=1}^{\infty}t_r=\frac{1}{4}(1)=\frac{1}{4}=\frac{m}{n}$$

$\therefore m+n=1+4=5$

Question: Rectangle having vertices (1, 2) & (2, 6) is circumscribed by circle with one of its diameter along $2x - y + 4 = 0$ find area.

Answer: 16.00

Solution:



$$AB = \sqrt{(6-2)^2 + (3-2)^2} = \sqrt{4^2 + 2^2} = \sqrt{20}$$

$$OB = \left|\frac{2\times 3 - 6\times 4}{\sqrt{2^2 + 1^2}}\right| = \frac{4}{\sqrt{5}}$$

$$BC = 2OB = \frac{8}{\sqrt{5}}$$

Area $= AB \times BC = \sqrt{20} \times \frac{8}{\sqrt{5}} = 16$

Question: $\int \frac{(x^2+1)e^x}{(x+1)^2} = f(x)e^x$. Find $\frac{d^3 f}{dx^3}$ at $x = 1$.

Answer: $\frac{3}{4}$

Solution:

$$g(x) = \left(\frac{x^2+1}{(x+1)^2}\right)e^x = \left(\frac{x^2+x-x+1}{(x+1)^2}\right)e^x$$

$$= \left(\frac{x}{x+1} + \frac{1-x}{(1+x)^2}\right)e^x$$

$$= \left(\frac{x}{x+1} + \frac{1}{(1+x)^2} - \frac{(x+1-1)}{(1+x)^2}\right)e^x$$

$$= \left(\frac{x}{x+1} + \frac{1}{(1+x)^2} - \frac{1}{1+x} + \frac{1}{(1+x)^2}\right)e^x$$

$$\int g(x) = \int\left(\frac{x}{1+x} + \frac{1}{(1+x)^2}\right)e^x + \left(\frac{-1}{1+x} + \frac{1}{(1+x)^2}\right)e^x$$

$$= e^x\left(\frac{x}{1+x} - \frac{1}{1+x}\right)$$

$$\Rightarrow f(x) = \frac{x-1}{x+1}$$

$$f'(x) = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$f''(x) = \frac{-4}{(x+1)^3}$$

$$f'''(x) = \frac{12}{(x+1)^4}$$

$$f'''(1) = \frac{12}{2^4} = \frac{3}{4}$$

Question: $\begin{vmatrix} 14 & 28 & -14 \\ -14 & 14 & 28 \\ 28 & -14 & 14 \end{vmatrix} = |aadj(adj(A))|$ then $|A| = ?$

Answer: ± 14

Solution:

Given, $\begin{vmatrix} 14 & 28 & -14 \\ -14 & 14 & 28 \\ 28 & -14 & 14 \end{vmatrix} = |aadj(adj(A))|$

$$|A|^{(3-1)^2} = 14(196+392) - 28(-196-784) - 14(196-329)$$

$$= 8232 + 27440 + 1862$$

$$|A|^4 = 38416 = 14^4$$

$$|A| = 14$$

Question: Find area of polygon formed by non-real roots of $\bar{z} = iz^2$.

Answer: $\frac{(3\sqrt{3})}{4}$

Solution:

Let $z = x + iy$

$$x - iy = i(x+iy)^2$$

$$x - iy = i(x^2 - y^2 + 2ixy)$$

$$x - iy = -2xy + i(x - y^2)$$

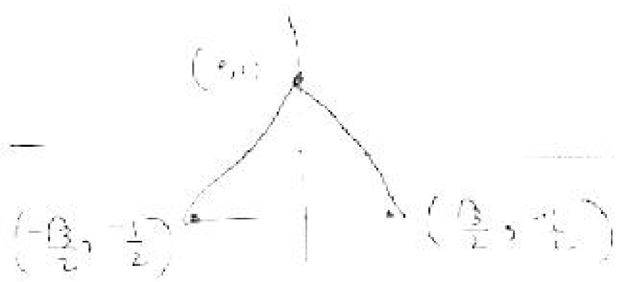
$\Rightarrow x = -2xy$ and $-y = x^2 - y^2$

$\Rightarrow x = 0$ or $y = \frac{1}{2}$

When $x = 0,\ y = 0$ and 1

When $y = \frac{-1}{2},\ x = \pm\sqrt{\frac{3}{2}}$

$x = (0,0), (0,1), \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right), \left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$



Area $= \frac{1}{2}\left(\sqrt{3}\right)\left(\frac{3}{2}\right) = \frac{3\sqrt{3}}{4}$

Question: $\frac{dy}{dx} = \frac{2^{x-y}\left(2^y - 1\right)}{2^{x-1}}$, $y(1) = 1$. Find $y(2)$.

Answer: ()

Solution:

$\frac{dy}{dx} = \frac{2^{x-y}\left(2^y - 1\right)}{2^{x-1}}$

$\Rightarrow \frac{dy}{dx} = \frac{2^{-y}\left(2^y - 1\right)}{2^{-1}}$

$\Rightarrow \frac{dy}{1 - 2^{-y}} = 2\,dx$

$\Rightarrow \int \frac{2^y\,dy}{2^y - 1} = \int 2\,dx$

$\Rightarrow \frac{1}{\ln 2}\ln\left(2^y - 1\right) = 2x + C$

$y(1) = 1$

$\Rightarrow \frac{\ln 1}{\ln 2} = 2 + C \Rightarrow C = -2$

$\Rightarrow \frac{1}{\ln 2}\ln\left(2^y - 1\right) = 2x - 2$

At $x = 2$

$\Rightarrow \ln\left(2^y - 1\right) = (\ln 2)(2 \times 2 - 2)$

$\Rightarrow \ln\left(2^{y}-1\right)=2\ln 2$

$\Rightarrow \ln\left(2^{y}-1\right)=\ln 4$

$\Rightarrow 2^{y}-1=4$

$\Rightarrow 2^{y}=5$

$\Rightarrow y=\log_2 5$

Question: $x_1, x_2, x_3, x_4, x_5 \in \{1, 2, \ldots, 18\}$ are arranged such that $x_1 > x_2 > x_3 > x_4 > x_5$ then find probability of $x_2 = 7$ & $x_4 = 11$.

Answer: $\frac{126}{{}^{18}C_5}$

Solution:

Total number of cases $= {}^{18}C_5$

$\underline{\quad} \quad \underline{7} \quad \underline{\quad} \quad \underline{11} \quad \underline{\quad}$

$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$

${}^{6}C_1 \qquad {}^{3}C_1 \qquad {}^{7}C_1$

Favourable number of cases $= 6\times 3\times 7 = 126$

Probability $= \frac{126}{{}^{18}C_5}$

Question: Find coefficient of x^{10} in $\left(\frac{\sqrt{x}}{5^{\frac{1}{4}}}+\frac{\sqrt{5}}{x^{\frac{1}{3}}}\right)^{60}$ is $5^{n}\times l$, then $n=?$

Answer: 6.00

Solution:

Given, $\left(\frac{\sqrt{x}}{5^{\frac{1}{4}}}+\frac{\sqrt{5}}{x^{\frac{1}{3}}}\right)^{60}$

$T_{r+1} = {}^{60}C_r \left(\frac{\sqrt{x}}{5^{\frac{1}{4}}}\right)^{60-r}\left(\frac{\sqrt{5}}{x^{\frac{1}{3}}}\right)^{r}$

$= {}^{60}C_r\, 5^{\frac{r-60}{4}}\, x^{\frac{60-r}{2}}\, 5^{\frac{r}{2}}\, x^{\frac{-r}{3}}$

$= {}^{60}C_r\, 5^{\frac{3r-60}{4}} \cdot x^{\frac{180-5r}{6}}$

Now, we need x^{10},

$\therefore \dfrac{180-5r}{6}=10$

$180-5r=60$

$120=5r$

$r=24$

$\therefore$ coefficient of x^{10} will be $={}^{60}C_{24}\,5^3$

$=\dfrac{60!}{24!36!}\cdot 5^3$

Now, exponent of 5 in $60!=\left[\dfrac{60}{5}\right]+\left[\dfrac{60}{5^2}\right]+\left[\dfrac{60}{5^3}\right]+\ldots$

$=12+2+0=14$

Exponent of 5 in $24!=\left[\dfrac{24}{5}\right]+\left[\dfrac{24}{5^2}\right]=4$

Exponent of 5 in $36!=\left[\dfrac{36}{5}\right]+\left[\dfrac{36}{5^2}\right]=7$

$\therefore$ Coefficient of x^{10} will be $=l\times 5^6$

$\therefore n=6$

Question: We have 11 identical blue balls & 5 red balls. Find number of ways to arrange these 16 balls such that minimum 2 balls are kept in between 2 red balls.

Answer: ${}^8C_5\cdot 5!$

Solution:

Number of ways to arrange blue ball

$x_1+x_2+x_3+x_4+x_5+x_6=11$

$x_1\ge 0,\ x_2\ge 2, x_3\ge 2,\ x_4\ge 2,\ x_5\ge 2,\ x_6\ge 0$

$t_2=x_2-2,\ t_3=x_3-2,\ t_4=x_4-2$

$\Rightarrow x_1+t_2+t_3+t_4+t_5+x_6\ge 3$

$\Rightarrow {}^{3+6-1}C_{6-1}={}^8C_5$

Ways to arrange red ball $=5!$

Total number of ways $={}^8C_5\cdot 5!$

Question: $\int_{2}^{-2}\dfrac{\left|x^3-x\right|}{e^{|x|x}+1}=?$

Answer: 6.00

Solution:

$I=\int_{-2}^{2}\dfrac{|x|\left(x^2+1\right)}{1+e^{x|x|}}dx \qquad \ldots(\text{i})$

$$I = \int_{-2}^{2} \frac{|x|\left(x^2+1\right)}{1+e^{-x|x|}} dx \qquad ....(\text{ii})$$

Adding (i) & (ii),

$$2I = \int_{-2}^{2} \frac{|x|\left(x^2+1\right)\left(1+e^{x|x|}\right)}{1+e^{x|x|}} dx$$

$$\Rightarrow 2I = 2\int_{0}^{2} \left(x^3+x\right) dx$$

$$\Rightarrow I = \left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^2 = 4+2=6$$