STA
2019

1. (a) Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers converging to a finite real number a, as $n \to \infty$. Define

$$y_n = \begin{cases} x_n - \frac{1}{n} & \text{if } n = 3k, \\ 2x_n & \text{if } n = 3k-1, \\ \frac{3x_n+1}{3|x_n|+1} & \text{if } n = 3k-2, \end{cases}$$

for $k = 1, 2, \ldots$. Find the values of a for which the sequence has three distinct limit points.
(b) Let $a_n = \int_0^1 (1-x^2)^n dx$ for $n \geq 1$. Find $\lim_{n\to\infty} \sqrt[n]{a_n}$. Also show that $\sum_{n=1}^{\infty} a_n$ diverges. [6+6]=12

2. Let f be a real-valued, continuous, strictly increasing function defined on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. Let g be the inverse function of f. Prove that

$$\int_0^1 f(x)dx + \int_0^1 g(y)dy = 1.$$

[12]

3. (a) Let $\boldsymbol{x}$ be the $n \times 1$ vector with $x_i = i$ for $i = 1, \ldots, n$. Find the determinant of $I + \boldsymbol{x}\boldsymbol{x}^T$, where I is the identity matrix of order n.
(b) Let A and G be matrices of order $m \times n$ and $n \times m$, respectively, such that $AGA = A$. Show that the determinant of $I + AG$ is non-zero.

[6+6]=12

4. Suppose that two distinct positive integers are chosen randomly from 1 to 50. What is the probability that their difference is divisible by 3? [12]

5. Consider a bivariate random vector (X_1, X_2) having joint density given by

$$f(x_1, x_2) = \begin{cases} 4x_1x_2, & \text{if } 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Define two new random variables as $Y_1 = X_1/X_2$ and $Y_2 = X_1X_2$.

a) Find the support of the joint distribution of (Y_1, Y_2) and represent it graphically.

b) Derive the joint distribution of (Y_1, Y_2).

c) Using (b), or otherwise, calculate $P(Y_1 > 2, Y_2 > 1/4)$.

[4+3+5]=12

6. Suppose $X_1, X_2, \ldots$ are independent and identically distributed random variables with $P(X_i = 1) = \frac{1}{4} = P(X_i = -1)$ and $P(X_i = 0) = \frac{1}{2}$ for all $i = 1, 2, \ldots$. Define $S_n = \sum_{i=1}^{n} X_i$ and $U_n = \text{sgn}(S_n)$, for $n \geq 1$, where the sgn function is given by

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Find $\lim_{n\to\infty} P(U_n \leq u)$ for $u \in \mathbb{R}$, and hence identify the limiting distribution of U_n. [12]

7. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed having a common density f_θ where $\theta \in \Theta \subseteq \mathbb{R}$. Let $\widehat{\theta}$ be the unique maximum likelihood estimator (MLE) of θ; note that $\widehat{\theta}$ solves the likelihood equation. Let $T(X_1, \ldots, X_n)$ be an efficient estimator of $\tau(\theta)$, a one-to-one function of the parameter θ, in the sense that T is unbiased for $\tau(\theta)$ and its variance attains the corresponding Cramer-Rao lower bound. Prove that T must be an MLE of $\tau(\theta)$. [12]

8. Suppose that a n-variate random vector $X = (X_1, X_2, \ldots, X_n)^T \sim N_n(\mu, I_n - \frac{1}{n}J_n)$, where J_n is $n \times n$ matrix with each entry 1. Define the multivariate log-normal random vector $Y = (Y_1, Y_2, \ldots, Y_n)^T$ through the relations $X_k = \log Y_k$ for $k = 1, \ldots, n$. Prove that the covariance matrix of Y can be expressed as DBD for some $n \times n$ matrix B and some diagonal matrix D. Find these matrices. [12]

9. It is believed that the number of daily hospital admissions on the average depends on whether it is in weekdays or in weekends. Based on records of daily admission counts from a hospital over 52 weeks, suggest a suitable model and corresponding analysis to test this belief. Write your notation and all the assumptions clearly, including limitation(s), if any.

[10+2]=12

10. Consider a randomized block design with v treatments and b blocks with $v < b$. But, suppose that observations under treatment i in block i are missing for $i = 1, 2, \ldots, v$, and the resulting design is denoted by $\mathcal{D}$.

a) What are block sizes in $\mathcal{D}$? Are the treatments equi-replicated in $\mathcal{D}$?

b) Is the design $\mathcal{D}$ connected? Justify your answer.

c) Using (b), or otherwise, prove that the design $\mathcal{D}$ is not orthogonal.

[2+4+6]=12