JEE-Main-24-06-2022-Shift-2 (Memory Based)

MATHEMATICS

Question: Find the sum of roots of $\left(e^{2x}-4\right)\left(6e^{2x}-5e^{x}+1\right)=0$

Options:
(a) $\ln 6$
(b) $\ln 3$
(c) $-\ln 3$
(d) $\ln 2$
Answer: (c)
Solution:
Let $e^{x}=t \Rightarrow \left(t^{2}-4\right)\left(6t^{2}-5t+1\right)=0$

$\left(t^{2}-4\right)(3t-1)(2t-1)=0$

$\Rightarrow t=-2,\frac{1}{3},\frac{1}{2},2$

$\because t>0 \Rightarrow \therefore t=\frac{1}{3},\frac{1}{2},2 \Rightarrow e^{x}=\frac{1}{3},\frac{1}{2},2$

$x=\ln\left(\frac{1}{3}\right),\ln\left(\frac{1}{2}\right),\ln(2)$

$\therefore$ sum $=\ln\left[\frac{1}{3}\times\frac{1}{2}\times 2\right]=\ln\left(\frac{1}{3}\right)=-\ln 3$

Question: $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n^{2}}{\left(n^{2}+r^{2}\right)(n+r)}$:

Options:
(a) $\frac{1}{4}\ln 2+\frac{\pi}{8}$
(b) $\frac{1}{4}\ln 2-\frac{\pi}{8}$
(c) $-\frac{1}{4}\ln 2-\frac{\pi}{8}$
(d)
Answer: (a)
Solution:
$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\left[1+\left(\frac{r}{n}\right)^{2}\right]\left[1+\left(\frac{r}{n}\right)\right]}=\int_{0}^{1}\frac{dx}{\left(1+x^{2}\right)(1+x)}$$

$$=\frac{1}{2}\int_0^1\frac{dx}{1+x}-\frac{1}{2}\int_0^1\left(\frac{x-1}{x^2+1}\right)dx$$

$$=\frac{1}{2}\left[\log(1+x)-\frac{1}{2}\log(1+x^2)+\tan^{-1}(x)\right]_0^1$$

$$=\frac{1}{2}\left[\log 2-\frac{1}{2}\log 2+\frac{\pi}{4}\right]$$

$$=\frac{1}{4}\log 2+\frac{\pi}{8}$$

Question: If $\Delta_r=\begin{vmatrix}2^{r-1} & \dfrac{(r+1)!}{\left(1+\dfrac{1}{r}\right)} & 4r^3-2nr \\ a & b & c \\ 2^n-1 & (n+1)!-1 & n^3(n+1)\end{vmatrix}, n\in N$ then $\sum_{r=1}^{n}\Delta_r=$

Options:

(a) abc

(b) $(n+3)!$

(c) 0

(d) $a(n!)+b.2^n+c$

Answer: (c)

Solution:

$$\sum_{r=1}^{n}2^{r-1}=1+2+2^2+\ldots+2^{n-1}=1(2^n-1)=2^n-1$$

$$\sum_{r=1}^{n}(4r^3-2nr)=4\left[\frac{n(n+1)}{2}\right]^2-2n\left[\frac{n(n+1)}{2}\right]$$

$$=n^2(n+1)^2-n^2(n+1)$$

$$=n^3(n+1)$$

$$\sum_{r=1}^{n}\frac{(r+1)!}{1+\frac{1}{r}}=\sum_{r=1}^{n}r\times r!=\sum_{r=1}^{n}[(r+1)!-r!]$$

$$=(n+1)!-1$$

$$\sum_{r=1}^{n}\Delta_r=\begin{vmatrix}\sum_{r=1}^{n}2^{r-1} & \sum_{r=1}^{n}\frac{(r+1)!}{\left(1+\frac{1}{r}\right)} & \sum_{r=1}^{n}\left(4^3-2nr\right)\\ a & b & c\\ 2^n-1 & (n+1)!-1 & n^3(n+1)\end{vmatrix}$$

$$=\begin{vmatrix}2^n-1 & (n+1)!-1 & n^3(n+1)\\ a & b & x\\ 2^n-1 & (n+1)!-1 & n^3(n+1)\end{vmatrix}$$

$=0$

Question: Find remainder when: $1+3+3^2+3^3+....+3^{2021}$ is divided by 50.

Answer: 4.00

Solution:

$1+3+3^2+3^3+....+3^{2021}=\dfrac{3^{2022}-1}{2}=\dfrac{9^{1011}-1}{2}$

$=\dfrac{(10-1)^{1011}-1}{2}=\dfrac{{}^{1011}C_0(10)^{1011}+.....-{}^{1011}C_{1009}(10)^2+10110-1-1}{2}$

$=\dfrac{100k+10108}{2}=50k+5054=50p+4$

$\therefore$ Remainder $=4$

Question: $1,2,3,4,5,6,9\rightarrow$ seven digit number multiple of 11?

Answer: 432.00

Solution:

Sum of digit at even place $=x$

Sum of digit at odd place $=y$

$\because x+y=30,\ \therefore x-y=0$

$x-y=\pm 11$

Only possibility is $x+y=30$ and $x-y=0$

$\therefore x=15,\ y=15$

So digit at even place is $\{1,5,9\},\{2,4,9\},\{4,5,6\}$

Odd place is $\{2,3,4,6\},\{1,3,5,6\},\{1,2,3,9\}$ respectively.

$\therefore$ Total number $=3(3!\cdot 4!)=18\times 24=144\times 3=432$

Question: $\cos\left(x+\dfrac{\pi}{3}\right)\cos\left(x-\dfrac{\pi}{3}\right)=\dfrac{1}{4}\cos^2(2x)$. Find number of solutions $[-3\pi,3\pi]$?

Answer: 7.00

Solution:

Given, $\cos\left(x+\frac{\pi}{3}\right)\cos\left(x-\frac{\pi}{3}\right)=\frac{1}{4}\cos^2(2x)$

$\Rightarrow \cos^2 x-\sin^2\frac{\pi}{3}=\frac{1}{4}\cos^2 2x$

$\Rightarrow \cos^2 x-\frac{3}{4}=\frac{1}{4}\cos^2 2x$

$\Rightarrow 4\cos^2 x-3=\cos^2 2x$

$\Rightarrow 4\cos^2 x-3=\left(2\cos^2 x-1\right)^2$

$\Rightarrow 4\cos^2 x-3=4\cos^4 x+1-4\cos^2 x$

$\Rightarrow 4\cos^2 x-8\cos^2 x+4=0$

$\Rightarrow \cos^4 x-2\cos^2 x+1=0$

$\Rightarrow \left(\cos^2 x-1\right)^2=0$

$\Rightarrow \cos^2 x=1$

$\cos x=\pm 1$

In $[-3\pi,3\pi]$

$\cos x=1,$ at $x=0,-2\pi,2\pi$

$\cos=-1,\ at\ x=\pi,-\pi,-3\pi,3\pi$

So, total 7 solutions

Question: $x^7-2x+3=p(x)$. Find number of real roots.

Answer: 1.00

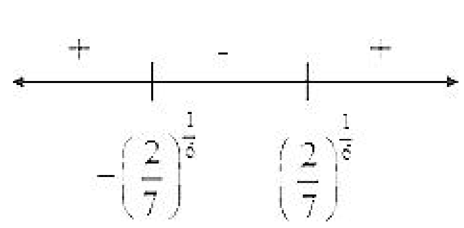
Solution:

$p(x)=x^7-2x+3$

$p'(x)=7x^6-2$

$p'(x)=0\Rightarrow x=\pm\left(\frac{2}{7}\right)^{\frac{1}{6}}$

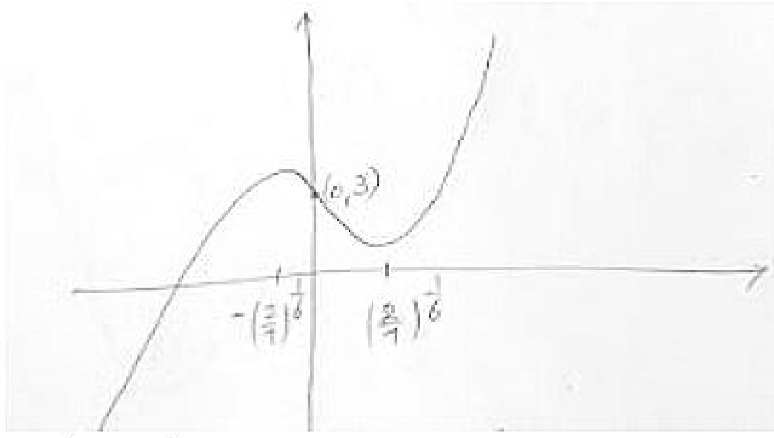
$p'(x)\rightarrow$



$p''(x)=42x^5$

$p''(x)>0$ for $x>0$ and $p''(x)<0$ for $x<0$

Graph of $p(x)$



$$p\left(\left(\frac{2}{7}\right)^{\frac{1}{6}}\right)=\left(\frac{2}{7}\right)^{\frac{7}{6}}-2\left(\frac{2}{7}\right)^{\frac{1}{6}}+3>0$$

$$p\left(-\left(\frac{2}{7}\right)^{\frac{1}{6}}\right)=-\left(\frac{2}{7}\right)^{\frac{7}{6}}+2\left(\frac{2}{7}\right)^{\frac{1}{6}}+3>0$$

Hence, number of real roots = 1

Question: Let a triangle of maximum area be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{4}=1,$ such that the area is $6\sqrt{3}.$ Find the eccentricity.

Answer: $\frac{\sqrt{3}}{2}$

Solution:

Maximum area of triangle inscribed in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{3\sqrt{3}}{4}ab$

$$\therefore\ \frac{3\sqrt{3}}{4}a\cdot 2=6\sqrt{3}\Rightarrow a=4$$

$$\therefore\ e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{12}{16}}=\frac{\sqrt{3}}{2}$$

Question: $(x-h)^2+(y-k)^2=r^2$ the arc touches x-axis at (1, 0) and $k>0$. $x+y=0$ intersect arc at 2 points chord length 2 unit. Find value of $h+k+r$?

Answer: 7.00

Solution:

$(x-h)^2+(y-k)^2=r^2$ touches x-axis at (1, 0)

$\therefore (1-h)^2+k^2=r^2\Rightarrow h^2+k^2-2h+1=r^2$

$\therefore h=1$ and $k=r$

Distance from $(1,k)$ on $x+y=0$ is $d=\left|\frac{1+k}{\sqrt{2}}\right|$

$$\therefore r^2=1+d^2=1+\left(\frac{1+r}{\sqrt{2}}\right)^2$$

$$2\left(r^2-1\right)=r^2+1+2r$$

$$r^2-2r-3=0$$

$$\Rightarrow (r-3)(r+1)=0$$

$$\Rightarrow r=3=k$$

$$\therefore\ h+k+r=1+3+3=7$$

Question: If S is given as $\{S:1,2,3,4....100\}$ then find the number of value of S such that {H.C.F of 24 & S is 1}

Answer: 1633.00

Solution:

Sum $=(1+2+....+100)-(2+4+....100)-(3+6+9.....99)+(6+12+18+....96)$

$$=\left(\frac{100\times101}{2}\right)-\left(\frac{2\times50\times51}{2}\right)-\left(\frac{3\times33\times34}{2}\right)+6\left(\frac{16\times17}{2}\right)$$

$$=5050-2550-1683+816=1633$$

Question: If it is given that $x*y=x^2+y^3$. Now if $(x*1)*1$ and $x*(1*1)$ both are equal then find the value of $2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$?

Answer: $\frac{\pi}{3}$

Solution:

$$\left(x^2+1^3\right)^2+1^3=x\left(1^2+1^3\right)=x^2+2^3$$

$$x^4+2+2x^2=x^2+8$$

$$x^4+x^2-6=0$$

$$\Rightarrow \left(x^2+3\right)\left(x^2-2\right)=0$$

$$\Rightarrow x=\pm\sqrt{2}$$

$$\therefore 2\sin^{-1}\left[\frac{4+2-2}{4+2+2}\right]=2\sin^{-1}\left[\frac{4}{8}\right]=2\times\frac{\pi}{6}=\frac{\pi}{3}$$

Question: pdf of x is

x	0	1	2	3	4
$P(x)$	k	$2k$	$3k$	$4k$	$5k$

Find $P\left(\frac{1<x<4}{x\le 2}\right)$

Answer: $\frac{1}{2}$

Solution:

$$P\left(\frac{1<x<4}{x\le 2}\right)=\frac{P(x=2)}{P(x\le 2)}$$

$$=\frac{P(x=2)}{P(x=0)+P(x=1)+P(x=2)}$$

$$=\frac{3k}{k+2k+3k}$$

$$=\frac{3k}{6k}=\frac{1}{2}$$

Question: Find minimum of $3x+4y$; $x, y \in R^+$ such that $x^3y^2=2^{13}$

Answer: 40.00

Solution:

$$x^3\cdot y^2=2^{13}$$

$$\therefore\ 3x+4y \Rightarrow \frac{x+x+x+2y+2y}{5}\ge\sqrt[5]{x^3.4y^2}$$

$$\Rightarrow \frac{3x+4y}{5}\ge\sqrt[5]{2^{15}}\ge 2^3\ge 8$$

$$\Rightarrow 3x+4y\ge 40$$

$$\Rightarrow (3x+4y)_{\min}=40$$

Question: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dx}{(e^x+1)(\sin^6 x+\cos^6 x)}$

Answer: π

Solution:

Given, $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dx}{(e^x+1)(\sin^6 x+\cos^6 x)}$ ....(1)

Applying property

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\left(e^{-x}+1\right)\left(\sin^6 x+\cos^6 x\right)}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x dx}{\left(e^x+1\right)\left(\sin^6 x+\cos^6 x\right)} \qquad ....(2)$$

Adding (1) and (2)

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(e^x+1\right)dx}{\left(e^x+1\right)\left(\sin^6 x+\cos^6 x\right)}$$

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{\sin^6 x+\cos^6 x}$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{1-3\sin^2 x\cos^2 x}$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{1-\frac{3}{4}\left(\sin 2x\right)^2}$$

Put $2x = t$

$2dx = dt$

$$I = \frac{1}{2}\int_0^{\pi} \frac{dt}{1-\frac{3}{4}\sin^2 t}$$

$$I = 2\int_0^{\pi} \frac{dt}{4-3\sin^2 t}$$

$$I = 2\int_0^{\pi} \frac{\sec^2 dt}{4\sec^2 t-3\tan^2 t}$$

$$= 2\int_0^{\pi} \frac{\sec^2 t\,dt}{4+\tan^2 t}$$

$$I = 2\cdot 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 t\,dt}{4+\tan^2 t}$$

$$I = 4\int_0^{\frac{\pi}{2}} \frac{\sec^2 t\,dt}{4+\tan^2 t}$$

$$I = 4\left[\frac{1}{2}\tan^{-1}\left(\tan t\right)\right]_0^{\frac{\pi}{2}}$$

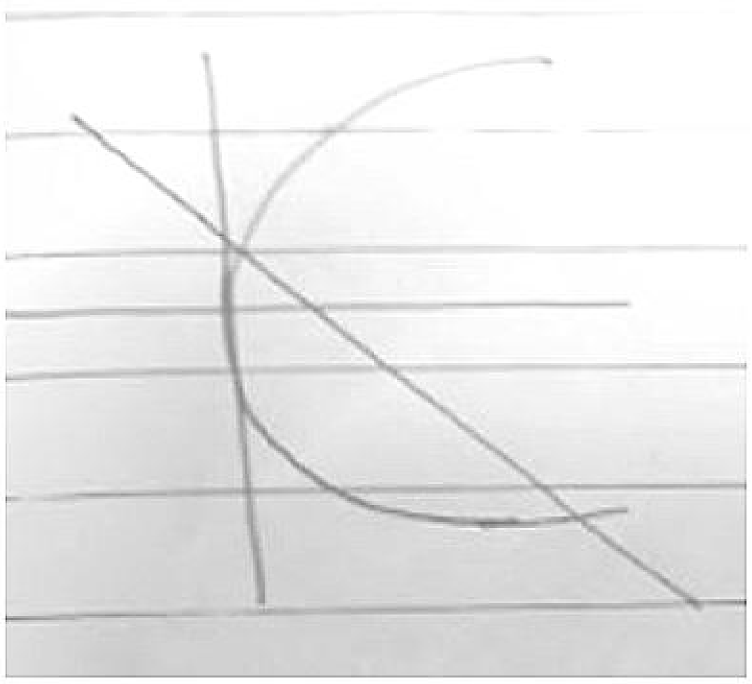
$$= 2\left[\frac{\pi}{2}-0\right]$$

$= \pi$

Question: Find area bounded by $y^2 = 2x$ & $x + y = 4$

Answer: 18.00

Solution:



$y^2 = 2(4-y) \Rightarrow y^2 + 2y - 8 = 0$

$(y+4)(y-2) = 0 \Rightarrow y = -4, 2$

$\therefore$ Area $= \int_{-4}^{2} \left((4-y) - \frac{y^2}{2} \right) dy$

$= \left[4y - \frac{y^2}{2} - \frac{y^3}{6} \right]_{-4}^{2}$

$= 24 + 6 - 12$

$= 18$