VECTOR ALGEBRA

$1 + 2 + 2 + 3 + 3 = 11m$

definition   section formula derivation

Vector : A quantity that has magnitude and direction.

Eg: $\xrightarrow{\vec{a}}$ from A (initial point) to B (terminal point)

$\vec{a}$ has the initial point A moving towards the terminal point B and hence $\vec{a} = \vec{AB}$ (not $\vec{BA}$).

$\vec{a}$ in 3D space has 3 components and is written as

$\vec{a} = a_1 i + a_2 j + a_3 k$

Here : Scalar components : a_1, a_2, a_3

Vector components ; $a_1 i, a_2 j, a_3 k$

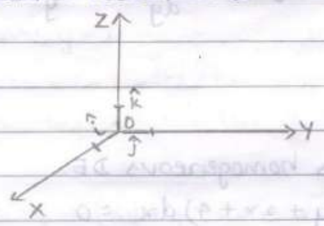
If $\vec{a} = a_1 i + a_2 j + a_3 k$, $\vec{b} = b_1 i + b_2 j + b_3 k$

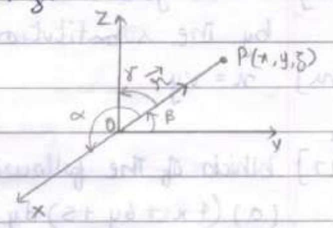
then they are position vectors of points A and B, then

$\vec{AB} = \vec{OB} - \vec{OA} = (b_1 - a_1) i + (b_2 - a_2) j + (b_3 - a_3) k$

$\vec{BA} = \vec{OA} - \vec{OB} = (a_1 - b_1) i + (a_2 - b_2) j + (a_3 - b_3) k$

Consider a point $P(x, y, z)$ in a 3D space. Then its position vector is $\vec{OP} = xi + yj + zk$





- $\vec{r} = xi + yj + zk = \vec{OP}$
- $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$
- $\hat{i}$ is unit vector on x-axis, $\hat{j}$ is unit vector on y-axis and $\hat{k}$ is unit vector on z-axis.
- The direction angles of any vector made with the coordinate axes are α, β, γ respectively

- cosines of these angles give the direction cosines (DCs) of the given vector and are represented by l, m, n.
  Hence, $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$
- $l^2 + m^2 + n^2 = 1$
  $\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
- $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$
- For a given vector: $\vec{a} = a_1 i + a_2 j + a_3 k$
  → $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
  → Direction ratios (Dr's):
  $a = a_1$, $b = a_2$, $c = a_3$
  → Unit vector $= \hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

  → DC's: $\dfrac{a_1}{|\vec{a}|}, \dfrac{a_2}{|\vec{a}|}, \dfrac{a_3}{|\vec{a}|}$

Eg: If $\vec{a} = i + j + k$, $\vec{b} = 2i - j + 2k$, $\vec{c} = 2i + 3j - 6k$

(i) $|\vec{a}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$|\vec{b}| = \sqrt{4+1+4} = \sqrt{9} = 3$ $\boxed{|(1,2,2)| = 3}$

$|\vec{c}| = \sqrt{4+9+36} = \sqrt{49} = 7$ $\boxed{|2,3,6| = 7}$

(ii) $3\vec{a} - 2\vec{c} = 3i + 3j + 3k - (4i + 6j - 12k)$
$= -i - 3j + 15k$

(iii) Unit vector of $\vec{c} - \vec{a}$

$\vec{c} - \vec{a} = i + 2j - 7k$

Unit vect of $\vec{c} - \vec{a} = \dfrac{\vec{c} - \vec{a}}{|\vec{c} - \vec{a}|} = \dfrac{i + 2j - 7k}{\sqrt{1+4+49}} = \dfrac{i + 2j - 7k}{\sqrt{54}}$

DC's are $= \dfrac{1}{\sqrt{54}}, \dfrac{2}{\sqrt{54}}, \dfrac{-7}{\sqrt{54}}$

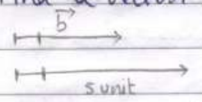
(iv) Find unit vector of $\vec{a}$ and hence its DC's

Unit vector of $\vec{a} = \hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{i + j + k}{\sqrt{3}}$

DC's are: $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

** Since the DC's are all equal, the vector $i+j+k$ is equally inclined to all the coordinate axes i.e, $\alpha = \beta = \gamma$

(v) Find a vector in the direction of $\vec{b}$ of magnitude 5.



vector along $\vec{b}$ of magnitude $5 = 5\,\hat{b}$

$$= 5\frac{\vec{b}}{|\vec{b}|} = 5\frac{(2i-j+2k)}{\sqrt{4+1+4}}$$

$$= \frac{10i-5j+10k}{3}$$

** Types of vectors:

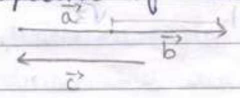
Zero vector: A vector whose initial and terminal points coincide is called a zero vector or a null vector and is denoted as $\vec{0}$. It may be regarded as having any direction.

$\vec{AA}$, $\vec{BB}$ ... represent $\vec{0}$.

Unit vector: A vector whose magnitude is unity (1 unit) is called a unit vector. The unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$.

Coinitial vectors: Two or more vectors having the same initial points are called coinitial vectors.

** Collinear vectors: Two or more vectors are said to be collinear is they are parallel to the same line, irrespective of their magnitudes and directions.



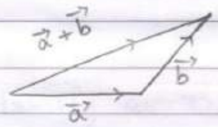
Parallel vectors :

** Negative of a vector : A vector whose magnitude is same as that of a given vector but has opposite direction of given vector, is called negative of the given vector.

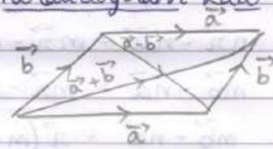
If given vector is $\overrightarrow{AB}$ then its negative vector is $\overrightarrow{BA}$ and is written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Equal vectors : Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal if they have same magnitude and direction regardless of positions of their initial points and is written as $\vec{A} = \vec{B}$.

Triangle Law of addition :



Parallelogram Law of addition :



** Section formula :

(a) For internal division : $\vec{r} = \dfrac{m\vec{b}+n\vec{a}}{m+n}$

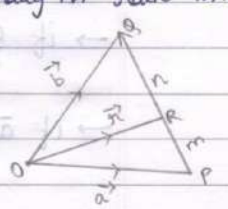
(b) For external division : $\vec{r} = \dfrac{m\vec{b}-n\vec{a}}{m-n}$

Let P, Q be two points with PV's $\vec{a}, \vec{b}$ and let point R with PV $\vec{r}$ divide $\overrightarrow{PQ}$ in the ratio m:n.

case (i) : When point R divides $\overrightarrow{PQ}$ internally in ratio m:n.

$$\frac{\overrightarrow{PR}}{\overrightarrow{RQ}} = \frac{m}{n}$$

$$\frac{\overrightarrow{OR}-\overrightarrow{OP}}{\overrightarrow{OQ}-\overrightarrow{OR}} = \frac{m}{n}$$



$$\frac{\vec{r}-\vec{a}}{\vec{b}-\vec{r}} = \frac{m}{n}$$

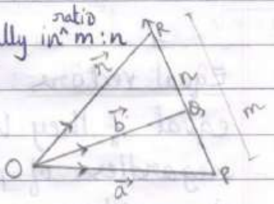
$$n\vec{r}-n\vec{a} = m\vec{b}-m\vec{r}$$

$$m\vec{r}+n\vec{r} = m\vec{b}+n\vec{a}$$

$$\vec{r}(m+n) = m\vec{b}+n\vec{a}$$

$$\boxed{\vec{r} = \frac{m\vec{b}+n\vec{a}}{m+n}}$$

case (ii): When R divides $\vec{PQ}$ externally in ratio $m:n$



$$\frac{\vec{PR}}{\vec{QR}} = \frac{m}{n}$$

$$\frac{\vec{OR}-\vec{OP}}{\vec{OR}-\vec{OQ}} = \frac{m}{n}$$

$$\frac{\vec{r}-\vec{a}}{\vec{r}-\vec{b}} = \frac{m}{n}$$

$$n\vec{r}-n\vec{a} = m\vec{r}-m\vec{b}$$

$$m\vec{b}-n\vec{a} = m\vec{r}-n\vec{r}$$

$$m\vec{b}-n\vec{a} = \vec{r}(m-n)$$

$$\boxed{\vec{r} = \frac{m\vec{b}-n\vec{a}}{m-n}}$$

NOTE:
* $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ [vector addition is commutative]
* $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ [vector addition is associative]
* zero vector is identity element under vector addition
  i.e. $\vec{a}+\vec{0} = \vec{0}+\vec{a} = \vec{a}$

** * Consider two vectors:
  $\vec{a} = a_1 i + a_2 j + a_3 k$, $\vec{b} = b_1 i + b_2 j + b_3 k$ then
  $\rightarrow$ if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \Rightarrow \vec{a} \parallel \vec{b}$

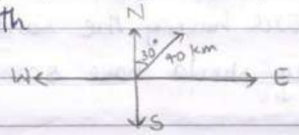
  $\rightarrow$ if $\vec{a} = \lambda\vec{b} \Rightarrow \vec{a} \parallel \vec{b}$ and
  $\vec{a}, \vec{b}$ are collinear vectors.

Date :
Page No :

* In the vector : $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
- a_1, a_2, a_3 are direction ratios of $\vec{a}$ and are called scalar components.
- $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ = magnitude of $\vec{a}$
- $\frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|}$ are DC's of $\vec{a}$
- Unit vector : $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

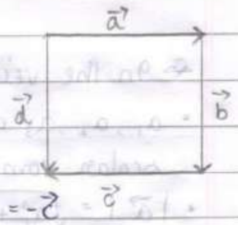
EXERCISE 10.1

1] Represent graphically a displacement of 40 km, 30° east of north



2] Classify as scalars and vectors:
i] 10 kg - weight - scalar
ii] 2 m, north-west - displacement - vector
iii] 40° - temperature - scalar
iv] 40 W - power - scalar
v] 10^{-19} c - charge - scalar
vi] 20 m/s² - acceleration - vector

3] Classify as scalars and vectors:
i] Time period - scalar
ii] Distance - scalar
iii] Force - vector
iv] Velocity - vector
v] Work done - scalar



4] Identify the vectors in diagram :
i] coinitial vectors : $\vec{a}$ & $\vec{d}$
ii] equal vectors : $\vec{b}$, $\vec{d}$
iii] collinear but not equal : $\vec{a}, \vec{c}$ $\because \vec{a} = -\vec{c}$

5] True or False :
i] Vector $\vec{a}$ and $-\vec{a}$ are collinear – T
ii] Two collinear vectors are always equal in magnitude – F
Eg: $\vec{a} = i+j+k$; $\vec{b} = 2i+2j+2k$ $\Rightarrow \vec{b} = 2\vec{a}$
$\therefore \vec{a} \parallel \vec{b} \Rightarrow \vec{a}, \vec{b}$ are collinear with different magnitudes
iii] Two vectors having same magnitude are collinear – F
$\because$ they may have different directions.
iv] Two collinear vectors having the same magnitude are equal – F ($\because$ they should have same direction also)

EXERCISE 10.2

Eg 4] Find the values of x, y, z so that the vectors $\vec{a} = xi+2j+zk$ and $\vec{b} = 2i+yj+k$ are equal.
Sol] $x = 2$, $y = 2$, $z = 1$

Eg 5] Let $\vec{a} = i+2j$, $\vec{b} = 2i+j$. is ~~ma~~ $|\vec{a}| = |\vec{b}|$; are $\vec{a}$ and $\vec{b}$ equal ?
Sol] $$\left.\begin{aligned} |\vec{a}| &= \sqrt{1+4} = \sqrt{5} \\ |\vec{b}| &= \sqrt{4+1} = \sqrt{5} \end{aligned}\right\} \Rightarrow |\vec{a}| = |\vec{b}| = \sqrt{5}$$
$\vec{a} \neq \vec{b}$ $\because$ their components are different.

Eg 6] Find unit vector in the direction of $\vec{a} = 2i+3j+k$
unit vector : $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{2i+3j+k}{\sqrt{4+9+1}} = \dfrac{2i+3j+k}{\sqrt{14}}$
$$= \frac{2}{\sqrt{14}}i + \frac{3}{\sqrt{14}}j + \frac{1}{\sqrt{14}}k$$

** Eg 7] Find a vector in the direction of $\vec{a} = i - 2j$ that has magnitude 7 units.

Sol] Vector along $\vec{a}$ of magnitude 7 $= 7\hat{a} = 7\frac{\vec{a}}{|\vec{a}|}$

$$= \frac{7(i-2j)}{\sqrt{1+4}} = \frac{7i - 14j}{\sqrt{5}} = \boxed{\frac{7}{\sqrt{5}}i - \frac{14}{\sqrt{5}}j}$$

Eg 8) Write the Dr's of $\vec{a} = i + j - 2k$ and hence calculate its DC's.

Sol] $|\vec{a}| = \sqrt{1+1+4} = \sqrt{6}$

(i) DR's are : $a = 1, b = 1, c = -2$

(ii) DC's are : $l = \frac{a_1}{|\vec{a}|} = \boxed{\frac{1}{\sqrt{6}}}$, $m = \frac{a_2}{|\vec{a}|} = \boxed{\frac{1}{\sqrt{6}}}$, $n = \frac{a_3}{|\vec{a}|} = \boxed{\frac{-2}{\sqrt{6}}}$

$\Rightarrow$ DC's are $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}$

Eg 8] Find unit vector in the direction of sum of the vectors $\vec{a} = 2i + 2j - 5k$, $\vec{b} = 2i + j + 3k$.

Sol] Sum $= \vec{a} + \vec{b}$

$= 4i + 3j - 2k$

Unit vector of $\vec{a} + \vec{b} = \frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \frac{4i + 3j - 2k}{\sqrt{16+9+4}} = \frac{4i + 3j - 2k}{\sqrt{29}}$

(1m) Eg 10] Find vector joining the points P(2,3,0) and Q(−1,−2,−4) directed from P to Q.

Sol] $\vec{PQ} = \vec{OQ} - \vec{OP} = -3i - 5j - 4k$

Eg 11] Consider points P, Q with PV's $\vec{OP} = 3\vec{a} - 2\vec{b}$ and $\vec{OQ} = \vec{a} + \vec{b}$. Find PV of R that divides $\vec{PQ}$ in the ratio 2:1 (i) internally (ii) externally.

Sol] (i) For internal division

$$\vec{r} = \frac{m\vec{b}+n\vec{a}}{m+n}$$

$$\vec{r} = \frac{2(\vec{a}+\vec{b})+1(3\vec{a}-2\vec{b})}{2+1} = \frac{5\vec{a}+0\vec{b}}{3} = \frac{5\vec{a}}{3}$$

(ii) For external division : $\vec{r} = \frac{m\vec{b}-n\vec{a}}{m-n}$

$$\vec{r} = \frac{2(\vec{a}+\vec{b})-1(3\vec{a}-2\vec{b})}{2-1} = \frac{-\vec{a}+4\vec{b}}{1} = -\vec{a}+4\vec{b}$$

(2m) Eg 12] ST the points $A(2i-j+k)$, $B(i-3j-5k)$, $C(3i-4j-4k)$ are vertices of a right angle Δ^{le}.

Sol] $\vec{AB} = \vec{OB} - \vec{OA} = -i-2j-6k$; $|\vec{AB}| = \sqrt{1+4+36} = \sqrt{41}$

$\vec{BC} = \vec{OC} - \vec{OB} = 2i-j+k$; $|\vec{BC}| = \sqrt{4+1+1} = \sqrt{6}$

$\vec{CA} = \vec{OA} - \vec{OC} = -i+3j+5k$; $|\vec{CA}| = \sqrt{1+9+25} = \sqrt{35}$

Clearly, $(\sqrt{35})^2 + (\sqrt{6})^2 = (\sqrt{41})^2$

$\Rightarrow |\vec{CA}|^2 + |\vec{BC}|^2 = |\vec{AB}|^2$

$\Rightarrow ABC$ is a right angle Δ^{le}.

17] ST the points A, B, C with PV's $\vec{OA} = 3i-4j-4k$, $\vec{OB} = 2i-j+k$, $\vec{OC} = i-3j-5k$ form the vertices of $\perp\Delta^{le}$.

$\vec{AB} = \vec{OB} - \vec{OA} = -i+3j+5k$; $|\vec{AB}| = \sqrt{1+9+25} = \sqrt{35}$

$\vec{BC} = \vec{OC} - \vec{OB} = -i-2j-6k$; $|\vec{BC}| = \sqrt{1+4+36} = \sqrt{41}$

$\vec{CA} = \vec{OA} - \vec{OC} = 2i-j+k$; $|\vec{CA}| = \sqrt{4+1+1} = \sqrt{6}$

Clearly, $(\sqrt{35})^2 + (\sqrt{6})^2 = (\sqrt{41})^2$

$\Rightarrow |\vec{AB}|^2 + |\vec{CA}|^2 = |\vec{BC}|^2$

$\Rightarrow ABC$ is a right angle Δ^{le}.

1] Compute the mag of vectors:

i] $\vec{a} = i+j+k \Rightarrow |\vec{a}| = \sqrt{1+1+1} = \sqrt{3}$

ii] $\vec{b} = 2i-7j-3k \Rightarrow |\vec{b}| = \sqrt{4+49+9} = \sqrt{62}$

iii] $\vec{c} = \frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k \Rightarrow |\vec{c}| = \sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 1$

2] Write two different vectors having same magnitude.

$\vec{a} = 2i-j$ and $\vec{b} = i+2j$

3] Write two different vectors with same direction.

$\vec{a} = i+2j+3k$ and $\vec{b} = 2i+4j+6k$

$\vec{a} = \lambda \vec{b}$

4] Find the values of x, y so that the vectors $\vec{a} = 2i+3j$ and $\vec{b} = xi+yj$ are equal.

$x = 2, y = 3$

5] Find the scalar and vector components of vector with initial point $(2,1)$ and terminal point $(-5,7)$.

Sol] $\vec{AB} = \vec{OB} - \vec{OA} = -7i+6j$

Scalar components are : $-7, 6$

Vector " " : $-7i, 6j$

6] Find the sum of the vectors $\vec{a} = i-2j+k$, $\vec{b} = -2i+4j+5k$, $\vec{c} = i-6j-7k$

Sum $= \vec{a}+\vec{b}+\vec{c} = 0i-4j-k = -4j-k$

7] Find the unit vector in the direction of $\vec{a} = i+j+2k$.

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{i+j+2k}{\sqrt{1+1+4}} = \frac{1}{\sqrt{6}}i + \frac{1}{\sqrt{6}}j + \frac{2}{\sqrt{6}}k$

8] Find the unit vector in the direction of $\vec{PQ}$ with $P(1,2,3)$, $Q \equiv (4,5,6)$

Sol] $\vec{PQ} = \vec{OQ} - \vec{OP} = 3i + 3j + 3k = 3(i+j+k)$

unit vector along $\vec{PQ} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{3(i+j+k)}{3\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k$

9] ~~Let~~ $\vec{a} = 2i - j + 2k$, $\vec{b} = -i + j - k$ find the unit vector in the direction of $\vec{a} + \vec{b}$.

Sol] $\vec{a} + \vec{b} = i + k$

unit vector of $\vec{a} + \vec{b} = \frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \frac{i+k}{\sqrt{2}} = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}k$

10] Find a vector in the direction of $5i - j + 2k$ which has magnitude 8 units.

Sol] Let $\vec{a} = 5i - j + 2k$

~~unit~~ vector along $\vec{a}$ ~~along~~ $= 8\hat{a} = \frac{8|\vec{a}|}{|\vec{a}|} = \frac{8(5i - j + 2k)}{\sqrt{25+1+4}}$

$= \frac{40}{\sqrt{30}}i - \frac{8}{\sqrt{30}}j + \frac{16}{\sqrt{30}}k$

11] ST the vectors $\vec{a} = 2i - 3j + 4k$ and $\vec{b} = -4i + 6j - 8k$ are collinear.

Clearly, $\vec{b} = -2(\vec{a}) \Rightarrow \vec{a}, \vec{b}$ are collinear vectors.

(or) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{-1}{2} \Rightarrow \vec{a} \parallel \vec{b}$ or $\vec{a}, \vec{b}$ are collinear.

12] Find the DC's of the vector $i + 2j + 3k$.

$|\vec{a}| = \sqrt{1+4+9} = \sqrt{14}$

DC's are : $l = \frac{1}{\sqrt{14}}$, $m = \frac{2}{\sqrt{14}}$, $n = \frac{3}{\sqrt{14}}$

13] Find DC's of the vector joining $A(1,2,-3)$ and $B(-1,-2,1)$ directed from A to B.

Sol] $\vec{AB} = \vec{OB} - \vec{OA} = -2i - 4j + 4k$

$\vec{AB} = 2(-i - 2j + 2k)$

$|\vec{AB}| = 2\sqrt{1+4+4} = 6$

DC's are $-\frac{2}{6}, \frac{-4}{6}, \frac{4}{6}$ i.e, $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$

14] ST the vector $i+j+k$ is equally inclined to the axis OX, OY and OZ.

Sol] Gn: $\vec{a} = i + j + k$

$|\vec{a}| = \sqrt{1+1+1} = \sqrt{3}$

DC's are: $l = \frac{1}{\sqrt{3}}$, $m = \frac{1}{\sqrt{3}}$, $n = \frac{1}{\sqrt{3}}$

$\Rightarrow$ DC's are all equal i.e $\boxed{\alpha = \beta = \gamma}$

$\therefore$ The gn. vector is equally inclined to the coordinate axes

15] Find the PV of pt R which divides line joining P and Q with PV's $i + 2j - k$ and $-i + j + k$ resp. in the ratio 2 : 1
(a) internally (b) externally.

Sol] Gn: $\vec{OP} = \vec{a} = i + 2j - k$; $\vec{OQ} = \vec{b} = -i + j + k$; $m : n = 2 : 1$

(i) Internal division:

$$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n} = \frac{2(-i+j+k) + 1(i+2j-k)}{2+1}$$

$$\vec{r} = \frac{-i + 4j + k}{3}$$

(ii) Externally:

$$\vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n} = \frac{2(-i+j+k) - 1(i+2j-k)}{2-1}$$

$$\vec{r} = \frac{-3i + 3k}{1}$$

16] Find the PV of mid point of the vector joining $P(2,3,4)$ and $Q(4,1,-2)$.

Sol] Gn: $\vec{a} = \vec{OP} = 2i + 3j + 4k$

$\vec{b} = \vec{OQ} = 4i + j - 2k$

mid point $= \vec{r} = \dfrac{\vec{a}+\vec{b}}{2} = \dfrac{6i + 4j + 2k}{2} = 3i + 2j + k$

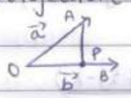
<u>Product of Two vectors</u>:

SCALAR PRODUCT OR DOT PRODUCT :

* Dot product of two ~~pro~~ vectors gives a constant and hence called scalar product.
* $\forall \vec{a}, \vec{b} \in V,\ \vec{a}\cdot\vec{b} \notin V$

  $\therefore$ dot product is not binary operation on set of vectors V

  i.e. 'V' is not closed under '$\cdot$'.
* $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (dot product is commutative)
* $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
* $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
* ** $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$
* $\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$
* Projection of $\vec{a}$ onto $\vec{b}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$



; OP = projection

* $i\cdot i = j\cdot j = k\cdot k = 1$
* $i\cdot j = j\cdot k = k\cdot i = 0$
* $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

Eg13] Find the angle b/w 2 vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 when $\vec{a}\cdot\vec{b} = 1$

Sol] $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{1\cdot 2} = \frac{1}{2} = \cos 60°$

$\Rightarrow \theta = \frac{\pi}{3}$

Eg14] Find θ b/w the vectors $\vec{a} = i+j-k$, $\vec{b} = i-j+k$

Sol) $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(i+j-k)\cdot(i-j+k)}{\sqrt{1+1+1}\cdot\sqrt{1+1+1}}$

$\cos\theta = \frac{1-1-1}{\sqrt{3}\cdot\sqrt{3}} = \frac{-1}{3}$

$\Rightarrow \theta = \cos^{-1}\left(-\frac{1}{3}\right)$

Eg15] If $\vec{a} = 5i-j-3k$, $\vec{b} = i+3j-5k$. ST the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are $\perp^{le}$

Sol] Gn: $\vec{a} = 5i-j-3k$; $\vec{b} = i+3j-5k$

$\vec{a}+\vec{b} = 6i+2j-8k$

$\vec{a}-\vec{b} = 4i-4j+2k$

$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 24-8-16 = 24-24 = 0$

$\Rightarrow (\vec{a}+\vec{b}) \perp (\vec{a}-\vec{b})$

Eg16] Find the projection of $\vec{a} = 2i+3j+2k$ on the $\vec{b} = i+2j+k$

Sol] Projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \frac{(2i+3j+2k)\cdot(i+2j+k)}{\sqrt{1+4+1}}$

$= \frac{2+6+2}{\sqrt{6}} = \frac{10}{\sqrt{6}}$ units

Eg17] Find $|\vec{a}-\vec{b}|$ if two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a}\cdot\vec{b} = 4$

Sol] Consider $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$

$= 2^2 + 3^2 - 2(4)$

$= 4 + 9 - 8 = 5$

$\therefore |\vec{a}-\vec{b}| = \sqrt{5}$

Eg 18] If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 8$. Find $|\vec{x}|$

Sol] Gn: $|\vec{a}| = 1$

$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 8$

$|\vec{x}|^2 - |\vec{a}|^2 = 8$

$|\vec{x}|^2 - 1 = 8$

$|\vec{x}|^2 = 9$

$|\vec{x}| = 3$

Eg 21] ST the points $A(-2i+3j+5k)$, $B(i+2j+3k)$, $C(7i-k)$ are collinear.

Sol] $\vec{AB} = \vec{OB} - \vec{OA} = 3i - j - 2k$; $|\vec{AB}| = \sqrt{9+1+4} = \sqrt{14}$

$\vec{BC} = \vec{OC} - \vec{OB} = 6i - 2j - 4k$; $|\vec{BC}| = \sqrt{36+4+16} = 2\sqrt{14}$

$\vec{CA} = \vec{OA} - \vec{OC} = -9i + 3j + 6k$; $|\vec{CA}| = \sqrt{81+9+36} = 3\sqrt{14}$

Clearly, $\sqrt{14} + 2\sqrt{14} = 3\sqrt{14}$

$\Rightarrow |\vec{AB}| + |\vec{BC}| = |\vec{CA}|$

$\Rightarrow$ gn. points are collinear.

(OR)

DR's of $\vec{AB}$ and $\vec{BC}$ are proportional

$\Rightarrow \vec{AB} \parallel \vec{BC}$ but pt. 'B' is common

$\therefore$ The gn. pts are collinear.

*] If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ PT $\vec{a}$ & $\vec{b}$ are $\perp^{le}$

Sol] sbs $|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$

$|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$

$4\vec{a}\cdot\vec{b} = 0$

$\vec{a}\cdot\vec{b} = 0$

$\Rightarrow \vec{a}$ is $\perp^{le}$ $\vec{b}$

EXERCISE 10.3

1] Find the angle b/w 2 vectors $\vec{a}$ and $\vec{b}$ with mag. $\sqrt{3}$ & 2, and $\vec{a}\cdot\vec{b} = \sqrt{6}$

Sol] Gn: $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$, $\vec{a}\cdot\vec{b} = \sqrt{6}$

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}\,2} = \frac{\sqrt{3}\sqrt{2}}{\sqrt{3}\,2} = \frac{1}{\sqrt{2}}$

$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

2] Find the angle b/w the vectors $i-2j+3k$ and $3i-2j+k$

Sol] Gn: $\vec{a} = i-2j+3k$, $\vec{b} = 3i-2j+k$

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{3+4+3}{\sqrt{1+4+9}\sqrt{9+4+1}} = \frac{10}{14} = \frac{5}{7}$

$\theta = \cos^{-1}\left(\frac{5}{7}\right)$

3] Find the projection of the vector $i-j$ on $i+j$

Sol) Gn: $\vec{a} = i-j$, $\vec{b} = i+j$

Proj of $\vec{a}$ on $\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{1-1}{\sqrt{2}} = 0$

4] Find the proj. of the vector $i+3j+7k$ on $7i-j+8k$.

Sol] Gn: $\vec{a} = i+3j+7k$, $\vec{b} = 7i-j+8k$

Proj. of $\vec{a}$ on $\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}$

$\begin{array}{r} 65 \\ 49 \\ \hline 114 \end{array}$

5] ST each of the 3 vectors is a unit vector, also show that they are mutually $\perp^{r}$ to each other.

$\underbrace{\frac{1}{7}(2i+3j+6k)}_{\vec{a}}$, $\underbrace{\frac{1}{7}(3i-6j+2k)}_{\vec{b}}$, $\underbrace{\frac{1}{7}(6i+2j-3k)}_{\vec{c}}$

$|\vec{a}| = \frac{1}{7}\sqrt{4+9+36} = \frac{\sqrt{49}}{7} = \frac{7}{7} = 1 \Rightarrow \vec{a}$ is unit vector.

$|\vec{b}| = \frac{1}{7}\sqrt{9+36+4} = \frac{\sqrt{49}}{7} = \frac{7}{7} = 1 \Rightarrow \vec{b}$ is unit vector

$|\vec{c}| = \frac{1}{7}\sqrt{36+4+9} = \frac{\sqrt{49}}{7} = \frac{7}{7} = 1 \Rightarrow \vec{c}$ is unit vector.

Consider $\vec{a}\cdot\vec{b} = \frac{1}{49}(6-18+12) = \frac{18-18}{49} = 0 \Rightarrow \vec{a}\perp\vec{b} \rightarrow ①$

$\vec{b}\cdot\vec{c} = \frac{1}{49}(18-12-6) = 0 \Rightarrow \vec{b}\perp\vec{c} \rightarrow ②$

$\vec{c}\cdot\vec{a} = \frac{1}{49}(12+6-18) = 0 \Rightarrow \vec{c}\perp\vec{a} \rightarrow ③$

①, ②, ③ $\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are mutually $\perp^{le}$.

6] $|\vec{a}|, |\vec{b}|$ if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$

Sol] Gn: $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 8$

$|\vec{a}|^2 - |\vec{b}|^2 = 8$

$(8|\vec{b}|)^2 - |\vec{b}|^2 = 8$

$64|\vec{b}|^2 - |\vec{b}|^2 = 8$

$63|\vec{b}|^2 = 8$

$|\vec{b}| = \sqrt{\frac{8}{63}}$ units

$|\vec{a}| = 8|\vec{b}|$

$= 8\sqrt{\frac{8}{63}}$ units

7] Evaluate $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$

Sol] $= 6(\vec{a}\cdot\vec{a}) + 21(\vec{a}\cdot\vec{b}) - 10(\vec{b}\cdot\vec{a}) - 35(\vec{b}\cdot\vec{b})$

$= 6|\vec{a}|^2 + 21(\vec{a}\cdot\vec{b}) - 10(\vec{a}\cdot\vec{b}) - 35|\vec{b}|^2$

$= 6|\vec{a}|^2 + 11(\vec{a}\cdot\vec{b}) - 35|\vec{b}|^2$

8] Find mag. of 2 vectors $\vec{a}$ and $\vec{b}$ having the same mag such that angle b/w them is 60° and their scalar product is half.

Date :
Page No :

Sol] $|\vec{a}| = |\vec{b}|$, $\theta = 60^\circ$, $\vec{a}\cdot\vec{b} = \frac{1}{2}$

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\cos 60^\circ = \frac{1}{2|\vec{a}||\vec{a}|} = \frac{1}{2|\vec{a}|^2}$

$\frac{|\vec{a}|^2}{\cancel{2}} = \frac{1}{\cancel{2}}$

$|\vec{a}| = 1 = |\vec{b}|$

9] Find $|\vec{x}|$ if for a unit vector $\vec{a}$ $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 12$

Sol] Gn: $|\vec{a}| = 1$

$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 12$

$|\vec{x}|^2 - |\vec{a}|^2 = 12$

$|\vec{x}|^2 = 13$

$|\vec{x}| = \sqrt{13}$

**10] If $\vec{a} = 2i+2j+3k$, $\vec{b} = -i+2j+k$, $\vec{c} = 3i+j$ such that $(\vec{a}+\lambda\vec{b}) \perp \vec{c}$. Find λ.

Sol] $(\vec{a}+\lambda\vec{b}) \perp \vec{c} \Rightarrow (\vec{a}+\lambda\vec{b})\cdot\vec{c} = 0$

$[(2i+2j+3k)+\lambda(-i+2j+k)]\cdot(3i+j) = 0$

$[(2-\lambda)i + (2+2\lambda)j + (3+\lambda)k]\cdot(3i+j) = 0$

$3(2-\lambda) + 1(2+2\lambda) = 0$

$6-3\lambda+2+2\lambda = 0$

$8-\lambda = 0$

$\boxed{\lambda = 8}$

**13] If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c} = 0$. Find the value of $(\vec{a}\cdot\vec{b})+(\vec{b}\cdot\vec{c})+(\vec{c}\cdot\vec{a})$

Sol] Gn: $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

$\vec{a}+\vec{b}+\vec{c} = 0$

$(\vec{a}+\vec{b}+\vec{c})^2 = 0$ sbs

$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$(1+1+1)+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\therefore\ \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=\frac{-3}{2}$

16] ST the points $A(1,2,7)$, $B(2,6,3)$, $C(3,10,-1)$ are collinear

$\vec{AB}=\vec{OB}-\vec{OA}=i+4j-4k$; $|\vec{AB}|=\sqrt{1+16+16}=\sqrt{33}$

$\vec{BC}=\vec{OC}-\vec{OB}=i+4j-4k$; $|\vec{BC}|=\sqrt{1+16+16}=\sqrt{33}$

$\vec{CA}=\vec{OA}-\vec{OC}=2i+8i-8k$; $|\vec{CA}|=\sqrt{4+64+64}=2\sqrt{33}$

$\because \sqrt{33}+\sqrt{33}=2\sqrt{33}$

$|\vec{AB}|+|\vec{BC}|=|\vec{CA}|$

$\Rightarrow$ Gn pts are collinear.

17] ST the vectors $2i-j+k$, $i-3j-5k$ and $3i-4j-4k$ form the vertices of a $L^{le}\ \Delta^{le}$.

$|\vec{AB}|=\vec{OB}-\vec{OA}=-i-2j-6k$; $|\vec{AB}|=\sqrt{1+4+36}=\sqrt{41}$

$|\vec{BC}|=\vec{OC}-\vec{OB}=2i-j+k$; $|\vec{BC}|=\sqrt{4+1+1}=\sqrt{6}$

$|\vec{CA}|=\vec{OA}-\vec{OC}=-i+3j+5k$; $|\vec{CA}|=\sqrt{1+9+25}=\sqrt{35}$

$(\sqrt{35})^2+(\sqrt{6})^2=(\sqrt{41})^2$

$\Rightarrow |\vec{BC}|^2+|\vec{CA}|^2=|\vec{AB}|^2$

$\Rightarrow$ A, B, C are vertices of $L^{le}\ \Delta^{le}$

(1m) ** 18] If $\vec{a}$ is a non zero vector of magnitude "a" and λ a non-zero scalar, then $\lambda\vec{a}$ is a unit vector if

(a) $\lambda=1$ (b) $\lambda=-1$, (c) $a=|\lambda|$ (d) $a=\frac{1}{|\lambda|}$

$|\vec{a}|=a$, $\lambda\vec{a}$ is unit vector

$\Rightarrow |\lambda\vec{a}|=1$

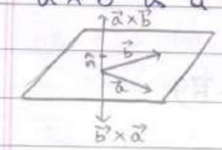
$|\lambda||\vec{a}|=1$

$|\lambda|a=1$

$a=\frac{1}{|\lambda|}$

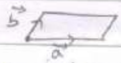
CROSS PRODUCT / VECTOR PRODUCT:

* Cross product of two vectors gives a vector and hence called vector product.
* $\forall\ \vec{a}, \vec{b} \in V,\ (\vec{a} \times \vec{b}) \in V$.
  $\therefore$ 'X' product is binary operation on V
  Also 'V' is closed under 'X' product.
* $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$
* $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$
* $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
* $\vec{a} \times \vec{b}$ is a vector $\perp^r$ to both $\vec{a}$ and $\vec{b}$



* Here vector $\vec{a} \times \vec{b}$ is $\perp^r$ to entire plane and hence $\perp^r$ to all the vectors present in the plane.
* $\hat{n}$ is unit vector on $\vec{a} \times \vec{b}$

* $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
* $\vec{a} \times \vec{a} = 0$
* $i \times i = j \times j = k \times k = 0$
* $i \times j = k$; $j \times k = i$; $k \times i = j$
  $j \times i = -k$; $k \times j = -i$; $i \times k = -j$
* $|\vec{a} \times \vec{b}|$ = Ar. of parallelogram



  (where $\vec{a}$ and $\vec{b}$ are adjacent sides)
* $\frac{1}{2}|\vec{a} \times \vec{b}|$ = Ar. of triangle.

* $\frac{1}{2}|\vec{AB} \times \vec{AC}|$ = Ar. of triangle (Gn: vertices A, B, C)

EXERCISE 10.4

Eg 22] Find $|\vec{a}\times\vec{b}|$ if $\vec{a} = 2i+j+3k$, $\vec{b} = 3i+5j-2k$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix} = i(-17) - j(-13) + k(7)$$

$$= -17i + 13j + 7k$$

$$|\vec{a}\times\vec{b}| = \sqrt{289+169+49} = \sqrt{507}$$

** Eg 23) Find a unit vector $\perp^{le}$ to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$

Gn: $\vec{a} = i+j+k$, $\vec{b} = i+2j+3k$

Sol] $\vec{a}+\vec{b} = 2i+3j+4k$

$\vec{a}-\vec{b} = 0i-j-2k$

Vector $\perp^{r}$ to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ $= (\vec{a}+\vec{b})\times(\vec{a}-\vec{b})$

$$= \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = i(-6+4) - j(-4) + k(-2)$$

$$= -2i + 4j - 2k$$

$$= 2(-i+2j-k)$$

Its unit vector $= \dfrac{\cancel{2}(-i+2j-k)}{\cancel{2}\sqrt{1+4+1}} = \dfrac{-i+2j-k}{\sqrt{6}}$

$$= \frac{-1}{\sqrt{6}}i + \frac{2}{\sqrt{6}}j - \frac{1}{\sqrt{6}}k$$

Eg 24] Find area of Δ^{le} having pts $A(1,1,1)$, $B(1,2,3)$, $C(2,3,1)$ as its vertices.

Sol] $\vec{AB} = \vec{OB} - \vec{OA} = 0i+j+2k$

$\vec{AC} = \vec{OC} - \vec{OA} = i+2j+0k$

Consider $\vec{AB}\times\vec{AC} = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = i(-4) - j(-2) + k(-1)$

$= -4i + 2j - k$

Ar of Δ^{le} = $\frac{1}{2}|\vec{AB} \times \vec{AC}|$ = $\frac{1}{2}\sqrt{16+4+1}$ = $\frac{\sqrt{21}}{2}$ sq. units

Eg 25] Find ar of llgm whose adjacent sides are $\vec{a} = 3i + j + 4k$, $\vec{b} = i - j + k$.

Sol] $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = i(5) - j(-1) + k(-4)$
$= 5i + j - 4k$

Ar of llgm = $|\vec{a} \times \vec{b}| = \sqrt{25+1+16} = \sqrt{42}$ sq. units

9] Find ar of Δ^{le} with vertices A(1, 1, 2), B(2, 3, 5), C(1, 5, 5).

Sol] Ar of Δ^{le} = $\frac{1}{2}|\vec{AB} \times \vec{AC}|$

$\vec{AB} = \vec{OB} - \vec{OA} = i + 2j + 3k$
$\vec{AC} = \vec{OC} - \vec{OA} = 0i + 4j + 3k$

$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix} = i(-6) - j(3) + k(4)$
$= -6i - 3j + 4k$

$|\vec{AB} \times \vec{AC}| = \sqrt{36+9+16} = \sqrt{61}$

$\therefore$ Ar of Δ^{le} = $\frac{1}{2}\sqrt{60}$ = $\frac{\sqrt{61}}{2}$ sq. units

10] Find ar of llgm whose adjacent sides are determined by $\vec{a} = i - j + 3k$, $\vec{b} = 2i - 7j + k$

Sol) $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = i(20) - j(-5) + k(-5)$
$= 20i + 5j - 5k$

Ar of llgm = $|\vec{a} \times \vec{b}| = 5\sqrt{\cancel{400}\ 16+1+1} = 5\sqrt{18}$ sq. units

12] Ar of a $\square^{le}$ having vertices A, B, C, D with PV's
$\underset{\vec{OA}}{-i + \frac{1}{2}j + 4k}$, $\underset{\vec{OB}}{i + \frac{1}{2}j + 4k}$, $\underset{\vec{OC}}{i - \frac{1}{2}j + 4k}$, $\underset{\vec{OD}}{-i - \frac{1}{2}j + 4k}$ is

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 4

$\vec{AB} = \vec{OB} - \vec{OA} = 2i + 0j + 0k = 2i$

$\vec{AC} = \vec{OC} - \vec{OA} = 0i - j + 0k = -j$

Ar of rec = Ar of llgm = $|\vec{AB} \times \vec{AC}|$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = i(0) + j(0) + k(-2) = -2k$$

Ar of llgm = $|\vec{AB} \times \vec{AC}| = \sqrt{4}$ sq. units $= 2$ sq. units

1] Find $|\vec{a} \times \vec{b}|$ if $\vec{a} = i - 7j + 7k$ $\vec{b} = 3i - 2j + 2k$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix} = i(0) - j(-19) + k(19)$$
$$= 19j + 19k$$
$$= 19(j + k)$$

$|\vec{a} \times \vec{b}| = \sqrt{(19)^2 + (19)^2} = 19\sqrt{2}$

2] Find unit vector $\perp^r$ to each of vector $(\vec{a} + \vec{b}), (\vec{a} - \vec{b})$.

Gn: $\vec{a} = 3i + 2j + 2k$, $\vec{b} = i + 2j - 2k$

Sol] $\vec{a} + \vec{b} = 4i + 4j + 0k$

$\vec{a} - \vec{b} = 2i + 0j + 4k$

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} i & j & k \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = i(16) - j(16) + k(-8)$$
$$= 16i - 16j - 8k$$

Its unit vector $= \frac{16i - 16j - 8k}{\sqrt{(16)^2 + (16)^2 + (8)^2}} = \frac{16i - 16j - 8k}{\sqrt{576}}$

$$= \frac{16i - 16j - 8k}{24}$$

$$= \frac{2}{3}i - \frac{2}{3}j - \frac{1}{3}k$$

Scalar triple product : (STP)

$(\cdot, \times)$, Box product $= [\vec{a}\ \vec{b}\ \vec{c}]$

The dot product of $\vec{a}$ and $\vec{b}\times\vec{c}$ is called STP written as

$\vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}]$

* $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} = [\vec{a}, \vec{b}, \vec{c}]$
* $[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
* $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}] = -[\vec{b}\ \vec{a}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$
* $[\vec{a}\ \vec{a}\ \vec{b}] = 0 = [\vec{a}\ \vec{b}\ \vec{b}] = [\vec{a}\ \vec{c}\ \vec{c}]$
* Geometrically $[\vec{a}\ \vec{b}\ \vec{c}]$ = Vol. of parallelopiped.
* $\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$ = Vol. of tetrahedron
* If $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, 3 vectors are coplanar.
* If $[\vec{AB}\ \vec{AC}\ \vec{AD}] = 0$, gn. 4 points are coplanar.

EXERCISE 10.5

1] Find $[\vec{a}\ \vec{b}\ \vec{c}]$ if $\vec{a} = i - 2j + 3k$, $\vec{b} = 2i - 3j + k$, $\vec{c} = 3i + j - 2k$

sol] $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \\ 3 & 1 & -2 \end{vmatrix} = 1(6-1) + 2(-4-3) + 3(2+9)$
$= 5 - 14 + 33$
$= 24$

2] ST the vectors $\vec{a} = i - 2j + 3k$, $\vec{b} = -2i + 3j - 4k$, $\vec{c} = i - 3j + 5k$ are coplanar.

sol] Consider $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 1(15-12) + 2(-10+4) + 3(6-3)$
$= 3 - 12 + 9$
$= 0$

$\because [\vec{a}\ \vec{b}\ \vec{c}] = 0$

Gn vectors are coplanar.

** 3] Find λ if the vectors $i-j+k$, $3i+j+2k$ and $i+\lambda j-3k$ are coplanar.

Sol] Gn: $\vec{a} = i-j+k$, $\vec{b} = 3i+j+2k$, $\vec{c} = i+\lambda j-3k$

Gn: $\vec{a}, \vec{b}, \vec{c}$ are coplanar

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\begin{vmatrix} 1 & -1 & 1 \\ 3 & 1 & 2 \\ 1 & \lambda & -3 \end{vmatrix} = 0$$

$1(-3-2\lambda) + 1(-11) + 1(3\lambda-1) = 0$

$-3-2\lambda-11+3\lambda-1 = 0$

$-15+\lambda = 0$

$\boxed{\lambda = 15}$

5] ST the four points with PV's $\overrightarrow{OA} = 4i+8j+12k$, $\overrightarrow{OB} = 2i+4j+6k$, $\overrightarrow{OC} = 3i+5j+4k$ and $\overrightarrow{OD} = 5i+8j+5k$ are coplanar.

Sol] $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2i-4j-6k$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -i-3j-8k$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = i+0j-7k$

Consider $[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \begin{vmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & -7 \end{vmatrix}$

$= 2\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 8 \\ 1 & 0 & -7 \end{vmatrix}$

$= 2\{1(-21) - 2(-15) + 3(-3)\}$

$= 2(-21+30-9) = 0$

$\because [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0$, gn 4 pts are coplanar.

** 6] Find x such that the 4 points $A(3,2,1)$, $B(4,x,5)$, $C(4,2,-2)$ and $D(6,5,-1)$ are coplanar.

Sol] $\vec{AB} = \vec{OB} - \vec{OA} = i + (x-2)j + 4k$

$\vec{AC} = \vec{OC} - \vec{OA} = i + 0j - 3k$

$\vec{AD} = \vec{OD} - \vec{OA} = 3i + 3j - 2k$

Given 4 pts are coplanar

$\Rightarrow (\vec{AB}\ \vec{AC}\ \vec{AD}] = 0$

$\Rightarrow \begin{vmatrix} 1 & x-2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix}$ $\Rightarrow 1(9) - (x-2)(7) + 4(3) = 0$

$\Rightarrow 9 - 7x + 14 + 12 = 0$

$\Rightarrow 35 - 7x = 0$

$\Rightarrow x = \frac{35}{7} = \boxed{5 = x}$

Eg 26] Find $\vec{a}\cdot(\vec{b}\times\vec{c})$ if $\vec{a} = 2i+j+3k$, $\vec{b} = -i+2j+k$, $\vec{c} = 3i+j+2$

Sol] $\vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

$= 2(4-1) - 1(-2-3) + 3(-1-6)$

$= 6 + 5 - 21$

$= -10$

Eg 28] Find λ if the vectors $\vec{a} = i+3j+k$, $\vec{b} = 2i-j-k$, $\vec{c} = \lambda i + 7j + 3k$ are coplanar.

Sol] Gn: 3 vectors are coplanar

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ \lambda & 7 & 3 \end{vmatrix} = 0$

$1(-3+7) - 3(6+\lambda) + 1(14+\lambda) = 0$

$4 - 18 - 3\lambda + 14 + \lambda = 0$

$-2\lambda = 0$

$\boxed{\lambda = 0}$

Eg 29) ST the 4 pts A, B, C, D with PV's $4i+5j+k$, $-(j+k)$, $3i+9j+4k$ and $4(-i+j+k)$ are coplanar.

Sol) Gn: $\vec{OA} = 4i+5j+k$

$\vec{OB} = -j-k$

$\vec{OC} = 3i+9j+4k$

$\vec{OD} = -4i+4j+4k$

$\vec{AB} = \vec{OB}-\vec{OA} = -4i-6j-2k$

$\vec{AC} = \vec{OC}-\vec{OA} = -i+4j+3k$

$\vec{AD} = \vec{OD}-\vec{OA} = -8i-j+3k$

Consider $[\vec{AB}\ \vec{AC}\ \vec{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$

$= -2\begin{vmatrix} 2 & 3 & 1 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$

$= -2\{2(12+3)-3(-3+24)+1(1+32)\}$

$= -2\{30-63+33\}$

$= 0$

Eg 30) PT $[(\vec{a}+\vec{b})(\vec{b}+\vec{c})(\vec{c}+\vec{d})] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

LHS $= [\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{d}]$

$= (\vec{a}+\vec{b})\cdot\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{d})\}$

$= (\vec{a}+\vec{b})\cdot\{(\vec{b}\times\vec{c})+(\vec{b}\times\vec{d})+(\vec{c}\times\vec{c})+(\vec{c}\times\vec{d})\}$

$= \vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{d})+\vec{a}\cdot(\vec{c}\times\vec{d})+$

$\vec{b}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{d})+\vec{b}\cdot(\vec{c}\times\vec{d})$

$= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{d}]+[\vec{a}\ \vec{c}\ \vec{d}]+[\vec{b}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{b}\ \vec{d}]+[\vec{b}\ \vec{c}\ \vec{d}]$

Eg 30) PT $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

LHS $= [\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$

$= (\vec{a}+\vec{b})\cdot\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$

$$= (\vec{a}+\vec{b})\cdot\{(\vec{b}\times\vec{c}) + (\vec{b}\times\vec{a}) + (\vec{c}\times\vec{c})^{0} + (\vec{c}\times\vec{a})\}$$

$$= \vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{a}\cdot(\vec{b}\times\vec{a}) + \vec{a}\cdot(\vec{c}\times\vec{a}) +$$
$$\vec{b}\cdot(\vec{b}\times\vec{c}) + \vec{b}\cdot(\vec{b}\times\vec{a}) + \vec{b}\cdot(\vec{c}\times\vec{a})$$

$$= [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{a}\,\vec{b}\,\vec{a}] + [\vec{a}\,\vec{c}\,\vec{a}] + [\vec{b}\,\vec{b}\,\vec{c}]^{0} + [\vec{b}\,\vec{b}\,\vec{a}]^{0} + [\vec{b}\,\vec{c}\,\vec{a}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{a}\,\vec{b}\,\vec{c}]$$

$$= 2[\vec{a}\,\vec{b}\,\vec{c}]$$

= RHS

Eg 31] PT $[\vec{a}\ \vec{b}\ (\vec{c}+\vec{d})] = [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{d}]$

$$\text{LHS} = [\vec{a}\ \vec{b}\ \ \vec{c}+\vec{d}]$$
$$= \vec{a}\cdot[\vec{b}\times(\vec{c}+\vec{d})]$$
$$= \vec{a}\cdot\{(\vec{b}\times\vec{c}) + (\vec{b}\times\vec{d})\}$$
$$= \vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{a}\cdot(\vec{b}\times\vec{d})$$
$$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{d}]$$

= RHS

Eg 29] 3 vectors $\vec{a}, \vec{b}, \vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=0$
Evaluate the quantity $\mu = (\vec{a}\cdot\vec{b}) + (\vec{b}\cdot\vec{c}) + (\vec{c}\cdot\vec{a})$ if $|\vec{a}| = 1$
$|\vec{b}| = 4$ and $|\vec{c}| = 2$.

Sol] Gn: $\vec{a}+\vec{b}+\vec{c} = 0$

Squaring on both sides,

$$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$$
$$(1 + 4^2 + 2^2) + 2\mu = 0$$
$$21 + 2\mu = 0$$
$$\mu = \frac{-21}{2}$$

Eg 28] Let $\vec{a}, \vec{b}, \vec{c}$ be 3 vectors such that $|\vec{a}| = 3$, $|\vec{b}| = 4$, $|\vec{c}| = 5$ and each one of them being $\perp^{r}$ to the sum of the other two. Find $|\vec{a}+\vec{b}+\vec{c}|$.

Sol] Gn: Each vector is ⊥^{r} to sum of the other two

$\vec{a} \perp (\vec{b}+\vec{c})$	$\vec{b} \perp (\vec{c}+\vec{a})$	$\vec{c} \perp^{r} (\vec{a}+\vec{b})$
$\vec{a}\cdot(\vec{b}+\vec{c}) = 0$	$\vec{b}\cdot(\vec{c}+\vec{a}) = 0$	$\vec{c}\cdot(\vec{a}+\vec{b}) = 0$
$\vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = 0 \rightarrow ①$	$\vec{b}\cdot\vec{c} + \vec{b}\cdot\vec{a} = 0 \rightarrow ②$	$\vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} = 0 \rightarrow ③$

① + ② + ③ => $2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0 \rightarrow ④$

Consider $|(\vec{a}+\vec{b}+\vec{c})|^2$

$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\overbrace{(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})}^{0}$$

$\therefore |\vec{a}+\vec{b}+\vec{c}|^2 = 9 + 16 + 25 = 50$

$|\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} + 5\sqrt{2}$

*] PT the vectors $\vec{a}-\vec{b}$, $\vec{b}-\vec{c}$, $\vec{c}-\vec{a}$ are coplanar.

Sol] $[\vec{a}-\vec{b} \quad \vec{b}-\vec{c} \quad \vec{c}-\vec{a}]$

$= (\vec{a}-\vec{b})\cdot[(\vec{b}-\vec{c})\times(\vec{c}-\vec{a})]$

$= (\vec{a}-\vec{b})\cdot[(\vec{b}\times\vec{c}) - (\vec{b}\times\vec{a}) - 0 + (\vec{c}\times\vec{a})]$

$= \vec{a}(\vec{b}\times\vec{c}) - \cancel{\vec{a}(\vec{b}\times\vec{a})}^{0} + \cancel{\vec{a}(\vec{a}\times\vec{a})}^{0} - \cancel{\vec{b}(\vec{b}\times\vec{c})}^{0} + \cancel{\vec{b}(\vec{b}\times\vec{a})}^{0} - \vec{b}(\vec{c}\times\vec{a}$

$= \vec{a}(\vec{b}\times\vec{c}) - \vec{b}(\vec{c}\times\vec{a})$

$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}]$

$= 0$

* $|\vec{a}\times\vec{b}|^2 + |\vec{a}\cdot\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$