MTA
2019

Notation

$\mathbb{Z}$ = the set of integers

$\mathbb{N} = \{n \in \mathbb{Z} : n \geq 1\}$

$\mathbb{R}$ = the set of real numbers

$\mathbb{Q}$ = the set of rational numbers

$\mathbb{C}$ = the set of complex numbers

---

(1) (a) Let $\Delta = \{(t_0, t_1, t_2) \in \mathbb{R}^3 : t_0 + t_1 + t_2 = 1 \text{ and } t_i \geq 0 \text{ for } i = 0, 1, 2\}$. Prove that the function $f : [0,1] \times [0,1] \to \Delta$ defined by

$$f(x_1, x_2) = \begin{cases} (x_1, x_2 - x_1, 1 - x_2) & \text{if } x_1 \leq x_2 \\ (x_2, x_1 - x_2, 1 - x_1) & \text{if } x_2 \leq x_1 \end{cases}$$

is continuous.

(b) Prove that $f(A \times B)$ is closed if A and B are closed subsets of $[0, 1]$.

(2) Show that

$$\lim_{n\to\infty} \int_\alpha^\infty \sqrt{n} e^{-nx^2} \mathrm{d}x = \int_\alpha^\infty \lim_{n\to\infty} \sqrt{n} e^{-nx^2} \mathrm{d}x$$

for $\alpha > 0$ but not for $\alpha = 0$.

(3) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function such that f has a zero of order $N \geq 1$ at the origin. Show that $|f(z)| \leq |z|^N$ for all $z \in \mathbb{D}$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that $f(x) = 0$ if and only if $x \in \mathbb{Z}$. Suppose the function $x : \mathbb{R} \to \mathbb{R}$ satisfies $x'(t) = f(x(t))$ for all $t \in \mathbb{R}$.
(a) If $\mathbb{Z} \cap \{x(t) : t \in \mathbb{R}\}$ is non-empty, then show that x is constant.
(b) If $\mathbb{Z} \cap \{x(t) : t \in \mathbb{R}\}$ is the empty set, then show that $\lim_{t\to\infty} x(t)$ exists and is an integer.

(5) (a) Let X be a Banach space, $x_0 \in X$ and $\varphi_0 \in X^*$. Define $T : X^* \to X^*$ by $T(\psi) = \psi(x_0)\varphi_0$ for $\psi \in X^*$. Prove that T is compact.

(b) Using part (a) or otherwise, prove that given a two-variable polynomial function a, the operator $A : L^\infty([0,1], m) \to L^\infty([0,1], m)$ (where m denotes the Lebesgue measure) defined by

$$Af(x) = \int_0^1 a(x,y) f(y) \mathrm{d}y$$

is compact.

(6) Let $f : \mathbb{R} \to [0,1]$ be a continuously differentiable function satisfying $f^2(x) + (f'(x))^2 \geq 1$ for all $x \in \mathbb{R}$ and $f(0) = 1$. Prove that there exists $t > 0$ such that $f'(t) = 0$.
Hint: Note that f attains its maximum at 0.

(7) Let $\{x_n\}_{n\geq 1}$ be a sequence in $\mathbb{R}$ and $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers satisfying $a_n \uparrow \infty$ as $n \to \infty$. Further, suppose $\sum_{n=1}^{\infty} \frac{x_n}{a_n}$ converges. Then, show that $\frac{1}{a_n} \sum_{k=1}^{n} x_k \longrightarrow 0$ as $n \longrightarrow \infty$.

(8) If f is an entire function such that $\iint_{\mathbb{R}^2} |f(x+iy)| \, dx dy < \infty$, then prove that $f \equiv 0$.

(9) Let $f : [0,1] \to \mathbb{R}$ be a monotonically increasing function (not necessarily continuous) with $f(0) = 0$ and $f(1) = 1$. Suppose μ denotes the Borel measure on $[0,1]$ such that $\mu((a,b])$ is the cardinality of the set

$$\left\{ x \in [0,1] \,\middle|\, a < \lim_{h \to 0^+} [f(x+h) - f(x)] \leq b \right\} \quad \text{for all } 0 \leq a < b \leq 1 .$$

Prove that $\int_0^1 t^p \, \mathrm{d}\mu < \infty$ for all $p > 1$.

Hint: Consider $\int_0^1 t^{p-1} \mu([t,1]) \, \mathrm{d}t$.

(10) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K}$ be a closed subspace of $\mathcal{H}$. Given any bounded linear functional $f : \mathcal{K} \to \mathbb{C}$, prove that there exists a **unique** extension $\widetilde{f} : \mathcal{H} \to \mathbb{C}$ of f as a bounded linear functional satisfying $\left\| \widetilde{f} \right\| = \|f\|$.