

MATHEMATICS

1. An urn contains 6 black and 9 red balls. Four balls are drawn from the urn twice without replacement. The probability that first four balls are black & 2nd four balls are red in colour is:

(1) $\frac{3}{765}$ (2) $\frac{6}{715}$ (3) $\frac{3}{715}$ (4) $\frac{6}{615}$

Ans. (3)

Sol. $\frac{{}^6C_4}{{}^{15}C_4} \times \frac{{}^9C_4}{{}^{11}C_4} = \frac{3}{715}$

2. A line $x + y = 0$ touches the circle $(x - \alpha)^2 + (y - \beta)^2 = 50$, $\alpha, \beta, > 0$. The distance of origin from its points of contact is $4\sqrt{2}$. Find $\alpha^2 + \beta^2$.

Ans. 82

Sol. Point of contact is $(0 + 4\sqrt{2} \cos 135^\circ, 0 + 4\sqrt{2} \sin 135^\circ) = (-4, 4)$

$$\left| \frac{\alpha + \beta}{\sqrt{2}} \right| = 5\sqrt{2} \quad \alpha + \beta = 10 \dots(i)$$

$$(-4 - \alpha)^2 + (4 - \beta)^2 = 50$$

$$(\alpha + 4)^2 + (4 - 10 + \alpha)^2 = 50$$

$$(\alpha + 4)^2 + (\alpha - 6)^2 = 50$$

$$\alpha = 1, \beta = 9$$

$$\alpha^2 + \beta^2 = 82$$

Also point of contact is $(4, -4)$

Satisfying this point of contact in the equation of circle we get

$$(4 - \alpha)^2 + (-4 - \beta)^2 = 50$$

$$(4 - \alpha)^2 + (\beta + 4)^2 = 50$$

$$(4 - \alpha)^2 + (14 - \alpha)^2 = 50$$

$$\Rightarrow \alpha = 9, \beta = 1$$

$$\alpha^2 + \beta^2 = 82$$

3. Let $2\tan^2x - 5\secx - 1 = 0$ has 7 solutions in $x \in \left[0, \frac{n\pi}{2}\right]$, then the minimum value of n is N find

$$\sum_{k=1}^N \frac{k}{2^k}$$

(1) $2 \cdot \left(\frac{2^{13}-1}{2^{13}}\right) - \frac{13}{2^{13}}$

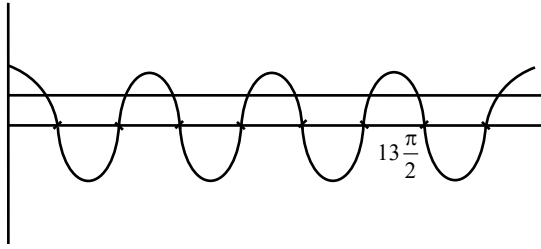
(2) $\left(\frac{2^{13}-1}{2^{13}}\right) - \frac{13}{2^{13}}$

(3) $2 \cdot \left(\frac{2^{13}-1}{2^{13}}\right) + \frac{13}{2^{14}}$

(4) $2 \cdot \left(\frac{2^{13}-1}{2^{13}}\right) - \frac{13}{2^{14}}$

Ans. (1)

Sol.



$$2\tan^2 x - 5\sec x - 1 = 0$$

$$2\sec^2 x - 5\sec x - 3 = 0$$

$$2\sec^2 x - 6\sec x + \sec x - 3 = 0$$

$$(2\sec x + 1)(\sec x - 3) = 0$$

$$\sec x = 3, -\frac{1}{2}$$

$$\Rightarrow \sec x = 3$$

$$\Rightarrow \cos x = \frac{1}{3}$$

For 7 solutions, $n = 13 = N$

$$\text{so } \sum_{k=1}^{13} \frac{k}{2^k}$$

$$\text{let } S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{13}{2^{13}}$$

$$\frac{1}{2}S = \frac{1}{2^2} + \frac{2}{2^3} + \dots + \frac{13}{2^{14}}$$

$$\frac{1}{2}S = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{13}} - \frac{13}{2^{14}}$$

$$\frac{1}{2}S = \frac{1}{2} \cdot \frac{\left(1 - \frac{1}{2^{13}}\right)}{1 - \frac{1}{2}} - \frac{13}{2^{14}}$$

$$S = 2 \cdot \left(\frac{2^{13} - 1}{2^{13}}\right) - \frac{13}{2^{13}}$$

4. The vertices of a triangle are $A(1, 2, 2)$, $B(2, 1, 2)$ & $C(2, 2, 1)$. The perpendicular distance of its orthocentre from the given sides are l_1 , l_2 & l_3 . Find the value of $l_1^2 + l_2^2 + l_3^2$.

- (1) 1 (2) $\frac{1}{2}$ (3) $\frac{1}{3}$ (4) $\frac{1}{4}$

Ans. (2)

Sol. ΔABC is equilateral

\therefore orthocentre & centroid will be same $\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$

midpoint of AB is $\left(\frac{3}{2}, \frac{3}{2}, 2\right)$

$$\Rightarrow \ell_1 = \sqrt{\frac{1}{36} + \frac{1}{36} + \frac{1}{9}}$$

$$= \frac{1}{\sqrt{6}} = \ell_2 = \ell_3$$

5. Let two sets A and B having 'm' & 'n' elements respectively such that difference of the number of subsets of A and that of B is 56, then (m, n) is

- (1) (8, 3) (2) (8, 5) (3) (6, 3) (4) (7, 4)

Ans. (3)

Sol. $2^m - 2^n = 56; m > n$

$$\Rightarrow 2^n(2^{m-n} - 1) = 8(2^3 - 1)$$

$$\Rightarrow m = 6, n = 3$$

6. If 'A' is a square matrix of order '2' such that roots of the equation $\det(A - \lambda I) = 0$ are 1 and -3, then sum of diagonal elements of matrix 'A²' is

- (1) 2 (2) -3 (3) 9 (4) 10

Ans. (4)

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\therefore |A - \lambda I| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (a + d)\lambda + ad - bc = 0$$

$$\text{Sum of the roots} = a + d = 2$$

$$\text{Product of roots} = ad - bc = -3$$

$$\text{Now } A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{bmatrix}$$

$$\therefore \text{tr}(A^2) = a^2 + d^2 + 2bc = (a + d)^2 - 2(ad - bc) = 4 + 6 = 10$$

7. Let $\tan^{-1}x + \tan^{-1}2x = \frac{\pi}{4}; x > 0$, then number of positive values of x is/are

- (1) 0 (2) 1 (3) 2 (4) 3

Ans. (2)

Sol. $\tan^{-1} \left(\frac{x+2x}{1-2x^2} \right) = \frac{\pi}{4}$

$$\Rightarrow \frac{3x}{1-2x^2} = 1 \quad \Rightarrow x = \frac{-3 \pm \sqrt{17}}{4}$$

$$\therefore x = \frac{\sqrt{17}-3}{4}$$

8. Find the coefficient of x^{2012} in $(1-x)^{2008} \cdot (1+x+x^2)^{2007}$

Ans. (0)

Sol. Coefficient of x^{2012} in $(1-x^3)^{2007} \cdot (1-x)$

Coefficient of x^{2012} in $(1-x^3)^{2007} - x(1-x^3)^{2007}$

Coefficient of x^{2012} in ${}^{2007}C_{r_1} (-x^3)^{r_1} + {}^{2007}C_{r_2} (-1)^{r_2} x^{3r_2+1}$

$3r_1 = 2012$ Which is not possible for any $r_1 \in \mathbb{w}$

and $3r_2 + 1 = 2012$ also not possible for any $r_2 \in \mathbb{w}$

\therefore no term containing x^{2012}

\therefore Coefficient of x^{2012} is 0

9. An ellipse is passing through foci of hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and product of their eccentricities is

1, then the length of chord of ellipse passing through $(0, 2)$ and parallel to x-axis is

(1) $\frac{5\sqrt{5}}{3}$ (2) $\frac{3}{5\sqrt{5}}$ (3) $\frac{10\sqrt{5}}{3}$ (4) $\frac{20\sqrt{5}}{3}$

Ans. (3)

Sol. $\frac{x^2}{16} - \frac{y^2}{9} = 1 \quad \Rightarrow e_H = \frac{5}{4} \quad \Rightarrow e_E = \frac{4}{5}$

Ellipse is passing through $(\pm 5, 0)$

$$\therefore \text{ellipse: } \frac{x^2}{25} + \frac{y^2}{9} = 1$$

and points of chord: $\left(\pm \frac{5\sqrt{5}}{3}, 2 \right)$

$$\therefore \text{Length of chord} = \frac{10\sqrt{5}}{3}$$

10. If α and $\frac{1}{\alpha}$ are two complex numbers which satisfy the equations $|z - z_0|^2 = 4$ and $|z - z_0|^2 = 16$ respectively, where $z_0 = 1 + i$, then the value of $5|\alpha|^2$ is

Ans. (1)

Sol. $|\alpha - z_0|^2 = 4$

$$\Rightarrow (\alpha - z_0) (\bar{\alpha} - \bar{z}_0) = 4$$

$$|\alpha|^2 - \alpha \bar{z}_0 - \bar{\alpha} z_0 + |z_0|^2 = 4$$

$$|\alpha|^2 - \alpha \bar{z}_0 - \bar{\alpha} z_0 = 2 \dots (i)$$

$$(ii) \left| \frac{1}{\bar{\alpha}} - z_0 \right|^2 = 16 \Rightarrow (1 - \bar{\alpha} z_0) (1 - \alpha \bar{z}_0) = 16|\alpha|^2$$

$$\Rightarrow 1 - \alpha \bar{z}_0 - \bar{\alpha} z_0 + |\alpha|^2 \cdot 2 = 16|\alpha|^2 \dots (ii)$$

$$\text{from (i) \& (ii) } -1 - |\alpha|^2 = 2 - 16|\alpha|^2 \Rightarrow 15|\alpha|^2 = 3 \Rightarrow 5|\alpha|^2 = 1$$

11. Let $x^2 - x - 1 = 0$ has roots α and β such that $S_n = 2023 \alpha^n + 2024 \beta^n$, then

(1) $S_{12} = S_{11} - S_{10}$

(2) $S_{12} = S_{10} - S_{11}$

(3) $S_{12} = S_{10} + S_{11}$

(4) $S_{12} = -S_{10} - S_{11}$

Ans. (3)

Sol. $S_n = 2023 \alpha^n + 2024 \beta^n$

$$\Rightarrow S_n - S_{n-1} - S_{n-2} = 0$$

$$\Rightarrow S_{12} = S_{11} + S_{10}$$

12. For the series $20, 19\frac{1}{4}, 18\frac{1}{2}, \dots, -129\frac{1}{4}$, the 20th term from end is

(1) -115

(2) -119

(3) -117

(4) -120

Ans. (1)

Sol. T_{20} for $a = -129\frac{1}{4} = -\frac{517}{4}$, $d = \frac{3}{4}$

$$T_{20} = -\frac{517}{4} + 19 \cdot \frac{3}{4} = -\frac{460}{4} = -115$$

13. $\int \frac{x^8 - x^2}{(x^{12} + 3x^6 + 1) \tan^{-1}\left(x^3 + \frac{1}{x^3}\right)} dx =$

(1) $\frac{1}{3} \ln \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) + C$

(2) $\ln \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) + C$

(3) $\tan^{-1}\left(x^3 + \frac{1}{x^3}\right) + C$

(4) None of these

Ans. (1)

Sol.
$$\int \frac{x^8 - x^2}{(x^{12} + 3x^6 + 1) \tan^{-1}\left(x^3 + \frac{1}{x^3}\right)} dx =$$

Let $\tan^{-1}\left(x^3 + \frac{1}{x^3}\right) = t$

$$\frac{1}{1 + \left(x^3 + \frac{1}{x^3}\right)^2} \left(3x^2 - \frac{3}{x^4}\right) dx = dt$$

$$\frac{x^6}{x^{12} + 2x^6 + 1 + x^6} \times \frac{3x^6 - 3}{x^4} dx = dt$$

$$\frac{1}{3} \frac{1}{t} dt = \frac{1}{3} \ell n(t) + C$$

$$= \frac{1}{3} \ell n \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) + C$$

14. If $\lim_{x \rightarrow 0} \frac{\alpha \sin x + \beta \cos x + \ln(1-x) + 3}{3 \tan^2 x} = \frac{1}{3}$ find $2\alpha - \beta$

(1) 10

(2) 7

(3) 8

(4) 5

Ans. (4)

Sol.
$$\lim_{x \rightarrow 0} \frac{\alpha \sin x + \beta \cos x + \ln(1-x) + 3}{\frac{3 \tan^2 x}{x^2}} = \frac{1}{3}$$

$$\beta + 3 = 0 \Rightarrow \beta = -3$$

$$\lim_{x \rightarrow 0} \frac{\alpha \cos x - \beta \sin x - \frac{1}{1-x}}{2x}$$

$$\alpha - 1 = 0 \Rightarrow \alpha = 1$$

$$\text{so } 2\alpha - \beta = 5$$

15. Let $a_1, a_2, a_3, \dots, a_{15}$ are 15 observations having mean and variance as 12 & 9 respectively. One of the observation which was 12, misread as 10. The correct mean and variance are μ and σ^2 respectively, then $15(\mu + \mu^2 + \sigma^2)$

(1) 2521

(2) 2522

(3) 2518

(4) 2621

Ans. (1)

Sol. old mean $12 = \frac{\sum x_i}{n} \Rightarrow 12 = \frac{a_1 + a_2 + \dots + a_{14} + 10}{15}$

$$\sum_{i=1}^{14} a_i = 170$$

old variance $= 9 \Rightarrow 9 + (12)^2 = \frac{a_1^2 + a_2^2 + \dots + a_{14}^2 + 10^2}{15}$

$$\sum_{i=1}^{14} a_i^2 = 2195$$

new mean $(\mu) = \frac{\sum_{i=1}^{14} a_i + 12}{15} = \frac{170 + 12}{15} = \frac{182}{15}$

new variance (σ^2)

$$\sigma^2 + \mu^2 = \frac{\sum_{i=1}^{14} a_i^2 + 12^2}{15} = \frac{2339}{15}$$

$$\sigma^2 + \mu^2 + \mu = \frac{2339}{15} + \frac{182}{15} = \frac{2521}{15}$$

$$15(\sigma^2 + \mu^2 + \mu) = 2521$$

16. Values of α for which $\begin{vmatrix} 1 & \frac{3}{2} & \alpha + \frac{3}{2} \\ 1 & \frac{1}{3} & \alpha + \frac{1}{3} \\ 2\alpha + 3 & 3\alpha + 1 & 0 \end{vmatrix} = 0$ lies in the interval

(1) $(0, 3)$

(2) $(-3, 0)$

(3) $(-2, 1)$

(4) $(-2, 0)$

Ans. (2)

Sol. $C_3 \rightarrow C_3 - (\alpha C_1 + C_2)$

$$\begin{vmatrix} 1 & \frac{3}{2} & 0 \\ 1 & \frac{1}{3} & 0 \\ 2\alpha + 3 & 3\alpha + 1 & -(2\alpha^2 + 6\alpha + 1) \end{vmatrix} = 0$$

$$\Rightarrow \frac{7}{6}(2\alpha^2 + 6\alpha + 1) = 0$$

$$\Rightarrow \alpha = \frac{-3 + \sqrt{7}}{2}, \frac{-3 - \sqrt{7}}{2}$$

17. Let $g(x) = 3f\left(\frac{x}{3}\right) + f(3-x)$, where $f'(x) > 0$ and $x \in (0, 3)$, $g(x)$ is decreasing in $x \in (0, \alpha)$ and increasing in $(\alpha, 3)$, then 8α is

Ans. (18)

Sol. $g'(x) = 3 \cdot \frac{1}{3} f'\left(\frac{x}{3}\right) - f'(3-x) = f'\left(\frac{x}{3}\right) - f'(3-x)$

$g(x)$ is decreasing $g'(x) < 0$

$$f'\left(\frac{x}{3}\right) < f'(3-x)$$

$\because f'(x) > 0 \Rightarrow f'(x)$ is increasing

$$\frac{x}{3} < 3-x$$

$$\frac{4x}{3} < 3 \Rightarrow x < \frac{9}{4}$$

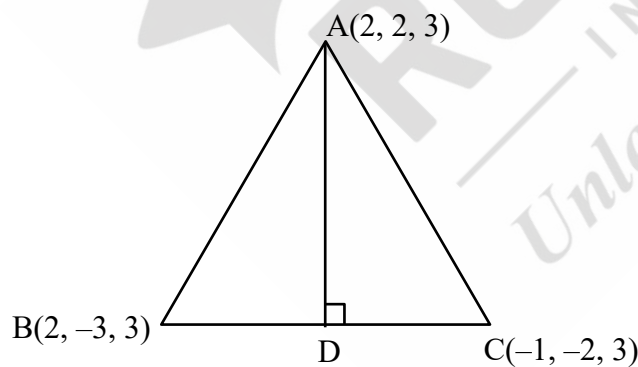
$$\alpha = \frac{9}{4}$$

$$\therefore 8\alpha = 18$$

18. Let ΔABC have vertices $A(2, 2, 3)$, $B(2, -3, 3)$ $C(-1, -2, 3)$ and length of internal angle bisector of angle A is ℓ , then the value of $2\ell^2$ is

Ans. (45)

Sol.



$$\overline{AB} = -5j$$

$$\overline{AC} = -3i - 4j$$

$$|\overline{AB}| = |\overline{AC}|$$

$$D\left(\frac{-1+2}{2}, \frac{-2-3}{2}, \frac{3+3}{2}\right)$$

$$D\left(\frac{1}{2}, \frac{-5}{2}, 3\right)$$

$$AD = \sqrt{\left(\frac{1}{2} - 2\right)^2 + \left(2 + \frac{5}{2}\right)^2 + (3 - 3)^2}$$

$$\ell = \sqrt{\frac{9}{4} + \frac{81}{4}}$$

$$2\ell^2 = 45$$

$$= \sqrt{\frac{90}{4}} = \sqrt{\frac{45}{2}}$$

19. Let $S_1 = \frac{|4!|}{(4!)^{3!}}$ and $S_2 = \frac{|5!|}{(5!)^{4!}}$, then

(1) $S_1 \in \mathbb{N}$ and $S_2 \notin \mathbb{N}$

(2) $S_1 \in \mathbb{N}$ and $S_2 \in \mathbb{N}$

(3) $S_1 \notin \mathbb{N}$ and $S_2 \in \mathbb{N}$

(2) $S_1 \notin \mathbb{N}$ and $S_2 \notin \mathbb{N}$

Ans. (2)

Sol. Make 6 groups of 4 each

$$24 \rightarrow (4, 4, 4, 4, 4, 4)$$

$$\text{Number of ways of making groups} = \frac{24!}{(4!)^6 \cdot 6!} = I_1$$

$$\frac{(24)!}{(4!)^6} = \frac{|4!|}{(4!)^{3!}} = (6! I_1)$$

$$S_1 \in \mathbb{N}$$

$$(5!) \rightarrow (5, 5, 5, 5, \dots, 5(24 \text{ times}))$$

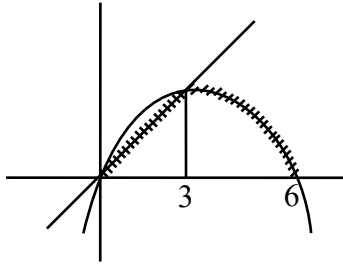
$$\frac{|5!|}{(5!)^{24} \cdot 24!} = I_2 \Rightarrow S_2 = (24!) I_2$$

$$\text{Hence } S_2 \in \mathbb{N}$$

20. Let area bounded by $y = \min.(3x, 6x - x^2)$; $y \geq 0$ is A, then 2A is

Ans. (63)

Sol.



$$2x = 6x - x^2$$

$$A = \frac{1}{2} \times 3 \times 9 + \int_3^6 \sqrt{6x - x^2} dx$$

$$A = \frac{27}{2} + \int_3^6 \sqrt{9 - (x-3)^2} dx$$

$$A = \frac{27}{2} + \left(\frac{x-3}{2} \sqrt{9 - (x-3)^2} + \frac{9}{2} \sin^{-1} \left(\frac{x-3}{3} \right) \right)_4^6$$

$$A = \frac{27}{2} + \frac{9\pi}{4}$$

$$12A = 162 + 27\pi$$

21. Let $(x^2 - 4)dy = y(y - 3)dx$ satisfying $y(4) = \frac{3}{2}$ then $y(10)$ is equal to

(1) $\frac{3}{1-8^{\frac{1}{4}}}$

(2) $\frac{3}{1+8^{\frac{1}{4}}}$

(3) $\frac{3}{1+2^{\frac{1}{4}}}$

(4) $\frac{3}{1-2^{\frac{1}{4}}}$

Ans. (2)

Sol. $= \frac{1}{3} \int \frac{y - (y-3)}{y(y-3)} dy = \frac{1}{4} \int \frac{(x+2) - (x-2)}{(x+2)(x-2)} dx$

$$\frac{1}{3} (\ell n |y-3| - \ell n |y|) = \frac{1}{4} (\ell n |x-2| - \ell n |x+2|) + C$$

$$\frac{1}{3} \ell n \left| \frac{y-3}{y} \right| = \frac{1}{4} \left(\ell n \left| \frac{x-2}{x+2} \right| \right) + C$$

$$\frac{1}{3} \ell n \left| \frac{\frac{3}{2} - 3}{\frac{3}{2}} \right| = \frac{1}{4} \ell n \left(\frac{4-2}{4+2} \right) + C$$

$$C = \frac{1}{4} \ell n 3$$

$$\frac{1}{3} \ell n \left| \frac{y-3}{y} \right| = \frac{1}{4} \ell n \left| \frac{x-2}{x+2} \right| + \frac{1}{4} \ell n 3$$

$$\Rightarrow x = 10$$

$$\frac{1}{3} \ell n \left| \frac{y-3}{y} \right| = \frac{1}{4} \ell n \left| \frac{2}{3} \right| + \frac{1}{4} \ell n 3$$

$$\frac{1}{3} \ell n \left| \frac{y-3}{y} \right| = \frac{1}{4} \ell n 2$$

$$\ell n \left| \frac{y-3}{y} \right| = \ell n 2^{\frac{3}{4}}$$

$$\left| \frac{y-3}{y} \right| = 2^{\frac{3}{4}}$$

$$-y + 3 = 8^{\frac{1}{4}} y$$

$$y = \frac{3}{1 + 8^{\frac{1}{4}}}$$

22. Three lines $2x - y - 3 = 0$, $6x + 3y + 4 = 0$, $\alpha x + 2y + 4 = 0$ does not form triangle then find $[\Sigma \alpha^2]$ (where $[\cdot]$ denotes the greatest integer function)

Ans. (32)

Sol. If two lines are parallel

$$\frac{2}{\alpha} = \frac{-1}{2} \Rightarrow \alpha = -4$$

$$\frac{6}{\alpha} = \frac{3}{2} \Rightarrow \alpha = 4$$

If lines are concurrent

$$\begin{vmatrix} 2 & -1 & -3 \\ 6 & 3 & 4 \\ \alpha & 2 & 4 \end{vmatrix} = 0$$

$$2(12 - 8) + 1(24 - 4\alpha) - 3(12 - 3\alpha) = 0$$

$$8 + 24 - 4\alpha - 36 + 9\alpha = 0$$

$$5\alpha = 4 \Rightarrow \alpha = \frac{4}{5}$$

$$\Sigma \alpha^2 = 16 + 16 + \frac{16}{25}$$

$$[\Sigma \alpha^2] = 32$$

23. Let $f(x) = \int_0^x g(t) \log \left(\frac{1-t}{1+t} \right) dt$, (where $g(x)$ is cont. odd function).

If $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(f(x) + \frac{x^2 \cos x}{1+e^x} \right) dx = \left(\frac{\pi}{\alpha} \right)^2 - \alpha$, then find α

Ans. $\alpha = 2$

Sol. $I = \int_0^{\pi/2} \left(f(x) + f(-x) + \frac{x^2 \cos x}{1+e^x} + \frac{x^2 \cos x}{1+e^{-x}} \right) dx = \int_0^{\pi/2} (f(x) + f(-x) + x^2 \cos x) dx \dots(i)$

Now $f(-x) = \int_0^{-x} g(t) \log \left(\frac{1-t}{1+t} \right) dt$

$t = -p$

$= \int_0^x -g(-p) \log \left(\frac{1+p}{1-p} \right) dp = -f(x)$

$\therefore (i)$ becomes $I = \int_0^{\pi/2} x^2 \cos x dx = x^2 \sin x - 2 \int x \sin x dx = (x^2 \sin x - 2)(-x \cos x + \sin x) \Big|_0^{\pi/2}$

$= \frac{\pi^2}{4} - 2 \Rightarrow \frac{\pi^2}{4} - 2$