Practice for the GRE Math Subject Test!

One Practice Test and Solutions

by

Charles Rambo

Contents

Preface

Thank you for checking out *Practice for the GRE Math Subject Test!* I hope you find the practice test and its solutions to be useful resources.

The GR1268, GR0568, and GR9768 are by far your best resources for math GRE subject test preparation. However, many students find that they need more material than just the three tests. The older tests are good resources but they are too easy, and *Cracking the GRE Mathematics Subject Test* only has one full-length practice test. As a result, I decided that it would be a good idea to write one more test.

This practice test and its solutions were heavily influenced by a variety of sources. I looked at the GR1268, GR0568, and GR8767 exams a lot while writing the problems. Some questions are very close to problems in those tests. I also used Rudin's *Principles of Mathematical Analysis*, Stewart's *Calculus*, *Counterexamples in Analysis* by Gelbaum and Olmsted, and *Counterexamples in Topology* by Steen and Seebach. Wikipedia was extensively used for theorems, and Wikimedia Commons is to thank for all the hippie images within this text.

A few words of gratitude are in order. The students of my tutoring business deserve a lot of thanks for finding some great problems and asking me about them. I am also indebted to the readers that emailed me about errors. They are fixed now, and the result is a better product for everybody else.

My apologies for any further errors. This is a self-published book for a relatively small audience, which makes hiring an editor impractical. If you find some errors please email me at `charles.tutoring@gmail.com`.

Please check out my other math GRE booklet, which contains the solutions to the GR1268, GR0568, and GR9768 exams. It is titled *GRE Mathematics Subject Test Solutions: Exams GR1268, GR0568, and GR9768*. I suspect that it will be of utility to you while you work through the GR1268, GR0568, and GR9768 tests.

I would like to encourage you to take a look at my website

`rambotutoring.com`.

It has a lot of math GRE material that you may be interested in.

Lastly, good luck on the exam, and I hope you find graduate school to be remarkably groovy!

Charles Rambo

Escondido, California
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Chapter 1

Questions

The questions begin on the next page. To save paper, less space is provided within this booklet than what is allotted on the actual test. As such, you will need scratch paper of your own. Furthermore, if you would like to simulate the actual test taking experience, you will need a scantron. We encourage you to try to solve the test while being timed because good time management is a critical component of success on the math GRE. Peace out!

MATHEMATICS TEST

Time—170 minutes
66 Questions

Directions: Each of the questions or incomplete statements below is followed by five suggested answers or completions. In each case, select the one that is best and then completely fill in the corresponding space on the answer sheet.

Computation and scratch work may be done on a separate sheet of paper.

In this test:

(1) All logarithms with an unspecified base are natural logarithms, that is, with base e.

(2) The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

---

1. $\lim_{n\to\infty}\left(n - n\cos\frac{1}{\sqrt{n}}\right)\sin\frac{1}{\sqrt{n}} =$

(A) $-\infty$ (B) 0 (C) $\frac{1}{2}$ (D) 1 (E) ∞

---

2. Let $f(4) = 3$, $f'(4) = -2$. If $g(x) = \frac{xf(x^2)}{x^2+1}$, then $g'(2) =$

(A) $-\frac{89}{25}$

(B) $-\frac{13}{8}$

(C) $\frac{13}{8}$

(D) $\frac{75}{16}$

(E) $\frac{89}{25}$

3. Which of the following is an equation of a tangent line of $f(x) = x^3 - 3x^2 + 4x - 3$ at an inflection point of f?

(A) $y = x - 2$

(B) $y = 13x - 8$

(C) $y = -x - 2$

(D) $y = -13x + 4$

(E) $y = 13x + 2$

---

4. $\int_e^{e^2} \log x \, dx =$

(A) $-e^2$ (B) $-e$ (C) 1 (D) e (E) e^2

---

5. Suppose $y = f(x)$ and $\frac{dy}{dx} = x - \frac{y}{2}$. If $f(1) = 2$, then $\lim_{x \to 1} \frac{f(x) - 2}{(x-1)^2} =$

(A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1

---

6. Determine the volume of the parallelepiped which has edges parallel to and the same lengths as the position vectors $\mathbf{u} = (0, 2, 0)$, $\mathbf{v} = (1, -1, 0)$, and $\mathbf{w} = (-2, 2, 1)$.

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$ (C) 2 (D) 15 (E) $13\sqrt{6}$

7. Consider the system

$$\begin{cases} y &= 2 \\ y &= a(x-b)^2 + c, \end{cases}$$

where a, b, and c are real numbers. For which of the following values of a, b, and c is there a solution to the system of equations?

(A) $a = -9$, $b = -4$, and $c = -5$

(B) $a = 7$, $b = -10$, and $c = 6$

(C) $a = 1$, $b = -6$, and $c = -4$

(D) $a = 2$, $b = 9$, and $c = 4$

(E) $a = -10$, $b = -10$, and $c = -6$

---

8. The lateral surface area of a cone is 6π and its slant height is 6. What is the radius of the cone's base?

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) $\frac{12}{\pi}$

---

9. What is the measure of the angle between $\mathbf{u} = (2, 0, 2)$ and $\mathbf{v} = \left(3\sqrt{2}, -6\sqrt{3}, 3\sqrt{2}\right)$ in xyz-space?

(A) 0° (B) 30° (C) 45° (D) 60° (E) 90°

---

10. If U and V are 3 dimensional subspaces of $\mathbb{R}^5$, what are the possible dimensions of $U \cap V$?

(A) 0 (B) 1 (C) 0 or 1 (D) 1, 2, or 3 (E) 0, 1, 2, or 3

11. What is the absolute minimum value of $f(x) = \dfrac{x^2 - 6x}{1 + |x+1|}$?

(A) -2 (B) -1 (C) 0 (D) $\dfrac{1}{4}$ (E) 7

---

12. Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}$ such that $T\begin{pmatrix} 1 \\ -3 \end{pmatrix} = 5$ and $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2$. Then $T\begin{pmatrix} 0 \\ 2 \end{pmatrix} =$

(A) -4 (B) $-\dfrac{7}{2}$ (C) -1 (D) 3 (E) $\dfrac{9}{2}$

---

13. If $f(x) = (1-x)^{17}e^{2x}$, then $f^{(17)}(1) =$

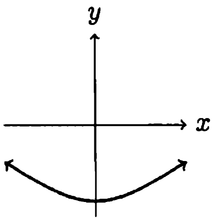
(A) $-\left(17!2^{17}e^2\right)$ (B) $-\left(17!e^2\right)$ (C) 0 (D) $17!2^{17}e^2$ (E) $17!e^2$

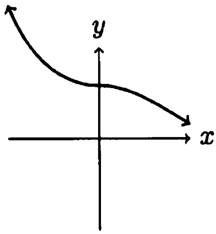
---

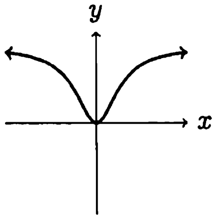
14. A class of 37 is to be divided into teams, and each student in the class must be a member of exactly one team. However, each student dislikes three of their classmates. Dislike between students need not be mutual. How many teams must be created so that no student is the teammate of someone they dislike?

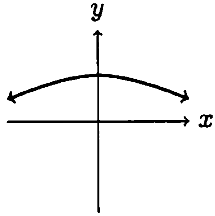
(A) 4 (B) 7 (C) 10 (D) 13 (E) 16

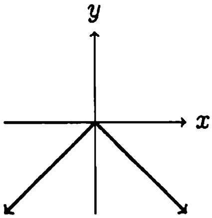
15. Which of the following could be the graph of a solution of $\frac{dy}{dx} = x\,|y|$?

(A) 

(B) 

(C) 

(D) 

(E) 

16. If $f(3) = 5$, $f(10) = 1$, and $\int_3^{10} f(x)\,dx = 20$, $\int_1^5 f^{-1}(x)\,dx =$

(A) 5 (B) 12 (C) 13 (D) 25 (E) 31

17. Suppose $S = [0,1] \times [1,3]$. Then $\iint_S xy^2 \, dA =$

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) 1 (D) $\frac{9}{2}$ (E) $\frac{13}{3}$

---

18. Suppose $\mathcal{F}$ is the set of functions such that $f(x) = \dfrac{ax+b}{cx+d}$, where the coefficients a, b, c, and d are real numbers and $ad - bc = 1$. Which of the following are TRUE?

I. For f and g in $\mathcal{F}$ $f \circ g = g \circ f$.

II. There is a function i in $\mathcal{F}$ such that $i \circ f = f \circ i$ for all f in $\mathcal{F}$.

III. For all f, g, and h in $\mathcal{F}$ $f \circ (g \circ h) = (f \circ g) \circ h$.

(A) I only

(B) II only

(C) I and II only

(D) II and III only

(E) I, II, and III

GO ON TO THE NEXT PAGE.

19. Find the area contained within the Lemniscate curve

$$r^2 = 4\sin 2\theta$$

but outside the circle $r = \sqrt{2}$.

(A) $\sqrt{3} - \frac{\pi}{3}$

(B) $2\sqrt{3} - \frac{2\pi}{3}$

(C) $2\sqrt{3} - \frac{\pi\sqrt{2}}{3}$

(D) $2\sqrt{3} - \frac{\pi}{3}$

(E) $2 - \frac{\pi}{3}$

---

20. Let f be a differentiable real-valued function such that $f(3) = 7$ and $f'(x) \geq x$ for all positive x. What is the maximum possible value of $\int_0^3 f(x)\,dx$?

(A) 0

(B) $\frac{9}{2}$

(C) 12

(D) $\frac{25}{2}$

(E) 21

GO ON TO THE NEXT PAGE.

21. Suppose $f : (0,1) \to (0,1]$. Which of the following could be TRUE?

I. f is one-to-one and onto.

II. The image of f is compact.

III. f is continuous, one-to-one, and onto.

(A) I only

(B) II only

(C) I and II only

(D) I and III only

(E) I, II, and III

---

22. Suppose

$$f(x) = \begin{cases} \frac{2|x|}{5}, & -1 \le x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $\int_{-\infty}^{\infty} x f(x)\, dx$.

(A) $\frac{14}{15}$ (B) 1 (C) $\frac{6}{5}$ (D) $\frac{7}{3}$ (E) 3

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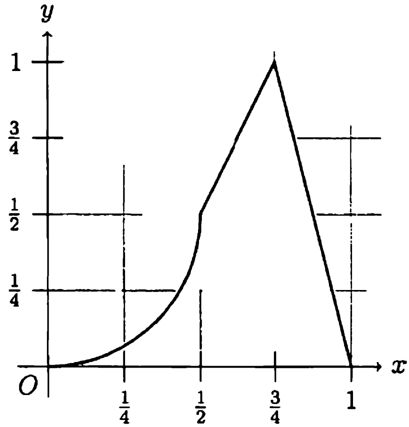
23. For which value of n are there exactly two abelian groups of order n up to isomorphism.

(A) 4 (B) 7 (C) 8 (D) 12 (E) none of these



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24. Above is the graph of $y = f(x)$. If $f(1+x) = f(x)$ for all real x, then $f'(25\pi) =$

(A) -16

(B) 0

(C) undefined

(D) 2

(E) not uniquely determined by the information given

---

25. The convergent sequence $\{x_n\}$ is defined by the recursive relationship $x_1 = 1$ and $x_{n+1} = \sqrt{15 - 2x_n}$ for all positive integers n. Then $\lim_{n\to\infty} x_n =$

(A) -5 (B) -3 (C) 0 (D) 3 (E) 5

GO ON TO THE NEXT PAGE.

x	$f(x)$	$f'(x)$	$g(x)$
-6	-5	1	-3
-2	$-\frac{5}{2}$	$\frac{1}{4}$	0
2	0	1	6
6	2	3	$\frac{13}{2}$

26. If f and g in the table above are inverses, $(g' \circ g)(0) =$

(A) -1

(B) $-\frac{1}{2}$

(C) $\frac{1}{3}$

(D) 1

(E) 4

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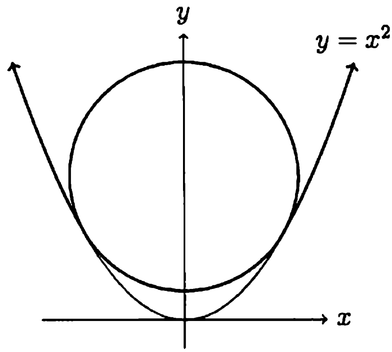

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27. A circle of radius 1 is tangent to $y = x^2$ at two points. Find the area bounded by the parabola and circle.

(A) $\frac{2}{3} - \frac{\pi}{6}$

(B) $\sqrt{3} - \frac{11\pi}{24}$

(C) $\frac{3\sqrt{3}}{4} - \frac{\pi}{3}$

(D) $\frac{2}{3} - \frac{\pi}{12}$

(E) $\frac{11}{6} - \frac{\pi}{6}$

28. Find the arc length of the curve C from the point $(8, 1)$ to the point $(8e, e^2 - 8)$. Suppose $C = \{(x, y) \in \mathbb{R}^2 | \ x = 8e^{t/2}$ and $y = e^t - 4t\}$.

(A) $7 - e^2$ (B) $7 - e$ (C) $e^2 - 7$ (D) $e + 7$ (E) $e^2 + 7$

---

29. Consider a segment of length 10. Points A and B are chosen randomly such that A and B divide the segment into three smaller segments. What is the probability that the three smaller segments could form the sides of a triangle?

(A) 0 (B) 10% (C) 25% (D) 50% (E) 80%

---

30. A discrete graph is complete if there is an edge connecting any pair of two vertices. How many edges does a complete graph with 10 vertices have?

(A) 10 (B) 20 (C) 25 (D) 45 (E) 90

---

31. Suppose P is the set of polynomials with coefficients in $\mathbb{Z}_5$ and degree less than or equal to 7. If the operator D sends $p(x)$ in P to its derivative $p'(x)$, what are the dimensions of the null space n and range r of D?

(A) $n = 1$ and $r = 6$

(B) $n = 1$ and $r = 7$

(C) $n = 2$ and $r = 5$

(D) $n = 2$ and $r = 6$

(E) $n = 3$ and $r = 5$

32. If a = 20 and b = 28 are the inputs of the following algorithm, what is the result?

```
input(a)
input(b)
   begin
      if a > b
         set max = a
         set min = b
      else
         set max = b
         set min = a
      while min > 0
         begin
            set r = max mod min
            replace max = min
            replace min = r
         end
      print a*b/max
   end
```

(A) 4 (B) 5 (C) 7 (D) 140 (E) 560

---

33. Let $\varphi(k)$ be a proposition which is either true or false depending on the integer k. Suppose that if $\varphi(k)$ is false then so is $\varphi(k-1)$. If there is some k_0 such that $\varphi(k_0)$ is true, what is the strongest conclusion that can be drawn?

(A) $\varphi(k)$ is true for all k.

(B) $\varphi(k_0+1)$ is true.

(C) $\varphi(k_0-1)$ is ture.

(D) $\varphi(k)$ is true for $k \leq k_0$.

(E) $\varphi(k)$ is true for $k \geq k_0$.

34. Define

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Let $I = \{(x, f(x)) \in \mathbb{R}^2 \mid -1 \leq x \leq 1\}$. Which of the following are TRUE?

I. The set I is connected.

II. The set I is path connected.

III. The set I is compact.

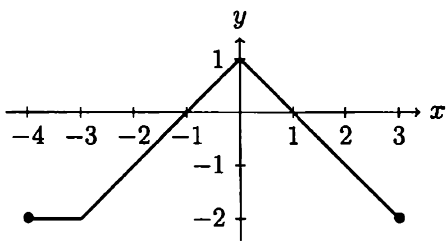
(A) I only

(B) III only

(C) I and II only

(D) I and III only

(E) I, II, and III



35. The figure above shows the graph of the function f. Suppose that $g(x) = \int_0^x f(t)\,dt$. The absolute maximum of g is

(A) $g(-4)$ (B) $g(-3)$ (C) $g(-1)$ (D) $g(1)$ (E) $g(3)$

36. Let $f(x) = \int_0^{x^2} \sqrt{t} \sin \frac{1}{t}\, dt$, and let $I = [-1, 0) \cup (0, 1]$. Which of the following are TRUE?

I. f is bounded on the set I.

II. f' is bounded on the set I.

III. f'' is bounded on the set I.

(A) I only

(B) II only

(C) I and II only

(D) I and III only

(E) I, II, and III

---

37. Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Then A^{500} is

(A) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(B) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(D) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(E) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

GO ON TO THE NEXT PAGE.

38. The region bounded by the x-axis and the function

$$f(x) = \frac{x}{1+x^3}$$

is rotated about the x-axis. What is the volume of the solid generated?

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$ (C) π (D) 2π (E) ∞

---

39. Suppose

for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x and y in D

$$|x-y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \varepsilon.$$

Consider the following statements.

A: For all $\varepsilon > 0$ there is a $\delta > 0$

B: For all x and y in D

C: $|x-y| < \delta$

D: $|f(x) - f(y)| \geq \varepsilon$

Using the letters listed above, the proposition originally stated is which of the following? Denote "not" by $\neg$.

(A) $A\Big(B(C \text{ or } D)\Big)$

(B) $A\Big(B(\neg C \text{ and } D)\Big)$

(C) $\neg A\Big(B(\neg C \text{ or } D)\Big)$

(D) $A\Big(\neg B(\neg C \text{ or } D)\Big)$

(E) $A\Big(B(\neg C \text{ or } \neg D)\Big)$

GO ON TO THE NEXT PAGE.

40. The radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(nx)^n}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n}$ is

(A) 0 (B) $\frac{2}{e^2}$ (C) $\frac{2}{e}$ (D) $\frac{e}{2}$ (E) ∞

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41. Suppose V is a real vector space of finite dimension n. Call the set of matrices from V into itself $\mathcal{M}(V)$. Let T be in $\mathcal{M}(V)$. Consider the two subspaces

$$\mathcal{U} = \{X \in \mathcal{M}(V) : TX = XT\} \quad \text{and} \quad \mathcal{W} = \{TX - XT | X \in \mathcal{M}(V)\}.$$

Which of the following must be TRUE?

I. If V has a basis containing only eigenvectors of T then $\mathcal{U} = \mathcal{M}(V)$.

II. $\dim(\mathcal{U}) + \dim(\mathcal{W}) = n^2$.

III. $\dim(\mathcal{U}) < n$.

(A) I only

(B) II only

(C) III only

(D) I and II only

(E) I, II, and III

---

42. If the finite group G contains a subgroup of order five but no element other than the identity is its own inverse, then the order of G could be

(A) 8 (B) 20 (C) 30 (D) 35 (E) 42

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43. If $\zeta = e^{\frac{2\pi}{5}i}$, then $3 + 3\zeta + 12\zeta^2 + 12\zeta^3 + 12\zeta^4 + 9\zeta^5 + 5\zeta^6 =$

(A) $-4e^{\frac{2\pi i}{5}}$ (B) $-4e^{\frac{4\pi i}{5}}$ (C) 0 (D) $4e^{\frac{2\pi i}{5}}$ (E) $4e^{\frac{4\pi i}{5}}$

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44. Suppose A is a 3×3 matrix such that

$$\det(A - \lambda I) = -\lambda^3 + 3\lambda^2 + \lambda - 3,$$

where I is the 3×3 identity matrix. Which of the following are TRUE of A?

I. The trace of A is 3.

II. The determinate of A is -3.

III. The matrix A has eigenvalues -3 and 1.

(A) I only

(B) II only

(C) III only

(D) I and II only

(E) I, II, and III

---

45. Find the general solution of

$$2\frac{d^2y}{dx^2} + 9\frac{dy}{dx} \quad 35y = 0.$$

(A) $y = C_1 e^{-7x} + C_2 e^{\frac{5}{2}x}$

(B) $y = C_1 e^{-\frac{5}{2}x} + C_2 e^{7x}$

(C) $y = C_1 e^{-7x} + C_2 e^{5x}$

(D) $y = C_1 e^{-5x} + C_2 e^{7x}$

(E) $y = C_1 \cos(5x) + C_2 \cos(7x)$

GO ON TO THE NEXT PAGE.

46. Let $f(x,y) = x^3 - y^3 + 3x^2y - x$ for all real x and y. Which of the following is TRUE of f?

(A) There is an absolute minimum at $\left(\frac{1}{3}, \frac{1}{3}\right)$.

(B) There is a relative maximum at $\left(-\frac{1}{3}, -\frac{1}{3}\right)$.

(C) There is a saddle point at $\left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$.

(D) There is an absolute maximum at $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.

(E) All critical values are on the line $y = x$.

---

47. Approximate the difference $\log(1.1) - p(1.1)$, where $p(x) = x - 1 - \frac{1}{2}(x-1)^2$.

(A) $-\frac{1}{3} \times 10^{-3}$

(B) $-\frac{1}{4} \times 10^{-4}$

(C) $\frac{3}{8} \times 10^{-5}$

(D) $\frac{1}{4} \times 10^{-4}$

(E) $\frac{1}{3} \times 10^{-3}$

48. Suppose today is Wednesday. What day of the week will it be $10^{10^{10}}$ days from now?

(A) Sunday

(B) Monday

(C) Tuesday

(D) Wednesday

(E) Thursday

---

49. It takes Kate k days to write a math GRE practice test. It takes John j days to write a math GRE practice test. If Kate and John work on a practice test in alternating 2-day shifts, it takes them 10 days when Kate starts and 10.5 days when John starts. How long would it take the two to complete a practice test if Kate and John worked simultaneously?

(A) $\frac{9}{2}$ days (B) 5 days (C) $\frac{41}{8}$ days (D) $\frac{36}{7}$ days (E) 6 days

---

50. In the complex plane, let C be the circle $|z + 2| = 3$ with positive (counterclockwise) orientation. Then $\int_C \frac{dz}{z^3(z-2)} =$

(A) $-\frac{\pi i}{4}$ (B) 0 (C) $\frac{3\pi i}{8}$ (D) $\frac{7\pi i}{8}$ (E) $2\pi i$

---

51. Compute

$$\int_0^{\pi/4} \log\left|\cos\left(\theta - \frac{\pi}{4}\right)\right| - \log|\cos\theta| \; d\theta.$$

(A) $-\infty$ (B) -1 (C) 0 (D) 1 (E) ∞

52. Suppose the real-valued function f has a continuous derivative for all values of x in $\mathbb{R}$. Which of the following must be FALSE?

I. For some closed interval $[a, b]$ and every natural number N there exists an x in $[a, b]$ such that $|f(x)| > N$.

II. For each c there are exactly two solutions of $f(x) = c$.

III. The limit $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$ if and only if $\lim_{x\to\infty} f'(x) = \infty$.

(A) I only

(B) II only

(C) III only

(D) I and II only

(E) I, II, and III

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53. Water drips out of a hole at the vertex of an upside down cone at a rate of 3 cm^3 per minute. The cone's height and radius are 2 cm and 1 cm, respectively. At what rate does the height of the water change when the water level is half a centimeter below the top of the cone? The volume of a cone is $V = \frac{\pi}{3}r^2h$, where r is the radius and h is the height of the cone.

(A) $-\frac{48}{\pi}$ cm/min

(B) $-\frac{4}{3\pi}$ cm/min

(C) $-\frac{8}{3\pi}$ cm/min

(D) $-\frac{24}{\pi}$ cm/min

(E) $-\frac{16}{3\pi}$ cm/min

54. Suppose f is an analytic function of the complex variable $z = x + iy$ where x and y are real variables. If

$$f(z) = g(x,y) + e^y i \sin x$$

and $g(x,y)$ is a real-valued function of x and y, what is the value of

$$g\left(\frac{\pi}{2}, 7\right) - g(0,0)?$$

(A) $1 + e^7$ (B) $1 - e^7$ (C) 1 (D) $e^7 - 1$ (E) $2 - 2e^7$

---

55. Suppose A and B are $n \times n$ matrices with real entries. Which of the follow are TRUE?

I. The trace of A^2 is nonnegative.

II. If $A^2 = A$, then the trace of A is nonnegative.

III. The trace of AB is the product of the traces of A and B.

(A) II only

(B) III only

(C) I and II only

(D) II and III only

(E) I, II, and III

56. Consider the independent random variables X_i such that either $X_i = 0$ or $X_i = 1$ and each event is equally as likely. Let

$$X = X_1 + X_2 + \ldots + X_{100}.$$

Which of the following values is largest?

(A) $\text{Var}(X)$

(B) $100P\left(|X - 50| > 25\right)$

(C) $\sum_{k=0}^{100} k \binom{100}{k} \left(\frac{1}{2}\right)^k$

(D) $100P\left(X \geq 60\right)$

(E) 30

57. $\lim_{n\to\infty} \frac{1}{n} + \frac{1}{2+n} + \frac{1}{4+n} + \ldots + \frac{1}{3n} =$

(A) $\frac{1}{2}\log 2$

(B) $\frac{3}{4}$

(C) $\log\sqrt{3}$

(D) 1

(E) 2

58. Suppose $A, A_k \subseteq \mathbb{R}$ where k is any positive integer. Which of the following must be TRUE?

I. If A is closed, then A is compact.

II. If for each sequence $\{a_k\}$ with terms in A there is a strictly increasing function $\alpha : \mathbb{Z} \to \mathbb{Z}$ such that $\lim_{k\to\infty} a_{\alpha(k)}$ is in A, then A is compact.

III. If $B = \bigcup_{k=1}^{\infty} A_k$, then $\overline{B} = \bigcup_{k=1}^{\infty} \overline{A_k}$.

(A) I only

(B) II only

(C) III only

(D) II and III only

(E) I, II, and III

---

59. A point (x, y) in $\mathbb{R}^2$ is chosen randomly within the region described by the inequality $0 < |x| + |y| < 1$. What is the probability that $2(x + y) > 1$?

(A) 0 (B) $\frac{1}{4}$ (C) $\frac{\sqrt{2}}{4}$ (D) $\frac{1}{\sqrt{2}}$ (E) $\frac{3}{4}$

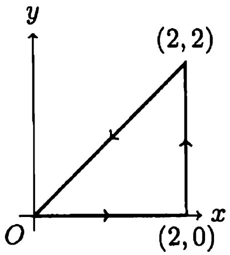
---

60. Let $\mathbf{F} = \left(y, -x, \frac{3}{\pi}\right)$ be a vector field in xyz-space. What is the work done by $\mathbf{F}$ on a particle that moves along the path given by $(\cos t, \sin t, t^2)$ where t goes from 0 to $\frac{\pi}{2}$?

(A) $-\frac{\pi}{2}$ (B) $-\frac{\pi}{4}$ (C) 0 (D) $\frac{\pi}{4}$ (E) $\frac{\pi}{2}$

61. There are 25 suitcases, 5 of which are damaged. Three suitcases are selected at random. What is the probability that exactly 2 are damaged?

(A) $\frac{2}{69}$ (B) $\frac{1}{30}$ (C) $\frac{2}{23}$ (D) $\frac{12}{125}$ (E) $\frac{3}{25}$



62. Let C be the positively oriented path shown above. Compute

$$\oint_C x\sin(x^2)\,dx + (3e^{y^2} - 2x)\,dy.$$

(A) −4 (B) −2 (C) 0 (D) 2 (E) 4

63. Find the point on $3x - 2y + z = 4$ which is closest to the origin.

(A) $(1, 2, 5)$

(B) $\left(\frac{6}{7}, -\frac{2}{7}, \frac{6}{7}\right)$

(C) $\left(\frac{5}{6}, \frac{4}{3}, \frac{25}{6}\right)$

(D) $(1, -3, -5)$

(E) $\left(\frac{6}{7}, -\frac{4}{7}, \frac{2}{7}\right)$

64. For each positive integer n, let f_n be the function defined on the interval $[0,1]$ by $f_n(x) = \frac{nx}{1+nx^2}$. Which of the following statements are TRUE?

I. The sequence $\{f_n\}$ converges pointwise on $[0,1]$ to a limit function f.

II. The sequence $\{f_n\}$ converges uniformly on $[0,1]$ to a limit function f.

III. $\left| \int_0^1 f_n(x)\,dx - \int_{\frac{1}{n}}^1 \lim_{k\to\infty} f_k(x)\,dx \right| \to 0$ as $n \to \infty$.

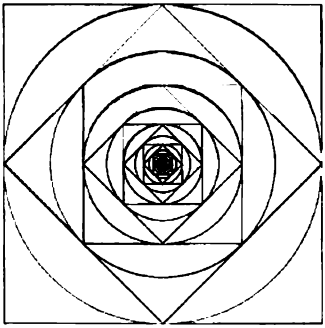
(A) I only

(B) III only

(C) I and II only

(D) I and III only

(E) I, II, and III



65. The pattern in the figure above continues infinitely into the page. If the outer most square has sides of length 1, what is the total gray area of the figure?

(A) $1 - \frac{\pi}{4}$

(B) $2 - \frac{\pi}{3}$

(C) $2 - \frac{\pi}{2}$

(D) $\frac{1}{2}$

(E) $\frac{1 + \pi}{4}$

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1

---

66. Suppose multiplication between 1, **i**, **j**, and **k** are as defined above. Which of the following are rings?

I. $\{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b,$ and c are rational$\}$

II. $\left\{\frac{m}{2^n} \middle| m \text{ is an integer and } n \text{ is a non-negative integer}\right\}$

III. $\{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c,$ and d are real$\}$

(A) I only

(B) I and II only

(C) I and III only

(D) II and III only

(E) I, II, and III

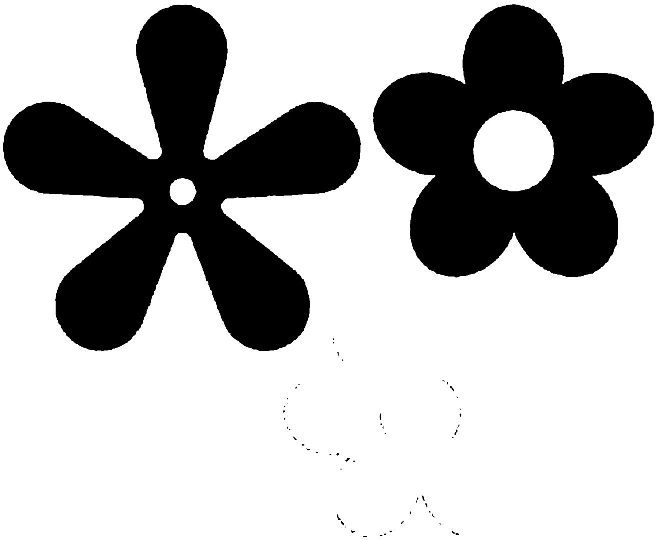
STOP

If you finished before time is called, you may check your work on this test.

Chapter 2

Answers

Turn the page to see the answers. Hang loose!



1. B
2. A
3. A
4. E
5. D
6. C
7. C
8. B
9. D
10. D
11. A
12. B
13. B
14. B
15. A
16. D
17. E
18. D
19. B
20. C
21. C
22. A
23. A
24. D
25. D
26. C
27. C
28. E
29. C
30. D
31. D
32. D
33. E
34. A
35. A
36. C
37. B
38. A
39. E
40. C
41. B
42. D
43. A
44. D
45. A
46. E
47. E
48. A
49. D
50. A
51. C
52. D
53. E
54. C
55. A
56. C
57. C
58. B
59. B
60. D
61. C
62. A
63. E
64. A
65. C
66. E

Chapter 3

Solutions

Solution 1.

Two well know *limits* from Calculus are

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \quad \text{and} \quad \lim_{x\to 0}\frac{1-\cos x}{x}=0.$$

Our task is to deconstruct the limit in the question into factors where we can use the above:

$$\lim_{n\to\infty} n\left(1-\cos\frac{1}{\sqrt{n}}\right)\sin\frac{1}{\sqrt{n}}=\lim_{n\to\infty}\frac{1-\cos\dfrac{1}{\sqrt{n}}}{\dfrac{1}{\sqrt{n}}}\lim_{n\to\infty}\frac{\sin\dfrac{1}{\sqrt{n}}}{\dfrac{1}{\sqrt{n}}}.$$

If we let $x=1/\sqrt{n}$, we see that

$$\begin{aligned}\lim_{n\to\infty} n\left(1-\cos\frac{1}{\sqrt{n}}\right)\sin\frac{1}{\sqrt{n}} &= \lim_{x\to 0+}\frac{1-\cos x}{x}\lim_{x\to 0+}\frac{\sin x}{x}\\ &=0\cdot 1\\ &=0.\end{aligned}$$

Therefore, the solution is (B).

■

Solution 2.

We will use *derivative rules* from Calculus to compute g', particularly the product and quotient rules. We have

$$g'(x) = \frac{(x^2+1)\Big(f(x^2) + 2x^2 f'(x^2)\Big) - 2x^2 f(x^2)}{(x^2+1)^2}.$$

It follows that

$$\begin{aligned} g'(2) &= \frac{5\Big(f(4) + 8f'(4)\Big) - 8f(4)}{25} \\ &= \frac{5\Big(3 + 8(-2)\Big) - 8(3)}{25} \\ &= -\frac{89}{25}. \end{aligned}$$

Alternatively, we could have computed g' for a particular f that satisfies the given criteria, e.g. $f(x) = -2x + 11$. Regardless of our approach, the solution is (A). ■

Solution 3.

Let us compute our derivatives:

$$f'(x) = 3x^2 - 6x + 4 \quad \text{and} \quad f''(x) = 6x - 6.$$

It follows that an *inflection point* occurs at $x = 1$ because the second derivative changes signs there.

To find a linear equation, we need a point and a slope. We obtain those respective values through evaluation of $f(1)$ and $f'(1)$:

$$f(1) = 1 - 3 + 4 - 3 = -1$$

and

$$f'(1) = 3 - 6 + 4 = 1.$$

Hence, an equation of a line which is tangent to f at the inflection point $(1, -1)$ is $y = x - 2$. Choose (A) and move on! ■

Solution 4.

Using *integration by parts* with $u = \log x$ and $dv = dx$, we see

$$\begin{aligned}\int_e^{e^2} \log x \, dx &= x \log x \Big|_e^{e^2} - \int_e^{e^2} x \cdot \frac{1}{x} \, dx \\ &= x \log x \Big|_e^{e^2} - \int_e^{e^2} dx \\ &= x \log x - x \Big|_e^{e^2} \\ &= e^2 \log e^2 - e^2 - e \log e + e \\ &= 2e^2 - e^2 - e + e \\ &= e^2.\end{aligned}$$

Thus, the answer is (E). If you are unfamiliar with the *logarithm properties* we used above, see the glossary for details. ■

Solution 5.

Notice that our limit is in the 0/0 indeterminate form, which means we can use *L'Hôspital's rule*:

$$\lim_{x\to 1} \frac{f(x) - 2}{(x-1)^2} \overset{LH}{=} \lim_{x\to 1} \frac{f'(x)}{2(x-1)}.$$

Because

$$f'(1) = \frac{dy}{dx}\Big|_{(1,2)} = 1 - \frac{2}{2} = 0,$$

we are in another 0/0 indeterminate form. As a result, we will use L'Hôspital's rule a second time:

$$\lim_{x\to 1} \frac{f'(x)}{2(x-1)} \overset{LH}{=} \lim_{x\to 1} \frac{f''(x)}{2}.$$

All that is left is to find $f''(x)$. We know

$$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(x - \frac{y}{2}\right) = 1 - \frac{1}{2}\frac{dy}{dx}.$$

It follows that

$$f''(1) = 1 - \frac{1}{2}\frac{dy}{dx}\Big|_{(1,2)} = 1 - \frac{1}{2}(0) = 1.$$

Hence,

$$\lim_{x\to 1}\frac{f(x)-2}{(x-1)^2}=\lim_{x\to 1}\frac{f''(x)}{2}=\frac{f''(1)}{2}=\frac{1}{2}.$$

We conclude that (D) is correct. ■

Solution 6.

There is a theorem from linear algebra which says the *volume of a parallelepiped* determined by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is

$$\pm\det\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\\ \mathbf{w}\end{pmatrix}$$

where the vectors are row vectors and the $\pm$ makes the determinate positive.

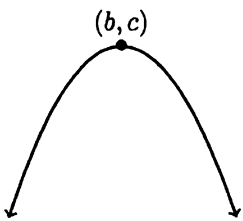
We have

$$\det\begin{pmatrix}0&2&0\\1&-1&0\\-2&2&1\end{pmatrix}=0\begin{vmatrix}-1&0\\2&1\end{vmatrix}-2\begin{vmatrix}1&0\\-2&1\end{vmatrix}+0\begin{vmatrix}1&-1\\-2&2\end{vmatrix}$$
$$=-2.$$

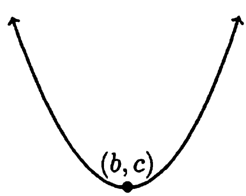
So, the volume must be 2. Select (C)! ■

Solution 7.

Consider the graph of $y=a(x-b)^2+c$. Recall that if $a>0$ the parabola opens upward and if $a<0$ the parabola opens downward. Furthermore, the equation of the vertex is (b,c).



$a<0$

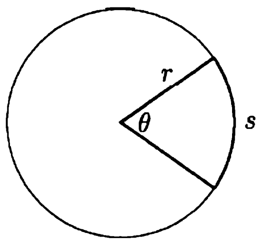


$a>0$

With this in mind, we know that the graph of $y = a(x-b)^2 + c$ intersects $y = 2$ if

$$\begin{gathered} a > 0 \text{ and } c < 2, \\ a < 0 \text{ and } c > 2, \\ \text{or} \\ a = 0 \text{ and } c = 2. \end{gathered}$$

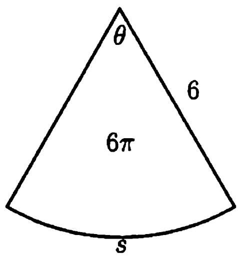
The only option which satisfies this criteria is (C). ■



Solution 8.

Consider the diagram above. Recall that the arc length $s = r\theta$, where θ is the radian measure of the interior angle. Furthermore, the area of the sector A inclosed by the two radii and s is $A = r^2\theta/2$.

Now, imagine that we slice the lateral surface area of our cone along a slant height. The result looks like the following.



This is a sector whose interior angle θ can be found via the equation

$$\frac{6^2\theta}{2} = 6\pi \quad \text{implies} \quad \theta = \frac{\pi}{3}.$$

We can find the circumference of the base by computing the arc length

$$s = 6\left(\frac{\pi}{3}\right) = 2\pi.$$

If r is the radius of the base, then

$$2\pi r = 2\pi \quad \text{implies} \quad r = 1.$$

We conclude that the answer is 1, and we pick (B). ■

Solution 9.

Recall that

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Let us compute the pieces:

$$\mathbf{u} \cdot \mathbf{v} = 6\sqrt{2} + 0 + 6\sqrt{2} = 12\sqrt{2}, \quad |\mathbf{u}| = \sqrt{2^2 + 2^2} = 2\sqrt{2},$$

and

$$|\mathbf{v}| = \sqrt{\left(3\sqrt{2}\right)^2 + \left(-6\sqrt{3}\right)^2 + \left(3\sqrt{2}\right)^2} = \sqrt{144} = 12.$$

It follows that

$$12\sqrt{2} = (2\sqrt{2})(12)\cos\theta \quad \text{implies} \quad \cos\theta = \frac{1}{2}.$$

The angle measure $\theta = 60°$ satisfies our equation, so we conclude that (D) is correct. ■

Solution 10.

We know that

$$\dim(U) + \dim(V) - \dim(U \cap V) \leq \dim(\mathbb{R}^5).$$

It follows that $3 + 3 - \dim(U \cap V) \leq 5$, so $\dim(U \cap V) \geq 1$. It is easy to produce examples where the dimension of $U \cap V$ is 1, 2, or

3. Hence, (D) is the correct answer. ■

Solution 11. ---

Since

$$|x+1| = \begin{cases} x+1, & x \geq -1 \\ -x-1, & x < -1, \end{cases}$$

we see that

$$f(x) = \begin{cases} \dfrac{x^2-6x}{1+x+1}, & x \geq -1 \\ \\ \dfrac{x^2-6x}{1-x-1}, & x < -1. \end{cases} = \begin{cases} \dfrac{x^2-6x}{x+2}, & x \geq -1 \\ -x+6, & x < -1. \end{cases}$$

It follows that

$$f'(x) = \begin{cases} \dfrac{(x+2)(2x-6)-(x^2-6x)(1)}{(x+2)^2}, & x > -1 \\ -1, & x < -1 \end{cases}$$

$$= \begin{cases} \dfrac{(x+6)(x-2)}{(x+2)^2}, & x > -1 \\ -1, & x < -1. \end{cases}$$

Because

$$\lim_{x \to -1^+} f'(x) = -15 \quad \text{and} \quad \lim_{x \to -1^-} f'(x) = -1,$$

the derivative f' is undefined at $x = -1$, due to the fact that *derivatives have no "simple discontinuities"*. Therefore, our critical numbers are $x = -1$ and $x = 2$.

Using f', we can deduce that f has an absolute minimum. In particular,

$$f'(x) < 0 \quad \text{for} \quad x < -1$$

tells us that f is always decreasing when $x < -1$. Since

$$f'(x) > 0 \quad \text{for} \quad x > 2,$$

we know f is always increasing when $x > 2$.

Let us test our critical points because those are our only candidates for the absolute minimum:

$$f(-1) = \frac{(-1)^2 - 6(-1)}{1 + |-1+1|} = 7 \quad \text{and} \quad f(2) = \frac{2^2 - 6(2)}{1 + |2+1|} = -2.$$

We conclude that the absolute minimum is -2. Select (A) and continue. ■

Solution 12.

If we can find constants a and b such that

$$a\begin{pmatrix} 1 \\ -3 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

we will be almost done because

$$\begin{aligned} T\begin{pmatrix} 0 \\ 2 \end{pmatrix} &= T\left(a\begin{pmatrix} 1 \\ -3 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \\ &= aT\begin{pmatrix} 1 \\ -3 \end{pmatrix} + bT\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 5a - 2b. \end{aligned}$$

Let us find our a and b:

$$a\begin{pmatrix} 1 \\ -3 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

implies the system

$$\begin{cases} a + b &= 0 \\ -3a + b &= 2 \end{cases}$$

is true. Solving the system gives $a = -1/2$ and $b = 1/2$.

Thus,

$$T\begin{pmatrix} 0 \\ 2 \end{pmatrix} = 5\left(-\frac{1}{2}\right) - 2\left(\frac{1}{2}\right) = -\frac{7}{2}.$$

The answer must be (B)! ■

Solution 13.
We have

$$f'(x) = -17(1-x)^{16}e^{2x} + 2(1-x)^{17}e^{2x}.$$

Similarly

$$\begin{aligned} f''(x) &= 17(16)(1-x)^{15}e^{2x} - 2(17)(1-x)^{16}e^{2x} - 2(17)(1-x)^{16}e^{2x} \\ &\quad + 2^2(1-x)^{17}e^{2x} \\ &= 17(16)(1-x)^{15}e^{2x} - 4(17)(1-x)^{16}e^{2x} + 2^2(1-x)^{17}e^{2x}. \end{aligned}$$

From here, we see

$$f^{(k)}(x) = (-1)^k \frac{17!}{(17-k)!}(1-x)^{17-k}e^{2x} + \text{other terms}.$$

Since all the "other terms" have factors of $1-x$ raised to powers greater than $17-k$,

$$f^{(17)}(1) = (-1)^{17}\frac{17!}{0!}e^2 + 0 = -(17!e^2).$$

Fill in the bubble for (B) and continue. ■

Solution 14.
Consider the worst case scenario:

> A student dislikes three classmates, and three different classmates dislike this student.

Since this student cannot be teammates with any of the six, there would need to be seven groups to choose from. Choice (B) must be correct. ■

Solution 15.

Notice

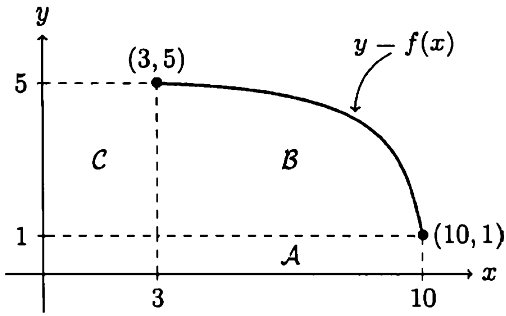
$$\frac{dy}{dx} < 0 \text{ for } x < 0, \quad \frac{dy}{dx} > 0 \text{ for } x > 0, \quad \text{and} \quad \frac{dy}{dx} = 0 \text{ for } x = 0$$

as long as $y \neq 0$. That leaves (A) and (C).

We can see that

$$\frac{dy}{dx} \to \infty \quad \text{as} \quad x \to \infty,$$

when $y \neq 0$. This eliminates (C). We shade in (A). ■



Solution 16.

Consider regions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ as shown above. We want to find

$$\int_1^5 f^{-1}(y)\, dy = \text{area}\,(\mathcal{B}) + \text{area}\,(\mathcal{C}).$$

Notice that

$$\int_3^{10} f(x)\, dx = \text{area}\,(\mathcal{A}) + \text{area}\,(\mathcal{B}) = 20.$$

Because $\mathcal{A}$ and $\mathcal{C}$ are rectangles,

$$\text{area}\,(\mathcal{A}) = 7(1) = 7 \quad \text{and} \quad \text{area}\,(\mathcal{C}) = 4(3) = 12.$$

So,

$$7 + \text{area}\,(\mathcal{B}) = 20 \quad \text{implies} \quad \text{area}\,(\mathcal{B}) = 13.$$

Therefore,

$$\int_1^5 f^{-1}(y)\, dy = 13 + 12 = 25.$$

The answer is (D)! ■

Solution 17.

Let us compute our integral:

$$\begin{aligned}\iint_S xy\, dA &= \int_{y=1}^{3} \int_{x=0}^{1} xy^2\, dxdy \\ &= \int_1^3 y^2\, dy \int_0^1 x\, dx \\ &= \left[\frac{y^3}{3}\right]_1^3 \left[\frac{x^2}{2}\right]_0^1 \\ &= \left(\frac{26}{3}\right)\left(\frac{1}{2}\right) \\ &= \frac{13}{3}.\end{aligned}$$

We conclude that (E) is correct. ■

Solution 18.

Option I is false. Consider

$$f(x) := x + 1 \quad \text{and} \quad g(x) := -\frac{1}{x}.$$

A quick check shows that both functions are in $\mathcal{F}$. But

$$(f \circ g)(x) = -\frac{1}{x} + 1 \quad \text{and} \quad (g \circ f)(x) = -\frac{1}{x+1}.$$

From here, it is clear $f \circ g \neq g \circ f$.

Option II is true. The function $i(x) := x$ is in $\mathcal{F}$ and $f \circ i = i \circ f$ for all f.

Option III is true as well. Function composition is associative generally. Therefore, it must be associative in $\mathcal{F}$.

The correct answer must be (D). Select it and continue! ■

Solution 19.

The area contained within the curve defined by the polar equation $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta,$$

when there is no overlapping area.

To find where $r = \sqrt{2}$ intersects $r^2 = 4\sin 2\theta$, let

$$\left(\sqrt{2}\right)^2 = 4\sin 2\theta.$$

So,

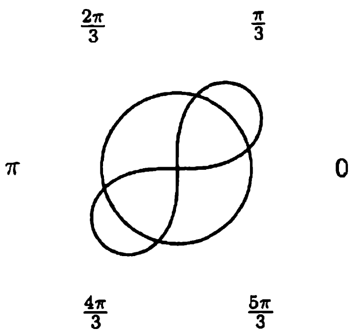
$$\sin 2\theta = \frac{1}{2} \quad \text{implies} \quad 2\theta = \frac{\pi}{6} + 2\pi n \text{ or } \frac{5\pi}{6} + 2\pi n.$$

for $n = 0, 1, -1, 2, -2, \ldots$. It follows that

$$\theta = \frac{\pi}{12} + \pi n \quad \text{or} \quad \frac{5\pi}{12} + \pi n$$

for $n = 0, 1, -1, 2, -2, \ldots$.

Let us graph our curves.



Due to symmetry, the total area is double the area between $\theta = \pi/12$ and $\theta = 5\pi/12$. Hence, it is

$$\begin{aligned}\int_{\pi/12}^{5\pi/12} 4\sin 2\theta - 2\ d\theta &= -2\cos 2\theta - 2\theta\Big|_{\pi/12}^{5\pi/12} \\ &= -2\cos\frac{5\pi}{6} - \frac{5\pi}{6} + 2\cos\frac{\pi}{6} + \frac{\pi}{6} \\ &= -2\left(-\frac{\sqrt{3}}{2}\right) - \frac{5\pi}{6} + 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6} \\ &= 2\sqrt{3} - \frac{2\pi}{3}.\end{aligned}$$

Fill in (B)! See the glossary for a list of *sine and cosine values in quadrant I.* ■

Solution 20.

One of the well know *integration properties* from Calculus says that for x in the interval $[a, b]$, we have

$$f(x) \geq g(x) \quad \text{implies} \quad \int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx.$$

We will use this property twice. Since $f'(x) \geq x$,

$$\int_x^3 f'(t)\ dt \geq \int_x^3 t\ dt.$$

Let us compute the left side of the inequality. Note that $f(3) = 7$. Therefore, the *Fundamental theorem of Calculus* tells us

$$\int_x^3 f'(x)\ dt = f(3) - f(x) = 7 - f(x).$$

Now the right side of the inequality:

$$\int_x^3 t\ dt = \frac{t^2}{2}\Big|_x^3 = \frac{9}{2} - \frac{x^2}{2}.$$

Hence,

$$7 - f(x) \geq \frac{9}{2} - \frac{x^2}{2} \quad \text{implies} \quad \frac{5}{2} + \frac{x^2}{2} \geq f(x).$$

It follows that

$$\int_0^3 \frac{5}{2} + \frac{x^2}{2}\, dx \geq \int_0^3 f(x)\, dx$$

Since

$$\int_0^3 \frac{5}{2} + \frac{x^2}{2}\, dx = 12,$$

we conclude

$$\int_0^3 f(x)\, dx \leq 12.$$

The answer must be (C). ■

Solution 21.

Option I is true. The map $f : (0,1) \to (0,1]$ such that

$$f(x) := \begin{cases} \dfrac{1}{n-1}, & x = \dfrac{1}{n} \text{ for } n \in \mathbb{N} \\ x, & \text{otherwise} \end{cases}$$

is one-to-one and onto.

Option II is correct because the image of f could be compact. For example, define $f(x) := 1/2$. Then its image is $\{1/2\}$, which is a compact set.

Option III is false. If f were continuous, one-to-one, and onto, there would have to be some value x_1 in the interval $(0,1)$ such that $f(x_1) = 1$. It would follow that f is increasing on some interval (a, x_1) and decreasing on some interval (x_1, b), where

$$0 < a < x_1 < b < 1.$$

However, this is a contradiction of the one-to-one assumption.

Answer (C) is correct. Mark it and move on! ■

Solution 22.

Let us compute our integral:

$$\begin{aligned}\int_{-\infty}^{\infty} xf(x)\,dx &= \int_{-1}^{2} \frac{2x|x|}{5}\,dx \\ &= \int_{-1}^{0} -\frac{2x^2}{5}\,dx + \int_{0}^{2} \frac{2x^2}{5}\,dx \\ &= -\frac{2x^3}{15}\Big|_{-1}^{0} + \frac{2x^3}{15}\Big|_{0}^{2} \\ &= -\frac{2}{15} + \frac{16}{15} \\ &= \frac{14}{15}.\end{aligned}$$

The answer is (A). ■

Solution 23.

Due to the *Fundamental theorem of finitely generated abelian groups*, there are two abelian groups of order 4. They are $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. The same theorem tells us that the others have more or fewer abelian groups, e.g. there is only one abelian group of order 7, specifically $\mathbb{Z}_7$, and there are three abelian groups of order 8, specifically $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We select (A). ■

Solution 24.

We know $25\pi \approx 25(3.142) = 78.55$. By applying the property $f(1+x) = f(x)$ a total of n times we know $f(n+x) = f(x)$ for n a positive integer. Therefore, $f(78+0.55) = f(0.55)$. So, $f'(25\pi)$ is the slope of the segment whose endpoints are $(1/2, 1/2)$ and $(3/4, 1)$. It follows that

$$f'(25\pi) = \frac{1 - 1/2}{3/4 - 1/2} = 2.$$

We conclude that the answer is (D). ■

Solution 25.

Since $\{x_n\}$ converges, say to x,

$$\lim_{n\to\infty} x_{n+1} - \sqrt{15 - 2x_n} = 0 \quad \text{implies} \quad x - \sqrt{15-2x} = 0$$

It follows that

$$\begin{aligned} x &= \sqrt{15-2x} \\ \Rightarrow \qquad x^2 &= 15 - 2x \\ \Rightarrow \qquad x^2 + 2x - 15 &= 0 \\ \Rightarrow \qquad (x+5)(x-3) &= 0. \end{aligned}$$

This means $x = -5$ or $x = 3$. Because the range of the square-root function is the set of nonnegative values, $x = -5$ is an extraneous solution. The correct answer must be 3, so we select (D). ■

Solution 26.

The *inverse function theorem* theorem from Calculus tells us that

$$(g' \circ f)(x) = \frac{1}{f'(x)}.$$

Since $f(2) = 0$, we know $g(0) = 2$. It follows that $(g' \circ g)(0) = g'(2)$. Notice that $f(6) = 2$, which implies

$$(g' \circ g)(0) = (g' \circ f)(6) = \frac{1}{f'(6)} = \frac{1}{3}.$$

The answer must be (C). ■

Solution 27.

The equation of the circle is

$$x^2 + (y-k)^2 = 1$$

for some k.

At the points of tangency, the circle and $y = x^2$ have the same slope. Let us compute each derivative. We will differentiate the circle equation using implicit differentiation. This yields

$$2x + 2(y-k)y' = 0 \quad \text{implies} \quad y' = -\frac{x}{y-k}.$$

Since the derivative of $y = x^2$ is $y' = 2x$, at the point of tangency,

$$2x = -\frac{x}{y-k} \quad \text{implies} \quad y - k = -\frac{1}{2}.$$

From our knowledge of the unit circle from Precalculus, we conclude that the points of intersection are

$$\left(-\frac{\sqrt{3}}{2}, k - \frac{1}{2}\right) \quad \text{and} \quad \left(\frac{\sqrt{3}}{2}, k - \frac{1}{2}\right).$$

Because the point of tangency lies on $y = x^2$,

$$k - \frac{1}{2} = \left(\frac{\sqrt{3}}{2}\right)^2 \quad \text{implies} \quad k = \frac{5}{4}.$$

Solving $x^2 + (y - 5/4)^2 = 1$ for y yields

$$y = \frac{5}{4} \pm \sqrt{1 - x^2}.$$

Since we are only concerned with the bottom half of the circle, we need only consider $y = 5/4 - \sqrt{1 - x^2}$. Hence, we just need to evaluate

$$\begin{aligned}\int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{5}{4} - \sqrt{1-x^2} - x^2\, dx &= 2\int_0^{\sqrt{3}/2} \frac{5}{4} - \sqrt{1-x^2} - x^2\, dx \\ &= 2\int_0^{\sqrt{3}/2} \frac{5}{4} - x^2\, dx - 2\int_0^{\sqrt{3}/2} \sqrt{1-x^2}\, dx\end{aligned}$$

Let us consider the two integrals separately. First,

$$2\int_0^{\sqrt{3}/2} \frac{5}{4} - x^2\, dx = 2\left[\frac{5}{4}x - \frac{x^3}{3}\right]_0^{\sqrt{3}/2} = 2\left(\frac{5\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - 0\right) = \sqrt{3}.$$

We need to use a trigonometric substitution for the other integral.

Let $x = \sin\theta$. Then $dx = \cos\theta\ d\theta$. So,

$$2\int_0^{\sqrt{3}/2} \sqrt{1-x^2}\ dx = 2\int_{x=0}^{\sqrt{3}/2} \sqrt{1-\sin^2\theta}\ \cos\theta\ d\theta$$
$$= 2\int_{\theta=0}^{\pi/3} \cos^2\theta\ d\theta$$

One of the *power reduction identities* tells us

$$\cos^2\theta = \frac{1}{2}\left(1+\cos 2\theta\right).$$

It follows that

$$2\int_0^{\sqrt{3}/2} \sqrt{1-x^2}\ dx = \int_0^{\pi/3} 1+\cos 2\theta\ d\theta$$
$$= \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/3}$$
$$= \frac{\pi}{3} + \frac{\sqrt{3}}{4}$$

Therefore,

$$\int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{5}{4} - \sqrt{1-x^2} - x^2\ dx = \sqrt{3} - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) = \frac{3\sqrt{3}}{4} - \frac{\pi}{3}.$$

This is option (C). This problem can also be done using only

$$\int_{-\sqrt{3}/2}^{\sqrt{3}/2} x^2\ dx = \frac{\sqrt{3}}{4}$$

and some trigonometry. We have had bad luck with this approach so we opted to solve the problem using more Calculus. ■

Solution 28.

For parametric equations $x = f(t)$ and $y = g(t)$, the *arc length* from $t = a$ to $t = b$ is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Let us find the pieces:

$$x = 8e^{t/2} \quad \text{implies} \quad \frac{dx}{dt} = 4e^{t/2};$$

$$y = e^t - 4t \quad \text{implies} \quad \frac{dy}{dt} = e^t - 4.$$

If $x = 8$ and $y = 1$, then $t = 0$. If $x = 8e$ and $y = e^2 - 8$, then $t = 2$. Hence, the arc length is

$$\begin{aligned}
\int_0^2 \sqrt{\left(4e^{t/2}\right)^2 + \left(e^t - 4\right)^2}\, dt &= \int_0^2 \sqrt{16e^t + e^{2t} - 8e^t + 16}\, dt \\
&= \int_0^2 \sqrt{e^{2t} + 8e^t + 16}\, dt \\
&= \int_0^2 \sqrt{(e^t + 4)^2}\, dt \\
&= \int_0^2 e^t + 4\, dt \\
&= e^t + 4t\Big|_0^2 \\
&= e^2 + 7.
\end{aligned}$$

Select answer (E) and move on! ■

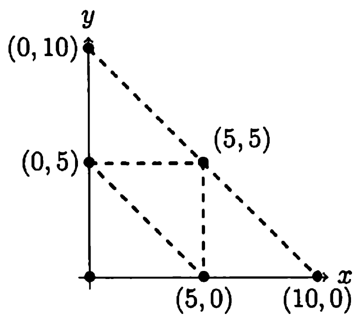
Solution 29.

Suppose the length of the segment farthest to the left is x and the length of the middle segment is y. Then the length of the right segment is $10 - x - y$. Under this formulation, the sample space is $\{(x, y) \in \mathbb{R}^2 | \; x + y < 10, \; x, y > 0\}$.

The three segments can be bent into a triangle if and only if the triangle inequality holds. As a result, we have that

$$\begin{aligned} x + y &> 10 - x - y & \implies \quad x + y &> 5 \\ x + (10 - x - y) &> y & \implies \quad 5 &> y \\ y + (10 - x - y) &> x & \implies \quad 5 &> x. \end{aligned}$$

Let us graph this situation.



From here, it is clear the area of the sample space is $10(10)/2 = 50$ and the area of the event that the sides of the segment could be bent to form a triangle is $5(5)/2 = 25/2$. Hence, the probability that the event occurs is

$$\frac{25/2}{50} = \frac{1}{4},$$

which is 25%. The answer is (C). ■

Solution 30.

This is simply a combination problem. How many ways are there to choose pairs of two vertices, where the order in which we choose the two vertices is irrelevant? The answer is

$$ {}_{10}C_2 = \frac{10!}{2!8!} = 45. $$

It is time to pick (D). ■

Solution 31.

The dimension of

$$ P = \{a_0 + a_1x + a_2x^2 + \ldots + a_7x^7 |\ a_i \in \mathbb{Z}_5\} $$

is 8. Furthermore, notice that D sends terms of the form $a_0 + a_5x^5$ to zero. As a result, the null space of D is $\{a_0 + a_5x^5 |\ a_i \in \mathbb{Z}_5\}$, which has dimension $n = 2$. Due to the *rank nullity theorem*, the rank must be $r = 8 - 2 = 6$. Therefore, (D) is correct. ■

Solution 32.

Let us go through the steps when `a = 20` and `b = 28`. Since `a < b`, we have `max = 28` and `min = 20`. Then we enter the while loop. Within the first iteration of the while loop we get `r = 8`, and we get a new `max` of 20 and a new `min` of 8. For the next iteration, we get `max = 8` and `min = 4`. And in the next iteration we get `max = 4` and `min = 0`. This is where the while loop terminates, since `min = 0` is not greater than 0. Hence, the algorithm prints `20*28/4 = 140`. This result corresponds to option (D). ■

Solution 33.

Because $\varphi(k_0)$ is true, so is $\varphi(k_0 + 1)$. If $\varphi(k_0 + 1)$ is true, so is $\varphi(k_0 + 2)$, which implies $\varphi(k_0 + 3)$ is true too, etc. It follows that $\varphi(k)$ is true for $k \geq k_0$. We have no knowledge as to whether $\varphi(k_0 - 1)$ is true or false. Hence, (E) is the correct answer. ■

Solution 34.

The function f is called the "topologist's sine curve". It is a famous counterexample to the notion that *connectedness* implies *path connectedness*. We will continue as though we do not know this.

Option I is true. It is clear that the sets

$$I_+ = \{(x, f(x)) \in \mathbb{R}^2 |\ 0 < x \leq 1\}$$

and

$$I_- = \{(x, f(x)) \in \mathbb{R}^2 |\ -1 \leq x < 0\}$$

are both connected, so the only point that could be an issue is $(0,0)$. However,

$$\overline{I}_+ = I_+ \cup \Big(\{0\} \times [-1,1]\Big)$$

and

$$\overline{I}_- = I_- \cup \Big(\{0\} \times [-1,1]\Big),$$

both contain $(0,0)$, so it would be impossible to write $I = A \cup B$ where A and B are nonempty and $\overline{A} \cap B = A \cap \overline{B} = \varnothing$.

Option II is false. Since $\lim_{x \to 0} \sin(1/x)$ does not exist, I is not the graph of a continuous function, so it is not path connected.

Option III is not true either. Because $\overline{I} = I \cup \Big(\{0\} \times [-1,1]\Big) \neq I$, the set I is not closed. Due to the *Heine-Borel theorem*, it is therefore not compact.

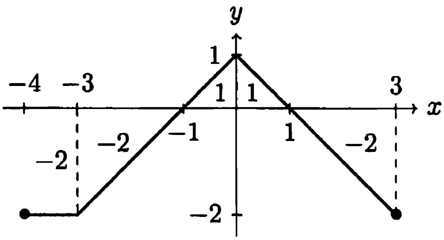
We conclude that the correct answer is (A). ■

Solution 35.

The value

$$\int_0^x f(t)\, dt$$

is the net signed area between f and the x-axis for $x > 0$, and it is the opposite of the net signed area for $x < 0$. As such, our strategy will be to break the interval $[-4,3]$ into smaller regions in which we can easily compute the signed area.



With the picture above in mind, it follows that

$$g(-4) = -(-2-2+1) = 3, \qquad g(-3) = -(-2+1) = 1,$$

$$g(-1) = -1, \qquad g(1) = 1,$$

and

$$g(3) = 1 + (-2) = -1.$$

From here, it is clear that the absolute maximum is $g(-4)$, so we fill in the bubble for (A). ■

Solution 36.

Option I is true. It is clear

$$-\sqrt{t} \le \sqrt{t}\sin\frac{1}{t} \le \sqrt{t}$$

for t in the interval $(0,1]$. Note x in I implies x^2 is in $(0,1]$. Since $-\sqrt{t}$ and $\sqrt{t}$ are bounded over the interval $(0,1]$, so are the signed areas under them. It follows that the area under $\sqrt{t}\sin(1/t)$ for $0 < t \le 1$ is bounded as well.

Option II is also true, because f' is bounded. Due to the *Fundamental theorem of Calculus*

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\int_0^{x^2} \sqrt{t}\sin\frac{1}{t}\,dt\right) \\ &= \sqrt{x^2}\sin\left(\frac{1}{x^2}\right)\cdot 2x \\ &= 2x|x|\sin\frac{1}{x^2}. \end{aligned}$$

From here, it is clear that $-2 \le f'(x) \le 2$ for x in I.

Option III is false. For $x > 0$,

$$f''(x) = 4x\sin\frac{1}{x^2} - \frac{4}{x}\cos\frac{1}{x^2}.$$

Due to the second term of $f''(x)$, the function is unbounded within the interval $(0,1]$.

Since I and II are true and option III is false, we are ready to conclude that (C) is correct. ■

Solution 37.

Our task is to see if there is an observable pattern to the sequence A, A^2, A^3,.... We have

$$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

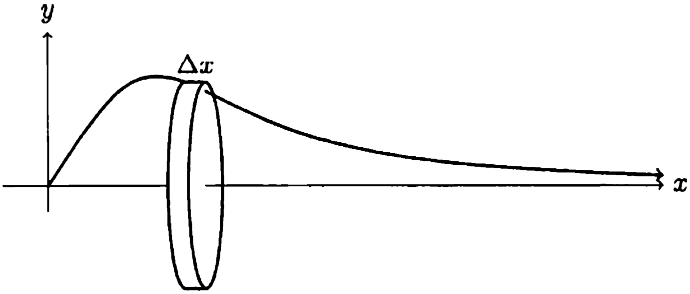
and

$$\begin{aligned} A^3 &= A^2 A \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

From here, it is not too tough to see that

$$A^{500} = A^{500 \pmod 3} = A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Therefore, the correct answer must be (B).



Solution 38.

Using the *disk method*, we see that

$$\Delta V = \pi \left(\frac{x}{1+x^3}\right)^2 \Delta x,$$

where x is in the interval $[0, \infty)$. This implies that the volume

$$V = \pi \int_0^\infty \left(\frac{x}{1+x^3}\right)^2 dx = \pi \int_0^\infty \frac{x^2}{(1+x^3)^2}\, dx.$$

A u-substation is a good path forward. Let $u = 1 + x^3$. Then $du = 3x^2 dx$. Hence,

$$\begin{aligned} V &= \frac{\pi}{3} \int_{x=0}^\infty \frac{du}{u^2}\, du \\ &= \frac{\pi}{3} \int_{u=1}^\infty \frac{du}{u^2}\, du \\ &= \frac{\pi}{3} \left[-\frac{1}{u}\right]_1^\infty \\ &= \frac{\pi}{3}. \end{aligned}$$

The volume is $\pi/3$, so the answer is (A). ∎

Solution 39.

Since $\neg D$ is $|f(x) - f(y)| < \varepsilon$, we see that the statement converts to

$$A\Big(B\,(C \Rightarrow \neg D)\Big).$$

Unfortunately, this is not a choice. However, $C \Rightarrow \neg D$ is logically equivalent $\neg C$ or $\neg D$. Let us prove this with a truth table:

C	D	$\neg C$	$\neg D$	$C \Rightarrow \neg D$	$\neg C$ or $\neg D$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Hence, the statement becomes

$$A\Big(B\,(\neg C \text{ or } \neg D)\Big).$$

The answer must be (E). ∎

Solution 40.

Let us use the *ratio test*:

$$\lim_{n\to\infty}\left|\frac{\dfrac{\big((n+1)x\big)^{n+1}}{2\cdot 4\cdot \ldots \cdot 2n\cdot 2(n+1)}}{\dfrac{(nx)^n}{2\cdot 4\cdot \ldots \cdot 2n}}\right| = \lim_{n\to\infty}\left|\frac{\big((n+1)x\big)^{n+1}}{2\cdot 4\cdot \ldots \cdot 2n\cdot 2(n+1)}\cdot\frac{2\cdot 4\cdot \ldots \cdot}{(nx)^n}\right.$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}x^{n+1}}{2(n+1)}\cdot\frac{1}{n^n x^n}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)^n|x|}{2n^n}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\frac{|x|}{2}$$

$$= \frac{e|x|}{2}.$$

The series converges when $e|x|/2 < 1$. This implies $|x| < 2/e$. So, the radius of convergence is $2/e$. This is (C). ∎

Solution 41.

Option I is false. This is equivalent to the claim that every diagonalizable matrix commutes with any matrix of the same dimensions. We can easily disprove this claim with a counterexample. Consider

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Clearly, T is diagonalizable using the standard basis of $\mathbb{R}^2$. However,

$$TX = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad XT = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Option II is true. Consider the map $\varphi_T : \mathcal{M}(V) \to \mathcal{M}(V)$ such that $\varphi_T : X \mapsto TX - XT$. The map φ_T is linear. To prove this consider X and Y in $\mathcal{M}(V)$, and α and β in $\mathbb{R}$. Then

$$\begin{aligned} \varphi_T\left(\alpha X + \beta Y\right) &= T\left(\alpha X + \beta Y\right) - \left(\alpha X + \beta Y\right)T \\ &= \alpha TX + \beta TY - \alpha XT - \beta YT \\ &= \alpha\left(TX - XT\right) + \beta\left(TY - YT\right) \\ &= \alpha\varphi_T(X) + \beta\varphi_T(Y). \end{aligned}$$

So, φ_T is linear. Furthermore, notice that the null space of φ_T is $\mathcal{U}$ and the range of φ_T is $\mathcal{W}$. Since the dimension of $\mathcal{M}(V)$ is n^2, the *rank nullity theorem* tells us that

$$\dim(\mathcal{U}) + \dim(\mathcal{W}) = n^2.$$

Option III is not necessarily true. Indeed, if T is the $n \times n$ identity matrix, then $\dim(\mathcal{U}) = n^2$.

Since I and III are false and II is true, (B) is correct. ■

Solution 42.

Lagrange's theorem says that the order of a subgroup must divide the order of the entire group. It follows that (A) and (E) are out because 5 does not divide their orders. If a group has a subgroup order 2, then the non-identity element of the subgroup must be its own inverse, which means that G cannot have a subgroup of order 2. *Cauchy's group theorem* tells us that a group of even order must have a subgroup of order 2. As a result, (B) and (C) are excluded because those orders are even. By the process of elimination, the answer is (D). Indeed, $\mathbb{Z}_{35}$ has no non-identity element which is its own inverse. ■

Solution 43.

Let us figure out an identity which we can use to simplify the expression. Due to *Euler's formula*, we know $e^{2\pi i} = 1$. This implies $\zeta^5 = 1$. Furthermore,

$$1 - \zeta^5 = 0 \quad \text{implies} \quad (1-\zeta)(1+\zeta+\zeta^2+\zeta^3+\zeta^4) = 0.$$

Since $\zeta \neq 1$, it follows that

$$1+\zeta+\zeta^2+\zeta^3+\zeta^4 = 0.$$

We are ready to simplify our sum:

$$\begin{aligned}
&3 + 3\zeta + 12\zeta^2 + 12\zeta^3 + 12\zeta^4 + 9\zeta^5 + 5\zeta^6 \\
&\qquad = 3(1+\zeta+\zeta^2+\zeta^3+\zeta^4) + 9\zeta^2 + 9\zeta^3 + 9\zeta^4 + 9\zeta^5 + 5\zeta^6 \\
&\qquad = 3(0) + 9\zeta^2(1+\zeta+\zeta^2+\zeta^3+\zeta^4) - 4\zeta^6 \\
&\qquad = 9(0) - 4\zeta^6 \\
&\qquad = -4\left(e^{2\pi i/5}\right)^6 \\
&\qquad = -4e^{12\pi i/5} \\
&\qquad = -4e^{2\pi i}e^{2\pi i/5} \\
&\qquad = -4e^{2\pi i/5}.
\end{aligned}$$

This is (A), so select it! ■

Solution 44.

Option I is true. If the *characteristic polynomial* of an $n \times n$ matrix A is

$$\det(A - \lambda I) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0,$$

then the *trace* of A is $-a_{n-1}/a_n$. In our case, it follows that the trace of A is $-3/(-1) = 3$.

Option II is true too. Indeed,

$$\det(A) = \det(A - 0I) = -(0)^3 + 3(0)^2 + 0 - 3 = -3.$$

Option III is false because -3 is not a zero of the characteristic polynomial:

$$-(-3)^3 + 3(-3)^2 + (-3) - 3 = 48.$$

Therefore, (D) is the correct choice. ■

Solution 45.

Suppose that a solution of the differential equation is of the form $y = e^{mx}$. Then

$$\frac{dy}{dx} = me^{mx} \quad \text{and} \quad \frac{d^2y}{dx^2} = m^2e^{mx}.$$

So,

$$2m^2e^{mx} + 9me^{mx} - 35e^{mx} = 0 \quad \text{implies} \quad 2m^2 + 9m - 35 = 0.$$

Solving yields $m = 5/2$ or $m = -7$.

It follows that $y = e^{-7x}$ and $y = e^{5x/2}$ are solutions. Because any linear combination of the two solutions is also a solution, the general solution is

$$y = C_1e^{-7x} + C_2e^{5x/2},$$

for C_1 and C_2 real numbers. This corresponds to (A). ■

Solution 46.

The first partial derivatives are

$$f_x(x,y) = 3x^2 + 6xy - 1 \quad \text{and} \quad f_y(x,y) = -3y^2 + 3x^2.$$

Let $f_x(x,y) = 0$ and $f_y(x,y) = 0$. Using the second equation, we have

$$-3y^2 + 3x^2 = 0 \quad \text{implies} \quad y = -x \text{ or } y = x.$$

If $y = x$ and $f_x(x,y) = 0$, then

$$9x^2 - 1 = 0 \quad \text{implies} \quad x = \pm\frac{1}{3}.$$

So, two critical points are $(1/3, 1/3)$ and $(-1/3, -1/3)$.

If $y = -x$ and $f_x(x,y) = 0$, then

$$-3x^2 - 1 = 0 \quad \text{implies} \quad x^2 = -\frac{1}{3}.$$

This is impossible, so this case adds no new critical points.

Since there were no critical points on $y = -x$, all critical points lie on the line $y = x$. This leads us to conclude that (E) is correct.

Since we are not being timed, we will use the *second derivatives test* to analyze the critical numbers. The second partial derivatives of f are $f_{xx}(x,y) = 6x + 6y$, $f_{xy}(x,y) = 6x$, $f_{yx}(x,y) = 6x$, and $f_{yy}(x,y) = -6y$. So, the Hessian matrix has determinant

$$\begin{vmatrix} 6x + 6y & 6x \\ 6x & -6y \end{vmatrix} = -36y(x+y) - 36x^2.$$

We are ready to examine each point:

$(\frac{1}{3}, \frac{1}{3})$: $H(1/3, 1/3) = -12 < 0$, which means there is a saddle point at $(1/3, 1/3)$.

$(-\frac{1}{3}, -\frac{1}{3})$: $H(-1/3, -1/3) = -12 < 0$, which means there is a saddle point at $(-1/3, -1/3)$.

■

Solution 47.

Notice that $p(x)$ is a second degree Taylor series of $f(x) = \log(x)$ whose center is 1. Furthermore, because the infinite Taylor series for $f(1.1)$ is an alternating series, has terms decreasing in magnitude, and the term of highest degree of $p(1.1)$ is negative, it must be that

$$\log(1.1) - p(1.1) > 0.$$

Taylor's theorem tells us that we can bound the difference by means of the Lagrange error bound:

$$\log(1.1) - p(1.1) \leq \frac{\sup_I |f'''|}{3!}(1.1-1)^3,$$

where I is the interval with endpoints 1 and 1.1. Since $f'''(x) = 2/x^3$, its supremum over I is $f'''(1) = 2$. Hence, the difference is less than

$$\frac{2}{3!}(1.1-1)^3 = \frac{1}{3}(0.1)^3 = \frac{1}{3}(10)^{-3}.$$

This is option (E). ■

Solution 48.

Our challenge is to find

$$10^{10^{10}} \equiv 3^{10^{10}} \pmod 7.$$

Due to *Fermat's little theorem*, we know

$$q^{p-1} \equiv 1 \pmod p,$$

where p and q are co-prime. That means $3^6 \equiv 1 \pmod 7$, which implies $3^{6m} \equiv 1 \pmod 7$ for all integers m. Furthermore,

$$10^{10} - 4 = 9999999996 = 6\,(\text{some integer}).$$

Thus,

$$\begin{aligned} 10^{10^{10}} &\equiv 3^{10^{10}} \\ &\equiv 3^{10^{10}-4} \cdot 3^4 \\ &\equiv 1 \cdot 3^4 \\ &\equiv 81 \\ &\equiv 4 \pmod 7. \end{aligned}$$

So, $10^{10^{10}}$ days from Wednesday will be the same day of the week as four days from Wednesday. Hence, it will be Sunday $10^{10^{10}}$ days from Wednesday. Select (A)!

■

Kate	2
John	2
Kate	2
John	2
Kate	2

John	2
Kate	2
John	2
Kate	2
John	2
Kate	0.5

Solution 49.

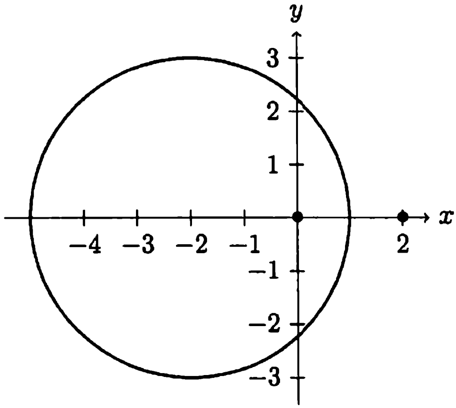
The tables above show the alternating two-day shifts when Kate and John start, respectively. It is clear Kate's rate is $1/k$ and John's rate is $1/j$. Hence, we have the system

$$\begin{cases} \dfrac{6}{k} + \dfrac{4}{j} & = 1 \\ \\ \dfrac{4.5}{k} + \dfrac{6}{j} & = 1. \end{cases}$$

Solving the system shows that $k = 9$ and $j = 12$. To find the number of days it takes the two of them to write a test when they work together, we need to solve the following equation for t:

$$\frac{t}{9} + \frac{t}{12} = 1.$$

The solution is $t = 36/7$ days, which corresponds to (D). ■



Solution 50.
Due to the *Cauchy's residue theorem*

$$\begin{aligned}\int_C \frac{dz}{z^3(z-2)} &= 2\pi i \operatorname{Res}(f,0)\\ &= \frac{2\pi i}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(\frac{1}{z-2}\right)\\ &= \pi i \lim_{z\to 0}\frac{2}{(z-2)^3}\\ &= -\frac{\pi i}{4}.\end{aligned}$$

Select (A). ■

Solution 51.
Using *integration properties* from Calculus, we have

$$\begin{aligned}\int_0^{\pi/4} \log\left|\cos\left(\theta-\frac{\pi}{4}\right)\right| - \log|\cos\theta|\ d\theta &= \int_0^{\pi/4} \log\left|\cos\left(\theta-\frac{\pi}{4}\right)\right|\ d\theta\\ &\quad - \int_0^{\pi/4} \log|\cos\theta|\ d\theta.\end{aligned}$$

Let us rewrite the first integral on the right side. Let $u = \theta - \pi/4$. Then $du = d\theta$. So,

$$\int_0^{\pi/4} \log\left|\cos\left(\theta - \frac{\pi}{4}\right)\right| \, d\theta = \int_{\theta=0}^{\pi/4} \log|\cos u| \, du$$
$$= \int_{u=-\pi/4}^{0} \log|\cos u| \, du$$

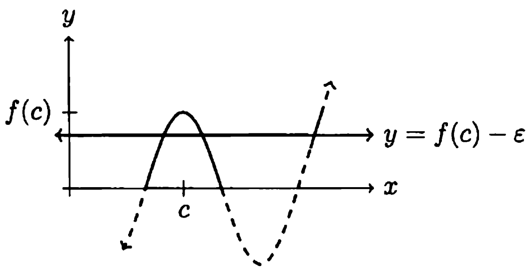
Since cosine is even the net signed area under the curve of $\log|\cos\theta|$ for θ in the interval $[-\pi/4, 0]$ is equal to the area under the curve for θ in $[0, \pi/4]$. Hence,

$$\int_{-\pi/4}^{0} \log|\cos u| \, du - \int_0^{\pi/4} \log|\cos\theta| \, d\theta = 0.$$

Choose (C) and move on! ■

Solution 52.

Option I must be false. If f is continuous, then the image of a compact set is compact. As a result, $f([a,b])$ is compact because $[a,b]$ is compact. Due to the *Heine-Borel theorem*, a subset of $\mathbb{R}$ is closed and bounded if and only if it is compact. This implies $f([a,b])$ is bounded, which contradicts I.



Option II has to be false too. If $f(x) = 0$ has two solutions, then there is a value of x say c where the derivative switches signs. It follows that there is a relative extremum at $x = c$. But then option II says there would be another value of x such that $f(x) = f(c)$. However, this implies that $f(x) = f(c) - \varepsilon$ or $f(x) = f(c) + \varepsilon$ has three or more solutions for $\varepsilon > 0$ sufficiently small. The graph illustrates the idea when there is a relative maximum at $(c, f(c))$, which would require $f(x) = f(c) - \varepsilon$ to have at least three solutions for $\varepsilon > 0$ small enough.

Option III is out because it is true. We know

$$\lim_{x \to \infty} \frac{f(x)}{x} = \infty \quad \text{implies} \quad \lim_{x \to \infty} f'(x) = \infty$$

is true because it follows from *L'Hôspital's rule*. Let us consider the other direction. Suppose $\lim_{x \to \infty} f'(x) = \infty$. Due to the *Mean value theorem*, there exists an x_n in the interval $(n, 2n)$ such that

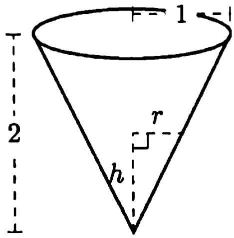
$$\frac{f(2n) - f(n)}{n} = f'(x_n) \quad \text{implies} \quad f(2n) = nf'(x_n) + f(n).$$

It follows that

$$\begin{aligned} \lim_{x \to \infty} \frac{f(x)}{x} &= \lim_{n \to \infty} \frac{f(2n)}{2n} \\ &= \lim_{n \to \infty} \frac{nf'(x_n) + f(n)}{2n} \\ &= \frac{1}{2} \lim_{n \to \infty} f'(x_n) + \frac{f(n)}{n}. \end{aligned}$$

Because $f'(x_n) \to \infty$ as $n \to \infty$, either the last limit must go to ∞ or $f(n)/n$ goes to $-\infty$. The latter scenario is impossible because $f(n)$ is increasing for n sufficiently large due to the fact that $f'(n) \to \infty$ as $n \to \infty$.

Since I and II are in and III is out, the answer is (D). ■



Solution 53.

This is a classic related rates problem. We know

$$\frac{dV}{dt} = -3.$$

and we want to find

$$\left.\frac{dh}{dt}\right|_{h=3/2}.$$

Because r and h are proportional

$$\frac{r}{h} = \frac{1}{2} \quad \text{implies} \quad r = \frac{h}{2}.$$

It follows that

$$V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

So,

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \quad \text{implies} \quad -3 = \frac{\pi}{4}\left(\frac{3}{2}\right)^2 \left.\frac{dh}{dt}\right|_{h=3/2}.$$

Therefore,

$$\left.\frac{dh}{dt}\right|_{h=3/2} = -\frac{16}{3\pi}.$$

The answer is (E). ■

Solution 54.

The *necessary and sufficient condition for a function to be analytic* is that for

$$f(x+iy) = u(x,y) + iv(x,y),$$

we have $u_x = v_y$ and $u_y = -v_x$. So,

$$g_x(x,y) = e^y \sin x \quad \text{and} \quad g_y(x,y) = -e^y \cos x.$$

We can use "partial integration" to find $g(x,y)$:

$$g(x,y) = \int e^y \sin x \, dx \qquad g(x,y) = \int -e^y \cos x \, dy$$

$$= -e^y \cos x + h_1(y) \qquad = -e^y \cos x + h_2(x).$$

The functions h_1 and h_2 are functions of y and x, respectively, because in "partial integration" we consider the variable we are not integrating with respect to a constant.

By inspection of our two equations for g, we see

$$g(x,y) = -e^y \cos x + C$$

for some real number C.

It follows that

$$g\left(\frac{\pi}{2}, 7\right) - g(0,0) = -e^7 \cos\frac{\pi}{2} + e^0 \cos 0 = 1.$$

We conclude that (C) is correct. ■

Solution 55.

Option I is false. Consider

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{implies} \quad \operatorname{tr}\left(A^2\right) = -2,$$

where $\operatorname{tr}(A^2)$ denotes the *trace* of A^2.

Option II is true. Consider an arbitrary vector $\mathbf{v}$. Then

$$\mathbf{v} = A\mathbf{v} + (\mathbf{v} - A\mathbf{v}).$$

If $A^2\mathbf{v} = A\mathbf{v}$, this means that we can deconstruct any vector $\mathbf{v}$ into the sum of a vector invariant under A and a vector which is sent to the null space of A. It follows that there is a basis for A which contains only eigenvectors with eigenvalues of 1 or 0. So, the trace must be nonnegative.

Option III is false. Consider 2×2 matrices $A = I$ and $B = I$:

$$\operatorname{tr}(A) = 2, \quad \operatorname{tr}(B) = 2, \quad \text{and} \quad \operatorname{tr}(AB) = 2.$$

We conclude that only II is correct, so we select (A). ∎

Solution 56.

After a bit of thought, it is clear that the random variable

$$X = X_1 + X_2 + \ldots + X_{100}$$

has a *binomial distribution*. As a result, its mean, variance, and standard deviation are

$$\mu = 100\left(\frac{1}{2}\right) = 50, \quad \mathrm{Var}(X) = 100\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 25,$$

and $\sigma = \sqrt{25} = 5$, respectively.

Let us go through our choices:

(A) Var(X)=25

(B) This is the probability that X is more than five standard deviations from the mean. It is highly unlikely for an outcome to be so far from the mean, so (B) is about 0.

(C) The mean of our binomial distribution is by definition

$$\sum_{k=0}^{100} k\binom{100}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{100-k}.$$

We know the mean of our distribution is 50. Furthermore, since

$$k\binom{100}{k}\left(\frac{1}{2}\right)^k \geq k\binom{100}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{100-k}$$

for $k = 0, 1, \ldots, 100$, it follows that

$$\sum_{k=0}^{100} k\binom{100}{k}\left(\frac{1}{2}\right)^k \geq \sum_{k=0}^{100} k\binom{100}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{100-k} = 50.$$

(D) Since the probability $X \leq \mu$ is equal to the probability that $X \geq \mu$, we know $100P(X \geq 60)$ is less than 50. Statistics buffs may know that $100P(X \geq 60) \approx 2.3$ due to the "empirical rule," which you do not need to know for the math GRE subject test.

(D) 30 is smaller than 50.

Hence, the answer must be (C) ■

Solution 57.

Our first task is to rewrite the sum using summation notation:

$$\lim_{n\to\infty} \frac{1}{n} + \frac{1}{2+n} + \frac{1}{4+n} + \ldots + \frac{1}{3n} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{2k+n}.$$

It is easier to convert this sum into an integral and then evaluate. As a result, we will think of the sum as a Riemann sum. Let us rewrite it a bit more from this perspective:

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{2k+n} &= \lim_{n\to\infty} \frac{1}{n} + \sum_{k=1}^{n} \frac{1}{1+2k/n} \cdot \frac{1}{n} \\ &= 0 + \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2} \cdot \frac{1}{1+2k/n} \cdot \frac{2}{n} \\ &= \frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+2k/n} \cdot \frac{2}{n}\end{aligned}$$

Let

$$f(x) := \frac{1}{x}, \quad x_k := 1 + \frac{2k}{n}, \quad \text{and} \quad \Delta x := \frac{2}{n}.$$

It follows that $a = 1$. Since our Δx implies that the length of the interval is 2, we see that $b = 3$. Hence,

$$\begin{aligned}\frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+2k/n} \cdot \frac{2}{n} &= \frac{1}{2} \int_1^3 \frac{1}{x}\, dx \\ &= \frac{1}{2} \log x \Big|_1^3 \\ &= \frac{1}{2} \log 3 - 0 \\ &= \log \sqrt{3}.\end{aligned}$$

The answer is (C). A list *logarithm properties* is located in the glossary. ■

Solution 58.

Option I is false. A subset of $\mathbb{R}$ must be both closed and bounded for it to be compact, due to the *Heine-Borel theorem*. The classic example of a bounded set which is not closed, and therefore not compact, is $(0,1)$ because the cover

$$\left\{ \left(0, 1 - \frac{1}{n}\right) \subset \mathbb{R} \,\middle|\, n = 2, 3, 4, \ldots \right\}$$

has no finite subcover.

Option II is true. What is described is sequential compactness, which is equivalent to compactness in $\mathbb{R}$. Indeed, A must be closed and bounded if each sequence has a convergent subsequence. If A were not bounded then for each n we could pick an x_n in A such that $|x_n| > n$, and no subsequence of $\{x_n\}$ would converge. If A were not closed, then it would not contain a limit point of A, say y, and for each n we would pick a y_n in the intersection of A and the ball of center y and radius $1/n$; all subsequences of $\{y_n\}$ would converge to y.

Option III is false. Consider $A_k = \{1/k\}$. We see

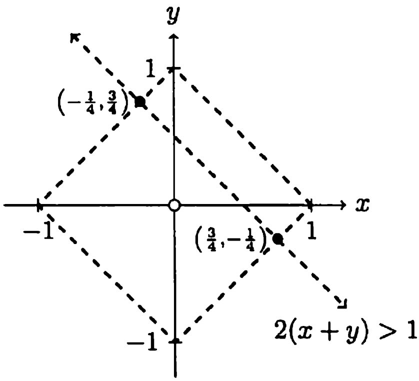
$$B = \left\{1,\ \frac{1}{2},\ \frac{1}{3}, \ldots \right\}.$$

It is clear that 0 is in $\overline{B}$. However, $\overline{A_k} = \{1/k\}$, which implies

$$\bigcup_{k=1}^{\infty} \overline{A_k} = B.$$

Since B does not contain 0, $\overline{B} \neq B$.

Fill in the bubble for (B). ∎



Solution 59.

Drawing a picture, we see that the sample space is the interior of a square with a hole in the center. The vertices of the square are $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$. A little math shows that the area of the sample space is 2 because the sides have length $\sqrt{2}$.

The event $2(x+y) > 1$ forms a rectangle within the sample space. Two of its vertices are $(1,0)$ and $(0,1)$. The other two can be found by solving the systems

$$\begin{cases} 2(x+y) & = 1 \\ -x+y & = 1 \end{cases} \quad \text{and} \quad \begin{cases} 2(x+y) & = 1 \\ -x+y & = -1. \end{cases}$$

Their solutions are $(-1/4, 3/4)$ and $(3/4, -1/4)$. Using the distance formula, we see that the sides of the rectangle have lengths $\sqrt{2}/4$ and $\sqrt{2}$. Hence, the area within the sample space where $2(x+y) > 1$ is $1/2$

We conclude that the probability of $2(x+y) > 1$ within our sample space is

$$\frac{1/2}{2} = \frac{1}{4}.$$

Select (B) and move on. ■

Solution 60.

The *work* done by a vector field $\mathbf{F}$ when an object moves along a smooth path γ from $t = a$ to $t = b$ is

$$W = \int_\gamma \mathbf{F} \cdot d\gamma = \int_a^b \mathbf{F}(\gamma) \cdot \gamma' \, dt.$$

Therefore, the work done by $\mathbf{F}$ is

$$\begin{aligned} W &= \int_0^{\pi/2} \left(\sin t, -\cos t, \frac{3}{\pi} \right) \cdot (-\sin t, \cos t, 2t) \, dt \\ &= \int_0^{\pi/2} -\sin^2 t - \cos^2 t + \frac{6}{\pi} t \, dt \\ &= \int_0^{\pi/2} -1 + \frac{6}{\pi} t \, dt \\ &= -t + \frac{3}{\pi} t^2 \Big|_0^{\pi/2} \\ &= \frac{\pi}{4}. \end{aligned}$$

This is (D)! ■

Solution 61.

Let us perform the calculations. There are

$$\binom{25}{3} = \frac{25!}{3!22!} = 2300$$

ways to choose 3 of the 25 suitcases. The respective numbers of ways to choose 2 of the 5 damaged and 1 of the 20 undamaged suitcases are

$$\binom{5}{2} = \frac{5!}{2!3!} = 10 \quad \text{and} \quad \binom{20}{1} = \frac{20!}{1!19!} = 20.$$

Hence, the probability of choosing 2 damaged and 1 undamaged suitcase is

$$\frac{10 \cdot 20}{2300} = \frac{2}{23}.$$

This corresponds to choice (C). ■

Solution 62.

Suppose the interior of C is D. Since the area of D is 2, *Green's theorem* tells us

$$\begin{aligned}\oint_C x\sin x^2\,dx + (3e^{y^2} - 2x)\,dy &= \iint_D -2 - 0\,dA \\ &= -2\,(2) \\ &= -4.\end{aligned}$$

Pick option (A). ■

Solution 63.

We want to minimize $\sqrt{x^2+y^2+z^2}$ subject to the constraint $3x - 2y + z = 4$. Since the square root is monotonic $\sqrt{x^2+y^2+z^2}$ is minimized at the same point as $f(x,y,z) := x^2+y^2+z^2$.

Therefore, our task is the following.

$$\begin{array}{lrl} \text{minimum} & f(x,y,z) & = x^2+y^2+z^2 \\ \text{subject to} & 3x-2y+z & = 4 \end{array}$$

Using the *method of Lagrange multipliers*, we see

$$2x = 3\lambda, \quad 2y = -2\lambda, \quad \text{and} \quad 2z = \lambda.$$

It follows that

$$x = 3z \quad \text{and} \quad y = -2z.$$

We have

$$g\,(3z,-2z,z) = 4 \quad \text{implies} \quad z = \frac{2}{7}.$$

So, the only extreme value is $(6/7, -4/7, 2/7)$. Because there is a minimum distance from the origin, this must be the point which produces that result. We conclude that (E) is correct. ■

Solution 64.

Option I is true. Indeed,

$$\lim_{n\to\infty} \frac{nx}{1+nx^2} = f(x) = \begin{cases} \frac{1}{x}, & 0 < x \le 1 \\ 0, & x = 0. \end{cases}$$

Option II is false. The function f_n is continuous for each n and the *uniform convergence theorem* tells us that f would have to be continuous if the sequence were to converge uniformly. Since f is not continuous, $\{f_n\}$ does not converge uniformly.

Option III is false. We have

$$\begin{aligned} \int_0^1 f_n(x)\,dx &= \int_0^1 \frac{nx}{1+nx^2} \\ &= \frac{1}{2}\log(1+nx^2)\Big|_0^1 \\ &= \frac{1}{2}\log(1+n) \end{aligned}$$

and

$$\begin{aligned} \int_{1/n}^1 \lim_{n\to\infty} f_n(x)\,dx &= \int_{1/n}^1 \frac{1}{x}\,dx \\ &= \log x\Big|_{1/n}^1 \\ &= \log n. \end{aligned}$$

It follows that

$$\begin{aligned} \left|\int_0^1 f_n(x)\,dx - \int_{1/n}^1 \lim_{n\to\infty} f_n(x)\,dx\right| &= \left|\frac{1}{2}\log(1+n) - \log n\right| \\ &= \left|\log\sqrt{1+n} - \log n\right| \\ &= \left|\log\frac{\sqrt{1+n}}{n}\right|. \end{aligned}$$

As $n \to \infty$,

$$\frac{\sqrt{1+n}}{n} \to 0 \quad \text{implies} \quad \left|\log\frac{\sqrt{1+n}}{n}\right| \to \infty.$$

Since only I was correct, we select (A). A list of *logarithm properties* is located in the glossary. ■

Solution 65.

The outermost gray area is

$$1-\pi\left(\frac{1}{2}\right)^2=1-\frac{\pi}{4}.$$

The second largest square and circle are similar to the largest square and circle. The scale factor from the larger to the smaller is $1/\sqrt{2}$. This implies the scale factor between their areas is $\left(1/\sqrt{2}\right)^2=1/2$. Indeed, the scale factor between any two consecutive areas is $1/2$. It follows that the total gray area is

$$\sum_{k=1}^{\infty}\left(1-\frac{\pi}{4}\right)\left(\frac{1}{2}\right)^{k-1}=\left(1-\frac{\pi}{4}\right)\cdot\frac{1}{1-1/2}=2-\frac{\pi}{2}.$$

This is (C), so choose it. ■

Solution 66.

Option I is a *ring*. All the properties are fairly clear, except multiplication. Let us take a look at the product of two elements in our set:

$$\begin{aligned}\left(a_1+b_1\sqrt[3]{2}+c_1\sqrt[3]{4}\right)&\left(a_2+b_2\sqrt[3]{2}+c_2\sqrt[3]{4}\right)\\ &=a_1a_2+(a_1b_2+b_1a_2)\sqrt[3]{2}+(a_1c_2+c_1a_2+b_1b_2)\sqrt[3]{4}\\ &\quad+(b_1c_2+c_1b_2)\sqrt[3]{8}+c_1c_2\sqrt[3]{16}\\ &=a_1a_2+2b_1c_2+2c_1b_2+(a_1b_2+b_1a_2+2c_1c_2)\sqrt[3]{2}\\ &\quad+(a_1c_2+c_1a_2+b_1b_2)\sqrt[3]{4}.\end{aligned}$$

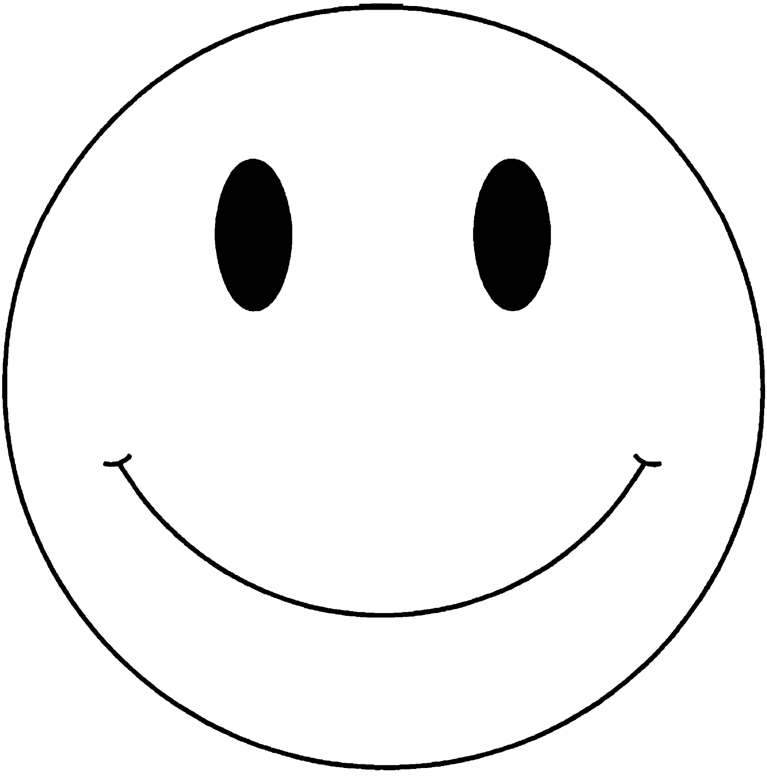
Option II is also right. Addition and multiplication seem the most suspicious. However,

$$\frac{m_1}{2^{n_1}}+\frac{m_2}{2^{n_2}}=\frac{m_1 2^{n_2}+m_2 2^{n_1}}{2^{n_1+n_2}}\quad\text{and}\quad\frac{m_1}{2^{n_1}}\cdot\frac{m_2}{2^{n_2}}=\frac{m_1m_2}{2^{n_1+n_2}}$$

are both in our set.

Option III is the infamous quaternions, which form a ring.

Since all three sets are rings, the answer is (E). ■



Glossary

Antiderivatives Useful antiderivatives.

- $\displaystyle\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
- $\displaystyle\int e^u\, du = e^u + C$
- $\displaystyle\int \frac{du}{u} = \log|u| + C$
- $\displaystyle\int \sin u\, du = -\cos u + C$
- $\displaystyle\int \cos u\, du = \sin u + C$
- $\displaystyle\int \tan u\, du = -\log|\cos u| + C$
- $\displaystyle\int \frac{du}{1+u^2} = \text{Arctan}\, u + C$

Arc length

- The arc length of the curve from $x = a$ to $x = b$ described by $y = f(x)$ is

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dt}\right)^2}\, dx.$$

- The arc length of the curve from $t = a$ to $t = b$ described by $(x, y) = \big(f(t), g(t)\big)$ is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

- The arc length of the curve from $\theta = \alpha$ to $\theta = \beta$ described by the polar equation $r = f(\theta)$ is

$$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Area with polar equations Consider the polar equation $r = f(\theta)$. The area between its graph and the pole from $\theta = \alpha$ to $\theta = \beta$ is

$$\frac{1}{2}\int_\alpha^\beta r^2\, d\theta,$$

as long as there is no overlapping area.

Basis The set $\mathcal{B}$ is a basis of a vector space V over a field $\mathbb{F}$ if and only if the following hold.

- The set $\mathcal{B}$ is nonempty.
- Every element in V can be written as a linear combination of elements in $\mathcal{B}$.
- The elements of $\mathcal{B}$ are linearly independent.

Binomial distribution Suppose n independent trials are conducted, each of which can either end in success or failure. Let p be the probability success. Then the probability of exactly k trials ending in success is

$$\binom{n}{k} p^k (1-p)^{n-k}.$$

Furthermore, in a binomial distribution:

- The mean is $\mu = np$.

- The variance is $\sigma^2 = np(1-p)$.
- The standard deviation is $\sigma = \sqrt{np(1-p)}$.

Cauchy's group theorem Suppose G is a group and the prime p divides the order of G. Then there is a cyclic subgroup of G of order p.

Cauchy's residue theorem Suppose U is a simply connected open subset of $\mathbb{C}$ and f is a function holomorphic on $U \setminus \{a_1, a_2, \ldots, a_n\}$. Let C be a positively oriented simple closed curve whose graph is contained in U, and suppose $a_1, a_2, \ldots, a_n$ are inside of C. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, a_k),$$

where for a_k a pole of order m

$$\operatorname{Res}(f, a_k) = \frac{1}{(m-1)!} \lim_{z \to c} \frac{d^{m-1}}{dz^{m-1}} \Big((z - a_k)^m f(z) \Big).$$

Characteristic polynomial The characteristic polynomial of an $n \times n$ matrix A is

$$p(\lambda) = \det(A - \lambda I).$$

The zeros of $p(\lambda)$ are the eigenvalues of A. If

$$p(\lambda) = a_0 + a_1\lambda + \ldots + a_{n-1}\lambda^{n-1} + a_n\lambda^n,$$

the trace of A is $-a_{n-1}/a_n$ and $\det(A) = a_0$. If A has n linearly independent eigenvectors, then a_0 is the product of their eigenvalues.

Compact Consider the set X under some topology. A collection $\mathcal{U}$ of open sets is said to be an "open cover" of X if and only if

$$X \subseteq \bigcup_{U \in \mathcal{U}} U.$$

The set X is compact if and only if every open cover $\mathcal{U}$ has a finite subcover $\{U_1, U_2, \ldots, U_n\} \subseteq \mathcal{U}$ such that

$$X \subseteq U_1 \cup U_2 \cup \ldots \cup U_n.$$

Connectedness A topological space X is connected if there are no open, disjoint, and nonempty subsets A and B of X such that $A \cup B$ is equal to X. A subset U of X is connected if $U = A \cup B$ implies $\overline{A} \cap B \neq \varnothing$ and $A \cap \overline{B} \neq \varnothing$, whenever A and B are nonempty.

Derivative rules Suppose that f and g are differentiable on some domain D. Assume c and n are constants.

- Constant rule:
$$\frac{d}{dx}(c) = 0.$$
- Constant multiple rule:
$$(c \cdot f)'(x) = c \cdot f'(x).$$
- Power rule:
$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$
- Sum and difference rules:
$$(f \pm g)'(x) = f'(x) \pm g'(x).$$
- Product rule:
$$(f \cdot g)'(x) = f(x)g'(x) + f'(x)g(x).$$
- Quotient Rule:
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2},$$
where $g(x) \neq 0$.
- Chain rule:
$$(f \circ g)'(x) = f'\big(g(x)\big)g'(x).$$

Derivatives Useful derivatives.

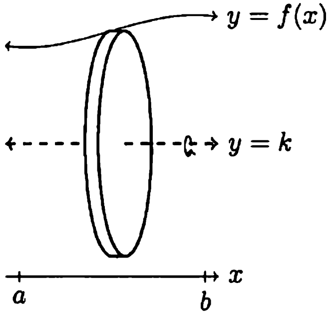
- $\dfrac{d}{dx}u^n = nu^{n-1}u'$
- $\dfrac{d}{dx}e^u = u'e^u$
- $\dfrac{d}{dx}\log|u| = \dfrac{u'}{u}$
- $\dfrac{d}{dx}\sin u = u'\cos u$
- $\dfrac{d}{dx}\cos u = -u'\sin u$
- $\dfrac{d}{dx}\tan u = u'\sec^2 u$
- $\dfrac{d}{dx}\text{Arctan } u = \dfrac{u'}{1+u^2}$

Derivatives have no "simple discontinuities" Suppose f is differentiable on the open interval (a, b) and c is in (a, b). Then

$$\lim_{x\to c^-} f'(x) = \lim_{x\to c^+} f'(x),$$

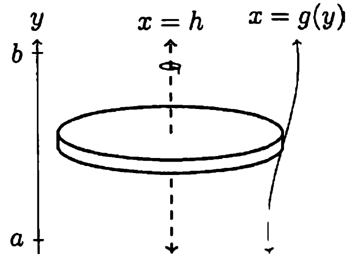
if both limits exist.

Disk method Consider the solid generated by rotating the region between $y = k$ and $y = f(x)$ about $y = k$. Then its volume from $x = a$ to $x = b$ is



$$\pi \int_a^b \left(f(x) - k\right)^2 dx$$

The rotation about $x = h$ of the region between $x = h$ and $x = g(y)$ generates a three dimensional object. Its volume from $y = a$ to $y = b$ is



$$\pi \int_a^b (g(y) - h)^2 \, dy$$

e

$$e := \sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n.$$

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ for all θ in $\mathbb{C}$.

Fermat's little theorem Suppose p is a prime number and a is an integer. Then

$$a^p \equiv a \pmod{p}.$$

If p does not divide a,

$$a^{p-1} \equiv 1 \pmod{p}.$$

First derivative test Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on the open interval (a, b) and differentiable on $(a, b) \setminus \{c\}$, where $a < c < b$.

- If $f'(x) > 0$ for x in (a, c) and $f'(x) < 0$ for x in (c, b), then $f(c)$ is a relative maximum.
- If $f'(x) < 0$ for x in (a, c) and $f'(x) > 0$ for x in (c, b), then $f(c)$ is a relative minimum.

In other words, if f' switches from positive to negative at c then $f(c)$ is a relative maximum, and if f' switches from negative to positive at c then $f(c)$ is a relative minimum.

Fundamental counting principle Suppose there are n_1 ways for an event to occur, and n_2 ways for another independent event to occur. Then there are

$$n_1 \cdot n_2$$

ways for the two events to occur. More generally, if there are n_i ways for the i-th independent event to occur, where $i = 1, 2, \ldots, m$, there are

$$n_1 \cdot n_2 \cdot \ldots \cdot n_m$$

ways for the consecutive occurrence of the m events to occur.

Fundamental theorem of Calculus Suppose f is continuous on the closed interval $[a, b]$. Then

$$\int_a^b f(x)\, dx = F(b) - F(a),$$

where $F'(x) = f(x)$. Another version of this theorem, often called the "Second fundamental theorem of Calculus" states that

$$\frac{d}{dx}\left(\int_a^x f(t)\, dt\right) = f(x),$$

when f is continuous on an interval containing a and x.

Fundamental theorem of finitely generated abelian groups Let G be a finitely generated abelian group. It is isomorphic to an expression of the form

$$\mathbb{Z}^k \times \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \ldots \mathbb{Z}_{p_n^{\alpha_m}},$$

where $k, \alpha_1, \alpha_2, \ldots, \alpha_m$ are whole numbers and $p_1, p_2, \ldots, p_m$ are primes which are not necessarily distinct. Alternatively, G is isomorphic to an expression of the form

$$\mathbb{Z}^k \times \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \ldots \times \mathbb{Z}_{r_n},$$

where $k, r_1, r_2, \ldots, r_n$ are whole numbers and r_i divides r_{i+1} for all $i = 1, 2, \ldots, n-1$. The values of k and each r_i are uniquely determined by G.

Green's theorem Let C be a piecewise smooth simple closed curve in the xy-plane, which is oriented counterclockwise. Suppose D is the region bounded by C. Assume L and M are functions of x and y and have continuous partial derivatives on an open region containing D. Then

$$\oint_C L\,dx + M\,dy = \iint_D \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\,dA.$$

Group The set G together with a binary operation $\cdot$ is a group if and only if the following properties of G and $\cdot$ hold:

- Closed: a and b in G implies $a \cdot b$ in G.
- Associative: for all a, b, and c in G, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- Contains the identity element: there is an element e such that $e \cdot a = a \cdot e = a$ for all a in G.
- Contains inverse elements: for all a in G there is a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Heine-Borel theorem For all positive integers n, a set in $\mathbb{R}^n$ is closed and bounded if and only if it is compact.

Inflection point Suppose f is a twice differentiable real-valued function on the set $(a, b) \setminus \{c\}$, where $a < c < b$. A point $(c, f(c))$ is an inflection point of the graph of f if and only if $f''(x) < 0$ for x in (a, c) and $f''(x) > 0$ for x in (c, b), or $f''(x) > 0$ for x in (a, c) and $f''(x) < 0$ for x in (c, b). In other words, $(c, f(c))$ is an inflection point if and only if f'' switches signs at c.

Integration by parts Suppose u and v are differentiable functions of x. Then

$$\int u\,dv = uv - \int v\,du.$$

Integration properties Suppose f and g are integrable real-valued functions over the closed interval $[a, b]$. Let α and β be in $\mathbb{R}$, and let c be in $[a, b]$. Then

- $\int_a^b \alpha f(x) + \beta g(x)\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx$
- $\int_a^b f(x)\, dx = - \int_b^a f(x)\, dx$
- $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$
- $\int_c^c f(x)\, dx = 0$
- $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$, whenever $f(x) \leq g(x)$ for x in $[a, b]$

Inverse function theorem Suppose f is one-to-one and has a continuous derivative f' within some connected open neighborhood of $x = a$. Further, assume the graph of f within this neighborhood contains the point (a, b). Then

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

L'Hôspital's rule Let f and g be functions differentiable on $(a, b) \setminus \{c\}$, and $g(x) \neq 0$ for all x in $(a, b) \setminus \{c\}$, where c is in (a, b). Assume

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$$

or

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \pm\infty.$$

Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

Lagrange's theorem Suppose G is a finite group and H is a subgroup of G. The order of H divides the order of G.

Limits Some well known limits from Calculus.

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$

- $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$
- $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$
- $\lim_{x \to \infty} x^{1/x} = 1$

Logarithm properties The GRE assumes log is base e <u>not</u> base 10.

- $\int \frac{du}{u} = \log |u| + C$
- $\log x = y \iff e^y = x$
- $\log(e^x) = x$ and $e^{\log x} = x$
- $\log 1 = 0$
- $\log e = 1$
- $\log(xy) = \log x + \log y$
- $\log\left(\frac{x}{y}\right) = \log x - \log y$
- $\log x^y = y \log x$

Maclaurin series Suppose the n-th derivative of f exists and is continuous. The Maclaurin polynomial of degree n for f is

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

If f is infinitely differentiable, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Well known Maclaurin series include:

- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

- $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$
- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, where $-1 < x < 1$

Mean value theorem Suppose f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Method of Lagrange multipliers Suppose $f(x, y, z)$ and $g(x, y, z)$ have continuous first order partial derivatives, and there is a constant k such that $g(x, y, z) = k$. The relative extrema of f occur at points (x, y, z) that satisfy

$$f_x(x, y, x) = \lambda g_x(x, y, z), \qquad f_y(x, y, z) = \lambda g_y(x, y, z),$$
$$\text{and} \quad f_z(x, y, z) = \lambda g_z(x, y, z)$$

for some λ in $\mathbb{R}$.

Necessary and sufficient condition for a function to be analytic The function $f(x + iy) = u(x, y) + iv(x, y)$ is analytic if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Path connectedness A topological space X is path connected if and only if for all elements x_0 and x_1 of X, there is a continuous function $f : [0, 1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. If a topological space is path connected, then it is connected. However, the converse is false.

Power reduction identities Suppose θ is in $\mathbb{R}$.

- $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$
- $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$

- $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$

Probability properties Let X be the sample space, and A and B be events in X.

- $P(X) = 1$
- $P(\varnothing) = 0$
- $0 \leq P(A) \leq 1$
- $P(X \setminus A) = 1 - P(A)$
- $P(B) \leq P(A)$ if $B \subseteq A$
- $P(A \setminus B) = P(A) - P(A \cap B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cap B) = P(A) \cdot P(B)$ if A and B are independent events

Pythagorean identities Suppose θ is in $\mathbb{R}$. Then

$$\cos^2\theta + \sin^2\theta = 1, \quad 1+\tan^2\theta = \sec^2\theta, \quad \text{and} \quad 1+\cot^2\theta = \csc^2\theta.$$

Rank nullity theorem Suppose V is a finite dimensional vector space and let $T : V \to W$ be a linear map. Then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

Ratio test Consider the series $S := \sum_{n=1}^{\infty} a_n$ and the limit $L := \lim_{n\to\infty} \left| \frac{a_{\text{?}}}{\text{?}} \right.$

- If $L < 1$ then S converges absolutely.
- If $L > 1$ then S does not converge.
- If $L = 1$ or L does not exist, then the test is inconclusive.

Ring A set R is a ring if and only if it is an abelian (commutative) group under $+$ and the following properties of R and $\cdot$ hold

- Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, and c in R.

- Distributive on the right: $a \cdot (b + c) = a \cdot b + a \cdot c$ for all a, b, and c in R.
- Distributive on the left: $(b+c) \cdot a = b \cdot a + c \cdot a$ for all a, b, and c in R.

Second derivatives test Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous second order partial derivatives in some $E \subseteq \mathbb{R}^2$. Suppose the point (a, b) in E is a critical point, i.e. $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Define

$$H_f(x, y) := \det \begin{pmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{pmatrix}$$
$$= f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2 .$$

- If $f_{xx}(a, b) > 0$ and $H_f(a, b) > 0$, then $f(a, b)$ is a relative minimum.
- If $f_{xx}(a, b) < 0$ and $H_f(a, b) > 0$, then $f(a, b)$ is a relative maximum.
- If $H_f(a, b) < 0$, then (a, b) is a saddle point.
- If $H_f(a, b) = 0$, then the test gives no information.

Sine and cosine values in quadrant I To convert the radian measures in the first row to degrees, simply multiply $180°/\pi$.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1

Slope and concavity of curves with parametric equations Suppose $x = f(t)$ and $y = g(t)$ are twice differentiable real-valued functions and t is a real number. At the point corresponding to t, the slope of the curve described by $\{(x(t), y(t)) \in \mathbb{R}^2 | \ t \text{ real}\}$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$

when $dx/dt \neq 0$. Furthermore, at the point corresponding to t, the concavity of the curve $\{(x(t), y(t)) \in \mathbb{R}^2 | \ t \text{ real}\}$ is

$$\frac{d^2y}{dx^2} = \frac{d^2y/dtdx}{dx/dt},$$

where $dx/dt \neq 0$.

Summation formulas

- $\sum_{k=1}^{n} a_k + b_k = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
- $\sum_{k=1}^{n} c \cdot a_k = c \sum_{k=1}^{n} a_k$
- $\sum_{k=1}^{n} 1 = n$
- $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$
- $\sum_{k=1}^{n} a_k = \frac{n(a_1 + a_n)}{2}$, where $\sum a_k$ is an arithmetic series
- $\sum_{k=1}^{n} a_1 r^{k-1} = \frac{a_1(1-r^n)}{1-r}$, where $r \neq 1$
- $\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r}$, where $|r| < 1$

Taylor's theorem Let f be a real-valued function defined on some set which contains the interval $[a, b]$. Suppose $f^{(n)}$ is continuous on $[a, b]$ and $f^{(n+1)}$ exists on the open interval (a, b), where n is a positive integer. Then for each x and c in $[a, b]$ there is a z between x and c such that

$$f(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} + \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Hence, f can be approximated by the polynomial

$$\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

and the error is less than or equal to the Lagrange error bound of

$$\frac{\sup_{z\in I}|f^{(n+1)}(z)|}{(n+1)!}|x-c|^{n+1},$$

where I is the open interval with endpoints x and c.

Trace The trace of a matrix A is defined to be the sum of the entries in the main diagonal of A. If tr denotes the trace function then $\text{tr}(AB) = \text{tr}(BA)$ for all square matrices A and B where the dimensions are equal.

Uniform continuity Consider the metric spaces (X, ρ) and (Y, σ). A function $f : X \to Y$ is uniformly continuous on $U \subseteq X$ if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sigma\left(f(x_1), f(x_2)\right) < \varepsilon \quad \text{whenever} \quad \rho(x_1, x_2) < \delta,$$

for all x_1 and x_2 in U.

Uniform convergence theorem Suppose $\{f_n\}$ is a sequence of continuous functions that converge pointwise to the function f. If $\{f_n\}$ converges uniformly to f on an interval U, then f is continuous on U.

Volume of a parallelepiped The volume of the parallelepiped determined by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is

$$\pm\det\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\\ \mathbf{w}\end{pmatrix}$$

where the vectors are row vectors in $\mathbb{R}^3$ and the $\pm$ makes the determinate positive.

Work Let $C := \{\gamma(t) | \; a \leq t \leq b\}$, where $\gamma : \mathbb{R} \to \mathbb{R}^n$ is differentiable in each coordinate. Then the work done by a vector field $\boldsymbol{F}$ over C is

$$W = \int_C \boldsymbol{F} \bullet \, d\gamma = \int_a^b \boldsymbol{F} \bullet \gamma'(t) \; dt.$$