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2011 MA

Test Paper Code: MA

Time: 3 Hours **Maximum Marks: 300**

INSTRUCTIONS

1. This question-cum-answer booklet has **40** pages and has **29** questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.
2. Write your **Registration Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on Page No. **7**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used:
   (a) For each correct answer, you will be awarded **6 (Six)** marks.
   (b) For each wrong answer, you will be awarded **-2 (Negative two)** marks.
   (c) Multiple answers to a question will be treated as a wrong answer.
   (d) For each un-attempted question, you will be awarded **0 (Zero)** marks.
   (e) Negative marks for objective part will be carried over to total marks.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing last only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Clip board, log tables, slide rule, calculator, cellular phone and electronic gadgets in any form are NOT allowed.**
11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.
12. Refer to special instructions/useful data on the reverse.

2011 MA

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

REGISTRATION NUMBER						

Name:

Test Centre:

Do not write your Registration Number or Name anywhere else in this question-cum-answer booklet.

I have read all the instructions and shall abide by them.

……………………………………………...

Signature of the Candidate

I have verified the information filled by the Candidate above.

……………………………………………...

Signature of the Invigilator

Special Instructions/ Useful Data

$\mathbb{R}$: The set of all real numbers

$\mathbb{N}$: The set of all natural numbers, that is, the set of all positive integers 1, 2, …. .

$\mathbb{Z}$: The set of all integers

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IMPORTANT NOTE FOR CANDIDATES

- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-29 (subjective questions) carry *fifteen* marks each.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 7 only.**

Q.1 Let $a_n = \sum_{k=1}^{n} \frac{n}{n^2+k}$, for $n \in \mathbb{N}$. Then the sequence $\{a_n\}$ is

(A) Convergent
(B) Bounded but not convergent
(C) Diverges to ∞
(D) Neither bounded nor diverges to ∞

Q.2 The number of real roots of the equation $x^3 + x - 1 = 0$ is

(A) 0 (B) 1 (C) 2 (D) 3

Q.3 The value of $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+kn}}$ is

(A) $2(\sqrt{2}-1)$ (B) $2\sqrt{2}-1$ (C) $2-\sqrt{2}$ (D) $\frac{1}{2}(\sqrt{2}-1)$

Q.4 Let V be the region bounded by the planes $x=0,\ x=2,\ y=0,\ z=0$ and $y+z=1$. Then the value of the integral $\iiint_V y\, dx\, dy\, dz$ is

(A) $\frac{1}{2}$ (B) $\frac{4}{3}$ (C) 1 (D) $\frac{1}{3}$

Q.5 The solution $y(x)$ of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ satisfying the conditions $y(0)=4,\ \frac{dy}{dx}(0) = 8$ is

(A) $4e^{2x}$ (B) $(16x+4)e^{-2x}$ (C) $4e^{-2x}+16x$ (D) $4e^{-2x}+16xe^{2x}$

Q.6 If y^a is an integrating factor of the differential equation $2xy\,dx - (3x^2 - y^2)\,dy = 0$, then the value of a is

(A) -4 (B) 4 (C) -1 (D) 1

Q.7 Let $\vec{F} = a\,y\,\hat{i} + z\,\hat{j} + x\,\hat{k}$ and C be the positively oriented closed curve given by $x^2 + y^2 = 1$, $z = 0$. If $\oint_C \vec{F} \cdot d\vec{r} = \pi$, then the value of a is

(A) -1 (B) 0 (C) $\frac{1}{2}$ (D) 1

Q.8 Consider the vector field $\vec{F} = (a\,x + y + a)\,\hat{i} + \hat{j} - (x + y)\,\hat{k}$, where a is a constant. If $\vec{F} \cdot \text{curl}\,\vec{F} = 0$, then the value of a is

(A) -1 (B) 0 (C) 1 (D) $\frac{3}{2}$

Q.9 Let G denote the group of all 2×2 invertible matrices with entries from $\mathbb{R}$. Let

$$H_1 = \{A \in G : \det(A) = 1\} \text{ and } H_2 = \{A \in G : A \text{ is upper triangular}\}.$$

Consider the following statements:

P : H_1 is a normal subgroup of G
Q : H_2 is a normal subgroup of G.

Then

(A) Both P and Q are true
(B) P is true and Q is false
(C) P is false and Q is true
(D) Both P and Q are false

Q.10 For $n \in \mathbb{N}$, let $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$. Then the number of units of $\mathbb{Z}/11\mathbb{Z}$ and $\mathbb{Z}/12\mathbb{Z}$, respectively, are

(A) 11, 12 (B) 10, 11 (C) 10, 4 (D) 10, 8

Q.11 Let A be a 3×3 matrix with trace (A) = 3 and det (A) = 2. If 1 is an eigenvalue of A, then the eigenvalues of the matrix $A^2 - 2I$ are

(A) $1,\ 2(i-1),\ -2(i+1)$ (B) $-1,\ 2(i-1),\ 2(i+1)$
(C) $1,\ 2(i+1),\ -2(i+1)$ (D) $-1,\ 2(i-1),\ -2(i+1)$

Q.12 Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, where $n \geq 2$. For $k \leq n$, let

$$E=\{v_1,v_2,...,v_k\}\subseteq \mathbb{R}^n \text{ and } F=\{Tv_1,Tv_2,...,Tv_k\}.$$

Then

(A) If E is linearly independent, then F is linearly independent
(B) If F is linearly independent, then E is linearly independent
(C) If E is linearly independent, then F is linearly dependent
(D) If F is linearly independent, then E is linearly dependent

Q.13 For $n \neq m$, let $T_1:\mathbb{R}^n \to \mathbb{R}^m$ and $T_2:\mathbb{R}^m \to \mathbb{R}^n$ be linear transformations such that T_1T_2 is bijective. Then

(A) rank $(T_1) = n$ and rank $(T_2) = m$ (B) rank $(T_1) = m$ and rank $(T_2) = n$
(C) rank $(T_1) = n$ and rank $(T_2) = n$ (D) rank $(T_1) = m$ and rank $(T_2) = m$

Q.14 The set of all x at which the power series $\sum_{n=1}^{\infty} \frac{n}{(2n+1)^2}(x-2)^{3n}$ converges is

(A) $[-1, 1)$ (B) $[-1,1]$ (C) $[1, 3)$ (D) $[1, 3]$

Q.15 Consider the following subsets of $\mathbb{R}$:

$$E=\left\{\frac{n}{n+1} : n\in\mathbb{N}\right\}, \qquad F=\left\{\frac{1}{1-x} : 0\leq x<1\right\}.$$

Then

(A) Both E and F are closed (B) E is closed and F is NOT closed
(C) E is NOT closed and F is closed (D) Neither E nor F is closed

Space for rough work

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Space for rough work

Space for rough work

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Answer Table for Objective Questions

Write the Code of your chosen answer only in the 'Answer' column against each Question No. Do not write anything else on this page.

Question Number	Answer	Do not write in this column
01		
02		
03		
04		
05		
06		
07		
08		
09		
10		
11		
12		
13		
14		
15		

FOR EVALUATION ONLY

Number of Correct Answers		Marks	(+)
Number of Incorrect Answers		Marks	(–)
Total Marks in Questions 1-15			( )

Q.16 (a) Let $\{a_n\}$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} a_n$ converges, and let $\{k_n\}$ be a strictly increasing sequence of positive integers. Show that $\sum_{n=1}^{\infty} a_{k_n}$ also converges. (9)

(b) Suppose $f : [0,1] \to \mathbb{R}$ is differentiable and $f'(x) \leq 1$ at every $x \in (0,1)$. If $f(0) = 0$ and $f(1) = 1$, show that $f(x) = x$ for all $x \in [0,1]$. (6)

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Q.17 (a) Suppose f is a real valued function defined on an open interval I and differentiable at every $x \in I$. If $[a, b] \subset I$ and $f'(a) < 0 < f'(b)$, then show that there exists $c \in (a,b)$ such that $f(c) = \min_{a \leq x \leq b} f(x)$. (9)

(b) Let $f : (a, b) \to \mathbb{R}$ be a twice differentiable function such that f'' is continuous at every point in (a, b). Prove that

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x)$$

for every $x \in (a, b)$. (6)

Q.18 Find all critical points of the following function and check whether the function attains maximum or minimum at each of these points:

$$u(x, y) = x^4 + y^4 - 2x^2 - 2y^2 + 4xy, \qquad (x, y) \in \mathbb{R}^2.$$

. (15)

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Q.19 (a) Let $\varphi:[a,b]\to\mathbb{R}$ be differentiable and $[c,d] = \{\varphi(x): a \le x \le b\}$, and let $f:[c,d]\to\mathbb{R}$ be continuous. Let $g:[a,b]\to\mathbb{R}$ be defined by $g(x) = \int_c^{\varphi(x)} f(t)\,dt$ for $x \in [a,b]$. Then show that g is differentiable and $g'(x) = f(\varphi(x))\,\varphi'(x)$ for all $x \in [a,b]$. (9)

(b) If $f:[0,1]\to\mathbb{R}$ is such that $\int_0^{\sin x} f(t)dt = \frac{\sqrt{3}}{2}x$ for all $x \in \mathbb{R}$, then find $f\left(\frac{1}{2}\right)$. (6)

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Q.20 Find the area of the surface of the solid bounded by the cone $z = 3 - \sqrt{x^2 + y^2}$ and the paraboloid $z = 1 + x^2 + y^2$. (15)

Q.21 Obtain the general solution of each of the following differential equations:

(a) $y - x\dfrac{dy}{dx} = \dfrac{dy}{dx}\, y^2 e^y$. (6)

(b) $\dfrac{dy}{dx} = \dfrac{x+2y+8}{2x+y+7}$. (9)

Q.22 (a) Determine the values of $b>1$ such that the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 0, \quad 1<x<b$$

satisfying the conditions $y(1) = 0 = y(b)$ has a nontrivial solution. (9)

(b) Find $v(x)$ such that $y(x)=e^{4x}\, v(x)$ is a particular solution of the differential equation

$$\frac{d^2 y}{dx^2} - 8\frac{dy}{dx} + 16y = \left(2x + 11x^{10} + 21x^{20}\right)e^{4x}.$$

(6)

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Q.23 (a) Change the order of integration in the double integral

$$\int_{-1}^{2}\left(\int_{-x}^{2-x^2} f(x,y)\,dy\right)dx.$$

(6)

(b) Let $\vec{F}=(x^2-xy^2)\hat{i}+y^2\,\hat{j}$. Using Green's theorem, evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, where C is the positively oriented closed curve which is the boundary of the region enclosed by the $x-$axis and the semi-circle $y=\sqrt{1-x^2}$ in the upper half plane.

(9)

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Q.24 (a) If $\vec{F} = (x^2 + y - 4)\hat{i} + 3xy\,\hat{j} + (2xz + z^2)\hat{k}$, then evaluate the surface integral $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$, where S is the surface of the cone $z = 1 - \sqrt{x^2 + y^2}$ lying above the $xy-$plane and $\hat{n}$ is the unit normal to S making an acute angle with $\hat{k}$. (9)

(b) Show that the series $\sum_{n=1}^{\infty} \frac{x}{\sqrt{n}(1 + n^p x^2)}$ converges uniformly on $\mathbb{R}$ for $p > 1$. (6)

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Q.25 (a) Find a value of c such that the following system of linear equations has no solution:

$$x+2y+3z=1,$$
$$3x+7y+cz=2,$$
$$2x+cy+12z=3.$$

(6)

(b) Let V be the vector space of all polynomials with real coefficients of degree at most n, where $n \geq 2$. Considering elements of V as functions from $\mathbb{R}$ to $\mathbb{R}$, define

$$W = \left\{ p \in V : \int_0^1 p(x)dx = 0 \right\}.$$

Show that W is a subspace of V and $\dim(W) = n$.

(9)

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Q.26 (a) Let A be a 3×3 real matrix with $\det(A)=6$. Then find $\det(\operatorname{adj} A)$. (6)

(b) Let v_1 and v_2 be non-zero vectors in $\mathbb{R}^n$, $n\geq 3$, such that v_2 is not a scalar multiple of v_1. Prove that there exists a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ such that $T^3 = T$, $Tv_1 = v_2$, and T has at least three distinct eigenvalues. (9)

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Q.27 (a) If E is a subset of $\mathbb{R}$ that does not contain any of its limit points, then prove that E is a countable set. (9)

(b) Let $f:(a,b)\to\mathbb{R}$ be a continuous function. If f is uniformly continuous, then prove that there exists a continuous function $g:[a,b]\to\mathbb{R}$ such that $g(x)=f(x)$ for all $x\in(a,b)$. (6)

Q.28 (a) On $\mathbb{R}^3$, define a binary operation $*$ as follows: For $(x, y, t), (x', y', t')$ in $\mathbb{R}^3$,

$$(x, y, t) * (x', y', t') = \left(x + x',\ y + y',\ t + t' + \frac{1}{2}(x'y - xy') \right).$$

Then show that $(\mathbb{R}^3, *)$ is a group, and find its center. (9)

(b) For $k \in \mathbb{N}$, let $k\mathbb{Z} = \{ kn : n \in \mathbb{Z} \}$. For any $m, n \in \mathbb{N}$, show that $I = m\mathbb{Z} \cap n\mathbb{Z}$ is an ideal of $\mathbb{Z}$. Further, find the generators of I. (6)

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Q.29 Let G be a group of order p^2, where p is a prime number. Let $x \in G$. Prove that $\{y \in G : xy = yx\} = G$. (15)

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2011 MA Objective Part (Question Number 1 – 15)	
Total Marks	**Signature**

Subjective Part					
Question Number	Marks		Question Number	Marks	
16			23		
17			24		
18			25		
19			26		
20			27		
21			28		
22			29		
Total Marks in Subjective Part					

Total (Objective Part)	:	
Total (Subjective Part)	:	
Grand Total	:	
Total Marks (in words)	:	
Signature of Examiner(s)	:	
Signature of Head Examiner(s)	:	
Signature of Scrutinizer	:	
Signature of Chief Scrutinizer	:	
Signature of Coordinating Head Examiner	:	