SENIOR SECONDARY SCHOOL

MATHEMATICS

SENIOR SECONDARY SCHOOL MATHEMATICS

FOR CLASS 12

[In accordance with the latest CBSE syllabus]

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Preface

It gives me great pleasure in presenting the new edition of this book. In this edition, the modifications have been dictated by the changes in the CBSE syllabus. The structure and the methods used in the previous editions, which have been appreciated by teachers using the book in classroom conditions, remain unchanged.

The main consideration in writing the book was to present the considerable requirements of the syllabus in as simple a manner as possible. Special attention has been paid to the gradation of problems. This will help students gain confidence in problem-solving.

One problem faced by students is the lack of a comprehensive and carefully selected set of solved problems in textbooks of this kind. I have given due weightage to this aspect. Each set of solved examples is followed by a comprehensive exercise section in which students will get enough questions for practice. Hints have been given to the more difficult questions. Students should take their help as a last resort.

I have received many suggestions and letters of appreciation from teachers all over the country. I thank them all for contributing in the improvement of the book and for their encouragement. I hope they will like this edition as well. And as always, I would like to hear their views on the book.

R S Aggarwal

Mathematics Syllabus

For Class 12

UNIT I. Relations and Functions

1. **Relations and Functions** **15 Periods**

   Types of relations: reflexive, symmetric, transitive and equivalence relations. One to one and onto functions, composite functions, inverse of a function. Binary operations.

2. **Inverse Trigonometric Functions** **15 Periods**

   Definition, range, domain, principal value branches. Graphs of inverse trigonometric functions. Elementary properties of inverse trigonometric functions.

UNIT II. Algebra

1. **Matrices** **25 Periods**

   Concept, notation, order, equality, types of matrices, zero matrix, transpose of a matrix, symmetric and skew-symmetric matrices. Addition, multiplication and scalar multiplication of matrices, simple properties of addition, multiplication and scalar multiplication. Non-commutativity of multiplication of matrices and existence of nonzero matrices whose product is the zero matrix (restrict to square matrices of order 2). Concept of elementary row and column operations. Invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).

2. **Determinants** **25 Periods**

   Determinant of a square matrix (up to 3×3 matrices), properties of determinants, minors, cofactors and applications of determinants in finding the area of a triangle. Adjoint and inverse of a square matrix. Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

UNIT III. Calculus

1. **Continuity and Differentiability** **20 Periods**

   Continuity and differentiability, derivative of composite functions, chain rule, derivatives of inverse trigonometric functions, derivative of implicit functions. Concept of exponential and logarithmic functions.

   Derivatives of logarithmic and exponential functions. Logarithmic differentiation, derivative of functions expressed in parametric forms. Second-order derivatives. Rolle's and Lagrange's Mean Value Theorems (without proof) and their geometric interpretation.

2. **Applications of Derivatives** **10 Periods**

Applications of derivatives: rate of change of bodies, increasing/decreasing functions, tangents and normals, use of derivatives in approximation, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-life situations).

3. **Integrals** **20 Periods**

Integration as inverse process of differentiation. Integration of a variety of functions by substitution, by partial fractions and by parts, simple integrals of the following type to be evaluated:

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}},$$

$$\int \frac{(px + q)}{ax^2 + bx + c} dx, \int \frac{(px + q)}{\sqrt{ax^2 + bx + c}} dx, \int \sqrt{a^2 \pm x^2} dx, \int \sqrt{x^2 - a^2} dx,$$

$$\int \sqrt{ax^2 + bx + c}\, dx, \int (px + q) \sqrt{ax^2 + bx + c}\, dx.$$

Definite integrals as a limit of a sum, Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.

4. **Applications of the Integrals** **15 Periods**

Applications in finding the area under simple curves, especially lines, areas of circles/parabolas/ellipses (in standard form only). Area between the two above-said curves (the region should be clearly identifiable).

5. **Differential Equations** **15 Periods**

Definition, order and degree, general and particular solutions of a differential equation. Formation of a differential equation whose general solution is given. Solution of differential equations by method of separation of variables, homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type:

$\frac{dy}{dx} + py = q$, where p and q are functions of x or constants.

$\frac{dx}{dy} + px = q$, where p and q are functions of y or constants.

UNIT IV. Vectors and Three-dimensional Geometry

1. **Vectors** **15 Periods**

   Vectors and scalars, magnitude and direction of a vector. Direction cosines and direction ratios of a vector. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, components of a vector, addition of vectors, multiplication of a vector by a scalar, position vector of a point dividing a line segment in a given ratio. Scalar (dot) product of vectors, projection of a vector on a line. Vector (cross) product of vectors. Scalar triple product of vectors.

2. **Three-dimensional Geometry** **15 Periods**

   Direction cosines and direction ratios of a line joining two points. Cartesian and vector equation of a line, coplanar and skew lines, shortest distance between two lines. Cartesian and vector equation of a plane. Angle between (i) two lines, (ii) two planes, (iii) a line and a plane. Distance of a point from a plane.

UNIT V. Linear Programming

1. **Linear Programming** **20 Periods**

   Introduction, related terminology such as constraints, objective function, optimization, different types of linear programming (L.P.) problems, mathematical formulation of L.P. problems, graphical method of solution for problems in two variables, feasible and infeasible regions, feasible and infeasible solutions, optimal feasible solutions (up to three non-trivial constraints).

UNIT VI. Probability

1. **Probability** **30 Periods**

   Conditional probability, multiplication theorem on probability, independent events, total probability, Bayes's theorem, Random variable and its probability distribution, mean and variance of random variable. Repeated independent (Bernoulli) trials and Binomial distribution.

Weightage

	Topic	Marks
1.	Relations and Functions	10
2.	Algebra	13
3.	Calculus	44
4.	Vectors and 3-Dimensional Geometry	17
5.	Linear Programming	06
6.	Probability	10
	Total	100

Type	No. of Questions	Marks of Each	Total Marks
Very-Short-Answer	4	1	4
Short-Answer	8	2	16
Long-Answer I	11	4	44
Long-Answer II	6	6	36
Total	29		100

Sl. No.	Division of Types of Questions	Marks of Each			
		VSA (1 Mark)	SA (2 Marks)	LA–I (4 Marks)	LA–II (6 Marks)
1.	Remembering	2	2	2	1
2.	Understanding	1	3	4	2
3.	Application	1	–	3	2
4.	HOTS	–	3	1	–
5.	Evaluation	–	–	1*	1
	Total	4	8	11	6

* VBQ

Contents

1. RELATIONS

In class 11 we discussed about the Cartesian product of sets. Now, we extend our ideas to relation in a set and then in next chapter we shall be taking up functions.

RELATION IN A SET

A relation R in a set A is a subset of $A \times A$.

Thus, R is a relation in a set $A \Leftrightarrow R \subseteq A \times A$.

If $(a, b) \in R$ *then we say that a is related to b and write,* $a\ R\ b$.

If $(a, b) \notin R$ *then we say that a is not related to b and write,* $a \not{R}\ b$.

Example Let $A = \{1, 2, 3, 4, 5, 6\}$ and let R be a relation in A, given by

$$R = \{(a, b) : a - b = 2\}.$$

Then, $R = \{(3, 1), (4, 2), (5, 3), (6, 4)\}$.
Clearly, $3\ R\ 1$, $4\ R\ 2$, $5\ R\ 3$ and $6\ R\ 4$.
But, $1 \not{R}\ 3$, $2 \not{R}\ 4$, $5 \not{R}\ 6$, etc.

DOMAIN AND RANGE OF A RELATION

Let R be a relation in a set A. Then, the set of all first coordinates of elements of R is called the domain of R, written as dom (R) and the set of all second coordinates of R is called the range of R, written as range (R).

$\therefore$ dom $(R) = \{a : (a, b) \in R\}$ and range $(R) = \{b : (a, b) \in R\}$.

Example Let $A = \{1, 2, 3, 4, \ldots, 15, 16\}$ and let R be a relation in A, given by $R = \{(a, b) : b = a^2\}$.

Then, $R = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$.
$\therefore$ dom $(R) = \{1, 2, 3, 4\}$ and range $(R) = \{1, 4, 9, 16\}$.

Some Particular Types of Relations

EMPTY RELATION (Or VOID RELATION) *A relation R in a set A is called an empty relation, if no element of A is related to any element of A and we denote such a relation by* ϕ.

Thus, $R = \phi \subseteq A \times A$.

Example Let $A = \{1, 2, 3, 4, 5\}$ and let R be a relation in A, given by $R = \{(a, b) : a - b = 6\}$.

Clearly, no element $(a, b) \in A \times A$ satisfies the property $a - b = 6$.

$\therefore$ R is an empty relation in A.

UNIVERSAL RELATION *A relation R in a set A is called a universal relation, if each element of A is related to every element of A.*

Thus, $R = (A \times A) \subseteq (A \times A)$ is the universal relation on A.

Example Let $A = \{1, 2, 3\}$. Then,

$R = (A \times A) = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ is the universal relation in A.

IDENTITY RELATION *The relation $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A.*

Example Let $A = \{1, 2, 3\}$. Then,

$I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A.

VARIOUS TYPES OF RELATIONS

Let A be a nonempty set. Then, a relation R on A is said to be

(i) **reflexive** if $(a, a) \in R$ for each $a \in A$,
i.e., if $a\ R\ a$ for each $a \in A$.

(ii) **symmetric** if $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$,
i.e., if $a\ R\ b \Rightarrow b R a$ for all $a, b \in A$.

(iii) **transitive** if $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$,
i.e., if $a\ R\ b$ and $b\ R\ c \Rightarrow a\ R\ c$.

EQUIVALENCE RELATION *A relation R in a set A is said to be an equivalence relation if it is reflexive, symmetric and transitive.*

SOLVED EXAMPLES

EXAMPLE 1 *Let A be the set of all triangles in a plane and let R be a relation in A, defined by $R = \{(\triangle_1, \triangle_2) : \triangle_1 \cong \triangle_2\}$.*
Show that R is an equivalence relation in A.

SOLUTION The given relation satisfies the following properties:

(i) *Reflexivity*

Let a be an arbitrary triangle in A. Then,

$$\triangle \cong \triangle \Rightarrow (\triangle, \triangle) \in R \text{ for all values of } \triangle \text{ in } A.$$

$\therefore$ R is reflexive.

(ii) *Symmetry*

Let $\triangle_1, \triangle_2 \in A$ such that $(\triangle_1, \triangle_2) \in R$. Then,

$$(\triangle_1, \triangle_2) \in R \Rightarrow \triangle_1 \cong \triangle_2$$
$$\Rightarrow \triangle_2 \cong \triangle_1$$
$$\Rightarrow (\triangle_2, \triangle_1) \in R.$$

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $\triangle_1, \triangle_2, \triangle_3 \in A$ such that $(\triangle_1, \triangle_2) \in R$ and $(\triangle_2, \triangle_3) \in R$. Then, $(\triangle_1, \triangle_2) \in R$ and $(\triangle_2, \triangle_3) \in R$

$$\Rightarrow \triangle_1 \cong \triangle_2 \text{ and } \triangle_2 \cong \triangle_3$$
$$\Rightarrow \triangle_1 \cong \triangle_2$$
$$\Rightarrow (\triangle_1, \triangle_3) \in R.$$

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

EXAMPLE 2 *Let A be the set of all lines in xy-plane and let R be a relation in A, defined by*

$$R = \{(L_1, L_2) : L_1 \,||\, L_2\}.$$

Show that R is an equivalence relation in A.
Find the set of all lines related to the line $y = 3x + 5$.

SOLUTION The given relation satisfies the following properties:

(i) *Reflexivity*

Let L be an arbitrary line in A. Then,

$$L \,||\, L \Rightarrow (L, L) \in R \;\forall\; L \in A.$$

Thus, R is reflexive.

(ii) *Symmetry*

Let $L_1, L_2 \in A$ such that $(L_1, L_2) \in R$. Then,

$$(L_1, L_2) \in R \Rightarrow L_1 \,||\, L_2$$
$$\Rightarrow L_2 \,||\, L_1$$
$$\Rightarrow (L_2, L_1) \in R.$$

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $L_1, L_2, L_3 \in A$ such that $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$. Then, $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$

$$\Rightarrow L_1 \,||\, L_2 \text{ and } L_2 \,||\, L_3$$

$\Rightarrow\ L_1 \parallel L_3$

$\Rightarrow\ (L_1, L_3) \in R.$

$\therefore$ R is transitive.

Thus R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

The family of lines parallel to the line $y = 3x + 5$ is given by $y = 3x + k$, where k is real.

EXAMPLE 3 *Let Z be the set of all integers and let R be a relation in Z, defined by*

$$R = \{(a, b) : (a - b) \text{ is even}\}.$$

Show that R is an equivalence relation in Z.

SOLUTION Here, R satisfies the following properties:

(i) *Reflexivity*
Let a be an arbitrary element of Z.

Then, $(a - a) = 0$, which is even.

$\therefore$ $(a, a) \in R\ \forall\ a \in Z.$

So, R is reflexive.

(ii) *Symmetry*
Let $a, b \in Z$ such that $(a, b) \in R$. Then,

$(a, b) \in R \Rightarrow (a - b)$ is even

$\Rightarrow -(a - b)$ is even

$\Rightarrow (b - a)$ is even

$\Rightarrow (b, a) \in R.$

$\therefore$ R is symmetric.

(iii) *Transitivity*
Let $a, b, c \in Z$ such that $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow (a - b)$ is even and $(b - c)$ is even

$\Rightarrow \{(a - b) + (b - c)\}$ is even

$\Rightarrow (a - c)$ is even

$\Rightarrow (a, c) \in R.$

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation in Z.

EXAMPLE 4 *Let A be the set of all lines in a plane and let R be a relation in A defined by*

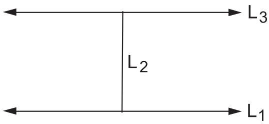
$$R = \{(L_1, L_2) : L_1 \perp L_2\}.$$

Show that R is symmetric but neither reflexive nor transitive.

SOLUTION Clearly, any line L cannot be perpendicular to itself.

$\therefore$ $(L, L) \notin R$ for any $L \in A$.

So, R is not reflexive.



Again, let $(L_1, L_2) \in R$. Then,

$$(L_1, L_2) \in R \Rightarrow L_1 \perp L_2$$
$$\Rightarrow L_2 \perp L_1$$
$$\Rightarrow (L_2, L_1) \in R.$$

$\therefore$ R is symmetric.

Now, let $L_1, L_2, L_3 \in A$ such that $L_1 \perp L_2$ and $L_2 \perp L_3$.

Then, clearly L_1 is not perpendicular to L_3.

Thus, $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$, but $(L_1, L_3) \notin R$.

$\therefore$ R is not transitive.

Hence, R is symmetric but neither reflexive nor transitive.

EXAMPLE 5 *Let S be the set of all real numbers and let R be a relation in S defined by*

$$R = \{(a, b) : (1 + ab) > 0\}.$$

Show that R is reflexive and symmetric but not transitive.

SOLUTION Let a be any real number. Then,

$(1 + aa) = (1 + a^2) > 0$ shows that $(a, a) \in R \; \forall \; a \in S$.

$\therefore$ R is reflexive.

Also, $(a, b) \in R \Rightarrow (1 + ab) > 0$

$\Rightarrow (1 + ba) > 0$ [$\because$ $ab = ba$]

$\Rightarrow (b, a) \in R$.

$\therefore$ R is symmetric.

In order to show that R is not transitive, consider $(-1, 0)$ and $(0, 2)$.

Clearly, $(-1, 0) \in R$, since $[1 + (-1) \times 0] > 0$.

And, $(0, 2) \in R$, since $[1 + 0 \times 2] > 0$.

But, $(-1, 2) \notin R$, since $[1 + (-1) \times 2]$ is not greater than 0.

Hence, R is reflexive and symmetric but not transitive.

EXAMPLE 6 *Let S be the set of all real numbers and let R be a relation in S, defined by*

$$R = \{(a, b) : a \leq b\}.$$

Show that R is reflexive and transitive but not symmetric.

SOLUTION Here, R satisfies the following properties:

(i) *Reflexivity*

Let a be an arbitrary real number.

Then, $a \le a \Rightarrow (a, a) \in R$.

Thus, $(a, a) \in R \; \forall \; a \in S$.

$\therefore$ R is reflexive.

(ii) *Transitivity*

Let a, b, c be real numbers such that $(a, b) \in R$ and $(b, c) \in R$.

Then, $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a \le b$ and $b \le c$

$\Rightarrow$ $a \le c$

$\Rightarrow$ $(a, c) \in R$.

$\therefore$ R is transitive.

(iii) *Nonsymmetry*

Clearly, $(4, 5) \in R$ since $4 \le 5$.

But, $(5, 4) \notin R$ since $5 \le 4$ is not true.

$\therefore$ R is not symmetric.

EXAMPLE 7 *Let S be the set of all real numbers and let R be a relation in S, defined by*

$R = \{(a, b) : a \le b^2\}$.

Show that R satisfies none of reflexivity, symmetry and transitivity.

SOLUTION (i) *Nonreflexivity*

Clearly, $\frac{1}{2}$ is a real number and $\frac{1}{2} \le \left(\frac{1}{2}\right)^2$ is not true.

$\therefore$ $\left(\frac{1}{2}, \frac{1}{2}\right) \notin R$.

Hence, R is not reflexive.

(ii) *Nonsymmetry*

Consider the real numbers $\frac{1}{2}$ and 1.

Clearly, $\frac{1}{2} \le 1^2 \Rightarrow \left(\frac{1}{2}, 1\right) \in R$.

But, $1 \le \left(\frac{1}{2}\right)^2$ is not true and so $\left(1, \frac{1}{2}\right) \notin R$.

Thus, $\left(\frac{1}{2}, 1\right) \in R$ but $\left(1, \frac{1}{2}\right) \in R$.

Hence, R is not symmetric.

(iii) *Nontransitivity*

Consider the real numbers 2, –2 and 1.

Clearly, $2 \le (-2)^2$ and $-2 \le (1)^2$ but $2 \le 1^2$ is not true.

Thus, $(2, -2) \in R$ and $(-2, 1) \in R$, but $(2, 1) \notin R$.

Hence, R is not transitive.

EXAMPLE 8 *Let S be the set of all real numbers and let R be a relation in S, defined by* $R = \{(a, b) : a \le b^3\}$.

Show that R satisfies none of reflexivity, symmetry and transitivity.

SOLUTION (i) *Nonreflexivity*

Clearly, $\frac{1}{2}$ is a real number and $\frac{1}{2} \le \left(\frac{1}{2}\right)^3$ is not true.

$\therefore \quad \left(\frac{1}{2}, \frac{1}{2}\right) \notin R.$

Hence, R is not reflexive.

(ii) *Nonsymmetry*

Take the real numbers $\frac{1}{2}$ and 1.

Clearly, $\frac{1}{2} \le 1^3$ is true and therefore, $\left(\frac{1}{2}, 1\right) \in R$.

But, $1 \le \left(\frac{1}{2}\right)^3$ is not true and so $\left(1, \frac{1}{2}\right) \notin R$.

Hence, R is not symmetric.

(iii) *Nontransitivity*

Consider the real numbers $3, \frac{3}{2}$ and $\frac{4}{3}\cdot$

Clearly, $3 \le \left(\frac{3}{2}\right)^3$ and $\frac{3}{2} \le \left(\frac{4}{3}\right)^3$ but $3 \le \left(\frac{4}{3}\right)^3$ is not true.

Thus, $\left(3, \frac{3}{2}\right) \in R$ and $\left(\frac{3}{2}, \frac{4}{3}\right) \in R$, but $\left(3, \frac{4}{3}\right) \notin R$.

Hence, R is not transitive.

Thus, R satisfies none of reflexivity, symmetry and transitivity.

EXAMPLE 9 *Let N be the set of all natural numbers and let R be a relation in N, defined by*

$$R = \{(a, b)\} : a \text{ is a factor of } b\}.$$

Then, show that R is reflexive and transitive but not symmetric.

SOLUTION Here, R satisfies the following properties:

(i) *Reflexivity*
Let a be an arbitrary element of N.

Then, clearly, a is a factor of a.

$\therefore \ (a, \ a) \in R \ \forall \ a \in N$.

So, R is reflexive.

(ii) *Transitivity*
Let $a, b, c \in N$ such that $(a, b) \in R$ and $(b, c) \in R$.

Now, $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ (a is a factor of b) and (b is a factor of c)

$\Rightarrow$ $b = ad$ and $c = be$ for some $d, e \in N$

$\Rightarrow$ $c = (ad)\ e = a(de)$ [by associative law]

$\Rightarrow$ a is a factor of c

$\Rightarrow$ $(a, c) \in R$.

$\therefore \ (a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

Hence, R is transitive.

(iii) *Nonsymmetry*
Clearly, 2 and 6 are natural numbers and 2 is a factor of 6.

$\therefore \ (2, 6) \in R$.

But, 6 is not a factor of 2.

$\therefore \ (6, 2) \notin R$.

Thus, $(2, 6) \in R$ and $(6, 2) \notin R$.

Hence, R is not symmetric.

EXAMPLE 10 *Let N be the set of all natural numbers and let R be a relation in N, defined by*

$R = \{(a, \ b) : a \text{ is a multiple of } b\}$.

Show that R is reflexive and transitive but not symmetric.

SOLUTION Here R satisfies the following properties:

(i) *Reflexivity*
Let a be an arbitrary element of N.

Then, $a = (a \times 1)$ shows that a is a multiple of a.

$\therefore \ (a, a) \in R \ \forall \ a \in N$.

So, R is reflexive.

(ii) *Transitivity*
Let $a, b, c \in N$ such that $(a, b) \in R$ and $(b, c) \in R$.

Now, $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ (a is a multiple of b) and (b is a multiple of c)

$\Rightarrow$ $a = bd$ and $b = ce$ for some $d \in N$ and $e \in N$

$\Rightarrow$ $a = (ce)d$

$\Rightarrow$ $a = c(ed)$

$\Rightarrow$ a is a multiple of c

$\Rightarrow$ $(a, c) \in R$.

$\therefore$ $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

Hence, R is transitive.

(iii) *Nonsymmetry*
Clearly, 6 and 2 are natural numbers and 6 is a multiple of 2.

$\therefore$ $(6, 2) \in R$.

But, 2 is not a multiple of 6.

$\therefore$ $(2, 6) \notin R$.

Thus, $(6, 2) \in R$ and $(2, 6) \notin R$.

Hence, R is not symmetric.

EXAMPLE 11 *Let X be a nonempty set and let S be the collection of all subsets of X. Let R be a relation in S, defined by*

$$R = \{(A, B) : A \subset B\}.$$

Show that R is transitive but neither reflexive nor symmetric.

SOLUTION Clearly, R satisfies the following properties:

(i) *Transitivity*
Let $A, B, C \in S$ such that $(A, B) \in R$ and $(B, C) \in R$.

Now, $(A, B) \in R$ and $(B, C) \in R$

$\Rightarrow$ $A \subset B$ and $B \subset C$

$\Rightarrow$ $A \subset C$

$\Rightarrow$ $(A, C) \in R$.

$\therefore$ R is transitive.

(ii) *Nonreflexivity*
Let A be any set in S.

Then, $A \not\subset A$ shows that $(A, A) \notin R$.

$\therefore$ R is not reflexive.

(iii) *Nonsymmetry*

Now $(A, B) \in R \Rightarrow A \subset B$

$\Rightarrow B \not\subset A$

$\Rightarrow (B, A) \notin R.$

$\therefore$ R is not symmetric.

Hence, R is transitive but neither reflexive nor symmetric.

EXAMPLE 12 *Give an example of a relation which is*

(i) reflexive and transitive but not symmetric;
(ii) symmetric and transitive but not reflexive;
(iii) reflexive and symmetric but not transitive;
(iv) symmetric but neither reflexive nor transitive;
(v) transitive but neither reflexive nor symmetric.

SOLUTION Let $A = \{1, 2, 3\}$.

Then, it is easy to verify that the relation

(i) $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$

is reflexive and transitive.

R_1 is not symmetric, since

$(1, 2) \in R$ and $(2, 1) \notin R$.

(ii) $R_2 = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$

is symmetric and transitive.

But, R_2 is not reflexive, since $(3, 3) \notin R_2$.

(iii) $R_3 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$

is reflexive and symmetric.

But, R_3 is not transitive, since

$(1, 2) \in R_3$, $(2, 3) \in R_3$ but $(1, 3) \notin R_3$.

(iv) $R_4 = \{(2, 2), (3, 3), (1, 2), (2, 1)\}$

is symmetric.

But, R_4 is not reflexive since $(1, 1) \notin R_4$.

Also, R_4 is not transitive, as

$(1, 2) \in R_4$ and $(2, 1) \in R_4$ but $(1, 1) \in R_4$.

(v) $R_5 = \{(2, 2), (3, 3), (1, 2)\}$

is transitive.

But, R_5 is not reflexive, since $(1, 1) \notin R$.

And, R_5 is not symmetric as $(1, 2) \in R_5$ but $(2, 1) \notin R_5$.

EXAMPLE 13 *Let N be the set of all natural numbers and let R be a relation on* $N \times N$, *defined by*

$$(a, b)\ R\ (c, d) \Leftrightarrow ad = bc.$$

Show that R is an equivalence relation.

SOLUTION Here R satisfies the following properties:

(i) *Reflexivity*

Let $(a, b) \Rightarrow R$. Then,

$(a, b)\ R\ (a, b)$, since $ab = ba$

[by commutative law of multiplication on N].

Thus, $(a, b)\ R\ (a, b)\ \forall\ (a, b) \in R$.

$\therefore$ R is reflexive.

(ii) *Symmetry*

Let $(a, b)\ R\ (c, d)$. Then,

$(a, b)\ R\ (c, d) \Rightarrow ad = bc$

$\Rightarrow bc = ad$

$\Rightarrow cb = da$

[by commutativity of multiplication on N]

$\Rightarrow (c, d)\ R\ (a, b)$.

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then,

$ad = bc$ and $cf = de$

$\Rightarrow adcf = bcde$

$\Rightarrow (af)(cd) = (be)(cd)$

$\Rightarrow af = be$ [by cancellation law]

$\Rightarrow (a, b)\ R\ (e, f)$.

$\therefore$ $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$.

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

EXAMPLE 14 *If R_1 and R_2 be two equivalence relations on a set A, prove that $R_1 \cap R_2$ is also an equivalence relation on A.*

SOLUTION Let R_1 and R_2 be two equivalence relations on a set A.

Then, $R_1 \subseteq A \times A,\ R_2 \subseteq A \times A \Rightarrow (R_1 \cap R_2) \subseteq A \times A$.

So, $(R_1 \cap R_2)$ is a relation on A.

This relation on A satisfies the following properties.

(i) *Reflexivity*

R_1 is reflexive and R_2 is reflexive

$\Rightarrow (a, a) \in R_1$ and $(a, a) \in R_2$ for all $a \in A$

$\Rightarrow (a, a) \in R_1 \cap R_2$ for all $a \in A$

$\Rightarrow R_1 \cap R_2$ is reflexive.

(ii) *Symmetry*

Let (a, b) be an arbitrary element of $R_1 \cap R_2$. Then,

$(a, b) \in R_1 \cap R_2$

$\Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2$

$\Rightarrow (b, a) \in R_1$ and $(b, a) \in R_2$

[$\because$ R_1 is symmetric and R_2 is symmetric]

$\Rightarrow (b, a) \in R_1 \cap R_2$.

This shows that $R_1 \cap R_2$ is symmetric.

(iii) *Transitivity*

$(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$

$\Rightarrow (a, b) \in R_1,\ (a, b) \in R_2$, and $(b, c) \in R_1,\ (b, c) \in R_2$

$\Rightarrow \{(a, b) \in R_1,\ (b, c) \in R_1\}$, and $\{(a, b) \in R_2,\ (b, c) \in R_2\}$

$\Rightarrow (a, c) \in R_1$ and $(a, c) \in R_2$

[$\because$ R_1 is transitive and R_2 is transitive]

$\Rightarrow (a, c) \in R_1 \cap R_2$.

This shows that $(R_1 \cap R_2)$ is transitive.

Thus, $R_1 \cap R_2$ is reflexive, symmetric and transitive.

Hence, $R_1 \cap R_2$ is an equivalence relation.

EXAMPLE 15 *Give an example to show that the union of two equivalence relations on a set A need not be an equivalence relation on A.*

SOLUTION Let R_1 and R_2 be two relations on a set $A = \{1, 2, 3\}$, given by

$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$.

Then, it is easy to verify that each one of R_1 and R_2 is an equivalence relation.

But, $R_1 \cup R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\}$ is not transitive, as

$$(3, 1) \in R_1 \cup R_2 \text{ and } (1, 2) \in R_1 \cup R_2 \text{ but } (3, 2) \notin R_1 \cup R_2.$$

Hence, $(R_1 \cup R_2)$ is not an equivalence relation.

EQUIVALENCE CLASSES *Let R be an equivalence relation in a set A and let $a \in A$. Then, the set of all those elements of A which are related to a, is called the equivalence class determined by a and it is denoted by $[a]$.*

Thus, $[a] = \{b \in A : (a, b) \in R\}$.

Two equivalence classes are either disjoint or identical.

An Important Result *An equivalence relation R on a set A partitions the set into mutually disjoint equivalence classes.*

EXAMPLE 16 *On the set Z of all integers, consider the relation*

$$R = \{(a, b) : (a - b) \text{ is divisible by } 3\}.$$

Show that R is an equivalence relation on Z.
Also find the partitioning of Z into mutually disjoint equivalence classes.

SOLUTION The relation R on Z satisfies the following properties:

(i) *Reflexivity*

Let $a \in Z$.

Then, $(a - a) = 0$, which is divisible by 3.

$\therefore \quad a \, R \, a \; \forall \; a \in Z$.

So, R is reflexive.

(ii) *Symmetry*

Let $a, b \in Z$ such that $a \, R \, b$. Then,

$$\begin{aligned} a \, R \, b &\Rightarrow (a - b) \text{ is divisible by } 3 \\ &\Rightarrow -(a - b) \text{ is divisible by } 3 \\ &\Rightarrow (b - a) \text{ is divisible by } 3 \\ &\Rightarrow b \, R \, a. \end{aligned}$$

$\therefore \quad a \, R \, b \Rightarrow b \, R \, a \; \forall \; a, b \in Z$.

So, R is symmetric.

(iii) *Transitivity*

Let $a, b, c \in Z$ such that $a \, R \, b$ and $b \, R \, c$. Then,

$$\begin{aligned} a \, R \, b, \; b \, R \, c &\Rightarrow (a - b) \text{ is divisible by } 3 \\ &\qquad \text{and } (b - c) \text{ is divisible by } 3 \\ &\Rightarrow [(a - b) + (b - c)] \text{ is divisible by } 3 \\ &\Rightarrow (a - c) \text{ is divisible by } 3. \end{aligned}$$

Thus, $a \, R \, b, \; b \, R \, c \Rightarrow a \, R \, c \; \forall \; a, b, c \in Z$.

$\therefore$ R is an equivalence relation on Z.

Now, let us consider [0], [1] and [2].

We have:

$$[0] = \{x \in Z : x \; R \; 0\}$$
$$= \{x \in Z : (x - 0) \text{ is divisible by } 3\}$$
$$= \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}.$$

$\therefore$ $[0] = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$.

Similarly, $[1] = \{x \in Z : x \; R \; 1\}$

$$= \{x \in Z : (x - 1) \text{ is divisible by } 3\}$$
$$= \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}.$$

$\therefore$ $[1] = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$.

And, $[2] = \{x \in Z : x \; R \; 2\}$

$$= \{x \in Z : (x - 2) \text{ is divisible by } 3\}$$
$$= \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}.$$

$\therefore$ $[2] = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$.

Clearly, [0], [1] and [2] are mutually disjoint

and $Z = [0] \cup [1] \cup [2]$.

EXAMPLE 17 *Let* $A = \{x \in Z : 0 \leq x \leq 12\}$.
Show that $R = \{(a, b) : |a - b|$ *is a multiple of* $4\}$ *is*
(i) reflexive, (ii) symmetric and (iii) transitive.
Find the set of elements related to 1. **[CBSE 2010]**

SOLUTION Clearly, $A = \{0, 1, 2, 3, 4, \ldots, 10, 11, 12\}$.

Here, R satisfies the following properties.

(i) *Reflexivity*

Let a be an arbitrary element of A. Then,

$a - a = 0$, which is a multiple of 4.

$\therefore$ $a R a$ for all $a \in A$.

So, R is reflexive.

(ii) *Symmetry*

Let $a R b$. Then,

$$a R b \Rightarrow |a - b| \text{ is a multiple of } 4$$
$$\Rightarrow |-(a - b)| \text{ is a multiple of } 4$$
$$\Rightarrow |b - a| \text{ is a multiple of } 4$$
$$\Rightarrow b \; R \; a.$$

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $a\,R\,b$ and $b\,R\,c$. Then,

$a\ R\ b,\ b\ R\ c$

$\Rightarrow$ $|a - b|$ is a multiple of 4 and $|b - c|$ is a multiple of 4.

Let $|a - b| = 4k_1$ and $|b - c| = 4k_2$. Then,

$$|a - c| = |(a - b) - (b - c)| = |4k_1 - 4k_2|$$

$= |4(k_1 - k_2)| = 4\,|k_1 - k_2|$, which is a multiple of 4.

$\therefore$ $a\ R\ b,\ b\ R\ c \Rightarrow a\ R\ c$. So, R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

Now, $[1] = \{x \in A :\ x\,R\,1\}$

$= \{x \in A :\ |x - 1|$ is a multiple of $4\}$

$= \{1, 5, 9\}$.

Hence, the required set is $\{1, 5, 9\}$.

EXAMPLE 18 *Let* $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *and* R *be a relation in* $A \times A$, *defined by* $(a, b)\,R\,(c, d) \Leftrightarrow a + d = b + c$ *for all* (a, b) *and* $(c, d) \in A \times A$. *Prove that* R *is an equivalence relation. Also obtain the equivalence class determined by* $(2, 5)$. **[CBSE 2014]**

SOLUTION (i) *Reflexivity*

Let $(a, b) \in A \times A$. Then,

$$(a, b) \in A \times A \Rightarrow a, b \in A$$

$$\Rightarrow a + b = b + a$$

$$\Rightarrow (a, b)\ R\ (a, b).$$

$\therefore$ R is reflexive.

(ii) *Symmetry*

Let $(a, b)\ R\,(c, d)$. Then,

$$(a, b)\ R\ (c, d) \Rightarrow a + d = b + c$$

$$\Rightarrow c + b = d + a$$

$$\Rightarrow (c, d)\ R\ (a, b).$$

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $(a, b)\ R\ (c, d)$ and $(c, d)\,R\,(e, f)$. Then,

$(a, b)\ R\ (c, d)$ and $(c, d)\,R\,(e, f)$

$\Rightarrow$ $a + d = b + c$ and $c + f = d + e$

$\Rightarrow$ $a + d + c + f = b + c + d + e$

$\Rightarrow$ $a + f = b + e$

$\Rightarrow$ $(a, b)\ R\ (e, f)$.

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

$$[(2, 5)] = \{(a, b) : (2, 5)\ R\ (a, b)\}$$
$$= \{(a, b) : 2 + b = 5 + a\} = \{(a, b) : b - a = 3\}$$
$$= \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}.$$

EXERCISE 1A

Very-Short-Answer Questions

1. Find the domain and range of the relation
$$R = \{(-1, 1), (1, 1), (-2, 4), (2, 4)\}.$$
2. Let $R = \{(a, a^3) : a$ is a prime number less than $5\}$.
Find the range of R. **[CBSE 2014]**
3. Let $R = \{(a, a^3) : a$ is a prime number less than $10\}$.
Find (i) R (ii) dom (R) (iii) range (R).
4. Let $R = \{x, y) : x + 2y = 8\}$ be a relation on N.
Write the range of R. **[CBSE 2014]**
5. Let $R = \{(a, b) : a, b \in N$ and $a + 3b = 12\}$.
Find the domain and range of R.
6. Let $R = \{(a, b) : b = |a - 1|, a \in Z$ and $|a| < 3\}$.
Find the domain and range of R.
7. Let $R = \left\{\left(a, \frac{1}{a}\right) : a \in N \text{ and } 1 < a < 5\right\}$.
Find the domain and range of R.
8. Let $R = \{(a, b) : a, b \in N$ and $b = a + 5, a < 4\}$.
Find the domain and range of R.
9. Let S be the set of all sets and let $R = \{(A, B) : A \subset B)\}$, i.e., A is a proper subset of B. Show that R is (i) transitive (ii) not reflexive (iii) not symmetric.
10. Let A be the set of all points in a plane and let O be the origin. Show that the relation $R = \{(P, Q) : P, Q \in A$ and $OP = OQ\}$ is an equivalence relation.
11. On the set S of all real numbers, define a relation $R = \{(a, b) : a \le b\}$.
Show that R is (i) reflexive (ii) transitive (iii) not symmetric .
12. Let $A = \{1, 2, 3, 4, 5, 6\}$ and let $R = \{(a, b) : a, b \in A$ and $b = a + 1\}$.
Show that R is (i) not reflexive, (ii) not symmetric and (iii) not transitive.

ANSWERS (EXERCISE 1A)

1. dom $(R) = \{-1, 1, -2, 2\}$ and range $(R) = \{1, 4\}$

2. range $(R) = \{8, 27\}$

3. (i) $R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$ (ii) dom $(R) = \{2, 3, 5, 7\}$

(iii) range $(R) = \{8, 27, 125, 343\}$

4. $\{3, 2, 1\}$

5. dom $(R) = \{3, 6, 9\}$ and range $(R) = \{3, 2, 1\}$

6. dom $(R) = \{-2, -1, 0, 1, 2\}$ and range $(R) = \{3, 2, 1, 0\}$

7. dom $(R) = \{2, 3, 4\}$ and range $(R) = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$

8. dom $(R) = \{1, 2, 3\}$ and range $(R) = \{6, 7, 8\}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 1A)

2. $R = \{(2, 8), (3, 27)\}$

4. $R = \{(2, 3), (4, 2), (6, 1)\}$

5. $R = \{(3, 3), (6, 2), (9, 1)\}$

6. Clearly, a is an integer such that $-3 < a < 3$.

$\therefore \quad R = \{(-2, 3), (-1, 2), (0, 1), (1, 0)(2, 1)\}$.

7. $R = \left\{\left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right), \left(4, \frac{1}{4}\right)\right\}$

8. $R = \{(1, 6), (2, 7), (3, 8)\}$.

9. (i) $(A \subset B$ and $B \subset C) \Rightarrow (A \subset C)$. So, R is transitive.

(ii) Clearly, $A \subset C$ is not true. So, R is not reflexive.

(iii) If $A \subset B$ then $B \subset C$ is not true. So, R is not symmetric.

10. Let O be the origin and let P, Q, X be any three points in a plane.

Then, (i) $OP = OP$ is always true. So, R is reflextive.

(ii) $OP = OQ \Rightarrow OQ = OP$. So, R is symmetric.

(iii) $(OP = OQ$ and $OQ = OX) \Rightarrow (OP = OX)$. So, R transitive.

Hence, R is an equivalence relation.

11. (i) For all $a \in R$, $a \le a$ is alway true. So, R is reflexive.

(ii) For all $a, b, c \in R$, we have $(a \le b, b \le c) \Rightarrow (a \le c)$.

$\therefore \quad R$ is transitive.

(iii) But R is not symmetrtic, as $3 \le 4$ is true, while $4 \le 3$ is not true.

12. (i) $(1, 1) \notin R$ as $1 = 1 + 1$ is not true.

(ii) $(2 = 1 + 1) \Rightarrow 1\ R\ 2$. But $1 = 2 + 1$ is not true. So, 2 is not related to 1.

$\therefore \quad R$ is not symmetric.

(iii) $1\ R\ 2$ and $2\ R\ 3$. But, 1 is not related to 3.

EXERCISE 1B

1. Define a relation on a set. What do you mean by the domain and range of a relation. Give an example.
2. Let A be the set of all triangles in a plane. Show that the relation $R = \{(\triangle_1, \triangle_2) : \triangle_1 \sim \triangle_2\}$ is an equivalence relation on A.
3. Let $R = \{(a, b) : a, b \in Z \text{ and } (a + b) \text{ is even}\}$.
   Show that R is an equivalence relation on Z.
4. Let $R = \{(a, b) : a, b \in Z \text{ and } (a - b) \text{ is divisible by } 5\}$.
   Show that R is an equivalence relation on Z.
5. Show that the relation R defined on the set $A = \{1, 2, 3, 4, 5\}$, given by $R = \{(a, b) : |a - b| \text{ is even}\}$ is an equivalence relation. **[CBSE 2009]**
6. Show that the relation R on $N \times N$, defined by
   $$(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c$$
   is an equivalent relation. **[CBSE 2010]**
7. Let S be the set of all real numbers and let
   $$R = \{(a, b) : a, b \in S \text{ and } a = \pm b\}.$$
   Show that R is an equivalence relation on S.
8. Let S be the set of all points in a plane and let R be a relation in S defined by $R = \{(A, B) : d(A, B) < 2 \text{ units}\}$, where $d(A, B)$ is the distance between the points A and B.
   Show that R is reflexive and symmetric but not transitive.
9. Let S be the set of all real numbers. Show that the relation $R = \{(a, b) : a^2 + b^2 = 1\}$ is symmetric but neither reflextive nor transitive. **[CBSE 2008]**
10. Let $R = \{(a, b) : a = b^2\}$ for all $a, b \in N$.
    Show that R satisfies none of reflexivity, symmetry and transitivity.
11. Show that the relation $R = \{(a, b) : a > b\}$ on N is transitive but neither reflexive nor symmetric.
12. Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$.
    Show that R is reflexive but neither symmetric nor transitive.
13. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (3, 2)\}$. Show that R is reflexive and transitive but not symmetric.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 1B)

3. (i) Let $a \in Z$. Then, $a + a = 2a$, which is even. So, $a\ R\ a$.

   (ii) Let $a\ R\ b$. Then,

   $$\begin{aligned} a\ R\ b &\Rightarrow a + b \text{ is even} \\ &\Rightarrow b + a \text{ is even} \\ &\Rightarrow b\ R\ a. \end{aligned}$$

(iii) Let $a\ R\ b$ and $b\ R\ c$. Then,

$a\,R\,b$ and $b\,R\,c \Rightarrow (a+b)$ is even and $(b+c)$ is even.

$\therefore \quad (a+c) = \{(a+c)+2b\} - 2b$

$= \{(a+b)+(b+c)-2b\}$, which is even

$\Rightarrow \quad a\ R\ c$ [$\because$ each of $(a+b)$, $(b+c)$ and $2b$ is even].

4. $(a-c) = (a-b)+(b-c)$.

5. (i) $|a-a| = 0$, which is even. So, $a\ R\ a$.

(ii) $a\ R\ b \Rightarrow |a-b|$ is even

$\Rightarrow |-(a-b)|$ is even

$\Rightarrow |b-a|$ is even

$\Rightarrow b\ R\ a$.

(iii) $a\ R\ b$, $b\ R\ c \Rightarrow |a-b|$ is even and $|b-c|$ is even

$\Rightarrow (a-b)$ is even and $(b-c)$ is even

$\Rightarrow \{(a-b)+(b-c)\}$ is even

$\Rightarrow (a-c)$ is even

$\Rightarrow |a-c|$ is even

$\Rightarrow a\ R\ c$.

8. (i) $d(A, A) = 0 < 2$. So, $A\,R\,A$.

(ii) $A\ R\ B \Rightarrow d(A, B) < 2$

$\Rightarrow d(B, A) < 2 \quad [\because d(B, A) = d(A, B)]$

$\Rightarrow B\ R\ A$.

(iii) Consider the points $A(0, 0)$, $B(1.5, 0)$ and $C(3, 0)$ on the x-axis.

Then, $d(A, B)1.5$, $d(B, C) = 1.5$ and $d(A, C) = 3$.

$\therefore \quad A\ R\ B$ and $B\ R\ C$. But, A is not related to C.

So, R is not transitive.

9. (i) R is symmetric, since

$a\ R\ b \Rightarrow a^2 + b^2 = 1$

$\Rightarrow b^2 + a^2 = 1 \Rightarrow b\ R\ a$.

(ii) R is not reflexive, since 1 is not related to 1, as

$1^2 + 1^2 = 1$ is not true.

(iii) Clearly, $\frac{1}{2}\ R\ \frac{\sqrt{3}}{2}$ and $\frac{\sqrt{3}}{2}\ R\ \frac{1}{2}$.

But, $\frac{1}{2}$ is not related to $\frac{1}{2}$ as $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \neq 1$.

$\therefore \quad R$ is not transitive.

10. (i) $2 \neq 2^2 \Rightarrow$ 2 is not related to 2.

(ii) $4 = 2^2 \Rightarrow 4\,R\,2$. But $2 \neq 4^2$. So, $2 \not R\, 4$.

(iii) $16\,R\,4$, $4\,R\,2$. But 16 is not related to 2, as $16 \neq 2^2$.

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. Let $A = \{1, 2, 3\}$ and let $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 2), (1, 2)\}$. Then, R is
   (a) reflexive and symmetric but not transitive
   (b) reflexive and transitive but not symmetric
   (c) symmetric and transitive but not reflexive
   (d) an equivalence relation
2. Let $A = \{a, b, c\}$ and let $R = \{(a, a), (a, b), (b, a)\}$. Then, R is
   (a) reflexive and symmetric but not transitive
   (b) reflexive and transitive but not symmetric
   (c) symmetric and transitive but not reflexive
   (d) an equivalence relation
3. Let $A = \{1, 2, 3\}$ and let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$. Then, R is
   (a) reflexive and symmetric but not transitive
   (b) symmetric and transitive but not reflexive
   (c) reflexive and transitive but not symmetric
   (d) an equivalence relation
4. Let S be the set of all straight lines in a plane. Let R be a relation on S defined by $a\ R\ b \Leftrightarrow a \perp b$. Then, R is
   (a) reflexive but neither symmetric nor transitive
   (b) symmetric but neither reflexive nor transitive
   (c) transitive but neither reflexive nor symmetric
   (d) an equivalence relation
5. Let S be the set of all straight lines in a plane. Let R be a relation on S defined by $a\ R\ b \Leftrightarrow a \,||\, b$. Then, R is
   (a) reflexive and symmetric but not transitive
   (b) reflexive and transitive but not symmetric
   (c) symmetric and transitive but not reflexive
   (d) an equivalence relation
6. Let Z be the set of all integers and let R be a relation on Z defined by $a\ R\ b \Leftrightarrow (a - b)$ is divisible by 3. Then, R is
   (a) reflexive and symmetric but not transitive
   (b) reflexive and transitive but not symmetric
   (c) symmetric and transitive but not reflexive
   (d) an equivalence relation

7. Let R be a relation on the set N of all natural numbers, defined by $a\ R\ b \Leftrightarrow a$ is a factor of b. Then, R is

(a) reflexive and symmetric but not transitive

(b) reflexive and transitive but not symmetric

(c) symmetric and transitive but not reflexive

(d) an equivalence relation

8. Let Z be the set of all integers and let R be a relation on Z defined by $a\ R\ b \Leftrightarrow a \geq b$. Then, R is

(a) symmetric and transitive but not reflexive

(b) reflexive and symmetric but not transitive

(c) reflexive and transitive but not symmetric

(d) an equivalence relation

9. Let S be the set of all real numbers and let R be a relation on S defined by $a\ R\ b \Leftrightarrow |a| \leq b$. Then, R is

(a) reflexive but neither symmetric nor transitive

(b) symmetric but neither reflexive nor transitive

(c) transitive but neither reflexive nor symmetric

(d) none of these

10. Let S be the set of all real numbers and let R be a relation on S, defined by $a\ R\ b \Leftrightarrow |a - b| \leq 1$. Then, R is

(a) reflexive and symmetric but not transitive

(b) reflexive and transitive but not symmetric

(c) symmetric and transitive but not reflexive

(d) an equivalence relation

11. Let S be the set of all real numbers and let R be a relation on S, defined by $a\ R\ b \Leftrightarrow (1 + ab) > 0$. Then, R is

(a) reflexive and symmetric but not transitive

(b) reflexive and transitive but not symmetric

(c) symmetric and transitive but not reflexive

(d) none of these

12. Let S be the set of all triangles in a plane and let R be a relation on S defined by $\Delta_1 S\ \Delta_2 \Leftrightarrow \Delta_1 \equiv \Delta_2$. Then, R is

(a) reflexive and symmetric but not transitive

(b) reflexive and transitive but not symmetric

(c) symmetric and transitive but not reflexive

(d) an equivalence relation

13. Let S be the set of all real numbers and let R be a relation on S defined by $a\ R\ b \Leftrightarrow a^2 + b^2 = 1$. Then, R is

(a) symmetric but neither reflexive nor transitive

(b) reflexive but neither symmetric nor transitive

(c) transitive but neither reflexive nor symmetric

(d) none of these

14. Let R be a relation on $N \times N$, defined by $(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c$. Then, R is

(a) reflexive and symmetric but not transitive

(b) reflexive and transitive but not symmetric

(c) symmetric and transitive but not reflexive

(d) an equivalence relation

15. Let A be the set of all points in a plane and let O be the origin. Let $R = \{(P, Q) : OP = OQ\}$. Then, R is

(a) reflexive and symmetric but not transitive

(b) reflexive and transitive but not symmetric

(c) symmetric and transitive but not reflexive

(d) an equivalence relation

16. Let Q be the set of all rational numbers and $*$ be the binary operation, defined by $a * b = a + 2b$, then

(a) $*$ is commutative but not associative

(b) $*$ is associative but not commutative

(c) $*$ is neither commutative nor associative

(d) $*$ is both commutative and associative

17. Let $a * b = a + ab$ for all $a, b \in Q$. Then,

(a) $*$ is not a binary composition

(b) $*$ is not commutative

(c) $*$ is commutative but not associative

(d) $*$ is both commutative and associative

18. Let Q^+ be the set of all positive rationals. Then, the operation $*$ on Q^+ defined by $a * b = \frac{ab}{2}$ for all $a, b \in Q^+$ is

(a) commutative but not associative

(b) associative but not commutative

(c) neither commutative nor associative

(d) both commutative and associative

19. Let Z be the set of all integers and let $a * b = a - b + ab$. Then, $*$ is
   (a) commutative but not associative
   (b) associative but not commutative
   (c) neither commutative nor associative
   (d) both commutative and associative
20. Let Z be the set of all integers. Then, the operation $*$ on Z defined by $a * b = a + b - ab$ is
   (a) commutative but not associative
   (b) associative but not commutative
   (c) neither commutative nor associative
   (d) both commutative and associative
21. Let Z^+ be the set of all positive integers. Then, the operation $*$ on Z^+ defined by $a * b = a^b$ is
   (a) commutative but not associative
   (b) associative but not commutative
   (c) neither commutative nor associative
   (d) both commutative and associative
22. Define $*$ on $Q - \{-1\}$ by $a * a + b + ab$. Then, $*$ on $Q - \{-1\}$ is
   (a) commutative but not associative
   (b) associative but not commutative
   (c) neither commutative nor associative
   (d) both commutative and associative

ANSWERS (OBJECTIVE QUESTIONS)

1. (b) **2.** (c) **3.** (a) **4.** (b) **5.** (d) **6.** (d) **7.** (b) **8** (c) **9.** (c) **10.** (a)
11. (a) **12.** (d) **13.** (a) **14.** (d) **15.** (d) **16.** (c) **17.** (b) **18.** (d) **19.** (c) **20.** (d)
21. (c) **22.** (d)

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

1. R is reflexive and transitive but not symmetric.
2. R is symmetric and transitive but not reflexive.
3. $(1, 2) \in R$ and $(2, 3) \in R$. But, $(1, 3) \notin R$. So, R is not transitive.
4. $a \perp a$ is not true. So, R is not reflexive.
$a \perp b$ and $b \perp c$ do not imply $a \perp c$. So, R is not transitive.
But, $a \perp b \Rightarrow b \perp a$ is always true.
9. (i) $|-3| \leq -3$ is not true. So, R is not reflexive.

(ii) $|-1| \le 1 \Rightarrow (-1)\ R\ 1$. But $|1| \le -1$ is not true.

$\therefore$ R is not symmetric.

(iii) $a\ R\ b,\ b\ R\ c \Rightarrow |a| \le b$ and $|b| \le c \Rightarrow |a| \le c$.

$\therefore$ R is transitive.

10. (i) $|a - a| = 0 \le 1$ is always true.

(ii) $a\ R\ b \Rightarrow |a - b| \le 1 \Rightarrow |-(a-b)| \le 1 \Rightarrow |b - a| \le 1 \Rightarrow b\ R\ a$.

(iii) $2\ R\ 1$ and $1\ R\ \frac{1}{2}\cdot$

But, 2 is not related to $\frac{1}{2}\cdot$ So, R is not transitive.

11. (i) $a\ R\ a$, since $(1 + a^2) > 0$.

(ii) $a\ R\ b \Rightarrow (1 + ab) > 0 \Rightarrow (1 + ba) > 0 \Rightarrow b\ R\ a$.

(iii) Let $a = \frac{-1}{2},\ b = \frac{1}{2}$ and $c = R$.

Then, $a\ R\ b$ and $b\ R\ c$. But, a is not related to c.

13. (i) $(1^2 + 1^2) \ne 1$. So, $1\ R\ 1$ is not true.

(ii) $a\ R\ b \Rightarrow a^2 + b^2 = 1 \Rightarrow b^2 + a^2 = 1 \Rightarrow b\ R\ a$.

(iii) $1\ R\ 0$ and $0\ R\ 1$. But, 1 is not related to 1.

2. FUNCTIONS

FUNCTION *Let A and B be two nonempty sets. Then, a rule f which associates to each element $x \in A$, a unique element, denoted by $f(x)$ of B, is called a function from A to B and we write,*

$$f : A \in B.$$

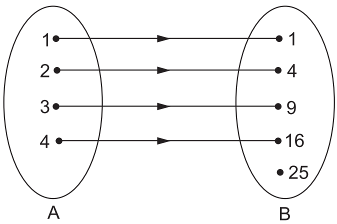
*$f(x)$ is called the **image** of x, while x is called the **pre-image** of $f(x)$.*

Domain, Codomain and Range of a Function

Let $f : A \to B$. Then, A is called the *domain* of f and B is called the *codomain* of f.

And, $f(A) = \{f(x) : x \in A\}$ is called the *range* of f.

Example 1 Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16, 25\}$.



Consider the rule $f : A \to B : f(x) = x^2 \;\forall\; x \in A$.

Then, each element in A has its unique image in B.

So, f is a function from A to B.

$$f(1) = 1^2 = 1,\;\; f(2) = 2^2 = 4,\;\; f(3) = 3^2 = 9,\;\; f(4) = 4^2 = 16.$$

Dom $(f) = \{1, 2, 3, 4\} = A$, codomain $(f) = \{1, 4, 9, 16, 25\} = B$ and range $(f) = \{1, 4, 9, 16\}$.

Clearly, $25 \in B$ does not have its pre-image in A.

Example 2 Let N be the set of all natural numbers.

Let $f : N \to N : f(x) = 2x \;\forall\; x \in N$.

Then, every element in N has its unique image in N.

So, f is a function from N to N.

Clearly, $f(1) = 2$, $f(2) = 4$, $f(3) = 6$ ..., and so on.

Dom $(f) = N$, codomain $(f) = N$,

range $(f) = \{2, 4, 6, 8, 10, ...\}$.

Various Types of Functions

MANY-ONE FUNCTION *A function $f : A \to B$ is said to be many-one if two or more than two elements in A have the same image in B.*

Example Let $A = \{-1, 1, 2, 3\}$ and $B = \{1, 4, 9\}$.

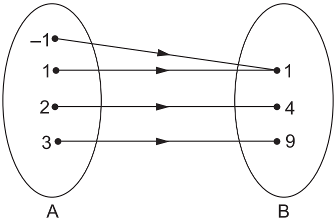
Let $f : A \to B : f(x) = x^2 \;\forall\; x \in A$.

Then, each element in A has a unique image under f in B.

$\therefore$ f is a function from A to B such that

$f(-1) = (-1)^2 = 1;\; f(1) = 1^2 = 1;$

$f(2) = 2^2 = 4$ and $f(3) = 3^2 = 9$.



Clearly, two different elements, namely –1 and 1, have the same image $1 \in B$.

Hence, f is many-one.

One-One or Injective Function

A function $f : A \to B$ is said to be one-one if distinct elements in A have distinct images in B.

f is one-one when $f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2$.

Example Let N be the set of all natural numbers.

Let $f : N \to N : f(x) = 2x \;\forall\; x \in N$.

Then, $f(x_1) = f(x_2) \;\Rightarrow\; 2x_1 = 2x_2$

$\Rightarrow\; x_1 = x_2$.

$\therefore$ f is one-one.

Onto or Surjective Function

A function $f : A \to B$ is said to be onto if every element in B has at least one pre-image in A.

Thus, if f is onto then for each $y \in B$ $\exists$ at least one element $x \in A$ such that $y = f(x)$.

Also, f is onto $\Leftrightarrow$ range$(f) = B$.

Example Let N be the set of all natural numbers and let E be the set of all even natural numbers.

Let $f : N \to E : f(x) = 2x \;\forall\; x \in N$.

Then, $y = 2x \;\Rightarrow\; x = \frac{1}{2}y$.

Thus, for each $y \in E$ there exists $\frac{1}{2}y \in N$ such that

$$f\left(\frac{1}{2}y\right) = \left(2 \times \frac{1}{2}y\right) = y.$$

$\therefore$ f is onto.

INTO FUNCTION *A function $f : A \to B$ is said to be into if there exists even a single element in B having no pre-image in A.*

Clearly, f is into $\Leftrightarrow$ range $(f) \subset B$.

Example Let $A = \{2, 3, 5, 7\}$ and $B = \{0, 1, 3, 5, 7\}$.

Let $f : A \to B : f(x) = (x - 2)$. Then,

$f(2) = (2 - 2) = 0$, $f(3) = (3 - 2) = 1$, $f(5) = (5 - 2) = 3$ and $f(7) = (7 - 2) = 5$.

Thus, every element in A has a unique image in B.

Now, $\exists\ 7 \in B$ having no pre-image in A.

$\therefore$ f is into.

Note that range $(f) = \{0, 1, 3, 5\} \subset B$.

Bijective Function

A one-one onto function is said to be bijective or a one-to-one correspondence.

CONSTANT FUNCTION *A function $f : A \to B$ is called a constant function if every element of A has the same image in B.*

Example Let $A = \{1, 2, 3\}$ and $B = \{5, 7, 9\}$.

Let $f : A \to B : f(x) = 5$ for all $x \in A$.

Clearly, every element in A has the same image.

So, f is a constant function.

REMARK The range of a constant function is a singleton set.

IDENTITY FUNCTION *The function $I_A : A \to A : I_A(x) = x\ \forall\ x \in A$ is called an identity function on A.*

Domain $(I_A) = A$ and range $(I_A) = A$.

EQUAL FUNCTIONS *Two functions f and g having the same domain D are said to be equal if $f(x) = g(x)\ \forall\ x \in D$.*

SOLVED EXAMPLES

EXAMPLE 1 *Let $f : N \to N : f(x) = 2x$ for all $x \in N$. Show that f is one-one and into.*

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Let $y = 2x$. Then, $x = \frac{y}{2}\cdot$

If we put $y = 3$ then $x = \frac{3}{2} \notin N$.

Thus, $3 \in N$ has no pre-image in N.

$\therefore$ f is into.

Hence, f is one-one and into.

EXAMPLE 2 *Show that the function* $f: R \to R: f(x) = x^2$ *is neither one-one nor onto.*

SOLUTION We have $f(-1) = (-1)^2 = 1$ and $f(1) = 1^2 = 1$.

Thus, two different elements in R have the same image.

$\therefore$ f is not one-one.

If we consider –1 in the codomain R, then it is clear that there is no element in R whose image is –1.

$\therefore$ f is not onto.

Hence, f is neither one-one nor onto.

EXAMPLE 3 *Show that the function* $f: R \to R: f(x) = |x|$ *is neither one-one nor onto.*

SOLUTION We have $f(-1) = |-1| = 1$ and $f(1) = |1| = 1$.

Thus, two different elements in R have the same image.

$\therefore$ f is not one-one.

If we consider –1 in the codomain R, then it is clear that there is no real number x whose modulus is –1.

Thus, $-1 \in R$ has no pre-image in R.

$\therefore$ f is not onto.

Hence, f is neither one-one nor onto.

EXAMPLE 4 *For any real number x, we define*

$[x]$ = *greatest integer less than or equal to x.*

Prove that the greatest integer function $f: R \to R: f(x) = [x]$ *is neither one-one nor onto.*

SOLUTION Clearly, $[1.2] = 1$ and $[1.3] = 1$.

$\therefore$ $f(1.2) = 1$ and $f(1.3) = 1$.

Thus, two different real numbers have the same image.

$\therefore$ f is not one-one.

Clearly, there is no real number x such that

$$f(x) = [x] = 1.1.$$

So, f is not onto.

Hence, f is neither one-one nor onto.

EXAMPLE 5 *Let R_0 be the set of all nonzero real numbers.*
Show that $f : R_0 \to R_0 : f(x) = \frac{1}{x}$ is a one-one onto function.

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Again, $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$.

Now, if y is a nonzero real number, then $x = \left(\frac{1}{y}\right)$ is a nonzero real number such that $f\left(\frac{1}{y}\right) = y$.

Thus, each y in R_0 has its pre-image in R_0.

So, f is onto.

Hence, f is one-one onto.

EXAMPLE 6 *Show that the function $f : R \to R : f(x) = x^3$ is one-one and onto.*

SOLUTION We have

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow x_1^3 = x_2^3 \\ &\Rightarrow (x_1^3 - x_2^3) = 0 \\ &\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0 \\ &\Rightarrow (x_1 - x_2)\left[\left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2\right] = 0 \\ &\Rightarrow (x_1 - x_2) = 0 \qquad \left[\because \left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2 \neq 0\right] \\ &\Rightarrow x_1 = x_2. \end{aligned}$$

$\therefore$ f is one-one.

Let $y \in R$ and let $y = x^3$. Then, $x = y^{1/3} \in R$.

Thus, for each y in the codomain R there exists $y^{1/3}$ in R such that $f(y^{1/3}) = (y^{1/3})^3 = y$.

$\therefore$ f is onto.

Hence, f is one-one onto.

EXAMPLE 7 *Show that the function $f : R \to R : f(x) = 3 - 4x$ is one-one onto and hence bijective.*

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow 3 - 4x_1 = 3 - 4x_2$$

$\Rightarrow \quad -4x_1 = -4x_2 \Rightarrow x_1 = x_2.$

$\therefore \quad f$ is one-one.

Now, let $y = 3 - 4x$. Then, $x = \dfrac{(3-y)}{4}$.

Thus, for each $y \in R$ (codomain of f), there exists $x = \dfrac{(3-y)}{4} \in R$ such that $f(x) = f\left(\dfrac{3-y}{4}\right) = \left\{3 - 4 \cdot \dfrac{(3-y)}{4}\right\} = 3 - (3-y) = y.$

This shows that every element in codomain of f has its pre-image in dom (f).

$\therefore \quad f$ is onto.

Hence, the given function is bijective.

EXAMPLE 8 *Show that the function $f : N \to N$, defined by*

$$f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$$

is one-one and onto. **[CBSE 2012]**

SOLUTION Suppose $f(x_1) = f(x_2)$.

Case 1 *When x_1 is odd and x_2 is even*

In this case, $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 - 1$

$\Rightarrow x_2 - x_1 = 2.$

This is a contradiction, since the difference between an odd integer and an even integer can never be 2.

Thus, in this case, $f(x_1) \neq f(x_2)$.

Similarly, when x_1 is even and x_2 is odd, then $f(x_1) \neq f(x_2)$.

Case 2 *When x_1 and x_2 are both odd*

In this case, $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1$

$\Rightarrow x_1 = x_2.$

$\therefore f$ is one-one.

Case 3 *When x_1 and x_2 are both even*

In this case, $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1$

$\Rightarrow x_1 = x_2.$

$\therefore \quad f$ is one-one.

In order to show that f is onto, let $y \in N$ (the codomain).

Case 1 *When y is odd*

In this case, $(y + 1)$ is even.

$\therefore \quad f(y+1) = (y+1) - 1 = y.$

Case 2 *When y is even*

In this case, $(y - 1)$ is odd.

$\therefore \quad f(y-1) = y - 1 + 1 = y.$

Thus, each $y \in N$ (codomain of f) has its pre-image in dom (f).

$\therefore$ f is onto.

Hence, f is one-one onto.

EXAMPLE 9 *Show that $f : N \to N$, defined by*

$$f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

is a many-one onto function. **[CBSE 2012C]**

SOLUTION We have

$$f(1) = \frac{(1+1)}{2} = \frac{2}{2} = 1 \quad \text{and} \quad f(2) = \frac{2}{2} = 1.$$

Thus, $f(1) = f(2)$ while $1 \neq 2$.

$\therefore$ f is many-one.

In order to show that f is onto, consider an arbitrary element $n \in N$.

If n is odd then $(2n - 1)$ is odd, and

$$f(2n-1) = \frac{(2n-1+1)}{2} = \frac{2n}{2} = n.$$

If n is even then $2n$ is even and

$$f(2n) = \frac{2n}{2} = n.$$

Thus, for each $n \in N$ (whether even or odd) there exists its pre-image in N.

$\therefore$ f is onto.

Hence, f is many-one onto.

EXAMPLE 10 *Show that the signum function $f : R \to R$, defined by*

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is neither one-one nor onto.

SOLUTION Clearly, $f(2) = 1$ and $f(3) = 1$.

Thus, $f(2) = f(3)$ while $2 \neq 3$.

$\therefore$ f is not one-one.

Range (f) = $\{1, \; 0, \; -1\} \subset R$.

So, f is into.

Hence, f is neither one-one nor onto.

EXAMPLE 11 *Let $A = R - \{3\}$ and $B = R - \{1\}$.*
Let $f : A \to B : f(x) = \dfrac{x-2}{x-3}$ for all values of $x \in A$.
Show that f is one-one and onto. **[CBSE 2012]**

SOLUTION f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$
$$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$$
$$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$$
$$\Rightarrow x_1 = x_2.$$

Let $y \in B$ such that $y = \dfrac{x-2}{x-3}$.

Then, $(x-3)\,y = (x-2) \Rightarrow x = \dfrac{(3y-2)}{(y-1)}$.

Clearly, x is defined when $y \neq 1$.

Also, $x = 3$ will give us $1 = 0$, which is false.

$\therefore\ x \neq 3$.

And, $f(x) = \dfrac{\left(\dfrac{3y-2}{y-1} - 2\right)}{\left(\dfrac{3y-2}{y-1} - 3\right)} = y.$

Thus, for each $y \in B$, there exists $x \in A$ such that $f(x) = y$.

$\therefore\ f$ is onto.

Hence, f is one-one onto.

EXAMPLE 12 *Let A and B be two nonempty sets. Show that the function $f : (A \times B) \to (B \times A) : f(a, b) = (b, a)$ is a bijective function.*

SOLUTION f is one-one, since

$$f(a_1, b_1) = f(a_2, b_2) \Rightarrow (b_1, a_1) = (b_2, a_2)$$
$$\Rightarrow a_1 = a_2 \text{ and } b_1 = b_2$$
$$\Rightarrow (a_1, b_1) = (a_2, b_2).$$

In order to show that f is onto, let (b, a) be an arbitrary element of $(B \times A)$.

Then, $(b, a) \in (B \times A) \Rightarrow b \in B$ and $a \in A$
$$\Rightarrow (a, b) \in (A \times B).$$

Thus, for each $(b, a) \in (B \times A)$, there exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$.

$\therefore\ f$ is onto.

Thus, f is one-one onto and hence bijective.

EXAMPLE 13 *Find the domain and range of the real function*

$$f(x) = \sqrt{9 - x^2}.$$

SOLUTION It is clear that $f(x) = \sqrt{9 - x^2}$ is not defined when $(9 - x^2) < 0$, i.e., when $x^2 > 9$, i.e., when $x > 3$ or $x < -3$.

$\therefore$ dom $(f) = \{x \in R : -3 \le x \le 3\}$.

Also, $y = \sqrt{9 - x^2} \Rightarrow y^2 = (9 - x^2)$

$\Rightarrow x = \sqrt{9 - y^2}$.

Clearly, x is not defined when $(9 - y^2) < 0$.

But, $(9 - y^2) < 0 \Rightarrow y^2 > 9$

$\Rightarrow y > 3$ or $y < -3$

$\therefore$ range $(f) = \{y \in R : -3 \le y \le 3\}$.

EXAMPLE 14 *Find the domain and range of the real function, defined by*

$$f(x) = \frac{1}{(1 - x^2)}.$$

SOLUTION Clearly, $\frac{1}{(1 - x^2)}$ is not defined when $1 - x^2 = 0$,

i.e., when $x = \pm 1$.

$\therefore$ dom $(f) = R - \{-1, 1\}$.

Also, $y = \frac{1}{(1 - x^2)} \Rightarrow (1 - x^2) = \frac{1}{y} \Rightarrow x = \sqrt{1 - \frac{1}{y}}$.

Clearly, x is not defined when $\left(1 - \frac{1}{y}\right) < 0$ or $1 < \frac{1}{y}$ or $y < 1$.

$\therefore$ range $(f) = R - \{y \in R : y < 1\} = \{y \in R : y \ge 1\}$.

EXAMPLE 15 *Consider a function $f : X \to Y$ and define a relation R in X by $R = \{(a, b) : f(a) = f(b)\}$. Show that R is an equivalence relation.*

SOLUTION Here, R satisfies the following properties:

(i) *Reflexivity*
Let $a \in X$. Then,

$$f(a) = f(a) \Rightarrow (a, a) \in R.$$

$\therefore$ R is reflexive.

(ii) *Symmetry*
Let $(a, b) \in R$. Then,

$$(a, b) \in R \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow (b, a) \in R.$$

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$$(a, b) \in R, \ (b, c) \in R$$

$$\Rightarrow f(a) = f(b) \text{ and } f(b) = f(c)$$

$$\Rightarrow f(a) = f(c)$$

$$\Rightarrow (a, c) \in R.$$

$\therefore$ R is transitive.

Hence, R is an equivalence relation.

EXERCISE 2A

1. Define a function. What do you mean by the domain and range of a function? Give examples.
2. Define each of the following:
   (i) injective function (ii) surjective function
   (iii) bijective function (iv) many-one function
   (v) into function
   Give an example of each type of functions.
3. Give an example of a function which is
   (i) one-one but not onto (ii) one-one and onto
   (iii) neither one-one nor onto (iv) onto but not one-one.
4. Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} 2x + 3, & \text{when } x < -2 \\ x^2 - 2, & \text{when } -2 \le x \le 3 \\ 3x - 1, & \text{when } x > 3. \end{cases}$$

   Find (i) $f(2)$ (ii) $f(4)$ (iii) $f(-1)$ (iv) $f(-3)$.
5. Show that the function $f : R \to R : f(x) = 1 + x^2$ is many-one into.
6. Show that the function $f : R \to R : f(x) = x^4$ is many-one and into.
7. Show that the function $f : R \to R : f(x) = x^5$ is one-one and onto.
8. Let $f : \left[0, \frac{\pi}{2}\right] \to R : f(x) = \sin x$ and $g : \left[0, \frac{\pi}{2}\right] \to R : g(x) = \cos x$. Show that each one of f and g is one-one but $(f + g)$ is not one-one.
9. Show that the function
   (i) $f : N \to N : f(x) = x^2$ is one-one into.
   (ii) $f : Z \to Z : f(x) = x^2$ is many-one into.

10. Show that the function
(i) $f : N \to N : f(x) = x^3$ is one-one into
(ii) $f : Z \to Z : f(x) = x^3$ is one-one into

11. Show that the function $f : R \to R : f(x) = \sin x$ is neither one-one nor onto.

12. Prove that the function $f : N \to N : f(n) = (n^2 + n + 1)$ is one-one but not onto.

13. Show that the function $f : N \to Z$, defined by

$$f(n) = \begin{cases} \frac{1}{2}(n-1), & \text{when } n \text{ is odd} \\ -\frac{1}{2}n, & \text{when } n \text{ is even} \end{cases}$$

is both one-one and onto.

14. Find the domain and range of the function
$f : R \to R : f(x) = x^2 + 1.$

15. Which of the following relations are functions? Give reasons. In case of a function, find its domain and range.
(i) $f = \{(-1, 2), (1, 8), (2, 11), (3, 14)\}$
(ii) $g = \{(1, 1), (1, -1), (4, 2), (9, 3), (16, 4)\}$
(iii) $h = \{(a, b), (b, c), (c, b), (d, c)\}$

16. Find the domain and range of the real function, defined by $f(x) = \dfrac{x^2}{(1 + x^2)}$.

Show that f is many-one.

17. Show that the function

$$f : R \to R : f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$

is many-one into.

Find (i) $f\left(\frac{1}{2}\right)$ (ii) $f(\sqrt{2})$ (iii) $f(\pi)$ (iv) $f(2 + \sqrt{3})$.

ANSWERS (EXERCISE 2A)

4. (i) 2 (ii) 11 (iii) –1 (iv) –3

14. dom $(f) = R$ and range $(f) = \{y \in R : y \geq 1\}$

15. (i) f is a function, dom $(f) = \{-1, 1, 2, 3\}$ and range $(f) = \{2, 8, 11, 14\}$
(ii) g is not a function
(iii) h is a function, dom $(h) = \{a, b, c, d\}$ and range $(h) = \{b, c\}$

17. (i) 1 (ii) –1 (iii) –1 (iv) –1

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 2A)

2. (i) $f: N \to N: f(x) = 2x$ is an injective function, as

$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.

(ii) Let $A = \{1, -1, 2, 3\}$ and $B = \{1, 4, 9\}$.

Then, $f: A \to B: f(x) = x^2$ is surjective, since each element of B has at least one pre-image in A.

(iii) Let E be the set of all even natural numbers.

Then, $f: N \to E: f(x) = 2x$ is one-one and onto.

f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2.$$

f is onto, since for each $y \in E$, there exists $\frac{1}{2}y \in N$ such that $f\left(\frac{1}{2}y\right) = y$.

(iv) Example given in (ii) is many-one.

(v) Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9, 16\}$.

Then, $f: A \to B: f(x) = x^2$ is an into function, since range $(f) = \{1, 4, 9\} \subset B$.

5. $f(-1) = 2 = f(1)$. So, f is many-one.

$-1 \in R$ has no pre-image in R. So, f is into.

6. $f(-1) = f(1) = 1$. So, f is many-one.

$-1 \in R$ has no pre-image in R. So, f is into.

7. $f(x_1) = f(x_2) \Rightarrow x_1^5 = x_2^5 \Rightarrow x_1 = x_2$. So, f is one-one

for each $y \in R \; \exists \; y^{1/5} \in R$ s.t. $f(y^{1/5}) = y$.

8. If $x_1 \neq x_2$ and $x_1, x_2 \in \left[0, \frac{\pi}{2}\right]$ then $\sin x_1 \neq \sin x_2$ and $\cos x_1 \neq \cos x_2$.

$\therefore$ f is one-one and g is one-one.

But $(f + g)(x) = f(x) + g(x) = \sin x + \cos x$.

$\therefore$ $(f + g)(0) + (\sin 0 + \cos 0) = 1$ and $(f + g)\left(\frac{\pi}{2}\right) = \left(\sin \frac{\pi}{2} + \cos \frac{\pi}{2}\right) = 1$.

9. (i) $f(a) = f(b) \Rightarrow a^2 = b^2 \Rightarrow a = b$ $[\because a, b \in N]$

$\therefore$ f is one-one.

Clearly, $2 \in N$ [codomain (f)] has no pre-image in N.

$\therefore$ f is into.

(ii) $f(-1) = f(1) = 1$. So, f is many-one.

$2 \in N$ [codomain (f)] has no pre-image in N.

$\therefore$ f is into.

10. (i) $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ $[\because x_1, x_2 \in N]$.

$\therefore$ f is one-one.

$2 \in N$. But, $2^{1/3} \notin N$. So, f is into.

(ii) $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ $[\because x_1, x_2 \in Z]$

$2 \in Z$. But, $2^{1/3} \notin Z$. So, f is into.

11. We know that $\sin(0) = 0$ and $\sin(\pi) = 0$.

Thus, 0 and π have the same image.

So, f is many-one.

Range $(f) = [-1, 1] \subset R$. Hence, f is into.

So, f is neither one-one nor onto.

12. $f(n_1) = f(n_2) \Rightarrow n_1^2 + n_1 + 1 = n_2^2 + n_2 + 1$

$$\Rightarrow (n_1^2 - n_2^2) + (n_1 - n_2) = 0$$

$$\Rightarrow (n_1 - n_2)(n_1 + n_2 + 1) = 0$$

$$\Rightarrow n_1 - n_2 = 0 \Rightarrow n_1 = n_2.$$

$\therefore$ f is one-one.

But, $f(n) = 1 \Rightarrow n^2 + n + 1 = 1 \Rightarrow n^2 + n = 0$

$$\Rightarrow n(n+1) = 0 \Rightarrow n = 0 \text{ or } n = -1.$$

And, none of 0 and –1 is a natural number.

Thus, $1 \in N$ has no pre-image in N.

$\therefore$ f is into.

14. Dom $(f) = R$. Also, $y = x^2 + 1 \Rightarrow x = \sqrt{y-1}$.

x is defined when $y - 1 \geq 0$, i.e., $y \geq 1$.

$\therefore$ range $(f) = \{y \in R : y \geq 1\}$.

15. g is not a function, since 1 has two images under g.

16. When x is real, $1 + x^2 \neq 0$. So, dom $(f) = R$.

$$y = \frac{x^2}{(1+x^2)} \Rightarrow x^2(1-y) = y \Rightarrow x = \sqrt{\frac{y}{1-y}}.$$

For x to be real, $\frac{y}{(1-y)} \geq 0$ and $(1-y) \neq 0$.

$\therefore$ range $(f) = \{y \in R : 0 \leq y < 1\}$.

Also, 1 and –1 have the same image $\left(\frac{1}{2}\right)$.

Composition of Functions

Let $f : A \to B$ and $g : B \to C$ be two given functions. Then, the composition of f and g, denoted by $g \circ f$ is the function, defined by

$$(g \circ f) : A \to C : (g \circ f)(x) = g\{f(x)\} \ \forall \ x \in A.$$

Clearly, dom $(g \circ f)$ = dom (f).

Also, $g \circ f$ is defined only when range $(f) \subseteq$ dom (g).

REMARK $(f \circ g)$ is defined only when range $(g) \subseteq$ dom (f).
And, dom $(f \circ g)$ = dom (g).

SOLVED EXAMPLES

EXAMPLE 1 *Let* $f:\{1, 3, 4\} \to \{1, 2, 5\}$ *and* $g:\{1, 2, 5\} \to \{1, 3\}$ *be defined as* $f = \{(1, 2), (3, 5), (4, 1)\}$ *and* $g = \{(1, 3), (2, 3), (5, 1)\}$. *Find* $(g \circ f)$ *and* $(f \circ g)$.

SOLUTION Here range $(f) = \{1, 2, 5\}$ and dom $(g) = \{1, 2, 5\}$.

Clearly, range $(f) \subseteq$ dom (g).

$\therefore$ $(g \circ f)$ is defined and dom $(g \circ f) =$ dom $(f) = \{1, 3, 4\}$.

Now, $(g \circ f)(1) = g\{f(1)\} = g(2) = 3$;

$(g \circ f)(3) = g\{f(3)\} = g(5) = 1$;

$(g \circ f)(4) = g\{f(4)\} = g(1) = 3$.

Hence, $(g \circ f) = \{(1, 3), (3, 1), (4, 3)\}$.

Again, range $(g) = \{1, 3\}$ and dom $(f) = \{1, 3, 4\}$.

Clearly, range $(g) \subseteq$ dom (f).

$\therefore$ $(f \circ g)$ is defined and dom $(f \circ g) =$ dom $(g) = \{1, 2, 5\}$.

Now, $(f \circ g)(1) = f\{g(1)\} = f(3) = 5$;

$(f \circ g)(2) = f\{g(2)\} = f(3) = 5$;

$(f \circ g)(5) = f\{g(5)\} = f(1) = 2$.

Hence, $(f \circ g) = \{(1, 5), (2, 5), (5, 2)\}$.

EXAMPLE 2 *Let R be the set of all real numbers. Let* $f: R \to R: f(x) = \cos x$ *and let* $g: R \to R: g(x) = 3x^2$. *Show that* $(g \circ f) \neq (f \circ g)$.

SOLUTION Let x be an arbitrary real number. Then,

$$(g \circ f)(x) = g\{f(x)\} = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x.$$

$$(f \circ g)(x) = f\{g(x)\} = f(3x^2) = \cos(3x^2).$$

Taking $x = 0$, we have

$$(g \circ f)(0) = 3\cos^2 0 = (3 \times 1) = 3.$$

$$(f \circ g)(0) = \cos(3 \times 0) = \cos 0 = 1.$$

$\therefore$ $(g \circ f)(0) \neq (f \circ g)(0)$.

Hence, $g \circ f \neq f \circ g$.

EXAMPLE 3 *Let R be the set of all real numbers. Let* $f: R \to R: f(x) = \sin x$ *and* $g: R \to R: g(x) = x^2$. *Prove that* $g \circ f \neq f \circ g$.

SOLUTION Let x be an arbitrary real number. Then,

$$(g \circ f)(x) = g\{f(x)\} = g(\sin x) = (\sin x)^2.$$

$$(f \circ g)(x) = f\{g(x)\} = f(x^2) = \sin x^2.$$

Clearly, $(\sin x)^2 \neq \sin x^2$.

Hence, $g \circ f \neq f \circ g$.

EXAMPLE 4 *Let* $f: R \to R: f(x) = 8x^3$ *and* $g: R \to R: g(x) = x^{1/3}$. *Find* $(g \circ f)$ *and* $(f \circ g)$ *and show that* $g \circ f \neq f \circ g$.

SOLUTION Let $x \in R$. Then, we have

$$(g \circ f)(x) = g\{f(x)\} = g(8x^3) = (8x^3)^{1/3} = 2x.$$

$$(f \circ g)(x) = f\{g(x)\} = f(x^{1/3}) = 8(x^{1/3})^3 = 8x.$$

$\therefore \quad g \circ f \neq f \circ g$.

EXAMPLE 5 *Let* $f: R \to R: f(x) = (x^2 - 3x + 2)$, *find* $(f \circ f)(x)$.

SOLUTION We have

$$(f \circ f)(x) = f\{f(x)\} = f(x^2 - 3x + 2) = f(y), \text{ where } y = (x^2 - 3x + 2)$$

$$= (y^2 - 3y + 2)$$

$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$

$$= (x^4 - 6x^3 + 10x^2 - 3x).$$

EXAMPLE 6 *If* $f: R \to R: f(x) = (3 - x^3)^{1/3}$, *show that* $(f \circ f)(x) = x$.

SOLUTION We have

$$(f \circ f)(x) = f\{f(x)\} = f(3 - x^3)^{1/3}$$

$$= f(y), \quad \text{where } y = (3 - x^3)^{1/3}$$

$$= (3 - y^3)^{1/3} = [3 - (3 - x^3)]^{1/3} \quad [\because \; y^3 = (3 - x^3)]$$

$$= (x^3)^{1/3} = x.$$

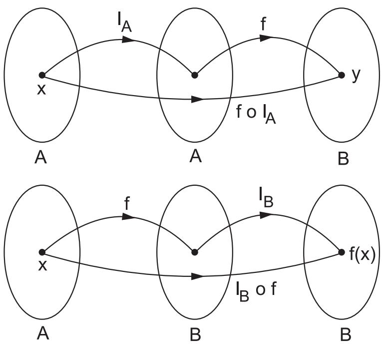
Hence, $(f \circ f)(x) = x$.

EXAMPLE 7 *Let* $f: A \to B$, *and let* I_A *and* I_B *be identity functions on* A *and* B *respectively. Prove that* $(f \circ I_A) = f$ *and* $(I_B \circ f) = f$.

SOLUTION Let $x \in A$ and let $f(x) = y$. Then,

$$(f \circ I_A)(x) = f\{I_A(x)\} = f(x) \qquad [\because \; I_A(x) = x].$$

$\therefore \quad (f \circ I_A) = f$.



And, $(I_B \circ f)(x) = I_B\{f(x)\}$

$= I_B(y)$ [$\because$ $f(x) = y$]

$= y$ [$\because$ $I_B(y) = y$]

$= f(x)$ [$\because$ $y = f(x)$].

$\therefore$ $(I_B \circ f) = f$.

Hence, $(f \circ I_A) = f$ and $(I_B \circ f) = f$.

EXAMPLE 8 (Associativity) *Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Then, prove that $(h \circ g) \circ f = h \circ (g \circ f)$.*

SOLUTION Let $x \in A$. Then,

$$\{(h \circ g) \circ f\}(x) = (h \circ g)\{f(x)\}$$
$$= h[g\{f(x)\}]$$
$$= h[(g \circ f)(x)]$$
$$= \{h \circ (g \circ f)\}(x).$$

$\therefore$ $(h \circ g) \circ f = h \circ (g \circ f)$.

EXAMPLE 9 *Let $f : Z \to Z : f(n) = 3n$ and let $g : Z \to Z$, defined by*

$$g(n) = \begin{cases} \dfrac{n}{3}, & \text{if } n \text{ is a multiple of } 3 \\ 0, & \text{if } n \text{ is not a multiple of } 3. \end{cases}$$

Show that $g \circ f = I_Z$ and $f \circ g \neq I_Z$.

SOLUTION Let n be an arbitrary element of Z. Then,

$$(g \circ f)(n) = g\{f(n)\}$$
$$= g(3n) = \frac{3n}{3} = n$$
$$= I_Z(n).$$

$\therefore$ $(g \circ f) = I_Z$.

Also, we have

$(f \circ g)(1) = f\{g(1)\}$

$= f(0)$ [$\because$ $g(1) = 0$]

$= (3 \times 0) = 0$ [$\because$ $f(n) = 3n$].

$I_Z(1) = 1$ [$\because$ $I_Z(x) = x \ \forall \ x \in Z$].

$\therefore$ $f \circ g \neq I_Z$.

EXAMPLE 10 *Let $A = R - \left\{\dfrac{3}{5}\right\}$ and $B = R - \left\{\dfrac{7}{5}\right\}$.*

Let $f : A \to B : f(x) = \dfrac{7x + 4}{5x - 3}$ and $g : B \to A : g(y) = \dfrac{3y + 4}{5y - 7}$.

Show that $(g \circ f) = I_A$ and $(f \circ g) = I_B$.

SOLUTION Let $x \in A$. Then,

$$(g \circ f)(x) = g[f(x)]$$
$$= g\left(\frac{7x+4}{5x-3}\right)$$
$$= g(y), \text{ where } y = \frac{7x+4}{5x-3} \quad \ldots \text{(i)}$$
$$= \frac{3y+4}{5y-7} = \frac{3\left(\frac{7x+4}{5x-3}\right)+4}{5\left(\frac{7x+4}{5x-3}\right)-7} \quad \text{[using (i)]}$$
$$= \frac{(21x+12+20x-12)}{(5x-3)} \times \frac{(5x-3)}{(35x+20-35x+21)}$$
$$= \frac{41x}{41} = x = I_A(x).$$

$\therefore \quad (g \circ f) = I_A$.

Again, let $y \in B$. Then,

$$(f \circ g)(y) = f[g(y)]$$
$$= f\left(\frac{3y+4}{5y-7}\right)$$
$$= f(z), \text{ where } z = \frac{3y+4}{5y-7} \quad \ldots \text{(ii)}$$
$$= \frac{7z+4}{5z-3} = \frac{7\left(\frac{3y+4}{5y-7}\right)+4}{5\left(\frac{3y+4}{5y-7}\right)-3}$$
$$= \frac{(21y+28+20y-28)}{(5y-7)} \times \frac{(5y-7)}{(15y+20-15y+21)}$$
$$= \frac{41y}{41} = y = I_B(y).$$

$\therefore \quad (f \circ g) = I_B$.

Hence, $(g \circ f) = I_A$ and $(f \circ g) = I_B$.

EXAMPLE 11 *Let $f: A \to B$ and $g: B \to A$ such that $(g \circ f) = I_A$. Show that f is one-one and g is onto.*

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow g\{f(x_1)\} = g\{f(x_2)\}$$
$$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$$
$$\Rightarrow I_A(x_1) = I_A(x_2)$$
$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

In order to show that g is onto, let $a \in A$ and let $f(a) = b \in B$.

Then, $g(b) = g[f(a)] = (g \circ f)(a)$

$= I_A(a)$ $[\because g \circ f = I_A]$.

Thus, for each $a \in A$, there exists $b \in B$ such that $g(b) = a$.

$\therefore$ g is onto.

EXERCISE 2B

1. Let $A = \{1, 2, 3, 4\}$. Let $f : A \to A$ and $g : A \to A$, defined by $f = \{(1, 4), (2, 1), (3, 3), (4, 2)\}$ and
$$g = \{(1, 3), (2, 1), (3, 2), (4, 4)\}.$$
Find (i) $g \circ f$ (ii) $f \circ g$ (iii) $f \circ f$.
2. Let $f : \{3, 9, 12\} \to \{1, 3, 4\}$ and $g : \{1, 3, 4, 5\} \to \{3, 9\}$ be defined as $f = \{(3, 1), (9, 3), (12, 4)\}$ and
$$g = \{(1, 3), (3, 3), (4, 9), (5, 9)\}.$$
Find (i) $(g \circ f)$ (ii) $(f \circ g)$.
3. Let $f : R \to R : f(x) = x^2$ and $g : R \to R : g(x) = (x + 1)$.
Show that $(g \circ f) \neq (f \circ g)$.
4. Let $f : R \to R : f(x) = (2x + 1)$ and $g : R \to R : g(x) = (x^2 - 2)$.
Write down the formulae for
(i) $(g \circ f)$ (ii) $(f \circ g)$ (iii) $(f \circ f)$ (iv) $(g \circ g)$.
5. Let $f : R \to R : f(x) = (x^2 + 3x + 1)$ and $g : R \to R : g(x) = (2x - 3)$. Write down the formulae for
(i) $g \circ f$ (ii) $f \circ g$ (iii) $g \circ g$.
6. Let $f : R \to R : f(x) = |x|$, prove that $f \circ f = f$.
7. Let $f : R \to R : f(x) = x^2$, $g : R \to R : g(x) = \tan x$ and $h : R \to R : h(x) = \log x$.
Find a formula for $h \circ (g \circ f)$.
Show that $[h \circ (g \circ f)] \sqrt{\dfrac{\pi}{4}} = 0$.
8. Let $f : R \to R : f(x) = (2x - 3)$ and $g : R \to R : g(x) = \dfrac{1}{2}(x + 3)$.
Show that $(f \circ g) = I_R = (g \circ f)$.
9. Let $f : Z \to Z : f(x) = 2x$. Find $g : Z \to Z : g \circ f = I_Z$.
10. Let $f : N \to N : f(x) = 2x$, $g : N \to N : g(y) = 3y + 4$ and $h : N \to N : h(z) = \sin z$.
Show that $h \circ (g \circ f) = (h \circ g) \circ f$.

11. If f be a greatest integer function and g be an absolute value function, find the value of $(f \circ g)\left(\frac{-3}{2}\right) + (g \circ f)\left(\frac{4}{3}\right)$. **[CBSE 2007]**

12. Let $f : R \to R : f(x) = x^2 + 2$ and $g : R \to R : g(x) = \frac{x}{x-1}$, $x \neq 1$. Find $f \circ g$ and $g \circ f$ and hence find $(f \circ g)(2)$ and $(g \circ f)(-3)$. **[CBSE 2014]**

ANSWERS (EXERCISE 2B)

1. (i) $(g \circ f) = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$
(ii) $(f \circ g) = \{(1, 3), (2, 4), (3, 1), (4, 2)\}$
(iii) $(f \circ f) = \{(1, 2), (2, 4), (3, 3), (4, 1)\}$

2. (i) $(g \circ f) = \{(3, 3), (9, 3), (12, 9)\}$
(ii) $(f \circ g) = \{(1, 1), (3, 1), (4, 3), (5, 3)\}$

4. (i) $(g \circ f)(x) = (4x^2 + 4x - 1)$ (ii) $(f \circ g)(x) = (2x^2 - 3)$
(iii) $(f \circ f)(x) = (4x + 3)$ (iv) $(g \circ g)(x) = (x^4 - 4x^2 + 2)$

5. (i) $(g \circ f)(x) = (2x^2 + 6x - 1)$ (ii) $(f \circ g)(x) = (4x^2 - 6x + 1)$
(iii) $(g \circ g)(x) = (4x - 9)$

7. $[h \circ (g \circ f)](x) = \log(\tan x^2)$ **11.** 2

12. $(f \circ g)(x) = \frac{x^2}{(x-1)^2} + 2$, $(g \circ f)(x) = \frac{x^2+2}{x^2+1}$;

$(f \circ g)(2) = 6$, $(g \circ f)(-3) = \frac{11}{10}$.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 2B)

9. Let $x \in Z$. Then,

$$(g \circ f) = I_Z \Rightarrow (g \circ f)(x) = I_Z(x)$$
$$\Rightarrow g[f(x)] = x$$
$$\Rightarrow g(2x) = x$$
$$\Rightarrow g(y) = \frac{1}{2}y \quad [\text{where } 2x = y].$$

Thus, $g : Z \to Z : g(y) = \frac{1}{2}y$.

11. $(f \circ g)\left(\frac{-3}{2}\right) = f\left\{g\left(\frac{-3}{2}\right)\right\} = f\left(\left|\frac{-3}{2}\right|\right) = f\left(\frac{3}{2}\right) = \left[\frac{3}{2}\right] = 1.$

$(g \circ f)\left(\frac{4}{3}\right) = g\left\{f\left(\frac{4}{3}\right)\right\} = g\left[\frac{4}{3}\right] = g(1) = |1| = 1.$

Required sum $= (1 + 1) = 2$.

12. $(f \circ g)(x) = f\{g(x)\} = f\left(\frac{x}{x-1}\right) = \frac{x^2}{(x-1)^2} + 2.$

$$(g \circ f)(x) = g\{f(x)\} = g(x^2+2) = \frac{x^2+2}{x^2+2-1} = \frac{x^2+2}{x^2+1}.$$

Invertible Function

Let $f : A \to B$. If there exists a function $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$ then f is called an invertible function and g is called the inverse of f. We write, $f^{-1} = g$.

REMARK Clearly, $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$.

Example Let $f : R \to R : f(x) = 2x + 3$.

Let $y = f(x)$. Then,

$$y = f(x) \Rightarrow y = 2x + 3$$
$$\Rightarrow x = \frac{1}{2}(y-3)$$
$$\Rightarrow f^{-1}(y) = \frac{1}{2}(y-3) \; [\because y = f(x) \Rightarrow x = f^{-1}(y)].$$

Thus, we define:

$$f^{-1} : R \to R : f^{-1}(y) = \frac{1}{2}(y-3).$$

THEOREM 1 *If $f : A \to B$ is one-one onto then prove that f is an invertible function.*

PROOF Let $y \in B$. Then, f being one-one onto, there exists a unique $x \in A$ such that $f(x) = y$.

We define $g : B \to A : g(y) = x$. Then,

$$(g \circ f)(x) = g[f(x)]$$
$$= g(y) \quad [\because f(x) = y]$$
$$= x \quad [\because g(y) = x]$$
$$= I_A(x).$$

$\therefore \; (g \circ f) = I_A$.

$$(f \circ g)(y) = f[g(y)]$$
$$= f(x) \quad [\because g(y) = x]$$
$$= y \quad [\because f(x) = y]$$
$$= I_B(y).$$

$\therefore \; (f \circ g) = I_B$.

Hence, f is invertible and $f^{-1} = g$.

THEOREM 2 *If $f : A \to B$ is an invertible function, then prove that f is one-one onto.*

PROOF Let $f : A \to B$ be an invertible function. Then, there exists a function $g : B \to A$ such that

$$g \circ f = I_A \quad \text{and} \quad f \circ g = I_B.$$

Now, $f(x_1) = f(x_2)$

$\Rightarrow g\{f(x_1)\} = g\{f(x_2)\}$

$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow I_A(x_1) = I_A(x_2) \quad [\because g \circ f = I_A]$

$\Rightarrow x_1 = x_2.$

$\therefore$ f is one-one.

Let $y \in B$. Then, $g(y) \in A$. Let $g(y) = x$. Then,

$g(y) = x$

$\Rightarrow f\{g(y)\} = f(x)$

$\Rightarrow (f \circ g)(y) = f(x)$

$\Rightarrow I_B(y) = f(x) \quad [\because f \circ g = I_B]$

$\Rightarrow y = f(x).$

Thus, for each $y \in B$ there exists $x \in A$ such that $y = f(x)$.

$\therefore$ f is onto.

Hence, f is one-one onto.

REMARK f is invertible $\Leftrightarrow$ f is one-one onto.

SOLVED EXAMPLES

EXAMPLE 1 *Let $f : R \to R : f(x) = 4x + 3$ for all $x \in R$. Show that f is invertible and find f^{-1}.*

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow 4x_1 + 3 = 4x_2 + 3$$
$$\Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Again, $y = 4x + 3 \Rightarrow x = \frac{(y-3)}{4}$.

Now, if $y \in R$ (codomain of f) then there exists $x = \frac{(y-3)}{4} \in R$

such that $f(x) = f\left(\frac{y-3}{4}\right) = \left\{4 \cdot \frac{(y-3)}{4} + 3\right\} = y$.

$\therefore$ f is onto.

Thus, f is one-one onto and therefore invertible.

Now, $y = f(x) \Rightarrow y = 4x + 3$

$$\Rightarrow x = \frac{(y-3)}{4}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{(y-3)}{4} \qquad [\because \quad f(x) = y \Rightarrow x = f^{-1}(y)].$$

Thus, we define:

$$f^{-1} : R \to R : f^{-1}(y) = \frac{(y-3)}{4} \text{ for all } y \in R.$$

EXAMPLE 2 *Let R_+ be the set of all positive real numbers. Let $f : R_+ \to [4, \infty[: f(x) = x^2 + 4$. Show that f is invertible and find f^{-1}.*

SOLUTION We have

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow x_1^2 + 4 = x_2^2 + 4 \\ &\Rightarrow x_1^2 = x_2^2 \\ &\Rightarrow x_1^2 - x_2^2 = 0 \\ &\Rightarrow (x_1 - x_2)(x_1 + x_2) = 0 \\ &\Rightarrow x_1 - x_2 = 0 \qquad [\because \quad (x_1 + x_2) \neq 0] \\ &\Rightarrow x_1 = x_2. \end{aligned}$$

$\therefore$ f is one-one.

Now, $y = x^2 + 4 \Rightarrow x = \sqrt{y-4}$.

For each $y \in [4, \infty[$ there exists $x = \sqrt{y-4}$ in R_+ such that $f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = (y-4) + 4 = y$.

$\therefore$ f is onto.

Thus, f is one-one onto and therefore invertible.

$$\begin{aligned} \text{Now, } y = f(x) &\Rightarrow y = x^2 + 4 \\ &\Rightarrow x = \sqrt{y-4} \\ &\Rightarrow f^{-1}(y) = \sqrt{y-4}. \end{aligned}$$

$\therefore$ $f^{-1} : [4, \infty[\to R_+ : f^{-1}(y) = \sqrt{y-4}$.

EXAMPLE 3 *Let R^+ be the set of all positive real numbers. Let $f : R^+ \to R^+ : f(x) = e^x$ for all $x \in R^+$. Show that f is invertible and hence find f^{-1}.*

SOLUTION f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2.$$

Now, for each $y \in R^+$, there exists a positive real number, namely $\log y$ such that

$$f(\log y) = e^{\log y} = y.$$

$\therefore$ f is onto.

Thus, f is one-one onto and hence invertible.

We define:

$$f^{-1} : R^+ \to R^+ : f^{-1}(y) = \log y \text{ for all } y \in R^+.$$

EXAMPLE 4 *Let* $A = \left\{x : x \in R,\ \frac{-\pi}{2} \le x \le \frac{\pi}{2}\right\}$ *and* $B = \{y : y \in R, -1 \le y \le 1\}$. *Show that the function* $f : A \to B : f(x) = \sin x$ *is invertible and hence find* f^{-1}.

SOLUTION Here, $A = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and $B = [-1, 1]$.

Also, $f : A \to B : f(x) = \sin x$.

f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow \sin x_1 = \sin x_2$$
$$\Rightarrow x_1 = x_2 \quad \left\{\because x_1, x_2 \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right\}.$$

$\therefore$ f is one-one.

Also, range $(f) = [-1, 1] = B$. So, f is onto.

Thus, f is one-one onto and hence invertible.

Now, $y = f(x) \Rightarrow y = \sin x$

$$\Rightarrow x = \sin^{-1} y$$
$$\Rightarrow f^{-1}(y) = \sin^{-1} y.$$

Thus, we define:

$$f^{-1} : [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] : f^{-1}(y) = \sin^{-1} y.$$

EXAMPLE 5 *Let* $f : N \to Y : f(x) = x^2$, *where* $Y = \text{range}(f)$. *Show that f is invertible and find* f^{-1}.

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$$
$$\Rightarrow x_1^2 - x_2^2 = 0$$
$$\Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$$
$$\Rightarrow x_1 - x_2 = 0 \quad [\because x_1 + x_2 \ne 0]$$
$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Since range $(f) = Y$, so f is onto.

Thus, f is one-one onto and therefore invertible.

Let $y \in Y$. Then, there exists $x \in N$ such that $f(x) = y$.

Now, $y = f(x) \Rightarrow y = x^2$

$\Rightarrow x = \sqrt{y}$

$\Rightarrow f^{-1}(y) = \sqrt{y}$ $[\because f(x) = y \Rightarrow x = f^{-1}(y)]$.

Thus, we define

$$f^{-1} : Y \to N : f^{-1}(y) = \sqrt{y}.$$

EXAMPLE 6 *Let* $f : [-1, 1] \to Y : f(x) = \dfrac{x}{(x+2)}$, $x \neq -2$ *and* $Y = range(f)$. *Show that f is invertible and find* f^{-1}.

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1}{x_1 + 2} = \frac{x_2}{x_2 + 2}$$
$$\Rightarrow x_1x_2 + 2x_1 = x_1x_2 + 2x_2$$
$$\Rightarrow 2(x_1 - x_2) = 0$$
$$\Rightarrow x_1 - x_2 = 0$$
$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Since range $(f) = Y$, so f is onto.

Thus, f is one-one onto and therefore invertible.

Let $y \in Y$. Then, there exists $x \in [-1, 1]$ such that $f(x) = y$.

Now, $y = f(x) \Rightarrow y = \dfrac{x}{(x+2)}$

$\Rightarrow x = \dfrac{2y}{(1-y)}$

$\Rightarrow f^{-1}(y) = \dfrac{2y}{(1-y)}$.

Thus, we define:

$$f^{-1} : [-1, 1] \to Y : f^{-1}(y) = \frac{2y}{(1-y)}, \ y \neq 1.$$

EXAMPLE 7 *Let* $f : N \to Y : f(x) = 4x^2 + 12x + 15$ *and* $Y = range(f)$. *Show that f is invertible and find* f^{-1}. **[CBSE 2013C]**

SOLUTION f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$$
$$\Rightarrow 4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0$$
$$\Rightarrow (x_1^2 - x_2^2) + 3(x_1 - x_2) = 0$$
$$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 3) = 0$$
$$\Rightarrow x_1 - x_2 = 0 \quad [\because x_1 + x_2 + 3 \neq 0]$$
$$\Rightarrow x_1 = x_2.$$

Also, range $(f) = Y$. So, f is onto.

Thus, f is one-one onto and therefore invertible.

Let $y \in Y$. Then, f being onto, there exists x such that $y = f(x)$.

Now, $y = f(x) \Rightarrow y = 4x^2 + 12x + 15$

$$\Rightarrow y = (2x+3)^2 + 6$$
$$\Rightarrow (2x+3) = \sqrt{y-6}$$
$$\Rightarrow x = \frac{1}{2}(\sqrt{y-6} - 3)$$
$$\Rightarrow f^{-1}(y) = \frac{1}{2}(\sqrt{y-6} - 3).$$

Thus, we define:

$$f^{-1} : Y \to N : f^{-1}(y) = \frac{1}{2}(\sqrt{y-6} - 3).$$

EXAMPLE 8 *Let $f : R \to R : f(x) = 10x + 7$. Find the function $g : R \to R$ such that $g \circ f = f \circ g = I_R$.* **[CBSE 2011]**

SOLUTION Clearly, $g = f^{-1}$ ... (i)

Now, $f(x_1) = f(x_2) \Rightarrow 10x_1 + 7 = 10x_2 + 7$

$$\Rightarrow 10x_1 = 10x_2$$
$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Now, $y = f(x) \Rightarrow y = 10x + 7$

$$\Rightarrow x = \frac{(y-7)}{10}.$$

Clearly, for each $y \in R$ (codomain of f) there exists $x \in R$ such that

$$f(x) = f\left(\frac{y-7}{10}\right) = \left\{10 \cdot \left(\frac{y-7}{10}\right) + 7\right\} = y.$$

$\therefore$ f is onto.

Thus, f is one-one onto and therefore, f^{-1} exists.

We define: $f^{-1} : R \to R : f^{-1}(y) = \dfrac{y-7}{10}$.

Hence, $g : R \to R : g(y) = \dfrac{y-7}{10}$ [using (i)].

EXAMPLE 9 *Let $f : W \to W : f(n) = \begin{cases} (n-1), & \text{when } n \text{ is odd} \\ (n+1), & \text{when } n \text{ is even.} \end{cases}$*

Show that f is invertible. Find f^{-1}. **[CBSE 2012]**

SOLUTION Let $f(n_1) = f(n_2)$.

Case 1 *When n_1 is odd and n_2 is even*

In this case, $f(n_1) = f(n_2) \Rightarrow n_1 - 1 = n_2 + 1$

$\Rightarrow n_1 - n_2 = 2.$

If n_1 is odd and n_2 is even, then $(n_1 - n_2) \neq 2$.

Thus, we arrive at a contradiction.

So, in this case, $f(n_1) \neq f(n_2)$.

Similarly, when n_1 is even and n_2 is odd, then $f(n_1) \neq f(n_2)$.

Case 2 *When n_1 and n_2 are both odd*

In this case, $f(n_1) = f(n_2) \Rightarrow n_1 - 1 = n_2 - 1$

$\Rightarrow n_1 = n_2.$

Case 3 *When n_1 and n_2 are both even*

In this case, $f(n_1) = f(n_2) \Rightarrow n_1 + 1 = n_2 + 1$

$\Rightarrow n_1 = n_2.$

Thus, from all the cases, we get $f(n_1) = f(n_2) \Rightarrow n_1 = n_2$.

$\therefore$ f is one-one.

Now, we show that f is onto.

Let $n \in W$.

Case 1 *When n is odd*

In this case, $(n - 1)$ is even

and $f(n-1) = (n-1) + 1 = n$. ... (i)

Case 2 *When n is even*

In this case, $(n + 1)$ is odd

and $f(n+1) = (n+1) - 1 = n$. ... (ii)

Thus, each $n \in W$ has its pre-image in W.

$\therefore$ f is onto.

Thus, f is one-one onto and hence invertible.

Clearly, we have

$$f^{-1}(n) = \begin{cases} (n-1), & \text{when } n \text{ is odd} \\ (n+1), & \text{when } n \text{ is even} \end{cases} \quad \text{[using (i) and (ii)].}$$

EXAMPLE 10 *Let $A = \{1, 2, 3\}$ and let $f : A \to A$, defined by*

$f = \{(1, 2), (2, 3), (3, 1)\}$.

Find f^{-1}, if it exists.

SOLUTION We have $f(1) = 2$, $f(2) = 3$ and $f(3) = 1$.

Dom $(f) = \{1, 2, 3\} = A$ and range $(f) = \{1, 2, 3\} = A$.

Clearly, different elements in A have different images.

$\therefore$ f is one-one.

Range $(f) = A \Rightarrow f$ is onto.

Thus, f is one-one onto and therefore invertible.

Now, $f(1) = 2$, $f(2) = 3$ and $f(3) = 1$

$\Rightarrow$ $f^{-1}(2) = 1$, $f^{-1}(3) = 2$ and $f^{-1}(1) = 3$.

Hence, $f^{-1} = \{(2, 1), (3, 2), (1, 3)\}$.

Some Results on Invertible Functions

THEOREM 1 *Prove that an invertible function has a unique inverse.*

PROOF Let $f : A \rightarrow B$, which is one-one onto and therefore, invertible.
If possible, let it have two inverses, say g and h.

Then, $(f \circ g) = I_B$ and $(f \circ h) = I_B$

$\Rightarrow$ $(f \circ g)(y) = (f \circ h)(y)$ [each $= I_B(y)$]

$\Rightarrow$ $f\{g(y)\} = f\{h(y)\}$ for all $y \in B$

$\Rightarrow$ $g(y) = h(y)$ for all $y \in B$ [$\because$ f is one-one].

$\therefore$ $g = h$.

Hence, f has a unique inverse.

THEOREM 2 *Let f be an invertible function. Then, prove that*

$$(f^{-1})^{-1} = f.$$

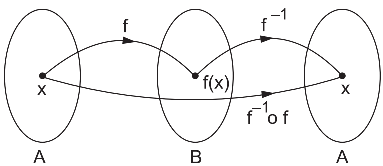
PROOF Let $f : A \rightarrow B$, which is invertible.
In order to prove that $(f^{-1})^{-1} = f$, it is sufficient to show that

$$f^{-1} \circ f = I_A \text{ and } f \circ f^{-1} = I_B.$$

Clearly $f : A \rightarrow B$ is one-one onto.

$\therefore$ $f^{-1} : B \rightarrow A$ is one-one onto.

Let $x \in A$ and let $f(x) = y$. Then, $f^{-1}(y) = x$.



$\therefore$ $(f^{-1} \circ f)(x) = f^{-1}\{f(x)\}$

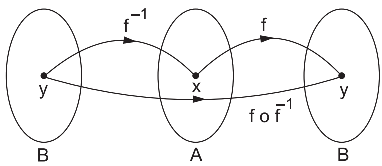
$= f^{-1}(y)$ [$\because$ $f(x) = y$]

$= x$

$= I_A(x)$.

$\therefore$ $f^{-1} \circ f = I_A$.

Again, let $y \in B$.



Then, f being onto, there exists $x \in A$ such that $f(x) = y$
and therefore, $f^{-1}(y) = x$.

$\therefore \quad (f \circ f^{-1})(y) = f\{f^{-1}(y)\}$
$= f(x) \quad [\because f^{-1}(y) = x]$
$= y$
$= I_B(y).$

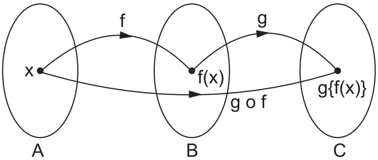
$\therefore \quad f \circ f^{-1} = I_B.$

Thus, $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$.

Hence, $(f^{-1})^{-1} = f$.

THEOREM 3 *Let $f: A \to B$ and $g: B \to C$ be one-one onto functions. Prove that $(g \circ f): A \to C$ which is one-one onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROOF Let $f: A \to B$ be one-one onto and $g: B \to C$ be one-one onto.



We first show that $g \circ f$ is one-one onto.

$(g \circ f)$ is one-one, since

$(g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow g\{f(x_1)\} = g\{f(x_2)\}$

$\Rightarrow f(x_1) = f(x_2) \quad [\because g$ is one-one$]$

$\Rightarrow x_1 = x_2 \quad [\because f$ is one-one$]$.

Let $z \in C$. Then, g being onto, there exists $y \in B$ such that $g(y) = z$.

Now, f being onto, there exists $x \in A$ such that $f(x) = y$.

$\therefore \quad z = g(y)$
$= g\{f(x)\} \quad [\because y = f(x)]$
$= (g \circ f)(x).$

Thus, for each $z \in C$, there exists $x \in A$ such that $(g \circ f)(x) = z$.

$\therefore \quad (g \circ f)$ is onto.

Thus, $(g \circ f)$ is one-one onto.

Now, $f(x) = y \Rightarrow f^{-1}(y) = x$.

And, $g(y) = z \Rightarrow g^{-1}(z) = y$.

Also, $(g \circ f)(x) = z \Rightarrow (g \circ f)^{-1}(z) = x$.

$\therefore \quad (f^{-1} \circ g^{-1})(z) = f^{-1}\{g^{-1}(z)\}$
$= f^{-1}(y) \quad [\because g^{-1}(z) = y]$
$= x \quad [\because f^{-1}(y) = x]$
$= (g \circ f)^{-1}(z).$

Hence, $(g \circ f)^{-1} = (f^{-1} \circ g^{-1})$.

EXERCISE 2C

Very-Short-Answer Questions

1. Prove that the function $f : R \to R : f(x) = 2x$ is one-one and onto.
2. Prove that the function $f : N \to N : f(x) = 3x$ is one-one and into.
3. Show that the function $f : R \to R : f(x) = x^2$ is neither one-one nor onto.
4. Show that the function $f : N \to N : f(x) = x^2$ is one-one and into.
5. Show that the function $f : R \to R : f(x) = x^4$ is neither one-one nor onto.
6. Show that the function $f : Z \to Z : f(x) = x^3$ is one-one and into.
7. Let R_0 be the set of all nonzero real numbers. Then, show that the function $f : R_0 \to R_0 : f(x) = \frac{1}{x}$ is one-one and onto.
8. Show that the function $f : R \to R : f(x) = 1 + x^2$ is many-one into.
9. Let $f : R \to R : f(x) = \frac{2x - 7}{4}$ be an invertible function. Find f^{-1}. **[CBSE 2008C]**
10. Let $f : R \to R : f(x) = 10x + 3$. Find f^{-1}.
11. $f : R \to R : f(x) = \begin{cases} 1, \text{ if } x \text{ is rational} \\ -1, \text{ if } x \text{ is rational.} \end{cases}$

    Show that f is many-one and into.
12. Let $f(x) = x + 7$ and $g(x) = x - 7$, $x \in R$. Find $(f \circ g)(7)$. **[CBSE 2008]**
13. Let $f : R \to R$ and $g : R \to R$ defined by $f(x) = x^2$ and $g(x) = (x + 1)$. Show that $g \circ f \neq f \circ g$.
14. Let $f : R \to R : f(x) = (3 - x^3)^{1/3}$. Find $f \circ f$. **[CBSE 2010]**
15. Let $f : R \to R : f(x) = 3x + 2$, find $f\{f(x)\}$. **[CBSE 2010C]**
16. Let $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down $g \circ f$. **[CBSE 2014C]**
17. Let $A = \{1, 2, 3, 4\}$ and $f = \{(1, 4), (2,1)\ (3, 3), (4, 2)\}$. Write down $(f \circ f)$.
18. Let $f(x) = 8x^3$ and $g(x) = x^{1/3}$. Find $g \circ f$ and $f \circ g$.
19. Let $f : R \to R : f(x) = 10x + 7$. Find the function $g : R \to R : g \circ f = f \circ g = I_g$. **[CBSE 2011]**
20. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. State whether f is one-one. **[CBSE 2011]**

ANSWERS (EXERCISE 2C)

9. $f^{-1}(y) = \frac{1}{2}(4y+7)$ for all $y \in R$ **10.** $f^{-1}(y) = \frac{1}{10}(y-3)$ for all $y \in R$

12. 7 **14.** $(f \circ f)(x) = x$ **15.** $f\{f(x)\} = (9x+8)$

16. $(g \circ f) = \{(1, 3), (3, 1), (4, 3)\}$ **17.** $f \circ f = \{(1, 2), (2, 4), (3, 3), (4, 1)\}$

18. $(g \circ f)(x) = 2x$ and $(f \circ g)(x) = 8x$ **19.** $g(x) = \frac{1}{10}(x-7)$ for all $x \in R$

20. Yes

HINTS TO THE GIVEN QUESTIONS (EXERCISE 2C)

1. (i) $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. So, f is one-one.

   (ii) Let $y = 2x$. Then, $x = \frac{1}{2}y$.

   Thus for each y in codomain R, there exists $\frac{1}{2}y$ such that

   $$f\left(\frac{1}{2}y\right) = \left(2 \times \frac{1}{2}y\right) = y.$$

   $\therefore$ f is onto.

2. (i) $f(x_1) = f(x_2) \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2$. So, f is one-one.

   (ii) If we consider 2 in codomain N, there is no natural number whose image is 2. So, f is into.

3. (i) Clearly, $f(1) = 1^2 = 1$ and $f(-1) = (-1)^2 = 1$.

   So, f is many-one.

   (ii) If we consider –1 in the codomain R then there is no element in R whose square is –1.

   $\therefore$ $-1 \in R$ has no pre-image in R. So, f is many-one into.

4. (i) $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$ [$\because$ $x_1, x_2 \in N$].

   $\therefore$ f is one-one.

   (ii) If we consider 2 in the codomain N, then $\sqrt{2} \in N$ and $f(\sqrt{2}) = (\sqrt{2})^2 = 2$.

   So, f is into.

5. (i) $f(1) = 1^4 = 1$ and $f(-1) = (-1)^4 = 1$.

   So, f is many-one.

   (ii) If we consider –1 in the codomain R then there exists no $x \in R$ such that $f(x) = x^4 = -1$. So, f is into.

6. (i) Let $x_1, x_2 \in Z$ and $x_1 \neq x_2$. Then, $x_1 \neq x_2 \Rightarrow x_1^3 \neq x_2^3 \Rightarrow f(x_1) \neq f(x_2)$.

   (ii) Let $2 \in Z$. Then, there exists no $x \in Z$ such that $x^3 = 2$.

   Thus, $2 \in Z$ has no pre-image in Z. So, f is into.

7. (i) $f(x_1)=f(x_2) \Rightarrow \frac{1}{x_1}=\frac{1}{x_2} \Rightarrow x_1=x_2$. So, f is one-one.

(ii) $y=\frac{1}{x} \Rightarrow x=\frac{1}{y}$.

Thus, for each y in codomain R_0, there exists $\frac{1}{y}$ in domain R_0 such that $f\left(\frac{1}{y}\right)=\frac{1}{\left(\frac{1}{y}\right)}=y$. So, f is onto.

8. (i) $f(-1)=1+(-1)^2=2$ and $f(1)=(1+1^2)=2$.

So, f is many-one.

(ii) $y=(1+x^2) \Rightarrow x=\sqrt{y-1}$.

So, when $y<1$, then $\sqrt{y-1}$ is imaginary.

In particular, $0 \in R$ has no pre-image in R.

$\therefore$ f is into.

9. $y=\frac{2x-7}{4} \Rightarrow x=\frac{1}{2}(4y+7)$

$$\Rightarrow f^{-1}(y)=\frac{4y+7}{2} \text{ for all } y \in R.$$

10. $y=10x+3 \Rightarrow x=\frac{y-3}{10}$

$$\Rightarrow f^{-1}(y)=\frac{(y-3)}{10}.$$

11. (i) Since all rationals have the same image, namely 1, so f is many-one.

(ii) Range $(f)=\{-1\ 1\} \subset R$. So, f is into.

12. $(f \circ g)(7)=f\{g(7)\}=f(7-7)=f(0)=(0+7)=7$.

13. $(g \circ f)(x)=g\{f(x)\}=g(x^2)=(x^2+1)$.

$(f \circ g)(x)=f\{g(x)\}=f(x+1)=(x+1)^2$.

Hence, $g \circ f \neq f \circ g$.

14. $(f \circ f)(x)=f\{f(x)\}=f\{(3-x^3)^{1/3}\}=f(y)$, where $y=(3-x^3)^{1/3}$

$$=(3-y^3)^{1/3}=\{3-(3-x^3)\}^{1/3}=(x^3)^{1/3}=x.$$

$\therefore$ $(f \circ f)(x)=x$.

15. $f\{f(x)\}=f(3x+2)=3(3x+2)+2=(9x+8)$.

16. Dom $(g \circ f)=$ Dom $(f)=\{1, 3, 4\}$.

$(g \circ f)(1)=g\{f(1)\}=g(2)=3$.

$(g \circ f)(3)=g\{f(3)\}=g(5)=1$.

$(g \circ f)(4)=g\{f(4)\}=g(1)=3$.

$\therefore$ $g \circ f=\{(1, 3), (3, 1), (4, 3)\}$.

17. $(f \circ f)(1)=f\{f(1)\}=f(4)=2$.

$(f \circ f)(2)=f\{f(2)\}=f(1)=4$.

$(f \circ f)(3) = f\{f(3)\} = f(3) = 3.$

$(f \circ f)(4) = f\{f(4)\} = f(2) = 1.$

$\therefore \quad f \circ f = \{(1, 2), (2, 4), (3, 3), (4, 1)\}.$

18. $(g \circ f)(x) = g[f(x)] = g(8x^3) = g(y)$, where $y = 8x^3$

$$= y^{1/3} = (8x^3)^{1/3} = 2x.$$

$(f \circ g)(x) = f[g(x)] = f(x^{1/3}) = f(y)$, where $y = x^{1/3}$

$$= 8y^3 = 8(x^{1/3})^3 = 8x^{\left(\frac{1}{3} \times 3\right)} = 8x.$$

19. $g \circ f = I_R \Rightarrow (g \circ f)(x) = I_R(x) = x$

$$\Rightarrow g\{f(x)\} = x$$

$$\Rightarrow g\{10x + 7\} = x.$$

Put $10x + 7 = y$. Then, $x = \frac{(y-7)}{10}$.

$\therefore \quad g(y) = \frac{(y-7)}{10}$.

Hence, $g : R \to R : g(x) = \frac{1}{10}(x - 7)$ for all $x \in R$.

20. $f(1) = 4, f(2) = 5$ and $f(3) = 6$.

Thus, different elements in A have different images in B.

Hence, f is one-one.

EXERCISE 2D

1. Let $A = \{2, 3, 4, 5\}$ and $B = \{7, 9, 11, 13\}$, and let $f = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$. Show that f is invertible and find f^{-1}.
2. Show that the function $f : R \to R : f(x) = 2x + 3$ is invertible and find f^{-1}.
3. Let $f : Q \to Q : f(x) = 3x - 4$. Show that f is invertible and find f^{-1}.
4. Let $f : R \to R : f(x) = \frac{1}{2}(3x + 1)$. Show that f is invertible and find f^{-1}.
5. If $f(x) = \frac{(4x + 3)}{(6x - 4)}$, $x \neq \frac{2}{3}$, show that $(f \circ f)(x) = x$ for all $x \neq \frac{2}{3}$. Hence, find f^{-1}.
6. Show that the function f on $A = R - \left\{\frac{2}{3}\right\}$, defined as $f(x) = \frac{4x + 3}{6x - 4}$ is one-one and onto. Hence, find f^{-1}. **[CBSE 2013]**
7. Show that the function f on $A = R - \left\{\frac{-4}{3}\right\}$ into itself, defined by $f(x) = \frac{4x}{(3x + 4)}$ is one-one and onto. Hence, find f^{-1}.

8. Let R_+ be the set of all positive real numbers. Show that the function $f: R_+ \to [-5, \infty[: f(x) = (9x^2 + 6x - 5)$ is invertible. Find f^{-1}.

9. Let $f: N \to R : f(x) = 4x^2 + 12x + 15$. Show that $f: N \to$ range (f) is invertible. Find f^{-1}. **[CBSE 2010, '13C]**

10. Let $A = R - \{2\}$ and $B = R - \{1\}$. If $f: A \to B : f(x) = \dfrac{x-1}{x-2}$, show that f is one-one and onto. Hence, find f^{-1}. **[CBSE 2013]**

11. Let f and g be two functions from R into R, defined by $f(x) = |x| + x$ and $g(x) = |x| - x$ for all $x \in R$. Find $f \text{ o } g$ and $g \text{ o } f$. **[CBSE 2014C]**

ANSWERS (EXERCISE 2D)

1. $f^{-1} = \{(7, 2), (9, 3), (11, 4), (13, 5)\}$ **2.** $f^{-1}(y) = \dfrac{1}{2}(y - 3)$

3. $f^{-1}(y) = \dfrac{1}{3}(y + 4)$ **4.** $f^{-1}(y) = \dfrac{(2y - 1)}{3}$

5. $f^{-1}(y) = \dfrac{(4y + 3)}{(6y - 4)}$ **6.** $f^{-1}(y) = \dfrac{(4y + 3)}{6y - 4}$

7. $f^{-1}(y) = \dfrac{(4y)}{(4 - 3y)}$ **8.** $f^{-1}(y) = \dfrac{\sqrt{y + 6} - 1}{3}$

9. $f^{-1}(y) = \dfrac{\sqrt{y - 6} - 3}{2}$ **10.** $f^{-1}(y) = \dfrac{2y - 1}{y - 1}$

11. $(f \text{ o } g)(x) = |x| - x$ and $(g \text{ o } f)(x) = |x| - x$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 2D)

1. Clearly $f(2) = 7, f(3) = 9, f(4) = 11$ and $f(5) = 13$.
Thus, different elements in A have different images in B.
So, f is one-one.
Range $(f) = \{7, 9, 11, 13\} = B$. So, f is onto.
$f^{-1} = \{(7, 2), (9, 3), (11, 4), (13, 5)\}$.

2. $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.
$\therefore$ f is one-one.
If $y \in R$ then there exists $x = \dfrac{y - 3}{2} \in R$ such that

$$f(x) = f\left(\frac{y-3}{2}\right) = \left\{2 \cdot \frac{(y-3)}{2} + 3\right\} = y.$$

$\therefore$ f is onto.

$$y = f(x) \Rightarrow y = 2x + 3$$
$$\Rightarrow x = \frac{1}{2}(y - 3) \Rightarrow f^{-1}(y) = \frac{1}{2}(y - 3).$$

5. $$(f \circ f)(x) = f\{f(x)\} = f\left\{\frac{4x + 3}{6x - 4}\right\} = \frac{4\left(\frac{4x + 3}{6x - 4}\right) + 3}{6\left(\frac{4x + 3}{6x - 4}\right) - 4}$$
$$= \frac{(16x + 12 + 18x - 12)}{(24x + 18 - 24x + 16)} = \frac{34x}{34} = x = I(x).$$

$\therefore \quad f \circ f = I \Rightarrow f^{-1} = f.$

6. $$f(x_1) = f(x_2) \Rightarrow \frac{4x_1 + 3}{6x_1 - 4} = \frac{4x_2 + 3}{6x_2 - 4}$$
$$\Rightarrow (4x_1 + 3)(6x_2 - 4) = (6x_1 - 4)(4x_2 + 3)$$
$$\Rightarrow 24x_1x_2 - 16x_1 + 18x_2 - 12 = 24x_1x_2 + 18x_1 - 16x_2 - 12$$
$$\Rightarrow 34x_1 = 34x_2 \Rightarrow x_1 = x_2.$$

$\therefore \quad f$ is one-one.

Let y be an arbitrary element of A. Then,

$$f(x) = y \Rightarrow \frac{4x + 3}{6x - 4} = y$$
$$\Rightarrow (6x - 4)y = 4x + 3 \Rightarrow 6xy - 4y = 4x + 3$$
$$\Rightarrow 6xy - 4x = 4y + 3 \Rightarrow x(6y - 4) = (4y + 3)$$
$$\Rightarrow x = \frac{4y + 3}{6y - 4}.$$

Thus, for each $y \in A$, there exists $x = \frac{4y + 3}{6y - 4}$ such that

$$f(x) = f\left(\frac{4y + 3}{6y - 4}\right) = \frac{4 \cdot \left(\frac{4y + 3}{6y - 4}\right) + 3}{6 \cdot \left(\frac{4y + 3}{6y - 4}\right) - 4}$$
$$= \frac{16y + 12 + 18y - 12}{24y + 18 - 24y + 16} = \frac{34y}{34} = y.$$

$\therefore \quad f$ is onto.

Now, $f(x) = y \Rightarrow x = f^{-1}(y)$.

$\therefore \quad f^{-1}(y) = \frac{4y + 3}{6y - 4}$, for all $y \in A$.

7. $$f(x_1) = f(x_2) \Rightarrow \frac{4x_1}{3x_1 + 4} = \frac{4x_2}{3x_2 + 4} \Rightarrow \frac{x_1}{3x_1 + 4} = \frac{x_2}{3x_2 + 4}$$
$$\Rightarrow x_1(3x_2 + 4) = x_2(3x_1 + 4)$$
$$\Rightarrow 3x_1x_2 + 4x_1 = 3x_1x_2 + 4x_2$$
$$\Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2.$$

$\therefore \quad f$ is one-one.

Let y be an arbitrary element of A. Then,

$$f(x) = y \Rightarrow \frac{4x}{3x+4} = y$$
$$\Rightarrow 3xy + 4y = 4x \Rightarrow 4x - 3xy = 4y$$
$$\Rightarrow x(4-3y) = 4y \Rightarrow x = \frac{4y}{(4-3y)}.$$

Thus, for each $y \in A$, there exists an $x \in A$ such that

$$f(x) = f\left(\frac{4y}{4-3y}\right) = \frac{4\cdot\left(\frac{4y}{4-3y}\right)}{3\cdot\left(\frac{4y}{4-3y}\right)+4}$$
$$= \frac{16y}{12y+16-12y} = \frac{16y}{16} = y.$$

$\therefore$ f is onto.

Now, $f(x) = y \Rightarrow x = f^{-1}(y)$.

$$\therefore \quad f^{-1}(y) = \frac{4y}{4-3y}.$$

8. $y = (3x+1)^2 - 6 \Rightarrow x = \dfrac{\sqrt{y+6}-1}{3} \Rightarrow f^{-1}(y) = \dfrac{\sqrt{y+6}-1}{3}.$

9. $f(x_1) = f(x_2) \Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$

$$\Rightarrow 4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0 \Rightarrow (x_1^2 - x_2^2) + 3(x_1 - x_2) = 0$$
$$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 3) = 0 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2.$$

Since $f : N \to \text{range}(f)$, so f is onto.

Now $4x^2 + 12x + 15 = y \Rightarrow (2x+3)^2 + 6 = y$

$$\Rightarrow (2x+3) = \sqrt{y-6} \Rightarrow x = \frac{\sqrt{y-6}-3}{2}.$$

$$\therefore \quad f^{-1}(y) = \frac{\sqrt{y-6}-3}{2}.$$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. $f : N \to N : f(x) = 2x$ is

   (a) one-one and onto (b) one-one and into
   (c) many-one and onto (d) many-one and into

2. $f : N \to N : f(x) = x^2 + x + 1$ is

   (a) one-one and onto (b) one-one and into
   (c) many-one and onto (d) many-one and into

3. $f : R \to R : f(x) = x^2$ is

(a) one-one and onto (b) one-one and into

(c) many-one and onto (d) many-one and into

4. $f : R \to R : f(x) = x^3$ is

(a) one-one and onto (b) one-one and into

(c) many-one and onto (d) many-one and into

5. $f : R^+ \to R^+ : f(x) = e^x$ is

(a) many-one and into (b) many-one and onto

(c) one-one and into (d) one-one and onto

6. $f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1] : f(x) = \sin x$ is

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

7. $f : R \to R : f(x) = \cos x$ is

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

8. $f : C \to R : f(z) = |z|$ is

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

9. Let $A = R - \{3\}$ and $B = R - \{1\}$. Then, $f : A \to B : f(x) = \frac{(x-2)}{(x-3)}$ is

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

10. Let $f : N \to N : f(n) = \begin{cases} \frac{1}{2}(n+1), \text{ when } n \text{ is odd} \\ \frac{n}{2}, \text{ when } n \text{ is even.} \end{cases}$

Then, f is

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

11. Let A and B be two non-empty sets and let $f : (A \times B) \to (B \times A) : f(a, b) = (b, a)$. Then, f is

(a) one-one and onto (b) one-one and into

(c) many-one and onto (d) many-one and into

12. Let $f : Q \to Q : f(x) = (2x + 3)$. Then, $f^{-1}(y) = ?$

(a) $(2y - 3)$ (b) $\frac{1}{(2y-3)}$ (c) $\frac{1}{2}(y - 3)$ (d) none of these

13. Let $f : R - \left\{\frac{-4}{3}\right\} \to R - \left\{\frac{4}{3}\right\} : f(x) = \frac{4x}{(3x+4)}$ · Then, $f^{-1}(y) = ?$

(a) $\frac{4y}{(4-3y)}$ (b) $\frac{4y}{(4y+3)}$ (c) $\frac{4y}{(3y-4)}$ (d) none of these

14. Let $f : N \to X : f(x) = 4x^2 + 12x + 15$. Then, $f^{-1}(y) = ?$

(a) $\frac{1}{2}(\sqrt{y-4} + 3)$ (b) $\frac{1}{2}(\sqrt{y-6} - 3)$

(c) $\frac{1}{2}(\sqrt{y-4} + 5)$ (d) none of these

15. If $f(x) = \frac{(4x+3)}{(6x-4)}, x \neq \frac{2}{3}$ then $(f \circ f)(x) = ?$

(a) x (b) $(2x-3)$ (c) $\frac{4x-6}{3x+4}$ (d) none of these

16. If $f(x) = (x^2 - 1)$ and $g(x) = (2x+3)$ then $(g \circ f)(x) = ?$

(a) $(2x^2 + 3)$ (b) $(3x^2 + 2)$ (c) $(2x^2 + 1)$ (d) none of these

17. If $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ then $f(x) = ?$

(a) x^2 (b) $(x^2 - 1)$ (c) $(x^2 - 2)$ (d) none of these

18. If $f(x) = \frac{1}{(1-x)}$ then $(f \circ f \circ f)(x) = ?$

(a) $\frac{1}{(1-3x)}$ (b) $\frac{x}{(1+3x)}$ (c) x (d) none of these

19. If $f(x) = \sqrt[3]{3 - x^3}$ then $(f \circ f)(x) = ?$

(a) $x^{1/3}$ (b) x (c) $(1 - x^{1/3})$ (d) none of these

20. If $f(x) = x^2 - 3x + 2$ then $(f \circ f)(x) = ?$

(a) x^4 (b) $x^4 - 6x^3$ (c) $x^4 - 6x^3 + 10x^2$ (d) none of these

21. If $f(x) = 8x^3$ and $g(x) = x^{1/3}$ then $(g \circ f)(x) = ?$

(a) x (b) $2x$ (c) $\frac{x}{2}$ (d) $3x^2$

22. If $f(x) = x^2$, $g(x) = \tan x$ and $h(x) = \log x$ then $\{h \circ (g \circ f)\}\left(\sqrt{\frac{\pi}{4}}\right) = ?$

(a) 0 (b) 1 (c) $\frac{1}{x}$ (d) $\frac{1}{2} \log \frac{\pi}{4}$

23. If $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$ then $(g \circ f) = ?$

(a) $\{(3, 1), (1, 3), (3, 4)\}$ (b) $\{(1, 3), (3, 1), (4, 3)\}$

(c) $\{(3, 4), (4, 3), (1, 3)\}$ (d) $\{(2, 5), (5, 2), (1, 5)\}$

24. Let $f(x) = \sqrt{9 - x^2}$. Then, dom $(f) = ?$

(a) $[-3, 3]$ (b) $(-\infty, -3]$

(c) $[3, \infty)$ (d) $(-\infty, -3] \cup (4, \infty)$

25. Let $f(x) \sqrt{\frac{x-1}{x-4}}$ $\cdot$ Then, dom $(f) = ?$

(a) $[1, 4)$ (b) $[1, 4]$

(c) $(-\infty, 4]$ (d) $(-\infty, 1] \cup (4, \infty)$

26. Let $f(x) = e^{\sqrt{x^2-1}} \cdot \log(x-1)$. Then, dom $(f) = ?$

(a) $(-\infty, 1]$ (b) $[-1, \infty)$

(c) $(1, \infty)$ (d) $(-\infty, -1] \cup (1, \infty)$

27. Let $f(x) = \frac{x}{(x^2-1)}$ $\cdot$ Then, dom $(f) = ?$

(a) R (b) $R - \{1\}$ (c) $R - \{-1\}$ (d) $R - \{-1, 1\}$

28. Let $f(x) = \frac{\sin^{-1} x}{x}$ $\cdot$ Then, dom $(f) = ?$

(a) $(-1, 1)$ (b) $[-1, 1]$ (c) $[-1, 1] - \{0\}$ (d) none of these

29. Let $f(x) = \cos^{-1} 2x$. Then, dom $(f) = ?$

(a) $[-1, 1]$ (b) $\left[\frac{-1}{2}, \frac{1}{2}\right]$ (c) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (d) $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

30. Let $f(x) = \cos^{-1}(3x - 1)$. Then, dom $(f) = ?$

(a) $\left(0, \frac{2}{3}\right)$ (b) $\left[0, \frac{2}{3}\right]$ (c) $\left[\frac{-2}{3}, \frac{2}{3}\right]$ (d) none of these

31. Let $f(x) = \sqrt{\cos x}$. Then, dom $(f) = ?$

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[\frac{3\pi}{2}, 2\pi\right]$

(c) $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$ (d) none of these

32. Let $f(x) = \sqrt{\log(2x - x^2)}$. Then, dom $(f) = ?$

(a) $(0, 2)$ (b) $[1, 2]$ (c) $(-\infty, 1]$ (d) none of these

33. Let $f(x) = x^2$. Then, dom (f) and range (f) are respectively

(a) R and R (b) R^+ and R^+ (c) R and R^+ (d) R and $R - \{0\}$

34. Let $f(x) = x^3$. Then, dom (f) and range (f) are respectively

(a) R and R (b) R^+ and R^+

(c) R and R^+ (d) R^+ and R

35. Let $f(x) = \log(1-x) + \sqrt{x^2-1}$. Then, dom $(f) = ?$

(a) $(1, \infty)$ (b) $(-\infty, -1]$ (c) $[-1, 1)$ (d) $(0, 1)$

36. Let $f(x) = \dfrac{1}{(1-x^2)}$ · Then, range $(f) = ?$

(a) $(-\infty, 1]$ (b) $[1, \infty)$ (c) $[-1, 1]$ (d) none of these

37. Let $f(x) = \dfrac{x^2}{(1+x^2)}$ · Then, range $(f) = ?$

(a) $[1, \infty)$ (b) $[0, 1)$ (c) $[-1, 1]$ (d) $(0, 1]$

38. The range of $f(x) = x + \dfrac{1}{x}$ is

(a) $[-2, 2]$ (b) $[2, \infty)$ (c) $(-\infty, -2]$ (d) none of these

39. The range of $f(x) = a^x$, where $a > 0$ is

(a) $]-\infty, 0]$ (b) $]-\infty, 0)$ (c) $[0, \infty)$ (d) $(0, \infty)$

ANSWERS (OBJECTIVE QUESTIONS)

1. (b)	**2.** (b)	**3.** (d)	**4.** (a)	**5.** (d)	**6.** (b)	**7.** (c)	**8** (c)	**9.** (b)	**10.** (d)
11. (a)	**12.** (c)	**13.** (a)	**14.** (b)	**15.** (a)	**16.** (c)	**17.** (c)	**18.** (c)	**19.** (b)	**20.** (d)
21. (b)	**22.** (a)	**23.** (b)	**24.** (a)	**25.** (d)	**26.** (c)	**27.** (d)	**28.** (c)	**29.** (b)	**30.** (b)
31. (c)	**32.** (d)	**33.** (c)	**34.** (a)	**35.** (b)	**36.** (b)	**37.** (b)	**38.** (d)	**39.** (d)	

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

1. $2x = 3 \Rightarrow x = \dfrac{3}{2} \notin N$. So, f is into.

2. $f(x_1) = f(x_2) \Rightarrow x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$

$$\Rightarrow (x_1^2 - x_2^2) + (x_1 - x_2) = 0 \Rightarrow (x_1 - x_2)(x_1 + x_2 + 1) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

$f(x) = 1 \Rightarrow x^2 + x + 1 = 1 \Rightarrow x(x+1) = 0 \Rightarrow x = 0$ or $x = -1$.

And, none of 0 and –1 is in N. So, f is into.

5. $f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2$. So, f is one-one.

For each $x \in R^+$ $\exists \log x \in R^+$ s.t. $f(\log x) = x$.

So, f is onto.

7. $\cos(2\pi - \theta) = \cos\theta \Rightarrow f$ is many-one.

Range $(f) = [-1, 1] \subset R \Rightarrow f$ is into.

8. $i \neq -i$. But $f(i) = f(-i) = 1$. So, f is many-one.

$-1 \in R$ having no pre-image in C. So, f is into.

9. $f(x_1) = f(x_2) \Rightarrow \dfrac{(x_1 - 2)}{(x_1 - 3)} = \dfrac{(x_2 - 2)}{(x_3 - 3)} \Rightarrow x_1 = x_2$. So, f is one-one.

Let $\dfrac{x-2}{x-3} = y$. Then, $x = \dfrac{3y-2}{y-1}$ · Clearly, $y \neq 1$ and $x \neq 3$.

$\therefore$ $f(x) = y$ and so f is onto.

10. $f(1) = f(2)$ shows that f is many-one.

If n is odd then $(2n-1)$ is odd and $f(2n-1) = n$.

If n is even then $2n$ is even and $f(2n) = n$.

$\therefore$ f is onto.

12. $y = 2x + 3 \Rightarrow x = \dfrac{1}{2}(y-3) \Rightarrow f^{-1}(y) = \dfrac{1}{2}(y-3)$.

13. $y = \dfrac{4x}{3x+4} \Rightarrow x = \dfrac{4y}{(4-3y)} \Rightarrow f^{-1}(y) = \dfrac{4y}{(4-3y)}$ ·

14. $y = 4x^2 + 12x + 15 = (2x+3)^2 + 6 \Rightarrow x = \dfrac{1}{2}(\sqrt{y-6} - 3)$

$\therefore$ $f^{-1}(y) = \dfrac{1}{2}(\sqrt{y-6} - 3)$.

15. $f(x) = \dfrac{4x+3}{6x-4} = y$ (say).

Then $f(y) = \dfrac{4y+3}{6y-4} = \dfrac{4\left(\dfrac{4x+3}{6x-4}\right) + 3}{6\left(\dfrac{4x+3}{6x-4}\right) - 4} = \dfrac{34x}{34} = x$.

$\Rightarrow$ $f[f(x)] = x \Rightarrow (f \circ f)(x) = x$.

16. $(g \circ f)(x) = g[f(x)] = g(x^2 - 1)$

$= 2(x^2 - 1) + 3 = (2x^2 + 1)$.

17. Let $x + \dfrac{1}{x} = z$. Then,

$$f(z) = f\left(x + \frac{1}{x}\right) = \left(x^2 + \frac{1}{x^2}\right) = \left(x + \frac{1}{x}\right)^2 - 2 = (z^2 - 2).$$

$\Rightarrow$ $f(x) = (x^2 - 2)$.

18. $(f \circ f)(x) = f\{f(x)\} = f\left(\dfrac{1}{1-x}\right) = \dfrac{1}{\left(1 - \dfrac{1}{1-x}\right)} = \dfrac{1-x}{-x} = \dfrac{x-1}{x}$

$\Rightarrow$ $\{f \circ (f \circ f)\}(x) = f\{(f \circ f)(x)\} = f\left(\dfrac{x-1}{x}\right) = \dfrac{1}{1 - \dfrac{x-1}{x}} = x$.

19. $(f \circ f)(x) = f\{f(x)\} - \{(3 - x^3)^{1/3}\} = f(y)$, where $y = (3 - x^3)^{1/3}$

$= (3 - y^3)^{1/3} = [3 - \{3 - x^3\}]^{1/3} = (x^3)^{1/3} = x$.

20. $(f\text{o}f)(x) = f\{f(x)\} = f(x^2 - 3x + 2) = f(y)$, where $y = (x^2 - 3x + 2)$

$$= y^2 - 3y + 2 = (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$

$$= (x^4 - 6x^3 + 10x^2 - 3x).$$

21. $(g\text{o}f)(x) = g[f(x)] = g(8x^3) = (8x^3)^{1/3} = 2x.$

22. $\{h\text{o}(g\text{o}f)\}(x) = (h\text{o}g)\{f(x)\} = (h\text{o}g)(x^2)$

$$= h\{g(x^2)\} = h(\tan x^2) = \log(\tan x^2).$$

$\therefore \quad \{h\text{o}(g\text{o}f)\}\sqrt{\frac{\pi}{4}} = \log\left(\tan\frac{\pi}{4}\right) = \log 1 = 0.$

23. $\text{Dom}(g\text{o}f) = \text{dom}(f) = \{1, 3, 4\}.$

$(g\text{o}f)(1) = g\{f(1)\} = g(2) = 3, (g\text{o}f)(3) = g\{f(3)\} = g(5) = 1$

$(g\text{o}f)(4) = g\{f(4)\} = g(1) = 3$

$\therefore \quad g\text{o}f = \{(1,3), (3,1), (4,3)\}.$

24. $f(x)$ is defined only when $9 - x^2 \geq 0 \Rightarrow x^2 \leq 9 \Rightarrow -3 \leq x \leq 3.$

$\therefore \quad \text{dom}(f) = [-3, 3].$

25. $f(x)$ is defined when $x - 4 \neq 0$ and $\frac{x-1}{x-4} \geq 0$

$\Rightarrow \quad x \neq 4$ and $(x \geq 4$ or $x \leq 1) \Rightarrow (x > 4$ or $x \leq 1)$

$\Rightarrow \quad \text{dom}(f) = (-\infty, 1] \cup (4, \infty).$

26. $f(x)$ is defined only when $(x^2 - 1) \geq 0$ and $(x - 1) > 0$

$\Rightarrow \quad (x-1)(x+1) \geq 0$ and $(x-1) > 0 \Rightarrow x + 1 \geq 0$ and $x - 1 > 0 \Rightarrow x > 1$

$\therefore \quad \text{dom}(f) = (1, \infty).$

27. $f(x)$ is not defined when $(x^2 - 1) = 0$, i.e., when $(x-1)(x+1) = 0$,

i.e., when $x = 1$ or $x = -1$.

$\therefore \quad \text{dom}(f) = R - \{1, -1\}.$

28. $\frac{\sin^{-1} x}{x}$ is defined only when $x \neq 0$ and $x \in [-1, 1]$.

$\therefore \quad \text{dom}(f) = [-1, 1] - \{0\}.$

29. $\sin^{-1} 2x$ is defined only when $2x \in [-1, 1] \Rightarrow x \in \left[\frac{-1}{2}, \frac{1}{2}\right].$

30. $\cos^{-1}(3x - 1)$ is defined only when $(3x - 1) \in [-1, 1]$

$\Rightarrow \quad 3x \in [0, 2] \Rightarrow x \in \left[0, \frac{2}{3}\right] \Rightarrow \text{dom}(f) = \left[0, \frac{2}{3}\right].$

31. $f(x)$ is defined only when $\cos x \geq 0$

$\Rightarrow \quad x$ lies in 1st or 4th quadrant

$\Rightarrow \quad \text{dom}(f) = \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right].$

32. $f(x)$ is defined only when $\log(2x - x^2) \geq 0$

$\Rightarrow\ (2x - x^2) \geq 1 \Rightarrow\ (1 + x^2 - 2x) \leq 0 \Rightarrow\ (1 - x)^2 \leq 0 \Rightarrow\ (1 - x) = 0 \Rightarrow\ x = 1.$

$\therefore$ dom $(f) = \{1\}$.

33. $f(x) = x^2$ is clearly defined for each $x \in R$. So, dom $(f) = R$.

$y = x^2 \Rightarrow\ x = \pm\sqrt{y}.$

When $y < 0$, there is no real value of x. So, $y \geq 0$.

$\therefore$ range $(f) = R^+$.

34. $f(x) = x^3$ is defined for each $x \in R$. So, dom $(f) = R$.

For each $y \in R$, $y^{1/3} \in R$ and so $x = y^{1/3}$ is real.

$\therefore$ range $(f) = R$.

35. Let $f(x) = g(x) + h(x)$, where $g(x) = \log(1 - x)$ and $h(x) = \sqrt{x^2 - 1}$.

$g(x)$ is defined only when $1 - x > 0 \Rightarrow\ x < 1$. So, dom $(g) = (-\infty, 1)$.

$h(x)$ is defined only when $x^2 - 1 \geq 0 \Rightarrow\ x \geq 1$ or $x \leq -1$.

$\therefore$ dom $(h) = (-\infty, -1] \cup [1, \infty)$.

$\therefore$ dom $(f) =$ dom $(g) \cap$ dom $(h) = (-\infty, -1]$.

36. $y = \dfrac{1}{(1 - x^2)} \Rightarrow\ x = \sqrt{1 - \dfrac{1}{y}}.$

Clearly, x is not defined when $y = 0$ or $1 - \dfrac{1}{y} < 0$, i.e., $y = 0$ or $y < 1$.

$\therefore$ range $(f) = [1, \infty)$.

37. $y = \dfrac{x^2}{(1 + x^2)} \Rightarrow\ x = \sqrt{\dfrac{y}{1 - y}}.$

Clearly, x is defined only when $\dfrac{y}{(1 - y)} \geq 0$, and $(1 - y) \neq 0$, i.e., when $0 \leq y < 1$.

So, range $(f) = [0, 1)$.

38. $y = \dfrac{x^2 + 1}{x} \Rightarrow\ x^2 - xy + 1 = 0 \Rightarrow\ x = \dfrac{y \pm \sqrt{y^2 - 4}}{2}.$

x is defined when $(y^2 - 4) \geq 0 \Rightarrow\ y^2 \geq 4 \Rightarrow\ y \geq 2$ or $y \leq -2$.

$\therefore$ range $(f) = (-\infty, -2] \cup [2, \infty)$.

39. Clearly, $a^x > 0$ whatever may be the value of x.

$\therefore$ range $(f) = (0, \infty)$.

3. BINARY OPERATIONS

CLOSURE PROPERTY An operation $*$ on a nonempty set S is said to satisfy the closure property, if

$$a \in S,\ b \in S \ \Rightarrow\ a * b \in S \text{ for all } a,\, b \in S.$$

Also, in this case, we say that S is closed for $*$.

An operation $*$ on a nonempty set S, satisfying the closure property is known as a *binary operation*.

EXAMPLE 1 (i) Addition on the set N of all natural numbers is a binary operation, since

$$a \in N,\ b \in N \ \Rightarrow\ a + b \in N \text{ for all } a,\, b \in N.$$

(ii) Multiplication on N is a binary operation, since

$$a \in N,\ b \in N \ \Rightarrow\ a \times b \in N \text{ for all } a,\, b \in N.$$

Similarly, addition as well as multiplication is a binary operation on each one of the sets Z, Q, R and C of integers, rationals, reals and complex numbers respectively.

EXAMPLE 2 Let S be a nonempty set and $P(S)$ be its power set. Then, the union operation on $P(S)$ is a binary operation, since

$$A \in P(S),\ B \in P(S) \ \Rightarrow\ A \cup B \in P(S) \text{ for all } A,\, B \in P(S).$$

Similarly, intersection on $P(S)$ is a binary operation, as

$$A \in P(S),\ B \in P(S) \ \Rightarrow\ A \cup B \in P(S) \text{ for all } A,\, B \in P(S).$$

EXAMPLE 3 Subtraction on N is not a binary operation, since

$$3 \in N,\, 5 \in N \text{ but } (3 - 5) = -2 \notin N.$$

Subtraction on the set Z of all integers is a binary operation, since

$$a \in Z,\ b \in Z \ \Rightarrow\ a - b \in Z \text{ for all } a,\ b \in Z.$$

EXAMPLE 4 Addition on the set S of all irrationals is not a binary operation, since

$$2 + \sqrt{3} \in S,\ 2 - \sqrt{3} \in S \text{ but } (2 + \sqrt{3}) + (2 - \sqrt{3}) = 4 \notin S.$$

Multiplication on the set S of all irrationals is not a binary operation, since

$$\sqrt{2} \in S,\ -\sqrt{2} \in S \text{ but } (\sqrt{2})(-\sqrt{2}) = -2 \notin S.$$

EXAMPLE 5 Let N be the set of all natural numbers. Then, the exponential operation $(a,\ b) \Rightarrow a^b$ is a binary operation on N, since

$$a \in N,\ b \in N \ \Rightarrow\ a^b \in N \text{ for all } a,\ b \in N.$$

Let Z be the set of all integers. The exponential operation $(a, b) \to a^b$ is not a binary operation on Z, since $2 \in Z$, $-3 \in Z$ but $2^{-3} = \frac{1}{2^3} = \frac{1}{8} \notin Z$.

Properties of a binary operation

(i) **Associative law** A binary operation $*$ on a nonempty set S is said to be associative, if

$$(a * b) * c = a * (b * c) \text{ for all } a, b, c \in S.$$

(ii) **Commutative law** A binary operation $*$ on a nonempty set S is said to be commutative, if

$$a * b = b * a \text{ for all } a, b \in S.$$

(iii) **Distributive law** Let $*$ and $\circ$ be two binary operations on a nonempty set S. We say that $*$ is distributive over $\circ$, if

$$a * (b \circ c) = (a * b) \circ (a * c) \text{ for all } a, b, c \in S.$$

EXAMPLE 1 Let R be the set of all real numbers. Then,

(i) addition on R satisfies the closure property, the associative law and the commutative law,

(ii) multiplication on R satisfies the closure property, the associative law and the commutative law,

(iii) multiplication distributes addition on R, since

$$a \cdot (b + c) = a \cdot b + a \cdot c \text{ for all } a, b, c \in R.$$

EXAMPLE 2 Let Z be the set of all integers. Then, subtraction on Z is clearly a binary operation. But, it is neither commutative nor associative, as

$$(3 - 5) \neq (5 - 3) \text{ and } (3 - 4) - 5 \neq 3 - (4 - 5).$$

IDENTITY ELEMENT Let $*$ be a binary operation on a nonempty set S. An element $e \in S$, if it exists such that

$$a * e = e * a = a \text{ for all } a \in S,$$

is called an identity element of S, with respect to $*$.

EXAMPLE 1 (i) For addition on R, zero is the identity element in R, since

$$a + 0 = 0 + a = a \text{ for all } a \in R.$$

(ii) For multiplication on R, 1 is the identity element in R, since

$$a \times 1 = 1 \times a = a \text{ for all } a \in R.$$

EXAMPLE 2 Let $P(S)$ be the power set of a nonempty set S. Then, ϕ is the identity element for union on $P(S)$ as

$$A \cup \phi = \phi \cup A = A \text{ for all } A \in P(S).$$

Also, S is the identity element for intersection on $P(S)$, since

$$A \cap S = S \cap A = A \text{ for all } A \in P(S).$$

INVERSE OF AN ELEMENT Let $*$ be a binary operation on a nonempty set S and let e be the identity element.

Let $a \in S$. We say that a is invertible if there exists an element $b \in S$ such that

$$a * b = b * a = e.$$

Also, in this case, b is called the inverse of a, and we write, $a^{-1} = b$.

EXAMPLE 1 *Consider addition on Z.*

Clearly, the additive identity is 0, since

$$a + 0 = 0 + a = a \text{ for all } a \in Z.$$

Also, corresponding to each $a \in Z$, there exists $-a \in Z$ such that

$$a + (-a) = (-a) + a = 0.$$

Thus, the *additive inverse* of a is $-a$.

EXAMPLE 2 *Consider multiplication on Z.*

Clearly, 1 is the multiplicative identity on Z, since

$$a \times 1 = 1 \times a = a \text{ for all } a \in Z.$$

Since $1 \times 1 = 1$, so the multiplicative inverse of 1 is 1.

Since $(-1) \times (-1) = 1$, so the multiplicative inverse of (-1) is (-1).

No integer other than 1 and -1 has its multiplicative inverse in Z.

SOLVED EXAMPLES

EXAMPLE 1 *Let Z be the set of all integers. Then, addition on Z satisfies the following properties:*

(i) *Closure Property*

We know that the sum of two integers is always an integer, i.e., $a \in Z,\ b \in Z \Rightarrow a + b \in Z$ for all $a, b \in Z$.

(ii) *Associative Law*

For all $a, b, c \in Z$, we have

$$(a + b) + c = a + (b + c).$$

(iii) *Commutative Law*

For all $a, b \in Z$, we have

$$a + b = b + a.$$

(iv) *Existence of Additive Identity*

Clearly, $0 \in Z$ is the additive identity, since

$$0 + a = a + 0 = a \text{ for all } a \in Z.$$

(v) *Existence of Additive Inverse*

For each $a \in Z$, there exists $-a \in Z$ such that

$$a + (-a) = (-a) + a = 0.$$

So, $-a$ is the additive inverse of a.

EXAMPLE 2 *Let R_0 be the set of all nonzero real numbers. Then, multiplication on R_0 satisfies the following properties:*

(i) *Closure Property*

We know that the product of two nonzero real numbers is a nonzero real number.

Thus, $a \in R_0,\ b \in R_0 \Rightarrow ab \in R_0$ for all $a,\ b \in R_0$.

(ii) *Associative Law*

For all $a,\ b,\ c \in R_0$, we have $(ab)c = a(bc)$.

(iii) *Commutative Law*

For all $a,\ b \in R_0$, we have $ab = ba$.

(iv) *Existence of Multiplicative Identity*

Clearly, $1 \in R_0$ is the multiplicative identity, since

$$1 \times a = a \times 1 \text{ for all } a \in R_0.$$

(v) *Existence of Multiplicative Inverse*

For each $a \in R_0$ there exists $\frac{1}{a} \in R_0$ such that

$$a \times \frac{1}{a} = \frac{1}{a} \times a = 1.$$

Thus, the multiplicative inverse of a is $\frac{1}{a}\cdot$

EXAMPLE 3 *Show that the operation $*$ on Z, defined by*

$$a * b = a + b + 1 \text{ for all } a,\ b \in Z$$

satisfies (i) the closure property, (ii) the associative law and (iii) the commutative law.

(iv) Find the identity element in Z.

(v) What is the inverse of an element $a \in Z$?

SOLUTION (i) *Closure Property*

Let $a \in Z,\ \ b \in Z$. Then,

$$a * b = a + b + 1.$$

Now, $a \in Z,\ \ b \in Z \Rightarrow a + b \in Z$

$\Rightarrow a + b + 1 \in Z$.

$\therefore$ $*$ on Z satisfies the closure property.

(ii) *Associative Law*

For all $a,\ b,\ c \in Z$, we have:

$$(a * b) * c = (a + b + 1) * c$$
$$= (a + b + 1) + c + 1$$
$$= a + b + c + 2.$$

$$a * (b * c) = a * (b + c + 1)$$
$$= a + (b + c + 1) + 1$$
$$= a + b + c + 2.$$

$\therefore$ $(a * b) * c = a * (b * c)$.

(iii) *Commutative Law*

For all $a, b \in Z$, we have

$$a * b = a + b + 1$$
$$= b + a + 1 \qquad [\because \quad a + b = b + a]$$
$$= b * a.$$

(iv) *Existence of Identity Element*

Let e be the identity element in Z.

Then, $a * e = a \;\Rightarrow\; a + e + 1 = a$
$\Rightarrow\; e = -1.$

Thus, $-1 \in Z$ is the identity element for $*$.

(v) *Existence of Inverse*

Let $a \in Z$ and let its inverse be b. Then,

$$a * b = -1 \;\Rightarrow\; a + b + 1 = -1$$
$$\Rightarrow\; b = -(2 + a).$$

Clearly, $2 \in Z,\; a \in Z \;\Rightarrow\; -(2 + a) \in Z$.
Thus, each $a \in Z$ has $-(2 + a) \in Z$ as its inverse.

EXAMPLE 4 *Show that the operation $*$ on $Q - \{1\}$, defined by*

$$a * b = a + b - ab \text{ for all } a,\; b \in Q - \{1\}$$

satisfies (i) the closure property, (ii) the associative law,
(iii) the commutative law.
(iv) What is the identity element?
(v) For each $a \in Q - \{1\}$, find the inverse of a.

SOLUTION (i) *Closure Property*

Let $a \in Q - \{1\}$ and $b \in Q - \{1\}$.

We know that Q is closed for addition, subtraction and multiplication.

$\therefore \quad a + b - ab \in Q.$

But, $a * b = 1 \Rightarrow\; a + b - ab = 1$
$\Rightarrow\; a(1 - b) = (1 - b) \;\Rightarrow\; a = 1,$

which is a contradiction since $1 \notin Q - \{1\}$.

$\therefore \quad a * b \neq 1.$

Thus, $a \in Q - \{1\}, b \in Q - \{1\} \;\Rightarrow\; a * b \in Q - \{1\}$.

$\therefore \quad *$ is a binary operation on $Q - \{1\}$.

(ii) *Associative Law*

Let $a,\; b,\; c \in Q - \{1\}$. Then,

$$(a * b) * c = (a + b - ab) * c$$
$$= (a + b - ab) + c - (a + b - ab)c$$
$$= (a + b + c) - (ab + bc + ac) + abc.$$

And, $a * (b * c) = a * (b + c - bc)$

$$= a + (b + c - bc) - a(b + c - bc)$$
$$= (a + b + c) - (ab + bc + ac) + abc.$$

$\therefore \quad (a * b) * c = a * (b * c)$.

Hence, $*$ is associative.

(iii) *Commutative Law*

Let $a, b \in Q - \{1\}$. Then,

$$a * b = a + b - ab$$
$$= b + a - ba \quad [\because + \text{ and } \cdot \text{ are commutative on } Q - \{1\}]$$
$$= b * a.$$

Hence, $*$ is commutative.

(iv) *Existence of Identity Element*

Let e be the identity element.

Then, for all $a \in Q - \{1\}$, we have

$$a * e = a \Rightarrow a + e - ae = a$$
$$\Rightarrow e(1 - a) = 0 \Rightarrow e = 0 \in Q - \{1\}.$$

Now, $a * 0 = a + 0 - a \times 0 = a$.

And, $0 * a = 0 + a - 0 \times a = a$.

Thus, 0 is the identity element in $Q - \{1\}$.

(v) *Existence of Inverse*

Let $a \in Q - \{1\}$ and let $a^{-1} = b$. Then,

$$a * b = 0 \Rightarrow a + b - ab = 0$$
$$\Rightarrow a = ab - b = (a - 1)b$$
$$\Rightarrow b = \frac{a}{(a-1)} \in Q - \{1\}.$$

$$\therefore \quad a^{-1} = \frac{a}{(a-1)} \in Q - \{1\}.$$

Thus, each $a \in Q - \{1\}$ has its inverse in $Q - \{1\}$.

EXAMPLE 5 *On the set N of all natural numbers, define the operation $*$ on N by*

$$m * n = \text{gcd}\,(m, n) \textit{ for all } m, n \in N.$$

Show that $*$ is commutative as well as associative.

SOLUTION (i) *Commutativity*

For all $m, n \in N$, we have $\text{gcd}\,(m, n) = \text{gcd}\,(n, m)$.

$\therefore \quad m * n = n * m \ \forall \ m, n \in N$.

Hence, $*$ is commutative on N.

(ii) *Associativity*

Let $m, n, p \in N$. Then,

$$(m * n) * p = [\text{gcd}\,(m, n)] * p$$
$$= \text{gcd}\,[\text{gcd}\,\{(m, n), p\}]$$
$$= \text{gcd}\,[\{m, \text{gcd}\,(n, p)\}]$$

[$\because$ gcd of three numbers = gcd {(gcd of any two, third)}]

$$= \text{gcd}\,(m, n * p) = m * (n * p).$$

Hence, $*$ is associative on N.

EXAMPLE 6 *Consider the set* $A = \{-1, 1\}$ *with multiplication operation. We may prepare its composition table as shown below.*

$\times$	1	–1
1	1	–1
–1	–1	1

Multiplication on A satisfies the following properties:

(i) *Closure Property*
Since all the entries of the composition table are in A, so A is closed for multiplication.

(ii) *Associative Law*
Since multiplication of integers is associative, in particular, multiplication on A is associative.

(iii) *Commutative Law*
Since every row of the table coincides with the corresponding column,

i.e., 1st row coincides with 1st column,

2nd row coincides with 2nd column.

So, multiplication is commutative on A.

(iv) *Existence of Identity*
Clearly, 1 is the identity element in A, since
$1 \times 1 = 1$ and $(-1) \times 1 = 1 \times (-1) = -1$.

(v) *Existence of Inverse*
It is clear from the table that
$1 \times 1 = 1 \Rightarrow$ inverse of 1 is 1,
$(-1) \times (-1) = 1 \Rightarrow$ inverse of (–1) is (–1).

NEW OPERATIONS

EXAMPLE 7 *Let* $A = \{1, 2, 3, 4, 5\}$. *Define an operation* $\wedge$ *by*
$a \wedge b = \min\{a, b\}$.
Then, we may prepare its composition table as given below.

$\vee$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Closure Property

Since all the entries of the composition table are from given set A, so A is closed for the operation $\wedge$.

Commutative Law

In the given table every row coincides with the corresponding column,

i.e., 1st row coincides with 1st column,
2nd row coincides with 2nd column, and so on.

$\therefore$ $\wedge$ on A is commutative.

EXAMPLE 8 *Let* $A = \{1, 2, 3, 4, 5\}$. *Define an operation* $\vee$ *by*

$a \vee b = \max\{a, b\}$.

Prepare its composition table.

Show that A is closed for the given operation and that the given operation is commutative.

SOLUTION We prepare the table as given below.

$\vee$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	3	4	5
3	3	3	3	4	5
4	4	4	4	4	5
5	5	5	5	5	5

Closure Property

Since all the entries of the composition table are from the given set, so closure property is satisfied.

Commutative Law

Clearly, every row coincides with the corresponding column. So, commutative law is satisfied.

EXERCISE 3A

Very-Short-Answer Questions

1. Let * be a binary operation on the set I of all integers, defined by $a * b = 3a + 4b - 2$. Find the value of $4 * 5$. **[CBSE 2011C]**
2. The binary operation * on R is defined by $a * b = 2a + b$. Find $(2 * 3) * 4$. **[CBSE 2012]**
3. Let * be a binary operation on the set of all nonzero real numbers, defined by $a * b = \frac{ab}{5}$. Find the value of x given that $2 * (x * 5) = 10$. **[CBSE 2014]**
4. Let $* : R \times R \to R$ be a binary operation given by $a * b = a + 4b^2$. Then, compute $(-5) * (2 * 0)$. **[CBSE 2014C]**

5. Let $*$ be a binary operation on the set Q of all rational numbers given as $a * b = (2a - b)^2$ for all $a, b \in Q$. Find $3 * 5$ and $5 * 3$. Is $3 * 5 = 5 * 3$?

[CBSE 2008C]

6. Let $*$ be a binary operation on N given by $a * b = \text{lcm}$ of a and b. Find the value of $20 * 16$.

   Is $*$ (i) commutative, (ii) associative? **[CBSE 2008C]**

7. If $*$ be the binary operation on the set Z of all integers defined by $a * b = (a + 3b^2)$, find $2 * 4$. **[CBSE 2009]**

8. Show that $*$ on Z^+ defined by $a * b = |a - b|$ is not a binary operation.

9. Let $*$ be a binary operation on N, defined by $a * b = a^b$ for all $a, b \in N$. Show that $*$ is neither commutative nor associative.

10. Let $a * b = \text{lcm}(a, b)$ for all values of $a, b \in N$.
    (i) Find $(12 * 16)$.
    (ii) Show that $*$ is commutative on N.
    (iii) Find the identity element in N.
    (iv) Find all invertible elements in N.

11. Let Q^+ be the set of all positive rational numbers.
    (i) Show that the operation $*$ on Q^+ defined by $a * b = \frac{1}{2}(a + b)$ is a binary operation.
    (ii) Show that $*$ is commutative.
    (iii) Show that $*$ is not associative. **[CBSE 2008]**

12. Show that the set $A = \{-1, 0, 1\}$ is not closed for addition.

13. Show that $*$ on $R - \{-1\}$, defined by $(a * b) = \dfrac{a}{(b+1)}$ is neither commutative nor associative. **[CBSE 2007]**

14. For all $a, b \in R$, we define $a * b = |a - b|$.
    Show that $*$ is commutative but not associative.

15. For all $a, b \in N$, we define $a * b = a^3 + b^3$.
    Show that $*$ is commutative but not associative.

16. Let X be a nonempty set and $*$ be a binary operation on $P(X)$, the power set of X, defined by $A * B = A \cap B$ for all $A, B \in P(X)$.
    (i) Find the identity element in $P(X)$.
    (ii) Show that X is the only invertible element in $P(X)$.

17. A binary operation $*$ on the set $\{0, 1, 2, 3, 4, 5\}$ is defined as

$$a * b = \begin{cases} a + b; & \text{if } a + b < 6 \\ a + b - 6; & \text{if } a + b \geq 6. \end{cases}$$

    Show that 0 is the identity for this operation and each element a has an inverse $(6 - a)$. **[CBSE 2011C]**

ANSWERS (EXERCISE 3A)

1. 30 **2.** 18 **3.** $x = 25$ **4.** 11
5. 1, 49; No **6.** $20 * 16 = 80$; (i) Yes (ii) Yes **7.** 50
10. (i) 48 (iii) 1 (iv) 1 **16.** (i) X

HINTS TO THE GIVEN QUESTIONS (EXERCISE 3A)

1. $4 * 5 = 3 \times 4 + 4 \times 5 - 2 = 12 + 20 - 2 = 30.$
2. $2 * 3 = 2 \times 2 + 3 = 4 + 3 = 7.$
   $\therefore \quad (2 * 3) * 4 = 7 * 4 = 2 \times 7 + 4 = 14 + 4 = 18.$
3. $2 * (x * 5) = 10 \Rightarrow 2 * \left(\frac{x \times 5}{5}\right) = 10$
   $\Rightarrow 2 * x = 10 \Rightarrow \frac{2 \times x}{5} = 10$
   $\Rightarrow 2x = 50 \Rightarrow x = 25.$
4. $(2 * 0) = 2 + 4 \times 0^2 = 2 + 4 \times 0 = 2 + 0 = 2.$
   $\therefore \quad (-5) * (2 * 0) = (-5) * 2 = (-5) + 4 \times 2^2 = (-5) + 16 = 11.$
5. $(3 * 5) = (2 \times 3 - 5)^2 = (6 - 5)^2 = 1^2 = 1.$
   $(5 * 3) = (2 \times 5 - 3)^2 = (10 - 3)^2 = 7^2 = 49.$
   $\therefore \quad (3 * 5) \neq (5 * 3).$
6. $(20 * 16) = \text{lcm}\ (20, 16) = 80.$
   (i) For all $a, b \in N$, we know that lcm $\{a, b\} = \{b, a\}$. So, $a * b = b * a$.
   $\therefore$ * is commutative on N.
   (ii) For all $a, b, c \in N$, we know that
   lcm $\{a, b$ and $c\}$ = lcm $\{(a, b)$ and $c\}$.
   $\therefore$ * is associative on N.
7. $(2 * 4) = (2 + 3 \times 4^2) = (2 + 3 \times 16) = (2 + 48) = 50.$
8. Let a be an arbitrary positive integer. Then,
   $a * a = |a - a| = 0$, which is not a positive integer.
   $\therefore$ * on Z^+ is not a binary operation.
9. (i) $2 * 5 = 2^5 = 32$ and $5 * 2 = 5^2 = 25$.
   Hence, $2 * 5 \neq 5 * 2$.
   (ii) $(2 * 1) * 4 = 2^1 * 4 = (2 * 4) = 2^4 = 16.$
   $2 * (1 * 4) = 2 * 1^4 = 2 * 1 = 2^1 = 2.$
   $\therefore \quad (2 * 1) * 4 \neq 2 * (1 * 4).$
10. (i) $12 * 16 = \text{lcm}\ (12, 16) = (4 \times 3 \times 4) = 48.$
   (ii) $a * b = \text{lcm}(a, b) = \text{lcm}(b, a) = b * a.$
   (iii) Let e be the identity element in N.

4	12, 16
	3, 4

Then, for all $a \in N$, we have

$$a * e = a \Rightarrow \text{lcm}(a, e) = a \Rightarrow e = 1.$$

$\therefore$ 1 is the identity element in N.

(iv) Let $a \in N$ be an arbitrary element.

Then, $a * b = 1 \Rightarrow \text{lcm}(a, b) = 1 \Rightarrow a = b = 1.$

$\therefore$ 1 is the only element in N, which is invertible.

11. (i) $a \in Q^+,\ b \in Q^+ \Rightarrow a + b \in Q^+$

$$\Rightarrow \frac{1}{2}(a + b) \in Q^+$$

$$\Rightarrow a * b \in Q^+.$$

(ii) $a * b = \frac{1}{2}(a + b) = \frac{1}{2}(b + a) = b * a.$

(iii) We have

$$(3 * 4) * 5 = \frac{1}{2}(3 + 4) * 5 = \left(\frac{7}{2} * 5\right) = \frac{1}{2}\left(\frac{7}{2} + 5\right) = \frac{17}{4}.$$

$$3 * (4 * 5) = \left(3 * \frac{9}{2}\right) = \frac{1}{2}\left(3 + \frac{9}{2}\right) = \frac{15}{4}.$$

$\therefore (3 * 4) * 5 \neq 3 * (4 * 5).$

12. $-1 \in A$ and $-1 \in A$. But $(-1) + (-1) = -2 \notin A$.

13. (i) $2 * 3 = \frac{2}{(3 + 1)} = \frac{2}{4} = \frac{1}{2}$ and $3 * 2 = \frac{3}{(2 + 1)} = \frac{3}{3} = 1.$

$\therefore$ $2 * 3 \neq 3 * 2$. So, $*$ is not commutative.

(ii) $(2 * 3) * 1 = \frac{1}{2} * 1 = \frac{1/2}{(1 + 1)} = \frac{1}{4}.$

$$(3 * 3) = \frac{3}{(1 + 1)} = \frac{3}{2}.$$

$$\therefore\ 2 * (3 * 1) = 2 * \frac{3}{2} = \frac{2}{\left(\frac{3}{2} + 1\right)} = \left(2 \times \frac{2}{5}\right) = \frac{4}{5}.$$

Thus, $(2 * 3) * 1 \neq 2 * (3 * 1).$

14. (i) For all $a, b \in R$ we have

$$a * b = |a - b| = |-(a - b)| = |b - a| = (b * a).$$

$\therefore$ $*$ is commutative.

(ii) We have

$$(2 * 3) * 4 = |2 - 3| * 4 = |-1| * 4 = 1 * 4 = |1 - 4| = |-3| = 3.$$

$$2 * (3 * 4) = 2 * |3 - 4| = 2 * |-1| = 2 * 1 = |2 - 1| = |1| = 1.$$

$\therefore (2 * 3) * 4 \neq 2 * (3 * 4).$

Hence, $*$ is not associative.

15. (i) For all $a, b \in N$ we have

$$a * b = a^3 + b^3 = b^3 + a^3 = b * a.$$

$\therefore$ $*$ is commutative.

(ii) $(1 * 2) * 3 = (1^3 + 2^3) * 3 = (9 * 3) = (9^3 + 3^3)$

$$= (729 + 27) = 756.$$

$1 * (2 * 3) = 1 * (2^3 + 3^3) = 1 * (8 + 27) = 1 * 35$

$= \{1^3 + (35)^3\}.$

$\therefore \ (1 * 2) * 3 \neq 1 * (2 * 3).$

16. (i) Since $A \cap X = A$ for all A in $P(X)$.

$\therefore$ X is the identity element.

(ii) Let A be invertible in $P(X)$ and let B be its inverse.

Then, $A \cap B = X$.

This is possible only when $A = B = X$.

$\therefore$ X is the only invertible element in $P(X)$ and its inverse is X.

17. $a * 0 = a + 0 = a \quad [\because \ 0 \leq a \leq 5]$.

So, 0 is the identity.

$a * (6 - a) = a + (6 - a) - 6 = 0 \quad [\because \ a + (6 - a) \geq 6]$.

$\therefore$ The inverse of a is $(6 - a)$.

EXERCISE 3B

1. Define $*$ on N by $m * n = \text{lcm}(m, n)$.
   Show that $*$ is a binary operation which is commutative as well as associative.
2. Define $*$ on Z by $a * b = a - b + ab$.
   Show that $*$ is a binary operation on Z which is neither commutative nor associative.
3. Define $*$ on Z by $a * b = a + b - ab$.
   Show that $*$ is a binary operation on Z which is commutative as well as associative.
4. Consider a binary operation on $Q - \{1\}$, defined by

   $$a * b = a + b - ab.$$

   (i) Find the identity element in $Q - \{1\}$.
   (ii) Show that each $a \in Q - \{1\}$ has its inverse.
5. Let Q_0 be the set of all nonzero rational numbers. Let $*$ be a binary operation on Q_0, defined by $a * b = \frac{ab}{4}$ for all $a, b \in Q_0$.
   (i) Show that $*$ is commutative and associative.
   (ii) Find the identity element in Q_0.
   (iii) Find the inverse of an element a in Q_0.
6. On the set Q^+ of all positive rational numbers, define an operation $*$ on Q^+ by $a * b = \frac{ab}{2}$ for all $a, b \in Q^+$.
   Show that
   (i) $*$ is a binary operation on Q^+,

(ii) $*$ is commutative,
(iii) $*$ is associative.
Find the identity element in Q^+ for $*$.
What is the inverse of $a \in Q^+$?

7. Let Q^+ be the set of all positive rational numbers.
   (i) Show that the operation $*$ on Q^+ defined by $a * b = \frac{1}{2}(a + b)$ is a binary operation.
   (ii) Show that $*$ is commutative.
   (iii) Show that $*$ is not associative.
8. Let Q be the set of all rational numbers. Define an operation $*$ on $Q - \{-1\}$ by $a * b = a + b + ab$.
   Show that
   (i) $*$ is a binary operation on $Q - \{-1\}$,
   (ii) $*$ is commutative,
   (iii) $*$ is associative,
   (iv) zero is the identity element in $Q - \{-1\}$ for $*$,
   (v) $a^{-1} = \left(\frac{-a}{1 + a}\right)$, where $a \in Q - \{-1\}$.
9. Let $A = N \times N$. Define $*$ on A by
   $$(a, b) * (c, d) = (a + c, b + d).$$
   Show that
   (i) A is closed for $*$,
   (ii) $*$ is commutative,
   (iii) $*$ is associative,
   (iv) identity element does not exist in A.
10. Let $A = \{1, -1, i, -i\}$ be the set of four 4th roots of unity. Prepare the composition table for multiplication on A and show that
    (i) A is closed for multiplication,
    (ii) multiplication is associative on A,
    (iii) multiplication is commutative on A,
    (iv) 1 is the multiplicative identity,
    (v) every element in A has its multiplicative inverse.

ANSWERS (EXERCISE 3B)

4. (i) 0 (ii) $a^{-1} = \frac{a}{(a-1)}$ **5.** (ii) 4 (iii) $a^{-1} = \frac{16}{a}$ **6.** $e = 2, a^{-1} = \frac{4}{a}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 3B)

1. We know that the lcm of two natural numbers is a natural number.
   So, N is closed for $*$.

We know that $\text{lcm}(m, n) = \text{lcm}(n, m)$. So, $m * n = n * m$.

$\text{lcm}(m, n, p) = \text{lcm}\{\text{lcm}(m, n), p\} = \text{lcm}\{m, \text{lcm}(n, p)\}$.

2. $a \in Z, b \in Z \Rightarrow (a - b) \in Z$ and $ab \in Z$

$$\Rightarrow \{(a-b) + ab\} \in Z \Rightarrow a - b + ab \in Z.$$

$\therefore$ Z is closed for $*$.

Show that $3 * 2 \neq 2 * 3$ and $(4 * 3) * 2 \neq 4 * (3 * 2)$.

4. (i) Let e be the identity element. Then, for all $a \in Q - \{1\}$, we have

$$a * e = a \Rightarrow a + e - ae = a$$

$$\Rightarrow e(1-a) = 0 \Rightarrow e = 0 \quad [\because a \neq 1].$$

$\therefore$ 0 is the identity element.

(ii) Let $a \in Q - \{1\}$ be an arbitrary element and let b be its inverse.

Then, $a * b = 0 \Rightarrow a + b - ab = 0 \Rightarrow ab - b = a$

$$\Rightarrow b(a-1) = a \Rightarrow b = \frac{a}{(a-1)}.$$

Thus, each $a \in Q - \{1\}$ has $\frac{a}{(a-1)}$ as its inverse.

5. (i) For all $a, b, c \in Q_0$, we have

$$a * b = \frac{ab}{4} = \frac{ba}{4} = b * a.$$

And, $(a * b) * c = \frac{ab}{4} * c = \frac{\frac{ab}{4} \cdot c}{4} = \frac{(ab)c}{16}$.

Also, $a * (b * c) = a * \frac{bc}{4} = \frac{a\left(\frac{bc}{4}\right)}{4} = \frac{a(bc)}{16}$.

But, $(ab)c = a(bc)$. Hence, $(a * b) * c = a * (b * c)$.

(ii) Let e be the identity element and let $a \in Q_0$. Then,

$$a * e = a \Rightarrow \frac{ae}{4} = a \Rightarrow e = 4.$$

$\therefore$ 4 is the identity element in Q_0.

(iii) Let $a \in Q_0$ and let its inverse be b. Then,

$$a * b = e \Rightarrow \frac{ab}{4} = 4 \Rightarrow b = \frac{16}{a} \in Q_0.$$

Thus, each $a \in Q_0$ has $\frac{16}{a}$ as its inverse.

6. $a * e = a \Rightarrow \frac{ae}{2} = a \Rightarrow e = 2.$

$a * b = 2 \Rightarrow \frac{ab}{2} = 2 \Rightarrow b = \frac{4}{a} \Rightarrow a^{-1} = \frac{4}{a}.$

9. (i) Let $(a, b) \in A$ and $(c, d) \in A$. Then, $a, b, c, d \in N$.

$(a, b) * (c, d) = (a + c, b + d) \in A \quad [\because a + c \in N, b + d \in N]$.

$\therefore$ A is closed for $*$.

(ii) $(a,\ b) * (c,\ d) = (a + c,\ b + d)$

$= (c + a,\ d + b)$ [$\because$ $a + c = c + a$ and $b + d = d + b$]

$= (c,\ d) * (a,\ b)$.

(iii) $[(a,\ b) * (c,\ d)] * (e, f) = (a + c,\ b + d) * (e, f)$

$= [(a + c) + e,\ (b + d) + f]$

$= [a + (c + e),\ b + (d + f)]$

$= (a,\ b) * (c + e,\ d + f)$

$= (a,\ b) * [(c,\ d) * (e,\ f)]$.

(iv) $(a,\ b) * (0,\ 0) = (a + 0,\ b + 0) = (a,\ b)$.

But, $(0,\ 0) \notin A$, since $0 \notin N$.

So, identity element does not belong to A.

10. We may prepare the composition table for multiplication on A as given below:

$\times$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Clearly, 1 is the identity element.

$(1)^{-1} = 1,\ (-1)^{-1} = -1,\ (i)^{-1} = -i$ and $(-i)^{-1} = i$.

4. INVERSE TRIGONOMETRIC FUNCTIONS

INVERTIBLE FUNCTIONS *A one-one onto function is called an invertible function.*

INVERSE OF A FUNCTION Let $f : X \to Y$ be a one-one onto function. Then, for each $y \in Y$, there exists a unique element $x \in X$ such that $f(x) = y$.

So, we define a new function, denoted by f^{-1}, called the *inverse* of f, as

$$f^{-1} : Y \to X : f^{-1}(y) = x \iff f(x) = y.$$

Clearly,

domain (f^{-1}) = range (f) and range (f^{-1}) = domain (f).

Trigonometric functions are, in general, not one-one onto.

Therefore, their inverses do not exist.

However, if we restrict their domains, we can make them one-one onto, enabling us to have their inverses.

EXAMPLE The sine function restricted to any of the intervals $[-\pi/2, \pi/2]$, $[3\pi/2, 5\pi/2]$, etc., is one-one onto and in each case, the range is $[-1, 1]$.

Therefore, we can define the inverse of the sine function, denoted by $\sin^{-1} x$, in each of these intervals. Corresponding to each such interval, we get a branch of $\sin^{-1} x$.

The branch $[-\pi/2, \pi/2]$ is called the *principal-value branch* and the value belonging to it is called the *principal value*.

We denote the inverse of the sine function as $\sin^{-1} x$, called *sine inverse x*.

$$\therefore \quad \sin^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } \sin^{-1} x = \theta \iff x = \sin \theta.$$

Thus, $\sin^{-1} x$ is a function whose domain is $[-1, 1]$ and range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Note that $\sin^{-1} x$ does not mean $(\sin x)^{-1}$.

Similarly, we can define the principal-value branches of the remaining five trigonometrical functions.

The following table shows the inverse trigonometric functions and their principal-value branches.

Function	Domain	Range (Principal value)
1. $y = \sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
2. $y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3. $y = \tan^{-1} x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
4. $y = \text{cosec}^{-1} x$	$R - (-1, 1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
5. $y = \sec^{-1} x$	$R - [-1, 1]$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
6. $y = \cot^{-1} x$	R	$(0, \pi)$

SOLVED EXAMPLES

EXAMPLE 1 *Find the principal value of each of the following:*

(i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ *(ii)* $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(iii) $\tan^{-1}(\sqrt{3})$ *(iv)* $\text{cosec}^{-1}(2)$

SOLUTION (i) We know that the range of the principal-value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta$. Then,

$$\sin\theta = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

Hence, the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$.

(ii) We know that the range of the principal-value branch of $\cos^{-1}$ is $[0, \pi]$.

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \theta$. Then,

$$\cos\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6} \in [0, \pi].$$

Hence, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

(iii) We know that the range of the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Let $\tan^{-1}(\sqrt{3}) = \theta$. Then,

$$\tan\theta = \sqrt{3} = \tan\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

Hence, the principal value of $\tan^{-1}(\sqrt{3})$ is $\frac{\pi}{3}$.

(iv) We know that the range of the principal-value branch of $\operatorname{cosec}^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Let $\operatorname{cosec}^{-1}(2) = \theta$. Then,

$$\operatorname{cosec}\theta = 2 = \operatorname{cosec}\frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}.$$

Hence, the principal value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

EXAMPLE 2 *Find the principal value of each of the following:*

(i) $\sin^{-1}\left(\frac{-1}{2}\right)$ *(ii)* $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ *(iii)* $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

(iv) $\tan^{-1}(-1)$ *(v)* $\operatorname{cosec}^{-1}(-2)$ *(vi)* $\tan^{-1}(-\sqrt{3})$

SOLUTION (i) We know that the principal-value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Let $\sin^{-1}\left(\frac{-1}{2}\right) = \theta$. Then,

$$\sin\theta = -\frac{1}{2} = \sin\left(\frac{-\pi}{6}\right), \text{ where } \frac{-\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

Hence, the principal value of $\sin^{-1}\left(\frac{-1}{2}\right)$ is $\frac{-\pi}{6}$.

(ii) We know that the principal-value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Let $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \theta$. Then,

$$\sin\theta = \frac{-\sqrt{3}}{2} = \sin\left(\frac{-\pi}{3}\right), \text{ where } \frac{-\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

Hence, the principal value of $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ is $\frac{-\pi}{3}$.

(iii) We know that the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Let $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \theta$. Then,

$$\tan\theta = \frac{-1}{\sqrt{3}} = \tan\left(\frac{-\pi}{6}\right), \text{ where } \frac{-\pi}{6} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

Hence, the principal value of $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ is $\frac{-\pi}{6}$.

(iv) We know that the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Let $\tan^{-1}(-1) = \theta$. Then,

$$\tan\theta = -1 = \tan\left(\frac{-\pi}{4}\right), \text{ where } \frac{-\pi}{4} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

Hence, the principal value of $\tan^{-1}(-1)$ is $\frac{-\pi}{4}$.

(v) We know that the principal-value branch of $\operatorname{cosec}^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Let $\operatorname{cosec}^{-1}(-2) = \theta$. Then,

$$\operatorname{cosec}\theta = -2 = \operatorname{cosec}\left(\frac{-\pi}{6}\right), \text{ where } \frac{-\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}.$$

Hence, the principal value of $\operatorname{cosec}^{-1}(-2)$ is $\frac{-\pi}{6}$.

(vi) We know that the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Let $\tan^{-1}(-\sqrt{3}) = \theta$. Then,

$$\tan\theta = -\sqrt{3} = \tan\left(\frac{-\pi}{3}\right), \text{ where } \frac{-\pi}{3} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

Hence, the principal value of $\tan^{-1}(-\sqrt{3})$ is $\frac{-\pi}{3}$.

EXAMPLE 3 *Find the principal value of each of the following:*

(i) $\cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$ *(ii)* $\cos^{-1}\left(\dfrac{-1}{2}\right)$ *(iii)* $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$

SOLUTION (i) We know that the range of the principal value of $\cos^{-1}$ is $[0, \pi]$.

Let $\cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right) = \theta$. Then,

$$\cos\theta = \frac{-1}{\sqrt{2}} = -\cos\frac{\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4}.$$

$$\therefore \quad \theta = \frac{3\pi}{4} \in [0, \pi].$$

Hence, the principal value of $\cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$ is $\dfrac{3\pi}{4}$.

(ii) We know that the range of the principal value of $\cos^{-1}$ is $[0, \pi]$.

Let $\cos^{-1}\left(\dfrac{-1}{2}\right) = \theta$. Then,

$$\cos\theta = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}.$$

$$\therefore \quad \theta = \frac{2\pi}{3} \in [0, \pi].$$

Hence, the principal value of $\cos^{-1}\left(\dfrac{-1}{2}\right)$ is $\dfrac{2\pi}{3}$.

(iii) We know that the range of the principal value of $\cot^{-1}$ is $(0, \pi)$.

Let $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right) = \theta$. Then,

$$\cot\theta = \frac{-1}{\sqrt{3}} = -\cot\frac{\pi}{3} = \cot\left(\pi - \frac{\pi}{3}\right) = \cot\frac{2\pi}{3}.$$

$$\therefore \quad \theta = \frac{2\pi}{3} \in (0, \pi).$$

Hence, the principal value of $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$ is $\dfrac{2\pi}{3}$.

EXAMPLE 4 *Find the principal value of each of the following:*

(i) $\cot^{-1}(-\sqrt{3})$ *(ii)* $\sec^{-1}(-\sqrt{2})$ *(iii)* $\operatorname{cosec}^{-1}(-1)$

SOLUTION (i) We know that the range of the principal value of $\cot^{-1}$ is $(0, \pi)$.

Let $\cot^{-1}(-\sqrt{3}) = \theta$. Then,

$$\cot \theta = -\sqrt{3} = -\cot \frac{\pi}{6} = \cot\left(\pi - \frac{\pi}{6}\right) = \cot \frac{5\pi}{6}.$$

$\therefore \quad \theta = \frac{5\pi}{6} \in (0, \ \pi).$

Hence, the principal value of $\cot^{-1}(-\sqrt{3})$ is $\frac{5\pi}{6}$.

(ii) We know that the range of principal value of $\sec^{-1}$ is $[0, \ \pi] - \left\{\frac{\pi}{2}\right\}$.

Let $\sec^{-1}(-\sqrt{2}) = \theta$. Then,

$$\sec \theta = -\sqrt{2} = -\sec \frac{\pi}{4} = \sec\left(\pi - \frac{\pi}{4}\right) = \sec \frac{3\pi}{4}.$$

$\therefore \quad \theta = \frac{3\pi}{4} \in [0, \ \pi] - \left\{\frac{\pi}{2}\right\}.$

Hence, the principal value of $\sec^{-1}(-\sqrt{2})$ is $\frac{3\pi}{4}$.

(iii) We know that the range of principal value of $\operatorname{cosec}^{-1}$ is $\left[\frac{-\pi}{2}, \ \frac{\pi}{2}\right] - \{0\}$.

Let $\operatorname{cosec}^{-1}(-1) = \theta$. Then, $\operatorname{cosec} \theta = -1$.

$$\operatorname{cosec} \theta = -1 = -\operatorname{cosec} \frac{\pi}{2} = \operatorname{cosec}\left(\frac{-\pi}{2}\right).$$

$\therefore \quad \theta = \frac{-\pi}{2} \in \left[\frac{-\pi}{2}, \ \frac{\pi}{2}\right] - \{0\}.$

Hence, the principal value of $\operatorname{cosec}^{-1}(-1)$ is $\frac{-\pi}{2}$.

EXAMPLE 5 *Find the value of* $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$. **[CBSE 2012C]**

SOLUTION We know that the ranges of principal values of $\cos^{-1}$ and $\sin^{-1}$ are $[0, \ \pi]$ and $\left[\frac{-\pi}{2}, \ \frac{\pi}{2}\right]$ respectively.

Let $\cos^{-1}\left(\frac{1}{2}\right) = \theta_1$. Then,

$$\cos \theta_1 = \frac{1}{2} = \cos \frac{\pi}{3} \quad \Rightarrow \quad \theta_1 = \frac{\pi}{3} \in [0, \ \pi].$$

Let $\sin^{-1}\left(\frac{1}{2}\right) = \theta_2$. Then,

$$\sin \theta_2 = \frac{1}{2} = \sin \frac{\pi}{6} \quad \Rightarrow \quad \theta_2 = \frac{\pi}{6} \in \left[\frac{-\pi}{2}, \ \frac{\pi}{2}\right].$$

$$\therefore \quad \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + \left(2 \times \frac{\pi}{6}\right) = \left(\frac{\pi}{3} + \frac{\pi}{3}\right) = \frac{2\pi}{3}.$$

EXAMPLE 6 *Find the value of* $\tan^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right)$.

SOLUTION We know that the ranges of principal values of $\tan^{-1}$, $\cos^{-1}$ and $\sin^{-1}$ are $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $[0, \pi]$ and $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ respectively.

Let $\tan^{-1}(1) = \theta_1$. Then,

$$\tan \theta_1 = 1 = \tan \frac{\pi}{4} \Rightarrow \theta_1 = \frac{\pi}{4} \in [0, \pi].$$

Let $\cos^{-1}\left(\frac{-1}{2}\right) = \theta_2$. Then,

$$\cos \theta_2 = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}.$$

$$\therefore \quad \theta_2 = \frac{2\pi}{3} \in [0, \pi].$$

Let $\sin^{-1}\left(\frac{-1}{2}\right) = \theta_3$. Then,

$$\sin \theta_3 = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right) \Rightarrow \theta_3 = \frac{-\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

$$\therefore \quad \tan^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right) = \left(\frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}\right) = \frac{3\pi}{4}.$$

EXAMPLE 7 *Find the value of* $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.

SOLUTION We know that the ranges of principal values of $\tan^{-1}$ and $\sec^{-1}$ are $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ respectively.

Let $\tan^{-1}\sqrt{3} = \theta_1$. Then,

$$\tan \theta_1 = \sqrt{3} = \tan\frac{\pi}{3} \Rightarrow \theta_1 = \frac{\pi}{3} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

Let $\sec^{-1}(-2) = \theta_2$. Then,

$$\sec \theta_2 = -2 = -\sec\frac{\pi}{3} = \sec\left(\pi - \frac{\pi}{3}\right) = \sec\frac{2\pi}{3}.$$

$$\therefore \quad \theta_2 = \frac{2\pi}{3} \in [0, \pi] - \left\{\frac{\pi}{2}\right\}.$$

Hence, $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \left(\frac{\pi}{3} - \frac{2\pi}{3}\right) = \frac{-\pi}{3}$.

EXERCISE 4A

1. Find the principal value of:

(i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (ii) $\sin^{-1}\left(\frac{1}{2}\right)$ (iii) $\cos^{-1}\left(\frac{1}{2}\right)$ (iv) $\tan^{-1}(1)$

(v) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (vi) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (vii) $\operatorname{cosec}^{-1}(\sqrt{2})$

2. Find the principal value of:

(i) $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ (iii) $\tan^{-1}(-\sqrt{3})$

(iv) $\sec^{-1}(-2)$ (v) $\operatorname{cosec}^{-1}(-\sqrt{2})$ (vi) $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

3. Evaluate $\cos\left\{\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right\}$.

HINT: $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)=\theta \Rightarrow \cos\theta=\frac{-\sqrt{3}}{2}=-\cos\frac{\pi}{6}=\cos\left(\pi-\frac{\pi}{6}\right)=\cos\frac{5\pi}{6} \Rightarrow \theta=\frac{5\pi}{6}$.

$\therefore$ given expression $=\cos\left(\frac{5\pi}{6}+\frac{\pi}{6}\right)=\cos\pi=-1$.

4. Evaluate $\sin\left\{\frac{\pi}{2}-\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right\}$.

HINT: Given exp. $=\sin\left\{\frac{\pi}{2}-\left(\frac{-\pi}{3}\right)\right\}=\sin\frac{5\pi}{6}=\sin\left(\pi-\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2}$.

ANSWERS (EXERCISE 4A)

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{6}$ (iii) $\frac{\pi}{3}$ (iv) $\frac{\pi}{4}$ (v) $\frac{\pi}{6}$ (vi) $\frac{\pi}{6}$ (vii) $\frac{\pi}{4}$

2. (i) $\frac{-\pi}{4}$ (ii) $\frac{5\pi}{6}$ (iii) $\frac{-\pi}{3}$ (iv) $\frac{2\pi}{3}$ (v) $\frac{-\pi}{4}$ (vi) $\frac{2\pi}{3}$

3. -1 **4.** $\frac{1}{2}$

PROPERTIES OF INVERSE FUNCTIONS

THEOREM 1 *Prove that:*

(i) $\sin^{-1}(\sin x) = x,\ x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos x) = x,\ x \in [0, \pi]$

(iii) $\tan^{-1}(\tan x) = x,\ x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\text{cosec}^{-1}(\text{cosec}\, x) = x,\ x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(v) $\sec^{-1}(\sec x) = x,\ x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

(vi) $\cot^{-1}(\cot x) = x,\ x \in (0, \pi)$

PROOF (i) Let $\sin x = y$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Then, $\sin^{-1} y = x \Rightarrow \sin^{-1}(\sin x) = x$ $[\because\ y = \sin x]$.

$\therefore\ \sin^{-1}(\sin x) = x$.

(ii) Let $\cos x = y$, where $x \in [0, \pi]$.

Then, $\cos^{-1} y = x \Rightarrow \cos^{-1}(\cos x) = x$ $[\because\ y = \cos x]$.

$\therefore\ \cos^{-1}(\cos x) = x$.

Similarly, the other results may be proved.

THEOREM 2 *Prove that:*

(i) $\sin(\sin^{-1} x) = x,\ x \in [-1, 1]$

(ii) $\cos(\cos^{-1} x) = x,\ x \in [-1, 1]$

(iii) $\tan(\tan^{-1} x) = x,\ x \in R$

(iv) $\text{cosec}(\text{cosec}^{-1} x) = x,\ x \in R - [-1, 1]$

(v) $\sec(\sec^{-1} x) = x,\ x \in R - [-1, 1]$

(vi) $\cot(\cot^{-1} x) = x,\ x \in R$

PROOF (i) Let $\sin^{-1} x = \theta$, where $x \in [-1, 1]$.

Then, $\sin\theta = x \Rightarrow \sin(\sin^{-1} x) = x$ $[\because\ \theta = \sin^{-1} x]$.

$\therefore\ \sin(\sin^{-1} x) = x$.

(ii) Let $\cos^{-1} x = \theta$, where $x \in [-1, 1]$.

Then, $\cos\theta = x \Rightarrow \cos(\cos^{-1} x) = x$ $[\because\ \theta = \cos^{-1} x]$.

$\therefore\ \cos(\cos^{-1} x) = x$.

Similarly, the other results may be proved.

THEOREM 3 *Prove that:*

$$(i)\ \sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x, (x \geq 1 \ \ or \ \ x \leq -1)$$

$$(ii)\ \cos^{-1}\frac{1}{x} = \sec^{-1}x, (x \geq 1 \ \ or \ \ x \leq -1)$$

$$(iii)\ \tan^{-1}\frac{1}{x} = \cot^{-1}x, \ \ (x > 0)$$

PROOF (i) $\sin^{-1}\frac{1}{x} = \theta \Rightarrow \sin\theta = \frac{1}{x}$

$$\Rightarrow \operatorname{cosec}\theta = x$$

$$\Rightarrow \theta = \operatorname{cosec}^{-1}x$$

$$\Rightarrow \sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x.$$

Hence, $\sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x.$

(ii) $\cos^{-1}\frac{1}{x} = \theta \Rightarrow \cos\theta = \frac{1}{x}$

$$\Rightarrow \sec\theta = x$$

$$\Rightarrow \theta = \sec^{-1}x$$

$$\Rightarrow \cos^{-1}\frac{1}{x} = \sec^{-1}x.$$

Hence, $\cos^{-1}\frac{1}{x} = \sec^{-1}x.$

(iii) $\tan^{-1}\frac{1}{x} = \theta \Rightarrow \tan\theta = \frac{1}{x}$

$$\Rightarrow \cot\theta = x$$

$$\Rightarrow \theta = \cot^{-1}x$$

$$\Rightarrow \tan^{-1}\frac{1}{x} = \cot^{-1}x.$$

Hence, $\tan^{-1}\frac{1}{x} = \cot^{-1}x.$

THEOREM 4 *Prove that:*

$$(i)\ \sin^{-1}(-x) = -\sin^{-1}x, \ \ x \in [-1, \ 1]$$

$$(ii)\ \tan^{-1}(-x) = -\tan^{-1}x, \ \ x \in R$$

$$(iii)\ \operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x, \ \ |x| \geq 1$$

PROOF (i) Let $\sin^{-1}(-x) = \theta$. Then,

$$\sin^{-1}(-x) = \theta \Rightarrow -x = \sin\theta$$

$$\Rightarrow x = -\sin\theta = \sin(-\theta)$$

$$\Rightarrow -\theta = \sin^{-1}x$$

$$\Rightarrow \theta = -\sin^{-1}x$$

$$\Rightarrow \sin^{-1}(-x) = -\sin^{-1}x.$$

$\therefore \quad \sin^{-1}(-x) = -\sin^{-1} x.$

Similarly, the other results may be proved.

THEOREM 5 *Prove that:*

(i) $\cos^{-1}(-x) = \pi - \cos^{-1} x, \quad x \in [-1, 1]$

(ii) $\sec^{-1}(-x) = \pi - \sec^{-1} x, \quad |x| \geq 1$

(iii) $\cot^{-1}(-x) = \pi - \cot^{-1} x, \quad x \in R$

PROOF (i) Let $\cos^{-1}(-x) = \theta$. Then,

$$\cos^{-1}(-x) = \theta \Rightarrow -x = \cos\theta$$
$$\Rightarrow x = -\cos\theta = \cos(\pi - \theta)$$
$$\Rightarrow \cos^{-1} x = (\pi - \theta) = \pi - \cos^{-1}(-x)$$
$$\Rightarrow \cos^{-1}(-x) = \pi - \cos^{-1} x.$$

$\therefore \quad \cos^{-1}(-x) = \pi - \cos^{-1} x.$

Similarly, the other results may be proved.

THEOREM 6 *Prove that:*

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \quad x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \quad x \in R$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}, \quad |x| \geq 1$

PROOF (i) Let $\sin^{-1} x = \theta$. Then,

$$\sin^{-1} x = \theta \Rightarrow x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$
$$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \theta$$
$$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$
$$\Rightarrow \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\cdot$$

$\therefore \quad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\cdot$

Similarly, the other results may be proved.

THEOREM 7 *Prove that:*

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, *if* $xy < 1$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, *if* $xy > -1$

(iii) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, *if* $|x| < 1$

PROOF (i) Let $xy < 1$, $\tan^{-1} x = \theta$ and $\tan^{-1} y = \phi$. Then,

$$\tan\theta = x \quad \text{and} \quad \tan\phi = y$$

$$\Rightarrow \tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{x + y}{1 - xy}$$

$$\Rightarrow \theta + \phi = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$$

$$\Rightarrow \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right).$$

$$\therefore \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right).$$

(ii) Let $xy > -1$.

On replacing y by $-y$ in (i), we get

$$\tan^{-1} x + \tan^{-1}(-y) = \tan^{-1}\left(\frac{x - y}{1 + xy}\right)$$

$$\Rightarrow \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x - y}{1 + xy}\right).$$

(iii) Let $|x| < 1$.

Replacing y by x in (i), we get

$$2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right).$$

THEOREM 8 *Prove that:*

$$(i)\ 2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1 + x^2}\right), \ |x| < 1$$

$$(ii)\ 2\tan^{-1} x = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right), \ |x| \geq 0$$

$$(iii)\ 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right), \ |x| < 1$$

PROOF Let $\tan^{-1} x = \theta$. Then, $x = \tan\theta$.

$$(i)\ \sin^{-1}\left(\frac{2x}{1 + x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right) \qquad [\because \ x = \tan\theta]$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\tan^{-1} x.$$

$$\therefore \ 2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1 + x^2}\right).$$

(ii) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$ [$\because$ $x = \tan\theta$]

$= \cos^{-1}(\cos 2\theta)$

$= 2\theta = 2\tan^{-1} x.$

$\therefore$ $2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right).$

(iii) $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$ [$\because$ $x = \tan\theta$]

$= \tan^{-1}(\tan 2\theta)$

$= 2\theta = 2\tan^{-1} x.$

$\therefore$ $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right).$

<u>THEOREM 9</u> *Prove that* $2\sin^{-1} x = \sin^{-1}[2x\sqrt{1-x^2}],\ \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}.$

PROOF Let $\sin^{-1} x = \theta$. Then, $x = \sin\theta$.

$\therefore$ $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\sin\theta \cdot \sqrt{(1-\sin^2\theta)}$

$= 2x\sqrt{1-x^2}$

$\Rightarrow$ $2\theta = \sin^{-1}\{2x\sqrt{1-x^2}\}$ $\Rightarrow$ $2\sin^{-1} x = \sin^{-1}\{2x\sqrt{1-x^2}\}.$

Hence, $2\sin^{-1} x = \sin^{-1}\{2x\sqrt{1-x^2}\}.$

<u>THEOREM 10</u> *Prove that* $2\cos^{-1} x = \cos^{-1}(2x^2-1),\ \frac{1}{\sqrt{2}} \le x \le 1.$

PROOF Let $\cos^{-1} x = \theta$. Then, $x = \cos\theta$.

$\therefore$ $\cos 2\theta = (2\cos^2\theta - 1) = (2x^2 - 1)$

$\Rightarrow$ $2\theta = \cos^{-1}(2x^2-1)$

$\Rightarrow$ $2\cos^{-1} x = \cos^{-1}(2x^2-1)$ [$\because$ $\theta = \cos^{-1} x$].

Hence, $2\cos^{-1} x = \cos^{-1}(2x^2-1).$

<u>THEOREM 11</u> *Prove that* $3\sin^{-1} x = \sin^{-1}(3x-4x^3),\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$

PROOF Put $\sin^{-1} x = \theta$. Then, $x = \sin\theta$.

Now, $\sin 3\theta = (3\sin\theta - 4\sin^3\theta) = (3x - 4x^3)$

$\Rightarrow$ $3\theta = \sin^{-1}(3x-4x^3)$ $\Rightarrow$ $3\sin^{-1} x = \sin^{-1}(3x-4x^3)$ [$\because$ $\theta = \sin^{-1} x$].

Hence, $3\sin^{-1} x = \sin^{-1}(3x-4x^3).$

THEOREM 12 *Prove that* $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x),\ x \in \left[\frac{1}{2}, 1\right]$.

PROOF Put $\cos^{-1}x = \theta$. Then, $x = \cos\theta$.

Now, $\cos 3\theta = (4\cos^3\theta - 3\cos\theta) = (4x^3 - 3x)$

$\Rightarrow 3\theta = \cos^{-1}(4x^3 - 3x) \Rightarrow 3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$ [$\because \theta = \cos^{-1}x$].

Hence, $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$.

THEOREM 13 *Prove that:*

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}$

(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\{xy - \sqrt{(1-x^2)(1-y^2)}\}$

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\{xy + \sqrt{(1-x^2)(1-y^2)}\}$

PROOF (i) Let $\sin^{-1}x = \theta_1$, and $\sin^{-1}y = \theta_2$. Then,

$\sin\theta_1 = x$ and $\sin\theta_2 = y$

$$\Rightarrow \sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

$$= \{\sin\theta_1 \cdot \sqrt{1-\sin^2\theta_2}\} + \{\sqrt{1-\sin^2\theta_1} \cdot \sin\theta_2\}$$

$$= \{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$$

$$\Rightarrow (\theta_1 + \theta_2) = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$$

$$\Rightarrow \sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}.$$

Hence, $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$.

The other results can be proved similarly.

THEOREM 14 *Prove that:*

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

(iii) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right) = \sec^{-1}(\sqrt{1+x^2}) = \cot^{-1}\left(\frac{1}{x}\right)$$

PROOF (i) Let $\sin^{-1} x = \theta$. Then, $\sin\theta = x$.

$$\therefore\quad \cos\theta = \sqrt{1-x^2},\quad \tan\theta = \frac{x}{\sqrt{1-x^2}},\quad \operatorname{cosec}\theta = \frac{1}{x},$$

$$\sec\theta = \frac{1}{\sqrt{1-x^2}} \text{ and } \cot\theta = \frac{\sqrt{1-x^2}}{x}.$$

$$\therefore\quad \theta = \cos^{-1}\sqrt{1-x^2},\quad \theta = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right),\quad \theta = \operatorname{cosec}^{-1}\frac{1}{x},$$

$$\theta = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) \text{ and } \theta = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right).$$

Hence, $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$$= \operatorname{cosec}^{-1}\frac{1}{x} = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right).$$

Similarly, the other results can be proved.

SOLVED EXAMPLES

EXAMPLE 1 *Find the value of:*

(i) $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$ *(ii)* $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ *(iii)* $\tan^{-1}\left(\tan\frac{\pi}{4}\right)$

SOLUTION We have

(i) $\sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$, since $\frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

(ii) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$, since $\frac{2\pi}{3} \in [0, \pi]$.

(iii) $\tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$, since $\frac{\pi}{4} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

EXAMPLE 2 *Find the value of:*

(i) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ *(ii)* $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ *(iii)* $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

SOLUTION (i) We know that the principal-value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \sin^{-1}\left(\sin\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$.

$$\text{Now, } \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left\{\sin\left(\pi - \frac{\pi}{3}\right)\right\}$$

$$= \sin^{-1}\left(\sin\frac{\pi}{3}\right) \quad [\because \sin(\pi - \theta) = \sin\theta]$$

$$= \frac{\pi}{3} \quad \left\{\because \frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right\}.$$

Hence, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$.

(ii) We know that the principal-value branch of $\cos^{-1}$ is $[0, \pi]$.

$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$.

$$\text{Now, } \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left\{\cos\left(2\pi - \frac{5\pi}{6}\right)\right\}$$

$$= \cos^{-1}\left\{\cos\frac{5\pi}{6}\right\} \quad \{\because \cos(2\pi - \theta) = \cos\theta\}$$

$$= \frac{5\pi}{6}.$$

Hence, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{5\pi}{6}$.

(iii) We know that the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \tan^{-1}\left(\tan\frac{3\pi}{4}\right) \neq \frac{3\pi}{4}$.

$$\text{Now, } \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left\{\tan\left(\pi - \frac{\pi}{4}\right)\right\}$$

$$= \tan^{-1}\left\{-\tan\frac{\pi}{4}\right\} \quad \left[\because \tan\left(\pi - \frac{\pi}{4}\right) = -\tan\frac{\pi}{4}\right]$$

$$= \tan^{-1}\left\{\tan\left(\frac{-\pi}{4}\right)\right\} \quad [\because -\tan\theta = \tan(-\theta)]$$

$$= \frac{-\pi}{4}.$$

Hence, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \frac{-\pi}{4}$.

EXAMPLE 3 *Find the value of:*

(i) $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ *(ii)* $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ *(iii)* $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

SOLUTION (i) We know that the principal-value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \quad \sin^{-1}\left(\sin\frac{3\pi}{5}\right) \neq \frac{3\pi}{5}$.

$$\text{Now, } \sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\left\{\sin\left(\pi - \frac{2\pi}{5}\right)\right\}$$

$$= \sin^{-1}\left(\sin\frac{2\pi}{5}\right) \quad [\because \sin(\pi - \theta) = \sin\theta]$$

$$= \frac{2\pi}{5} \quad \left\{\because \frac{2\pi}{5} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right\}.$$

Hence, $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \frac{2\pi}{5}$.

(ii) We know that the principal-value branch of $\cos^{-1}$ is $[0, \pi]$.

$\therefore \quad \cos^{-1}\left(\cos\frac{13\pi}{6}\right) \neq \frac{13\pi}{6}$.

$$\text{Now, } \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left\{\cos\left(2\pi + \frac{\pi}{6}\right)\right\}$$

$$= \cos^{-1}\left\{\cos\frac{\pi}{6}\right\} \quad [\because \cos(2\pi + \theta) = \cos\theta]$$

$$= \frac{\pi}{6}.$$

Hence, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \frac{\pi}{6}$.

(iii) We know that the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \quad \tan^{-1}\left(\tan\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$.

$$\text{Now, } \tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left\{\tan\left(\pi + \frac{\pi}{6}\right)\right\}$$

$$= \tan^{-1}\left(\tan\frac{\pi}{6}\right) \quad [\because \tan(\pi + \theta) = \tan\theta]$$

$$= \frac{\pi}{6}.$$

Hence, $\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right)=\dfrac{\pi}{6}\cdot$

EXAMPLE 4 *(i) Show that* $\sin^{-1}\left\{\sin\dfrac{3\pi}{4}\right\}\neq\dfrac{3\pi}{4}$ *and find its value.*

(ii) Show that $\cos^{-1}\left\{\cos\left(\dfrac{-\pi}{3}\right)\right\}\neq\dfrac{-\pi}{3}$ *and find its value.*

(iii) Show that $\tan^{-1}\left\{\tan\dfrac{5\pi}{6}\right\}\neq\dfrac{5\pi}{6}$ *and find its value.*

SOLUTION (i) We know that the principal-value branch of $\sin^{-1}$ is $\left[\dfrac{-\pi}{2},\dfrac{\pi}{2}\right]$.

$\therefore\quad \sin^{-1}\left\{\sin\dfrac{3\pi}{4}\right\}\neq\dfrac{3\pi}{4}\cdot$

$$\text{Now, }\sin^{-1}\left\{\sin\frac{3\pi}{4}\right\}=\sin^{-1}\left\{\sin\left(\pi-\frac{\pi}{4}\right)\right\}$$

$$=\sin^{-1}\left\{\sin\frac{\pi}{4}\right\}\quad\left[\because\ \sin\left(\pi-\frac{\pi}{4}\right)=\sin\frac{\pi}{4}\right]$$

$$=\frac{\pi}{4}\quad\left\{\because\ \frac{\pi}{4}\in\left[\frac{-\pi}{2},\ \frac{\pi}{2}\right]\right\}.$$

$\therefore\quad \sin^{-1}\left\{\sin\dfrac{3\pi}{4}\right\}=\dfrac{\pi}{4}\cdot$

(ii) We know that the principal-value branch of $\cos^{-1}$ is $[0,\ \pi]$.

$\therefore\quad \cos^{-1}\left\{\cos\left(\dfrac{-\pi}{3}\right)\right\}\neq\dfrac{-\pi}{3}\cdot$

$$\text{Now, }\cos^{-1}\left\{\cos\left(\frac{-\pi}{3}\right)\right\}=\cos^{-1}\left\{\cos\frac{\pi}{3}\right\}\quad[\because \cos(-\theta)=\cos\theta]$$

$$=\frac{\pi}{3}\quad\left[\because\ \frac{\pi}{3}\in[0,\ \pi]\right]\cdot$$

$\therefore\quad \cos^{-1}\left\{\cos\left(\dfrac{-\pi}{3}\right)\right\}=\dfrac{\pi}{3}\cdot$

(iii) We know that the principal-value branch of $\tan^{-1}$ is $\left(\dfrac{-\pi}{2},\dfrac{\pi}{2}\right)$.

$\therefore\quad \tan^{-1}\left\{\tan\dfrac{5\pi}{6}\right\}\neq\dfrac{5\pi}{6}\cdot$

$$\text{Now, }\tan^{-1}\left\{\tan\frac{5\pi}{6}\right\}=\tan^{-1}\left\{\tan\left(\pi-\frac{\pi}{6}\right)\right\}$$

$$=\tan^{-1}\left\{\tan\left(\frac{-\pi}{6}\right)\right\}$$

$$[\because\ \tan(\pi-\theta)=\tan(-\theta)]$$

$$= \frac{-\pi}{6} \quad \left[\because \; \frac{-\pi}{6} \in \left(\frac{-\pi}{2}, \; \frac{\pi}{2}\right)\right].$$

$$\therefore \quad \tan^{-1}\left\{\tan \frac{5\pi}{6}\right\} = \frac{-\pi}{6} \cdot$$

EXAMPLE 5 *Evaluate:*

(i) $\sin\left(\cos^{-1} \frac{3}{5}\right)$ (ii) $\cos\left(\tan^{-1} \frac{3}{4}\right)$

SOLUTION (i) Let $\cos^{-1} \frac{3}{5} = \theta$, where $\theta \in [0, \; \pi]$.

Then, $\cos \theta = \frac{3}{5} \cdot$

Since $\theta \in [0, \; \pi]$, we have $\sin \theta > 0$.

$$\therefore \quad \sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \cdot$$

Hence, $\sin\left(\cos^{-1} \frac{3}{5}\right) = \sin \theta = \frac{4}{5} \cdot$

(ii) Let $\tan^{-1} \frac{3}{4} = \theta$, where $\theta \in \left(\frac{-\pi}{2}, \; \frac{\pi}{2}\right) \cdot$

Then, $\tan \theta = \frac{3}{4} \cdot$

Since $\theta \in \left(\frac{-\pi}{2}, \; \frac{\pi}{2}\right)$, so $\cos \theta > 0$.

$$\therefore \quad \cos \theta = \frac{1}{\sec \theta} = \frac{1}{\sqrt{1 + \tan^2 \theta}} = \frac{1}{\sqrt{1 + \frac{9}{16}}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \cdot$$

Hence, $\cos\left(\tan^{-1} \frac{3}{4}\right) = \cos \theta = \frac{4}{5} \cdot$

EXAMPLE 6 *Evaluate:*

(i) $\sin\left\{\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right\}$ **[CBSE 2008]** (ii) $\sin\left(\frac{1}{2} \cos^{-1} \frac{4}{5}\right)$

(iii) $\sin(\cot^{-1} x)$ (iv) $\tan \frac{1}{2}\left(\cos^{-1} \frac{\sqrt{5}}{3}\right)$

SOLUTION (i) We know that $\sin^{-1}(-\theta) = -\sin^{-1}\theta$.

$$\therefore \quad \sin\left\{\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right\} = \sin\left\{\frac{\pi}{3} + \sin^{-1} \frac{1}{2}\right\}$$

$$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) \left[\because \; \sin^{-1} \frac{1}{2} = \frac{\pi}{6}\right]$$

$$= \sin \frac{\pi}{2} = 1.$$

(ii) Let $\cos^{-1}\frac{4}{5} = \theta$, where $\theta \in [0, \pi]$.

Then, $\cos\theta = \frac{4}{5}\cdot$

Since $\theta \in [0, \pi] \Rightarrow \frac{1}{2}\theta \in \left[0, \frac{\pi}{2}\right] \Rightarrow \sin\frac{1}{2}\theta > 0$.

$$\therefore \sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right) = \sin\frac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(\frac{4}{5}\right)}{2}} = \frac{1}{\sqrt{10}}\cdot$$

(iii) Let $\cot^{-1}x = \theta$. Then, $\theta \in [0, \pi]$.

$\therefore \sin(\cot^{-1}x) = \sin\theta > 0$.

$$\text{Now, } \sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+x^2}}\cdot$$

$$\therefore \sin(\cot^{-1}x) = \sin\theta = \frac{1}{\sqrt{1+x^2}}\cdot$$

(iv) Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$. Then, $\cos\theta = \frac{\sqrt{5}}{3}$, where $\theta \in [0, \pi]$.

$$\therefore \tan\frac{1}{2}\left(\cos^{-1}\frac{\sqrt{5}}{3}\right) = \tan\frac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{(1-\sqrt{5}/3)}{(1+\sqrt{5}/3)}}$$

$$= \sqrt{\frac{(3-\sqrt{5})}{(3+\sqrt{5})}\times\frac{(3-\sqrt{5})}{(3-\sqrt{5})}} = \frac{(3-\sqrt{5})}{2}\cdot$$

EXAMPLE 7 *Evaluate* $\sin\left[2\cos^{-1}\left(\frac{-3}{5}\right)\right]\cdot$

SOLUTION Let $\cos^{-1}\left(\frac{-3}{5}\right) = \theta$, where $\theta \in [0, \pi]$.

Then, $\cos\theta = \frac{-3}{5}\cdot$

Since $\theta \in [0, \pi]$, we have $\sin\theta > 0$.

$$\therefore \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}\cdot$$

$$\therefore \sin\left[2\cos^{-1}\left(\frac{-3}{5}\right)\right] = \sin 2\theta$$

$$= 2\sin\theta\cos\theta = \left\{2\times\frac{4}{5}\times\left(\frac{-3}{5}\right)\right\} = \frac{-24}{25}\cdot$$

EXAMPLE 8 *Evaluate* $\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)\cdot$

SOLUTION Let $\sin^{-1}\frac{3}{5}=A$ and $\sin^{-1}\frac{5}{13}=B$. Then,

$$A, B \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \cos A > 0 \text{ and } \cos B > 0.$$

$$\therefore \sin A = \frac{3}{5} \text{ and } \sin B = \frac{5}{13}$$

$$\Rightarrow \cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\text{and } \cos B = \sqrt{1-\sin^2 B} = \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}\cdot$$

$$\therefore \cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right) = \cos(A+B)$$

$$= \cos A \cos B - \sin A \sin B$$

$$= \left(\frac{4}{5}\times\frac{12}{13}\right)-\left(\frac{3}{5}\times\frac{5}{13}\right)$$

$$= \left(\frac{48}{65}-\frac{15}{65}\right) = \frac{33}{65}\cdot$$

EXAMPLE 9 *Find the value of* $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\}\cdot$

SOLUTION We have

$$\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\} = \tan^{-1}\left\{2\cos\left(2\times\frac{\pi}{6}\right)\right\} \quad \left[\because \sin^{-1}\frac{1}{2}=\frac{\pi}{6}\right]$$

$$= \tan^{-1}\left\{2\cos\frac{\pi}{3}\right\}$$

$$= \tan^{-1}\left(2\times\frac{1}{2}\right) = \tan^{-1}1 = \frac{\pi}{4}\cdot$$

EXAMPLE 10 *If* $\tan^{-1}\frac{4}{3}=\theta$, *find the value of* $\cos\theta$.

SOLUTION $\tan^{-1}\frac{4}{3}=\theta$, where $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\cdot$

$$\therefore \tan\theta = \frac{4}{3}\cdot$$

We know that $\cos\theta > 0$, when $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\cdot$

$$\therefore \quad \cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+\frac{16}{9}}} = \frac{3}{5}\cdot$$

EXAMPLE 11 *If* $\cot^{-1}\left(-\frac{1}{5}\right) = \theta$, *find the value of* $\sin\theta$.

SOLUTION Given: $\cot^{-1}\left(-\frac{1}{5}\right) = \theta$, where $\theta \in (0, \pi)$.

$$\therefore \quad \cot\theta = \frac{-1}{5}\cdot$$

$\sin\theta > 0$ in $(0, \pi)$.

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+\frac{1}{25}}} = \frac{5}{\sqrt{26}}\cdot$$

EXAMPLE 12 *Prove that* $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\frac{2}{9}\cdot$

SOLUTION We know that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $xy < 1$.

Let $x = \frac{1}{7}$ and $y = \frac{1}{13}\cdot$ Then, $xy = \frac{1}{91} < 1$.

$$\therefore \quad \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\left\{\frac{\left(\frac{1}{7}+\frac{1}{13}\right)}{1-\frac{1}{7}\times\frac{1}{13}}\right\} = \tan^{-1}\left(\frac{20}{90}\right) = \tan^{-1}\frac{2}{9}\cdot$$

EXAMPLE 13 *Prove that* $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}\cdot$ **[CBSE 2009, '12C]**

SOLUTION We have

$$\text{LHS} = \left\{\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5}\right\} - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left\{\frac{\left(\frac{3}{4}+\frac{3}{5}\right)}{\left(1-\frac{3}{4}\times\frac{3}{5}\right)}\right\} - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left(\frac{27}{11}\right) - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1} \frac{\left(\frac{27}{11} - \frac{8}{19}\right)}{\left(1 + \frac{27}{11} \times \frac{8}{19}\right)} = \tan^{-1}\left(\frac{425}{425}\right) = \tan^{-1} 1 = \frac{\pi}{4} = \text{RHS.}$$

$\therefore$ LHS = RHS.

EXAMPLE 14 *Prove that* $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}\cdot$

[CBSE 2008, '09C]

SOLUTION We have

$$\text{LHS} = \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5}\right) + \left(\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}\right)$$

$$= \tan^{-1}\frac{\left(\frac{1}{3} + \frac{1}{5}\right)}{\left(1 - \frac{1}{3} \times \frac{1}{5}\right)} + \tan^{-1}\frac{\left(\frac{1}{7} + \frac{1}{8}\right)}{\left(1 - \frac{1}{7} \times \frac{1}{8}\right)}$$

$$= \tan^{-1}\frac{\left(\frac{8}{15}\right)}{\left(\frac{14}{15}\right)} + \tan^{-1}\frac{\left(\frac{15}{56}\right)}{\left(\frac{55}{56}\right)}$$

$$= \tan^{-1}\frac{8}{14} + \tan^{-1}\frac{15}{55} = \tan^{-1}\frac{4}{7} + \tan^{-1}\frac{3}{11}$$

$$= \tan^{-1}\frac{\left(\frac{4}{7} + \frac{3}{11}\right)}{\left(1 - \frac{4}{7} \times \frac{3}{11}\right)} = \tan^{-1}\frac{\left(\frac{65}{77}\right)}{\left(\frac{65}{77}\right)} = \tan^{-1} 1 = \frac{\pi}{4} = \text{RHS.}$$

$\therefore$ LHS = RHS.

EXAMPLE 15 *Prove that* $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}\cdot$

SOLUTION We have

$$\text{LHS} = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\left(2 \times \frac{1}{2}\right)}{\left\{1 - \left(\frac{1}{2}\right)^2\right\}} + \tan^{-1}\frac{1}{7} \quad \left[\because\ 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)\right]$$

$$= \tan^{-1}\frac{1}{\left(\frac{3}{4}\right)} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1} \frac{\left(\frac{4}{3} + \frac{1}{7}\right)}{\left(1 - \frac{4}{3} \times \frac{1}{7}\right)} = \tan^{-1} \frac{\left(\frac{31}{21}\right)}{\left(\frac{17}{21}\right)} = \tan^{-1} \frac{31}{17} = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 16 *Prove that* $2\left(\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}\right) = \tan^{-1}\frac{4}{3}$.

SOLUTION We have

$$\text{LHS} = 2\tan^{-1}\frac{1}{4} + 2\tan^{-1}\frac{2}{9}$$

$$= \tan^{-1} \frac{\left(2 \times \frac{1}{4}\right)}{\left\{1 - \left(\frac{1}{4}\right)^2\right\}} + \tan^{-1} \frac{\left(2 \times \frac{2}{9}\right)}{\left\{1 - \left(\frac{2}{9}\right)^2\right\}}$$

$$= \tan^{-1} \frac{\left(\frac{1}{2}\right)}{\left(\frac{15}{16}\right)} + \tan^{-1} \frac{\left(\frac{4}{9}\right)}{\left(\frac{77}{81}\right)}$$

$$= \tan^{-1}\left(\frac{1}{2} \times \frac{16}{15}\right) + \tan^{-1}\left(\frac{4}{9} \times \frac{81}{77}\right)$$

$$= \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{36}{77}$$

$$= \tan^{-1} \frac{\left(\frac{8}{15} + \frac{36}{77}\right)}{\left(1 - \frac{8}{15} \times \frac{36}{77}\right)} = \tan^{-1} \frac{(616 + 540)}{(1155 - 288)}$$

$$= \tan^{-1}\left(\frac{1156}{867}\right) = \tan^{-1}\frac{4}{3} = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 17 *Prove that* $\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18 = \cot^{-1} 3$. **[CBSE 2008C, '14]**

SOLUTION We have

$$\text{LHS} = \cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18$$

$$= \left(\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}\right) + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1} \frac{\left(\frac{1}{7}+\frac{1}{8}\right)}{\left(1-\frac{1}{7}\times\frac{1}{8}\right)} + \tan^{-1}\frac{1}{18} = \tan^{-1}\frac{\left(\frac{15}{56}\right)}{\left(\frac{55}{56}\right)} + \tan^{-1}\frac{1}{18}$$

$$= \left(\tan^{-1}\frac{3}{11} + \tan^{-1}\frac{1}{18}\right) = \tan^{-1}\frac{\left(\frac{3}{11}+\frac{1}{18}\right)}{\left(1-\frac{3}{11}\times\frac{1}{18}\right)}$$

$$= \tan^{-1}\frac{\left(\frac{65}{198}\right)}{\left(\frac{195}{198}\right)} = \tan^{-1}\frac{1}{3} = \cot^{-1} 3 = \text{RHS.}$$

$\therefore$ LHS = RHS.

EXAMPLE 18 *Prove that* $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}\cdot$ **[CBSE 2012]**

SOLUTION Let $\sin^{-1}\frac{3}{5} = x$ and $\sin^{-1}\frac{8}{17} = y$. Then,

$$\sin x = \frac{3}{5} \text{ and } \sin y = \frac{8}{17}\cdot$$

$$\therefore \quad \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\text{and } \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}\cdot$$

$$\therefore \quad \cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$= \left(\frac{4}{5}\times\frac{15}{17}\right) + \left(\frac{3}{5}\times\frac{8}{17}\right) = \left(\frac{12}{17}+\frac{24}{85}\right) = \frac{84}{85}$$

$$\Rightarrow \quad x - y = \cos^{-1}\left(\frac{84}{85}\right) \quad \Rightarrow \quad \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}\cdot$$

EXAMPLE 19 *Prove that* $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5}\cdot$ **[CBSE 2011C]**

SOLUTION LHS $= \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}$, where $xy = \left(\frac{1}{4}\times\frac{2}{9}\right) = \frac{1}{18} < 1$

$$= \tan^{-1}\left\{\frac{\left(\frac{1}{4}+\frac{2}{9}\right)}{\left(1-\frac{1}{4}\times\frac{2}{9}\right)}\right\} = \tan^{-1}\left(\frac{17}{34}\right) = \tan^{-1}\frac{1}{2}\cdot$$

Now, let $\frac{1}{2}\cos^{-1}\frac{3}{5} = \theta$. Then, $\cos 2\theta = \frac{3}{5}\cdot$

$$\therefore \quad \tan\theta = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \sqrt{\frac{\left(1-\frac{3}{5}\right)}{\left(1+\frac{3}{5}\right)}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\Rightarrow \tan^{-1}\frac{1}{2} = \theta \Rightarrow \tan^{-1}\frac{1}{2} = \frac{1}{2}\cos^{-1}\frac{3}{5}.$$

Hence, $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5}.$

EXAMPLE 20 *Prove that* $\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{27}{11}.$

SOLUTION Let $\cos^{-1}\frac{4}{5} = \theta$. Then, $\cos\theta = \frac{4}{5}.$

$$\therefore \quad \tan\theta = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-\frac{16}{25}}}{(4/5)} = \left(\frac{3}{5}\times\frac{5}{4}\right) = \frac{3}{4}$$

$$\Rightarrow \theta = \tan^{-1}\frac{3}{4}.$$

$$\therefore \quad \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}.$$

$$\therefore \quad \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5}$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right) = \tan^{-1}\frac{27}{11}.$$

EXAMPLE 21 *Solve* $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}.$ **[CBSE 2008, '09, '12C, '13C]**

SOLUTION $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right) = \frac{\pi}{4} \Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{(1-6x^2)} = \tan\frac{\pi}{4} = 1 \Rightarrow 1-6x^2 = 5x$$

$$\Rightarrow 6x^2+5x-1=0 \Rightarrow 6x^2+6x-x-1=0$$

$$\Rightarrow 6x(x+1)-(x+1)=0 \Rightarrow (x+1)(6x-1)=0$$

$$\Rightarrow x=-1 \quad\text{or}\quad x=\frac{1}{6}.$$

$$\Rightarrow x=\frac{1}{6} \qquad [\because\ x=-1 \text{ makes LHS negative}].$$

EXAMPLE 22 *Solve* $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}\cdot$ **[CBSE 2008, '12C]**

SOLUTION We have

$$\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{\left\{\frac{(x-1)}{(x-2)}+\frac{(x+1)}{(x+2)}\right\}}{\left\{1-\frac{(x-1)}{(x-2)}\cdot\frac{(x+1)}{(x+2)}\right\}}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left\{\frac{(x-1)(x+2)+(x+1)(x-2)}{(x^2-4)-(x^2-1)}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left\{\frac{(x^2+x-2)+(x^2-x-2)}{-3}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \frac{2x^2-4}{-3}=\tan\frac{\pi}{4}=1 \Rightarrow 2x^2-4=-3$$

$$\Rightarrow 2x^2=1 \Rightarrow x^2=\frac{1}{2} \Rightarrow x=\pm\frac{1}{\sqrt{2}}\cdot$$

EXAMPLE 23 *Solve* $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}\cdot$ **[CBSE 2008C]**

SOLUTION We have

$$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$$

$$\Rightarrow \tan^{-1}\left\{\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right\}=\tan^{-1}\frac{8}{31}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{2-x^2}\right)=\tan^{-1}\frac{8}{31}$$

$$\Rightarrow \tan\left\{\tan^{-1}\left(\frac{2x}{2-x^2}\right)\right\}=\frac{8}{31} \Rightarrow \frac{2x}{2-x^2}=\frac{8}{31}$$

$$\Rightarrow 8x^2+62x-16=0 \Rightarrow 4x^2+31x-8=0$$

$$\Rightarrow (4x-1)(x+8)=0 \Rightarrow x=\frac{1}{4} \text{ or } x=-8.$$

But, $x=-8$ gives LHS $=\tan^{-1}(-7)+\tan^{-1}(-9)$, which is negative, while RHS is positive. So, $x=-8$ is not possible.

Hence, $x=\frac{1}{4}\cdot$

EXAMPLE 24 *Solve* $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x,\ (x>0).$ **[CBSE 2008C, '11, '14C]**

SOLUTION We have

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x,\ (x>0)$$

$$\Rightarrow\ \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow\ \frac{3}{2}\tan^{-1}x=\tan^{-1}1=\frac{\pi}{4}$$

$$\Rightarrow\ \tan^{-1}x=\left(\frac{\pi}{4}\times\frac{2}{3}\right)=\frac{\pi}{6}\ \Rightarrow\ x=\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}\cdot$$

Hence, $x=\frac{1}{\sqrt{3}}\cdot$

EXAMPLE 25 *Solve* $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec}x).$

SOLUTION We have

$$2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec}x)$$

$$\Rightarrow\ \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)=\tan^{-1}(2\operatorname{cosec}x)$$

$$\Rightarrow\ \tan\left[\tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right)\right]=2\operatorname{cosec}x$$

$$\Rightarrow\ \frac{2\cos x}{\sin^2 x}=2\operatorname{cosec}x\ \Rightarrow\ \cos x=\sin x$$

$$\Rightarrow\ \tan x=1\ \Rightarrow\ x=\frac{\pi}{4}\cdot$$

PROBLEMS BASED ON TRIGONOMETRIC FORMULAE

EXAMPLE 26 *Prove that* $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\frac{x}{2},\ \ x<\pi.$

SOLUTION We have

$$\text{LHS}=\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\tan^{-1}\sqrt{\frac{2\sin^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}}$$

$$=\tan^{-1}\left(\tan\frac{x}{2}\right)=\frac{x}{2}=\text{RHS}.$$

$$\therefore\ \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\frac{x}{2}\cdot$$

EXAMPLE 27 *Prove that* $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right) = \left(\dfrac{\pi}{4} - x\right), \; x < \pi.$

SOLUTION We have

$$\text{LHS} = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$$

$$= \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right) \text{ [dividing num. and denom. by } \cos x\text{]}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - x\right)\right\} = \left(\frac{\pi}{4} - x\right) = \text{RHS.}$$

$$\therefore \quad \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \left(\frac{\pi}{4} - x\right).$$

EXAMPLE 28 *Prove that* $\tan^{-1}\left(\dfrac{\cos x}{1 + \sin x}\right) = \left(\dfrac{\pi}{4} - \dfrac{x}{2}\right).$ **[CBSE 2012]**

SOLUTION We have

$$\text{LHS} = \tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$$

$$= \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\} = \left(\frac{\pi}{4} - \frac{x}{2}\right) = \text{RHS.}$$

$$\therefore \quad \tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right) = \left(\frac{\pi}{4} - \frac{x}{2}\right).$$

EXAMPLE 29 *Prove that* $\tan^{-1}\left(\dfrac{\cos x}{1 - \sin x}\right) = \left(\dfrac{\pi}{4} + \dfrac{x}{2}\right), \; x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right).$ **[CBSE 2012]**

SOLUTION We have

$$\text{LHS} = \tan^{-1}\left(\frac{\cos x}{1 - \sin x}\right)$$

$$= \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \tan^{-1}\left[\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}\right]$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \left(\frac{\pi}{4}+\frac{x}{2}\right) = \text{RHS}.$$

Hence, $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \left(\frac{\pi}{4}+\frac{x}{2}\right)$.

EXAMPLE 30 *Prove that* $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right) = 3\tan^{-1}\frac{x}{a}$.

SOLUTION Putting $x = a\tan\theta$, we get

$$\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right) = \tan^{-1}\left(\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3 - 3a^3\tan^2\theta}\right)$$

$$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta)$$

$$= 3\theta = 3\tan^{-1}\frac{x}{a}.$$

Hence, $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right) = 3\tan^{-1}\frac{x}{a}$.

EXAMPLE 31 *Prove that* $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x - a\sin x}\right) = \tan^{-1}\left(\frac{a}{b}\right) - x$.

SOLUTION We have

$$\text{LHS} = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x - a\sin x}\right) = \tan^{-1}\left\{\frac{\frac{a\cos x - b\sin x}{b\cos x}}{\frac{b\cos x - a\sin x}{b\cos x}}\right\}$$

[on dividing num. and denom. by $b\cos x$]

$$= \tan^{-1}\left\{\frac{\frac{a}{b} - \tan x}{1 - \frac{a}{b}\tan x}\right\} = \tan^{-1}\left(\frac{p-q}{1-pq}\right), \text{ where } \frac{a}{b} = p \text{ and } \tan x = q$$

$$= \tan^{-1} p - \tan^{-1} q = \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x)$$

$$= \left(\tan^{-1}\frac{a}{b} - x\right) = \text{RHS}.$$

Hence, $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x - a\sin x}\right) = \left(\tan^{-1}\frac{a}{b} - x\right)$.

EXAMPLE 32 *Prove that* $\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\} = \frac{x}{2},\ x \in \left(0, \frac{\pi}{4}\right)$.

[CBSE 2011, '14]

SOLUTION We have

$$\text{LHS} = \cot^{-1}\left\{\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})} \times \frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}\right\}$$

$$= \cot^{-1}\left\{\frac{(1+\sin x)+(1-\sin x)+2\sqrt{1-\sin^2 x}}{(1+\sin x)-(1-\sin x)}\right\}$$

$$= \cot^{-1}\left\{\frac{2(1+\cos x)}{2\sin x}\right\} = \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \cot^{-1}\left\{\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right\} = \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} = \text{RHS}.$$

Hence, LHS = RHS.

EXAMPLE 33 *Prove that* $\tan^{-1}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$. **[CBSE 2011, '14]**

SOLUTION Putting $x = \cos\theta$, we get

$$\text{LHS} = \tan^{-1}\left\{\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)}-\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)}+\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}\right\} = \tan^{-1}\left\{\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right\} \qquad \left[\text{dividing num. and denom. by } \cos\frac{\theta}{2}\right]$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\} = \left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= \left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right) = \text{RHS.}$$

Hence, LHS = RHS.

EXAMPLE 34 *Prove that* $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\tan^{-1}x.$

SOLUTION Putting $x = \tan\theta$, we get

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left\{\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\right\} = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x.$$

$$\therefore \quad \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\tan^{-1}x.$$

EXAMPLE 35 *Prove that* $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \sin^{-1}\frac{x}{a}.$

SOLUTION Putting $x = a\sin\theta$, we get

$$\text{LHS} = \tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right) = \tan^{-1}(\tan\theta)$$

$$= \theta = \sin^{-1}\frac{x}{a} = \text{RHS}.$$

$$\therefore\quad \tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \sin^{-1}\frac{x}{a}\cdot$$

<u>EXAMPLE 36</u> *Prove that* $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),\ x \in [0, 1]\cdot$ **[CBSE 2010]**

SOLUTION Putting $x = \tan^2\theta$, we get

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \frac{1}{2}\cos^{-1}(\cos 2\theta)$$

$$= \left(\frac{1}{2}\times 2\theta\right) = \theta = \tan^{-1}\sqrt{x} = \text{LHS}$$

$$[\because\quad x = \tan^2\theta \Rightarrow \tan\theta = \sqrt{x} \Rightarrow \theta = \tan^{-1}\sqrt{x}].$$

Hence, $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$.

<u>EXAMPLE 37</u> *Prove that* $\tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \left(\frac{\pi}{2} - \sec^{-1}x\right)\cdot$

SOLUTION Putting $x = \sec\theta$, we get

$$\tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{\sec^2\theta-1}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\tan\theta}\right) = \tan^{-1}(\cot\theta)$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\theta\right)\right\} = \left(\frac{\pi}{2}-\theta\right) = \left(\frac{\pi}{2}-\sec^{-1}x\right)\cdot$$

$$\therefore\quad \tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \left(\frac{\pi}{2} - \sec^{-1}x\right)\cdot$$

EXAMPLE 38 *Prove that* $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2.$

SOLUTION Putting $x^2=\cos 2\theta$, we get

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

[dividing num. and denom. by $\cos\theta$]

$$=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\theta\right)\right\}=\left(\frac{\pi}{4}+\theta\right)$$

$$=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2.$$

$$\therefore\ \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2.$$

EXAMPLE 39 *Prove that* $2\tan^{-1}\frac{1}{x}=\sin^{-1}\left(\frac{2x}{x^2+1}\right).$

SOLUTION Let $\tan^{-1}\frac{1}{x}=\theta$. Then, $\frac{1}{x}=\tan\theta \Rightarrow x=\cot\theta$.

$$\therefore\ \text{LHS}=2\tan^{-1}\frac{1}{x}=2\theta.$$

$$\text{RHS}=\sin^{-1}\left(\frac{2\cot\theta}{\cot^2\theta+1}\right)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$=\sin^{-1}(\sin 2\theta)=2\theta.$$

$\therefore$ LHS = RHS.

Hence, $2\tan^{-1}\frac{1}{x}=\sin^{-1}\left(\frac{2x}{x^2+1}\right).$

EXAMPLE 40 *Prove that* $\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1-x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}=\left(\frac{x+y}{1-xy}\right),$ *where* $|x|<1,\ y>0$ *and* $xy<1$. **[CBSE 2013]**

SOLUTION Putting $x = \tan\theta$ and $y = \tan\phi$, we get

$$\text{LHS} = \tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1-x^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}$$

$$= \tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)\right\}$$

$$= \tan\left\{\frac{1}{2}\sin^{-1}(\sin 2\theta) + \frac{1}{2}\cos^{-1}(\cos 2\phi)\right\}$$

$$= \tan\left\{\left(\frac{1}{2}\times 2\theta\right) + \left(\frac{1}{2}\times 2\phi\right)\right\}$$

$$= \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} = \frac{(x+y)}{(1-xy)} = \text{RHS}.$$

$$\therefore\quad \tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1-x^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\} = \frac{x+y}{1-xy}.$$

EXAMPLE 41 *Prove that* $\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right) = 0.$

SOLUTION We have

$$\text{LHS} = \tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \tan^{-1}\left(\frac{c-a}{1+ca}\right)$$

$$= (\tan^{-1}a - \tan^{-1}b) + (\tan^{-1}b - \tan^{-1}c) + (\tan^{-1}c - \tan^{-1}a)$$

$$= 0 = \text{RHS}.$$

$\therefore$ LHS = RHS.

SOME MORE EXAMPLES

EXAMPLE 1 *Prove that* $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}.$ **[CBSE 2010, '12]**

SOLUTION We have

$$\text{LHS} = \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13}$$

$$= \cos^{-1}\left\{\left(\frac{4}{5}\times\frac{12}{13}\right) - \sqrt{1-\left(\frac{4}{5}\right)^2}\cdot\sqrt{1-\left(\frac{12}{13}\right)^2}\right\}$$

$$= \cos^{-1}\left\{\frac{48}{65} - \sqrt{1-\left(\frac{16}{25}\right)}\cdot\sqrt{1-\frac{144}{169}}\right\}$$

$$= \cos^{-1}\left\{\frac{48}{65} - \sqrt{\frac{9}{25}} \cdot \sqrt{\frac{25}{169}}\right\}$$

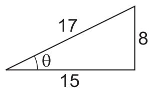
$$= \cos^{-1}\left\{\frac{48}{65} - \frac{3}{5} \times \frac{5}{13}\right\} = \cos^{-1}\left(\frac{48}{65} - \frac{15}{65}\right)$$

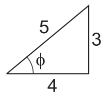
$$= \cos^{-1}\left(\frac{33}{65}\right) = \text{RHS.}$$

Hence, $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$.

EXAMPLE 2 *Prove that* $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{36}{85}$. **[CBSE 2014C]**

SOLUTION $\text{LHS} = \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$





$$= \cos^{-1}\frac{15}{17} + \cos^{-1}\frac{4}{5}$$

$$= \cos^{-1}\left\{\left(\frac{15}{17} \times \frac{4}{5}\right) - \sqrt{1 - \left(\frac{15}{17}\right)^2} \cdot \sqrt{1 - \left(\frac{4}{5}\right)^2}\right\}$$

$$= \cos^{-1}\left\{\frac{12}{17} - \sqrt{1 - \left(\frac{225}{289}\right)} \cdot \sqrt{1 - \frac{16}{25}}\right\}$$

$$= \cos^{-1}\left\{\frac{12}{17} - \sqrt{\frac{64}{289}} \cdot \sqrt{\frac{9}{25}}\right\} = \cos^{-1}\left\{\frac{12}{17} - \frac{8}{17} \times \frac{3}{5}\right\}$$

$$= \cos^{-1}\left\{\frac{12}{17} - \frac{24}{85}\right\} = \cos^{-1}\left(\frac{36}{85}\right) = \text{RHS.}$$

Hence, $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{36}{85}$.

EXAMPLE 3 *Prove that* $2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$. **[CBSE 2011C]**

SOLUTION We have

$$\text{LHS} = \tan^{-1}\frac{3}{4} - \tan\frac{17}{31}$$

$$= \tan^{-1}\left\{\frac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2}\right\} - \tan^{-1}\frac{17}{31}$$

$$= \tan^{-1}\frac{24}{7} - \tan^{-1}\frac{17}{31} = \tan^{-1}\left\{\frac{\left(\frac{24}{7} - \frac{17}{31}\right)}{1 + \left(\frac{24}{7} \times \frac{17}{31}\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{625}{217} \times \frac{217}{625}\right\} = \tan^{-1} 1 = \frac{\pi}{4} = \text{RHS}.$$

Hence, $2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}\cdot$

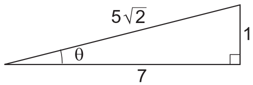
EXAMPLE 4 *Prove that* $2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{5\sqrt{2}}{7} + 2\tan^{-1}\frac{1}{8} = \frac{\pi}{4}\cdot$ **[CBSE 2014]**

SOLUTION Let $\sec^{-1}\frac{5\sqrt{2}}{7} = \theta$. Then, $\sec\theta = \frac{5\sqrt{2}}{7}\cdot$ So, $\cos\theta = \frac{7}{5\sqrt{2}}\cdot$

Draw a right triangle with base = 7 units

and hypotenuse = $5\sqrt{2}$ units.

$\therefore$ perpendicular = $\sqrt{(5\sqrt{2})^2 - 7^2} = \sqrt{50 - 49} = 1$ unit.



$\therefore$ $\tan\theta = \frac{1}{7} \Rightarrow \theta = \tan^{-1}\frac{1}{7}\cdot$

$\therefore$ $\text{LHS} = 2\left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right) + \tan^{-1}\frac{1}{7}$

$$= 2\cdot\tan^{-1}\left\{\frac{\left(\frac{1}{5} + \frac{1}{8}\right)}{\left(1 - \frac{1}{5} \times \frac{1}{8}\right)}\right\} + \tan^{-1}\frac{1}{7}$$

$$= 2\tan^{-1}\left(\frac{13}{40} \times \frac{40}{30}\right) + \tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left\{\frac{2 \times \frac{1}{3}}{1 - \left(\frac{1}{3}\right)^2}\right\} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\left(\frac{3}{4} + \frac{1}{7}\right)}{\left(1 - \frac{3}{4} \times \frac{1}{7}\right)} = \tan^{-1}\frac{25}{25} = \tan^{-1} 1$$

$$= \frac{\pi}{4} = \text{RHS}.$$

Hence, LHS = RHS.

EXAMPLE 5 *Prove that* $\cos^{-1} x + \cos^{-1}\left(\frac{x}{2} + \frac{1}{2}\sqrt{3 - 3x^2}\right) = \frac{\pi}{3}\cdot$ **[CBSE 2014C]**

SOLUTION Let $\cos^{-1} x = \theta$. Then, $x = \cos\theta$ and $\sqrt{1 - x^2} = \sin\theta$.

$\therefore$ LHS $= \cos^{-1} x + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1 - x^2}\right)$

$$= \theta + \cos^{-1}\left(\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta\right)$$

$$= \theta + \cos^{-1}\left\{\cos\left(\frac{\pi}{3} - \theta\right)\right\} = \theta + \left(\frac{\pi}{3} - \theta\right) = \frac{\pi}{3} = \text{RHS}.$$

Hence, LHS = RHS.

EXAMPLE 6 *Prove that* $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \sqrt{\frac{x^2 + 1}{x^2 + 2}}\cdot$ **[CBSE 2010]**

SOLUTION Let $\cot^{-1} x = \theta$. Then, $x = \cot\theta$.

$\therefore$ $\operatorname{cosec}\theta = \sqrt{1 + \cot^2\theta} = \sqrt{1 + x^2}$

$\Rightarrow$ $\sin\theta = \frac{1}{\sqrt{1 + x^2}}$

$\Rightarrow$ $\sin(\cot^{-1} x) = \frac{1}{\sqrt{1 + x^2}}$ $[\because\ \theta = \cot^{-1} x]$

$\Rightarrow$ $\tan^{-1}\{\sin(\cot^{-1} x)\} = \tan^{-1}\frac{1}{\sqrt{1 + x^2}} = \phi$ (say)

$\Rightarrow$ $\cos[\tan^{-1}\{\sin(\cot^{-1} x\}] = \cos\phi$. ... (i)

Now, $\tan^{-1}\frac{1}{\sqrt{1 + x^2}} = \phi \Rightarrow \tan\phi = \frac{1}{\sqrt{1 + x^2}}$

$\Rightarrow$ $\sec\phi = \sqrt{1 + \tan^2\phi} = \sqrt{1 + \frac{1}{(1 + x^2)}} = \sqrt{\frac{2 + x^2}{1 + x^2}}$

$\Rightarrow$ $\cos\phi = \sqrt{\frac{1 + x^2}{2 + x^2}}\cdot$ ... (ii)

From (i) and (ii), we get

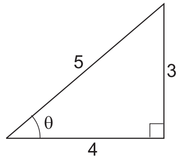
$$\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \sqrt{\frac{x^2 + 1}{x^2 + 2}}\cdot$$

EXAMPLE 7 *Prove that* $\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}\cdot$ **[CBSE 2012]**

SOLUTION Let $\sin^{-1}\frac{3}{5} = \theta$. Then, $\sin\theta = \frac{3}{5}\cdot$

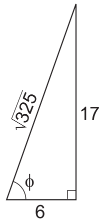
Draw a right triangle in which

perp. = 3 units,
hyp. = 5 units and
base = $\sqrt{(5)^2-(3)^2} = \sqrt{16} = 4$ units.



$\therefore \quad \tan\theta = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\frac{3}{4}$.

$\therefore \quad \cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \cos\left(\theta + \tan^{-1}\frac{2}{3}\right)$



$= \cos\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right) = \cos\left[\tan^{-1}\left\{\dfrac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right\}\right]$

$= \cos\left(\tan^{-1}\frac{17}{6}\right) = \cos\left\{\cos^{-1}\frac{6}{\sqrt{325}}\right\} = \frac{6}{5\sqrt{13}}$

$\left[\because \tan\phi = \frac{17}{6} \Rightarrow \cos\phi = \frac{6}{\sqrt{325}} = \frac{6}{5\sqrt{13}} \Rightarrow \phi = \cos^{-1}\frac{6}{5\sqrt{13}} \Rightarrow \cos\phi = \frac{6}{5\sqrt{13}}\right]$.

EXAMPLE 8 *Evaluate:*

(i) $\tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\}$ (ii) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$

SOLUTION (i) Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$. Then, $\cos\theta = \frac{\sqrt{5}}{3}$.

$\therefore \quad \tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\} = \tan\frac{\theta}{2}$

$$= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\dfrac{\left(1-\frac{\sqrt{5}}{3}\right)}{\left(1+\frac{\sqrt{5}}{3}\right)}} = \sqrt{\frac{(3-\sqrt{5})}{(3+\sqrt{5})}}$$

$$= \sqrt{\frac{(3-\sqrt{5})}{(3+\sqrt{5})}\times\frac{(3-\sqrt{5})}{(3+\sqrt{5})}} = \frac{(3-\sqrt{5})}{\sqrt{9-5}}$$

$$= \frac{3-\sqrt{5}}{\sqrt{4}} = \frac{(3-\sqrt{5})}{2}.$$

(ii) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$

$$= \tan\left\{\tan^{-1}\left(\dfrac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right) - \tan^{-1}1\right\}$$

$$= \tan\left\{\tan^{-1}\left(\frac{2}{5}\times\frac{25}{24}\right) - \tan^{-1} 1\right\}$$

$$= \tan\left\{\tan^{-1}\frac{5}{12} - \tan^{-1} 1\right\}$$

$$= \tan\left\{\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)\right\} = \tan\left\{\tan^{-1}\left(\frac{-7}{17}\right)\right\} = \frac{-7}{17}.$$

EXAMPLE 9 *Prove that* $\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}.$

SOLUTION Let $\cos^{-1}\frac{a}{b} = \theta$. Then, $\frac{a}{b} = \cos\theta$.

$$\therefore \quad \text{LHS} = \tan\left\{\frac{\pi}{4}+\frac{1}{2}\theta\right\} = \tan\left\{\frac{\pi}{4}-\frac{1}{2}\theta\right\}$$

$$= \frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)} + \frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}$$

$$= \frac{\left\{1+\tan\left(\frac{\theta}{2}\right)\right\}^2 + \left\{1-\tan\left(\frac{\theta}{2}\right)\right\}^2}{\left\{1-\tan^2\left(\frac{\theta}{2}\right)\right\}}$$

$$= 2\left\{\frac{1+\tan^2\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)}\right\} = \frac{2}{\cos\theta} = \frac{2b}{a} = \text{RHS}.$$

Hence, LHS = RHS.

EXAMPLE 10 *Prove that* $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}.$ **[CBSE 2011]**

SOLUTION We have

$$\text{LHS} = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}$$

$$= \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right)$$

$$= \frac{9}{4}\cos^{-1}\frac{1}{3} = \frac{9}{4}\cdot\sin^{-1}\sqrt{1-\frac{1}{9}}$$

$$= \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \text{RHS}.$$

Hence, LHS = RHS.

EXAMPLE 11 *Find the value of* $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)\cdot$ **[CBSE 2011]**

SOLUTION We have

$$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left\{\frac{\frac{x}{y}-1}{\frac{x}{y}+1}\right\}$$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \left\{\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}(1)\right\}$$

$$= \tan^{-1}(1) = \frac{\pi}{4}\cdot$$

EXAMPLE 12 *Prove that* $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5} = \frac{1}{2}\sin^{-1}\frac{4}{5}\cdot$ **[CBSE 2010C]**

SOLUTION We have

$$\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \tan^{-1}\left\{\frac{\frac{1}{4}+\frac{2}{9}}{1-\left(\frac{1}{4}\times\frac{2}{9}\right)}\right\} = \tan^{-1}\frac{1}{2}\cdot$$

Now, $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \Rightarrow \tan^{-1}x = \frac{1}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$\Rightarrow \tan^{-1}\frac{1}{2} = \frac{1}{2}\cos^{-1}\left(\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right) = \frac{1}{2}\cos^{-1}\frac{3}{5}\cdot$$

And, $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \Rightarrow \tan^{-1}x = \frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$\Rightarrow \tan^{-1}\frac{1}{2} = \frac{1}{2}\sin^{-1}\left(\frac{2\times\frac{1}{2}}{1+\frac{1}{4}}\right) = \frac{1}{2}\sin^{-1}\frac{4}{5}\cdot$$

Hence, $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5} = \frac{1}{2}\sin^{-1}\frac{4}{5}\cdot$

Problems on Trigonometric Equations

EXAMPLE 13 *Solve for x:* $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x).$ **[CBSE 2009, '10C]**

SOLUTION We have

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1}(2\operatorname{cosec} x \cot x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow 2\operatorname{cosec} x \cot x = 2\operatorname{cosec} x \Rightarrow \cot x = 1 \Rightarrow x = \frac{\pi}{4}\cdot$$

Hence, $x = \frac{\pi}{4}\cdot$

EXAMPLE 14 *Solve for x:* $2\tan^{-1}(\sin x) = \tan^{-1}(2\operatorname{cosec} x),\ x \neq \frac{\pi}{2}\cdot$ **[CBSE 2012]**

SOLUTION We have

$$2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\sin x}{1-\sin^2 x}\right) = \tan\,(2\sec x)$$

$$\Rightarrow \tan^{-1}(2\sec x \tan x) = \tan^{-1}(2\sec x)$$

$$\Rightarrow 2\sec x \tan x = 2\sec x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}\cdot$$

Hence, $x = \frac{\pi}{4}\cdot$

EXAMPLE 15 *If* $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1,$ *find the value of x.* **[CBSE 2014]**

SOLUTION We have

$$\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1$$

$$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1} x = \sin^{-1} 1 = \frac{\pi}{2}$$

$$\Rightarrow \cos^{-1} x = \left(\frac{\pi}{2} - \sin^{-1}\frac{1}{5}\right)$$

$$\Rightarrow \cos^{-1} x = \cos^{-1}\frac{1}{5} \Rightarrow x = \frac{1}{5}\cdot$$

Hence, $x = \frac{1}{5}\cdot$

EXAMPLE 16 *If* $\sin^{-1}\frac{x}{5}+\text{cosec}^{-1}\frac{5}{4}=\frac{\pi}{2}$ *then find the value of x.*

SOLUTION The given equation may be written as

$$\sin^{-1}\frac{x}{5}+\sin^{-1}\frac{4}{5}=\frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\frac{x}{5}=\left(\frac{\pi}{2}-\sin\frac{4}{5}\right)$$

$$\Rightarrow \sin^{-1}\frac{x}{5}=\cos^{-1}\frac{4}{5}=\sin^{-1}\sqrt{1-\left(\frac{4}{5}\right)^2}$$

$$=\sin^{-1}\sqrt{1-\frac{16}{25}}=\sin^{-1}\frac{3}{5}$$

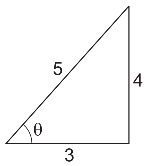
$$\Rightarrow \frac{x}{5}=\frac{3}{5} \Rightarrow x=3.$$

Hence, $x = 3$.

EXAMPLE 17 *Solve for x:* $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$. **[CBSE 2014]**

SOLUTION Let $\cot^{-1}\frac{3}{4}=\theta$. Then, $\cot\theta=\frac{3}{4}\cdot$

Draw a right triangle in which base = 3 units and perpendicular = 4 units.



Then, hypotenuse $=\sqrt{3^2+4^2}=\sqrt{25}=5$ units.

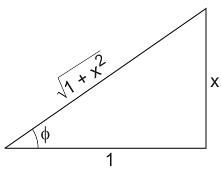
$\therefore \sin\theta=\frac{4}{5} \Rightarrow \theta=\sin^{-1}\frac{4}{5}\cdot$

So, $\sin\left(\cot^{-1}\frac{3}{4}\right)=\sin\left(\sin^{-1}\frac{4}{5}\right)=\frac{4}{5}\cdot$

Let $\tan^{-1}x=\phi$. Then, $\tan\phi=\frac{x}{1}\cdot$

Draw a right triangle in which perpendicular = x and base = 1.

Then, hypotenuse $=\sqrt{1+x^2}$.



$\therefore \cos\phi=\frac{1}{\sqrt{1+x^2}}\cdot$

So, $\cos(\tan^{-1}x)=\cos\phi=\frac{1}{\sqrt{1+x^2}}\cdot$

Thus, $\frac{1}{\sqrt{1+x^2}}=\frac{4}{5} \Rightarrow \frac{1}{(1+x^2)}=\frac{16}{25} \Rightarrow 16x^2=9$

$$\Rightarrow x^2=\frac{9}{16} \Rightarrow x=\pm\frac{3}{4}\cdot$$

Hence, $x=\pm\frac{3}{4}\cdot$

EXAMPLE 18 *If* $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$, *find the value of x.* **[CBSE 2014]**

SOLUTION The given equation may be written as

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)=\frac{\pi}{4}-\tan^{-1}\left(\frac{x+2}{x+4}\right)$$

$$=\tan^{-1}1-\tan^{-1}\left(\frac{x+2}{x+4}\right)\quad\left[\because\ \frac{\pi}{4}=\tan^{-1}1\right]$$

$$=\tan^{-1}\left\{\frac{1-\frac{x+2}{x+4}}{1+\frac{x+2}{x+4}}\right\}=\tan^{-1}\left(\frac{2}{2x+6}\right)$$

$$=\tan^{-1}\left(\frac{1}{x+3}\right).$$

$$\therefore\quad \frac{x-2}{x-4}=\frac{1}{x+3}\ \Rightarrow\ (x-2)(x+3)=(x-4)$$

$$\Rightarrow\ x^2+x-6=x-4\ \Rightarrow\ x^2=2\ \Rightarrow\ x=\pm\sqrt{2}.$$

Hence, $x=\pm\sqrt{2}$.

EXAMPLE 19 *Solve for x:* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3},\ -1<x<1.$ **[CBSE 2011C]**

SOLUTION The given equation may be written as

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$$

$$\Rightarrow\ 2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$$

$$\Rightarrow\ \tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{6}$$

$$\Rightarrow\ \frac{2x}{1-x^2}=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow\ (1-x^2)=2\sqrt{3}x$$

$$\Rightarrow\ x^2+2\sqrt{3}x-1=0$$

$$\Rightarrow\ x=\frac{-2\sqrt{3}\pm\sqrt{12+4}}{2}=\frac{-2\sqrt{3}\pm 4}{2}$$

$$=(-\sqrt{3}+2)\text{ or }(-\sqrt{3}-2).$$

But, $-1<x<1$. So, $x\neq(-\sqrt{3}-2)$.

Hence, $x=(2-\sqrt{3})$.

EXERCISE 4B

Very-Short-Answer Questions

Find the principal value of each of the following:

1. $\sin^{-1}\left(\frac{-1}{2}\right)$ 2. $\cos^{-1}\left(\frac{-1}{2}\right)$ 3. $\tan^{-1}(-1)$
4. $\sec^{-1}(-2)$ 5. $\operatorname{cosec}^{-1}(-\sqrt{2})$ 6. $\cot^{-1}(-1)$
7. $\tan^{-1}(-\sqrt{3})$ 8. $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ 9. $\operatorname{cosec}^{-1}(2)$

Find the principal value of each of the following:

10. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ 11. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ 12. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$
13. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ 14. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ 15. $\tan^{-1}\sqrt{3}-\cot^{-1}(-\sqrt{3})$
16. $\sin\left\{\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right\}$ 17. $\cot(\tan^{-1}x+\cot^{-1}x)$
18. $\operatorname{cosec}(\sin^{-1}x+\cos^{-1}x)$ 19. $\sin(\sec^{-1}x+\operatorname{cosec}^{-1}x)$
20. $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}$ 21. $\tan^{-1}1+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$
22. $\sin^{-1}\left\{\sin\frac{3\pi}{5}\right\}$ **[CBSE 2009]**

ANSWERS (EXERCISE 4B)

1. $\frac{-\pi}{6}$ **2.** $\frac{2\pi}{3}$ **3.** $\frac{-\pi}{4}$ **4.** $\frac{2\pi}{3}$ **5.** $\frac{5\pi}{4}$ **6.** $\frac{3\pi}{4}$ **7.** $\frac{-\pi}{3}$ **8.** $\frac{5\pi}{6}$
9. $\frac{\pi}{6}$ **10.** $\frac{\pi}{3}$ **11.** $\frac{-\pi}{4}$ **12.** $\frac{5\pi}{6}$ **13.** $\frac{\pi}{6}$ **14.** $\frac{\pi}{6}$ **15.** $\frac{5\pi}{6}$ **16.** 1
17. 0 **18.** 1 **19.** 1 **20.** $\frac{2\pi}{3}$ **21.** $\frac{3\pi}{4}$ **22.** $\frac{2\pi}{5}$

HINTS TO THE GIVEN QUESTIONS (EXERCISE 4B)

1. Range of $\sin^{-1}$ is $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$.

$\therefore\ \sin^{-1}\left(-\frac{1}{2}\right)=\theta \Rightarrow \sin\theta=-\frac{1}{2}=\sin\left(\frac{-\pi}{6}\right)$

$\Rightarrow\ \theta=\frac{-\pi}{6}$.

2. Range of $\cos^{-1}$ is $[0, \pi]$.

$$\therefore \quad \cos^{-1}\left(\frac{-1}{2}\right) = \theta \;\Rightarrow\; \cos\theta = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\Rightarrow \quad \theta = \frac{2\pi}{3}\cdot$$

3. Range of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \quad \tan^{-1}(-1) = \theta \;\Rightarrow\; \tan\theta = -1 = -\tan\frac{\pi}{4} = \tan\left(\frac{-\pi}{4}\right)$$

$$\Rightarrow \quad \theta = \frac{-\pi}{4}\cdot$$

4. Range of $\sec^{-1}$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$$\sec^{-1}(-2) = \theta \;\Rightarrow\; \sec\theta = -2 = -\sec\frac{\pi}{3} = \sec\left(\pi - \frac{\pi}{3}\right) = \sec\frac{2\pi}{3}$$

$$\Rightarrow \quad \theta = \frac{2\pi}{3}\cdot$$

5. Range of $\operatorname{cosec}^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

$$\therefore \quad \operatorname{cosec}^{-1}(-\sqrt{2}) = \theta \;\Rightarrow\; \operatorname{cosec}\theta = -\sqrt{2} = -\operatorname{cosec}\frac{\pi}{4} = \operatorname{cosec}\left(\pi + \frac{\pi}{4}\right) = \operatorname{cosec}\frac{5\pi}{4}$$

$$\Rightarrow \quad \theta = \frac{5\pi}{4}\cdot$$

6. Range of $\cot^{-1}$ is $(0, \pi)$.

$$\therefore \quad \cot^{-1}(-1) = \theta \;\Rightarrow\; \cot\theta = -1 = -\cot\frac{\pi}{4} = \cot\left(\pi - \frac{\pi}{4}\right) = \cot\frac{3\pi}{4} \;\Rightarrow\; \theta = \frac{3\pi}{4}\cdot$$

7. Range of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \quad \tan^{-1}(-\sqrt{3}) = \theta \;\Rightarrow\; \tan\theta = -\sqrt{3} = \tan\left(\frac{-\pi}{3}\right) \;\Rightarrow\; \theta = \frac{-\pi}{3}\cdot$$

8. Range of $\sec^{-1}$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$$\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right) = \theta \;\Rightarrow\; \sec\theta = \frac{-2}{\sqrt{3}} = -\sec\frac{\pi}{6} = \sec\left(\pi - \frac{\pi}{6}\right) = \sec\frac{5\pi}{6}$$

$$\Rightarrow \quad \theta = \frac{5\pi}{6}\cdot$$

9. Range of $\operatorname{cosec}^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

$$\therefore \quad \operatorname{cosec}^{-1}(2) = \theta \;\Rightarrow\; \operatorname{cosec}\theta = 2 = \operatorname{cosec}\frac{\pi}{6}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}\cdot$$

10. Range of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \quad \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left\{\sin\left(\pi - \frac{\pi}{3}\right)\right\}$$

$$= \sin^{-1}\left\{\sin\frac{\pi}{3}\right\} = \frac{\pi}{3}.$$

11. Range of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \quad \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left\{\tan\left(\pi - \frac{\pi}{4}\right)\right\}$$

$$= \tan^{-1}\left\{-\tan\frac{\pi}{4}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{-\pi}{4}\right)\right\} = \frac{-\pi}{4}.$$

12. Range of $\cos^{-1}$ is $[0, \pi]$.

$$\therefore \quad \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left\{\cos\left(2\pi - \frac{5\pi}{6}\right)\right\}$$

$$= \cos^{-1}\left\{\cos\frac{5\pi}{6}\right\} = \frac{5\pi}{6}.$$

13. Range of $\cos^{-1}$ is $[0, \pi]$.

$$\therefore \quad \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left\{\cos\left(2\pi + \frac{\pi}{6}\right)\right\}$$

$$= \cos^{-1}\left\{\cos\frac{\pi}{6}\right\} = \frac{\pi}{6}.$$

14. Range of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \quad \tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left\{\tan\left(\pi + \frac{\pi}{6}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\frac{\pi}{6}\right\} = \frac{\pi}{6}.$$

15. $\tan^{-1}\sqrt{3} = \theta_1 \;\Rightarrow\; \tan\theta_1 = \sqrt{3} = \tan\frac{\pi}{3} \;\Rightarrow\; \theta_1 = \frac{\pi}{3}.$

$$\cot^{-1}(-\sqrt{3}) = \theta_2 \;\Rightarrow\; \cot\theta_2 = -\sqrt{3} = -\cot\frac{\pi}{6} = \cot\left(\pi - \frac{\pi}{6}\right) = \cot\frac{5\pi}{6}$$

$$\Rightarrow \quad \theta_2 = \frac{5\pi}{6}.$$

$$\therefore \quad \tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3}) = \left(\frac{\pi}{3} - \frac{5\pi}{6}\right)$$

$$= \frac{-3\pi}{6} = \frac{-\pi}{2}.$$

16. Let $\sin^{-1}\left(-\frac{1}{2}\right)=\theta$. Then, $\sin\theta=-\frac{1}{2}=\sin\left(-\frac{\pi}{6}\right) \Rightarrow \theta=\frac{-\pi}{6}$.

$\therefore$ given exp. $=\sin\left[\frac{\pi}{3}-\left(\frac{-\pi}{6}\right)\right]=\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\sin\frac{3\pi}{6}=\sin\frac{\pi}{2}=1.$

17. $(\tan^{-1}a+\cot^{-1}a)=\frac{\pi}{2} \Rightarrow \cot(\tan^{-1}a+\cot^{-1}a)=\cot\frac{\pi}{2}=0.$

18. $(\sin^{-1}a+\cos^{-1}a)=\frac{\pi}{2} \Rightarrow \operatorname{cosec}(\sin^{-1}a+\cos^{-1}a)=\operatorname{cosec}\frac{\pi}{2}=1.$

19. $(\sec^{-1}a+\operatorname{cosec}^{-1}a)=\frac{\pi}{2} \Rightarrow \sin(\sec^{-1}a+\operatorname{cosec}^{-1}a)=\sin\frac{\pi}{2}=1.$

20. $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}=\left(\frac{\pi}{3}+2\times\frac{\pi}{6}\right)=\left(\frac{\pi}{3}+\frac{\pi}{3}\right)=\frac{2\pi}{3}.$

21. $\tan^{-1}1+\cos^{-1}\left(\frac{-1}{2}\right)+\sin^{-1}\left(\frac{-1}{2}\right)=\left\{\frac{\pi}{4}+\frac{2\pi}{3}+\left(\frac{-\pi}{6}\right)\right\}=\left(\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}\right)$

$$=\frac{(3\pi+8\pi-2\pi)}{12}=\frac{9\pi}{12}=\frac{3\pi}{4}.$$

22. See hint for Q. 10.

EXERCISE 4C

1. Prove that:

   (i) $\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x,\ x<1$

   (ii) $\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1)$

2. Prove that

   $\sin^{-1}(2x\sqrt{1-x^2})=2\sin^{-1}x,\ |x|\le\frac{1}{\sqrt{2}}.$ **[CBSE 2008]**

3. Prove that:

   (i) $\sin^{-1}(3x-4x^3)=3\sin^{-1}x,\ |x|\le\frac{1}{2}$

   (ii) $\cos^{-1}(4x^3-3x)=3\cos^{-1}x,\ \frac{1}{2}\le x\le 1$

   (iii) $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)=3\tan^{-1}x,\ |x|<\frac{1}{\sqrt{3}}$

   (iv) $\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ **[CBSE 2010]**

4. Prove that:

   (i) $\cos^{-1}(1-2x^2)=2\sin^{-1}x$

   (ii) $\cos^{-1}(2x^2-1)=2\cos^{-1}x$

(iii) $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right) = 2\cos^{-1} x$

(iv) $\cot^{-1}(\sqrt{1+x^2}-x) = \dfrac{\pi}{2} - \dfrac{1}{2}\cot^{-1} x$ **[CBSE 2007]**

5. Prove that:

(i) $\tan^{-1}\left(\dfrac{\sqrt{x}+\sqrt{y}}{1-\sqrt{xy}}\right) = \tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{y}$

(ii) $\tan^{-1}\left(\dfrac{x+\sqrt{x}}{1-x^{3/2}}\right) = \tan^{-1} x + \tan^{-1}\sqrt{x}$

(iii) $\tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right) = \dfrac{x}{2}$

6. Prove that:

(i) $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{2}{11} = \tan^{-1}\dfrac{3}{4}$

(ii) $\tan^{-1}\dfrac{2}{11} + \tan^{-1}\dfrac{7}{24} = \tan^{-1}\dfrac{1}{2}$

(iii) $\tan^{-1} 1 + \tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{2}$

(iv) $2\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{7} = \dfrac{\pi}{4}$

(v) $\tan^{-1} 2 - \tan^{-1} 1 = \tan^{-1}\dfrac{1}{3}$

(vi) $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$

(vii) $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{8} = \dfrac{\pi}{4}$ **[CBSE 2011, '12C]**

(viii) $\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{2}{9} = \dfrac{1}{2}\tan^{-1}\dfrac{4}{3}$

7. Prove that:

(i) $\cos^{-1}\dfrac{4}{5} + \cos^{-1}\dfrac{12}{13} = \cos^{-1}\dfrac{33}{65}$

(ii) $\sin^{-1}\dfrac{1}{\sqrt{5}} + \sin^{-1}\dfrac{2}{\sqrt{5}} = \dfrac{\pi}{2}$

(iii) $\cos^{-1}\dfrac{3}{5} + \sin^{-1}\dfrac{12}{13} = \sin^{-1}\dfrac{56}{65}$

(iv) $\cos^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{3}{5} = \sin^{-1}\dfrac{27}{11}$

(v) $\tan^{-1}\dfrac{1}{3} + \sec^{-1}\dfrac{\sqrt{5}}{2} = \dfrac{\pi}{4}$

(vi) $\sin^{-1}\dfrac{1}{\sqrt{17}} + \cos^{-1}\dfrac{9}{\sqrt{85}} = \tan^{-1}\dfrac{1}{2}$

(vii) $2\sin^{-1}\dfrac{3}{5} - \tan^{-1}\dfrac{17}{31} = \dfrac{\pi}{4}$

8. Solve for x:

(i) $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\dfrac{8}{31}$ **[CBSE 2008, '08C]**

(ii) $\tan^{-1}(2+x)+\tan^{-1}(2-x)=\tan^{-1}\dfrac{2}{3}$ (iii) $\cos(\sin^{-1}x)=\dfrac{1}{9}$

(iv) $\cos(2\sin^{-1}x)=\dfrac{1}{9}$ (v) $\sin^{-1}\dfrac{8}{x}+\sin^{-1}\dfrac{15}{x}=\dfrac{\pi}{2}$

9. Solve for x:

(i) $\cos(\sin^{-1}x)=\dfrac{1}{2}$ (ii) $\tan^{-1}x=\sin^{-1}\dfrac{1}{\sqrt{2}}$

(iii) $\sin^{-1}x-\cos^{-1}x=\dfrac{\pi}{6}$

ANSWERS (EXERCISE 4C)

8. (i) $x=\dfrac{1}{4}$ (ii) $x=3$ or $x=-3$ (iii) $x=\pm\dfrac{4\sqrt{5}}{9}$ (iv) $x=\pm\dfrac{2}{3}$ (v) $x=17$

9. (i) $x=\pm\dfrac{\sqrt{3}}{2}$ (ii) $x=1$ (iii) $x=\dfrac{\sqrt{3}}{2}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 4C)

1. (i) Put $x=\tan\theta$ (ii) LHS $=\tan^{-1}x+\dfrac{1}{\tan^{-1}(x+1)}$

2. Put $x=\sin\theta$

3. (i) Put $x=\sin\theta$ (ii) Put $x=\cos\theta$ (iii) Put $x=\tan\theta$ (iv) Use $\tan^{-1}A+\tan^{-1}B$

4. (i) Put $x=\sin\theta$ (ii) Put $x=\cos\theta$ (iii) Put $x=\cos\theta$

(iv) Putting $x=\cot\theta$, we get:

$$\cot^{-1}(\sqrt{1+x^2}-x)=\cot^{-1}(\operatorname{cosec}\theta-\cot\theta)$$

$$=\cot^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)=\cot^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$=\cot^{-1}\left(\tan\frac{\theta}{2}\right)=\cot^{-1}\left[\cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]$$

$$=\frac{\pi}{2}-\frac{\theta}{2}=\left(\frac{\pi}{2}-\frac{1}{2}\cot^{-1}x\right).$$

5. (i) Put $\sqrt{x}=\tan\theta$ and $\sqrt{y}=\tan\phi$ on each side.

(ii) Put $x=\tan\theta$ and $\sqrt{x}=\tan\phi$ on each side.

(iii) $\text{LHS} = \tan^{-1}\left\{\dfrac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}\right\} = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2} = \text{RHS}.$

6. (iv) $2\tan^{-1}\frac{1}{3} = \tan^{-1}\left\{\dfrac{2\times\frac{1}{3}}{1-\frac{1}{9}}\right\}.$

(v) Here $2\times 1 = 2 > -1$. Now, use $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$.

(vi) Here $2\times 3 = 6 > 1$. So, we use the formula

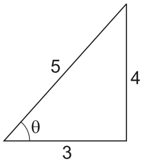
$$\tan^{-1}2 + \tan^{-1}3 = \pi + \tan^{-1}\left\{\frac{2+3}{1-(2\times 3)}\right\} = \pi + \tan^{-1}(-1) = \pi - \tan^{-1}1.$$

$\therefore\quad \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi.$

(viii) $$2\left(\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}\right) = 2\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right) = 2\tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}\left(\frac{2\times\frac{1}{2}}{1-\frac{1}{4}}\right) = \tan^{-1}\frac{4}{3}.$$

7. (iii) Let $\cos^{-1}\frac{3}{5} = \theta$.

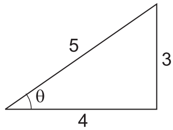


Then, base = 3, hypotenuse = 5.

So, perpendicular $= \sqrt{(5)^2 - 3^2} = \sqrt{16} = 4$.

$\therefore\quad \sin\theta = \frac{4}{5} \Rightarrow \theta = \sin^{-1}\frac{4}{5}.$

(iv) Let $\cos^{-1}\frac{4}{5} = \theta$.

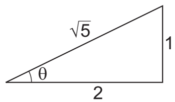


Then, base = 4, hypotenuse = 5.

$\therefore\quad$ perpendicular $= \sqrt{5^2 - 4^2} = \sqrt{9} = 3$.

$\therefore\quad \tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\frac{3}{4}.$

(v) Let $\sec^{-1}\frac{\sqrt{5}}{2} = \theta$. Then, $\sec\theta = \frac{\sqrt{5}}{2}$.

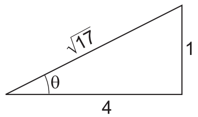


So, hypotenuse $= \sqrt{5}$, base = 2.

So, perpendicular $= \sqrt{(\sqrt{5})^2 - 2^2} = 1$.

$\therefore\quad \tan\theta = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\frac{1}{2}.$

(vi) Let $\sin^{-1}\frac{1}{\sqrt{4}}=\theta$. Then, $\sin\theta=\frac{1}{\sqrt{17}}$.

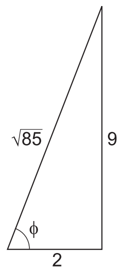


$\therefore$ perp. = 1 and hyp. = $\sqrt{17}$.

$\therefore$ base = $\sqrt{(\sqrt{17})^2-1^2}=\sqrt{16}=4$.

$\therefore$ $\tan\theta=\frac{1}{4}\Rightarrow\theta=\tan^{-1}\frac{1}{4}$.

Again, let $\cos^{-1}\frac{9}{\sqrt{85}}=\phi$. Then, $\cos\phi=\frac{9}{\sqrt{85}}$.

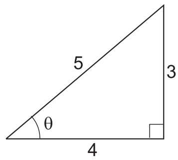


$\therefore$ base = 9, hyp. = $\sqrt{85}$.

So, perp. = $\sqrt{(\sqrt{85})^2-9^2}=\sqrt{4}=2$.

$\therefore$ $\tan\phi=\frac{2}{9}\Rightarrow\phi=\tan^{-1}\frac{2}{9}$.

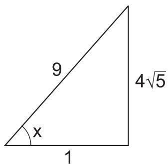
(vii) $\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$.



$$2\sin^{-1}\frac{3}{5}=2\tan^{-1}\frac{3}{4}=\tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}\right)$$

$$=\tan^{-1}\frac{24}{7}.$$

8. (iii) $\sin^{-1}x=\cos^{-1}\frac{1}{9}$



$$=\sin^{-1}\frac{4\sqrt{5}}{9}.$$

9. (i) $\cos(\sin^{-1}x)=\frac{1}{2}=\cos\frac{\pi}{3}$ or $\cos\frac{5\pi}{3}$ $\left[\text{i.e., }\cos\left(2\pi-\frac{\pi}{3}\right)\right]$

$\Rightarrow$ $\sin^{-1}x=\frac{\pi}{3}$ or $\sin^{-1}x=\frac{5\pi}{3}$

$\Rightarrow$ $x=\sin\frac{\pi}{3}$ or $x=\sin\frac{5\pi}{3}\Rightarrow x=\frac{\sqrt{3}}{2}$ or $\frac{-\sqrt{3}}{2}$.

GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

1. Graph of $\sin^{-1}x$

Let $f:[-1,1]\to\left[\frac{-\pi}{2},\frac{\pi}{2}\right]:f(x)=\sin^{-1}x$.

Here $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ is called the *principal-value branch* of $\sin^{-1}x$.

The other branches of $\sin^{-1}x$ are

$\left[\frac{\pi}{2},\frac{3\pi}{2}\right],\left[\frac{3\pi}{2},\frac{5\pi}{2}\right],\ldots$ and $\left[\frac{-3\pi}{2},\frac{-\pi}{2}\right],\left[\frac{-5\pi}{2},\frac{-3\pi}{2}\right]$, etc.

Table for $\sin^{-1} x$

x	0	$\frac{1}{2} = 0.5$	$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.41}{2} = 0.70$	$\frac{\sqrt{3}}{2} = \frac{1.73}{2} = 0.86$	1
$\sin^{-1} x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

Also, $\sin^{-1}(-x) = - \sin^{-1} x$.

$$\therefore \left(x = \frac{-1}{2} \Rightarrow \sin^{-1} x = \frac{-\pi}{6}\right),$$

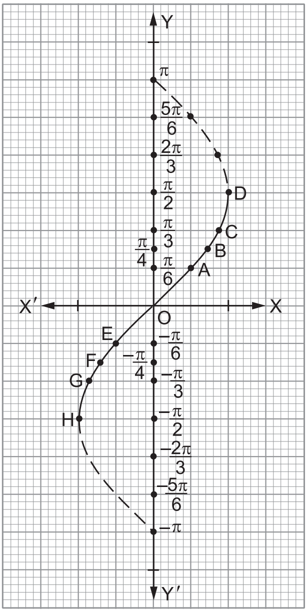
$$\left(x = -0.7 \Rightarrow \sin^{-1} x = \frac{-\pi}{4}\right),$$

$$\left(x = -0.86 \Rightarrow \sin^{-1} x = \frac{-\pi}{3}\right),$$

$$\left(x = -1 \Rightarrow \sin^{-1} x = \frac{-\pi}{2}\right).$$

On a graph paper, we plot the points

$O(0, 0)$, $A\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $B\left(0.7, \frac{\pi}{4}\right)$, $C\left(0.86, \frac{\pi}{3}\right)$, $D\left(1, \frac{\pi}{2}\right)$, $E\left(\frac{-1}{2}, \frac{-\pi}{6}\right)$, $F\left(-0.7, \frac{-\pi}{4}\right)$, $G\left(-0.86, \frac{-\pi}{3}\right)$ and $H\left(-1, \frac{-\pi}{2}\right)$.



Graph of $\sin^{-1} x$

Scale: Along the x-axis, 10 small divisions = 1
Along the y-axis, 5 small divisions = $\pi/6$

Join the points OA, AB, BC, CD, and OE, EF, FG, GH successively with a freehand to get the required graph, as shown in the given figure.

Moreover, we have

$$\sin \frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin \frac{\pi}{3} = 0.86.$$

$$\sin \frac{5\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin \frac{\pi}{6} = 0.5, \ \sin \pi = 0.$$

$$\sin\left(\frac{-2\pi}{3}\right) = -\sin \frac{2\pi}{3} = -0.86.$$

$$\sin\left(\frac{-5\pi}{6}\right) = -\sin \frac{5\pi}{6} = -0.5.$$

Now, we may extend the graph as shown in the figure.

2. Graph of $\cos^{-1}x$

Let $f : [-1, 1] \to [0, \pi] : f(x) = \cos^{-1} x$.

Here, $[0, \pi]$ is called the principal-value branch of $\cos^{-1} x$.

The other branches of $\cos^{-1} x$ are $[\pi, 2\pi]$, $[2\pi, 3\pi]$, … , $[-\pi, 0]$, $[-2\pi, -\pi]$, etc.

Table for $\cos^{-1}x$

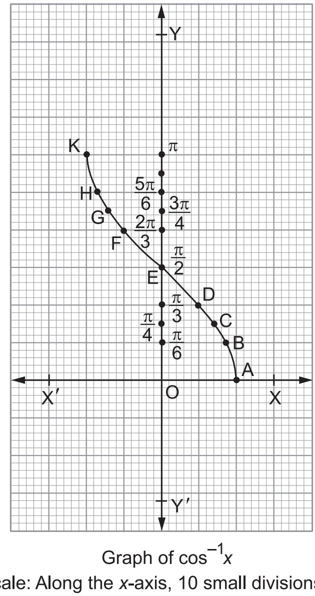
x	1	$\frac{\sqrt{3}}{2} = \frac{1.732}{2} = 0.87$	$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.41}{2} = 0.7$	$\frac{1}{2} = 0.5$	0	$\frac{-1}{2} = -0.5$	$\frac{-1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{-\sqrt{2}}{2} = \frac{-1.41}{2} = -0.70$	$\frac{-\sqrt{3}}{2} = \frac{-1.73}{2} = -0.86$	-1
$\cos^{-1} x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

On a graph paper, we plot the points

$$A(1, 0),\ B\left(0.87, \frac{\pi}{6}\right), C\left(0.7, \frac{\pi}{4}\right), D\left(0.5, \frac{\pi}{3}\right), E\left(0, \frac{\pi}{2}\right), F\left(-0.5, \frac{2\pi}{3}\right),$$

$$G\left(-0.7, \frac{3\pi}{4}\right), H\left(-0.86, \frac{5\pi}{6}\right) \text{ and } K(-1, \pi).$$

Join *AB*, *BC*, *CD*, *DE*, *EF*, *FG*, *GH* and *HK* successively freehand to obtain the graph of $\cos^{-1} x$, as shown in the given figure.



Graph of $\cos^{-1}x$

Scale: Along the x-axis, 10 small divisions = 1

Along the y-axis, 5 small divisions = $\pi/6$

3. Graph of $\tan^{-1}x$

Let $f : R \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) : f(x) = \tan^{-1} x.$

Here $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is the principal-value branch of $\tan^{-1} x.$

The other branches of $\tan^{-1} x$ are $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right), \ldots, \left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$, etc.

We take the positive values and then use $\tan^{-1}(-x) = -\tan^{-1} x.$

Table for $\tan^{-1}x$

x	0	$\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1.73}{3} = 0.58$	1	$\sqrt{3} = 1.73$
$\tan^{-1} x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

Also, $\tan^{-1}(-x) = -\ \tan^{-1} x.$

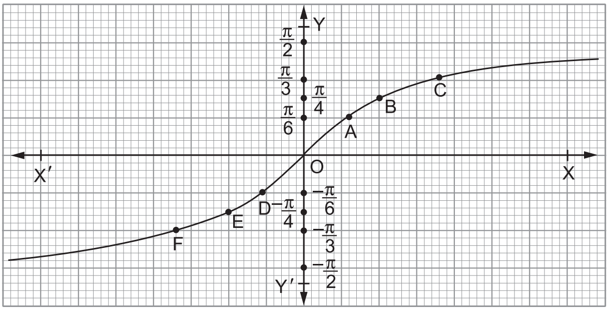
$\therefore\ \tan^{-1}(-0.58) = -\frac{\pi}{6},\ \tan^{-1}(-1) = -\frac{\pi}{4},\ \tan^{-1}(-1.73) = \frac{-\pi}{3}\cdot$

On a graph paper, we plot the points $O(0, 0)$, $A\left(0.58, \frac{\pi}{6}\right)$, $B\left(1, \frac{\pi}{4}\right)$, $C\left(1.73, \frac{\pi}{3}\right)$, and $D\left(-0.58, \frac{-\pi}{6}\right)$, $E\left(-1, \frac{-\pi}{4}\right)$, $F\left(-1.73, \frac{-\pi}{3}\right)\cdot$

Join OA, AB, BC, and OD, DE, EF successively freehand to get the graph.

Now, when $x \to \infty$, then $\tan^{-1} x \to \frac{\pi}{2}\cdot$

Also, when $x \to -\infty$, then $\tan^{-1} x \to \frac{-\pi}{2}\cdot$



Graph of $\tan^{-1}x$

Scale: Along the x-axis, 10 small divisions = 1

Along the y-axis, 5 small divisions = $\pi/6$

4. Graph of $\cot^{-1}x$

Let $f : R \to (0, \pi) : f(x) = \cot^{-1} x$.

Here, $(0, \pi)$ is the principal-value branch of $\cot^{-1} x$.

The other branches of $\cot^{-1} x$ are $(\pi, 2\pi), (2\pi, 3\pi), \ldots, (-\pi, 0)$, etc.

Table for $\cot^{-1} x$

x	$\sqrt{3} = 1.73$	1	$\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1.732}{3} = 0.58$	0
$\cot^{-1} x$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

Using $\cot(\pi - x) = -\cot x$, we have

$$\cot\frac{2\pi}{3} = \cot\left(\pi - \frac{\pi}{3}\right) = -\cot\frac{\pi}{3} = -0.58, \cot\frac{3\pi}{4} = \cot\left(\pi - \frac{\pi}{4}\right) = -\cot\frac{\pi}{4} = -1,$$

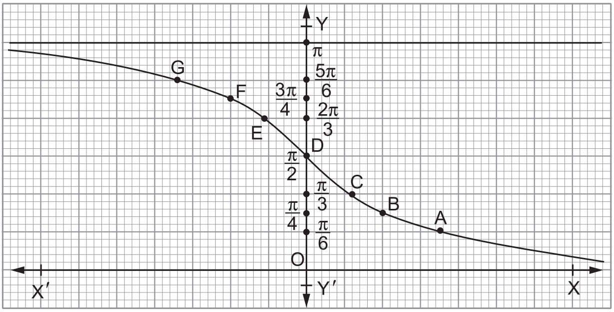
$$\cot\frac{5\pi}{6} = \cot\left(\pi - \frac{\pi}{6}\right) = -\cot\frac{\pi}{6} = -1.73.$$

On a graph paper, we plot the points $A\left(1.73, \frac{\pi}{6}\right), B\left(1, \frac{\pi}{4}\right), C\left(0.58, \frac{\pi}{3}\right),$ $D\left(0, \frac{\pi}{2}\right), E\left(-0.58, \frac{2\pi}{3}\right), F\left(-1, \frac{3\pi}{4}\right)$, and $G\left(-1.73, \frac{5\pi}{6}\right)$.

We join the points AB, BC, CD, DE, EF, FG successively to get the graph.

As $x \to \infty$, then $\cot^{-1} x \to 0$.

And, as $x \to -\infty$, then $\cot^{-1} x \to \pi$.



Graph of $\cot^{-1}x$

Scale: Along the x-axis, 10 small divisions = 1

Along the y-axis, 5 small divisions = $\pi/6$

5. Graph of $\sec^{-1}x$

Let $f : R - (-1, 1) \to [0, \pi] - \left\{\frac{\pi}{2}\right\} : f(x) = \sec^{-1}x.$

The other branches of $\sec^{-1}x$ are $[\pi, 2\pi] - \left\{\frac{3\pi}{2}\right\}, \ldots, [-\pi, 0] - \left\{\frac{-\pi}{2}\right\}$, etc.

Table for $\sec^{-1}x$

x	1	$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = \frac{2 \times 1.73}{3} = \frac{3.46}{3} = 1.15$	$\sqrt{2} = 1.41$	2
$\sec^{-1}x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

$\left\{\sec\frac{2\pi}{3} = -2 \Rightarrow \sec^{-1}(-2) = \frac{2\pi}{3}\right\}, \left\{\sec\frac{3\pi}{4} = -\sqrt{2} = -1.41 \Rightarrow \sec^{-1}(-1.41) = \frac{3\pi}{4}\right\},$

$\left\{\sec\frac{5\pi}{6} = \frac{-2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{-2\sqrt{3}}{3} = \frac{-2 \times 1.73}{3} = \frac{-3.46}{3} = -1.15 \Rightarrow \sec^{-1}(-1.15) = \frac{5\pi}{6}\right\},$

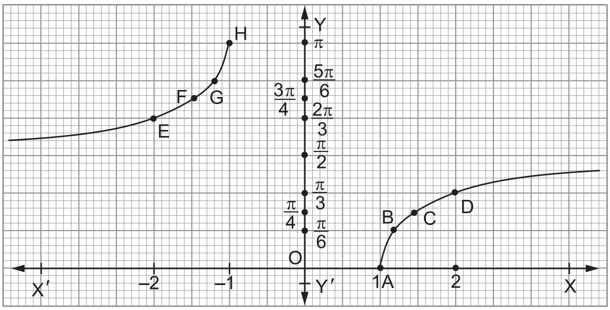
$\sec \pi = -1 \Rightarrow \sec^{-1}(-1) = \pi.$

On a graph paper, we plot the points

$A(1, 0), B\left(1.15, \frac{\pi}{6}\right), C\left(1.41, \frac{\pi}{4}\right), D\left(2, \frac{\pi}{3}\right)$ and

$E\left(-2, \frac{2\pi}{3}\right), F\left(-1.41, \frac{3\pi}{4}\right), G\left(-1.15, \frac{5\pi}{6}\right), H(-1, \pi).$

We join the points AB, BC, CD, DE, and HG, GF, FE.



Graph of $\sec^{-1}x$

Scale: Along the x-axis, 10 small divisions = 1

Along the y-axis, 5 small divisions = $\pi/6$

6. Graph of $\text{cosec}^{-1}x$

Let $f : R - (-1, 1) \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\} : f(x) = \text{cosec}^{-1}x.$

The other branches of $\text{cosec}^{-1}x$ are $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$, ..., etc.

Table for $\text{cosec}^{-1}x$

x	2	$\sqrt{2} = 1.41$	$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = \frac{2 \times 1.73}{3} = 1.15$	1
$\text{cosec}^{-1}x$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

Since $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, we have

$$\text{cosec}^{-1}\left(\frac{-\pi}{6}\right) = -2, \text{cosec}^{-1}\left(\frac{-\pi}{4}\right) = -1.41, \text{cosec}^{-1}\left(\frac{-\pi}{3}\right) = -1.15 \text{ and}$$

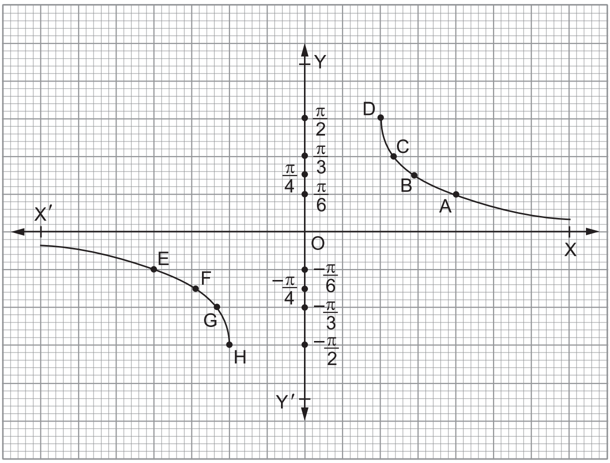
$$\text{cosec}^{-1}\left(\frac{-\pi}{2}\right) = -1.$$

On a graph paper, we plot the points $A\left(2, \frac{\pi}{6}\right)$, $B\left(1.41, \frac{\pi}{4}\right)$, $C\left(1.15, \frac{\pi}{3}\right)$, $D\left(1, \frac{\pi}{2}\right)$ and $E\left(-2, \frac{-\pi}{6}\right)$, $F\left(-1.41, \frac{-\pi}{4}\right)$, $G\left(-1.15, \frac{-\pi}{3}\right)$, $H\left(-1, \frac{-\pi}{2}\right)$.

Join the points as shown in the given figure, to get the graph.

Also as $x \to 0$ from +ve values, then $\text{cosec}^{-1}x \to \infty$.

As $x \to 0$ from –ve values, then $\text{cosec}^{-1}x \to -\infty$.



Graph of $\text{cosec}^{-1}x$

Scale: Along the x-axis, 10 small divisions = 1

Along the y-axis, 5 small divisions = $\pi/6$

EXERCISE 4D

Write down the interval for the principal-value branch of each of the following functions and draw its graph:

1. $\sin^{-1}$
2. $\cos^{-1} x$
3. $\tan^{-1} x$
4. $\cot^{-1} x$
5. $\sec^{-1} x$
6. $\operatorname{cosec}^{-1} x$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. The principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{7\pi}{6}$ (d) none of these

2. The principal value of $\operatorname{cosec}^{-1}(2)$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

3. The principal value of $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ is

(a) $\frac{-\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) $\frac{5\pi}{4}$

4. The principal value of $\sin^{-1}\left(\frac{-1}{2}\right)$ is

(a) $\frac{-\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{7\pi}{6}$ (d) none of these

5. The principal value of $\cos^{-1}\left(\frac{-1}{2}\right)$ is

(a) $\frac{-\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{\pi}{3}$

6. The principal value of $\tan^{-1}(-\sqrt{3})$ is

(a) $\frac{2\pi}{3}$ (b) $\frac{4\pi}{3}$ (c) $\frac{-\pi}{3}$ (d) none of these

7. The principal value of $\cot^{-1}(-1)$ is

(a) $\frac{-\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{5\pi}{4}$ (d) $\frac{3\pi}{4}$

8. The principal value of $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ is

(a) $\frac{\pi}{6}$ (b) $\frac{-\pi}{6}$ (c) $\frac{5\pi}{6}$ (d) $\frac{7\pi}{6}$

9. The principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$ is

(a) $\frac{-\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{5\pi}{4}$ (d) none of these

10. The principal value of $\cot^{-1}(-\sqrt{3})$ is

(a) $\frac{-\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $\frac{7\pi}{6}$ (d) $\frac{5\pi}{6}$

11. The value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is

(a) $\frac{2\pi}{3}$ (b) $\frac{5\pi}{3}$ (c) $\frac{\pi}{3}$ (d) none of these

12. The value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ is

(a) $\frac{13\pi}{6}$ (b) $\frac{7\pi}{6}$ (c) $\frac{5\pi}{6}$ (d) $\frac{\pi}{6}$

13. The value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ is

(a) $\frac{7\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{6}$ (d) none of these

14. The value of $\cot^{-1}\left(\cot\frac{5\pi}{4}\right)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{-\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) none of these

15. The value of $\sec^{-1}\left(\sec\frac{8\pi}{5}\right)$ is

(a) $\frac{2\pi}{5}$ (b) $\frac{3\pi}{5}$ (c) $\frac{8\pi}{5}$ (d) none of these

16. The value of $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{4\pi}{3}\right)$ is

(a) $\frac{\pi}{3}$ (b) $\frac{-\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) none of these

17. The value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ is

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{-\pi}{4}$ (d) none of these

18. $\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right) = ?$

(a) 0 (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

19. The value of $\sin\left(\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}\right)=?$

(a) 0 (b) 1 (c) –1 (d) none of these

20. If $x \neq 0$ then $\cos(\tan^{-1}x+\cot^{-1}x)=?$

(a) –1 (b) 1
(c) 0 (d) none of these

21. The value of $\sin\left(\cos^{-1}\frac{3}{5}\right)$ is

(a) $\frac{2}{5}$ (b) $\frac{4}{5}$ (c) $\frac{-2}{5}$ (d) none of these

22. $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=?$

(a) $\frac{4\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) π

23. $\tan^{-1}(\sqrt{3})-\sec^{-1}(-2)=?$

(a) $\frac{\pi}{3}$ (b) $\frac{-\pi}{3}$ (c) $\frac{5\pi}{3}$ (d) none of these

24. $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}=?$

(a) $\frac{2\pi}{3}$ (b) $\frac{3\pi}{2}$ (c) 2π (d) none of these

25. $\tan^{-1}1+\cos^{-1}\left(\frac{-1}{2}\right)+\sin^{-1}\left(\frac{-1}{2}\right)=?$

(a) π (b) $\frac{2\pi}{3}$ (c) $\frac{3\pi}{4}$ (d) $\frac{\pi}{2}$

26. $\tan\left[2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right]=?$

(a) $\frac{7}{17}$ (b) $\frac{-7}{17}$ (c) $\frac{7}{12}$ (d) $\frac{-7}{12}$

27. $\tan\frac{1}{2}\left(\cos^{-1}\frac{\sqrt{5}}{3}\right)=?$

(a) $\frac{(3-\sqrt{5})}{2}$ (b) $\frac{(3+\sqrt{5})}{2}$ (c) $\frac{(5-\sqrt{3})}{2}$ (d) $\frac{(5+\sqrt{3})}{2}$

28. $\sin\left(\cos^{-1}\frac{3}{5}\right)=?$

(a) $\frac{3}{4}$ (b) $\frac{4}{5}$ (c) $\frac{3}{5}$ (d) none of these

29. $\cos\left(\tan^{-1}\frac{3}{4}\right) = ?$

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{4}{9}$ (d) none of these

30. $\sin\left\{\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right\} = ?$

(a) 1 (b) 0 (c) $\frac{-1}{2}$ (d) none of these

31. $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right) = ?$

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{2}{\sqrt{5}}$ (c) $\frac{1}{\sqrt{10}}$ (d) $\frac{2}{\sqrt{10}}$

32. $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\} = ?$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) $\frac{2\pi}{3}$

33. If $\cot^{-1}\left(\frac{-1}{5}\right) = x$ then $\sin x = ?$

(a) $\frac{1}{\sqrt{26}}$ (b) $\frac{5}{\sqrt{26}}$ (c) $\frac{1}{\sqrt{24}}$ (d) none of these

34. $\sin^{-1}\left(\frac{-1}{2}\right) + 2\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = ?$

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) none of these

35. $\tan^{-1}(-1) + \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = ?$

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) $\frac{2\pi}{3}$

36. $\cot(\tan^{-1}x + \cot^{-1}x) = ?$

(a) 1 (b) $\frac{1}{2}$ (c) 0 (d) none of these

37. $\tan^{-1}1 + \tan^{-1}\frac{1}{3} = ?$

(a) $\tan^{-1}\frac{4}{3}$ (b) $\tan^{-1}\frac{2}{3}$ (c) $\tan^{-1}2$ (d) $\tan^{-1}3$

38. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = ?$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

39. $2\tan^{-1}\frac{1}{3} = ?$

(a) $\tan^{-1}\frac{3}{2}$ (b) $\tan^{-1}\frac{3}{4}$ (c) $\tan^{-1}\frac{4}{3}$ (d) none of these

40. $\cos\left(2\tan^{-1}\frac{1}{2}\right) = ?$

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{7}{8}$ (d) none of these

41. $\sin\left[2\tan^{-1}\frac{5}{8}\right]$

(a) $\frac{25}{64}$ (b) $\frac{80}{89}$ (c) $\frac{75}{128}$ (d) none of these

42. $\sin\left[2\sin^{-1}\frac{4}{5}\right]$

(a) $\frac{12}{25}$ (b) $\frac{16}{25}$ (c) $\frac{24}{25}$ (d) none of these

43. If $\tan^{-1}x = \frac{\pi}{4} - \tan^{-1}\frac{1}{3}$ then $x = ?$

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) none of these

44. If $\tan^{-1}(1 + x) + \tan^{-1}(1 - x) = \frac{\pi}{2}$ then $x = ?$

(a) 1 (b) –1 (c) 0 (d) $\frac{1}{2}$

45. If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$ then $(\cos^{-1}x + \cos^{-1}y) = ?$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) π (d) $\frac{2\pi}{3}$

46. $(\tan^{-1}2 + \tan^{-1}3) = ?$

(a) $\frac{-\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) π

47. If $\tan^{-1}x + \tan^{-1}3 = \tan^{-1}8$ then $x = ?$

(a) $\frac{1}{3}$ (b) $\frac{1}{5}$ (c) 3 (d) 5

48. If $\tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4}$ then $x = ?$

(a) $\frac{1}{2}$ or –2 (b) $\frac{1}{3}$ or –3 (c) $\frac{1}{4}$ or –2 (d) $\frac{1}{6}$ or –1

49. $\tan\left\{\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right\}=?$

(a) $\frac{13}{6}$ (b) $\frac{17}{6}$ (c) $\frac{19}{6}$ (d) $\frac{23}{6}$

50. $\cot^{-1}9+\operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}=?$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{3\pi}{4}$

51. Range of $\sin^{-1}x$ is

(a) $\left[0,\frac{\pi}{2}\right]$ (b) $[0,\pi]$ (c) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (d) none of these

52. Range of $\cos^{-1}x$ is

(a) $[0,\pi]$ (b) $\left[0,\frac{\pi}{2}\right]$ (c) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (d) none of these

53. Range of $\tan^{-1}x$ is

(a) $\left(0,\frac{\pi}{2}\right)$ (b) $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ (c) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (d) none of these

54. Range of $\sec^{-1}x$ is

(a) $\left[0,\frac{\pi}{2}\right]$ (b) $[0,\pi]$ (c) $[0,\pi]-\left\{\frac{\pi}{2}\right\}$ (d) none of these

55. Range of $\operatorname{cosec}^{-1}x$ is

(a) $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ (b) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (c) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$ (d) none of these

56. Domain of $\cos^{-1}x$ is

(a) $[0, 1]$ (b) $[-1, 1]$ (c) $[-1, 0]$ (d) none of these

57. Domain of $\sec^{-1}x$ is

(a) $[-1, 1]$ (b) $R-\{0\}$ (c) $R-[-1, 1]$ (d) $R-\{-1, 1\}$

ANSWERS (OBJECTIVE QUESTIONS)

1. (a) **2.** (b) **3.** (c) **4.** (a) **5.** (b) **6.** (c) **7.** (d) **8** (c) **9.** (a) **10.** (d)
11. (c) **12.** (d) **13.** (c) **14.** (a) **15.** (a) **16.** (b) **17.** (c) **18.** (c) **19.** (b) **20.** (c)
21. (b) **22.** (d) **23.** (b) **24.** (a) **25.** (c) **26.** (b) **27.** (a) **28.** (b) **29.** (b) **30.** (a)
31. (c) **32.** (b) **33.** (b) **34.** (c) **35.** (a) **36.** (c) **37.** (c) **38.** (b) **39.** (b) **40.** (a)
41. (b) **42.** (c) **43.** (a) **44.** (c) **45.** (b) **46.** (c) **47.** (b) **48.** (d) **49.** (b) **50.** (b)
51. (c) **52.** (a) **53.** (b) **54.** (c) **55.** (c) **56.** (b) **57.** (c)

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

1. Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = x$, where $x \in [0, \pi]$.

Then, $\cos x = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow x = \frac{\pi}{6}\cdot$

2. Let $\operatorname{cosec}^{-1}(2) = x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Then, $\operatorname{cosec} x = 2 \operatorname{cosec}\frac{\pi}{6} \Rightarrow x = \frac{\pi}{6}\cdot$

3. Let $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = x$, where $x \in [0, \pi]$.

Then, $\cos x = \frac{-1}{\sqrt{2}} = -\cos\frac{\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4} \Rightarrow x = \frac{3\pi}{4}\cdot$

4. Let $\sin^{-1}\left(\frac{-1}{2}\right) = x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\cdot$

Then, $\sin x = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right) \Rightarrow x = \frac{-\pi}{6}\cdot$

5. Let $\cos^{-1}\left(\frac{-1}{2}\right) = x$, where $x \in [0, \pi]$.

Then, $\cos x = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3} \Rightarrow x = \frac{2\pi}{3}\cdot$

6. Let $\tan^{-1}(-\sqrt{3}) = x$, where $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\cdot$

Then, $\tan x = -\sqrt{3} = -\tan\frac{\pi}{3} = \tan\left(\frac{-\pi}{3}\right) \Rightarrow x = \frac{-\pi}{3}\cdot$

7. Let $\cot^{-1}(-1) = x$, where $x \in [0, \pi]$.

Then, $\cot x = -1 = -\cot\frac{\pi}{4} = \cot\left(\pi - \frac{\pi}{4}\right) = \cot\frac{3\pi}{4} \Rightarrow x = \frac{3\pi}{4}\cdot$

8. Let $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right) = x$, where $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}\cdot$

Then, $\sec x = \frac{-2}{\sqrt{3}} = -\sec\frac{\pi}{6} = \sec\left(\pi - \frac{\pi}{6}\right) = \sec\frac{5\pi}{6} \Rightarrow x = \frac{5\pi}{6}\cdot$

9. Let $\operatorname{cosec}^{-1}(-\sqrt{2}) = x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Then, $\operatorname{cosec} x = -\sqrt{2} = -\operatorname{cosec}\frac{\pi}{4} = \operatorname{cosec}\left(\frac{-\pi}{4}\right) \Rightarrow x = \frac{-\pi}{4}\cdot$

10. Let $\cot^{-1}(-\sqrt{3}) = x$, where $x \in [0, \pi]$.

Then, $\cot x = -\sqrt{3} = -\cot\frac{\pi}{6} = \cot\left(\pi - \frac{\pi}{6}\right) = \cot\frac{5\pi}{6} \Rightarrow x = \frac{5\pi}{6}\cdot$

11. Let $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Then, $\sin x = \sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} \Rightarrow x = \frac{\pi}{3}$.

12. Let $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = x$, where $x \in [0, \pi]$.

Then, $\cos x = \cos\frac{13\pi}{6} = \cos\left(2\pi + \frac{\pi}{6}\right) = \cos\frac{\pi}{6} \Rightarrow x = \frac{\pi}{6}$.

13. Let $\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = x$, where $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Then, $\tan x = \tan\frac{7\pi}{6} = \tan\left(\pi + \frac{\pi}{6}\right) = \tan\frac{\pi}{6} \Rightarrow x = \frac{\pi}{6}$.

14. Let $\cot^{-1}\left(\cot\frac{5\pi}{4}\right) = x$, where $x \in [0, \pi]$.

Then, $\cot x = \cot\frac{5\pi}{4} = \cot\left(\pi + \frac{\pi}{4}\right) = \cot\frac{\pi}{4} \Rightarrow x = \frac{\pi}{4}$.

15. Let $\sec^{-1}\left(\sec\frac{8\pi}{5}\right) = x$, where $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$.

Then, $\sec x = \sec\frac{8\pi}{5} = \sec\left(2\pi - \frac{2\pi}{5}\right) = \sec\frac{2\pi}{5} \Rightarrow x = \frac{2\pi}{5}$.

16. Let $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{4\pi}{3}\right) = x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Then, $\operatorname{cosec} x = \operatorname{cosec}\frac{4\pi}{3} = \operatorname{cosec}\left(\pi + \frac{\pi}{3}\right) = -\operatorname{cosec}\frac{\pi}{3} = \operatorname{cosec}\left(\frac{-\pi}{3}\right)$.

$\therefore \quad x = \frac{-\pi}{3}$.

17. Let $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = x$, where $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Then, $\tan x = \tan\frac{3\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\left(\frac{-\pi}{4}\right) \Rightarrow x = \frac{-\pi}{4}$.

18. Let $\sin^{-1}\left(\frac{-1}{2}\right) = x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Then, $\sin x = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right) \Rightarrow x = \frac{-\pi}{6}$.

$\therefore \quad$ given exp. $= \frac{\pi}{3} - \left(\frac{-\pi}{6}\right) = \left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \frac{3\pi}{6} = \frac{\pi}{2}$.

19. $\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2} = \frac{\pi}{2} \Rightarrow \sin\left(\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}\right) = \sin\frac{\pi}{2} = 1.$

20. $\cos(\tan^{-1} x + \cot^{-1} x) = \cos\frac{\pi}{2} = 0.$

21. $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} \Rightarrow \cos^{-1}\frac{3}{5} = \sin^{-1}\sqrt{1-\frac{9}{25}} = \sin^{-1}\frac{4}{5}$.

$\therefore \quad \sin\left(\cos^{-1}\frac{3}{5}\right) = \sin\left(\sin^{-1}\frac{4}{5}\right) = \frac{4}{5}$.

22. $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$$= \cos^{-1}\left\{\cos\left(\pi - \frac{\pi}{3}\right)\right\} + \sin^{-1}\left\{\sin\left(\pi - \frac{\pi}{3}\right)\right\}$$

$$= \cos^{-1}\left(-\cos\frac{\pi}{3}\right) + \sin^{-1}\left\{\sin\frac{\pi}{3}\right\} = \cos^{-1}\left(\frac{-1}{2}\right) + \frac{\pi}{3}$$

$$= \left(\frac{2\pi}{3} + \frac{\pi}{3}\right) = \pi.$$

23. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \tan^{-1}\sqrt{3} - (\pi - \sec^{-1}2)$

$$= \frac{\pi}{3} - \pi + \frac{\pi}{3} = \frac{-\pi}{3}.$$

24. $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2} = \frac{\pi}{3} + \left(2\times\frac{\pi}{6}\right) = \left(\frac{\pi}{3} + \frac{\pi}{3}\right) = \frac{2\pi}{3}$.

25. $\tan^{-1}1 + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right) = \frac{\pi}{4} + \left(\pi - \cos^{-1}\frac{1}{2}\right) - \sin^{-1}\frac{1}{2}$

$$= \left(\frac{\pi}{4} + \pi - \frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{3\pi}{4}.$$

26. Let $2\tan^{-1}\frac{1}{5} = \theta$. Then, $\tan^{-1}\frac{1}{5} = \frac{1}{2}\theta \Rightarrow \tan\frac{1}{2}\theta = \frac{1}{5}$.

$$\therefore \quad \tan\theta = \frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta} = \frac{\left(2\times\frac{1}{5}\right)}{\left(1-\frac{1}{25}\right)} = \left(\frac{2}{5}\times\frac{25}{24}\right) = \frac{5}{12}.$$

$$\therefore \quad \tan\left[2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right] = \tan\left(\theta - \frac{\pi}{4}\right) = \frac{\tan\theta - \tan\frac{\pi}{4}}{1+\tan\theta\cdot\tan\frac{\pi}{4}}$$

$$= \frac{\left(\frac{5}{12} - 1\right)}{\left(1 + \frac{5}{12}\times 1\right)} = \frac{\left(\frac{-7}{12}\right)}{\left(\frac{17}{12}\right)} = \frac{-7}{17}.$$

27. Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$. Then, $\cos\theta = \frac{\sqrt{5}}{3}$.

$$\tan\frac{1}{2}\left(\cos^{-1}\frac{\sqrt{5}}{3}\right) = \tan\frac{1}{2}\theta = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$$

$$= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \left\{\frac{\left(1-\frac{\sqrt{5}}{3}\right)}{\left(1+\frac{\sqrt{5}}{3}\right)}\right\}^{1/2}$$

$$= \left(\frac{3-\sqrt{5}}{3+\sqrt{5}}\right)^{1/2} = \left\{\frac{3-\sqrt{5}}{3+\sqrt{5}} \times \frac{3-\sqrt{5}}{3-\sqrt{5}}\right\}^{1/2} = \frac{(3-\sqrt{5})}{2}.$$

28. Let $\cos^{-1}\frac{3}{5} = x$, where $x \in [0, \pi]$. Then, $\cos x = \frac{3}{5}\cdot$

$\therefore$ since $x \in [0, \pi]$, $\sin x > 0$.

$\therefore$ $\sin x = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \Rightarrow \sin\left(\cos^{-1}\frac{3}{5}\right) = \frac{4}{5}\cdot$

29. Let $\tan^{-1}\frac{3}{4} = x$, where $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\cdot$

$\therefore$ $\tan x = \frac{3}{4}$ and since $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, we have $\cos x > 0$.

$\therefore$ $\cos x = \frac{1}{\sec x} = \frac{1}{\sqrt{1+\tan^2 x}} = \frac{1}{\sqrt{1+\frac{9}{16}}} = \frac{4}{5}\cdot$

$\therefore$ $\cos\left(\tan^{-1}\frac{3}{4}\right) = \cos x = \frac{4}{5}\cdot$

30. $\sin\left\{\frac{\pi}{3} - \sin^{-3}\left(\frac{-1}{2}\right)\right\} = \sin\left\{\frac{\pi}{3} + \sin^{-1}\frac{1}{2}\right\} = \sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \sin\frac{\pi}{2} = 1.$

31. Let $\cos^{-1}\frac{4}{5} = x$, where $x \in [0, \pi]$. Then, $\cos x = \frac{4}{5}\cdot$

Since $x \in [0, \pi] \to \frac{1}{2}x \in \left[0, \frac{\pi}{2}\right] \Rightarrow \sin\frac{1}{2}x > 0.$

$$\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right) = \sin\frac{1}{2}x = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{\left(1-\frac{4}{5}\right)}{2}} = \frac{1}{\sqrt{10}}\cdot$$

32. $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\} = \tan^{-1}\left\{2\cos\left(2\times\frac{\pi}{6}\right)\right\}$

$$= \tan^{-1}\left\{2\cos\frac{\pi}{3}\right\} = \tan^{-1}\left\{2\times\frac{1}{2}\right\} = \tan^{-1}1 = \frac{\pi}{4}\cdot$$

33. $\cot^{-1}\left(\frac{-1}{5}\right) = x \Rightarrow \cot x = \frac{-1}{5}$, where $x \in (0, \pi)$.

$\sin x > 0$ in $(0, \pi)$.

$\sin x = \frac{1}{\operatorname{cosec} x} = \frac{1}{\sqrt{1+\cot^2 x}} = \frac{1}{\sqrt{1+\frac{1}{25}}} = \frac{5}{\sqrt{26}}\cdot$

34. Range of $\cos^{-1}$ is $[0, \pi]$.

$$\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = x \Rightarrow \cos x = \frac{-\sqrt{3}}{2} = -\cos\frac{\pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right) = \cos\frac{5\pi}{6} \Rightarrow x = \frac{5\pi}{6}.$$

$$\sin^{-1}\left(\frac{-1}{2}\right) + 2\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\sin^{-1}\frac{1}{2} + 2 \times \frac{5\pi}{6} = -\frac{\pi}{6} + \frac{5\pi}{3} = \frac{9\pi}{6} = \frac{3\pi}{2}.$$

35. Range of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\tan^{-1}(-1) = x \Rightarrow \tan x = -1 = -\tan\frac{\pi}{4} = \tan\left(\frac{-\pi}{4}\right) \Rightarrow x = \frac{-\pi}{4}$$

Range of $\cos^{-1}$ is $[0, \pi]$.

$$\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = y \Rightarrow \cos y = \frac{-1}{\sqrt{2}} = -\cos\frac{\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4} \Rightarrow y = \frac{3\pi}{4}.$$

$$\therefore \quad \text{given exp.} = -\frac{\pi}{4} + \frac{3\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}.$$

36. Given exp. $= \cot\frac{\pi}{2} = 0.$

37. $\tan^{-1} 1 + \tan^{-1}\frac{1}{3} = \tan^{-1}\left\{\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}}\right\} = \tan^{-1} 2.$

38. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\left\{\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}}\right\} = \tan^{-1} 1 = \frac{\pi}{4}.$

39. Use $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right).$

40. Use $2\tan^{-1} x = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right).$

41. Use $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1 + x^2}\right).$

42. Use $2\sin^{-1} x = \sin^{-1}[2x\sqrt{1 - x^2}].$

43. $\frac{\pi}{4} - \tan^{-1}\frac{1}{3} = \tan^{-1} 1 - \tan^{1}\frac{1}{3} = \tan^{-1}\left\{\frac{\left(1 - \frac{1}{3}\right)}{\left(1 + \frac{1}{3}\right)}\right\} = \tan^{-1}\frac{1}{2} \Rightarrow x = \frac{1}{2}.$

44. We know that $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \Rightarrow \tan^{-1} x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}.$

$$\therefore \quad \tan^{-1}(1+x) + \tan^{-1}(1-x) = \frac{\pi}{2} \Rightarrow (1-x) = \frac{1}{(1+x)}$$

$$\Rightarrow (1-x^2) = 1 \Rightarrow x = 0.$$

45. $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3} \Rightarrow \left(\frac{\pi}{2} - \cos^{-1} x\right) + \left(\frac{\pi}{2} - \cos^{-1} y\right) = \frac{2\pi}{3}\cdot$

$$\therefore \quad \cos^{-1} x + \cos^{-1} y = \left(\pi - \frac{2\pi}{3}\right) = \frac{\pi}{3}\cdot$$

46. $(x = 2, y = 3) \Rightarrow xy > 1.$

$$\therefore \quad \tan^{-1} 2 + \tan^{-1} 3 = \pi + \tan^{-1}\left(\frac{2+3}{1-2\times 3}\right) = \pi + \tan^{-1}(-1)$$

$$= \pi - \tan(1) = \left(\pi - \frac{\pi}{4}\right) = \frac{3\pi}{4}\cdot$$

47. $\tan^{-1} x + \tan^{-1} 3 = \tan^{-1} 8 \Rightarrow \tan^{-1}\left(\frac{3+x}{1-3x}\right) = \tan^{-1} 8.$

$$\therefore \quad \frac{3+x}{1-3x} = 8 \Rightarrow 3 + x = 8 - 24x \Rightarrow x = \frac{1}{5}\cdot$$

48. $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4} \Rightarrow \tan^{-1}\left(\frac{3x+2x}{1-6x^2}\right) = \frac{\pi}{4}$

$$\therefore \quad \frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1 \Rightarrow 6x^2 + 5x - 1 = 0$$

$$\Rightarrow (x+1)(6x-1) = 0$$

$$\Rightarrow x = -1 \text{ or } x = \frac{1}{6}\cdot$$

49. $\cos^{-1} x = \tan^{-1} \frac{\sqrt{1-x^2}}{x} \Rightarrow \cos^{-1}\frac{4}{5} = \tan^{-1} \frac{\sqrt{1-\frac{16}{25}}}{\left(\frac{4}{5}\right)} = \tan^{-1}\frac{3}{4}$

$$\therefore \quad \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3} = \tan^{-1}\frac{\left(\frac{3}{4}+\frac{2}{3}\right)}{\left(1-\frac{3}{4}\times\frac{2}{3}\right)} = \tan^{-1}\frac{17}{6}$$

$$\therefore \quad \text{given exp.} = \tan\left\{\tan^{-1}\frac{17}{6}\right\} = \frac{17}{6}\cdot$$

50. $\operatorname{cosec}^{-1} x = \cot^{-1}\sqrt{x^2-1} \Rightarrow \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4} = \cot^{-1}\sqrt{\frac{41}{16}-1} = \cot^{-1}\frac{5}{4}$

$$\therefore \quad \cot^{-1} 9 + \cot^{-1}\frac{5}{4} = \tan^{-1}\frac{1}{9} + \tan^{-1}\frac{4}{5} = \tan^{-1}\left(\frac{\frac{1}{9}+\frac{4}{5}}{1-\frac{1}{9}\times\frac{4}{5}}\right) = \tan^{-1} 1 = \frac{\pi}{4}\cdot$$

5. MATRICES

MATRIX *A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns) is called a matrix of order m by n, written as an* $m \times n$ *matrix.*

Such an array is enclosed by [ ] or ( ).

Each of the *mn* numbers constituting the matrix is called an *element* or an *entry* of the matrix.

Usually, we denote a matrix by a capital letter.

The plural of matrix is matrices.

Example (i) $A = \begin{bmatrix} 3 & 5 & -4 \\ 0 & 1 & 9 \end{bmatrix}$ is a matrix, having 2 rows and 3 columns. Its order is 2×3 and it has 6 elements.

(ii) $B = \begin{bmatrix} 9 & 4 & \sqrt{2} & -1 \\ 1 & 8 & -3 & 2 \\ 6 & 0 & 5 & 7 \end{bmatrix}$ is a matrix, having 3 rows and 4 columns. Its order is 3×4 and it has 12 elements.

How to Describe a Matrix

In order to locate the position of a particular element of a matrix, we have to specify the number of the row and that of the column in which the element occurs.

An element occurring in the ith row and jth column of a matrix A will be called the (i, j)*th element of* A*, to be denoted by* a_{ij}.

In general, an $m \times n$ matrix A may be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}.$$

EXAMPLE 1 *Consider the matrix* $A = \begin{bmatrix} 3 & -2 & 5 \\ 6 & 9 & 1 \end{bmatrix}$.

Clearly, the element in the 1st row and 2nd column is –2.

So, we write $a_{12} = -2$.

Similarly, $a_{11} = 3$; $a_{12} = -2$; $a_{13} = 5$; $a_{21} = 6$; $a_{22} = 9$ and $a_{23} = 1$.

EXAMPLE 2 *Construct a 3×2 matrix whose elements are given by $a_{ij} = (i + 2j)$.*

SOLUTION A 3×2 matrix has 3 rows and 2 columns.

In general, a 3×2 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3 \times 2.}$$

Thus $a_{ij} = (i + 2j)$ for $i = 1, 2, 3$ and $j = 1, 2$.

$\therefore \quad a_{11} = (1 + 2 \times 1) = 3;\ a_{12} = (1 + 2 \times 2) = 5;$

$a_{21} = (2 + 2 \times 1) = 4;\ a_{22} = (2 + 2 \times 2) = 6;$

$a_{31} = (3 + 2 \times 1) = 5;\ a_{32} = (3 + 2 \times 2) = 7.$

Hence, $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \\ 5 & 7 \end{bmatrix}_{3 \times 2.}$

EXAMPLE 3 *Construct a 2×3 matrix whose elements are given by $a_{ij} = \frac{1}{2}|5i - 3j|$.*

SOLUTION A 2×3 matrix has 2 rows and 3 columns.

In general, a 2×3 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{3 \times 2.}$$

Thus, $a_{ij} = \frac{1}{2}|5i - 3j|$ where $i = 1, 2$ and $j = 1, 2, 3$.

$\therefore \quad a_{11} = \frac{1}{2}|5 \times 1 - 3 \times 1| = \frac{1}{2} \cdot |2| = \frac{1}{2} \times 2 = 1;$

$a_{12} = \frac{1}{2}|5 \times 1 - 3 \times 2| = \frac{1}{2} \cdot |5 - 6| = \frac{1}{2} \cdot |-1| = \frac{1}{2} \times 1 = \frac{1}{2};$

$a_{13} = \frac{1}{2}|5 \times 1 - 3 \times 3| = \frac{1}{2} \cdot |5 - 9| = \frac{1}{2} \cdot |-4| = \frac{1}{2} \times 4 = 2;$

$a_{21} = \frac{1}{2}|5 \times 2 - 3 \times 1| = \frac{1}{2} \cdot |10 - 3| = \frac{1}{2} \cdot |7| = \frac{1}{2} \times 7 = \frac{7}{2};$

$a_{22} = \frac{1}{2}|5 \times 2 - 3 \times 2| = \frac{1}{2} \cdot |10 - 6| = \frac{1}{2} \cdot |4| = \frac{1}{2} \times 4 = 2;$

$a_{23} = \frac{1}{2}|5 \times 2 - 3 \times 3| = \frac{1}{2} \cdot |10 - 9| = \frac{1}{2} \cdot |1| = \frac{1}{2} \times 1 = \frac{1}{2} \cdot$

Hence, $A = \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ \frac{7}{2} & 2 & \frac{1}{2} \end{bmatrix} \cdot$

EXAMPLE 4 *If a matrix has 12 elements, what are the possible orders it can have?*

SOLUTION We know that a matrix of order $m \times n$ has mn elements.

Hence, all possible orders of a matrix having 12 elements are (12×1), (1×12), (6×2), (2×6), (4×3) and (3×4).

EXERCISE 5A

1. If $A = \begin{bmatrix} 5 & -2 & 6 & 1 \\ 7 & 0 & 8 & -3 \\ \sqrt{2} & \frac{3}{5} & 4 & 3 \end{bmatrix}$ then write

   (i) the number of rows in A, (ii) the number of columns in A,
   (iii) the order of the matrix A, (iv) the number of all entries in A,
   (v) the elements a_{23}, a_{31}, a_{14}, a_{33}, a_{22} of A.

2. Write the order of each of the following matrices:

   (i) $A = \begin{bmatrix} 3 & 5 & 4 & -2 \\ 0 & \sqrt{3} & -1 & \frac{4}{9} \end{bmatrix}$ (ii) $B = \begin{bmatrix} 6 & -5 \\ \frac{1}{2} & \frac{3}{4} \\ -2 & -1 \end{bmatrix}$

   (iii) $C = [7 \quad -\sqrt{2} \quad 5 \quad 0]$ (iv) $D = [8 \quad -3]$

   (v) $E = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ (vi) $F = [6]$

3. If a matrix has 18 elements, what are the possible orders it can have?
4. Find all possible orders of matrices having 7 elements.
5. Construct a 3×2 matrix whose elements are given by $a_{ij} = (2i - j)$.
6. Construct a 4×3 matrix whose elements are given by $a_{ij} = \frac{i}{j}$.
7. Construct a 2×2 matrix whose elements are $a_{ij} = \frac{(i + 2j)^2}{2}$. **[CBSE 2002, '07]**
8. Construct a 2×3 matrix whose elements are $a_{ij} = \frac{(i - 2j)^2}{2}$. **[CBSE 2002]**
9. Construct a 3×4 matrix whose elements are given by $a_{ij} = \frac{1}{2}|-3i + j|$.

ANSWERS (EXERCISE 5A)

1. (i) 3 (ii) 4 (iii) 3×4 (iv) 12 (v) $a_{23} = 8$, $a_{31} = \sqrt{2}$, $a_{14} = 1$, $a_{33} = 4$, $a_{22} = 0$

2. (i) (2×4) (ii) (3×2) (iii) (1×4) (iv) (1×2) (v) (3×1) (vi) (1×1)

3. (18×1), (1×18), (9×2), (2×9), (6×3), (3×6) **4.** (7×1), (1×7)

5. $A = \begin{bmatrix} 1 & 9 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 3 & \frac{3}{2} & 1 \\ 4 & 2 & \frac{4}{3} \end{bmatrix}$ **7.** $\begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$

8. $\begin{bmatrix} \frac{1}{2} & \frac{9}{2} & \frac{25}{2} \\ 0 & 2 & 8 \end{bmatrix}$ **9.** $\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$

Various Types of Matrices

ROW MATRIX *A matrix having only one row is known as a row matrix or a row vector.*

Examples (i) $A = [5 \;\; 18]$ is a row matrix of order 1×2.

(ii) $B = [2 \;\; \sqrt{5} \;\; -9 \;\; 0]$ is a row matrix of order 1×4.

COLUMN MATRIX *A matrix having only one column is known as a column matrix or a column vector.*

Examples (i) $A = \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a column matrix of order 3×1.

(ii) $B = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ is a column matrix of order 2×1.

ZERO OR NULL MATRIX *A matrix each of whose elements is zero is called a zero matrix or a null matrix.*

Examples The matrices $[0]$, $[0 \;\; 0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are null matrices of order (1×1), (1×2), (2×2) and (2×3) respectively.

SQUARE MATRIX *A matrix having the same number of rows and columns is called a square matrix.*

A matrix of order $(n \times n)$ is called a *square matrix of order n* or an *n-rowed square matrix.*

A matrix of order $m \times n$, where $m \neq n$, is called a *rectangular matrix.*

Examples (i) The matrix $\begin{bmatrix} 3 & 2 \\ 6 & -5 \end{bmatrix}$ is a 2-rowed square matrix.

(ii) The matrix $\begin{bmatrix} 5 & 3 & 6 \\ 7 & \sqrt{2} & -4 \\ -9 & \frac{1}{3} & 0 \end{bmatrix}$ is a 3-rowed square matrix.

DIAGONAL ELEMENTS OF A MATRIX *Let $A = [a_{ij}]_{m \times n}$ be an $m \times n$ matrix. Then, the elements a_{ij} for which $i = j$, are called the diagonal elements of A.*

Thus, the diagonal elements of $A = [a_{ij}]_{m \times n}$ are $a_{11}, a_{22}, a_{33}, a_{44}$, etc.

The line along which the diagonal elements lie is called the diagonal of the matrix.

Example Let $A = \begin{bmatrix} 3 & 2 & -1 \\ \sqrt{5} & \frac{5}{8} & 7 \\ 6 & -4 & \sqrt{2} \end{bmatrix}$

Then, the diagonal elements of A are:

$$a_{11} = 3, \; a_{22} = \frac{5}{8}, \; a_{33} = \sqrt{2}.$$

DIAGONAL MATRIX *A square matrix in which every nondiagonal element is zero is called a diagonal matrix.*

If $A = [a_{ij}]_{n \times n}$ be a diagonal matrix then $a_{ij} = 0$ when $i \neq j$ and we write it as

$$A = \text{diag}\,[a_{11}, \; a_{22}, \; a_{33}, \; \ldots, \; a_{nn}].$$

Example Let $A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Then, A is a diagonal matrix.

We may write it as, $A = \text{diag}\,[6, \; 4, \; -2]$.

SCALAR MATRIX *A square matrix in which every nondiagonal element is zero and all diagonal elements are equal is known as a scalar matrix.*

Examples (i) $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ is a scalar matrix of order 2.

(ii) $B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ is a scalar matrix of order 3.

UNIT MATRIX *A square matrix in which every nondiagonal element is* 0 *and every diagonal element is* 1 *is called a unit matrix or an identity matrix.*

Thus, a square matrix $[a_{ij}]_{n \times n}$ is a unit matrix if

$$a_{ij} = \begin{cases} 0 \text{ when } i \neq j, \\ 1 \text{ when } i = j. \end{cases}$$

A unit matrix of order n will be denoted by I_n or simply by I.

Examples (i) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

(ii) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

COMPARABLE MATRICES *Two matrices A and B are said to be comparable if they are of the same order, i.e., they have the same number of rows and the same number of columns.*

Example $A = \begin{bmatrix} 2 & -5 & 1 \\ 0 & 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 & 0 \\ 1 & 4 & -9 \end{bmatrix}$ are comparable matrices, each being of order (2×3).

EQUAL MATRICES *Two matrices A and B are said to be equal, written as $A = B$, if they are of the same order and their corresponding elements are equal.*

EXAMPLE 1 *Find x, y, z when* $\begin{bmatrix} 5 & 3 \\ x & 7 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 7 \end{bmatrix}$.

SOLUTION Since the corresponding elements of equal matrices are equal, we have

$$\begin{bmatrix} 5 & 3 \\ x & 7 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 7 \end{bmatrix} \Leftrightarrow x = 1, \ y = 5 \text{ and } z = 3.$$

EXAMPLE 2 *Find x, y, z, w when* $\begin{bmatrix} x - y & 2x + z \\ 2x - y & 3z + w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$. **[CBSE 2013]**

SOLUTION We know that in equal matrices, the corresponding elements are equal.

$$\therefore \begin{bmatrix} x - y & 2x + z \\ 2x - y & 3z + w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

$$\Leftrightarrow x - y = -1, \ 2x - y = 0, \ 2x + z = 5 \text{ and } 3z + w = 13.$$

Solving the first two equations, we get $x = 1$ and $y = 2$.

Putting $x = 1$ in $2x + z = 5$, we get $z = 3$.

Putting $z = 3$ in $3z + w = 13$, we get $w = 4$.

$\therefore\ x = 1,\ y = 2,\ z = 3$ and $w = 4$.

EXAMPLE 3 $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. *Why?*

SOLUTION Since the given null matrices are not comparable, they are not equal.

Operations on Matrices

ADDITION OF MATRICES *Let A and B be two comparable matrices, each of order $(m \times n)$. Then, their sum $(A + B)$ is a matrix of order $(m \times n)$, obtained by adding the corresponding elements of A and B.*

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then

$$A + B = [a_{ij} + b_{ij}]_{m \times n}.$$

REMARK For two matrices A and B, the sum $(A + B)$ exists only when A and B are comparable.

Example 1 If $A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ then A and B are matrices of order 2×2 and 2×3 respectively.

So, A and B are not comparable.

Hence, $A + B$ is not defined.

Example 2 Let $A = \begin{bmatrix} 5 & 0 & -2 \\ 3 & 2 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 & -6 \\ -1 & 0 & 4 \end{bmatrix}$.

Clearly, each one of A and B is a 2×3 matrix.

So, A and B are comparable matrices.

$\therefore\ A + B$ is defined.

Now, $A + B = \begin{bmatrix} 5+4 & 0+(-3) & -2+(-6) \\ 3+(-1) & 2+0 & -7+4 \end{bmatrix}$

$= \begin{bmatrix} 9 & -3 & -8 \\ 2 & 2 & -3 \end{bmatrix}$.

Some Results on Addition of Matrices

THEOREM 1 *Matrix addition is commutative,*
i.e., $A + B = B + A$ for all comparable matrices A and B.

PROOF Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then,

$$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$$
$$= [a_{ij} + b_{ij}]_{m \times n} \quad \text{[by the definition of addition of matrices]}$$
$$= [b_{ij} + a_{ij}]_{m \times n} \quad [\because \text{ addition of numbers is commutative}]$$
$$= [b_{ij}]_{m \times n} + [a_{ij}]_{m \times n} = B + A.$$

Hence, $A + B = B + A$.

THEOREM 2 *Matrix addition is commutative,*
i.e, $(A + B) + C = A + (B + C)$ *for all comparable matrices A, B, C.*

PROOF Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ and $C = [c_{ij}]_{m \times n}$. Then,

$$(A + B) + C = ([a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}) + [c_{ij}]_{m \times n}$$
$$= [a_{ij} + b_{ij}]_{m \times n} + [c_{ij}]_{m \times n}$$
$$= [(a_{ij} + b_{ij}) + c_{ij}]_{m \times n}$$
$$= [a_{ij} + (b_{ij} + c_{ij})]_{m \times n}$$
$$[\because \text{ addition of numbers is associative}]$$
$$= [a_{ij}]_{m \times n} + [b_{ij} + c_{ij}]_{m \times n}$$
$$= [a_{ij}]_{m \times n} + ([b_{ij}] + [c_{ij}]_{m \times n}) = A + (B + C).$$

Hence, $(A + B) + C = A + (B + C)$.

THEOREM 3 *If A is an* $m \times n$ *matrix and O is an* $m \times n$ *null matrix then*

$$A + O = O + A = A.$$

PROOF Let $A = [a_{ij}]_{m \times n}$ and $O = [b_{ij}]_{m \times n}$,
where $b_{ij} = 0$ for all suffixes i and j.

Then, $A + O = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$
$$= [a_{ij} + 0]_{m \times n} \quad [\because b_{ij} = 0]$$
$$= [a_{ij}]_{m \times n} = A.$$

$\therefore \quad A + O = A.$

Similarly, $O + A = A$.

Hence, $A + O = O + A = A$.

REMARK The null matrix O of order $m \times n$ is the additive identity in the set of all $m \times n$ matrices.

EXAMPLE 3 *Let* $A = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix}$ *and* $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *then verify that*
$A + O = O + A = A$.

SOLUTION Clearly, each one of A and O is a matrix of order (2×3).
So, $(A + O)$ and $(O + A)$ are both defined.

Now, $A + O = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 3+0 & 5+0 & 4+0 \\ 1+0 & 2+0 & -3+0 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} = A.$$

And, $O + A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 0+3 & 0+5 & 0+4 \\ 0+1 & 0+2 & 0+(-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} = A.$$

Hence, $A + O = O + A = A$.

NEGATIVE OF A MATRIX *Let $A = [a_{ij}]_{m \times n}$. Then, the negative of A is the matrix $(-A) = [-a_{ij}]_{m \times n}$, obtained by replacing each element of A with its corresponding additive inverse. $(-A)$ is called the additive inverse of A.*

EXAMPLE 4 *If $A = \begin{bmatrix} 3 & -2 & 0 \\ -5 & 7 & \sqrt{2} \end{bmatrix}$, find $(-A)$ and verify that*

$A + (-A) = (-A) + A = 0.$

SOLUTION Clearly, we have:

$$(-A) = \begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & -\sqrt{2} \end{bmatrix}$$

$$A + (-A) = \begin{bmatrix} 3 & -2 & 0 \\ -5 & 7 & \sqrt{2} \end{bmatrix} + \begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & -\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3+(-3) & -2+2 & 0+0 \\ -5+5 & 7+(-7) & \sqrt{2}+(-\sqrt{2}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

And, $(-A) + A = \begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & -\sqrt{2} \end{bmatrix} + \begin{bmatrix} 3 & -2 & 0 \\ -5 & 7 & \sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} -3+3 & 2+(-2) & 0+0 \\ 5+(-5) & -7+7 & -\sqrt{2}+\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $A + (-A) = (-A) + A = O$.

SUMMARY

Laws of Addition on Matrices

(i) $A + B = B + A$ [Commutative law]

(ii) $(A + B) + C = A + (B + C)$ [Associative law]

(iii) $A + O = O + A = A$

(iv) $A + (-A) = (-A) + A = O$

SOLVED EXAMPLES

EXAMPLE 1 *Let $A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 6 & -1 \end{bmatrix}$.*

Verify that $A + B = B + A$.

SOLUTION Here, A is a 2×3 matrix and B is a 2×3 matrix. So, A and B are comparable.

Therefore, $(A + B)$ and $(B + A)$ both exist and each is a 2×3 matrix.

$$\text{Now, } A + B = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 3 \\ 2 & 6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4 & 3+(-2) & 5+3 \\ -1+2 & 0+6 & 4+(-1) \end{bmatrix} = \begin{bmatrix} 6 & 1 & 8 \\ 1 & 6 & 3 \end{bmatrix}.$$

$$\text{And, } B + A = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 6 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2 & -2+3 & 3+5 \\ 2+(-1) & 6+0 & (-1)+4 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 8 \\ 1 & 6 & 3 \end{bmatrix}.$$

Hence, $A + B = B + A$.

EXAMPLE 2 *Let* $A = \begin{bmatrix} 1 & -2 \\ 5 & 4 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -3 & 5 \end{bmatrix}$ *and* $C = \begin{bmatrix} 4 & 3 \\ -2 & 2 \\ 1 & 6 \end{bmatrix}$.

Verify that $(A + B) + C = A + (B + C)$.

SOLUTION Clearly, each one of the matrices A, B, C is a (3×2) matrix. So, $(A + B) + C$ and $A + (B + C)$ are both defined and each one is a 3×2 matrix.

$$\text{Now, } (A + B) = \begin{bmatrix} 1 & -2 \\ 5 & 4 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3 & -2+1 \\ 5+0 & 4+2 \\ 3+(-3) & 0+5 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ 5 & 6 \\ 0 & 5 \end{bmatrix}.$$

$$\therefore \quad (A + B) + C = \begin{bmatrix} 4 & -1 \\ 5 & 6 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & 2 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4 & -1+3 \\ 5+(-2) & 6+2 \\ 0+1 & 5+6 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 8 \\ 1 & 11 \end{bmatrix}.$$

$$\text{Also, } (B + C) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & 2 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3+4 & 1+3 \\ 0+(-2) & 2+2 \\ -3+1 & 5+6 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ -2 & 4 \\ -2 & 11 \end{bmatrix}$$

$$\therefore \quad A + (B + C) = \begin{bmatrix} 1 & -2 \\ 5 & 4 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 4 \\ -2 & 4 \\ -2 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 1+7 & -2+4 \\ 5+(-2) & 4+4 \\ 3+(-2) & 0+11 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 8 \\ 1 & 11 \end{bmatrix}.$$

Hence, $(A + B) + C = A + (B + C)$.

EXAMPLE 3 *Find the additive inverse of the matrix* $A = \begin{bmatrix} 2 & -5 & 0 \\ 4 & 3 & -1 \end{bmatrix}$.

SOLUTION Clearly, the additive inverse of the given matrix A is the matrix $-A$, given by

$$-A = \begin{bmatrix} -2 & -(-5) & 0 \\ -4 & -3 & -(-1) \end{bmatrix} = \begin{bmatrix} -2 & 5 & 0 \\ -4 & -3 & 1 \end{bmatrix}.$$

SUBTRACTION OF MATRICES *If A and B are two comparable matrices then we define*

$$(A - B) = A + (-B).$$

EXAMPLE 4 *If* $A = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 7 & -9 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 8 & -4 \end{bmatrix}$, *find* $(A - B)$.

SOLUTION We have, $(-B) = \begin{bmatrix} -1 & -2 & 3 \\ -4 & -8 & 4 \end{bmatrix}$.

$$\therefore \quad (A - B) = A + (-B)$$

$$= \begin{bmatrix} 2 & -3 & 1 \\ 0 & 7 & -9 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 3 \\ -4 & -8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+(-1) & -3+(-2) & 1+3 \\ 0+(-4) & 7+(-8) & -9+4 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 4 \\ -4 & -1 & -5 \end{bmatrix}.$$

Hence, $(A - B) = \begin{bmatrix} 1 & -5 & 4 \\ -4 & -1 & -5 \end{bmatrix}$.

EXAMPLE 5 *If* $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$, *then find the value of* $(a - 2b)$. **[CBSE 2014]**

SOLUTION Comparing the corresponding elements of given equal matrices, we have $a + 4 = 2a + 2$, $3b = b + 2$ and $a - 8b = -6$.

From these equations, we get $a = 2$ and $b = 1$.

$\therefore \quad (a - 2b) = (2 - 2 \times 1) = (2 - 2) = 0.$

EXAMPLE 6 *If* $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, *then find the matrix* A. **[CBSE 2013]**

SOLUTION Clearly, we have

$$A = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & -1-2 & 4+1 \\ -2-0 & 1-4 & 3-9 \end{bmatrix} = \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}.$$

SCALAR MULTIPLICATION *Let A be a given matrix and k be a number. Then, the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k, to be denoted by kA.*

If A is an $(m \times n)$ matrix then kA is also an $(m \times n)$ matrix.

If $A = [a_{ij}]_{m \times n}$ then $kA = [k \cdot a_{ij}]_{m \times n}$.

EXAMPLE 7 *If* $A = \begin{bmatrix} 5 & 4 & -2 \\ 6 & -1 & 7 \end{bmatrix}$, *find (i)* $3A$ *(ii)* $\frac{1}{2}A$ *(iii)* $-2A$.

SOLUTION We have:

(i) $$3A = \begin{bmatrix} 3 \cdot 5 & 3 \cdot 4 & 3 \cdot (-2) \\ 3 \cdot 6 & 3 \cdot (-1) & 3 \cdot 7 \end{bmatrix} = \begin{bmatrix} 15 & 12 & -6 \\ 18 & -3 & 21 \end{bmatrix}.$$

(ii) $$\frac{1}{2}A = \begin{bmatrix} \frac{1}{2} \cdot 5 & \frac{1}{2} \cdot 4 & \frac{1}{2} \cdot (-2) \\ \frac{1}{2} \cdot 6 & \frac{1}{2} \cdot (-1) & \frac{1}{2} \cdot 7 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 2 & -1 \\ 3 & -\frac{1}{2} & \frac{7}{2} \end{bmatrix}.$$

(iii) $$-2A = (-2) \cdot A = \begin{bmatrix} (-2) \cdot 5 & (-2) \cdot 4 & (-2) \cdot (-2) \\ (-2) \cdot 6 & (-2) \cdot (-1) & (-2) \cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -8 & 4 \\ -12 & 2 & -14 \end{bmatrix}.$$

Some Properties of Scalar Multiplication

THEOREM 1 *If A and B are two matrices of the same order and k is a scalar then prove that* $k(A + B) = kA + kB$.

PROOF Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then,

$$\begin{aligned} k(A + B) &= k \cdot ([a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}) \\ &= k \cdot [a_{ij} + b_{ij}]_{m \times n} \quad \text{[by definition of addition of matrices]} \\ &= [k(a_{ij} + b_{ij})]_{m \times n} \quad \text{[by definition of scalar multiplication]} \\ &= [k \cdot a_{ij} + k \cdot b_{ij}]_{m \times n} \quad \text{[by distributive law]} \\ &= [k \cdot a_{ij}]_{m \times n} + [k \cdot b_{ij}]_{m \times n} \\ &= kA + kB. \end{aligned}$$

Hence, $k(A + B) = kA + kB$.

THEOREM 2 *If A is any matrix and k_1, k_2 are any scalars then prove that*
(i) $(k_1 + k_2)A = k_1A + k_2A$ *(ii)* $k_1(k_2A) = (k_1k_2)A$.

PROOF Let $A = [a_{ij}]_{m \times n}$. Then,

(i) $(k_1 + k_2)A = (k_1 + k_2) \cdot [a_{ij}]_{m \times n}$

$= [(k_1 + k_2) \cdot a_{ij}]_{m \times n}$ [by definition of scalar multiplication]

$= [k_1 \cdot a_{ij} + k_2 \cdot a_{ij}]_{m \times n}$ [by distributive law]

$= [k_1 \cdot a_{ij}]_{m \times n} + [k_2 \cdot a_{ij}]_{m \times n}$

[by definition of addition of matrices]

$= k_1A + k_2A$.

Hence, $(k_1 + k_2)A = k_1A + k_2A$.

(ii) $k_1(k_2A) = k_1[k_2 \cdot a_{ij}]_{m \times n} = [k_1(k_2 \cdot a_{ij})]_{m \times n}$

$= [(k_1k_2) \cdot a_{ij}]_{m \times n}$

[by associativity of multiplication in numbers]

$= (k_1k_2) \cdot [a_{ij}]_{m \times n} = (k_1k_2)\ A$.

Hence, $k_1(k_2A) = (k_1k_2)A$.

SUMMARY

I. If A and B are comparable matrices and k is a scalar then
$k(A + B) = (kA + kB)$.

II. If k_1, k_2 are scalars and A is any matrix then
(i) $(k_1 + k_2)A = (k_1A + k_2A)$
(ii) $k_1(k_2A) = (k_1k_2)A$

SOLVED EXAMPLES

EXAMPLE 1 *If* $A = \begin{bmatrix} 3 & 5 \\ 7 & -9 \end{bmatrix}$ *and* $B = \begin{bmatrix} 6 & -4 \\ 2 & 3 \end{bmatrix}$, *find* $(4A - 3B)$.

SOLUTION $(4A - 3B) = 4A + (-3B)$.

Now, $4A = \begin{bmatrix} 4 \cdot 3 & 4 \cdot 5 \\ 4 \cdot 7 & 4 \cdot (-9) \end{bmatrix} = \begin{bmatrix} 12 & 20 \\ 28 & -36 \end{bmatrix}$.

And, $-3B = (-3) \cdot B = \begin{bmatrix} (-3) \cdot 6 & (-3) \cdot (-4) \\ (-3) \cdot 2 & (-3) \cdot 3 \end{bmatrix} = \begin{bmatrix} -18 & 12 \\ -6 & -9 \end{bmatrix}$.

$\therefore \quad 4A - 3B = 4A + (-3B)$

$= \begin{bmatrix} 12 & 20 \\ 28 & -36 \end{bmatrix} + \begin{bmatrix} -18 & 12 \\ -6 & -9 \end{bmatrix}$

$= \begin{bmatrix} 12 + (-18) & 20 + 12 \\ 28 + (-6) & -36 + (-9) \end{bmatrix} = \begin{bmatrix} -6 & 32 \\ 22 & -45 \end{bmatrix}$.

Hence, $(4A - 3B) = \begin{bmatrix} -6 & 32 \\ 22 & -45 \end{bmatrix}$.

EXAMPLE 2 *Let* $A = \text{diag}\,[3, \; -5, \; 7]$ *and* $B = \text{diag}\,[-1, \; 2, \; 4]\cdot$
Find (i) $(A + B)$ *(ii)* $(A - B)$ *(iii)* $-5A$ *(iv)* $(2A + 3B)$.

SOLUTION We have

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

$$\therefore \quad \text{(i)} \;\; A + B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 11 \end{bmatrix}.$$

(ii) $(A - B) = A + (-B)$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$\text{(iii)} \;\; -5A = (-5)\cdot A = (-5)\cdot \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} -15 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -35 \end{bmatrix}.$$

$$\text{(iv)} \;\; 2A + 3B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 14 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 26 \end{bmatrix}.$$

EXAMPLE 3 *Simplify* $\cos\theta\cdot\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\cdot\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$.

[CBSE 2012]

SOLUTION We have

$$\cos\theta\cdot\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\cdot\begin{bmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \sin\theta\;\cos\theta \\ -\sin\theta\;\cos\theta & \cos s2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\;\cos\theta \\ \sin\theta\;\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\;\cos\theta + (-\sin\theta\;\cos\theta) \\ -\sin\theta\;\cos\theta + \sin\theta\;\cos\theta & \cos^2\theta + \sin^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

EXAMPLE 4 *If* $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, *find* $(x - y)$. **[CBSE 2014]**

SOLUTION We have

$$2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & 8 \\ 10 & 2x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6+1 & 8+y \\ 10+0 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow 2x + 1 = 5 \text{ and } 8 + y = 0$$

[comparing the corresponding elements]

$$\Rightarrow x = 2 \text{ and } y = -8.$$

$$\Rightarrow (x - y) = 2 - (-8) = 10.$$

EXAMPLE 5 *Find a matrix X, if* $X + \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}$.

SOLUTION Let $A = \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}$.

Then, the given matrix equation is $X + A = B$.

$\therefore \quad X + A = B \Rightarrow X = B + (-A)$

$$= \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix} + \begin{bmatrix} -4 & -6 \\ 3 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + (-4) & -6 + (-6) \\ 5 + 3 & -8 + (-7) \end{bmatrix} = \begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}.$$

Hence, $X = \begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}$.

EXAMPLE 6 *Find a matrix X such that* $2A + B + X = O$, *where* $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$. **[CBSE 2002C]**

SOLUTION We have

$$2A + B + X = O \Rightarrow X = -(2A + B).$$

Now, $(2A + B) = 2 \cdot \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & 4+(-2) \\ 6+1 & 8+5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 7 & 13 \end{bmatrix}.$$

$$\therefore \quad X = -(2A+B) = \begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}.$$

EXAMPLE 7 *Find matrices X and Y, if* $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ *and* $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

SOLUTION Adding the given matrices, we get

$$(X+Y)+(X-Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \quad 2X = \begin{bmatrix} 5+3 & 2+6 \\ 0+0 & 9+(-1) \end{bmatrix}$$

$$\Rightarrow \quad 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow X = \frac{1}{2} \cdot \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}.$$

On subtracting the given matrices, we get

$$(X+Y)-(X-Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \quad 2Y = \begin{bmatrix} 5-3 & 2-6 \\ 0-0 & 9-(-1) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$$

$$\Rightarrow \quad Y = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}.$$

Hence, $X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$.

EXERCISE 5B

1. If $A = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & -2 \\ 4 & -3 & 1 \end{bmatrix}$, verify that $(A+B) = (B+A)$.

2. If $A = \begin{bmatrix} 3 & 5 \\ -2 & 0 \\ 6 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 2 \\ 3 & -4 \\ 1 & 6 \end{bmatrix}$,

   verify that $(A+B)+C = A+(B+C)$.

3. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 & 4 \\ 5 & -3 & 2 \end{bmatrix}$, find $(2A - B)$.

4. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$. Find:

   (i) $A + 2B$ (ii) $B - 4C$ (iii) $A - 2B + 3C$

5. Let $A = \begin{bmatrix} 0 & 1 & -2 \\ 5 & -1 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 & -1 \\ 0 & -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -5 & 1 \\ -4 & 0 & 6 \end{bmatrix}$.
Compute $5A - 3B + 4C$.

6. If $5A = \begin{bmatrix} 5 & 10 & -15 \\ 2 & 3 & 4 \\ 1 & 0 & -5 \end{bmatrix}$, find A.

7. Find matrices A and B, if

$$A + B = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 4 & -6 \\ 7 & 3 & 8 \end{bmatrix} \text{ and } A - B = \begin{bmatrix} -5 & -4 & 8 \\ 11 & 2 & 0 \\ -1 & 7 & 4 \end{bmatrix}.$$

8. Find matrices A and B, if

$$2A - B = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} \text{ and } 2B + A = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{bmatrix}.$$

9. Find matrix X, if $\begin{bmatrix} 3 & 5 & -9 \\ -1 & 4 & -7 \end{bmatrix} + X = \begin{bmatrix} 6 & 2 & 3 \\ 4 & 8 & 6 \end{bmatrix}$.

10. If $A = \begin{bmatrix} -2 & 3 \\ 4 & 5 \\ 1 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ -7 & 3 \\ 6 & 4 \end{bmatrix}$, find a matrix C such that $A + B - C = O$.

11. Find the matrix X such that $2A - B + X = O$,
where $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$.

12. If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$, find a matrix C such that $(A + B + C)$ is a zero matrix.

13. If $A = \text{diag}\,[2, -5, 9]$, $B = \text{diag}\,[-3, 7, 14]$ and $C = \text{diag}\,[4, -6, 3]$, find:
(i) $A + 2B$ (ii) $B + C - A$ (iii) $2A + B - 5C$.

14. Find the values of x and y, when

(i) $\begin{bmatrix} x + y \\ x - y \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 2x + 5 & 7 \\ 0 & 3y - 7 \end{bmatrix} = \begin{bmatrix} x - 3 & 7 \\ 0 & -5 \end{bmatrix}$

(iii) $2\begin{bmatrix} x & 5 \\ 7 & y - 3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$ **[CBSE 2012]**

15. Find the value of $(x + y)$ from the following equation:

$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$ **[CBSE 2012, '13C]**

16. If $\begin{bmatrix} x-y & 2y \\ 2y+z & x+y \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 9 & 5 \end{bmatrix}$ then write the value of $(x+y+z)$. **[CBSE 2013C]**

ANSWERS (EXERCISE 5B)

3. $\begin{bmatrix} 8 & 2 & 0 \\ -3 & 7 & -8 \end{bmatrix}$ **4.** (i) $\begin{bmatrix} 4 & 10 \\ -1 & 12 \end{bmatrix}$ (ii) $\begin{bmatrix} 9 & -17 \\ -14 & -11 \end{bmatrix}$ (iii) $\begin{bmatrix} -6 & 13 \\ 16 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 5 & -6 & -3 \\ 9 & 1 & -11 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 2 & -3 \\ \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ \frac{1}{5} & 0 & -1 \end{bmatrix}$

7. $A = \begin{bmatrix} -2 & -2 & 5 \\ 8 & 3 & -3 \\ 3 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & -3 \\ -3 & 1 & -3 \\ 4 & -2 & 2 \end{bmatrix}$

8. $A = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 & 4 \\ 0 & 0 & -6 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -3 & 12 \\ 5 & 4 & 13 \end{bmatrix}$ **10.** $\begin{bmatrix} 3 & 5 \\ -3 & 8 \\ 7 & -2 \end{bmatrix}$

11. $\begin{bmatrix} -8 & -1 \\ 0 & -1 \end{bmatrix}$ **12.** $\begin{bmatrix} -3 & 4 & -1 \\ -3 & 0 & -1 \end{bmatrix}$

13. (i) diag [–4, 11, 37] (ii) diag [–1, 6, 8] (iii) diag [–19, 27, 17]

14. (i) $x = 6,\ y = 2$ (ii) $x = -8,\ y = \frac{2}{3}$ (iii) $x = 2,\ y = 4$

15. $x + y = 6$ **16.** 10

Multiplication of Matrices

For two given matrices A and B, we say that the product AB exists only when the number of rows in A equals the number of columns in B.

When AB exists, we say that A is conformable to B for multiplication.

PRODUCT OF MATRICES

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$ be two matrices such that the number of columns in A equals the number of rows in B.

Then, AB exists and it is an $(m \times p)$ matrix, given by

$$AB = [c_{ik}]_{m \times p}, \text{ where } c_{ik} = (a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots a_{in}b_{nk}) = \sum_{j=1}^{n} a_{ij}b_{jk}.$$

$\therefore$ (i, k)th element of AB

= sum of the products of corresponding elements of ith row of A and kth column of B.

SUMMARY

(i) If A is an $(m \times n)$ matrix and B is an $(n \times p)$ matrix then AB exists and it is an $(m \times p)$ matrix.

(ii) (i, k)th element of AB
= sum of the products of corresponding elements of ith row of A and kth column of B.

REMARKS For two given matrices A and B:
(i) AB may exist and BA may not exist;
(ii) BA may exist and AB may not exist;
(iii) AB and BA both may not exist;
(iv) AB and BA both may exist.

SOLVED EXAMPLES

EXAMPLE 1 *If* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ *and* $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$ *then show that* *AB and BA both exist. Find AB and BA.*

SOLUTION Here, A is a 2×3 matrix and B is a 3×2 matrix.

$\therefore$ AB exists and it is a 2×2 matrix.

Let $AB = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$. Then,

$c_{11} = (\text{1st row of } A) \times (\text{1st column of } B)$
$= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31};$

$c_{12} = (\text{1st row of } A) \times (\text{2nd column of } B)$
$= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32};$

$c_{21} = (\text{2nd row of } A) \times (\text{1st column of } B)$
$= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31};$

and $c_{22} = (\text{2nd row of } A) \times (\text{2nd column of } B)$

$$= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}.$$

$$\therefore \quad AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}.$$

Again, B is a 3×2 matrix and A is a 2×3 matrix. So, BA exists and it is a 3×3 matrix.

Proceeding as above, we get

$$BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}\,a_{11} + b_{12}\,a_{21} & b_{11}\,a_{12} + b_{12}\,a_{22} & b_{11}\,a_{13} + b_{12}\,a_{23} \\ b_{21}\,a_{11} + b_{22}\,a_{21} & b_{21}\,a_{12} + b_{22}\,a_{22} & b_{21}\,a_{13} + b_{22}\,a_{23} \\ b_{31}\,a_{11} + b_{32}\,a_{21} & b_{31}\,a_{12} + b_{32}\,a_{22} & b_{31}\,a_{13} + b_{32}\,a_{23} \end{bmatrix}.$$

EXAMPLE 2 *If* $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$, *find AB. Does BA exist?*

SOLUTION Here A is a 3×2 matrix and B is a 2×2 matrix.

Clearly, the number of columns in A equals the number of rows in B.

$\therefore$ AB exists and it is a 3×2 matrix.

$$\text{Now, } AB = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot(-1) + (-1)\cdot 2 & 2\cdot 3 + (-1)\cdot 1 \\ 3\cdot(-1) + 4\cdot 2 & 3\cdot 3 + 4\cdot 1 \\ 1\cdot(-1) + 5\cdot 2 & 1\cdot 3 + 5\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 5 \\ 5 & 13 \\ 9 & 8 \end{bmatrix}.$$

Further, B is a 2×2 matrix and A is a 3×2 matrix. So, the number of columns in B is not equal to the number of rows in A.

So, BA does not exist.

EXAMPLE 3 *Let* $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -2 & 1 \end{bmatrix}$. *Find AB and BA, and show that* $AB \neq BA$.

SOLUTION Here A is a 2×3 matrix and B is a 3×2 matrix.

So, AB exists and it is a 2×2 matrix.

$$\text{Now, } AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 2 + (-2) \cdot 4 + 3 \cdot (-2) & 1 \cdot 3 + (-2) \cdot 5 + 3 \cdot 1 \\ (-4) \cdot 2 + 2 \cdot 4 + 5 \cdot (-2) & (-4) \cdot 3 + 2 \cdot 5 + 5 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -4 \\ -10 & 3 \end{bmatrix}.$$

Again, B is a 3×2 matrix and A is a 2×3 matrix.

So, BA exists and it is a 3×3 matrix.

$$\text{Now, } BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 1 + 3 \cdot (-4) & 2 \cdot (-2) + 3 \cdot 2 & 2 \cdot 3 + 3 \cdot 5 \\ 4 \cdot 1 + 5 \cdot (-4) & 4 \cdot (-2) + 5 \cdot 2 & 4 \cdot 3 + 5 \cdot 5 \\ (-2) \cdot 1 + 1 \cdot (-4) & (-2) \cdot (-2) + 1 \cdot 2 & (-2) \cdot 3 + 1 \cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -6 & 6 & -1 \end{bmatrix}.$$

Hence, $AB \neq BA$.

EXAMPLE 4 *If* $[2x \quad 4] \begin{bmatrix} x \\ -8 \end{bmatrix} = O$, *find the positive value of x.* **[CBSE 2014C]**

SOLUTION The given matrix equation is $AB = O$, where A is a (1×2) matrix and B is a (2×1) matrix. So, AB is a (1×1) matrix.

So, O is a (1×1) matrix.

$$\therefore \quad [2x \quad 4] \begin{bmatrix} x \\ -8 \end{bmatrix} = [0]$$

$$\Rightarrow \quad 2x^2 - 32 = 0 \Rightarrow 2x^2 = 32$$

$$\Rightarrow \quad x^2 = 16 \Rightarrow x = \sqrt{16} = 4.$$

Hence, $x = 4$.

Properties of Matrix Multiplication

1. *Commutativity*

Matrix multiplication is not commutative in general.

PROOF Let A and B be two given matrices.

If AB exists then it is quite possible that BA may not exist.
For example, if A is a 3×2 matrix and B is a 2×2 matrix then clearly, AB exists but BA does not exist.

Similarly, if BA exists then AB may not exist.

For example, if A is a 2×3 matrix and B is a 2×2 matrix then clearly, BA exists but AB does not exist.

Further, if AB and BA both exist, they may not be comparable. For example, if A is a 2×3 matrix and B is a 3×2 matrix then clearly, AB as well as BA exists. But, AB is a 2×2 matrix while BA is a 3×3 matrix.

Again, if AB and BA both exist and they are comparable, even then they may not be equal.

For example, if $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ then AB and BA are both defined and each one is a 2×2 matrix.

But, $AB = \begin{bmatrix} 1 & 5 \\ 1 & 8 \end{bmatrix}$ and $BA = \begin{bmatrix} 3 & 5 \\ 3 & 6 \end{bmatrix}$.

This shows that $AB \neq BA$.

Hence, in general, $AB \neq BA$.

REMARKS (i) When $AB = BA$, we say that A and B commute.
(ii) When $AB = -BA$, we say that A and B anticommute.

2. *Associative law*

For any matrices A, B, C for which $(AB)C$ and $A(BC)$ *both exist, we have*

$$(AB)C = A(BC).$$

3. *Distributive laws of multiplication over addition*

We have:

(i) $A \cdot (B + C) = (AB + AC)$ *(ii)* $(A + B) \cdot C = (AC + BC)$

4. *The product of two nonzero matrices can be a zero matrix.*

Example Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

Then, $A \neq O$ and $B \neq O$.

And, $AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 2 + 1 \cdot 0 \\ 0 \cdot 1 + 2 \cdot 0 & 0 \cdot 2 + 2 \cdot 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Left zero divisor *If $AB = O$ and $A \neq O$ then A is called a left zero divisor of AB.*

Right zero divisor *If $AB = O$ and $B \neq O$ then B is called a right zero divisor of AB.*

5. ***If A is a given square matrix and I is an identity matrix of the same order as A then we have $A \cdot I = I \cdot A = A$.***

6. ***If A is a given square matrix and O is the null matrix of the same order as A then $O \cdot A = A \cdot O = O$.***

Positive Integral Powers of a Square Matrix

Let A be a square matrix of order n. Then, we define:

$$A^2 = A \cdot A;$$
$$A^3 = A \cdot A \cdot A = A^2 \cdot A;$$
$$A^4 = A \cdot A \cdot A \cdot A = A^3 \cdot A, \text{ and so on.}$$
$$\therefore \quad A^n = (A \cdot A \cdot A \cdot \ldots \cdot n \text{ times}).$$

THEOREM 1 *If A and B are square matrices of the same order then*

$$(A + B)^2 = A^2 + AB + BA + B^2.$$

Also, when $AB = BA$ then $(A + B)^2 = A^2 + 2AB + B^2$.

PROOF Let A and B be n-rowed square matrices.
Then, clearly, $(A + B)$ is a square matrix of order n.
So, $(A + B)^2$ is defined.

Now, $(A + B)^2 = (A + B) \cdot (A + B)$

$= A \cdot (A + B) + B \cdot (A + B)$ [by distributive law]

$= AA + AB + BA + BB$ [by distributive law]

$= A^2 + AB + BA + B^2$.

Hence, $(A + B)^2 = (A^2 + AB + BA + B^2)$.

Particular case *When $AB = BA$*

In this case, we have

$$(A + B)^2 = (A^2 + AB + AB + B^2) = (A^2 + 2AB + B^2) \quad [\because \; BA = AB].$$

THEOREM 2 *If A and B are square matrices of the same order then*

$$(A + B)(A - B) = A^2 - AB + BA - B^2.$$

Also, when $AB = BA$ then $(A + B)(A - B) = A^2 - B^2$.

PROOF We have:

$(A + B) \cdot (A - B) = A(A - B) + B(A - B)$ [by distributive law]

$= AA - AB + BA - BB$ [$\because \; A(B - C) = AB - AC$]

$= A^2 - AB + BA - B^2$.

Hence, $(A + B)(A - B) = A^2 - AB + BA - B^2$.

Particular case *When AB = BA*

In this case, $(A + B)(A - B) = (A^2 - B^2)$ $[\because\ BA = AB]$.

MATRIX POLYNOMIAL Let $f(x) = a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_{m-1}x + a_m$ be a polynomial of degree m and let A be a square matrix of order n. Then, we define

$$f(A) = a_0A^m + a_1A^{m-1} + a_2A^{m-2} + \ldots + a_{m-1}A + a_mI,$$

where I is a unit matrix of order n.

SOLVED EXAMPLES

EXAMPLE 1 *If* $A = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & 1 \\ 6 & 8 & 4 \end{bmatrix}$, *find AB and BA whichever exists.*

SOLUTION Here, A is a 2×2 matrix and B is a 2×3 matrix.

Clearly, the number of columns in A = number of rows in B.

$\therefore$ AB exists and it is a 2×3 matrix.

$$AB = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \\ 6 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5\cdot 3 + 4\cdot 6 & 5\cdot 5 + 4\cdot 8 & 5\cdot 1 + 4\cdot 4 \\ 2\cdot 3 + 3\cdot 6 & 2\cdot 5 + 3\cdot 8 & 2\cdot 1 + 3\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 15 + 24 & 25 + 32 & 5 + 16 \\ 6 + 18 & 10 + 24 & 2 + 12 \end{bmatrix} = \begin{bmatrix} 39 & 57 & 21 \\ 24 & 34 & 14 \end{bmatrix}.$$

Again, B is a 2×3 matrix and A is a 2×2 matrix.

$\therefore$ number of columns in $B \neq$ number of rows in A.

So, BA does not exist.

EXAMPLE 2 *If* $A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix}$ *then find AB and BA. Show that AB ≠ BA.*

SOLUTION Here, A is a 2×3 matrix and B is a 3×2 matrix

So, number of columns in A = number of rows in B.

$\therefore$ AB exists and it is a 2×2 matrix.

$$AB = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 2 + (-1)\cdot 4 + 3\cdot 1 & 2\cdot 3 + (-1)\cdot(-2) + 3\cdot 5 \\ -4\cdot 2 + 5\cdot 4 + 1\cdot 1 & -4\cdot 3 + 5\cdot(-2) + 1\cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4-4+3 & 6+2+15 \\ -8+20+1 & -12-10+5 \end{bmatrix} = \begin{bmatrix} 3 & 23 \\ 13 & -17 \end{bmatrix}.$$

Again, B is a 3×2 matrix and A is a 2×3 matrix.

So, number of columns in B = number of rows in A.

$\therefore$ BA exists and it is a 3×3 matrix.

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 2 + 3\cdot(-4) & 2\cdot(-1) + 3\cdot 5 & 2\cdot 3 + 3\cdot 1 \\ 4\cdot 2 + (-2)\cdot(-4) & 4\cdot(-1) + (-2)\cdot 5 & 4\cdot 3 + (-2)\cdot 1 \\ 1\cdot 2 + 5\cdot(-4) & 1\cdot(-1) + 5\cdot 5 & 1\cdot 3 + 5\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-12 & -2+15 & 6+3 \\ 8+8 & -4-10 & 12-2 \\ 2-20 & -1+25 & 3+5 \end{bmatrix} = \begin{bmatrix} -8 & 13 & 9 \\ 16 & -14 & 10 \\ -18 & 24 & 8 \end{bmatrix}.$$

Clearly, $AB \neq BA$.

EXAMPLE 3 *If* $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix}$ *and* $C = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix}$ *then verify that* $(AB)C = A(BC)$.

SOLUTION We have

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3-0-4 & 1-2+10 \\ 9+0-0 & 3+4+0 \\ -6+0-2 & -2+0+5 \end{bmatrix} = \begin{bmatrix} -1 & 9 \\ 9 & 7 \\ -8 & 3 \end{bmatrix}$$

$$\Rightarrow \quad (AB)C = \begin{bmatrix} -1 & 9 \\ 9 & 7 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+27 & -1+0 & 3-9 \\ 18+21 & 9+0 & -27-7 \\ -16+9 & -8+0 & 24-3 \end{bmatrix} = \begin{bmatrix} 25 & -1 & -6 \\ 39 & 9 & -34 \\ -7 & -8 & 21 \end{bmatrix}.$$

Also, $BC = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 6+3 & 3+0 & -9-1 \\ 0+6 & 0+0 & 0-2 \\ -4+15 & -2+0 & 6-5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & -10 \\ 6 & 0 & -2 \\ 11 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A(BC) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 3 & -10 \\ 6 & 0 & -2 \\ 11 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-6+22 & 3-0-4 & -10+2+2 \\ 27+12+0 & 9+0-0 & -30-4+0 \\ -18+0+11 & -6+0-2 & 20-0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -1 & -6 \\ 39 & 9 & -34 \\ -7 & -8 & 21 \end{bmatrix}.$$

Hence, $(AB)C = A(BC)$.

EXAMPLE 4 *If* $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 7 & 3 \end{bmatrix}$ *and* $C = \begin{bmatrix} 8 & 1 & -6 \\ 2 & -5 & 0 \end{bmatrix}$, *verify that* $A(B+C) = (AB + AC)$.

SOLUTION We have

$$A(B+C) = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \cdot \left\{ \begin{bmatrix} 1 & -2 & 5 \\ 0 & 7 & 3 \end{bmatrix} + \begin{bmatrix} 8 & 1 & -6 \\ 2 & -5 & 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 9 & -1 & -1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot 9 + 2 \cdot 2 & 3 \cdot (-1) + 2 \cdot 2 & 3 \cdot (-1) + 2 \cdot 3 \\ 1 \cdot 9 + 0 \cdot 2 & 1 \cdot (-1) + 0 \cdot 2 & 1 \cdot (-1) + 0 \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 1 & 3 \\ 9 & -1 & -1 \end{bmatrix}.$$

Now, $AB = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 7 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3 \cdot 1 + 2 \cdot 0 & 3 \cdot (-2) + 2 \cdot 7 & 3 \cdot 5 + 2 \cdot 3 \\ 1 \cdot 1 + 0 \cdot 0 & 1 \cdot (-2) + 0 \cdot 7 & 1 \cdot 5 + 0 \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 & 21 \\ 1 & -2 & 5 \end{bmatrix}.$$

And, $AC = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 & 1 & -6 \\ 2 & -5 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 3\cdot 8+2\cdot 2 & 3\cdot 1+2\cdot(-5) & 3\cdot(-6)+2\cdot 0 \\ 1\cdot 8+0\cdot 2 & 1\cdot 1+0\cdot(-5) & 1\cdot(-6)+0\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & -7 & -18 \\ 8 & 1 & -6 \end{bmatrix}.$$

$$\therefore \quad (AB+AC) = \begin{bmatrix} 3 & 8 & 21 \\ 1 & -2 & 5 \end{bmatrix} + \begin{bmatrix} 28 & -7 & -18 \\ 8 & 1 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 1 & 3 \\ 9 & -1 & -1 \end{bmatrix}.$$

Hence, $A(B+C) = AB + AC$.

EXAMPLE 5 *Give an example of two matrices A and B such that* $A \neq O$, $B \neq O$ *and* $AB = BA = O$.

SOLUTION Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & -1+1 \\ 1-1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{and } BA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $A \neq O$, $B \neq O$.

But, $AB = BA = O$.

EXAMPLE 6 *If* $A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ *and* $B = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$, *show that AB is a zero matrix if* θ *and* ϕ *differ by an odd multiple of* $\frac{\pi}{2}$.

SOLUTION We have

$$AB = \begin{bmatrix} \cos^2\theta & \cos\theta\ \sin\theta \\ \cos\theta\ \sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} \cos^2\phi & \cos\phi\ \sin\phi \\ \cos\phi\ \sin\phi & \sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\ \cos\phi\cdot\cos(\theta-\phi) & \cos\theta\ \sin\phi\cdot\cos(\theta-\phi) \\ \sin\theta\ \cos\phi\cdot\cos(\theta-\phi) & \sin\theta\ \sin\phi\cdot\cos(\theta-\phi) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \because\ (\theta-\phi) \text{ being an odd multiple of } (\pi/2), \\ \text{we have } \cos(\theta-\phi)=0 \end{bmatrix}.$$

EXAMPLE 7 *If* $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, *show that* $A^2 - 5A - 14I = O$. **[CBSE 2004]**

SOLUTION We have

$$A^2 = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot3+(-5)(-4) & 3\cdot(-5)+(-5)\cdot2 \\ -4\cdot3+2\cdot(-4) & -4\cdot(-5)+2\cdot2 \end{bmatrix} = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix};$$

$$-5A = (-5)\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -15 & 25 \\ 20 & -10 \end{bmatrix};$$

$$-14I = (-14)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix}.$$

$$\therefore\quad A^2 - 5A - 14I = A^2 + (-5)A + (-14I)$$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} + \begin{bmatrix} -15 & 25 \\ 20 & -10 \end{bmatrix} + \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} 29+(-15)+(-14) & -25+25+0 \\ -20+20+0 & 24+(-10)+(-14) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $A^2 - 5A - 14I = O$.

EXAMPLE 8 *If* $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, *find k so that* $A^2 = 8A + kI$. **[CBSE 2005C]**

SOLUTION We have

$$A^2 = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1-0 & 0+0 \\ -1-7 & 0+49 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix};$$

$$(8A + kI) = 8\cdot\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} + k\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

$$\therefore\quad A^2 = 8A + kI \Rightarrow \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

$$\Rightarrow 8+k=1 \text{ and } 56+k=49 \Rightarrow k=-7.$$

Hence, $k = -7$.

EXAMPLE 9 *If* $f(x) = x^2 - 5x + 7$ *and* $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, *find* $f(A)$. **[CBSE 2004C, '07C]**

SOLUTION We have $f(A) = A^2 - 5A + 7I$.

Now, $A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix};$$

$$-5A = \begin{bmatrix} (-5)\cdot 3 & (-5)\cdot 1 \\ (-5)\cdot(-1) & (-5)\cdot 2 \end{bmatrix} = \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix};$$

$$7I = 7\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}.$$

$\therefore \quad f(A) = A^2 - 5A + 7I$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8+(-15)+7 & 5+(-5)+0 \\ -5+5+0 & 3+(-10)+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $f(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

EXAMPLE 10 *Find the matrix A such that* $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \cdot A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$.

SOLUTION Clearly, the product is a 3×3 matrix and the prefactor is a 3×2 matrix. So, A must be a 2×3 matrix.

Let $A = \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}$.

Then, the given equation becomes

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

or $$\begin{bmatrix} 2x-u & 2y-v & 2z-w \\ x & y & z \\ -3x+4u & -3y+4v & -3z+4w \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}.$$

So, by the definition of equal matrices, we have

$$2x - u = -1;\ 2y - v = -8;\ 2z - w = -10;\ x = 1;\ y = -2;\ z = -5.$$

$\therefore \quad x = 1;\ y = -2;\ z = -5;\ u = 3;\ v = 4$ and $w = 0$.

Hence, $A = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$.

EXAMPLE 11 *Find the value of x, if*

$$[1 \;\; x \;\; 1] \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O.$$ **[CBSE 2006C]**

SOLUTION We have

$$[1 \;\; x \;\; 1]_{1 \times 3} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}_{3 \times 1} = O$$

$$\Rightarrow \quad [1 + 2x + 15 \quad 3 + 5x + 3 \quad 2 + x + 2] \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

$$\Rightarrow \quad [16 + 2x \quad 6 + 5x \quad 4 + x] \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

$\Rightarrow \quad [(16 + 2x) \cdot 1 + (6 + 5x) \cdot 2 + (4 + x) \cdot x] = O$

$\Rightarrow \quad (16 + 2x) + (12 + 10x) + (4x + x^2) = 0$

$\Rightarrow \quad x^2 + 16x + 28 = 0$

$\Rightarrow \quad (x + 14)(x + 2) = 0$

$\Rightarrow \quad x + 14 = 0$ or $x + 2 = 0$

$\Rightarrow \quad x = -14$ or $x = -2$.

Hence, $x = -14$ or $x = -2$.

EXAMPLE 12 *Solve for x and y, given that* $\begin{bmatrix} x & y \\ 3y & x \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. **[CBSE 2003C]**

SOLUTION We have

$$\begin{bmatrix} x & y \\ 3y & x \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x + 2y \\ 3y + 2x \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{cases} x + 2y = 3 & \text{... (i)} \\ 2x + 3y = 5 & \text{... (ii)} \end{cases}$$

Multiplying (i) by 2 and subtracting (ii) from it, we get $y = 1$.

Putting $y = 1$ in (i), we get $x = 1$.

Hence, $x = 1$ and $y = 1$.

EXAMPLE 13 *Let* $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ *and I is the identity matrix of order* 2.

Show that $(I+A) = (I-A)\cdot\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

SOLUTION Let $\tan\frac{\alpha}{2} = t$.

Then, $\cos\alpha = \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)} = \frac{1-t^2}{1+t^2}$

and $\sin\alpha = \frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)} = \frac{2t}{1+t^2}$.

$$\therefore \quad (I+A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}.$$

And, $(I-A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}$.

$$\therefore \quad (I-A)\cdot\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix} \begin{bmatrix} \frac{1-t^2}{1+t^2} & \frac{-2t}{1+t^2} \\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-t^2}{1+t^2} + \frac{2t^2}{1+t^2} & \frac{-2t}{1+t^2} + \frac{t(1-t^2)}{1+t^2} \\ \frac{-t(1-t^2)}{1+t^2} + \frac{2t}{1+t^2} & \frac{2t^2}{1+t^2} + \frac{1-t^2}{1+t^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} = (I+A).$$

Hence, $(I+A) = (I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

EXAMPLE 14 *If* $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ *then prove that*

$A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in N$. **[CBSE 2004, '05, '06]**

SOLUTION We shall prove the result by using the principle of mathematical induction.

When $n = 1$, we have

$$A^1 = \begin{bmatrix} \cos 1 \cdot \theta & \sin 1 \cdot \theta \\ -\sin 1 \cdot \theta & \cos 1 \cdot \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.$$

Thus, the result is true for $n = 1$.

Let the result be true for $n = k$.

Then, $A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$.

$\therefore \quad A^{k+1}$

$$= A \cdot A^k = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta \ \cos k\theta - \sin \theta \ \sin k\theta & \cos \theta \ \sin k\theta + \sin \theta \ \cos k\theta \\ -\sin \theta \ \cos k\theta - \cos \theta \ \sin k\theta & -\sin \theta \ \sin k\theta + \cos \theta \ \cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + k\theta) & \sin(\theta + k\theta) \\ -\sin(\theta + k\theta) & \cos(\theta + k\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Thus, the result is true for $n = (k + 1)$, whenever it is true for $n = k$.

Hence, $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$ for all values of $n \in N$.

EXAMPLE 15 *If* $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ *then prove that* $A^n = \begin{bmatrix} 1 + 2n & -4n \\ n & 1 - 2n \end{bmatrix}$ *for all values of* $n \in N$.

SOLUTION We shall prove the result by using the principle of mathematical induction.

When $n = 1$, we have

$$A^1 = \begin{bmatrix} 1 + 2 \cdot 1 & -4 \cdot 1 \\ 1 & 1 - 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 + 2 & -4 \\ 1 & 1 - 2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}.$$

Thus, the result is true for $n = 1$.

Let the result be true for $n = k$. Then, $A^k = \begin{bmatrix} 1 + 2k & -4k \\ k & 1 - 2k \end{bmatrix}$.

$\therefore \quad A^{k+1} = A \cdot A^k$

$$= \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 + 2k & -4k \\ k & 1 - 2k \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 6k - 4k & -12k - 4 + 8k \\ 1 + 2k - k & -4k - 1 + 2k \end{bmatrix} = \begin{bmatrix} 3 + 2k & -4k - 4 \\ k + 1 & 1 - 2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}.$$

Thus, the result is true for $n = (k+1)$, whenever it is true for $n = k$.

So, the result is true for all $n \in N$.

Hence, $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ for all values of $n \in N$.

EXAMPLE 16 *If* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, *prove that* $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$ *for all values of* $n \in N$.

SOLUTION We shall prove the result by using the principle of mathematical induction.

When $n = 1$, we have

$$A^1 = \begin{bmatrix} 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \end{bmatrix} = \begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Thus, the result is true for $n = 1$.

Let it be true for $n = k$. Then, $A^k = \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}.$

$$\therefore \quad A^{k+1} = A \cdot A^k$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3(3^{k-1}) & 3(3^{k-1}) & 3(3^{k-1}) \\ 3(3^{k-1}) & 3(3^{k-1}) & 3(3^{k-1}) \\ 3(3^{k-1}) & 3(3^{k-1}) & 3(3^{k-1}) \end{bmatrix} = \begin{bmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{bmatrix}.$$

Thus, the result is true for $n = (k+1)$, whenever it is true for $n = k$.

So, the result is true for all $n \in N$.

Hence, $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$ for all values of $n \in N$.

EXAMPLE 17 *If* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, *prove that for all* $n \in N$,

$$(aI + bA)^n = a^n I + na^{n-1} bA,$$

where I is the identity matrix of order 2.

SOLUTION We shall prove the result by mathematical induction.

When $n = 1$, we have:

$$\text{LHS} = (aI + bA)^1 = (aI + bA) = (a^1 I + 1a^0 bA) = \text{RHS}.$$

So, the result is true for $n = 1$.

Let it be true for $n = m$, so that

$$(aI + bA)^m = a^m I + ma^{m-1} bA \quad \ldots \text{(i)}$$

$$\therefore \quad (aI + bA)^{m+1} = (aI + bA) \cdot (aI + bA)^m$$

$$= (aI + bA) \cdot (a^m I + ma^{m-1} bA) \quad \text{[using (i)]}$$

$$= aI(a^m I + ma^{m-1} bA) + bA(a^m I + ma^{m-1} bA)$$

$$= a^{m+1} I + ma^m bA + a^m bA + ma^{m-1} b^2 A^2$$

$$[\because \; II = I, \; IA = A = AI]$$

$$= a^{m+1} + (m+1)\, a^m bA \quad [\because \; A^2 = 0].$$

This shows that the result is true for $n = (m+1)$, whenever it is true for $n = m$.

Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

EXAMPLE 18 *If* $A = \text{diag}\,[a, b, c]$, *show that* $A^n = \text{diag}\,[a^n, b^n, c^n]$ *for all* $n \in N$.

SOLUTION We shall prove the result by mathematical induction.

When $n = 1$, we have

$$A^1 = \text{diag}\,[a^1, \; b^1, \; c^1] = \text{diag}\,[a, \; b, \; c] = A.$$

So, the result is true for $n = 1$.

Let it be true for $n = m$, so that

$$A^m = \text{diag}\,[a^m, \; b^m, \; c^m] \quad \ldots \text{(i)}$$

$$\therefore \quad A^{m+1} = A \cdot A^m$$

$$= \text{diag}\,[a, \; b, \; c] \cdot \text{diag}\,[a^m, \; b^m, \; c^m] \; \text{[using (i)]}$$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \cdot \begin{bmatrix} a^m & 0 & 0 \\ 0 & b^m & 0 \\ 0 & 0 & c^m \end{bmatrix}$$

$$= \begin{bmatrix} a^{m+1} & 0 & 0 \\ 0 & b^{m+1} & 0 \\ 0 & 0 & c^{m+1} \end{bmatrix} = \text{diag}\,[a^{m+1}, b^{m+1}, c^{m+1}].$$

This shows that the result is true for $n = (m+1)$, whenever it is true for $n = m$.

Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

EXAMPLE 19 *If A is an $m \times n$ matrix and B is a matrix given in such a way that AB and BA are both defined, show that B is an $n \times m$ matrix.*

SOLUTION Since AB is defined, we have

number of rows in B = number of columns in $A = n$.

Again, since BA is defined, we have

number of columns in B = number of rows in $A = m$.

Hence, the order of B is $n \times m$.

EXAMPLE 20 *If A and B are two matrices given in such a way that AB and $A + B$ are both defined, show that A and B are square matrices of the same order.*

SOLUTION Since $(A + B)$ is defined, it follows that both A and B are of the same order, say $(m \times n)$.

Thus, order of A is $m \times n$ and order of B is $m \times n$.

But, AB is defined.

So, number of columns in A = number of rows in B.

Consequently, $n = m$.

$\therefore$ A and B are square matrices of the same order.

VALUE BASED QUESTION

EXAMPLE 21 *The cooperative store of a particular school has* 10 *dozen physics books,* 8 *dozen chemistry books and* 5 *dozen mathematics books. Their selling prices are* ₹ 65.70, ₹ 43.20 *and* ₹ 76.50 *respectively. Find by matrix method the total amount received by the store from selling all these items.*

SOLUTION We have

$$\begin{matrix} & \underline{\text{Number of books}} & \underline{\text{Rates}} \\ [\text{Total cost}] = & \begin{bmatrix} \text{physics} & \text{chemistry} & \text{mathematics} \\ 120 & 96 & 60 \end{bmatrix} & \begin{bmatrix} ₹\ 65.70 \\ ₹\ 43.20 \\ ₹\ 76.50 \end{bmatrix} \end{matrix}$$

$$= [₹\ (120 \times 65.70 + 96 \times 43.20 + 60 \times 76.50)]$$
$$= [₹\ 16621.20].$$

$\therefore$ the total amount realised = ₹ 16621.20.

EXERCISE 5C

1. Compute AB and BA, whichever exists when

(i) $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$

(ii) $A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \\ -3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & 2 \\ -3 & 4 & -5 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 0 & 1 & -5 \\ 2 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 0 & 5 \end{bmatrix}$

(iv) $A = [1 \;\; 2 \;\; 3 \;\; 4]$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(v) $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

2. Show that $AB \neq BA$ in each of the following cases:

(i) $A = \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

3. Show that $AB = BA$ in each of the following cases:

(i) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 10 & -4 & -1 \\ -11 & 5 & 0 \\ 9 & -5 & 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$

4. If $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, show that

$AB = A$ and $BA = B$.

5. If $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$, show that AB is a zero matrix.

6. For the following matrices, verify that $A(BC) = (AB)C$:

(i) $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 4 \\ 1 & -1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $C = [1 \quad -2]$

7. Verify that $A(B + C) = (AB + AC)$, when

(i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -3 \\ 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$.

8. If $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 & -4 \\ -2 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$;
verify that $A(B - C) = (AB - AC)$.

9. If $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$, show that $A^2 = O$.

10. If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, show that $A^2 = A$.

11. If $A = \begin{bmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$, show that $A^2 = I$.

12. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$, find $(3A^2 - 2B + I)$. **[CBSE 2005]**

13. If $A = \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$ then find $(-A^2 + 6A)$.

14. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $(A^2 - 5A + 7I) = O$. **[CBSE 2003C]**

15. Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^3 - 4A^2 + A = O$. **[CBSE 2005]**

16. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find k so that $A^2 = kA - 2I$. **[CBSE 2003]**

17. If $A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$, find $f(A)$, where $f(x) = x^2 - 2x + 3$. **[CBSE 2004C]**

18. If $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ and $f(x) = 2x^3 + 4x + 5$, find $f(A)$.

19. Find the values of x and y, when
$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$ **[CBSE 2003C]**

20. Solve for x and y, when
$$\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}.$$ **[CBSE 2005C]**

21. If $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find x and y such that $A^2 + xI = yA$. **[CBSE 2005]**

22. If $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the values of a and b such that $A^2 + aA + bI = O$. **[CBSE 2006]**

23. Find the matrix A such that $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \cdot A = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$.

24. Find the matrix A such that $A \cdot \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$.

25. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & -1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = (A^2 + B^2)$ then find the values of a and b.

26. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $F(x) \cdot F(y) = F(x + y)$.

27. If $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$, show that $A^2 = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$.

28. If $[1 \quad x \quad 1] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = O$, find x. **[CBSE 2005]**

29. If $[x \quad 4 \quad 1] \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ -1 \end{bmatrix} = O$, find x.

30. Find the values of a and b for which

$$\begin{bmatrix} a & b \\ -a & 2b \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$ **[CBSE 2003C]**

31. If $A = \begin{bmatrix} 3 & 4 \\ -4 & -3 \end{bmatrix}$, find $f(A)$, where $f(x) = x^2 - 5x + 7$. **[CBSE 2004C]**

32. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for all $n \in N$.

33. Give an example of two matrices A and B such that

$$A \neq O, \ B \neq O, \ AB = O \text{ and } BA \neq O.$$

34. Give an example of three matrices A, B, C such that

$$AB = AC \text{ but } B \neq C.$$

35. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$, find $(3A^2 - 2B + I)$. **[CBSE 2005]**

36. If $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & x \end{bmatrix}$, find the value of x. **[CBSE 2012]**

ANSWERS (EXERCISE 5C)

1. (i) $AB = \begin{bmatrix} -4 & 2 \\ -6 & 9 \\ 2 & 13 \end{bmatrix}$ and BA does not exist.

(ii) $BA = \begin{bmatrix} -2 & 2 \\ -8 & 8 \\ 10 & -10 \end{bmatrix}$ and AB does not exist.

(iii) $AB = \begin{bmatrix} -1 & -25 \\ -2 & 6 \end{bmatrix}$ and $BA = \begin{bmatrix} 6 & 13 & -5 \\ 0 & -1 & 5 \\ 10 & 20 & 0 \end{bmatrix}$

(iv) $AB = [30]$ and $BA = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$

(v) $AB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 4 & -20 \\ 38 & -10 \end{bmatrix}$ **13.** $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ **16.** $k = 1$ **17.** $\begin{bmatrix} 12 & -4 \\ -6 & 8 \end{bmatrix}$

18. $f(A)=\begin{bmatrix}-5 & 68\\ 136 & -141\end{bmatrix}$ **19.** $x=2,\ y=1$ **20.** $x=5,\ y=3$

21. $x=8,\ y=8$ **22.** $a=-4,\ b=1$ **23.** $A=\begin{bmatrix}1 & -4\\ 3 & -2\end{bmatrix}$ **24.** $A=\begin{bmatrix}-8 & 4\\ -19 & 12\end{bmatrix}$

25. $a=1,\ b=4.$ **28.** $x=\dfrac{-5}{3}$ **29.** $x=-2$ or $x=-1$

30. $a=1,\ b=-3$ **31.** $f(A)=\begin{bmatrix}-15 & -20\\ 20 & 15\end{bmatrix}$ **33.** $A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ B=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$

34. $A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ B=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\ C=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ **35.** $\begin{bmatrix}4 & -8\\ -22 & 134\end{bmatrix}$ **36.** $x=13$

Transpose of Matrices

TRANSPOSE OF A MATRIX *Let A be an $(m\times n)$ matrix. Then, the matrix obtained by interchanging the rows and columns of A is called the transpose of A, denoted by A' or A^T.*

Thus, if $A=[a_{ij}]_{m\times n}$ then $A'=[a_{ji}]_{n\times m}$.

REMARKS (i) If A is an $(m\times n)$ matrix then A' is an $(n\times m)$ matrix.
(ii) (i, j)th element of $A=(j, i)$th element of A'.

EXAMPLE If $A=\begin{bmatrix}2 & \sqrt{2} & 0\\ 3 & -2 & \frac{2}{5}\end{bmatrix}$, then $A'=\begin{bmatrix}2 & 3\\ \sqrt{2} & -2\\ 0 & \frac{2}{5}\end{bmatrix}$.

Some Results on Transpose of Matrices

THEOREM 1 *For any matrix A, prove that $(A')'=A$.*

PROOF Let $A=[a_{ij}]_{m\times n}$. Then,

A is an $m\times n$ matrix $\Rightarrow$ A' is an $n\times m$ matrix
$\Rightarrow$ $(A')'$ is an $m\times n$ matrix.

$\therefore$ A and $(A')'$ are matrices of the same order.

Also, (i, j)th element of $A=(j, i)$th element of A'
$=(i, j)$th element of $(A')'$.

$\therefore$ (i, j)th element of $A=(i, j)$th element of $(A')'$.

Thus, A and $(A')'$ are comparable matrices having their corresponding elements equal.

Hence, $(A')'=A$.

THEOREM 2 *If A is any matrix and k is a scalar, prove that $(kA)' = kA'$.*

PROOF Let $A = [a_{ij}]_{m \times n}$ be an $m \times n$ matrix and k be a scalar.

Then, A is an $m \times n$ matrix $\Rightarrow$ kA is an $m \times n$ matrix

$\Rightarrow$ $(kA)'$ is an $n \times m$ matrix.

Again, A is an $m \times n$ matrix $\Rightarrow$ A' is an $n \times m$ matrix

$\Rightarrow$ kA' is an $n \times m$ matrix.

Thus, $(kA)'$ and kA' are matrices of the same order.

Now, (j, i)th element of $(kA)' = (i, j)$th element of kA

$= k$ times (i, j)th element of A

$= k$ times (j, i)th element of A'

$= (j, i)$th element of kA'.

$\therefore$ (j, i)th element of $(kA)' = (j, i)$th element of kA'.

Thus, $(kA)'$ and kA' are comparable matrices having their corresponding elements equal.

Hence, $(kA)' = kA'$.

THEOREM 3 *If A and B are two matrices of the same order then prove that $(A + B)' = A' + B'$.*

PROOF Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then,

A is an $m \times n$ matrix and B is an $m \times n$ matrix

$\Rightarrow$ $(A + B)$ is an $m \times n$ matrix

$\Rightarrow$ $(A + B)'$ is an $n \times m$ matrix.

Also, A is an $m \times n$ matrix and B is an $m \times n$ matrix

$\Rightarrow$ A' is an $n \times m$ matrix and B' is an $n \times m$ matrix

$\Rightarrow$ $(A' + B')$ is an $n \times m$ matrix.

$\therefore$ $(A + B)'$ and $(A' + B')$ are comparable matrices.

Also, (j, i)th element of $(A + B')$

$= (i, j)$th element of $(A + B)$

$= (i, j)$th element of $A + (i, j)$th element of B

$= (j, i)$th element of $A' + (j, i)$th element of B'

$= (j, i)$th element of $(A' + B')$.

$\therefore$ (j, i)th element of $(A + B') = (j, i)$th element of $(A' + B')$.

Thus, $(A + B)'$ and $(A' + B')$ are comparable and their corresponding elements are equal.

Hence, $(A + B)' = A' + B'$.

THEOREM 4 *If A and B are square matrices of the same order then $(AB)' = B'A'$.*

REMARK The proof of this theorem is beyond the scope of this book.

SUMMARY

(i) $(A')' = A$ (ii) $(kA)' = kA'$

(iii) $(A + B)' = A' + B'$ (iv) $(AB)' = B'A'$

SOLVED EXAMPLES

EXAMPLE 1 *Let* $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 7 \end{bmatrix} \cdot$ *Verify that* $(A')' = A$.

SOLUTION We have

$$A' = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 7 \end{bmatrix}' = \begin{bmatrix} 2 & 0 \\ 3 & -5 \\ -1 & 7 \end{bmatrix}$$

$$\Rightarrow \quad (A')' = \begin{bmatrix} 2 & 0 \\ 3 & -5 \\ -1 & 7 \end{bmatrix}' = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 7 \end{bmatrix} = A.$$

Hence, $(A')' = A$.

EXAMPLE 2 *If* $A = \begin{bmatrix} 2 & 3 & -5 \\ 0 & -1 & 4 \end{bmatrix}$, *verify that* $(3A)' = 3A'$.

SOLUTION We have

$$3A = \begin{bmatrix} 3 \times 2 & 3 \times 3 & 3 \times (-5) \\ 3 \times 0 & 3 \times (-1) & 3 \times 4 \end{bmatrix} = \begin{bmatrix} 6 & 9 & -15 \\ 0 & -3 & 12 \end{bmatrix}.$$

$$\therefore \quad (3A)' = \begin{bmatrix} 6 & 9 & -15 \\ 0 & -3 & 12 \end{bmatrix}' = \begin{bmatrix} 6 & 0 \\ 9 & -3 \\ -15 & 12 \end{bmatrix}.$$

$$\text{Also,} \quad A' = \begin{bmatrix} 2 & 3 & -5 \\ 0 & -1 & 4 \end{bmatrix}' = \begin{bmatrix} 2 & 0 \\ 3 & -1 \\ -5 & 4 \end{bmatrix}.$$

$$\therefore \quad 3A' = 3 \cdot \begin{bmatrix} 2 & 0 \\ 3 & -1 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 9 & -3 \\ -15 & 12 \end{bmatrix}.$$

Hence, $(3A)' = 3A'$.

EXAMPLE 3 *Let* $A = \begin{bmatrix} 3 & 4 \\ -2 & 0 \\ 7 & -5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & -3 \\ 5 & 6 \\ -1 & 8 \end{bmatrix} \cdot$ *Verify that* $(A + B)' = A' + B'$.

SOLUTION We have

$$A+B=\begin{bmatrix}3&4\\-2&0\\7&-5\end{bmatrix}+\begin{bmatrix}2&-3\\5&6\\-1&8\end{bmatrix}$$

$$=\begin{bmatrix}3+2&4+(-3)\\-2+5&0+6\\7+(-1)&-5+8\end{bmatrix}=\begin{bmatrix}5&1\\3&6\\6&3\end{bmatrix}.$$

$$\therefore\quad (A+B)'=\begin{bmatrix}5&1\\3&6\\6&3\end{bmatrix}'=\begin{bmatrix}5&3&6\\1&6&3\end{bmatrix}.$$

$$\text{Also, } A'=\begin{bmatrix}3&4\\-2&0\\7&-5\end{bmatrix}'=\begin{bmatrix}3&-2&7\\4&0&-5\end{bmatrix}.$$

$$\text{And, } B'=\begin{bmatrix}2&-3\\5&6\\-1&8\end{bmatrix}'=\begin{bmatrix}2&5&-1\\-3&6&8\end{bmatrix}.$$

$$\therefore\quad (A'+B')=\begin{bmatrix}3&-2&7\\4&0&-5\end{bmatrix}+\begin{bmatrix}2&5&-1\\-3&6&8\end{bmatrix}$$

$$=\begin{bmatrix}3+2&-2+5&7+(-1)\\4+(-3)&0+6&-5+8\end{bmatrix}=\begin{bmatrix}5&3&6\\1&6&3\end{bmatrix}.$$

Hence, $(A+B)'=A'+B'$.

EXAMPLE 4 *If* $A=\begin{bmatrix}-3\\5\\2\end{bmatrix}$ *and* $B=[1\ \ 6\ \ -4]$ *then verify that* $(AB)'=B'A'$.

[CBSE 2002]

SOLUTION We have $A=\begin{bmatrix}-3\\5\\2\end{bmatrix}$ and $B=[1\ \ 6\ \ -4]$.

$$\therefore\quad AB=\begin{bmatrix}-3\\5\\2\end{bmatrix}[1\ \ 6\ \ -4]$$

$$=\begin{bmatrix}-3&-18&12\\5&30&-20\\2&12&-8\end{bmatrix}.$$

So, $(AB)' = \begin{bmatrix} -3 & 5 & 2 \\ -18 & 30 & 12 \\ 12 & -20 & -8 \end{bmatrix}$.

Also, $A' = [-3 \;\; 5 \;\; 2]$ and $B' = \begin{bmatrix} 1 \\ 6 \\ -4 \end{bmatrix}$.

$$\therefore \quad B'A' = \begin{bmatrix} 1 \\ 6 \\ -4 \end{bmatrix} [-3 \;\; 5 \;\; 2]$$

$$= \begin{bmatrix} -3 & 5 & 2 \\ -18 & 30 & 12 \\ 12 & -20 & -8 \end{bmatrix}.$$

Hence, $(AB)' = B'A'$.

SYMMETRIC AND SKEW-SYMMETRIC MATRICES

SYMMETRIC MATRIX *A square matrix A is said to be symmetric if $A' = A$.*

A is symmetric $\Leftrightarrow$ $a_{ji} = a_{ij}$.

EXAMPLE 1 *Consider the matrix* $A = \begin{bmatrix} 4 & \sqrt{2} \\ \sqrt{2} & \frac{1}{3} \end{bmatrix}$.

Then, $A' = \begin{bmatrix} 4 & \sqrt{2} \\ \sqrt{2} & \frac{1}{3} \end{bmatrix}' = \begin{bmatrix} 4 & \sqrt{2} \\ \sqrt{2} & \frac{1}{3} \end{bmatrix} = A$.

Hence, A is symmetric.

EXAMPLE 2 *Consider the matrix* $B = \begin{bmatrix} 6 & -7 & 4 \\ -7 & 3 & 0 \\ 4 & 0 & \sqrt{5} \end{bmatrix}$.

Then, $B' = \begin{bmatrix} 6 & -7 & 4 \\ -7 & 3 & 0 \\ 4 & 0 & \sqrt{5} \end{bmatrix}' = \begin{bmatrix} 6 & -7 & 4 \\ -7 & 3 & 0 \\ 4 & 0 & \sqrt{5} \end{bmatrix} = B$.

Hence, B is symmetric.

SKEW-SYMMETRIC MATRIX *A square matrix A is said to be skew-symmetric if $A' = -A$.*

REMARK A is skew-symmetric $\Leftrightarrow A' = -A \Leftrightarrow a_{ji} = -a_{ij}$

$\Leftrightarrow a_{ii} = -a_{ii} \Leftrightarrow 2a_{ii} = 0 \Leftrightarrow a_{ii} = 0.$

Thus, every diagonal element of a skew-symmetric matrix is zero.

EXAMPLE 1 *Let* $A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$. Then,

$$A' = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} = -A.$$

Hence, A is skew-symmetric.

EXAMPLE 2 *Let* $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$. Then,

$$A' = \begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix} = -A.$$

Hence, A is skew-symmetric.

SOME RESULTS ON SYMMETRIC AND SKEW-SYMMETRIC MATRICES

THEOREM 1 *Prove that the sum of two symmetric matrices is a symmetric matrix.*

PROOF Let A and B be symmetric matrices of the same order.
Then, $A' = A$ and $B' = B$.

$\therefore \quad (A + B)' = (A' + B') = (A + B) \qquad [\because \quad A' = A$ and $B' = B]$.

Hence, $(A + B)$ is symmetric.

THEOREM 2 *If A is a symmetric matrix then prove that kA is symmetric.*

PROOF Since A is symmetric, we have $A' = A$.
$\therefore \quad (kA)' = k \cdot A' = kA \quad [\because \quad A' = A]$.

This shows that kA is symmetric.

THEOREM 3 *Prove that the sum of two skew-symmetric matrices is a skew-symmetric matrix.*

PROOF Let A and B be two skew-symmetric matrices.
Then, $A' = -A$ and $B' = -B$.

$\therefore \quad (A + B)' = (A' + B') = (-A) + (-B) = -(A + B)$.

Hence, $(A + B)$ is skew-symmetric.

THEOREM 4 *If A is a skew-symmetric matrix then prove that kA is skew-symmetric.*

PROOF Since A is skew-symmetric, we have $A' = -A$.

$$\therefore \quad (kA)' = k \cdot A' = k \cdot (-A) \qquad [\because \quad A' = -A]$$
$$= -(k \cdot A).$$

Thus, $(kA)' = -(kA)$.

Hence, (kA) is skew-symmetric.

THEOREM 5 *For any square matrix A with real number entries, prove that (i) $(A + A')$ is symmetric and (ii) $(A - A')$ is skew-symmetric.*

PROOF Let A be a square matrix with real number entries. Then, we have:

(i) $(A + A')' = A' + (A')'$ $[\because \ (A + B)' = A' + B']$

$= A' + A$ $[\because \ (A')' = A]$

$= (A + A')$ $[\because \ (A + B) = (B + A)]$.

Thus, $(A + A')' = (A + A')$.

Hence, $(A + A')$ is symmetric.

(ii) $(A - A')' = A' - (A')'$ $[\because \ (A - B)' = A' - B']$

$= A' - A$ $[\because \ (A')' = A]$

$= -(A - A')$.

Thus, $(A - A')' = -(A - A')$

Hence, $(A - A')$ is skew-symmetric.

THEOREM 6 *Prove that every square matrix is expressible as the sum of a symmetric and a skew-symmetric matrix.*

PROOF Let A be any square matrix. Then, we can write

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q \text{ (say)},$$

where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$.

$$\text{Now, } P' = \left\{\frac{1}{2}(A + A')\right\}'$$

$$= \frac{1}{2}(A + A')' \qquad [\because \ (kA)' = kA']$$

$$= \frac{1}{2}\{A' + (A')'\} \qquad [\because \ (A + B)' = (A' + B')]$$

$$= \frac{1}{2}(A' + A) \qquad [\because \ (A')' = A]$$

$$= \frac{1}{2}(A + A') \qquad [\because \ (A' + A) = (A + A')]$$

$$= P.$$

Thus, $P' = P$ and therefore, P is symmetric.

$$\text{And, } Q' = \left\{\frac{1}{2}(A - A')\right\}'$$

$$= \frac{1}{2}(A - A') \qquad [\because \ (kA)' = kA']$$

$$= \frac{1}{2}\{A' - (A')'\} \qquad [\because \ (A - B)' = A' - B']$$

$$= \frac{1}{2}(A' - A) \qquad [\because \ (A')' = A]$$

$$= -\frac{1}{2}(A - A') = -Q.$$

Thus, $Q' = -Q$ and therefore, Q is skew-symmetric.

$\therefore$ $A = P + Q$, where P is symmetric and Q is skew-symmetric.

SOLVED EXAMPLES

EXAMPLE 1 *Express the matrix* $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ *as the sum of a symmetric matrix and a skew-symmetric matrix.*

SOLUTION We know that $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$, where $\frac{1}{2}(A + A')$ is symmetric and $\frac{1}{2}(A - A')$ is skew-symmetric.

Here, $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$.

$$\therefore \quad (A + A') = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 3 & -4 + 1 \\ 1 + (-4) & -1 + (-1) \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -3 & -2 \end{bmatrix}.$$

Let $P = \frac{1}{2}(A + A') = \frac{1}{2} \cdot \begin{bmatrix} 6 & -3 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 3 & \frac{-3}{2} \\ \frac{-3}{2} & -1 \end{bmatrix}$.

And, $(A - A') = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 3 - 3 & -4 - 1 \\ 1 - (-4) & -1 - (-1) \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 1 + 4 & -1 + 1 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}.$$

Let $Q = \frac{1}{2}(A - A') = \frac{1}{2} \cdot \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}$.

Then, $P' = \begin{bmatrix} 3 & \frac{-3}{2} \\ \frac{-3}{2} & -1 \end{bmatrix}' = \begin{bmatrix} 3 & \frac{-3}{2} \\ \frac{-3}{2} & -1 \end{bmatrix} = P$.

$\therefore$ P is symmetric.

And, $Q' = \begin{bmatrix} 0 & \frac{-5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}' = \begin{bmatrix} 0 & \frac{5}{2} \\ \frac{-5}{2} & 0 \end{bmatrix} = -Q$.

$\therefore$ Q is skew-symmetric.

$$\text{Now, } P + Q = \begin{bmatrix} 3 & \frac{-3}{2} \\ \frac{-3}{2} & -1 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-5}{2} \\ \frac{5}{2} & 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = A.$$

Hence, $A = P + Q$, where P is symmetric and Q is skew-symmetric.

<u>EXAMPLE 2</u> *Express the matrix* $A = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ *as the sum of a symmetric and a skew-symmetric matrix.* **[CBSE 2006]**

SOLUTION We have

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}.$$

$$\therefore \quad (A + A') = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix} + \begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 3+(-6) & 5+(-4) \\ -6+3 & 8+8 & 3+6 \\ -4+5 & 6+3 & 5+5 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix}.$$

$$\text{Let } P = \frac{1}{2}(A + A') = \frac{1}{2} \cdot \begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-3}{2} & \frac{1}{2} \\ \frac{-3}{2} & 8 & \frac{9}{2} \\ \frac{1}{2} & \frac{9}{2} & 5 \end{bmatrix}.$$

$$\text{And, } (A - A') = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 3-(-6) & 5-(-4) \\ -6-3 & 8-8 & 3-6 \\ -4-5 & 6-3 & 5-5 \end{bmatrix} = \begin{bmatrix} 0 & 3+6 & 5+4 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}.$$

$$\text{Let } Q = \frac{1}{2}(A - A') = \frac{1}{2}\cdot\begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{9}{2} & \frac{9}{2} \\ \frac{-9}{2} & 0 & \frac{-3}{2} \\ \frac{-9}{2} & \frac{3}{2} & 0 \end{bmatrix}.$$

$$P' = \begin{bmatrix} 1 & \frac{-3}{2} & \frac{1}{2} \\ \frac{-3}{2} & 8 & \frac{9}{2} \\ \frac{1}{2} & \frac{9}{2} & 5 \end{bmatrix}' = \begin{bmatrix} 1 & \frac{-3}{2} & \frac{1}{2} \\ \frac{-3}{2} & 8 & \frac{9}{2} \\ \frac{1}{2} & \frac{9}{2} & 5 \end{bmatrix} = P.$$

$\therefore$ P is symmetric.

$$\text{And, } Q' = \begin{bmatrix} 0 & \frac{9}{2} & \frac{9}{2} \\ \frac{-9}{2} & 0 & \frac{-3}{2} \\ \frac{-9}{2} & \frac{3}{2} & 0 \end{bmatrix}' = \begin{bmatrix} 0 & \frac{-9}{2} & \frac{-9}{2} \\ \frac{9}{2} & 0 & \frac{3}{2} \\ \frac{9}{2} & \frac{-3}{2} & 0 \end{bmatrix} = -Q.$$

$\therefore$ Q is skew-symmetric.

$$\text{Now, } (P+Q) = \begin{bmatrix} 1 & \frac{-3}{2} & \frac{1}{2} \\ \frac{-3}{2} & 8 & \frac{9}{2} \\ \frac{1}{2} & \frac{9}{2} & 5 \end{bmatrix} + \begin{bmatrix} 0 & \frac{9}{2} & \frac{9}{2} \\ \frac{-9}{2} & 0 & \frac{-3}{2} \\ \frac{-9}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix} = A.$$

Hence, $A = P + Q$,
where P is symmetric and Q is skew-symmetric.

EXAMPLE 3 *If A and B are symmetric matrices of the same order then show that AB is symmetric if and only if $AB = BA$.*

SOLUTION Let A and B be symmetric matrices.

Then, $A' = A$ and $B' = B$.

Let AB be symmetric. Then,

$AB = (AB)'$ [by definition]

$= B'A' = BA$ [$\because$ $B' = B$ and $A' = A$].

$\therefore$ $AB = BA$.

Conversely, let $AB = BA$. Then,

$$(AB)' = B'A'$$
$$= BA \quad [\because \ B' = B \text{ and } A' = A]$$
$$= AB \quad [\text{given}].$$

Thus, $(AB)' = AB$ and hence, AB is symmetric.

EXAMPLE 4 *If A and B are symmetric matrices, prove that $(AB - BA)$ is skew-symmetric.*

SOLUTION Let A and B be symmetric matrices.

Then, $A' = A$ and $B' = B$.

Now, $(AB - BA)' = (AB)' - (BA)'$

$$= (B'A') - (A'B')$$
$$= (BA - AB) \quad [\because \ A' = A \text{ and } B' = B]$$
$$= -(AB - BA).$$

Thus, $(AB - BA)' = -(AB - BA)$.

Hence, $(AB - BA)$ is skew-symmetric.

EXAMPLE 5 *If A is symmetric, show that $B'AB$ is symmetric.*

SOLUTION Let A be symmetric. Then, $A' = A$.

$\therefore \ (B'AB)' = B'A'(B')'$

$$= B'A'B \quad [\because \ (B')' = B]$$
$$= B'AB \quad [\because \ A' = A].$$

Thus, $(B'AB)' = B'AB$.

Hence, $(B'AB)$ is symmetric.

EXERCISE 5D

1. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 7 & -4 \end{bmatrix}$, verify that $(A')' = A$.

2. If $A = \begin{bmatrix} 3 & 5 \\ -2 & 0 \\ 4 & -6 \end{bmatrix}$, verify that $(2A)' = 2A'$.

3. If $A = \begin{bmatrix} 3 & 2 & -1 \\ -5 & 0 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -5 & -2 \\ 3 & 1 & 8 \end{bmatrix}$, verify that $(A + B)' = (A' + B')$.

4. If $P = \begin{bmatrix} 3 & 4 \\ 2 & -1 \\ 0 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} 7 & -5 \\ -4 & 0 \\ 2 & 6 \end{bmatrix}$, verify that $(P + Q)' = (P' + Q')$.

5. If $A = \begin{bmatrix} 4 & 1 \\ 5 & 8 \end{bmatrix}$, show that $(A + A')$ is symmetric. **[CBSE 2001]**

6. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, show that $(A - A')$ is skew-symmetric. **[CBSE 2001C]**

7. Show that the matrix $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ is skew-symmetric.

HINT: Show that $A' = -A$.

8. Express the matrix $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ as the sum of a symmetric matrix and a skew-symmetric matrix.

9. Express the matrix $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ as the sum of a symmetric matrix and a skew-symmetric matrix.

10. Express the matrix $A = \begin{bmatrix} -1 & 5 & 1 \\ 2 & 3 & 4 \\ 7 & 0 & 9 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

11. Express the matrix A as the sum of a symmetric and a skew-symmetric matrix, where

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 3 \\ 1 & -1 & 2 \end{bmatrix}.$$ **[CBSE 2005C]**

12. Express the matrix $A = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}$ as sum of two matrices such that one is symmetric and the other is skew-symmetric. **[CBSE 2008]**

13. For each of the following pairs of matrices A and B, verify that $(AB)' = (B'A')$:

(i) $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 3 & -1 \\ 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = [-2 \quad -1 \quad -4]$ **[CBSE 2002]**

(iv) $A = \begin{bmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ 2 & 1 \\ -1 & 0 \end{bmatrix}$

14. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A'A = I$.

15. If matrix $A = [1 \quad 2 \quad 3]$, write AA'. **[CBSE 2009]**

ANSWERS (EXERCISE 5D)

8. $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

9. $A = \begin{bmatrix} 3 & \frac{-3}{2} \\ \frac{-3}{2} & -1 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}$

10. $A = \begin{bmatrix} -1 & \frac{7}{2} & 4 \\ \frac{7}{2} & 3 & 2 \\ 4 & 2 & 9 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & -3 \\ \frac{-3}{2} & 0 & 2 \\ 3 & -2 & 0 \end{bmatrix}$

11. $A = \begin{bmatrix} 3 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-3}{2} & \frac{-1}{2} \\ \frac{3}{2} & 0 & 2 \\ \frac{1}{2} & -2 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} 3 & 3 & \frac{5}{2} \\ 3 & 1 & \frac{9}{2} \\ \frac{5}{2} & \frac{9}{2} & 7 \end{bmatrix} + \begin{bmatrix} 0 & -1 & \frac{5}{2} \\ 1 & 0 & \frac{-3}{2} \\ \frac{-5}{2} & \frac{3}{2} & 0 \end{bmatrix}$

15. [14]

Elementary Operations on Matrices

Given below are three row operations and three column operations on a matrix, which are called elementary operations or transformations.

EQUIVALENT MATRICES *Two matrices A and B are said to be equivalent if one is obtained from the other by one or more elementary operations and we write, $A \sim B$.*

THREE ELEMENTARY ROW OPERATIONS

(i) Interchange of any two rows The interchange of ith and jth rows is denoted by $R_i \leftrightarrow R_j$.

EXAMPLE Let $A = \begin{bmatrix} 3 & 2 & -1 \\ \sqrt{2} & 4 & 6 \\ 5 & -3 & 7 \end{bmatrix}$.

$$\text{Applying } R_2 \leftrightarrow R_3 \text{, we get } A \sim \begin{bmatrix} 3 & 2 & -1 \\ 5 & -3 & 7 \\ \sqrt{2} & 4 & 6 \end{bmatrix}.$$

(ii) Multiplication of the elements of a row by a nonzero number Suppose each element of ith row of a given matrix is multiplied by a nonzero number k.

Then, we denote it by $\boldsymbol{R_i \to k\, R_i}$.

EXAMPLE Let $A = \begin{bmatrix} 3 & 2 & -1 \\ \sqrt{3} & -5 & 6 \\ 1 & 8 & 4 \end{bmatrix}$.

$$\text{Applying } R_2 \to 4R_2 \text{, we get } A \sim \begin{bmatrix} 3 & 2 & -1 \\ 4\sqrt{3} & -20 & 24 \\ 1 & 8 & 4 \end{bmatrix}.$$

(iii) Multiplying each element of a row by a nonzero number and then adding them to the corresponding elements of another row Suppose each element of jth row of a matrix A is multiplied by a nonzero number k and then added to the corresponding elements of ith row.

We denote it by $\boldsymbol{R_i \to R_i + kR_j}$.

EXAMPLE Let $A = \begin{bmatrix} 2 & -1 & 5 \\ -3 & 4 & \sqrt{2} \\ 7 & 6 & 3 \end{bmatrix}$.

$$\text{Applying } R_1 \to R_1 + 2R_3 \text{, we get } A \sim \begin{bmatrix} 16 & 11 & 11 \\ -3 & 4 & \sqrt{2} \\ 7 & 6 & 3 \end{bmatrix}.$$

THREE ELEMENTARY COLUMN OPERATIONS

(i) Interchange of any two columns The interchange of ith and jth columns is denoted by $\boldsymbol{C_i \leftrightarrow C_j}$.

EXAMPLE Let $A = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 5 & 4 \\ 6 & 3 & \frac{1}{2} \end{bmatrix}$.

$$\text{Applying } C_1 \leftrightarrow C_2, \text{ we get } A \sim \begin{bmatrix} 1 & 2 & -3 \\ 5 & -1 & 4 \\ 3 & 6 & \frac{1}{2} \end{bmatrix}.$$

(ii) Multiplying each element of a column by a nonzero number Suppose each element of ith column of matrix A is multiplied by a nonzero number k.

Then, we write, $\boldsymbol{C_i \rightarrow kC_i}$.

EXAMPLE Let $A = \begin{bmatrix} 3 & 1 & -5 \\ \sqrt{2} & -2 & 4 \\ 6 & 2 & 8 \end{bmatrix}$.

$$\text{Applying } C_3 \rightarrow 2C_3, \text{ we get } A \sim \begin{bmatrix} 3 & 1 & -10 \\ \sqrt{2} & -2 & 8 \\ 6 & 2 & 16 \end{bmatrix}.$$

(iii) Multiplying each element of a column of a given matrix *A* by a nonzero number and then adding to the corresponding elements of another column

Suppose each element of jth column of a given matrix A is multiplied by a nonzero number k and then added to the corresponding elements of ith column.

Then, we write, $\boldsymbol{C_i \rightarrow C_i + kC_j}$.

EXAMPLE Let $A = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & 1 \\ 5 & -2 & 6 \end{bmatrix}$.

$$\text{Applying } C_3 \rightarrow C_3 + 2C_1, \text{ we get } A \sim \begin{bmatrix} 2 & 0 & 8 \\ -1 & 3 & -1 \\ 5 & -2 & 16 \end{bmatrix}.$$

INVERTIBLE MATRICES *A square matrix A of order n is said to be invertible if there exists a square matrix B of order n such that*

$$\boldsymbol{AB = BA = I}.$$

Also, then B is called the inverse of A and we write, $\boldsymbol{A^{-1} = B}$.

EXAMPLE Let $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & -15+15 \\ 2-2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

$$BA = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & 10-10 \\ -3+3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$\therefore \quad AB = BA = I.$

Hence, $A^{-1} = B.$

THEOREM 1 (Uniqueness of Inverse) *Every invertible square matrix has a unique inverse.*

PROOF Let A be an invertible square matrix of order n.
If possible, let B as well as C be the inverse of A.

Then, $AB = BA = I$ and $AC = CA = I$.

Now, $AC = I \Rightarrow B(AC) = B \cdot I = B,$

$BA = I \Rightarrow (BA)C = I \cdot C = C.$

But, $B(AC) = (BA)C$ [by associative law of multiplication]

$\therefore \quad B = C.$

Hence, an invertible matrix has a unique inverse.

INVERSE OF A MATRIX BY ELEMENTARY ROW OPERATIONS

Let A be a square matrix of order n.

We can write, $A = I \cdot A.$ ... (i)

Now, let a sequence of elementary row operations reduce $\mathbf{A}$ on LHS of (i) to I and I on RHS of (i) to a matrix B.

Then, $I = BA \Rightarrow I \cdot A^{-1} = (BA)A^{-1} = B(AA^{-1}) = BI$

$\Rightarrow A^{-1} = B.$

We can summarise the above method as given below.

METHOD **Step 1.** Write $A = I \cdot A.$

Step 2. By using elementary row operations on A, transform it into a unit matrix.

Step 3. In the same order we apply elementary operations on I to convert it into a matrix B.

Step 4. Then, $A^{-1} = B.$

REMARK If on applying one or more elementary row operations on A, we obtain all zeros in one or more rows, then we say that A^{-1} does not exist.

SOLVED EXAMPLES

EXAMPLE 1 *By using elementary row operations, find the inverse of the matrix* $A = \begin{bmatrix} 1 & -2 \\ 2 & -6 \end{bmatrix}$.

SOLUTION We have

$$\begin{bmatrix} 1 & -2 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \cdot A \quad [R_2 \to R_2 - 2R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & \frac{-1}{2} \end{bmatrix} \cdot A \qquad \left[R_2 \to \left(\frac{-1}{2}\right) R_2\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & -\frac{1}{2} \end{bmatrix} \cdot A \qquad [R_1 \to R_1 + 2R_2].$$

Hence, $A^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & -\frac{1}{2} \end{bmatrix}$.

EXAMPLE 2 *By using elementary row operations, find the inverse of the matrix*

$$A = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}.$$

SOLUTION We have

$$A = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot A \quad [R_1 \to R_1 + R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \cdot A \quad [R_1 \to (-1) \cdot R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix} \cdot A \quad [R_2 \to R_2 + 4R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & \frac{3}{2} \end{bmatrix} \cdot A \quad \left[R_2 \to \left(\frac{-1}{2}\right) R_2\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} \cdot A \qquad [R_1 \to R_1 + R_2].$$

Hence, $A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$.

EXAMPLE 3 *If* $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$, *show that* A^{-1} *does not exist.*

SOLUTION We have

$$\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 1 \end{bmatrix} \cdot A \quad \left[R_1 \to \frac{1}{6} R_1\right]$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ \frac{1}{3} & 1 \end{bmatrix} \cdot A \quad [R_2 \to R_2 + 2R_1].$$

Thus, we have all zeros in second row of the left-hand side matrix. Hence, A^{-1} does not exist.

EXAMPLE 4 *By using elementary row operations, find the inverse of the matrix*

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}.$$

SOLUTION We have

$$\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \cdot A \quad \begin{bmatrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & 4 \\ 0 & 9 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \cdot A \quad [R_2 \leftrightarrow R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & -1 & 4 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -2 & 0 & 1 \\ -15 & 1 & 9 \end{bmatrix} \cdot A \quad \begin{bmatrix} R_1 \to R_1 + 3R_2 \\ R_3 \to R_3 + 9R_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 2 & 0 & -1 \\ -15 & 1 & 9 \end{bmatrix} \cdot A \quad [R_2 \to (-1) \cdot R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 2 & 0 & -1 \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} \cdot A \quad \left[R_3 \to \frac{1}{25} R_3 \right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} \cdot A \quad \begin{bmatrix} R_1 \to R_1 - 10R_3 \\ R_2 \to R_2 + 4R_3 \end{bmatrix}.$$

Hence, $A^{-1} = \begin{bmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$.

EXAMPLE 5 *By using elementary row operations, find the inverse of the matrix*

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}.$$

SOLUTION We have

$$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \quad [R_1 \to R_1 - R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} \cdot A \quad \begin{bmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & \frac{3}{2} & 0 \\ -3 & 3 & 1 \end{bmatrix} \cdot A \quad \left\{ R_2 \to \frac{1}{2} R_2 \right\}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ -5 & 6 & 1 \end{bmatrix} \cdot A \quad \left\{ \begin{matrix} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + 2R_2 \end{matrix} \right\}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ \frac{-5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix} \cdot A \quad \left\{ R_3 \to \frac{1}{4} R_3 \right\}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-5}{8} & \frac{5}{4} & \frac{1}{8} \\ \frac{-3}{8} & \frac{3}{4} & \frac{-1}{8} \\ \frac{-5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix} \cdot A. \quad \left\{ \begin{matrix} R_1 \to R_1 + \frac{1}{2} R_3 \\ R_2 \to R_2 - \frac{1}{2} R_3 \end{matrix} \right\}$$

Hence, $A^{-1} = \begin{bmatrix} \frac{-5}{8} & \frac{5}{4} & \frac{1}{8} \\ \frac{-3}{8} & \frac{3}{4} & \frac{-1}{8} \\ \frac{-5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix}$.

EXAMPLE 6 *By using elementary row transformations, find the inverse of the matrix*

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}.$$

SOLUTION We have

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \quad [R_2 \to R_2 - 2R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \quad [R_1 \leftrightarrow R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 5 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \quad [R_2 \to R_2 - 2R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -5 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -2 & 0 \end{bmatrix} \cdot A \quad [R_2 \leftrightarrow R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 1 \\ 5 & -2 & 2 \end{bmatrix} \cdot A \quad \begin{bmatrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 2R_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \cdot A \quad \begin{bmatrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 3R_3 \end{bmatrix}.$$

Hence, $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$.

EXAMPLE 7 *If* $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & -2 & 2 \end{bmatrix}$, *show that* A^{-1} *does not exist.*

SOLUTION We have

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot A \quad \begin{bmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \cdot A \quad [R_3 \to R_3 + R_2].$$

Thus, we have all zeros in 3rd row of the left-hand side matrix.
Hence, A^{-1} does not exist.

EXERCISE 5E

Using elementary row transformations, find the inverse of each of the following matrices:

1. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ **[CBSE 2007]** 2. $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ 3. $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ 5. $\begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$ 6. $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ 8. $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$ 9. $\begin{bmatrix} 3 & 0 & 2 \\ 1 & 5 & 9 \\ 6 & 4 & 7 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$ 11. $\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$ 12. $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ **[CBSE 2011]**

13. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ **[CBSE 2008]** 14. $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$ **[CBSE 2009]** 15. $\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ **[CBSE 2012]**

ANSWERS (EXERCISE 5E)

1. $\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$ **2.** $\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$ **3.** $\frac{1}{17} \cdot \begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$

4. $\begin{bmatrix} \frac{5}{22} & \frac{3}{22} \\ \frac{-2}{11} & \frac{1}{11} \end{bmatrix}$ **5.** $\begin{bmatrix} \frac{1}{4} & 0 \\ \frac{-1}{10} & \frac{1}{5} \end{bmatrix}$ **6.** $\begin{bmatrix} \frac{-9}{2} & \frac{7}{3} \\ 4 & -3 \end{bmatrix}$

7. $\frac{1}{2} \cdot \begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$ **8.** $\frac{-1}{5} \cdot \begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$ **9.** $\frac{-1}{55} \cdot \begin{bmatrix} -1 & 8 & -10 \\ 47 & 9 & -25 \\ -26 & -12 & 15 \end{bmatrix}$

10. $\frac{1}{67} \cdot \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$ **11.** $\frac{-1}{8} \cdot \begin{bmatrix} 5 & -10 & -1 \\ 3 & -6 & 1 \\ 10 & -12 & -2 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$

13. $\begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$ **14.** $\begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix}$ **15.** $\begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}$

EXERCISE 5F

Very-Short-Answer Questions

1. Construct a 3×2 matrix whose elements are given by
$$a_{ij} = \frac{1}{2}(i - 2j)^2.$$

2. Construct a 2×3 matrix whose elements are given by
$$a_{ij} = \frac{1}{2}|-3i + j|.$$

3. If $\begin{bmatrix} x + 2y & -y \\ 3x & 4 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 6 & 4 \end{bmatrix}$, find the values of x and y. **[CBSE 2008C]**

4. Find the values of x and y, if
$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}.$$ **[CBSE 2008]**

5. If $x \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find the values of x and y.

6. If $\begin{bmatrix} x & 3x - y \\ 2x + z & 3y - w \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$, find the values of x, y, z, w.

7. If $\begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x + y \\ z + w & 3 \end{bmatrix} = 3\begin{bmatrix} x & y \\ z & w \end{bmatrix}$, find the values of x, y, z, w.

8. If $A = \text{diag}\,(3 \;\; -2 \;\; 5)$ and $B = \text{diag}\,(1 \;\; 3 \;\; -4)$, find $(A + B)$.

9. Show that $\cos\theta \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \cdot \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} = I.$

10. If $A = \begin{bmatrix} 1 & -5 \\ -3 & 2 \\ 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ -2 & 3 \end{bmatrix}$, find the matrix C such that $A + B + C$ is a zero matrix.

11. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ then find the least value of α for which $A + A' = I$.

12. Find the values of x and y for which
$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$ **[CBSE 2003]**

13. Find the values of x and y for which
$$\begin{bmatrix} x & y \\ 3y & x \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$ **[CBSE 2003C]**

14. If $A = \begin{bmatrix} 4 & 5 \\ 1 & 8 \end{bmatrix}$, show that $(A + A')$ is symmetric. **[CBSE 2001]**

15. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, show that $(A - A')$ is skew-symmetric. **[CBSE 2001]**

16. If $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$, find a matrix X such that

$$A + 2B + X = O.$$

17. If $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$, find a matrix X such that

$$3A - 2B + X = O.$$ **[CBSE 2000]**

18. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A'A = I$.

19. If A and B are symmetric matrices of the same order, show that $(AB - BA)$ is a skew-symmetric matrix.

20. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $f(x) = x^2 - 4x + 1$, find $f(A)$.

21. If the matrix A is both symmetric and skew-symmetric, show that A is a zero matrix.

ANSWERS (EXERCISE 5F)

1. $\begin{bmatrix} \frac{1}{2} & \frac{9}{2} \\ 0 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{5}{2} & 2 & \frac{3}{2} \end{bmatrix}$ **3.** $x = 2, y = -3$ **4.** $x = 3, y = 3$

5. $x = 3, y = -4$ **6.** $x = 3, y = 7, z = -2, w = 14$

7. $x = 2, y = 4, z = 1, w = 3$ **8.** diag (4 1 1)

10. $C = \begin{bmatrix} -4 & 4 \\ 1 & -1 \\ -2 & -1 \end{bmatrix}$ **11.** $\alpha = \frac{\pi}{3}$ **12.** $x = 2, y = 1$

13. $x = 1, y = 1$ **16.** $X = \begin{bmatrix} 0 & -1 \\ -4 & -11 \end{bmatrix}$

17. $X = \begin{bmatrix} -16 & -4 \\ 3 & -5 \end{bmatrix}$ **20.** $f(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

6. DETERMINANTS

Determinant of a Square Matrix

Corresponding to each square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix},$$

there is associated an expression, called the *determinant* of A, denoted by det A, or $|A|$, written as

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}.$$

A matrix is an arrangement of numbers and so it has no fixed value, while each determinant has a fixed value.

A determinant having n rows and n columns is known as a determinant of order n.

The determinants of nonsquare matrices are not defined.

VALUE OF A DETERMINANT OF ORDER 1 *The value of a determinant of a* (1×1) *matrix* $[a]$ *is defined as* $|a| = a$.

VALUE OF A DETERMINANT OF ORDER 2 *We define*

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{21}a_{12}).$$

EXAMPLE 1 *Evaluate:*

(i) $\begin{vmatrix} 6 & -3 \\ 7 & -2 \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

SOLUTION (i) $\begin{vmatrix} 6 & -3 \\ 7 & -2 \end{vmatrix} = 6(-2) - 7(-3) = -12 + 21 = 9.$

(ii) $$\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = (x^2 - x + 1)(x + 1) - (x + 1)(x - 1)$$
$$= (x^3 + 1) - (x^2 - 1) = (x^3 - x^2 + 2).$$

EXAMPLE 2 *If* $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, *find the value of x.* **[CBSE 2014]**

SOLUTION We have $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$

$\Leftrightarrow \quad (3x \times 4) - (-2) \times 7 = (8 \times 4) - (6 \times 7)$

$\Leftrightarrow \quad 12x + 14 = 32 - 42$

$\Leftrightarrow \quad 12x + 14 = -10$

$\Leftrightarrow \quad 12x = -24 \quad \Leftrightarrow \quad x = -2.$

Hence, $x = -2$.

EXAMPLE 3 *If* $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, *find the value of x.* **[CBSE 2013]**

SOLUTION We have $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$

$\Leftrightarrow \quad (x+1)(x+2) - (x-3)(x-1) = (4 \times 3) - 1 \times (-1)$

$\Leftrightarrow \quad (x^2 + 3x + 2) - (x^2 - 4x + 3) = 12 + 1$

$\Leftrightarrow \quad 7x - 1 = 13 \quad \Leftrightarrow \quad 7x = 14 \quad \Leftrightarrow \quad x = 2.$

Hence, $x = 2$.

EXAMPLE 4 *Show that* $\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix} = 1.$

SOLUTION We have $\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix}$

$= (\sin 10^\circ)(\cos 80^\circ) - (\sin 80^\circ)(-\cos 10^\circ)$

$= (\sin 10^\circ \cos 80^\circ + \cos 10^\circ \sin 80^\circ)$

$= \sin(10^\circ + 80^\circ) \quad [\because \ \sin A \cos B + \cos A \sin B = \sin(A + B)]$

$= \sin 90^\circ = 1.$

VALUE OF A DETERMINANT OF ORDER 3 OR MORE For finding the value of a determinant of order 3 or more, we need the following definitions.

MINOR OF a_{ij} IN $|A|$ *The minor of an element a_{ij} in $|A|$ is defined as the value of the determinant obtained by deleting the ith row and jth column of $|A|$, and it is denoted by M_{ij}.*

COFACTOR OF a_{ij} IN $|A|$ *The cofactor C_{ij} of an element a_{ij} in $|A|$ is defined as* $C_{ij} = (-1)^{i+j} \cdot M_{ij}$.

EXAMPLE 1 *Find the minors and cofactors of the elements of the determinant*

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

SOLUTION Let M_{ij} denote the minor of a_{ij} in Δ.

Now, a_{11} occurs in the 1st row and 1st column. So, in order to find the minor of a_{11}, we delete the 1st row and 1st column of Δ. The minor M_{11} of a_{11} is given by $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (a_{22}a_{33} - a_{32}a_{23})$.

Similarly, we have

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (a_{21}a_{33} - a_{31}a_{23});$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = (a_{21}a_{32} - a_{31}a_{22});$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{32} \end{vmatrix} = (a_{12}a_{33} - a_{32}a_{13}).$$

Similarly, we may obtain the minor of each of the remaining elements.

Now, if we denote the cofactor of a_{ij} by C_{ij} then

$$C_{11} = (-1)^{1+1} \cdot M_{11} = M_{11} = (a_{22}\,a_{33} - a_{32}\,a_{23});$$

$$C_{12} = (-1)^{1+2} \cdot M_{12} = -M_{12} = (a_{31}\,a_{23} - a_{21}\,a_{33});$$

$$C_{13} = (-1)^{1+3} \cdot M_{13} = M_{13} = (a_{21}\,a_{32} - a_{31}\,a_{22});$$

$$C_{21} = (-1)^{2+1} \cdot M_{21} = -M_{21} = (a_{32}\,a_{13} - a_{12}\,a_{33}).$$

Similarly, the cofactor of each of the remaining elements of Δ can be determined.

EXAMPLE 2 *Find the minor and cofactor of each element of* $\Delta = \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}.$

SOLUTION The minors of the elements of Δ are given by

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = -12;\; M_{12} = \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = 2;\; M_{13} = \begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix} = 23;$$

$$M_{21} = \begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} = -16;\; M_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4;\; M_{23} = \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} = 14;$$

$$M_{31} = \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix} = -4;\; M_{32} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = -6;\; M_{33} = \begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} = 11.$$

So, the cofactors of the corresponding elements of Δ are

$$C_{11} = (-1)^{1+1} \cdot M_{11} = M_{11} = -12;\; C_{12} = (-1)^{1+2} \cdot M_{12} = -M_{12} = -2;$$

$$C_{13} = (-1)^{1+3} \cdot M_{13} = M_{13} = 23;\ C_{21} = (-1)^{2+1} \cdot M_{21} = -M_{21} = 16;$$
$$C_{22} = (-1)^{2+2} \cdot M_{22} = M_{22} = -4;\ C_{23} = (-1)^{2+3} \cdot M_{23} = -M_{23} = -14;$$
$$C_{31} = (-1)^{3+1} \cdot M_{31} = M_{31} = -4;\ C_{32} = (-1)^{3+2} \cdot M_{32} = -M_{32} = 6;$$
$$C_{33} = (-1)^{3+3} \cdot M_{33} = M_{33} = 11.$$

Value of a Determinant

The value of a determinant is the sum of the products of elements of a row (or a column) with their corresponding cofactors.

We may expand a determinant by any arbitrarily chosen row or column.

Expansion of a Determinant

Expanding the given determinant by 1st row, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot (\text{its cofactor}) + a_{12} \cdot (\text{its cofactor}) + a_{13} \cdot (\text{its cofactor})$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \qquad [\because \ C_{12} = -M_{12}]$$

$$= a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} \cdot (a_{22}a_{33} - a_{32}a_{23}) - a_{12} \cdot (a_{21}a_{33} - a_{31}a_{23}) + a_{13} \cdot (a_{21}a_{32} - a_{31}a_{22}).$$

We may expand it by any row or column.

REMARK 1 If we expand a determinant by any row or column using minors, we keep in view the following symbols for a determinant of order three:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

REMARK 2 If a row or a column of a determinant consists of all zeros, the value of the determinant is zero.

EXAMPLE 1 *Evaluate* $\Delta = \begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix}$.

SOLUTION Expanding the given determinant by 1st row, we get

$$\Delta = 3 \cdot \begin{vmatrix} 2 & -3 \\ 1 & 7 \end{vmatrix} - 4 \cdot \begin{vmatrix} -6 & -3 \\ 8 & 7 \end{vmatrix} + 5 \cdot \begin{vmatrix} -6 & 2 \\ 8 & 1 \end{vmatrix}$$

$$= 3 \cdot (14 + 3) - 4 \cdot (-42 + 24) + 5 \cdot (-6 - 16) = 13.$$

EXAMPLE 2 *Expand the determinant* $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

SOLUTION Expanding by 1st column, we get

$$\Delta = a \cdot \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \cdot \begin{vmatrix} h & g \\ f & c \end{vmatrix} + g \cdot \begin{vmatrix} h & g \\ b & f \end{vmatrix}$$

$$= a(bc - f^2) - h \cdot (ch - fg) + g \cdot (fh - bg)$$

$$= abc - af^2 - ch^2 + fgh + fgh - bg^2$$

$$= (abc + 2fgh - af^2 - bg^2 - ch^2).$$

Properties of Determinants

The properties of a determinant serve the purpose of a useful tool for finding its value. We will mention these properties and verify them for a third-order determinant.

THEOREM 1 *The value of a determinant remains unchanged if its rows and columns are interchanged.*

PROOF Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by interchanging the rows and columns of Δ.

Then, $\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= a_1 \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \cdot \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \cdot \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

[expanded by 1st column]

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - a_3b_2)$$

$= \Delta$ [expanded by 1st row].

COROLLARY *If* A *is a square matrix then* $|A'| = |A|$.

THEOREM 2 *If two rows or columns of a determinant are interchanged then the determinant retains its absolute value but its sign is changed.*

PROOF Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by interchanging any two rows, say 1st and 3rd rows, of Δ. Then,

$$\Delta' = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

$$= a_3 \cdot \begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + a_1 \cdot \begin{vmatrix} b_3 & c_3 \\ b_2 & c_2 \end{vmatrix}$$

[expanded by 1st column]

$$= a_3(b_2c_1 - b_1c_2) - a_2(b_3c_1 - b_1c_3) + a_1(b_3c_2 - b_2c_3)$$
$$= -a_1(b_2c_3 - b_3c_2) + a_2(b_1c_3 - b_3c_1) - a_3(b_1c_2 - b_2c_1)$$
$$= -[a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)]$$
$$= -\Delta \quad \text{[expanded by 1st column].}$$

THEOREM 3 *If any two rows or columns of a determinant are identical then its value is zero.*

PROOF If we interchange the identical rows of the given determinant Δ then clearly there is no change in Δ. But, interchanging any two rows of a determinant changes its sign.

$$\therefore \quad \Delta = -\Delta \quad \Leftrightarrow \quad 2\Delta = 0, \text{ i.e., } \Delta = 0.$$

THEOREM 4 *If each element of a row or a column of a determinant is multiplied by a constant k then the value of the new determinant is k times the value of the original determinant.*

PROOF Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by multiplying each element of a row, say the second row of Δ, by k. Then,

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 \cdot \begin{vmatrix} kb_2 & kc_2 \\ b_3 & c_3 \end{vmatrix} - ka_2 \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & c_1 \\ kb_2 & kc_2 \end{vmatrix}$$

$$= a_1(kb_2c_3 - kb_3c_2) - ka_2(b_1c_3 - b_3c_1) + a_3(kb_1c_2 - kb_2c_1)$$
$$= k[a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)] = k\,\Delta.$$

COROLLARY *For a square matrix A of order n,* $|kA| = k^n \cdot |A|$.

THEOREM 5 *If any two rows or columns of a determinant are proportional then its value is zero.*

PROOF Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{vmatrix}$

$$= k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} \qquad \text{[by Theorem 4]}$$

$$= k \times 0 = 0 \qquad [\because \text{ 1st and 3rd rows are identical}].$$

THEOREM 6 *If each element of a row (or column) of a determinant is expressed as a sum of two or more terms then the determinant can be expressed as the sum of two or more determinants.*

PROOF Let $\Delta = \begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Then, on expanding Δ by first row, we get

$$\Delta = (a_1+\alpha_1)(b_2c_3-b_3c_2)-(b_1+\beta_1)(a_2c_3-a_3c_2) + (c_1+\gamma_1)(a_2b_3-a_3b_2)$$

$$= [a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)] + [\alpha_1(b_2c_3-b_3c_2)-\beta_1(a_2c_3-a_3c_2)+\gamma_1(a_2b_3-a_3b_2)]$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

THEOREM 7 *If to any row or column of a determinant, a multiple of another row or column is added, the value of the determinant remains the same.*

PROOF Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Let $\Delta' = \begin{vmatrix} a_1+ka_3 & b_1+kb_3 & c_1+kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Then, $\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & kb_3 & kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad [\because \text{ 2nd det. is 0, its 1st and 3rd rows being proportional}]$$

$$= \Delta.$$

THEOREM 8 *The sum of the products of the elements of any row (or column) of a determinant with cofactors of the corresponding elements of any other row (or column) is zero.*

PROOF Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$

Let us take the sum of the products of elements of first row with the cofactors of the corresponding elements of second row.

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23}$$
$$= a_{11}\cdot(-1)^{2+1}\cdot\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\cdot(-1)^{2+2}\cdot\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\cdot(-1)^{2+3}\cdot\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= -a_{11}(a_{12}a_{33} - a_{32}a_{13}) + a_{12}(a_{11}a_{33} - a_{31}a_{13}) - a_{13}(a_{11}a_{32} - a_{31}a_{12})$$
$$= 0.$$

To Find the Value of a Determinant

The main theme behind the simplification of a determinant lies in obtaining the maximum possible number of zeros in a row (or a column) by using the above properties and then to expand the determinant by that row (or column). We denote the 1st, 2nd, 3rd rows of a determinant by R_1, R_2, R_3 respectively and the 1st, 2nd, 3rd columns by C_1, C_2, C_3 respectively. We shall also express the

(i) interchange of the ith and jth rows by $R_i \leftrightarrow R_j$;

(ii) addition of k times the elements of the jth row with the corresponding elements of the ith row by $R_i \to R_i + kR_j$.

We use similar notations for operations on columns, replacing R by C.

EXAMPLES

Operation	Notation
(i) Interchange of 2nd and 3rd rows.	(i) $R_2 \leftrightarrow R_3$
(ii) Interchange of 1st and 3rd columns	(ii) $C_1 \leftrightarrow C_3$
(iii) Multiplying each element of 2nd row by (–5)	(iii) $R_2 \to (-5)R_2$
(iv) Multiplying each element of 1st column by $\frac{1}{3}$.	(iv) $C_1 \to \frac{1}{3}C_1$
(v) Multiplying each element of 3rd row by 6 and adding it to the corresponding element of 2nd row.	(v) $R_2 \to R_2 + 6R_3$
(vi) Multiplying each element of 2nd column by (–3) and adding it to the corresponding element of 1st column.	(vi) $C_1 \to C_1 + (-3)C_2$

SOLVED EXAMPLES (Short-Answer Questions)

EXAMPLE 1 *Evaluate* $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$. **[CBSE 2014C]**

SOLUTION Given determinant

$$= \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 2 \\ 3 & 8 & 3 \\ 5 & 9 & 5 \end{vmatrix} \quad [\text{applying } C_3 \to C_3 - 9C_2]$$

$$= 0 \qquad [\because\ C_1 \text{ and } C_3 \text{ are identical}].$$

EXAMPLE 2 *Prove that* $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0.$ **[CBSE 2009]**

SOLUTION Given determinant

$$= \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \quad [\text{applying } C_1 \to C_1 + C_2 + C_3]$$

$$= 0 \qquad [\because\ C_1 \text{ consists of all zeros}].$$

EXAMPLE 3 *Prove that* $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0.$

SOLUTION The given determinant

$$= \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} \quad [\text{applying } C_3 \to C_3 + C_2]$$

$$= (a+b+c) \cdot \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} \quad \text{[taking } (a+b+c) \text{ common from } C_3]$$

$$= (a+b+c) \times 0 = 0 \qquad [\because \; C_1 \text{ and } C_3 \text{ are identical}].$$

EXAMPLE 4 *Prove that* $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0.$

SOLUTION The given determinant

$$= \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix} \quad \text{[applying } C_3 \to C_3 + C_2]$$

$$= (ab+bc+ca) \cdot \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} \quad \text{[taking } (ab+bc+ca) \text{ common from } C_3]$$

$$= (ab+bc+ca) \times 0 = 0 \qquad [\because \; C_1 \text{ and } C_3 \text{ are identical}].$$

EXAMPLE 5 *Without expanding prove that* $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0.$

SOLUTION The given determinant

$$= \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} \quad [R_1 \to R_1 + R_2]$$

$$= (x+y+z) \cdot \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} \quad \text{[taking } (x+y+z) \text{ common from } R_1]$$

$$= (x+y+z) \times 0 = 0 \qquad [\because \; R_1 \text{ and } R_3 \text{ are identical}].$$

EXAMPLE 6 *If ω is a complex cube root of unity, prove that*

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0.$$

SOLUTION The given determinant

$$= \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & w \end{vmatrix}$$

$$= \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \quad [R_1 \to R_1 + R_2 + R_3]$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \quad [\because \ 1+\omega+\omega^2 = 0]$$

$= 0$ [$\because$ R_1 consists of all zeros].

EXAMPLE 7 *Show that* $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix} = 0.$ **[CBSE 2009]**

SOLUTION The given determinant

$$= \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$$

$$= (3x) \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix} \quad [\text{taking } 3x \text{ common from } R_3]$$

$= (3x) \times 0 = 0$ [$\because$ R_1 and R_3 are identical].

EXAMPLE 8 *Prove that* $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy.$

SOLUTION The given determinant

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & 0 & y \end{vmatrix} \quad [\text{applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$= 1 \cdot \begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix} = xy \quad [\text{expanding by } R_1].$$

EXAMPLE 9 *Prove that* $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ *is independent of* θ.

SOLUTION Let the given determinant be Δ. Then, expanding by first row, we get:

$$\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$$

$$= x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$$

$$= -x^3 - x + x(\sin^2\theta + \cos^2\theta) = -x^3 - x + x = -x^3,$$

which is independent of θ.

EXAMPLE 10 *Prove that* $\begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix} = 0.$

SOLUTION Let the given determinant be Δ. Then, expanding by first row, we get:

$$\Delta = (-\sin\alpha) \cdot \begin{vmatrix} -\sin\alpha & \sin\beta \\ \cos\alpha & 0 \end{vmatrix} + (-\cos\alpha) \cdot \begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & -\sin\beta \end{vmatrix}$$

$$= (-\sin\alpha) \cdot \{0 - \cos\alpha\sin\beta\} + (-\cos\alpha) \cdot (\sin\alpha\sin\beta - 0)$$

$$= (-\sin\alpha)(-\cos\alpha\sin\beta) + (-\cos\alpha)(\sin\alpha\sin\beta)$$

$$= (\sin\alpha\cos\alpha\sin\beta - \sin\alpha(\cos\alpha\sin\beta) = 0.$$

Hence, $\Delta = 0$.

EXAMPLE 11 *Prove that* $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0.$

SOLUTION Let the given determinant be Δ. Then, applying $R_2 \to R_2 - R_1$, we get:

$$\Delta = \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ x & y & z \\ x & y & z \end{vmatrix} = 0 \quad [\because \; R_2 \text{ and } R_3 \text{ are identical}].$$

EXAMPLE 12 *Without expanding the determinant, prove that*

$$\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix} = 0.$$

SOLUTION Let the given determinant be Δ. Then, applying $C_3 \to C_3 - C_2$, we get:

$$\Delta = \begin{vmatrix} a+b & 2a+b & a \\ 2a+b & 3a+b & a \\ 4a+b & 5a+b & a \end{vmatrix}$$

$$= \begin{vmatrix} a+b & a & a \\ 2a+b & a & a \\ 4a+b & a & a \end{vmatrix} \quad [\text{applying } C_2 \to C_2 - C_1]$$

$$= 0 \quad [\because \ C_2 \text{ and } C_3 \text{ are identical}].$$

Hence, $\Delta = 0$.

ILLUSTRATIVE EXAMPLES [Long-Answer Questions]

EXAMPLE 1 *Evaluate* $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$.

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

$$= \begin{vmatrix} 46 & 21 & 219 \\ 42 & 27 & 198 \\ 38 & 17 & 181 \end{vmatrix} \quad [C_1 \to (C_1 - C_3) \text{ and } C_2 \to (C_2 - C_3)]$$

$$= \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix} \quad [C_1 \to (C_1 - 2C_2) \text{ and } C_3 \to (C_3 - 10C_2)]$$

$$= \begin{vmatrix} 0 & 4 & -2 \\ 0 & 78 & -39 \\ 4 & 17 & 11 \end{vmatrix} \quad [R_1 \to (R_1 - R_3) \text{ and } R_2 \to (R_2 + 3R_3)]$$

$$= 2(39) \begin{vmatrix} 0 & 2 & -1 \\ 0 & 2 & -1 \\ 4 & 17 & 11 \end{vmatrix}$$

[taking 2 common from R_1 and 39 common from R_2]

$$= (78 \times 0) = 0 \quad [\because \ R_1 \text{ and } R_2 \text{ are identical}].$$

EXAMPLE 2 *Without expanding, prove that*

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0.$$ **[CBSE 2007]**

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} \quad [C_3 \to C_3 + (\sin\delta)C_1 - (\cos\delta)C_2]$$

$= 0$ [expanded by C_3].

Hence, $\Delta = 0$.

EXAMPLE 3 *Prove that* $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3.$ **[CBSE 2012]**

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$$

$$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix} \quad [R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 3R_1]$$

$$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix} \quad [R_3 \to R_3 - 3R_2]$$

$$= a \cdot \begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} \quad [\text{expanding along } C_1]$$

$= a \cdot (a^2 - 0) = a^3.$

Hence, $\Delta = a^3$.

EXAMPLE 4 *Prove that* $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2.$ **[CBSE 2011, '11C]**

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

$$= (abc) \cdot \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix} \quad \begin{bmatrix} \text{taking out } a, b, c \text{ common from} \\ C_1, C_2, C_3 \text{ respectively} \end{bmatrix}$$

$$= \begin{vmatrix} -a & 0 & 0 \\ b & 0 & 2b \\ c & 2c & 0 \end{vmatrix} \quad [C_2 \to (C_2 + C_1) \text{ and } C_3 \to (C_3 + C_1)]$$

$$= (abc) \cdot (-a) \begin{vmatrix} 0 & 2b \\ 2c & 0 \end{vmatrix} = (abc)(-a)(-4bc)$$

$$= 4a^2b^2c^2.$$

EXAMPLE 5 *Using properties of determinants, prove that*

$$\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & cb^2 & 0 \end{vmatrix} = 2a^3b^3c^3.$$

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & cb^2 & 0 \end{vmatrix}$$

$$= (a^2b^2c^2) \cdot \begin{vmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{vmatrix} \quad \begin{bmatrix} \text{taking } a^2, b^2, c^2 \text{ common from} \\ C_1, C_2 \text{ and } C_3 \text{ respectively} \end{bmatrix}$$

$$= (a^3b^3c^3) \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \quad \begin{bmatrix} \text{taking } a, b, c \text{ common from} \\ R_1, R_2 \text{ and } R_3 \text{ respectively} \end{bmatrix}$$

$$= (a^3b^3c^3) \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} \quad [R_3 \to R_3 - R_2]$$

$$= (a^3b^3c^3) \cdot (-1) \cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \quad [\text{expanded by } C_1]$$

$$= -(a^3b^3c^3)(-1-1) = 2a^3b^3c^3.$$

Hence, $\Delta = 2a^3b^3c^3$.

EXAMPLE 6 *Using properties of determinants, prove that*

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3.$$ **[CBSE 2009, '14]**

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$$

$$= \begin{vmatrix} x+y & x & x \\ 3x+2y & 2x & 0 \\ 7x+5y & 5x & 0 \end{vmatrix} \quad [R_2 \to (R_2 - 2R_1) \text{ and } R_3 \to (R_3 - 3R_1)]$$

$$= x \cdot \begin{vmatrix} 3x+2y & 2x \\ 7x+5y & 5x \end{vmatrix} \quad [\text{expanded by } C_3]$$

$$= x^2 \cdot \begin{vmatrix} 3x+2y & 2 \\ 7x+5y & 5 \end{vmatrix} \quad [\text{taking } x \text{ common from } C_2]$$

$$= x^2 \cdot [5(3x+2y) - 2(7x+5y)] = (x^2 \cdot x) = x^3.$$

Hence, $\Delta = x^3$.

EXAMPLE 7 *Prove that* $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad [\text{applying } R_2 \to (R_2 - R_1) \text{ and } R_3 \to (R_3 - R_1)]$$

$$= (b-a)(c-a) \cdot \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

[taking $(b-a)$ common from R_2 and $(c-a)$ common from R_3]

$$= (b-a)(c-a) \times 1 \cdot \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} \quad [\text{expanded by } C_1]$$

$$= (b-a)(c-a)\{(c+a) - (b+a)\}$$
$$= (b-a)(c-a)(c-b) = (a-b)(b-c)(c-a).$$

Hence, $\Delta = (a-b)(b-c)(c-a)$.

EXAMPLE 8 *Prove that* $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$

[CBSE 2009C, '11, '12, '13C]

SOLUTION Let the value of the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a^3-c^3 & b^3-c^3 & c^3 \end{vmatrix}$$

[applying $C_1 \to (C_1 - C_3)$ and $C_2 \to (C_2 - C_3)$]

$$= (a-c)(b-c) \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ac+c^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

[taking out $(a-c)$ and $(b-c)$ common from C_1 and C_2]

$$= (a-c)(b-c) \cdot 1 \cdot \begin{vmatrix} 1 & 1 \\ a^2+ac+c^2 & b^2+bc+c^2 \end{vmatrix} \quad \text{[expanded by } R_1\text{]}$$

$$= (a-c)(b-c) \cdot [(b^2+bc+c^2) - (a^2+ac+c^2)]$$

$$= (a-c)(b-c)[(b^2-a^2) + (bc-ac)]$$

$$= (a-c)(b-c)[(b^2-a^2) + (b-a)c]$$

$$= (a-c)(b-c)(b-a)(b+a+c)$$

$$= (a-b)(b-c)(c-a)(a+b+c).$$

Hence, $\Delta = (a-b)(b-c)(c-a)(a+b+c)$.

EXAMPLE 9 *Prove that*

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma).$$

[CBSE 2007C, '08, '10C, '11C, '12C]

SOLUTION Let the value of the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$$

$$= \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{vmatrix} \quad \text{[applying } R_3 \to (R_3 + R_1)\text{]}$$

$$= (\alpha+\beta+\gamma)\cdot\begin{vmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ 1 & 1 & 1\end{vmatrix} \quad [\text{taking } (\alpha+\beta+\gamma) \text{ common from } R_3]$$

$$= (\alpha+\beta+\gamma)\cdot\begin{vmatrix}\alpha-\gamma & \beta-\gamma & \gamma\\ \alpha^2-\gamma^2 & \beta^2-\gamma^2 & \gamma^2\\ 0 & 0 & 1\end{vmatrix}$$

[applying $C_1 \to (C_1 - C_3)$ and $C_2 \to (C_2 - C_3)$]

$$= (\alpha+\beta+\gamma)(\alpha-\gamma)(\beta-\gamma)\cdot\begin{vmatrix}1 & 1 & \gamma\\ \alpha+\gamma & \beta+\gamma & \gamma^2\\ 0 & 0 & 1\end{vmatrix}$$

[taking $(\alpha-\gamma)$ common from C_1 and $(\beta-\gamma)$ common from C_2]

$$= (\alpha+\beta+\gamma)(\alpha-\gamma)(\beta-\gamma)\cdot 1\cdot\begin{vmatrix}1 & 1\\ \alpha+\gamma & \beta+\gamma\end{vmatrix} \quad [\text{expanded by } R_3]$$

$$= (\alpha+\beta+\gamma)(\alpha-\gamma)(\beta-\gamma)[(\beta+\gamma)-(\alpha+\gamma)]$$

$$= (\alpha+\beta+\gamma)(\alpha-\gamma)(\beta-\gamma)(\beta-\alpha)$$

$$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma).$$

Hence, $\Delta = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$.

EXAMPLE 10 *Using properties of determinants, show that*

$$\begin{vmatrix}a+x & y & z\\ x & a+y & z\\ x & y & a+z\end{vmatrix} = a^2(a+x+y+z).$$ **[CBSE 2003, '14]**

SOLUTION Let the value of the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix}a+x & y & z\\ x & a+y & z\\ x & y & a+z\end{vmatrix}$$

$$= \begin{vmatrix}a+x+y+z & y & z\\ a+x+y+z & a+y & z\\ a+x+y+z & y & a+z\end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= (a+x+y+z)\cdot\begin{vmatrix}1 & y & z\\ 1 & a+y & z\\ 1 & y & a+z\end{vmatrix}$$

[taking $(a+x+y+z)$ common from C_1]

$$= (a+x+y+z)\cdot\begin{vmatrix}1 & y & z\\ 0 & a & 0\\ 0 & 0 & a\end{vmatrix} \quad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= (a+x+y+z)\cdot 1\cdot\begin{vmatrix}a & 0\\ 0 & a\end{vmatrix} \quad [\text{expanded by } C_1]$$

$$= a^2(a+x+y+z).$$

Hence, $\Delta = a^2(a+x+y+z)$.

EXAMPLE 11 *Using properties of determinants, prove that*

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2.$$ **[CBSE 2008C, '13]**

SOLUTION Let the value of the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix} \quad [\text{applying } C_1 \to (C_1 + C_2 + C_3)]$$

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix} \quad [\text{taking } (1+x+x^2) \text{ common from } C_1]$$

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x(1-x) \\ 0 & x(x-1) & (1-x)(1+x) \end{vmatrix}$$

$[\text{applying } R_2 \to (R_2 - R_1) \text{ and } R_3 \to (R_3 - R_1)]$

$$= (1+x+x^2)(1-x)^2 \cdot \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & -x & (1+x) \end{vmatrix}$$

$[\text{taking } (1-x) \text{ common from each of } R_2 \text{ and } R_3]$

$$= (1+x+x^2)(1-x)^2 \cdot 1 \cdot \begin{vmatrix} 1 & x \\ -x & (1+x) \end{vmatrix} \quad [\text{expanded by } C_1]$$

$$= (1+x+x^2)(1-x)^2 \cdot (1+x+x^2)$$

$$= \{(1-x)(1+x+x^2)\}^2 = (1-x^3)^2.$$

Hence, $\Delta = (1-x^3)^2$.

EXAMPLE 12 *Using the properties of determinants, prove that*

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3.$$ **[CBSE 2014]**

SOLUTION Let the value of the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

$$= \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x+y+z & x+y+z & x+y+z \end{vmatrix} \text{ [applying } R_3 \to (R_1+R_2+R_3)]$$

$$= (x+y+z) \cdot \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ 1 & 1 & 1 \end{vmatrix}$$

[taking $(x+y+z)$ common from R_3]

$$= (x+y+z) \cdot \begin{vmatrix} 0 & -(x+y+z) & 2y \\ (x+y+z) & (x+y+z) & z-x-y \\ 0 & 0 & 1 \end{vmatrix}$$

[applying $C_1 \to (C_1 - C_3)$ and $C_2 \to (C_2 - C_3)$]

$$= (x+y+z) \cdot 1 \cdot \begin{vmatrix} 0 & -(x+y+z) \\ (x+y+z) & (x+y+z) \end{vmatrix} \text{ [expanded by } R_3]$$

$$= (x+y+z)\{0+(x+y+z)^2\} = (x+y+z)(x+y+z)^2 = (x+y+z)^3.$$

Hence, $\Delta = (x+y+z)^3$.

EXAMPLE 13 *Using properties of determinants, prove that*

$$\begin{vmatrix} 3a & -a+b & -a+c \\ a-b & 3b & c-b \\ a-c & b-c & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca).$$ **[CBSE 2006, '13]**

SOLUTION Let the value of the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ a-b & 3b & c-b \\ a-c & b-c & 3c \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & c-b \\ a+b+c & b-c & 3c \end{vmatrix} \quad [C_1 \to (C_1+C_2+C_3)]$$

$$= (a+b+c) \cdot \begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & c-b \\ 1 & b-c & 3c \end{vmatrix}$$

[taking $(a+b+c)$ common from C_1]

$$= (a+b+c) \cdot \begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix}$$

[$R_2 \to (R_2 - R_1)$ and $R_3 \to (R_3 - R_1)$]

$$= (a+b+c) \cdot 1 \cdot \begin{vmatrix} 2b+a & a-b \\ a-c & 2c+a \end{vmatrix} \text{ [expanded by } C_1]$$

$$= (a+b+c)[(2b+a)(2c+a) - (a-c)(a-b)]$$

$$= (a+b+c)[(4bc+2ab+2ac+a^2)-(a^2-ab-ac+bc)]$$
$$= 3(a+b+c)(ab+bc+ca).$$

Hence, $\Delta = 3(a+b+c)(ab+bc+ca)$.

EXAMPLE 14 *Prove that*

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(a+b+c)(ab+bc+ca-a^2-b^2-c^2).$$

[CBSE 2004, '09C]

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \quad [R_1 \to (R_1+R_2+R_3)]$$

$$= 2(a+b+c) \cdot \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

[taking $(a+b+c)$ common from R_1]

$$= 2(a+b+c) \cdot \begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix} \quad \begin{bmatrix} C_2 \to (C_2-C_1) \text{ and} \\ C_3 \to (C_3-C_1) \end{bmatrix}$$

$$= 2(a+b+c) \cdot 1 \cdot \begin{vmatrix} b-c & b-a \\ c-a & c-b \end{vmatrix} \quad [\text{expanded by } R_1]$$

$$= 2(a+b+c) \cdot [(b-c)(c-b)-(c-a)(b-a)]$$
$$= 2(a+b+c)(ab+bc+ca-a^2-b^2-c^2).$$

Hence, $\Delta = 2(a+b+c)(ab+bc+ca-a^2-b^2-c^2)$.

EXAMPLE 15 *Using properties of determinants, prove that*

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$$ **[CBSE 2001, '04, '06C, '07]**

SOLUTION Let the value of the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad [R_1 \to (R_1+R_2+R_3)]$$

$$= (a+b+c) \cdot \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

[taking $(a+b+c)$ common from R_1]

$$= (a+b+c) \cdot \begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

[$C_2 \to (C_2 - C_1)$ and $C_3 \to (C_3 - C_1)$]

$$= (a+b+c) \cdot 1 \cdot \begin{vmatrix} -(a+b+c) & 0 \\ 0 & -(a+b+c) \end{vmatrix} \quad \text{[expanded by } R_1\text{]}$$

$$= (a+b+c)(a+b+c)^2 = (a+b+c)^3.$$

Hence, $\Delta = (a+b+c)^3$.

EXAMPLE 16 *Prove that*

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$$

[CBSE 2008, '12C, '14C]

SOLUTION Let the value of the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & -(a+b+c) & 0 \\ c & b+c+2a & b \\ 0 & -(a+b+c) & (a+b+c) \end{vmatrix} \quad \begin{Bmatrix} \text{by } R_1 \to (R_1 - R_2) \\ \text{and } R_3 \to (R_3 - R_2) \end{Bmatrix}$$

$$= (a+b+c)^2 \cdot \begin{vmatrix} 1 & -1 & 0 \\ c & b+c+2a & b \\ 0 & -1 & 1 \end{vmatrix}$$

[taking $(a+b+c)$ common from each one of R_1 and R_3]

$$= (a+b+c)^2 \cdot \begin{vmatrix} 1 & -1 & 0 \\ 0 & b+2c+2a & b \\ 0 & -1 & 1 \end{vmatrix} \quad [R_2 \to R_2 - cR_1]$$

$$= (a+b+c)^2 \cdot 1 \cdot \begin{vmatrix} b+2c+2a & b \\ -1 & 1 \end{vmatrix} \quad \text{[expanded by } C_1\text{]}$$

$$= (a+b+c)^2 \cdot (b+2c+2a+b) = 2(a+b+c)^3.$$

Hence, $\Delta = 2(a+b+c)^3$.

EXAMPLE 17 *Using properties of determinants, prove that*

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca).$$ **[CBSE 2011C, '13C]**

SOLUTION Let the value of the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$$

$$= \begin{vmatrix} a-c & b-c & c \\ a^2-c^2 & b^2-c^2 & c^2 \\ b(c-a) & a(c-b) & ab \end{vmatrix} \quad [C_1 \to (C_1 - C_3), C_2 \to (C_2 - C_3)]$$

$$= (a-c)(b-c) \cdot \begin{vmatrix} 1 & 1 & c \\ a+c & b+c & c^2 \\ -b & -a & ab \end{vmatrix}$$

[taking $(a-c)$ common from C_1 and $(b-c)$ common from C_2]

$$= (a-c)(b-c) \cdot \begin{vmatrix} 1 & 0 & 0 \\ a+c & b-a & -ca \\ -b & b-a & (a+c)b \end{vmatrix}$$

$[C_2 \to (C_2 - C_1), C_3 \to (C_3 - cC_1)]$

$$= (a-c)(b-c) \cdot 1 \cdot \begin{vmatrix} b-a & -ca \\ b-a & (a+c)b \end{vmatrix} \quad \{\text{expanded by } R_1\}$$

$$= (a-c)(b-c) \cdot (b-a) \begin{vmatrix} 1 & -ca \\ 1 & (a+c)b \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(ab+bc+ca).$$

Hence, $\Delta = (a-b)(b-c)(c-a)(ab+bc+ca)$.

EXAMPLE 18 *Using properties of determinants, prove that*

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc).$$ **[CBSE 2006, '09, '12C]**

SOLUTION Let the value of the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix} \quad [C_1 \to (C_1 + C_2 + C_3)]$$

$$= (a+b+c)\cdot\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$

[taking $(a + b + c)$ common from C_1]

$$= (a+b+c)\cdot\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix} \quad [R_3 \to R_3 - 2R_1]$$

$$= (a+b+c)\cdot 1\cdot\begin{vmatrix} b-c & c-a \\ c+a-2b & a+b-2c \end{vmatrix} \quad [\text{expanded by } C_1]$$

$$= (a+b+c)\cdot\begin{vmatrix} b-c & c-a \\ a-b & b-c \end{vmatrix} \quad [R_2 \to R_2 + R_1]$$

$$= (a+b+c)\cdot[(b-c)^2 - (a-b)(c-a)]$$

$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= (a^3+b^3+c^3-3abc).$$

Hence, $\Delta = (a^3 + b^3 + c^3 - 3abc)$.

EXAMPLE 19 *Prove that* $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz.$

SOLUTION Given determinant $= \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$

$$= \begin{vmatrix} 0 & -2x & -2x \\ z & z+x & x \\ y & x & x+y \end{vmatrix} \quad [R_1 \to R_1 - (R_2 + R_3)]$$

$$= (-2x)\cdot\begin{vmatrix} 0 & 1 & 1 \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$$

[taking $(-2x)$ common from R_1]

$$= (-2x)\cdot\begin{vmatrix} 0 & 0 & 1 \\ z & z & x \\ y & -y & x+y \end{vmatrix} \quad [C_2 \to (C_2 - C_3)]$$

$$= (-2x)\cdot 1\cdot\begin{vmatrix} z & z \\ y & -y \end{vmatrix} \quad [\text{expanded by } R_1]$$

$$= (-2x)\cdot 1\cdot(-yz - yz) = (-2x)(-2yz) = 4xyz.$$

EXAMPLE 20 *Prove that* $\begin{vmatrix} a^2 & bc & c^2+ac \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$ **[CBSE 2014]**

SOLUTION $\begin{vmatrix} a^2 & bc & c^2+ac \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

$$= (abc) \cdot \begin{vmatrix} a & c & c+a \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

[taking a, b, c common from C_1, C_2 and C_3 respectively]

$$= (abc) \cdot \begin{vmatrix} a & c & c+a \\ 0 & -2c & -2c \\ b & b+c & c \end{vmatrix} \quad [R_2 \to R_2 - (R_1 + R_3)]$$

$$= (abc) \cdot \begin{vmatrix} a & -a & c+a \\ 0 & 0 & -2c \\ b & b & c \end{vmatrix} \quad [C_2 \to C_2 - C_3]$$

$$= (abc)(2c) \cdot \begin{vmatrix} a & -a \\ b & b \end{vmatrix} \quad \text{[expanded by } R_2\text{]}$$

$$= 2abc^2 \cdot (ab + ab) = 2abc^2(2ab)$$

$$= 4a^2b^2c^2.$$

EXAMPLE 21 *If* a, b, c *are positive and unequal, show that the value of the determinant* $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ *is negative.*

SOLUTION The given determinant

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} \quad [C_1 \to (C_1 + C_2 + C_3)]$$

$$= (a+b+c) \cdot \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

$$= (a+b+c) \cdot \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} [R_2 \to (R_2 - R_1) \text{ and } R_3 \to (R_3 - R_1)]$$

$$= (a+b+c) \cdot [(c-b)(b-c) - (a-b)(a-c)] \qquad [\text{expanded by } C_1]$$

$$= (a+b+c) \cdot (-a^2 - b^2 - c^2 + ab + bc + ca)$$

$$= -\frac{1}{2}(a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$$

$$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2], \text{ which is negative}$$

$$[\because \ (a+b+c) > 0, (a-b)^2 > 0, (b-c)^2 > 0 \text{ and } (c-a)^2 > 0].$$

EXAMPLE 22 *Evaluate* $\begin{vmatrix} {}^mC_1 & {}^mC_2 & {}^mC_3 \\ {}^nC_1 & {}^nC_2 & {}^nC_3 \\ {}^pC_1 & {}^pC_2 & {}^pC_3 \end{vmatrix}$.

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} m & \frac{1}{2}m(m-1) & \frac{1}{6} \cdot m(m-1)(m-2) \\ n & \frac{1}{2}n(n-1) & \frac{1}{6} \cdot n(n-1)(n-2) \\ p & \frac{1}{2}p(p-1) & \frac{1}{6} \cdot p(p-1)(p-2) \end{vmatrix}$$

$$= \left(\frac{1}{2} \times \frac{1}{6} \times mnp\right) \cdot \begin{vmatrix} 1 & (m-1) & (m-1)(m-2) \\ 1 & (n-1) & (n-1)(n-2) \\ 1 & (p-1) & (p-1)(p-2) \end{vmatrix}$$

$$= \frac{1}{12} \cdot mnp \cdot \begin{vmatrix} 1 & m-1 & (m-1)(m-2) \\ 0 & n-m & (n-m)(n+m-3) \\ 0 & p-m & (p-m)(p+m-3) \end{vmatrix}$$

$$[R_2 \to (R_2 - R_1) \text{ and } R_3 \to (R_3 - R_1)]$$

$$= \frac{1}{12} \cdot (mnp)(n-m)(p-m) \cdot \begin{vmatrix} 1 & m-1 & (m-1)(m-2) \\ 0 & 1 & (n+m-3) \\ 0 & 1 & (p+m-3) \end{vmatrix}$$

[taking $(n-m)$ common from R_2 and $(p-m)$ common from R_3]

$$= \frac{1}{12} \cdot (mnp)(n-m)(p-m) \cdot 1 \cdot \begin{vmatrix} 1 & (n+m-3) \\ 1 & (p+m-3) \end{vmatrix}$$

$$= \frac{1}{12} \cdot (mnp)(n-m)(p-m)[(p+m-3) - (n+m-3)]$$

$$= \frac{1}{12} \cdot (mnp)(n-m)(p-m)(p-n)$$

$$= \frac{1}{12} \cdot (mnp)(m-n)(n-p)(p-m).$$

Hence, $\Delta = \frac{1}{12} \cdot (mnp)(m-n)(n-p)(p-m)$.

EXAMPLE 23 *Prove that* $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$.

[CBSE 2010, '10C]

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$[C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 - C_3]$

$$= \begin{vmatrix} (a+b+c)(b+c-a) & 0 & a^2 \\ 0 & (a+b+c)(c+a-b) & b^2 \\ (a+b+c)(c-a-b) & (a+b+c)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \cdot \begin{vmatrix} (b+c-a) & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

[taking $(a+b+c)$ common from C_1 and C_2 both]

$$= (a+b+c)^2 \cdot \begin{vmatrix} (b+c-a) & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \quad [R_3 \to R_3 - (R_1 + R_2)]$$

$$= (a+b+c)^2[(b+c-a)\{(c+a-b) \cdot 2ab + 2ab^2\} + a^2\{0 + 2b(c+a-b)\}]$$

$$= (a+b+c)^2[(b+c-a) \cdot 2ab\{(c+a-b+b)\} + 2a^2b(c+a-b)]$$

$$= 2ab(a+b+c)^2\{(b+c-a)(c+a) + a(c+a-b)\}$$

$$= 2ab(a+b+c)^2 \cdot \{bc + ab + c^2 + ac - ac - a^2 + ac + a^2 - ab\}$$

$$= 2ab(a+b+c)^2\{bc + c^2 + ac\} = 2abc(a+b+c)^3.$$

Hence, $\Delta = 2abc(a+b+c)^3$.

TYPE: EXPRESSING A GIVEN DET. AS SUM OF TWO DETERMINANTS

EXAMPLE 24 *Prove that* $\begin{vmatrix} b+c & a & b \\ c+a & c & a \\ a+b & b & c \end{vmatrix} = (a+b+c)(a-c)^2$. **[CBSE 2005C, '07]**

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} b+c & a & b \\ c+a & c & a \\ a+b & b & c \end{vmatrix}$$

$$= \begin{vmatrix} b & a & b \\ c & c & a \\ a & b & c \end{vmatrix} + \begin{vmatrix} c & a & b \\ a & c & a \\ b & b & c \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c & c & a \\ a & b & c \end{vmatrix} + \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & c & a \\ b & b & c \end{vmatrix}$$

[$R_1 \to (R_1 + R_2 + R_3)$ in each determinant]

$$= (a+b+c) \cdot \begin{vmatrix} 1 & 1 & 1 \\ c & c & a \\ a & b & c \end{vmatrix} + (a+b+c) \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & c & a \\ b & b & c \end{vmatrix}$$

[taking out $(a+b+c)$ common from R_1 in each determinant]

$$= (a+b+c) \cdot \begin{vmatrix} 1 & 0 & 0 \\ c & 0 & a-c \\ a & b-a & c-a \end{vmatrix} + (a+b+c) \cdot \begin{vmatrix} 1 & 0 & 0 \\ a & c-a & 0 \\ b & 0 & c-b \end{vmatrix}$$

[$C_2 \to (C_2 - C_1)$ and $C_3 \to (C_3 - C_1)$ in each]

$$= (a+b+c) \cdot 1 \cdot \begin{vmatrix} 0 & a-c \\ b-a & c-a \end{vmatrix} + (a+b+c) \cdot 1 \cdot \begin{vmatrix} c-a & 0 \\ 0 & c-b \end{vmatrix}$$

[each det. expanded by R_1]

$$= (a+b+c) \cdot [0 - (b-a)(a-c)] + (a+b+c)(c-a)(c-b)$$
$$= (a+b+c)(a-b)(a-c) - (a+b+c)(a-c)(c-b)$$
$$= (a+b+c)(a-c)\{(a-b) - (c-b)\}$$
$$= (a+b+c)(a-c)^2.$$

Hence, $\Delta = (a+b+c)(a-c)^2$.

EXAMPLE 25 *Prove that* $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0.$ **[CBSE 2005C]**

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & -bc \\ 1 & b & -ca \\ 1 & c & -ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} + \begin{vmatrix} 1 & a & -bc \\ 0 & b-a & c(b-a) \\ 0 & c-a & b(c-a) \end{vmatrix}$$

[$R_2 \to (R_2 - R_1)$ and $R_3 \to (R_3 - R_1)$ in each determinant]

$$= (b-a)(c-a) \cdot \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} + (b-a)(c-a) \cdot \begin{vmatrix} 1 & a & -bc \\ 0 & 1 & c \\ 0 & 1 & b \end{vmatrix}$$

$$= (b-a)(c-a) \cdot 1 \cdot \{(c+a)-(b+a)\} + (b-a)(c-a) \cdot 1 \cdot (b-c)$$

$$= (b-a)(c-a)(c-b) + (b-a)(c-a)(b-c)$$

$$= (a-b)(b-c)(c-a) - (a-b)(b-c)(c-a) = 0.$$

Hence, $\Delta = 0$.

EXAMPLE 26 *If* $x \neq y \neq z$ *and* $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ *then prove that* $xyz = -1$.

[CBSE 2008, '09C, '11]

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + (xyz) \cdot \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + (xyz)(-1)^2 \cdot \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

[interchanging the columns of the 2nd det. twice]

$$= (1+xyz) \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

$$= (1+xyz) \begin{vmatrix} x & x^2 & 1 \\ (y-x) & (y^2-x^2) & 0 \\ (z-x) & (z^2-x^2) & 0 \end{vmatrix} [R_2 \to (R_2 - R_1), R_3 \to (R_3 - R_1)]$$

$$= (1+xyz)(y-x)(z-x) \begin{vmatrix} x & x^2 & 1 \\ 1 & y+z & 0 \\ 1 & z+x & 0 \end{vmatrix}$$

$$= (1 + xyz)(y - x)(z - x) \cdot 1 \cdot \begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix} \quad \text{[expanding by } C_3\text{]}$$

$$= (1 + xyz)(y - x)(z - x)(z - y).$$

$\therefore \quad \Delta = 0 \Rightarrow (1 + xyz)(y - x)(z - x)(z - y) = 0$

$\Rightarrow (1 + xyz) = 0 \quad [\because (y - x) \neq 0, (z - x) \neq 0, (z - y) \neq 0]$

$\Rightarrow xyz = -1.$

Hence, $xyz = -1$.

EXAMPLE 27 *For any scalar p, prove that*

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1 + pxyz)(x - y)(y - z)(z - x). \quad \textbf{[CBSE 2010]}$$

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} \quad \text{[say } \Delta_1 + \Delta_2\text{]}$$

$$= (-1) \cdot \begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + p \cdot \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

[by $C_2 \leftrightarrow C_3$ in Δ_1] [taking p common from C_3 in Δ_2]

$$= (-1)(-1) \cdot \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + (pxyz) \cdot \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

[by $C_1 \leftrightarrow C_2$] [taking x, y, z common from R_1, R_2, R_3 resp.]

$$= (1 + pxyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (1 + pxyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$$

[by $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$= (1 + pxyz)(y - x)(z - x) \cdot \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

[taking $(y - x)$ common from R_2 and $(z - x)$ common from R_3]

$$= (1 + pxyz)(y - x)(z - x) \cdot 1 \cdot \begin{vmatrix} 1 & y + x \\ 1 & z + x \end{vmatrix}$$

$$= (1 + pxyz)\ (y - x)\ (z - x) \cdot \{(z + y) - (y + x)\}$$

$$= (1 + pxyz)\ (y - x)(z - x)(z - y)$$

$$= (1 + pxyz)\ (x - y)(y - z)(z - x) = \text{RHS.}$$

$\therefore$ LHS = RHS.

TYPE: SOME MORE PROBLEMS

EXAMPLE 28 *Prove that*

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = (abc)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right) = (bc + ca + ab + abc).$$

[CBSE 2004, '08, '09, '12, '14]

SOLUTION The given determinant

$$= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

$$= (abc) \cdot \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

[taking a, b, c common from R_1, R_2 and R_3 respectively]

$$= (abc)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right) \cdot \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

[applying $R_1 \to R_1 + R_2 + R_3$ and taking out $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right)$ common from R_1]

$$= (abc)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right) \cdot \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & \frac{1}{b} \\ -1 & -1 & \frac{1}{c}+1 \end{vmatrix}$$

[applying $C_1 \to (C_1 - C_3)$ and $C_2 \to (C_2 - C_3)$]

$$= (abc)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right) \cdot (1) \cdot \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} \quad \text{[expanding by 1st row]}$$

$$= (abc)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right) \cdot 1$$

$$= (abc)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right) = (bc + ca + ab + abc).$$

Hence, the result follows.

EXAMPLE 29 *Prove that* $\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix} = 0.$

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$$

$$= \frac{1}{abc} \cdot \begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & cba \\ 1 & c^3 & abc \end{vmatrix}$$ [multiplying R_1, R_2, R_3 by a, b, c respectively and dividing Δ by abc]

$$= \frac{1}{abc} \cdot abc \cdot \begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix}$$

$= (1 \times 0) = 0$ [$\because$ C_1 and C_3 are identical].

Hence, $\Delta = 0$.

EXAMPLE 30 *Prove that*

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = (1 + a^2 + b^2 + c^2).$$

[CBSE 2007, '08, '11C, '14]

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} a\left(a + \frac{1}{a}\right) & ab & ac \\ ab & \left(b + \frac{1}{b}\right)b & bc \\ ca & cb & \left(c + \frac{1}{c}\right)c \end{vmatrix}$$

$$= (abc) \cdot \begin{vmatrix} a + \frac{1}{a} & a & a \\ b & b + \frac{1}{b} & b \\ c & c & c + \frac{1}{c} \end{vmatrix}$$

[taking a, b, c common from C_1, C_2, C_3 respectively]

$$= \frac{(abc)}{(abc)} \cdot \begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

[applying $R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$ and dividing the whole det. by abc]

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

[by $R_1 \to R_1 + R_2 + R_3$]

$$= (1+a^2+b^2+c^2) \cdot \begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

[taking out $(1+a^2+b^2+c^2)$ common from R_1]

$$= (1+a^2+b^2+c^2) \cdot \begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix} \text{[by } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1\text{]}$$

$$= (1+a^2+b^2+c^2) \cdot 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \text{[expanded by } R_1\text{]}$$

$$= (1+a^2+b^2+c^2) \cdot 1 \cdot (1-0) = (1+a^2+b^2+c^2).$$

Hence, $\Delta = (1+a^2+b^2+c^2)$.

EXAMPLE 31 *Prove that* $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ac)^3$. **[CBSE 2007]**

SOLUTION We have

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b+abc & -abc & c^2b+abc \\ a^2c+abc & b^2c+abc & -abc \end{vmatrix}$$

[$R_1 \to aR_1$, $R_2 \to bR_2$ and $R_3 \to cR_3$ and dividing the det. by abc]

$$= \left(\frac{abc}{abc}\right)\begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

[taking out a, b, c common from C_1, C_2, C_3 respectively]

$$= \begin{vmatrix} ab+bc+ac & ab+bc+ac & ab+bc+ac \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix} \quad [R_1 \to (R_1 + R_2 + R_3)]$$

$$= (ab+bc+ac)\cdot\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$$= (ab+bc+ac)\cdot\begin{vmatrix} 0 & 0 & 1 \\ 0 & -(ab+bc+ac) & bc+ab \\ (ab+bc+ac) & (ab+bc+ac) & -ab \end{vmatrix}$$

[$C_1 \to (C_1 - C_3)$ and $C_2 \to (C_2 - C_3)$]

$= (ab+bc+ac)^3$ [expanding by R_1].

EXAMPLE 32 *Prove that*

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (a^2+b^2+c^2)(a-b)(b-c)(c-a)(a+b+c).$$

[CBSE 2012C]

SOLUTION Let the value of the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & -(b-c)^2 & bc \\ b^2 & -(c-a)^2 & ca \\ c^2 & -(a-b)^2 & ab \end{vmatrix} \quad [\text{applying } C_2 \to (C_2 - C_1)]$$

$$= (-1)\cdot\begin{vmatrix} a^2 & (b-c)^2 & bc \\ b^2 & (c-a)^2 & ca \\ c^2 & (a-b)^2 & ab \end{vmatrix}$$

$$= (-1)\cdot\begin{vmatrix} a^2 & b^2+c^2-2bc & bc \\ b^2 & c^2+a^2-2ca & ca \\ c^2 & a^2+b^2-2ab & ab \end{vmatrix}$$

$$= (-1)\cdot\begin{vmatrix} a^2 & a^2+b^2+c^2 & bc \\ b^2 & a^2+b^2+c^2 & ca \\ c^2 & a^2+b^2+c^2 & ab \end{vmatrix} \quad [C_2 \to C_2 + C_1 + 2C_3]$$

$$= (-1)(a^2+b^2+c^2) \cdot \begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$

$$= (-1)(a^2+b^2+c^2) \cdot \begin{vmatrix} a^2 & 1 & bc \\ b^2-a^2 & 0 & ca-bc \\ c^2-a^2 & 0 & ab-bc \end{vmatrix}$$

$$= (-1)(a^2+b^2+c^2) \cdot \begin{vmatrix} a^2 & 1 & bc \\ b^2-a^2 & 0 & (a-b)c \\ c^2-a^2 & 0 & (a-c)b \end{vmatrix}$$

$$= (-1)(a^2+b^2+c^2)(b-a)(c-a) \cdot \begin{vmatrix} a^2 & 1 & bc \\ b+a & 0 & -c \\ c+a & 0 & -b \end{vmatrix}$$

[taking out $(b-a)$ common from R_2 and $(c-a)$ common from R_3]

$$= (-1)(a^2+b^2+c^2)(b-a)(c-a)(-1) \cdot \begin{vmatrix} b+a & -c \\ c+a & -b \end{vmatrix}$$

$$= (a^2+b^2+c^2)(b-a)(c-a) \cdot (-1) \cdot \begin{vmatrix} b+a & c \\ c+a & b \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a-b)(c-a) \cdot \{b(b+a) - c(c+a)\}$$

$$= (a^2+b^2+c^2)(a-b)(c-a) \cdot \{(b^2-c^2) + (ab-ac)\}$$

$$= (a^2+b^2+c^2)(a-b)(c-a) \cdot \{(b^2-c^2) + a(b-c)\}$$

$$= (a^2+b^2+c^2)(a-b)(b-c)(c-a)(a+b+c).$$

Hence, $\Delta = (a^2+b^2+c^2)(a-b)(b-c)(c-a)(a+b+c)$.

TYPE: DETERMINANTS ON TRIGONOMETRY

EXAMPLE 33 *Prove that* $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix} = 1.$

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$$

$$= \frac{1}{\sin\alpha\cos\alpha} \cdot \begin{vmatrix} \sin\alpha\cos\alpha\cos\beta & \sin\alpha\cos\alpha\sin\beta & -\sin^2\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\alpha\cos\beta & \sin\alpha\cos\alpha\sin\beta & \cos^2\alpha \end{vmatrix}$$

[$R_1 \to (\sin\alpha)R_1$, $R_3 \to (\cos\alpha)R_3$ and dividing Δ by $(\sin\alpha\cos\alpha)$]

$$= \frac{1}{\sin\alpha\cos\alpha}\cdot\begin{vmatrix} 0 & 0 & -1 \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\alpha\cos\beta & \sin\alpha\cos\alpha\sin\beta & \cos^2\alpha \end{vmatrix}$$

$[R_1 \to (R_1 - R_3)]$

$$= \frac{1}{\sin\alpha\cos\alpha}\cdot(-1)\cdot\begin{vmatrix} -\sin\beta & \cos\beta \\ \sin\alpha\cos\alpha\cos\beta & \sin\alpha\cos\alpha\sin\beta \end{vmatrix}$$

$$= \frac{(\sin\alpha\cos\alpha)}{(\sin\alpha\cos\alpha)}\cdot(-1)\cdot\begin{vmatrix} -\sin\beta & \cos\beta \\ \cos\beta & \sin\beta \end{vmatrix}$$

[taking $\sin\alpha\cos\alpha$ common from R_2]

$$= (-1)\cdot[-\sin^2\beta - \cos^2\beta] = (\sin^2\beta + \cos^2\beta) = 1.$$

Hence, $\Delta = 1$.

EXAMPLE 34 *If $A + B + C = \pi$, prove that*

$$\begin{vmatrix} \sin(A+B+C) & \sin(A+C) & \cos C \\ -\sin B & 0 & \tan C \\ \cos(A+B) & \tan(B+C) & 0 \end{vmatrix} = 0.$$

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} \sin(A+B+C) & \sin(A+C) & \cos C \\ -\sin B & 0 & \tan C \\ \cos(A+B) & \tan(B+C) & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \sin\pi & \sin(\pi - B) & \cos C \\ -\sin B & 0 & \tan C \\ \cos(\pi - C) & \tan(\pi - A) & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ -\cos C & -\tan A & 0 \end{vmatrix}$$

[$\because$ $\sin\pi = 0$, $\sin(\pi - B) = \sin B$; $\cos(\pi - C) = -\cos C$ and $\tan(\pi - A) = -\tan A$]

$$= \sin B\cdot\begin{vmatrix} \sin B & \cos C \\ -\tan A & 0 \end{vmatrix} - \cos C\cdot\begin{vmatrix} \sin B & \cos C \\ 0 & \tan A \end{vmatrix}$$

{expanded by C_1}

$$= (\sin B\cdot\tan A\cos C) - (\sin B\tan A\cdot\cos C) = 0.$$

Hence, $\Delta = 0$.

EXAMPLE 35 *If $A + B + C = \pi$, show that*

$$\begin{vmatrix} \sin^2 A & \sin A\cos A & \cos^2 A \\ \sin^2 B & \sin B\cos B & \cos^2 B \\ \sin^2 C & \sin C\cos C & \cos^2 C \end{vmatrix} = -\sin(A - B)\sin(B - C)\sin(C - A).$$

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} 1 & (1/2)\sin 2A & (1/2)(1+\cos 2A) \\ 1 & (1/2)\sin 2B & (1/2)(1+\cos 2B) \\ 1 & (1/2)\sin 2C & (1/2)(1+\cos 2C) \end{vmatrix} \quad [C_1 \to C_1 + C_3]$$

$$= \frac{1}{4} \cdot \begin{vmatrix} 1 & \sin 2A & 1+\cos 2A \\ 1 & \sin 2B & 1+\cos 2B \\ 1 & \sin 2C & 1+\cos 2C \end{vmatrix}$$

$$= \frac{1}{4} \cdot \begin{vmatrix} 1 & \sin 2A & 1+\cos 2A \\ 0 & \sin 2B - \sin 2A & \cos 2B - \cos 2A \\ 0 & \sin 2C - \sin 2A & \cos 2C - \cos 2A \end{vmatrix} \quad \begin{Bmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{Bmatrix}$$

$$= \frac{1}{4} \cdot \begin{vmatrix} 1 & \sin 2A & 1+\cos 2A \\ 0 & 2\cos(A+B)\sin(B-A) & 2\sin(A+B)\sin(A-B) \\ 0 & 2\cos(A+C)\sin(C-A) & 2\sin(A+C)\sin(A-C) \end{vmatrix}$$

$$= \sin(A-B)\sin(A-C) \cdot \begin{vmatrix} 1 & \sin 2A & 1+\cos 2A \\ 0 & -\cos(A+B) & \sin(A+B) \\ 0 & -\cos(A+C) & \sin(A+C) \end{vmatrix}$$

[taking $2\sin(A-B)$ common from R_2 and $2\sin(A-C)$ common from R_3]

$$= \sin(A-B)\sin(A-C) \cdot \begin{vmatrix} 1 & \sin 2A & 1+\cos 2A \\ 0 & -\cos(\pi-C) & \sin(\pi-C) \\ 0 & -\cos(\pi-B) & \sin(\pi-B) \end{vmatrix}$$

$[\because \quad A+B+C=\pi]$

$$= \sin(A-B)\sin(A-C) \cdot \begin{vmatrix} 1 & \sin 2A & 1+\cos 2A \\ 0 & \cos C & \sin C \\ 0 & \cos B & \sin B \end{vmatrix}$$

$$= \sin(A-B)\sin(A-C)[\sin B\cos C - \cos B \sin C]$$

$$= \sin(A-B)\sin(A-C)\sin(B-C)$$

$$= -\sin(A-B)\sin(B-C)\sin(C-A).$$

Hence, $\Delta = -\sin(A-B)\sin(B-C)\sin(C-A)$.

EXAMPLE 36 *Prove that* $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0.$

SOLUTION Let the given determinant be Δ.

Then, applying $C_3 \to C_3 + (\sin\delta)C_1 - (\cos\delta)C_2$, we get

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} = 0 \quad [\because \text{ each element in } C_3 \text{ is } 0].$$

TYPE: EQUALITY OF TWO DETERMINANTS

EXAMPLE 37 *Without expanding prove that*

$$\begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}.$$

SOLUTION We have

$$\text{LHS} = \begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$$

$$= \begin{vmatrix} a-ax^2 & c-cx^2 & p-px^2 \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} \quad [\text{applying } R_1 \to R_1 - xR_2]$$

$$= \begin{vmatrix} a(1-x^2) & c(1-x^2) & p(1-x^2) \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$$

$$= (1-x^2)\begin{vmatrix} a & c & p \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$$

[taking out $(1-x^2)$ common from R_1]

$$= (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix} \quad [\text{applying } R_2 \to R_2 - xR_1]$$

= RHS.

Hence, LHS = RHS.

EXAMPLE 38 *Without expanding the determinant, prove that*

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

[CBSE 2001C]

SOLUTION We have

$$\text{LHS} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$

$$= \frac{1}{abc} \cdot \begin{vmatrix} a^2 & a^2 & abc \\ b^2 & b^2 & abc \\ c^2 & c^2 & abc \end{vmatrix}$$

[$R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$ and dividing the whole det. by abc]

$$= \frac{1}{abc} \cdot abc \cdot \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \quad \text{[taking } abc \text{ common from } C_3]$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \quad [C_2 \leftrightarrow C_3 \text{ and } C_1 \leftrightarrow C_2]$$

= RHS.

Hence, LHS = RHS.

EXAMPLE 39 *Using the properties of determinants, prove that*

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

[CBSE 2007, '10C]

SOLUTION We have

$$\text{LHS} = \begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$$

$$= \begin{vmatrix} -2c & b+c & c+a \\ -2a & c+a & a+b \\ -2b & a+b & b+c \end{vmatrix} \quad \text{[applying } C_1 \to C_1 - (C_2 + C_3)]$$

$$= (-2) \begin{vmatrix} c & b+c & c+a \\ a & c+a & a+b \\ b & a+b & b+c \end{vmatrix} \quad \text{[taking out (–2) common from } C_1]$$

$$= (-2) \cdot \begin{vmatrix} c & b & a \\ a & c & b \\ b & a & c \end{vmatrix} \quad \text{[applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$= 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \quad \text{[applying } C_1 \leftrightarrow C_3]$$

= RHS.

Hence, LHS = RHS.

EXAMPLE 40 *Show that*

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}.$$ **[CBSE 2008, '10C, '12C, '14]**

SOLUTION We have

$$\text{LHS} = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix}$$

[applying $C_1 \to (C_1 + C_2 + C_3)$ and taking out 2 common from C_1]

$$= 2\begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix} \quad [C_2 \to (C_2 - C_1),\ C_3 \to (C_3 - C_1)]$$

$$= 2(-1)(-1)\cdot\begin{vmatrix} a+b+c & b & c \\ p+q+r & q & r \\ x+y+z & y & z \end{vmatrix}$$

[taking out (–1) common from each one of C_2 and C_3]

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \text{RHS} \quad [\text{applying } C_1 \to C_1 - (C_2 + C_3)]$$

Hence, LHS = RHS.

EXAMPLE 41 *Show that* $\Delta = \Delta_1$, *where*

$$\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} \text{ and } \Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}.$$ **[CBSE 2014C]**

SOLUTION We have

$$\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} Ax & By & Cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} \quad [\text{interchanging the rows and columns}]$$

$$= (xyz)\begin{vmatrix} A & B & C \\ x & y & z \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \end{vmatrix}$$

[applying $C_1 \to \frac{1}{x}C_1, C_2 \to \frac{1}{y}C_2, C_3 \to \frac{1}{z}C_3$ and multiplying the whole determinant by xyz]

$$= \frac{(xyz)}{(xyz)} \cdot \begin{vmatrix} A & B & C \\ x & y & z \\ yz & zx & xy \end{vmatrix}$$

[applying $R_3 \to (xyz)R_3$ and dividing the whole determinant by xyz]

$$= \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix} = \Delta_1.$$

Hence, $\Delta = \Delta_1$.

EXAMPLE 42 *If* $\Delta_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$ *and* $\Delta_2 = \begin{vmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{vmatrix}$ *then without expanding* Δ_1 *and* Δ_2, *prove that* $\Delta_1 + \Delta_2 = 0$.

SOLUTION We have

$$\Delta_2 = \begin{vmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{vmatrix}$$

$$= (-1) \cdot \begin{vmatrix} q & -b & y \\ p & -a & x \\ r & -c & z \end{vmatrix} \quad \text{[taking (–1) common from } R_2]$$

$$= (-1)(-1) \cdot \begin{vmatrix} q & b & y \\ p & a & x \\ r & c & z \end{vmatrix} \quad \text{[taking (–1) common from } C_2]$$

$$= \begin{vmatrix} q & b & y \\ p & a & x \\ r & c & z \end{vmatrix} = \begin{vmatrix} q & p & r \\ b & a & c \\ y & x & z \end{vmatrix} \quad \text{[interchanging rows and columns]}$$

$$= (-1)\begin{vmatrix} p & q & r \\ a & b & c \\ x & y & z \end{vmatrix} \quad \text{[applying } C_1 \leftrightarrow C_2]$$

$$= (-1)(-1) \cdot \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \quad [\text{applying } R_1 \leftrightarrow R_2]$$

$$= \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

$$= (-1) \cdot \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = -\Delta_1 \quad [\text{applying } R_2 \leftrightarrow R_3].$$

Thus, $\Delta_2 = -\Delta_1$ and hence $\Delta_1 + \Delta_2 = 0$.

TYPE: EQUATIONS OF DETERMINANTS

EXAMPLE 43 *Solve for x:*

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0.$$

[CBSE 2004, '05, '11]

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix}$$

$$= \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} \quad [C_1 \to (C_1 + C_2 + C_3)]$$

$$= (3a-x) \cdot \begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} \quad [\text{taking } (3a-x) \text{ common from } C_1]$$

$$= (3a-x) \cdot \begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} \quad [R_2 \to (R_2 - R_1) \text{ and } R_3 \to (R_3 - R_1)]$$

$$= (3a-x) \cdot 1 \cdot \begin{vmatrix} 2x & 0 \\ 0 & 2x \end{vmatrix} \quad [\text{expanding by } C_1]$$

$$= 4(3a-x)x^2.$$

$$\therefore \quad \Delta = 0 \Leftrightarrow 4(3a-x)x^2 = 0$$

$$\Leftrightarrow x = 0 \quad \text{or} \quad x = 3a.$$

Hence, solution set $= \{0, 3a\}$.

EXAMPLE 44 *Solve* $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0.$ **[CBSE 2011]**

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$$

$$= \begin{vmatrix} x-2 & 1 & 2 \\ x-4 & -1 & -4 \\ x-8 & -11 & -40 \end{vmatrix} \quad [C_2 \to (C_2 - 2C_1),\ C_3 \to (C_3 - 3C_1)]$$

$$= \begin{vmatrix} x-2 & 1 & 2 \\ -2 & -2 & -6 \\ -6 & -12 & -42 \end{vmatrix} \quad [R_2 \to (R_2 - R_1),\ R_3 \to (R_3 - R_1)]$$

$$= (-2)\cdot(-6)\cdot \begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 7 \end{vmatrix}$$

$$= 12 \cdot \begin{vmatrix} x-3 & 1 & 2 \\ 0 & 1 & 3 \\ -1 & 2 & 7 \end{vmatrix} \quad [C_1 \to (C_1 - C_2)]$$

$$= 12\cdot[(x-3)(7-6) - 1\cdot(3-2)]$$
$$= 12\cdot(x-4).$$

$\therefore\ \ \Delta = 0 \iff 12(x-4) = 0 \iff x = 4.$

Hence, solution set = {4}.

EXAMPLE 45 *If* $a + b + c = 0$ *and* $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ *then show that*

$$x = 0 \ \ or \ \ x = \sqrt{(3/2)(a^2 + b^2 + c^2)}.$$

SOLUTION Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{vmatrix} \quad [C_1 \to (C_1 + C_2 + C_3)]$$

$$= (a+b+c-x)\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix}$$

$$= (a+b+c-x)\begin{vmatrix} 1 & c & b \\ 0 & b-x-c & a-b \\ 0 & a-c & c-x-b \end{vmatrix}$$

$[R_2 \to (R_2 - R_1)$ and $R_3 \to (R_3 - R_1)]$

$$= (a+b+c-x)[(b-x-c)(c-x-b)-(a-c)(a-b)]$$
$$= (a+b+c-x)[x^2-(a^2+b^2+c^2-ab-bc-ca)].$$

$$\text{Now, } \Delta = 0 \Leftrightarrow (a+b+c-x)[x^2-(a^2+b^2+c^2-ab-bc-ca)] = 0$$
$$\Leftrightarrow x = a+b+c \text{ or } x = \pm\sqrt{(a^2+b^2+c^2-ab-bc-ca)}$$
$$\Leftrightarrow x = 0 \text{ or } x = \pm\sqrt{(3/2)(a^2+b^2+c^2)}$$
$$[\because a+b+c=0 \Leftrightarrow (a^2+b^2+c^2) = -2(ab+bc+ca)$$
$$\Leftrightarrow (ab+bc+ca) = -(1/2)(a^2+b^2+c^2)].$$

EXERCISE 6A

Very-Short-Answer Questions

1. If A is a 2×2 matrix such that $|A| \neq 0$ and $|A| = 5$, write the value of $|4A|$.
2. If A is a 3×3 matrix such that $|A| \neq 0$ and $|3A| = k|A|$ then write the value of k. **[CBSE 2014]**
3. Let A be a square matrix of order 3, write the value of $|2A|$, where $|A| = 4$. **[CBSE 2012]**
4. If A_{ij} is the cofactor of the element a_{ij} of $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ then write the value of $(a_{32}A_{32})$. **[CBSE 2013]**
5. Evaluate $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$. **[CBSE 2008]**
6. Evaluate $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$. **[CBSE 2008]**
7. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x. **[CBSE 2014]**
8. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, write the value of x. **[CBSE 2014]**
9. If $\begin{vmatrix} 2x & x+3 \\ 2(x+1) & x+1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix}$, find the value of x. **[CBSE 2013C]**
10. If $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$, find the value of $3|A|$. **[CBSE 2011C]**
11. Evaluate $2\begin{vmatrix} 7 & -2 \\ -10 & 5 \end{vmatrix}$. **[CBSE 2009C]**

12. Evaluate $\begin{vmatrix} \sqrt{6} & \sqrt{5} \\ \sqrt{20} & \sqrt{24} \end{vmatrix}$. **[CBSE 2009C]**

13. Evaluate $\begin{vmatrix} 2\cos\theta & -2\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$. **[CBSE 2008]**

14. Evaluate $\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$.

15. Evaluate $\begin{vmatrix} \sin 60^\circ & \cos 60^\circ \\ -\sin 30^\circ & \cos 30^\circ \end{vmatrix}$.

16. Evaluate $\begin{vmatrix} \cos 65^\circ & \sin 65^\circ \\ \sin 25^\circ & \cos 25^\circ \end{vmatrix}$.

17. Evaluate $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$. **[CBSE 2011]**

18. Evaluate $\begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$.

19. Without expanding the determinant, prove that $\begin{vmatrix} 41 & 1 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix} = 0$.

SINGULAR MATRIX *A square matrix A is said to be singular if* $|A| = 0$.

*Also, A is called **nonsingular** if* $|A| \neq 0$.

20. For what value of x, the given matrix $A = \begin{bmatrix} 3-2x & x+1 \\ 2 & 4 \end{bmatrix}$ is a singular matrix? **[CBSE 2013C]**

21. Evaluate $\begin{vmatrix} 14 & 9 \\ -8 & -7 \end{vmatrix}$.

22. Evaluate $\begin{vmatrix} \sqrt{3} & \sqrt{5} \\ -\sqrt{5} & 3\sqrt{3} \end{vmatrix}$.

ANSWERS (EXERCISE 6A)

1. 80 **2.** $k = 27$ **3.** 32 **4.** 110 **5.** $x^3 - x^2 + 2$ **6.** $(a^2 + b^2 + c^2 + d^2)$
7. $x = -2$ **8.** $x = \pm 6$ **9.** $x = 1$ **10.** 6 **11.** 30 **12.** 2 **13.** 2 **14.** 1
15. 1 **16.** 0 **17.** 0 **18.** 8 **20.** $x = 1$ **21.** –26 **22.** 14

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 6A)

1. Since A is a 2×2 matrix, so $|4A| = (4 \times 4) \cdot |A|$.
2. Since A is a 3×3 matrix, so $|3A| = (3 \times 3 \times 3) \cdot |A|$.
4. $M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = (8 - 30) = -22.$

   So, $A_{32} = (-1)^{3+2} \cdot M_{32} = (-1)(-22) = 22.$

   $\therefore \ (a_{32}A_{32}) = (5 \times 22) = 110.$
5. $\Delta = (x^2 - x + 1)(x + 1) - (x + 1)(x - 1) = (x^3 + 1) - (x^2 - 1) = (x^3 - x^2 + 2).$
6. $\Delta = (a + ib)(a - ib) - (c + id)(-c + id)$
   $= \{a^2 - (ib)^2\} - \{(id)^2 - c^2\} = (a^2 + b^2) - (-d^2 - c^2) = (a^2 + b^2 + c^2 + d^2).$
7. $(3x \times 4) - 7 \times (-2) = 32 - 42 \Rightarrow 12x + 14 = -10 \Rightarrow 12x = -24 \Rightarrow x = -2.$
8. $2x^2 - 40 = 18 + 14 \Rightarrow 2x^2 = 32 + 40 = 72 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6.$
9. $2x(x + 1) - 2(x + 1)(x + 3) = 3 - 15$
   $\Rightarrow 2(x + 1)\{x - (x + 3)\} = -12 \Rightarrow -6(x + 1) = -12 \Rightarrow x + 1 = 2 \Rightarrow x = 1.$
10. $3|A| = 3 \cdot \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3 \cdot (6 - 4) = 3 \times 2 = 6.$
11. $2\Delta = 2(35 - 20) = (2 \times 15) = 30.$
12. $\Delta = (\sqrt{6} \times \sqrt{24}) - (\sqrt{20} \times \sqrt{5}) = \{\sqrt{6 \times 24} - \sqrt{20 \times 5}\}$
    $= \{\sqrt{144} - \sqrt{100}\} = (12 - 10) = 2.$
13. $\Delta = (2\cos^2\theta + 2\sin^2\theta) = 2(\cos^2\theta + \sin^2\theta) = (2 \times 1) = 2.$
14. $\Delta = (\cos^2\alpha + \sin^2\alpha) = 1.$
15. $\Delta = (\sin 60° \cos 30° + \cos 60° \sin 30°) = \sin(60° + 30°) = \sin 90° = 1.$
16. $\Delta = \{\cos 65° \cos 25° - \sin 65° \sin 25°\} = \cos(65° + 25°) = \cos 90° = 0.$
18. $\Delta = (-2) \cdot \begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix} = (-2)(12 - 16) = (-2) \times (-4) = 8.$
19. Applying $C_1 \to C_1 - 8C_3$, we get

    $$\Delta = \begin{vmatrix} 1 & 1 & 5 \\ 7 & 7 & 9 \\ 5 & 5 & 3 \end{vmatrix} = 0 \quad [\because \ C_1 \text{ and } C_2 \text{ are identical}].$$
20. A is singular $\Leftrightarrow |A| = 0$

    $$\Leftrightarrow \begin{vmatrix} 3 - 2x & x + 1 \\ 2 & 4 \end{vmatrix} = 0$$

    $$\Leftrightarrow 4(3 - 2x) - 2(x + 1) = 0 \Leftrightarrow 10x = 10 \Leftrightarrow x = 1.$$

EXERCISE 6B

Evaluate:

1. $\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$

2. $\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 63 & 54 & 46 \end{vmatrix}$

3. $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

4. $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$

Using properties of determinants prove that:

5. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a).$ **[CBSE 2006]**

6. $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a).$

7. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1.$ **[CBSE 2009]**

8. $\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z).$ **[CBSE 2014]**

9. $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x+2a)(x-a)^2.$

10. $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(x-4)^2.$ **[CBSE 2009, '11]**

11. $\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda-x)^2.$ **[CBSE 2014]**

12. $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3.$

13. $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$ **[CBSE 2013]**

14. $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix} = 3(x+y+z)(xy+yz+zx).$ **[CBSE 2013]**

15. $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x).$ **[CBSE 2010C, '11]**

16. $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3.$ **[CBSE 2006, '12C]**

17. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc.$ **[CBSE 2006, '12, '14C]**

18. $\begin{vmatrix} a & a+2b & a+2b+3c \\ 3a & 4a+6b & 5a+7b+9c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix} = -a^3.$

19. $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a).$

20. $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2-ac)(ax^2+2bxy+cy^2).$

21. $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = -4(a-b)(b-c)(c-a).$

22. $\begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix} = -8.$

23. $\begin{vmatrix} (m+n)^2 & l^2 & mn \\ (n+l)^2 & m^2 & ln \\ (l+m)^2 & n^2 & lm \end{vmatrix} = (l^2+m^2+n^2)(l-m)(m-n)(n-l).$

24. $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a^2+b^2+c^2)(a-b)(b-c)(c-a)(a+b+c).$

25. $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$

26. $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3.$ **[CBSE 2008, '09, '10C]**

27. $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2).$

28. $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0.$

29. $\begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$ **[CBSE 2010]**

30. $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix} = 0.$

31. $\begin{vmatrix} -a(b^2+c^2-a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2+a^2-b^2) & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2+b^2+c^2) \end{vmatrix} = (abc)(a^2+b^2+c^2)^3.$

32. $\begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix} = 0$, where α, β, γ are in AP. **[CBSE 2007]**

33. $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2.$

34. If $x \neq y \neq z$ and $\begin{vmatrix} x & x^3 & x^4-1 \\ y & y^3 & y^4-1 \\ z & z^3 & z^4-1 \end{vmatrix} = 0$, prove that

$xyz(xy+yz+zx) = (x+y+z).$

35. Prove that

$$\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = -(a-b)(b-c)(c-a)(a^2+b^2+c^2).$$

Without expanding the determinant, prove that:

36. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

37. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix}$

38. Show that $x = 2$ is a root of the equation $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & 2+x \end{vmatrix} = 0.$ **[CBSE 2007]**

Solve the following equations:

39. $\begin{vmatrix} 1 & x & x^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = 0$

40. $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ b & b & x+c \end{vmatrix} = 0$

41. $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$

42. $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$

43. $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

44. $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$

45. Prove that

$$\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2).$$ **[CBSE 2012C]**

ANSWERS (EXERCISE 6B)

1. –43 **2.** 132 **3.** 0 **4.** –8 **39.** $x = b$ or $x = c$ or $x = -(b+c)$

40. $x = 0$ or $x = -(a+b+c)$ **41.** $x = \frac{2}{3}$ or $x = \frac{11}{3}$ **42.** $x = 1$ or $x = -9$

43. $x = -9$ or $x = 2$ or $x = 7$ **44.** $x = 1$ or $x = 2$ or $x = -3$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 6B)

1. Apply $C_1 \to C_1 - 3C_2, C_3 \to C_3 - C_2$.
   Now, apply $C_2 \to C_2 - 13C_3$. Expand by R_2.
2. Apply $C_1 \to C_1 - C_2$ and $C_3 \to C_3 - C_2$.
   Take 3 common from C_1 and –4 common from C_3.
   Apply $C_2 \to C_2 - 26C_3$ and $C_3 \to C_3 - C_1$.
3. Apply $R_1 \to R_1 - 6R_3$.
4. Apply $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$. Now, apply $C_3 \to C_3 - C_2$.
5. Apply $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$.

6. Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 3R_1$.
7. Apply $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$.
8. Apply $C_1 \to C_1 + C_2 + C_3$ and take $(a + x + y + z)$ common from C_1.
9. Apply $C_1 \to C_1 + C_2 + C_3$ and take $(x + 2a)$ common from C_1.
10. Apply $R_1 \to R_1 + R_2 + R_3$ and take $(5x + 4)$ common from R_1.
11. Apply $R_1 \to R_1 + R_2 + R_3$ and take $(5x + \lambda)$ common from R_1.
12. Apply $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$ and then expand along C_3.
13. Apply $R_1 \to R_1 + R_2 + R_3$ and take $3(x + y)$ common from R_1.
14. Apply $C_1 \to C_1 + C_2 + C_3$ and take $(x + y + z)$ common from C_1.
15. Take x, y and z common from C_1, C_2, C_3 respectively.
16. Apply $C_1 \to C_1 + C_3$ and $C_2 \to C_2 - C_3$.
17. Apply $R_1 \to R_1 - (R_2 + R_3)$. Take (-2) common from R_1.
    Now, apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$.
18. Apply $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 6R_1$.
19. Apply $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$.
20. Apply $R_3 \to (R_3 - xR_1 - yR_2)$.
21. Apply $R_2 \to (R_2 - R_3)$ and take 4 common from R_2.
    Now, apply $R_3 \to R_3 - R_1 + 2R_2$.
22. Apply $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$. Then, apply $R_2 \to R_2 + 2R_1$.
    Now, $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$.
23. Apply $C_1 \to (C_1 + C_2 - 2C_3)$ and take $(l^2 + m^2 + n^2)$ common from C_1.
24. Apply $C_1 \to (C_1 + C_2 - 2C_3)$.
25. Apply $R_1 \to R_1 - (R_2 + R_3)$ and expand by C_1.
26. Apply $C_1 \to C_1 - bC_3$ and $C_2 \to C_2 + aC_3$.
    Now, take $(1 + a^2 + b^2)$ common from each of C_1 and C_2.
27. Apply $C_1 \to aC_1, C_2 \to bC_2, C_3 \to cC_3$ and divide the whole determinant by abc.
    Now, apply $C_1 \to (C_1 + C_2 + C_3)$.
28. Taking bc, ca and ab common from R_1, R_2 and R_3 respectively, we get

$$\Delta = (a^2b^2c^2) \cdot \begin{vmatrix} bc & 1 & \frac{1}{b} + \frac{1}{c} \\ ca & 1 & \frac{1}{c} + \frac{1}{a} \\ ab & 1 & \frac{1}{a} + \frac{1}{b} \end{vmatrix}.$$

    Now, $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$.
29. Apply $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$ and divide Δ by abc.
    Take a, b, c common from C_1, C_2 and C_3 respectively.
30. $$\Delta = \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$$

$$= (b-a)^2 \cdot \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$ [taking $(b-a)$ common from each of C_1 and C_3].

    Now, apply $C_2 \to (C_2 - C_1 + C_3)$.

31. Take a, b, c common from C_1, C_2 and C_3 respectively.
Now, apply $C_1 \to (C_1 + C_2 + C_3)$.

32. Apply $R_1 \to (R_1 + R_2 + R_3)$ and use $\alpha + \gamma = 2\beta$.
Then, R_1 and R_2 are proportional and hence $\Delta = 0$.

33. $\Delta = \begin{vmatrix} a^2 + 3a + 2 & a + 2 & 1 \\ a^2 + 5a + 6 & a + 3 & 1 \\ a^2 + 7a + 12 & a + 4 & 1 \end{vmatrix}$.

Now, apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$.

35. $\Delta = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} 1 & bc & a^3 \\ 1 & ca & b^3 \\ 1 & ab & c^3 \end{vmatrix}$.

36. Applying $R_1 \to aR_1, R_2 \to bR_2$ and $R_3 \to cR_3$, we get

$$\text{LHS} = \frac{1}{abc} \cdot \begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

37. Applying $C_3 \to C_3 - (a + b + c)C_1$, we get

$$\text{RHS} = \begin{vmatrix} 1 & bc & -a \\ 1 & ca & -b \\ 1 & ab & -c \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$= \frac{1}{abc} \cdot \begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \text{LHS}.$$

38. Apply $R_1 \to (R_1 - R_2)$ and take $(x - 2)$ common from R_1.

39. Apply $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$.
Take $(x - b)$ common from R_1.

40. Apply $C_1 \to C_1 + C_2 + C_3$ and take $(x + a + b + c)$ common from C_1.

41. Apply $C_1 \to C_1 + C_2 + C_3$ and take $(3x - 2)$ common from C_1.

42. Apply $C_1 \to C_1 + C_2 + C_3$ and take $(x + 9)$ common from C_1.

43. Apply $R_1 \to R_1 + R_2 + R_3$ and take $(x + 9)$ common from R_1.

44. Apply $R_1 \to R_1 - R_2$ and take $(x - 2)$ common from R_1.

45. Apply $C_1 \to aC_1$ and divide Δ by a.

$$\therefore \quad \Delta = \frac{1}{a} \cdot \begin{vmatrix} a^2 & b - c & c + b \\ a^2 + ac & b & c - a \\ a^2 - ab & b + a & c \end{vmatrix}$$

$$= \frac{1}{a} \begin{vmatrix} a^2 + b^2 + c^2 & b - c & c + b \\ a^2 + b^2 + c^2 & b & c - a \\ a^2 + b^2 + c^2 & b + a & c \end{vmatrix} \quad [\text{applying } C_1 \to C_1 + bC_2 + cC_3]$$

Applications of Determinants

AREA OF A TRIANGLE IN DETERMINANT FORM We know that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\triangle = \frac{1}{2}[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{(in determinant form).}$$

REMARK 1. Since area is a positive quantity, we always take the absolute value of the above determinant for the area.

REMARK 2. If three points A, B, C are collinear then $\text{ar}(\triangle ABC) = 0$.

CONDITION FOR COLLINEARITY OF THREE POINTS

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be three given points.

Then, A, B, C are collinear

$$\Leftrightarrow \text{ar}(\triangle ABC) = 0$$

$$\Leftrightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$$

$\therefore$ A, B, C are collinear $\Leftrightarrow x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$

SOLVED EXAMPLES

EXAMPLE 1 *Find the area of a triangle whose vertices are $A(-2, -3)$, $B(3, 2)$ and $C(-1, -8)$.*

SOLUTION Here, $(x_1 = -2, y_1 = -3)$, $(x_2 = 3, y_2 = 2)$ and $(x_3 = -1, y_3 = -8)$.

$$\therefore \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\cdot\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\cdot\begin{vmatrix} -2 & -3 & 1 \\ 5 & 5 & 0 \\ 1 & -5 & 0 \end{vmatrix}$$

$[R_2 \to (R_2 - R_1)$ and $R_3 \to (R_3 - R_1)]$

$$= \frac{1}{2}\cdot 1\cdot\begin{vmatrix} 5 & 5 \\ 1 & -5 \end{vmatrix} = \frac{1}{2}\times(-25 - 5) = -15.$$

Hence, $\text{ar}(\triangle ABC) = |-15| = 15$ square units.

EXAMPLE 2 *Show that the points* $A(a, b+c)$, $B(b, c+a)$ *and* $C(c, a+b)$ *are collinear.*

SOLUTION Points A, B, C are collinear $\Leftrightarrow$ ar$(\triangle ABC) = 0$.

$$\text{Now, ar}(\triangle ABC) = \frac{1}{2}\cdot\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2}\cdot\begin{vmatrix} a & a+b+c & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix} \quad [C_2 \to (C_2 + C_1)]$$

$$= \frac{1}{2}(a+b+c)\cdot\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}(a+b+c) \times 0 = 0 \quad [\because C_2 \text{ and } C_3 \text{ are identical}].$$

Hence, the given points are collinear.

EXAMPLE 3 *If the points* (a, b), (a', b') *and* $(a-a', b-b')$ *are collinear, show that* $ab' = a'b$.

SOLUTION The given points are collinear

$$\Leftrightarrow \begin{vmatrix} a & b & 1 \\ a' & b' & 1 \\ a-a' & b-b' & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} a & b & 1 \\ a'-a & b'-b & 0 \\ -a' & -b' & 0 \end{vmatrix} = 0 \quad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Leftrightarrow 1\cdot[(a'-a)(-b') + a'(b'-b)] = 0 \quad [\text{expanding by } C_3]$$

$$\Leftrightarrow ab' - a'b = 0$$

$$\Leftrightarrow ab' = a'b.$$

EXAMPLE 4 *Find the value of k in order that the points* $(5, 5)$, $(k, 1)$ *and* $(10, 7)$ *are collinear.*

SOLUTION The given points are collinear

$$\Leftrightarrow \begin{vmatrix} 5 & 5 & 1 \\ k & 1 & 1 \\ 10 & 7 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} 5 & 5 & 1 \\ k-5 & -4 & 0 \\ 5 & 2 & 0 \end{vmatrix} = 0 \quad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Leftrightarrow\ 1\cdot[2(k-5)+20]=0$$
$$\Leftrightarrow\ 2k+10=0\ \Leftrightarrow\ k=-5.$$

Hence, $k=-5$.

EXAMPLE 5 *Let $A(1, 3)$, $B(0, 0)$ and $C(k, 0)$ be three points such that $ar(\triangle ABC)=3$ sq units. Find the value of k.*

SOLUTION We have

$$\text{ar}(\triangle ABC)=3 \text{ sq units}$$
$$\Leftrightarrow\ \frac{1}{2}\cdot\begin{vmatrix}1&3&1\\0&0&1\\k&0&1\end{vmatrix}=\pm3\ \Leftrightarrow\ \begin{vmatrix}1&3&1\\0&0&1\\k&0&1\end{vmatrix}=\pm6$$
$$\Leftrightarrow\ (-1)\cdot\begin{vmatrix}1&3\\k&0\end{vmatrix}=\pm6\ \Leftrightarrow\ 3k=\pm6\ \Leftrightarrow\ k=\pm2.$$

Hence, $k=\pm2$.

EXAMPLE 6 *Find the equation of the line joining the points $A(1, 2)$ and $B(3, 6)$, using determinants.*

SOLUTION Let $P(x, y)$ be a point on AB.

Then, the points A, P and B are collinear.

$\therefore\ \text{ar}(\triangle APB)=0$

$$\Rightarrow\ \frac{1}{2}\cdot\begin{vmatrix}1&2&1\\x&y&1\\3&6&1\end{vmatrix}=0\ \Rightarrow\ \begin{vmatrix}1&2&1\\x&y&1\\3&6&1\end{vmatrix}=0$$
$$\Rightarrow\ \begin{vmatrix}1&2&1\\x&y&1\\0&0&-2\end{vmatrix}=0\quad[R_3\to R_3-3R_1]$$
$$\Rightarrow\ (-2)\cdot\begin{vmatrix}1&2\\x&y\end{vmatrix}=0\ \Rightarrow\ (y-2x)=0\ \Rightarrow\ y=2x.$$

Hence, the required equation is $y=2x$.

EXERCISE 6C

1. Find the area of the triangle whose vertices are:
   (i) $A(3, 8)$, $B(-4, 2)$ and $C(5, -1)$
   (ii) $A(-2, 4)$, $B(2, -6)$ and $C(5, 4)$
   (iii) $A(-8, -2)$, $B(-4, -6)$ and $C(-1, 5)$
   (iv) $P(0, 0)$, $Q(6, 0)$ and $R(4, 3)$
   (v) $P(1, 1)$, $Q(2, 7)$ and $R(10, 8)$ **[CBSE 2007]**
2. Use determinants to show that the following points are collinear.
   (i) $A(2, 3)$, $B(-1, -2)$ and $C(5, 8)$

(ii) $A(3, 8)$, $B(-4, 2)$ and $C(10, 14)$

(iii) $P(-2, 5)$, $Q(-6, -7)$ and $R(-5, -4)$

3. Find the value of k for which the points $A(3, -2)$, $B(k, 2)$ and $C(8, 8)$ are collinear.
4. Find the value of k for which the points $P(5, 5)$, $Q(k, 1)$ and $R(11, 7)$ are collinear.
5. Find the value of k for which the points $A(1, -1)$, $B(2, k)$ and $C(4, 5)$ are collinear.
6. Find the value of k for which the area of aABC having vertices $A(2, -6)$, $B(5, 4)$ and $C(k, 4)$ is 35 sq units.
7. If $A(-2, 0)$, $B(0, 4)$ and $C(0, k)$ be three points such that area of a ABC is 4 sq units, find the value of k.
8. If the points $A(a, 0)$, $B(0, b)$ and $C(1, 1)$ are collinear, prove that $\frac{1}{a} + \frac{1}{b} = 1$.

ANSWERS (EXERCISE 6C)

1. (i) 37.5 sq units (ii) 35 sq units (iii) 28 sq units (iv) 9 sq units (v) 23.5 sq units

3. $k = 5$ **4.** $k = -7$ **5.** $k = 1$ **6.** $k = -2$ or $k = 12$ **7.** $k = 0$ or $k = 8$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. $\begin{vmatrix} \cos 70° & \sin 20° \\ \sin 70° & \cos 20° \end{vmatrix} = ?$

(a) 1 (b) 0 (c) $\cos 50°$ (d) $\sin 50°$

2. $\begin{vmatrix} \cos 15° & \sin 15° \\ \sin 15° & \cos 15° \end{vmatrix} = ?$

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) none of these

3. $\begin{vmatrix} \sin 23° & -\sin 7° \\ \cos 23° & \cos 7° \end{vmatrix} = ?$

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $\sin 16°$ (d) $\cos 16°$

4. $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} = ?$

(a) $(a^2 + b^2 - c^2 - d^2)$ (b) $(a^2 - b^2 + c^2 - d^2)$

(c) $(a^2 + b^2 + c^2 + d^2)$ (d) none of these

5. If ω is a complex root of unity then $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = ?$

(a) 1 (b) –1 (c) 0 (d) none of these

6. If ω is a complex cube root of unity then the value of $\begin{vmatrix} 1 & \omega & 1+\omega \\ 1+\omega & 1 & \omega \\ \omega & 1+\omega & 1 \end{vmatrix}$ is

(a) 2 (b) 4 (c) 0 (d) –3

7. $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix} = ?$

(a) 8 (b) –8 (c) 16 (d) 142

8. $\begin{vmatrix} 1! & 2! & 3! \\ 2! & 3! & 4! \\ 3! & 4! & 5! \end{vmatrix} = ?$

(a) 2 (b) 6 (c) 24 (d) 120

9. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = ?$ **[CBSE 2009]**

(a) $(a+b+c)$ (b) $3(a+b+c)$ (c) $3abc$ (d) 0

10. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = ?$ **[CBSE 2009]**

(a) 0 (b) 1 (c) –1 (d) none of these

11. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = ?$ **[CBSE 2009]**

(a) $(a-b)(b-c)(c-a)$ (b) $-(a-b)(b-c)(c-a)$

(c) $(a-b)(b-c)(c-a)(a+b+c)$ (d) $abc(a-b)(b-c)(c-a)$

12. $\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix} = ?$

(a) 0

(b) 1

(c) $\sin(\alpha+\delta) + \sin(\beta+\delta) + \sin(\gamma+\delta)$

(d) none of these

13. If a, b, c be distinct positive real numbers then the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is

(a) positive (b) negative
(c) a perfect square (d) 0

14. $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = ?$

(a) 0 (b) x^3 (c) y^3 (d) none of these

15. $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = ?$

(a) $(a-1)$ (b) $(a-1)^2$ (c) $(a-1)^3$ (d) none of these

16. $\begin{vmatrix} a & a+2b & a+2b+3c \\ 3a & 4a+6b & 5a+7b+9c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix} = ?$

(a) a^3 (b) $-a^3$ (c) 0 (d) none of these

17. $\begin{vmatrix} b+c & a & b \\ c+a & c & a \\ a+b & b & c \end{vmatrix} = ?$

(a) $(a+b+c)(a-c)$ (b) $(a+b+c)(b-c)$
(c) $(a+b+c)(a-c)^2$ (d) $(a+b+c)(b-c)^2$

18. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = ?$

(a) $(x+y)$ (b) $(x-y)$ (c) xy (d) none of these

19. $\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix} = ?$

(a) $(a-b)(b-c)(c-a)$ (b) $-(a-b)(b-c)(c-a)$
(c) $(a+b)(b+c)(c+a)$ (d) none of these

20. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = ?$

(a) $4abc$ (b) $2(a+b+c)$ (c) $(ab+bc+ca)$ (d) none of these

21. $\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix} = ?$

(a) $a+b+c$ (b) $2(a+b+c)$ (c) $4abc$ (d) $a^2b^2c^2$

22. $\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix} = ?$

(a) –2 (b) 2 (c) x^2-2 (d) x^2+2

23. If $\begin{vmatrix} 5 & 3 & -1 \\ -7 & x & 2 \\ 9 & 6 & -2 \end{vmatrix} = 0$ then $x = ?$

(a) 0 (b) 6 (c) –6 (d) 9

24. The solution set of the equation $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ is

(a) {2, –3, 7} (b) {2, 7, –9} (c) [–2, 3, –7] (d) none of these

25. The solution set of the equation $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$ is

(a) {4} (b) {2, 4} (c) {2, 8} (d) {4, 8}

26. The solution set of the equation $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$ is

(a) $\{a, 0\}$ (b) $\{3a, 0\}$ (c) $\{a, 3a\}$ (d) none of these

27. The solution set of the equation $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$ is

(a) $\left\{\frac{2}{3}, \frac{8}{3}\right\}$ (b) $\left\{\frac{2}{3}, \frac{11}{3}\right\}$ (c) $\left\{\frac{3}{2}, \frac{8}{3}\right\}$ (d) none of these

28. The vertices of a a ABC are $A(-2, 4)$, $B(2, -6)$ and $C(5, 4)$. The area of a ABC is

(a) 17.5 sq units (b) 35 sq units (c) 32 sq units (d) 28 sq units

29. If the points $A(3, -2)$, $B(k, 2)$ and $C(8, 8)$ are collinear then the value of k is

(a) 2 (b) –3 (c) 5 (d) –4

ANSWERS (OBJECTIVE QUESTIONS)

1. (b)	**2.** (c)	**3.** (b)	**4.** (c)	**5.** (c)	**6.** (b)	**7.** (b)	**8.** (c)	**9.** (d)
10. (b)	**11.** (c)	**12.** (a)	**13.** (b)	**14.** (b)	**15.** (c)	**16.** (b)	**17.** (c)	**18.** (c)
19. (a)	**20.** (a)	**21.** (c)	**22.** (a)	**23.** (c)	**24.** (b)	**25.** (a)	**26.** (b)	**27.** (b)
28. (b)	**29.** (c)							

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

2. $\Delta = (\cos^2 15° - \sin^2 15°) = \cos(2 \times 15°) = \cos 30° = \frac{\sqrt{3}}{2}\cdot$

4. $\Delta = (a + ib)(a - ib) + (c - id)(c + id) = (a^2 + b^2 + c^2 + d^2)$.

5. Apply $R_1 \to (R_1 + R_2 + R_3)$ and use the result $(1 + \omega + \omega^2) = 0$.

6. $1 + \omega + \omega^2 = 0 \Rightarrow (1 + \omega) = -\omega^2$. Put $(1 + \omega) = -\omega^2$ and expand.

9. $R_1 \to (R_1 + R_2 + R_3)$ gives all zeros in R_1 and hence $\Delta = 0$.

10. Applying $R_2 \to (R_2 - 2R_1)$ and $R_3 \to (R_3 - 3R_1)$ and expanding by C_1, we get $\Delta = 1$.

12. $C_3 \to C_3 + (\sin\delta)C_1 - (\cos\delta)C_2$ makes all zeros in C_3 and hence $\Delta = 0$.

13. $C_1 \to (C_1 + C_2 + C_3)$ and take $(a + b + c)$ common from C_1.
Simplify to get
$$\Delta = -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= -\frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}, \text{ which is negative.}$$

14. Take x common from each of C_2 and C_3. Apply $R_3 \to R_3 - 2R_2$.

15. Apply $R_1 \to (R_1 - R_2)$ and $R_3 \to (R_3 - R_2)$. Expand by C_3.

16. Apply $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 6R_1$.

17. $\Delta = \begin{vmatrix} b & a & b \\ c & c & a \\ a & b & c \end{vmatrix} + \begin{vmatrix} c & a & b \\ a & c & a \\ b & b & c \end{vmatrix}$

Apply $R_1 \to R_1 + R_2 + R_3$ in each and simplify.

19. $\Delta = \begin{vmatrix} bc & b & 1 \\ ca & c & 1 \\ ab & a & 1 \end{vmatrix} + \begin{vmatrix} bc & c & 1 \\ ca & a & 1 \\ ab & b & 1 \end{vmatrix}\cdot$

20. Apply $R_1 \to R_1 - (R_2 + R_3)$.
Take (-2) common from R_1. Apply $R_2 \to (R_2 - R_1)$ and $R_3 \to (R_3 - R_1)$.

21. $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$. Take $(a - b)$ common from R_2 and $(c - a)$ common from R_3.

22. Apply $C_2 \to (C_2 - C_1)$ and $C_3 \to (C_3 - C_1)$.
Now, apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$.

23. Apply $C_1 \to (C_1 - C_3)$ and take $(x - 7)$ common from C_1.

25. Apply $C_2 \to (C_2 - 2C_1)$ and $C_3 \to (C_3 - 3C_1)$.
Now, apply $R_2 \to (R_2 - R_1)$ and $R_3 \to (R_3 - R_1)$.

26. Apply $C_1 \to (C_1 + C_2 + C_3)$ and take $(3a - x)$ common from C_1.

27. Apply $C_1 \to (C_1 + C_2 + C_3)$ and take $(3x - 2)$ common from C_1.

28. $\Delta = \frac{1}{2}\begin{vmatrix} -2 & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -2 & 4 & 1 \\ 4 & -10 & 0 \\ 7 & 0 & 0 \end{vmatrix} = 35$ sq units.

29. If $\Delta = \begin{vmatrix} 3 & -2 & 1 \\ k & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix}$ then we must have $\Delta = 0$.

7. ADJOINT AND INVERSE OF A MATRIX

ADJOINT OF A MATRIX *Let $A = [a_{ij}]$ be a square matrix of order n and let A_{ij} denote the cofactor of a_{ij} in $|A|$. Then, the adjoint of A, denoted by* adj A, *is defined as*

$$\text{adj } A = [A_{ji}]_{n \times n}.$$

Thus adj A is the transpose of the matrix of the corresponding cofactors of elements of $|A|$.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix},$$

where A_{ij} denotes the cofactor of a_{ij} in $|A|$.

EXAMPLE 1 *If* $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, *find* adj A.

SOLUTION Clearly, $|A| = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}$.

The cofactors of the elements of $|A|$ are given by

$$A_{11} = 3, \ A_{12} = -1;$$

$$A_{21} = -5, \ A_{22} = 2.$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

EXAMPLE 2 *If* $A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & 1 \\ -4 & 5 & 3 \end{bmatrix}$, *find* adj A.

SOLUTION We have, $|A| = \begin{vmatrix} 1 & -2 & 4 \\ 0 & 2 & 1 \\ -4 & 5 & 3 \end{vmatrix}$.

The cofactors of the elements of $|A|$ are given by

$$A_{11} = \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 1;\ A_{12} = -\begin{vmatrix} 0 & 1 \\ -4 & 3 \end{vmatrix} = -4;\ A_{13} = \begin{vmatrix} 0 & 2 \\ -4 & 5 \end{vmatrix} = 8;$$

$$A_{21} = -\begin{vmatrix} -2 & 4 \\ 5 & 3 \end{vmatrix} = 26;\ A_{22} = \begin{vmatrix} 1 & 4 \\ -4 & 3 \end{vmatrix} = 19;\ A_{23} = -\begin{vmatrix} 1 & -2 \\ -4 & 5 \end{vmatrix} = 3;$$

$$A_{31} = \begin{vmatrix} -2 & 4 \\ 2 & 1 \end{vmatrix} = -10;\ A_{32} = -\begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = -1;\ A_{33} = \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} = 2.$$

$$\therefore\ \text{adj}\, A = \begin{bmatrix} 1 & -4 & 8 \\ 26 & 19 & 3 \\ -10 & -1 & 2 \end{bmatrix}' = \begin{bmatrix} 1 & 26 & -10 \\ -4 & 19 & -1 \\ 8 & 3 & 2 \end{bmatrix}.$$

THEOREM 1 *If A is a square matrix of order n then prove that*

$$A \cdot (\text{adj}\, A) = (\text{adj}\, A) \cdot A = |A| \cdot I.$$

PROOF Let
$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$ Then,

$$\text{adj}\, A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ A_{k1} & A_{k2} & \dots & A_{kn} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}' = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{k1} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{k2} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{kn} & \dots & A_{nn} \end{bmatrix}.$$

Now, the (i, k)th element of $A \cdot (\text{adj}\, A) = a_{i1} A_{k1} + a_{i2} A_{k2} + \dots + a_{in} A_{kn}$

$$= \begin{cases} |A|, & \text{when } i = k \\ 0, & \text{when } i \neq k. \end{cases}$$

This shows that each diagonal element of $A \cdot (\text{adj}\, A)$ is $|A|$ and each one of its nondiagonal elements is 0.

$$\therefore \quad A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = |A| \cdot I.$$

Thus, $A(\text{adj } A) = |A| \cdot I.$

Similarly, $(\text{adj } A)A = |A| \cdot I.$

Hence, $A(\text{adj } A) = (\text{adj } A)A = |A| \cdot I.$

SUMMARY

For every square matrix A, we have

$$A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I.$$

EXAMPLE 3 *If* $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, *verify that*

$$A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I.$$

SOLUTION We have

$$A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = (15 - 7) = 8 \neq 0.$$

The cofactors of the elements of $|A|$ are given by

$A_{11} = 5, \; A_{12} = -7;$

$A_{21} = -1, \; A_{22} = 3.$

$$\therefore \quad (\text{adj } A) = \begin{bmatrix} 5 & -7 \\ -1 & 3 \end{bmatrix}' = \begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}.$$

$$\therefore \quad A \cdot (\text{adj } A) = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 15 - 7 & -3 + 3 \\ 35 - 35 & -7 + 15 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$= 8 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 8I = |A| \cdot I \qquad [\because \; |A| = 8].$$

$$\text{And, } (\text{adj } A) \cdot A = \begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 15 - 7 & 5 - 5 \\ -21 + 21 & -7 + 15 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$= 8 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 8I = |A| \cdot I \qquad [\because \ |A| = 8].$$

Hence, $A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I.$

EXAMPLE 4 *If* $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$, *verify that*

$$A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I.$$

SOLUTION We have

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = [1 \cdot (-28 + 30) - 1 \cdot (-18 - 0)] = 20.$$

Now, the cofactors of the elements of $|A|$ are given by

$$A_{11} = \begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix} = 2, \; A_{12} = -\begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix} = 21, \; A_{13} = \begin{vmatrix} 3 & 4 \\ 0 & -6 \end{vmatrix} = -18;$$

$$A_{21} = -\begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix} = 6, \; A_{22} = \begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix} = -7, \; A_{23} = -\begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix} = 6;$$

$$A_{31} = \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} = 4, \; A_{32} = -\begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} = -8, \; A_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4.$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}' = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}.$$

$$\text{So, } A \cdot (\text{adj } A) = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = 20 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |A| \cdot I.$$

Thus, $A \cdot (\text{adj } A) = |A| \cdot I.$

$$\text{Further, } (\text{adj } A) \cdot A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = 20 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |A| \cdot I.$$

Thus, $(\text{adj } A) \cdot A = |A| \cdot I.$

Hence, $A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I.$

SINGULAR AND NONSINGULAR MATRICES *A square matrix A is said to be (i) singular if* $|A| = 0$, *(ii) nonsingular if* $|A| \neq 0$.

Examples (i) Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$. Then,

$$|A| = \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} = (8 - 8) = 0 \text{ and therefore, } A \text{ is singular.}$$

(ii) Let $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then,

$$|B| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (4 - 6) = -2 \neq 0, \text{ and}$$

therefore, B is nonsingular.

INVERTIBLE MATRIX *A nonzero square matrix A of order n is said to be invertible if there exists a square matrix B of order n such that AB = BA = I.*

We say that the inverse of A is B and we write, $A^{-1} = B$.

RESULTS ON INVERTIBLE MATRICES

THEOREM 2 *An invertible matrix possesses a unique inverse.*

PROOF Let A be an invertible square of order n.

If possible, let B as well as C be the inverse of A.

Then, $AB = BA = I$ and $AC = CA = I$.

Now, $AC = I \Rightarrow B(AC) = B \cdot I = B$.

And, $BA = I \Rightarrow (BA)C = I \cdot C = C$.

But $B(AC) = (BA)C$ [by associative law of multiplication].

$\therefore \quad B = C$.

Hence, an invertible has a unique inverse.

THEOREM 3 *A square matrix A is invertible if and only if A is nonsingular, i.e., A is invertible* $\Leftrightarrow |A| \neq 0$.

PROOF Let A be an invertible square matrix of order n.

Then, there exists a square matrix B of order n such that

$$AB = BA = I.$$

Now, $AB = I \Rightarrow |AB| = |I|$

$\Rightarrow |A| \cdot |B| = 1$ [$\because$ $|AB| = |A| \cdot |B|$ and $|I| = 1$]

$\Rightarrow |A| \neq 0$ [$\because$ $ab = 1 \Rightarrow a \neq 0$ and $b \neq 0$].

This shows that A is nonsingular.

Conversely, Let A be nonsingular. Then, $|A| \neq 0$.

$\therefore \quad A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I$ and $|A| \neq 0$

$$\Rightarrow A \cdot \left(\frac{1}{|A|} \cdot \text{adj } A \right) = \left(\frac{1}{|A|} \cdot \text{adj } A \right) \cdot A = I$$

$$\Rightarrow \; A^{-1} = \frac{1}{|A|} \cdot \text{adj } A.$$

This shows that A is invertible.

Hence, A is invertible $\Leftrightarrow$ A is nonsingular.

FORMULA FOR FINDING A^{-1}

Let A be a square matrix such that $|A| \neq 0$.

Then, $A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A)$.

EXAMPLE 5 *Find the inverse of the matrix,* $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$.

SOLUTION We have

$$|A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix} = (14 - 12) = 2 \neq 0.$$

So, A^{-1} exists.

The cofactors of the elements of $|A|$ are given by

$A_{11} = 7, \;\; A_{12} = -(-4) = 4;$

$A_{21} = -(-3) = 3, \;\; A_{22} = 2.$

$$\therefore \;\; (\text{adj } A) = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}.$$

Hence, $A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A)$

$$= \frac{1}{2} \cdot \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & \frac{3}{2} \\ 2 & 1 \end{bmatrix}.$$

EXAMPLE 6 *Find the inverse of the matrix* $\begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix}$.

SOLUTION Let $\begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix}$. Then,

$$|A| = \begin{vmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 4 & -12 & 2 \\ -4 & 16 & -2 \end{vmatrix}$$

$[C_1 \to C_1 + 3C_3 \;\text{ and }\; C_2 \to C_2 - 10C_3]$

$$= (-1) \cdot (64 - 48) = -16 \neq 0.$$

Thus, $|A| \neq 0$ and therefore, A^{-1} exists.

Now, the cofactors of the elements of $|A|$ are given by

$$A_{11} = \begin{vmatrix} 8 & 2 \\ -4 & -2 \end{vmatrix} = -8,\ A_{12} = -\begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} = 0,\ A_{13} = \begin{vmatrix} -2 & 8 \\ 2 & -4 \end{vmatrix} = -8;$$

$$A_{21} = -\begin{vmatrix} -10 & -1 \\ -4 & -2 \end{vmatrix} = -16,\ A_{22} = \begin{vmatrix} 3 & -1 \\ 2 & -2 \end{vmatrix} = -4,\ A_{23} = -\begin{vmatrix} 3 & -10 \\ 2 & -4 \end{vmatrix} = -8;$$

$$A_{31} = \begin{vmatrix} -10 & -1 \\ 8 & 2 \end{vmatrix} = -12,\ A_{32} = -\begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} = -4,\ A_{33} = \begin{vmatrix} 3 & -10 \\ -2 & 8 \end{vmatrix} = 4.$$

$$\therefore \quad (\text{adj } A) = \begin{bmatrix} -8 & 0 & -8 \\ -16 & -4 & -8 \\ -12 & -4 & 4 \end{bmatrix}' = \begin{bmatrix} -8 & -16 & -12 \\ 0 & -4 & -4 \\ -8 & -8 & 4 \end{bmatrix}.$$

Hence, $A^{-1} = \dfrac{1}{|A|} \cdot (\text{adj } A)$

$$= \frac{1}{-16} \cdot \begin{bmatrix} -8 & -16 & -12 \\ 0 & -4 & -4 \\ -8 & -8 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

Some Results on Invertible Matrices

THEOREM 1 (Cancellation law) *Let A, B, C be three square matrices, each of order n such that AB = AC. If A is nonsingular then B = C.*

PROOF Let A, B, C be square matrices, each of order n such that $AB = AC$ and A is nonsingular.

Then, $|A| \neq 0$ and therefore, A^{-1} exists.

$$\therefore \quad AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$$
$$\Rightarrow (A^{-1}A)B = (A^{-1}A)C$$
$$\Rightarrow IB = IC$$
$$\Rightarrow B = C.$$

Thus, $AB = AC$ and A is nonsingular $\Rightarrow B = C$.

REMARK If $AB = AC$ and $|A| = 0$ then B and C are not necessarily equal.

Example Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Then, $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = (6 - 6) = 0.$

Here, $AB = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}$

and $AC = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}$.

Thus, $AB = AC$ but $B \neq C$.

THEOREM 2 (Reversal law) *If A and B are invertible square matrices of the same order then AB is also invertible and* $(AB)^{-1} = B^{-1}A^{-1}$.

PROOF Let A and B be two invertible square matrices, each of order n.

Then, $|A| \neq 0$ and $|B| \neq 0$.

$\therefore \quad |AB| = |A| \cdot |B| \neq 0$.

Thus, AB is invertible.

$$\begin{aligned} \text{Now,} \quad (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} && \text{[by associativity]} \\ &= (AI)A^{-1} && [\because BB^{-1} = I] \\ &= AA^{-1} = I && [\because AI = A]. \end{aligned}$$

$$\begin{aligned} \text{And,} \quad (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B && \text{[by associativity]} \\ &= B^{-1}(IB) && [\because A^{-1}A = I] \\ &= B^{-1}B = I && [\because IB = B]. \end{aligned}$$

$\therefore \quad (AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$.

Hence, $(AB)^{-1} = B^{-1}A^{-1}$.

THEOREM 3 *If A is an invertible square matrix then prove that A′ is also invertible and* $(A')^{-1} = (A^{-1})'$.

PROOF Let A be an invertible square matrix of order n.

Then, $|A| \neq 0$.

$\therefore \quad |A'| = |A| \neq 0$.

This shows that A' is invertible.

Now, $AA^{-1} = A^{-1}A = I$

$\Rightarrow \quad (AA^{-1})' = (A^{-1}A)' = I'$

$\Rightarrow \quad (A^{-1})' \cdot A' = A' \cdot (A^{-1})' = I \quad [\because (AB)' = B'A' \text{ and } I' = I]$

$\Rightarrow \quad (A')^{-1} = (A^{-1})' \quad [\because AB = BA = I \Rightarrow B^{-1} = A]$.

Hence, $(A')^{-1} = (A^{-1})'$.

THEOREM 4 *If A is an invertible symmetric then prove that* A^{-1} *is also symmetric.*

PROOF Let A be an invertible symmetric matrix. Then, $A' = A$.

We know that $(A^{-1})' = (A')^{-1}$.

$\therefore \quad (A^{-1})' = A^{-1} \qquad [\because \quad A' = A].$

Hence, A^{-1} is symmetric.

THEOREM 5 *If A and B are nonsingular matrices of the same order then prove that* $(\text{adj } AB) = (\text{adj } B) \cdot (\text{adj } A)$.

PROOF We have

$$|A| \neq 0 \text{ and } |B| \neq 0.$$

$$\therefore \quad |AB| = |A| \cdot |B| \neq 0.$$

So, $(AB)^{-1}$ exists.

Now, $A^{-1} = \dfrac{\text{adj } A}{|A|}$; $B^{-1} = \dfrac{\text{adj } B}{|B|}$ and $(AB)^{-1} = \dfrac{\text{adj } (AB)}{|AB|}$.

$$\therefore \quad \text{adj } (AB) = |AB| \cdot \frac{\text{adj } (AB)}{|AB|} = |A| \cdot |B| \cdot (AB)^{-1}$$

$$\left[\because \; |AB| = |A| \cdot |B| \text{ and } \frac{\text{adj } (AB)}{|AB|} = (AB)^{-1}\right]$$

$$= |A| \cdot |B| \cdot (B^{-1}A^{-1}) \qquad [\because \quad (AB)^{-1} = B^{-1}A^{-1}]$$

$$= |A| \cdot |B| \cdot \frac{\text{adj } B}{|B|} \cdot \frac{\text{adj } A}{|A|} = (\text{adj } B)\,(\text{adj } A).$$

Hence, $\text{adj } (AB) = (\text{adj } B)\,(\text{adj } A)$.

THEOREM 6 *For any square matrix A, prove that* $(\text{adj } A)' = \text{adj } A'$.

PROOF Let A be a square matrix of order n.
Then, each one of $(\text{adj } A)'$ and $(\text{adj } A')$ is a square matrix of order n.

Also, (i, j)th element of $(\text{adj } A)'$

$= (j, i)$th element of $(\text{adj } A)$

$=$ cofactor of (i, j)th element of A

$=$ cofactor of (j, i)th element of A'

$= (i, j)$th element of $(\text{adj } A')$.

Hence, $(\text{adj } A)' = (\text{adj } A')$.

THEOREM 7 *If A is a nonsingular square matrix of order n then prove that* $|\text{adj } A| = |A|^{n-1}$.

PROOF We have

$$A \cdot (\text{adj } A) = |A| \cdot I$$

$$\Rightarrow \quad |A \cdot (\text{adj } A)| = \big|\,|A| \cdot I\,\big| \qquad [\because \quad A = B \Rightarrow |A| = |B|]$$

$$\Rightarrow \quad |A|\,|\text{adj } A| = |A|^n |I| \qquad [\because \quad |kI| = k^n |I|]$$

$$\Rightarrow \quad |A| \cdot |\text{adj } A| = |A|^n \qquad [\because \quad |I| = 1]$$

$$\Rightarrow \quad |\text{adj } A| = |A|^{n-1}$$

Hence, $|\text{adj } A| = |A|^{n-1}$.

THEOREM 8 *If A is a nonsingular square matrix of order n then prove that*

$$\text{adj}\,(\text{adj}\,A) = |A|^{n-2}A.$$

PROOF For any nonsingular B of order n, we have

$$B(\text{adj}\,B) = |B| \cdot I.$$

Taking $B = \text{adj}\,A$, we get

$$(\text{adj}\,A)\,[\text{adj}\,(\text{adj}\,A)] = |\text{adj}\,A|\,I$$

$\Rightarrow$ $(\text{adj}\,A)\,[\text{adj}\,(\text{adj}\,A)] = |A|^{n-1}\,I$ $[\because \ |\text{adj}\,A| = |A|^{n-1}]$

$\Rightarrow$ $A(\text{adj}\,A)\,[\text{adj}\,(\text{adj}\,A)] = |A|^{n-1}\,A$ $[\because \ A\,I_n = A]$

$\Rightarrow$ $|A|\,I_n\,[\text{adj}\,(\text{adj}\,A)] = |A|^{n-1}\,A$ $[\because \ A(\text{adj}\,A) = |A|\,I]$

$\Rightarrow$ $[\text{adj}\,(\text{adj}\,A)] = |A|^{n-2}\,A.$

SOLVED EXAMPLES

EXAMPLE 1 *If* $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, *verify that* $(AB)^{-1} = B^{-1}A^{-1}$.

SOLUTION We have $|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = (15 - 14) = 1 \neq 0.$

Cofactors of the elements of $|A|$ are

$$A_{11} = 5, \ A_{12} = -7;$$
$$A_{21} = -2, \ A_{22} = 3.$$

$\therefore$ $\text{adj}\,A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}' = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}.$

Hence, $A^{-1} = \dfrac{1}{|A|} \cdot \text{adj}\,A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$ $[\because \ |A| = 1].$

Further, $|B| = \begin{vmatrix} 6 & 7 \\ 8 & 9 \end{vmatrix} = (54 - 56) = -2 \neq 0.$

Cofactors of the elements of $|B|$ are

$$B_{11} = 9, \ B_{12} = -8;$$
$$B_{21} = -7, \ B_{22} = 6.$$

$\therefore$ $\text{adj}\,B = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}' = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}.$

Hence, $B^{-1} = \dfrac{1}{|B|} \cdot \text{adj}\,B = -\dfrac{1}{2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}.$

Now, $|AB| = |A| \cdot |B| = 1 \times (-2) = -2 \neq 0,$

and $\text{adj}\,AB = (\text{adj}\,B) \cdot (\text{adj}\,A)$

$$= \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}.$$

$$\therefore \quad (AB)^{-1} = \frac{1}{|AB|} \cdot (\text{adj } AB) = \frac{1}{-2} \cdot \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}.$$

Also, $B^{-1}A^{-1} = -\frac{1}{2} \cdot \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = -\frac{1}{2} \cdot \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}.$

Hence, $(AB)^{-1} = B^{-1} A^{-1}$.

EXAMPLE 2 *Show that the matrix* $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ *satisfies the equation*

$A^2 - 4A - 5I = O$, *and hence find* A^{-1}. **[CBSE 2008]**

SOLUTION We leave it to the reader to show that $A^2 - 4A - 5I = O$.

Now, $A^2 - 4A - 5I = O$

$\Rightarrow \quad AA - 4A = 5I$

$\Rightarrow \quad (AA) \cdot A^{-1} - 4A \cdot A^{-1} = 5I \cdot A^{-1}$

$\Rightarrow \quad A(AA^{-1}) - 4I = 5A^{-1}$

$\Rightarrow \quad AI - 4I = 5A^{-1}$

$\Rightarrow \quad A - 4I = 5A^{-1}$

$\Rightarrow \quad A^{-1} = \frac{1}{5}(A - 4I).$

$$\therefore \quad A^{-1} = \frac{1}{5} \cdot \left\{ \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$= \frac{1}{5} \cdot \left\{ \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right\}$$

$$= \frac{1}{5} \cdot \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}.$$

EXAMPLE 3 *Find a matrix X such that* $X \cdot \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

SOLUTION Let $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

Here, $|A| = \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = (-3-2) = -5 \neq 0.$

So, A is nonsingular and therefore, invertible.
The given equation is $XA = B$.

Now, $XA = B \Rightarrow (XA)\cdot A^{-1} = BA^{-1}$

$$\Rightarrow X(AA^{-1}) = BA^{-1}$$
$$\Rightarrow XI = BA^{-1}$$
$$\Rightarrow X = BA^{-1}.$$

Now, cofactors of elements of $|A|$ are

$$A_{11} = -1, \quad A_{12} = -1;$$
$$A_{21} = -2, \quad A_{22} = 3.$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix}' = \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\cdot \text{adj } A = -\frac{1}{5}\cdot\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}.$$

Hence, $X = BA^{-1}$

$$= \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\cdot\left(-\frac{1}{5}\right)\cdot\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$$
$$= \left(-\frac{1}{5}\right)\cdot\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$$
$$= \left(-\frac{1}{5}\right)\cdot\begin{bmatrix} -5 & -5 \\ -5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

EXAMPLE 4 *Find the matrix A satisfying the equation*

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\cdot A\cdot\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

SOLUTION Let $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$.

Clearly, $|B| = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = (4-3) = 1 \neq 0.$

And, $|C| = \begin{vmatrix} -3 & 2 \\ 5 & -3 \end{vmatrix} = (9-10) = -1 \neq 0.$

This shows that B as well as C is invertible.

The given matrix equation is $BAC = I$.

Now, $BAC = I \Rightarrow B^{-1}\, BA\, CC^{-1} = B^{-1}\, I\, C^{-1}$

$$\Rightarrow I\, A\, I = B^{-1}\, C^{-1}$$
$$\Rightarrow A = B^{-1}\, C^{-1}.$$

Now, the cofactors of the elements of $|B|$ are

$$B_{11} = 2,\ \ B_{12} = -3;\ \ B_{21} = -1,\ \ B_{22} = 2.$$

$$\therefore \quad \text{adj } B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

So, $B^{-1} = \dfrac{1}{|B|} \cdot \text{adj } B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ $[\because\ |B| = 1]$.

Again, the cofactors of the elements of $|C|$ are

$$C_{11} = -3,\ \ C_{12} = -5;\ \ C_{21} = -2,\ \ C_{22} = -3.$$

$$\therefore \quad \text{adj } C = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}' = \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$$

$$\Rightarrow \quad C^{-1} = \frac{1}{|C|} \cdot (\text{adj } C) = \frac{1}{(-1)} \cdot \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$\Rightarrow \quad A = (B^{-1} C^{-1}) = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

EXAMPLE 5 *If* $A = \begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$, *verify that* $(\text{adj } A)^{-1} = (\text{adj } A^{-1})$.

SOLUTION We have, $|A| = \begin{vmatrix} 2 & -3 \\ 4 & 6 \end{vmatrix} = (12 + 12) = 24 \neq 0.$

Cofactors of the elements of $|A|$ are

$$A_{11} = 6,\ A_{12} = -4;\ A_{21} = 3,\ A_{22} = 2.$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 6 & -4 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}.$$

$$\text{So,}\ \ A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{1}{24} \cdot \begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ -\frac{1}{6} & \frac{1}{12} \end{bmatrix}.$$

$$\text{Now,}\ \ |A^{-1}| = \begin{vmatrix} \frac{1}{4} & \frac{1}{8} \\ -\frac{1}{6} & \frac{1}{12} \end{vmatrix} = \left(\frac{1}{48} + \frac{1}{48}\right) = \frac{1}{24}.$$

Cofactors of the elements of $|A^{-1}|$ are

$$C_{11} = \frac{1}{12},\ \ C_{12} = \frac{1}{6};\ \ C_{21} = -\frac{1}{8},\ \ C_{22} = \frac{1}{4}.$$

$$\therefore \quad \text{adj } A^{-1} = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ -\frac{1}{8} & \frac{1}{4} \end{bmatrix}' = \begin{bmatrix} \frac{1}{12} & \frac{-1}{8} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix}.$$

$$\therefore \quad (\text{adj } A)\,(\text{adj } A^{-1}) = \begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{12} & \frac{-1}{8} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$\text{And, } (\text{adj } A^{-1})\,(\text{adj } A) = \begin{bmatrix} \frac{1}{12} & \frac{-1}{8} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Thus, $(\text{adj } A)\,(\text{adj } A^{-1}) = (\text{adj } A^{-1})\,(\text{adj } A) = I.$

Hence, $(\text{adj } A)^{-1} = (\text{adj } A^{-1}).$

EXAMPLE 6 *If* $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, *verify that* $(\text{adj } A)^{-1} = (\text{adj } A^{-1}).$

SOLUTION We have

$$|A| = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 3 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$$= 1 \cdot (-4 - 9) = -13 \neq 0.$$

So, A^{-1} exists.

The cofactors of the elements of $|A|$ are

$$A_{11} = \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 14, A_{12} = -\begin{vmatrix} -2 & 1 \\ 1 & 5 \end{vmatrix} = 11, A_{13} = \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} = -5;$$

$$A_{21} = -\begin{vmatrix} -2 & 1 \\ 1 & 5 \end{vmatrix} = 11, A_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4, A_{23} = -\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = -3;$$

$$A_{31} = \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = -5, A_{32} = -\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -3, A_{33} = \begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix} = -1.$$

$$\therefore \quad (\text{adj } A) = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}' = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\cdot(\text{adj } A) = \frac{1}{-13}\cdot\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{-14}{13} & \frac{-11}{13} & \frac{5}{13} \\ \frac{-11}{13} & \frac{-4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}.$$

The cofactors of the elements of $|A^{-1}|$ are

$$C_{11} = \begin{vmatrix} \frac{-4}{13} & \frac{3}{13} \\ \frac{3}{13} & \frac{1}{13} \end{vmatrix} = \frac{-1}{13}, C_{12} = -\begin{vmatrix} \frac{-11}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{1}{13} \end{vmatrix} = \frac{2}{13},$$

$$C_{13} = \begin{vmatrix} \frac{-11}{13} & \frac{-4}{13} \\ \frac{5}{13} & \frac{3}{13} \end{vmatrix} = \frac{-1}{13};$$

$$C_{21} = -\begin{vmatrix} \frac{-11}{13} & \frac{5}{13} \\ \frac{3}{13} & \frac{1}{13} \end{vmatrix} = \frac{2}{13}, C_{22} = \begin{vmatrix} \frac{-14}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{1}{13} \end{vmatrix} = \frac{-3}{13},$$

$$C_{23} = -\begin{vmatrix} \frac{-14}{13} & \frac{-11}{13} \\ \frac{5}{13} & \frac{3}{13} \end{vmatrix} = \frac{-1}{13};$$

$$C_{31} = \begin{vmatrix} \frac{-11}{13} & \frac{5}{13} \\ \frac{-4}{13} & \frac{3}{13} \end{vmatrix} = \frac{-1}{13}, C_{32} = -\begin{vmatrix} \frac{-14}{13} & \frac{5}{13} \\ \frac{-11}{13} & \frac{3}{13} \end{vmatrix} = \frac{-1}{13},$$

$$C_{33} = \begin{vmatrix} \frac{-14}{13} & \frac{-11}{13} \\ \frac{-11}{13} & \frac{-4}{13} \end{vmatrix} = \frac{-5}{13}.$$

$$\therefore \quad (\text{adj } A^{-1}) = \begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & \frac{-1}{13} \\ \frac{2}{13} & \frac{-3}{13} & \frac{-1}{13} \\ \frac{-1}{13} & \frac{-1}{13} & \frac{-5}{13} \end{bmatrix}' = \begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & \frac{-1}{13} \\ \frac{2}{13} & \frac{-3}{13} & \frac{-1}{13} \\ \frac{-1}{13} & \frac{-1}{13} & \frac{-5}{13} \end{bmatrix}.$$

$$\therefore \quad (\text{adj } A)(\text{adj } A^{-1}) = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} \begin{bmatrix} \frac{-1}{13} & \frac{2}{13} & \frac{-1}{13} \\ \frac{2}{13} & \frac{-3}{13} & \frac{-1}{13} \\ \frac{-1}{13} & \frac{-1}{13} & \frac{-5}{13} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, $(\text{adj } A)\ (\text{adj } A^{-1}) = I$. Similarly, $(\text{adj } A^{-1})\ (\text{adj } A) = I$.

Hence, $(\text{adj } A)^{-1} = (\text{adj } A^{-1})$.

EXAMPLE 7 *If* $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, *verify that* $(A')^{-1} = (A^{-1})'$.

SOLUTION Given: $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, and therefore $A' = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$.

$$\therefore \quad |A| = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = (10 - 3) = 7 \neq 0.$$

So, A^{-1} exists.

The cofactors of the elements of $|A|$ are

$$A_{11} = 5,\ \ A_{12} = -1;\ \ A_{21} = -3,\ \ A_{22} = 2.$$

$$\therefore \quad (\text{adj } A) = \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A) = \frac{1}{7} \cdot \begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{7} & \frac{-3}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix}$$

$$\Rightarrow \quad (A^{-1})' = \begin{bmatrix} \frac{5}{7} & \frac{-1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{bmatrix}. \qquad \ldots \text{(i)}$$

Also, $|A'| = |A| = 7 \neq 0$.

So, $(A')^{-1}$ exists.

The cofactors of elements of $|A'|$ are

$$C_{11} = 5,\ \ C_{12} = -3;\ \ C_{21} = -1,\ \ C_{22} = 2.$$

$$\therefore \quad (\text{adj } A') = \begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix}' = \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow \quad (A')^{-1} = \frac{1}{|A'|}\cdot(\text{adj } A') = \frac{1}{7}\cdot\begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow \quad (A')^{-1} = \begin{bmatrix} \frac{5}{7} & \frac{-1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{bmatrix}. \qquad \ldots \text{(ii)}$$

Hence, from (i) and (ii), we get $(A')^{-1} = (A^{-1})'$.

EXAMPLE 8 *If* $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, *show that* $A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

SOLUTION We have

$$|A| = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix} = (1 + \tan^2 x) = \sec^2 x \neq 0.$$

So, A is invertible.

The cofactors of the elements of $|A|$ are

$$A_{11} = 1,\ A_{12} = \tan x;\ A_{21} = -\tan x,\ A_{22} = 1.$$

$$\therefore \quad (\text{adj } A) = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}' = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{\sec^2 x}\cdot\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \cos^2 x\cdot\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} = \begin{bmatrix} \cos^2 x & -\tan x \cos^2 x \\ \tan x \cos^2 x & \cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x & -\sin x \cos x \\ \sin x \cos x & \cos^2 x \end{bmatrix}$$

$$\Rightarrow \quad A'A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}\begin{bmatrix} \cos^2 x & -\sin x \cos x \\ \sin x \cos x & \cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x - \sin^2 x & -2\sin x \cos x \\ 2\sin x \cos x & -\sin^2 x + \cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}.$$

Hence, $A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

<u>EXAMPLE 9</u> *Let* $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $G(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$.

Show that $[F(\alpha)\cdot G(\beta)]^{-1} = G(-\beta)\cdot F(-\alpha)$.

SOLUTION We have

$$F(\alpha)\cdot F(-\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 & 0 \\ 0 & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Thus, $F(\alpha)\cdot F(-\alpha) = I \Rightarrow \{F(\alpha)\}^{-1} = F(-\alpha)$.

Similarly, $G(\beta)\cdot G(-\beta) = I \Rightarrow \{G(\beta)\}^{-1} = G(-\beta)$.

$\therefore \quad \{F(\alpha)\cdot G(\beta)\}^{-1} = \{G(\beta)\}^{-1}\cdot\{F(\alpha)\}^{-1}$ [by reversal law]

$= G(-\beta)\cdot F(-\alpha)$.

Hence, $\{F(\alpha)\cdot G(\beta)\}^{-1} = G(-\beta)\cdot F(-\alpha)$.

EXERCISE 7

Find the adjoint of the given matrix and verify in each case that $A\cdot(\text{adj } A) = (\text{adj } A)\cdot A = |A|\cdot I$.

1. $\begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

3. $\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

4. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 4 & 5 & 3 \\ 1 & 0 & 6 \\ 2 & 7 & 9 \end{bmatrix}$

9. $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, show that adj $A = A$.

11. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, show that adj $A = 3A'$.

Find the inverse of each of the matrices given below:

12. $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$

15. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, when $(ad - bc) \neq 0$

16. $\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

17. $\begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -1 \\ 2 & 6 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

19. $\begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix}$

20. $\begin{bmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

21. $\begin{bmatrix} 8 & -4 & 1 \\ 10 & 0 & 6 \\ 8 & 1 & 6 \end{bmatrix}$

22. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, show that $A^{-1} = \frac{1}{19} A$.

23. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, show that $A^{-1} = A^2$.

24. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, prove that $A^{-1} = A^3$.

25. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, show that $A^{-1} = A'$.

26. Let $D = \text{diag}\,[d_1, d_2, d_3]$, where none of d_1, d_2, d_3 is 0; prove that

$$D^{-1} = \text{diag}\,[d_1^{-1}, d_2^{-1}, d_3^{-1}].$$

27. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1} A^{-1}$.

28. If $A = \begin{bmatrix} 9 & -1 \\ 6 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 3 \\ 5 & -4 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1} A^{-1}$.

29. Compute $(AB)^{-1}$ when $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$.

30. Obtain the inverses of the matrices $\begin{bmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{bmatrix}$.

    And, hence find the inverse of the matrix $\begin{bmatrix} 1+pq & p & 0 \\ q & 1+pq & p \\ 0 & q & 1 \end{bmatrix}$.

31. If $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$, verify that $A^2 - 4A - I = O$, and hence find A^{-1}.

32. Show that the matrix $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$ satisfies the equation $x^2 + 4x - 42 = 0$ and hence find A^{-1}.

33. If $A = \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}$, show that $A^2 + 3A + 4I_2 = O$ and hence find A^{-1}.

34. If $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find x and y such that $A^2 + xI = yA$. Hence, find A^{-1}.

[CBSE 2005]

35. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find the value of λ so that $A^2 = \lambda A - 2I$. Hence, find A^{-1}.

[CBSE 2007]

36. Show that the $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ satisfies the equation $A^3 - A^2 - 3A - I = O$, and hence find A^{-1}.

37. Prove that: (i) adj $I = I$ (ii) adj $O = O$ (iii) $I^{-1} = I$.

ANSWERS (EXERCISE 7)

1. $\begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix}$ **2.** $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ **3.** $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

4. $\begin{bmatrix} 3 & 3 & 0 \\ -11 & 1 & 8 \\ -1 & -1 & 4 \end{bmatrix}$ **5.** $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ **6.** $\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

7. $\begin{bmatrix} -34 & 10 & 31 \\ 14 & 0 & -21 \\ 46 & -30 & -44 \end{bmatrix}$ **8.** $\begin{bmatrix} -42 & -24 & 30 \\ 3 & 30 & -21 \\ 7 & -18 & -5 \end{bmatrix}$ **9.** $\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ **13.** $\begin{bmatrix} \frac{3}{10} & \frac{-1}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$ **14.** $\begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{-1}{6} & \frac{1}{12} \end{bmatrix}$ **15.** $\frac{1}{(ad-bc)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

16. $\frac{1}{27} \cdot \begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$ **17.** $\frac{1}{32} \cdot \begin{bmatrix} 6 & 6 & 1 \\ -2 & -2 & 5 \\ 8 & -14 & 3 \end{bmatrix}$ **18.** $\frac{1}{5} \cdot \begin{bmatrix} -2 & 0 & 3 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$

19. $\frac{1}{4} \cdot \begin{bmatrix} -8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$ **20.** $\frac{1}{19} \cdot \begin{bmatrix} 1 & -6 & 1 \\ -5 & -8 & 14 \\ 3 & 1 & 3 \end{bmatrix}$

21. $\frac{1}{10} \cdot \begin{bmatrix} -6 & 25 & -24 \\ -12 & 40 & -38 \\ 10 & -40 & 40 \end{bmatrix}$ **29.** $\frac{1}{19} \cdot \begin{bmatrix} 16 & 12 & 1 \\ 21 & 11 & -7 \\ 10 & -2 & 3 \end{bmatrix}$

30. $\begin{bmatrix} 1 & -p & p^2 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ q^2 & -q & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -p & p^2 \\ -q & pq+1 & -qp^2-p \\ q^2 & -pq^2-q & p^2q^2+pq+1 \end{bmatrix}$

31. $\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$ **32.** $\frac{1}{42} \cdot \begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}$ **33.** $A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$

34. $x = 8,\ y = 8$ and $A^{-1} = \frac{1}{8} \cdot \begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}$ **35.** $\lambda = 1,\ A^{-1} = \frac{1}{2} \cdot \begin{bmatrix} -2 & 2 \\ -4 & 3 \end{bmatrix}$

36. $A^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 7)

29. Find A^{-1}. Then, $(AB)^{-1} = B^{-1}A^{-1}$.

30. Let the first two matrices be A and B. Then, the third matrix is AB. Now, $(AB)^{-1} = (B^{-1} \cdot A^{-1})$.

37. (i) $|I| = 1$. Cofactors of a_{ij} in $|I|$ are $A_{ij} = \begin{cases} 1, \text{ when } i = j \\ 0, \text{ when } i \neq j \end{cases}$

$\therefore \quad (\text{adj } I) = [A_{ij}]' = I' = I$.

(ii) $|O| = 0$ and cofactor of each element of $|O|$ is 0.

$\therefore \quad \text{adj } O = O$.

(iii) $I \cdot I = I \;\Rightarrow\; I^{-1} = I$.

8. SYSTEM OF LINEAR EQUATIONS

Solving a System of Linear Equations by Matrix Method

CONSISTENT SYSTEM OF EQUATIONS *A given system of equations is said to be consistent if it has one or more solutions.*

INCONSISTENT SYSTEM OF EQUATIONS *A given system of equations is said to be inconsistent if it has no solution.*

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3.$$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$.

Then, the given system can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

$\therefore \quad AX = B.$

CASE 1 *When* $|A| \neq 0$

In this case, A^{-1} exists.

$\therefore \quad AX = B \Rightarrow A^{-1}(AX) = A^{-1}B$

$\Rightarrow (A^{-1}A)X = A^{-1}B$ [by associative law]

$\Rightarrow I \cdot X = A^{-1}B$ [$\because A^{-1}A = I$]

$\Rightarrow X = A^{-1}B.$

Since A^{-1} is unique, the given system has a unique solution.

Thus, when $|A| \neq 0$, then the given system is consistent and it has a unique solution.

CASE 2 *When* $|A| = 0$ *and* $(\text{adj } A)B \neq O$

In this case, the given system has no solution and hence it is inconsistent.

CASE 3 *When* $|A| = 0$ *and* $(\text{adj } A)B = O$

In this case, the given system has infinitely many solutions.

SUMMARY

Let $AX = B$ be the given system of equations.

(i) If $|A| \neq 0$, the system has a unique solution.

(ii) If $|A| = 0$ and $(\text{adj } A)B \neq O$ then the given system has no solution.

(iii) If $|A| = 0$ and $(\text{adj } A)B = O$ then the system has infinitely many solutions.

SOLVED EXAMPLES

EXAMPLE 1 *Use matrix method to show that the system of equations*

$$2x + 5y = 7,$$
$$6x + 15y = 13$$

is inconsistent.

SOLUTION The given equations are

$$2x + 5y = 7, \quad \ldots \text{(i)}$$
$$6x + 15y = 13. \quad \ldots \text{(ii)}$$

Let $A = \begin{bmatrix} 2 & 5 \\ 6 & 15 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 13 \end{bmatrix}$.

Then, the given system in matrix form is $AX = B$.

Now, $|A| = \begin{vmatrix} 2 & 5 \\ 6 & 15 \end{vmatrix} = 0.$

The system will be inconsistent if $(\text{adj } A)B \neq O$.

The minors of the elements of $|A|$ are

$M_{11} = 15, \quad M_{12} = 6;$

$M_{21} = 5, \quad M_{22} = 2.$

So, the cofactors of the elements of $|A|$ are

$A_{11} = 15, \quad A_{12} = -6;$

$A_{21} = -5, \quad A_{22} = 2.$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 15 & -6 \\ -5 & 2 \end{bmatrix}' = \begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix}$$

$$\Rightarrow \quad (\text{adj } A)B = \begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 13 \end{bmatrix} = \begin{bmatrix} 105 - 65 \\ -42 + 26 \end{bmatrix} = \begin{bmatrix} 40 \\ -16 \end{bmatrix} \neq O.$$

Thus, $|A| = 0$ and $(\text{adj } A)B \neq O$.

Hence, the given system of equations is inconsistent.

EXAMPLE 2 *Use matrix method to show that the following system of equations is inconsistent:*

$$3x - y + 2z = 3,$$
$$2x + y + 3z = 5,$$
$$x - 2y - z = 1.$$

SOLUTION Let us take

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & -2 & -1 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}.$$

The given system in matrix form is $AX = B$.

$$\text{Now, } |A| = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= 3(-1 + 6) + 1 \cdot (-2 - 3) + 2 \cdot (-4 - 1)$$
$$= (15 - 5 - 10) = 0.$$

So, the system will be inconsistent if $(\text{adj } A) \cdot B \neq O$.

The minors of the elements of $|A|$ are

$M_{11} = 5, \ M_{12} = -5, \ M_{13} = -5;$
$M_{21} = 5, \ M_{22} = -5, \ M_{23} = -5;$
$M_{31} = -5, \ M_{32} = 5, \ M_{33} = 5.$

So, the cofactors of the elements of $|A|$ are

$A_{11} = 5, \ A_{12} = 5, \ A_{13} = -5;$
$A_{21} = -5, \ A_{22} = -5, \ A_{23} = 5;$
$A_{31} = -5, \ A_{32} = -5, \ A_{33} = 5.$

$$\therefore \quad (\text{adj } A) = \begin{bmatrix} 5 & 5 & -5 \\ -5 & -5 & 5 \\ -5 & -5 & 5 \end{bmatrix}' = \begin{bmatrix} 5 & -5 & -5 \\ 5 & -5 & -5 \\ -5 & 5 & 5 \end{bmatrix}$$

$$\Rightarrow \quad (\text{adj } A)B = \begin{bmatrix} 5 & -5 & -5 \\ 5 & -5 & -5 \\ -5 & 5 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \cdot 3 + (-5) \cdot 5 + (-5) \cdot 1 \\ 5 \cdot 3 + (-5) \cdot 5 + (-5) \cdot 1 \\ (-5) \cdot 3 + 5 \cdot 5 + 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 15 - 25 - 5 \\ 15 - 25 - 5 \\ -15 + 25 + 5 \end{bmatrix}$$

$$= \begin{bmatrix} -15 \\ -15 \\ 15 \end{bmatrix} \neq O.$$

Thus, $|A| = 0$ and $(\text{adj } A)B \neq O$.

Hence, the given system of equations is inconsistent.

EXAMPLE 3 *Show that the following system of equations is consistent and solve it:*

$$2x + 5y = 1,$$
$$3x + 2y = 7.$$

SOLUTION The given system of equations is

$2x + 5y = 1,$ ... (i)

$3x + 2y = 7.$ ... (ii)

Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$.

Then, the given system is $AX = B$.

Now, $|A| = \begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} = (4 - 15) = -11 \neq 0.$

Hence, the given system has a unique solution.

The minors of the elements of $|A|$ are

$M_{11} = 2,\ M_{12} = 3;$

$M_{21} = 5,\ M_{22} = 2.$

So, the cofactors of the elements of $|A|$ are

$A_{11} = 2,\ A_{12} = -3;$

$A_{21} = -5,\ A_{22} = 2.$

$$\therefore \quad (\text{adj } A) = \begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{-1}{11} \cdot \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{-2}{11} & \frac{5}{11} \\ \frac{3}{11} & \frac{-2}{11} \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-2}{11} & \frac{5}{11} \\ \frac{3}{11} & \frac{-2}{11} \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{-2}{11} + \frac{35}{11} \\ \frac{3}{11} - \frac{14}{11} \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$\Rightarrow \quad x = 3$ and $y = -1$.

Hence, $x = 3$ and $y = -1$.

EXAMPLE 4 *If* $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$, *find* A^{-1} *and hence solve the system of linear equations:*

$x + 2y - 3z = -4;\ 2x + 3y + 2z = 2;\ 3x - 3y - 4z = 11.$ **[CBSE 2012C]**

SOLUTION The given equations are $x + 2y - 3z = -4.$ ... (i)

$2x + 3y + 2z = 2,$ ... (ii)

$3x - 3y - 4z = 11.$ ... (iii)

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$.

So, the given system in matrix form is $AX = B$.

Now, $|A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -1 & 8 \\ 0 & -9 & 5 \end{vmatrix}$

$[R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1]$

$= 1 \cdot (-5 + 72) = 67 \neq 0.$

Thus, A is invertible.

So, the system has a unique solution, $X = A^{-1}B$.

Now, the cofactors of the elements of $|A|$ are

$A_{11} = -6,\quad A_{12} = 14,\quad A_{13} = -15;$
$A_{21} = 17,\quad A_{22} = 5,\quad A_{23} = 9;$
$A_{31} = 13,\quad A_{32} = -8,\quad A_{33} = -1.$

$\therefore \quad \text{adj } A = \begin{bmatrix} -6 & 14 & -15 \\ 17 & 5 & 9 \\ 13 & -8 & -1 \end{bmatrix}' = \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}.$

So, $\quad A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{1}{67} \cdot \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}.$

$\therefore \quad X = A^{-1} B$

or $\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67} \cdot \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$

$= \frac{1}{67} \cdot \begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$

$\therefore \quad x = 3,\ y = -2$ and $z = 1.$

EXAMPLE 5 *Using matrices, solve the following system of linear equations:*

$3x + 4y + 2z = 8,$
$2y - 3z = 3,$
$x - 2y + 6z = -2.$ **[CBSE 2006C]**

SOLUTION The given equations are

$3x + 4y + 2z = 8,$ … (i)

$2y - 3z = 3,$ … (ii)

$x - 2y + 6z = -2.$ … (iii)

Let $A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 2 & -3 \\ 1 & -2 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 3 \\ -2 \end{bmatrix}$.

So, the given system in matrix form is $AX = B$.

Now, $|A| = \begin{vmatrix} 3 & 4 & 2 \\ 0 & 2 & -3 \\ 1 & -2 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 10 & -16 \\ 0 & 2 & -3 \\ 1 & -2 & 6 \end{vmatrix}$ $[R_1 \to R_1 - 3R_3]$

$$= 1 \cdot (-30 + 32) = 2 \neq 0.$$

So, A is invertible.

Therefore, the given system has a unique solution, $X = A^{-1}B$.

Now, the minors of the elements of $|A|$ are

$M_{11} = 6,\ M_{12} = 3,\ M_{13} = -2;$
$M_{21} = 28,\ M_{22} = 16,\ M_{23} = -10;$
$M_{31} = -16,\ M_{32} = -9,\ M_{33} = 6.$

The cofactors of the elements of $|A|$ are

$A_{11} = 6,\quad A_{12} = -3,\quad A_{13} = -2;$
$A_{21} = -28,\quad A_{22} = 16,\quad A_{23} = 10;$
$A_{31} = -16,\quad A_{32} = 9,\quad A_{33} = 6.$

$$\therefore\quad (\text{adj } A) = \begin{bmatrix} 6 & -3 & -2 \\ -28 & 16 & 10 \\ -16 & 9 & 6 \end{bmatrix}' = \begin{bmatrix} 6 & -28 & -16 \\ -3 & 16 & 9 \\ -2 & 10 & 6 \end{bmatrix}$$

$$\Rightarrow\quad A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A)$$

$$= \frac{1}{2} \cdot \begin{bmatrix} 6 & -28 & -16 \\ -3 & 16 & 9 \\ -2 & 10 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -14 & -8 \\ \frac{-3}{2} & 8 & \frac{9}{2} \\ -1 & 5 & 3 \end{bmatrix}.$$

$$\therefore\quad X = A^{-1} B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -14 & -8 \\ \frac{-3}{2} & 8 & \frac{9}{2} \\ -1 & 5 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 - 42 + 16 \\ -12 + 24 - 9 \\ -8 + 15 - 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow x = -2,\ y = 3 \text{ and } z = 1.$$

Hence, $x = -2$, $y = 3$ and $z = 1$.

EXAMPLE 6 *Using matrices, solve the following system of equations:*

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4;$$
$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1;$$
$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2.\ (x, y, z \neq 0)$$ **[CBSE 2011]**

SOLUTION Putting $\frac{1}{x}=u$, $\frac{1}{y}=v$ and $\frac{1}{z}=w$, the given equations become:

$2u+3v+10w=4,$ … (i)

$4u-6v+5w=1,$ … (ii)

$6u+9v-20w=2.$ … (iii)

Let $A=\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, $Y=\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ and $B=\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$.

Then, the given system in matrix form is $AY=B$.

Now, $|A|=\begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$

$=\begin{vmatrix} 2 & 3 & 10 \\ 0 & -12 & -15 \\ 0 & 0 & -50 \end{vmatrix}$ $[R_2 \to R_2-2R_1;\ R_3 \to R_3-3R_1]$

$=(-50)\cdot(-24-0)=1200 \neq 0.$

Thus, A is invertible.

So, the given system has a unique solution, $Y=A^{-1}B$.

The minors of the elements of $|A|$ are

$M_{11}=75,\ M_{12}=-110,\ M_{13}=72;$

$M_{21}=-150,\ M_{22}=-100,\ M_{23}=0;$

$M_{31}=75,\ M_{32}=-30,\ M_{33}=-24.$

So, the cofactors of the elements of $|A|$ are

$A_{11}=75,\ A_{12}=110,\ A_{13}=72;$

$A_{21}=150,\ A_{22}=-100,\ A_{23}=0;$

$A_{31}=75,\ A_{32}=30,\ A_{33}=-24.$

$$\therefore\quad (\text{adj } A)=\begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}'=\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A) = \frac{1}{1200} \cdot \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}.$$

$$\therefore \quad Y = A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200} \cdot \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200} \cdot \begin{bmatrix} 300 + 150 + 150 \\ 440 - 100 + 60 \\ 288 + 0 - 48 \end{bmatrix} = \frac{1}{1200} \cdot \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{600}{1200} \\ \frac{400}{1200} \\ \frac{240}{1200} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$\Rightarrow \quad u = \frac{1}{2}, \; v = \frac{1}{3}, \; w = \frac{1}{5}$$

$$\Rightarrow \quad \frac{1}{x} = \frac{1}{2}, \; \frac{1}{y} = \frac{1}{3} \text{ and } \frac{1}{z} = \frac{1}{5}$$

$$\Rightarrow \quad x = 2, \; y = 3 \text{ and } z = 5.$$

Hence $x = 2$, $y = 3$ and $z = 5$.

<u>EXAMPLE 7</u> *Use the product* $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ *to solve the following system of equations:*

$$x - y + 2z = 1;$$
$$2y - 3z = 1;$$
$$3x - 2y + 4z = 2.$$

SOLUTION The given equations are

$$x - y + 2z = 1, \quad \text{... (i)}$$
$$2y - 3z = 1, \quad \text{... (ii)}$$
$$3x - 2y + 4z = 2. \quad \text{... (iii)}$$

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

Then, the given system in matrix form is $AX = B$.

Now, $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A \cdot \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} = I$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}.$$

Now, $AX = B$

$\Rightarrow \quad X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

$\Rightarrow \quad x = 0, \ y = 5$ and $z = 3$.

Hence, $x = 0, \ y = 5$ and $z = 3$.

EXAMPLE 8 *Given* $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$, *find AB and use this result in solving the following system of equations:*

$x - y + z = 4;$

$x - 2y - 2z = 9;$

$2x + y + 3z = 1.$ **[CBSE 2006C, '10C, '12C]**

SOLUTION The given equations are

$x - y + z = 4,$ … (i)

$x - 2y - 2z = 9,$ … (ii)

$2x + y + 3z = 1.$ … (iii)

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $C = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$.

Then, the given system of equations is $AX = C$.

Now, $AB = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$

$$= \begin{bmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= 8I$$

$\Rightarrow \quad A \cdot \left(\frac{1}{8} B \right) = I$

$\Rightarrow \quad A^{-1} = \frac{1}{8} B = \frac{1}{8} \cdot \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}.$

Now, $AX = C$

$\Rightarrow \quad X = A^{-1}C$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8} \cdot \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

$$= \frac{1}{8} \cdot \begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8} \cdot \begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$\Rightarrow \quad x = 3, \; y = -2$ and $z = -1$.

Hence, $x = 3, \; y = -2$ and $z = -1$.

EXAMPLE 9 *The sum of three numbers is* 6. *Twice the third number when added to the first number gives* 7. *On adding the sum of the second and third numbers to thrice the first number, we get* 12. *Find the numbers, using method.*

SOLUTION Let the first, second and third numbers be x, y, z respectively. Then,

$$x + y + z = 6, \qquad \ldots \text{(i)}$$

$$x + 2z = 7, \qquad \ldots \text{(ii)}$$

$$3x + y + z = 12. \qquad \ldots \text{(iii)}$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$.

Then, the given system in matrix form is $AX = B$.

Now, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & -2 \end{vmatrix}$ $\begin{bmatrix} R_2 \to R_2 - R_1; \\ R_3 \to R_3 - 3R_1 \end{bmatrix}$

$= 1 \cdot (2 + 2) = 4 \neq 0.$

$\therefore$ A is invertible.

So, the given system has a unique solution, $X = A^{-1}B$.

The minors of the elements of $|A|$ are

$M_{11} = -2,\ M_{12} = -5,\ M_{13} = 1;$

$M_{21} = 0,\ M_{22} = -2,\ M_{23} = -2;$

$M_{31} = 2,\ M_{32} = 1,\ M_{33} = -1.$

The cofactors of the elements of $|A|$ are

$A_{11} = -2,\ A_{12} = 5,\ A_{13} = 1;$

$A_{21} = 0,\ A_{22} = -2,\ A_{23} = 2;$

$A_{31} = 2,\ A_{32} = -1,\ A_{33} = -1.$

$$\therefore \quad (\text{adj } A) = \begin{bmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & -1 & -1 \end{bmatrix}' = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A) = \frac{1}{4} \cdot \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \cdot \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4} \cdot \begin{bmatrix} -12 + 0 + 24 \\ 30 - 14 - 12 \\ 6 + 14 - 12 \end{bmatrix} = \frac{1}{4} \cdot \begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$\Rightarrow \quad x = 3,\ y = 1,\ z = 2.$

Hence, the required numbers are 3, 1, 2.

EXAMPLE 10 *A school wants to award its students for the value of Honesty, Regularity and Hardwork with a total cash award of* ₹ 6000. *Three times the award money for Hardwork added to that given for Honesty amounts to* ₹11000. *The award money given for Honesty and Hardwork together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from the given three values, suggest one more value which the school must include for awards.* **[CBSE 2013]**

SOLUTION Let the amount of award for Honesty, Regularity and Hardwork be ₹x, ₹y and ₹z respectively. Then,

$$x + y + z = 6000, \quad \dots \text{(i)}$$

$$x + 0y + 3z = 11000 \quad \dots \text{(ii)}$$

and $x + z = 2y \Rightarrow x - 2y + z = 0. \quad \dots$ (iii)

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$.

Then, the matrix equation is $AX = B$.

$\therefore \quad X = A^{-1}B.$

Now, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -3 & 0 \end{vmatrix}$

$\{C_2 \to (C_2 - C_1) \text{ and } C_3 \to (C_3 - C_1)\}$

$$= 1 \cdot \begin{vmatrix} -1 & 2 \\ -3 & 0 \end{vmatrix} = (0 + 6) = 6 \neq 0.$$

$\therefore \quad A^{-1}$ exists.

Now, the cofactors of the elements of $|A|$ are given by

$$C_{11} = \begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix} = 6, \; C_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 2, \; C_{13} = \begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} = -2;$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -3, \; C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \; C_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3;$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3, \; C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2, \; C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

$$\therefore \quad (\text{adj } A) = \begin{bmatrix} 6 & 2 & -2 \\ -3 & 0 & 3 \\ 3 & -2 & -1 \end{bmatrix}^t = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A) = \frac{1}{6} \cdot \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B = \frac{1}{6} \cdot \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad X = \frac{1}{6} \cdot \begin{bmatrix} 36000 - 33000 + 0 \\ 12000 + \quad 0 \quad + 0 \\ -12000 + 33000 - 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

$\Rightarrow \quad x = 500,\ y = 2000$ and $z = 3500.$

Hence, the award money for Honesty, Regularity and Hardwork is ₹ 500, ₹ 2000 and ₹ 3500 respectively.

Apart from honesty, regularity and hardwork, the school must include an award for a student to be well-behaved.

EXERCISE 8A

Show that each one of the following systems of equations is inconsistent.

1. $x + 2y = 9;$
   $2x + 4y = 7.$

2. $2x + 3y = 5;$
   $6x + 9y = 10.$

3. $4x - 2y = 3;$
   $6x - 3y = 5.$

4. $6x + 4y = 5;$
   $9x + 6y = 8.$

5. $x + y - 2z = 5;$
   $x - 2y + z = -2;$
   $-2x + y + z = 4.$

6. $2x - y + 3z = 1;$
   $3x - 2y + 5z = -4;$
   $5x - 4y + 9z = 14.$

7. $x + 2y + 4z = 12;$
   $y + 2z = -1;$
   $3x + 2y + 4z = 4.$

8. $3x - y - 2z = 2;$
   $2y - z = -1;$
   $3x - 5y = 3.$

Solve each of the following systems of equations using matrix method.

9. $5x + 2y = 4;$
   $7x + 3y = 5.$

10. $3x + 4y - 5 = 0;$
    $x - y + 3 = 0.$

11. $x + 2y = 1;$
    $3x + y = 4.$

12. $5x + 7y + 2 = 0;$
    $4x + 6y + 3 = 0.$

13. $2x - 3y + 1 = 0;$
    $x + 4y + 3 = 0.$

14. $4x - 3y = 3;$
    $3x - 5y = 7.$

15. $2x + 8y + 5z = 5;$
    $x + y + z = -2;$
    $x + 2y - z = 2.$ **[CBSE 2009C]**

16. $x - y + z = 1;$
    $2x + y - z = 2;$
    $x - 2y - z = 4.$ **[CBSE 2006C]**

17. $3x + 4y + 7z = 4;$
$2x - y + 3z = -3;$
$x + 2y - 3z = 8.$ **[CBSE 2012]**

18. $x + 2y + z = 7;$
$x + 3z = 11;$
$2x - 3y = 1.$ **[CBSE 2005, '08, '11]**

19. $2x - 3y + 5z = 16;$
$3x + 2y - 4z = -4;$
$x + y - 2z = -3.$ **[CBSE 2005C]**

20. $x + y + z = 4;$
$2x - y + z = -1;$
$2x + y - 3z = -9.$ **[CBSE 2005]**

21. $2x - 3y + 5z = 11;$
$3x + 2y - 4z = -5;$
$x + y - 2z = -3.$ **[CBSE 2009]**

22. $x + y + z = 1;$
$x - 2y + 3z = 2;$
$5x - 3y + z = 3.$ **[CBSE 2004, '09C]**

23. $x + y + z = 6;$
$x + 2z = 7;$
$3x + y + z = 12.$ **[CBSE 2009]**

24. $2x + 3y + 3z = 5;$
$x - 2y + z = -4;$
$3x - y - 2z = 3.$ **[CBSE 2008C, '12]**

25. $4x - 5y - 11z = 12;$
$x - 3y + z = 1;$
$2x + 3y - 7z = 2.$ **[CBSE 2007]**

26. $x - y + 2z = 7;$
$3x + 4y - 5z = -5;$
$2x - y + 3z = 12.$ **[CBSE 2012]**

27. $6x - 9y - 20z = -4;$
$4x - 15y + 10z = -1;$
$2x - 3y - 5z = -1.$

28. $3x - 4y + 2z = -1;$
$2x + 3y + 5z = 7;$
$x + z = 2.$ **[CBSE 2011C]**

29. $x + y - z = 1;$
$3x + y - 2z = 3;$
$x - y - z = -1.$ **[CBSE 2004]**

30. $2x + y - z = 1;$
$x - y + z = 2;$
$3x + y - 2z = -1.$ **[CBSE 2004C]**

31. $x + 2y + z = 4;$
$-x + y + z = 0;$
$x - 3y + z = 4.$ **[CBSE 2012C]**

32. $x - y - 2z = 3;$
$x + y = 1;$
$x + z = -6.$

33. $5x - y = -7;$
$2x + 3z = 1;$
$3y - z = 5.$

34. $x - 2y + z = 0;$
$y - z = 2;$
$2x - 3z = 10.$

35. $x - y = 3;$
$2x + 3y + 4z = 17;$
$y + 2z = 7.$ **[CBSE 2003C, '07C]**

36. $4x + 3y + 2z = 60;$
$x + 2y + 3z = 45;$
$6x + 2y + 3z = 70.$ **[CBSE 2011]**

37. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. **[CBSE 2007C, '08C]**

Using A^{-1}, solve the following system of equations:
$2x - 3y + 5z = 11;$
$3x + 2y - 4z = -5;$
$x + y - 2z = -3.$

38. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$, find A^{-1}.

Using A^{-1}, solve the following system of linear equations:

$$2x + y + z = 1;$$
$$x - 2y - z = \frac{3}{2};$$
$$3y - 5z = 9.$$

HINT: Here $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$.

39. If $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$, find AB.

Hence, solve the system of equations:

$x - 2y = 10,\ 2x + y + 3z = 8$ and $-2y + z = 7$. **[CBSE 2011]**

HINT: $AB = (11)\,I \Rightarrow A\left(\frac{1}{11}B\right) = I \Rightarrow A^{-1} = \left(\frac{1}{11}\right)B.$

Using matrices, solve the following system of equations.

40. $\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10,\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10,\ \frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13$ **[CBSE 2007C]**

41. $\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 4;\ \frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 0;\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2.\ (x, y, z \neq 0)$

VALUE BASED QUESTIONS

42. The sum of three numbers is 2. If twice the second number is added to the sum of first and third, we get 1. On adding the sum of second and third numbers to five times the first, we get 6. Find the three numbers by using matrices.

43. The cost of 4 kg potato, 3 kg wheat and 2 kg rice is ₹ 60. The cost of 1 kg potato, 2 kg wheat and 3 kg rice is ₹ 45. The cost of 6 kg potato, 2 kg wheat and 3 kg rice is ₹ 70. Find the cost of each item per kg by matrix method.

44. An amount of ₹ 5000 is put into three investments at 6%, 7% and 8% per annum respectively. The total annual income from these investments is ₹ 358. If the total annual income from first two investments is ₹ 70 more than the income from the third, find the amount of each investment by the matrix method.

HINT: Let these investments be ₹ x, ₹ y and ₹ z respectively.

Then, $x + y + z = 5000,$ … (i)

$$\frac{6x}{100} + \frac{7y}{100} + \frac{8z}{100} = 358 \Rightarrow 6x + 7y + 8z = 35800 \quad \text{… (ii)}$$

and, $$\frac{6x}{100} + \frac{7y}{100} = \frac{8z}{100} + 70 \Rightarrow 6x + 7y - 8z = 7000. \quad \text{… (iii)}$$

45. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to 3, 2 and 1 students respectively with a total award money of ₹ 1,600. School B wants to spend ₹ 2,300 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is ₹ 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award. **[CBSE 2014]**

HINT: By the given data, we have

$$\left.\begin{aligned} 3x + 2y + z &= 1600 \\ 4x + y + 3z &= 2300 \\ x + y + z &= 900 \end{aligned}\right\}$$

ANSWERS (EXERCISE 8A)

9. $x = 2, \; y = -3$ **10.** $x = -1, \; y = 2$ **11.** $x = \frac{7}{5}, \; y = \frac{-1}{5}$

12. $x = \frac{9}{2}, \; y = \frac{-7}{2}$ **13.** $x = \frac{-13}{11}, \; y = \frac{-5}{11}$ **14.** $x = \frac{-6}{11}, \; y = \frac{-19}{11}$

15. $x = -3, y = 2, z = -1$ **16.** $x = 1, y = -1, z = -1$ **17.** $x = 1, y = 2, z = -1$

18. $x = 2, y = 1, z = 3$ **19.** $x = 2, y = 1, z = 3$ **20.** $x = 2, y = 1, z = 3$

21. $x = 1, y = 2, z = 3$ **22.** $x = \frac{1}{2}, y = 0, z = \frac{1}{2}$ **23.** $x = 3, y = 1, z = 2$

24. $x = 1, y = 2, z = -1$ **25.** $x = -1, y = -1, z = -1$ **26.** $x = 1, y = 2, z = 5$

27. $x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{1}{5}$ **28.** $x = 3, y = 2, z = -1$ **29.** $x = 2, y = 1, z = 2$

30. $x = 1, y = 2, z = 3$ **31.** $x = 2, y = 0, z = 2$ **32.** $x = -2, y = 3, z = -4$

33. $x = -1, y = 2, z = 1$ **34.** $x = 2, y = 0, z = -2$ **35.** $x = 2, y = -1, z = 4$

36. $x = 5, y = 8, z = 8$ **37.** $x = 1, y = 2, z = 3$ **38.** $x = 1, y = \frac{1}{2}, z = \frac{-3}{2}$

39. $x = 4, y = -3, z = 1$ **40.** $x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{1}{5}$ **41.** $x = \frac{1}{2}, y = -1, z = 1$

42. 1, –1, 2 **43.** ₹ 5, ₹ 8, ₹ 8 **44.** ₹ 1000, ₹ 2200, ₹ 1800

45. ₹ 200 for sincerity, ₹ 300 for truthfulness and ₹ 400 for helpfulness.

One more value may be like honesty, kindness, etc.

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. If A and B are 2-rowed square matrices such that $(A+B)=\begin{bmatrix}4 & -3\\ 1 & 6\end{bmatrix}$ and $(A-B)=\begin{bmatrix}-2 & -1\\ 5 & 2\end{bmatrix}$ then $AB=?$

   (a) $\begin{bmatrix}-7 & 5\\ 1 & -5\end{bmatrix}$ (b) $\begin{bmatrix}7 & -5\\ 1 & 5\end{bmatrix}$ (c) $\begin{bmatrix}7 & -1\\ 5 & -5\end{bmatrix}$ (d) $\begin{bmatrix}7 & -1\\ -5 & 5\end{bmatrix}$

2. If $\begin{bmatrix}3 & -2\\ 5 & 6\end{bmatrix}+2A=\begin{bmatrix}5 & 6\\ -7 & 10\end{bmatrix}$ then $A=?$

   (a) $\begin{bmatrix}1 & 3\\ -5 & 4\end{bmatrix}$ (b) $\begin{bmatrix}-1 & 5\\ -3 & 4\end{bmatrix}$ (c) $\begin{bmatrix}1 & 4\\ -6 & 2\end{bmatrix}$ (d) none of these

3. If $A=\begin{bmatrix}2 & 0\\ -3 & 1\end{bmatrix}$ and $B=\begin{bmatrix}4 & -3\\ -6 & 2\end{bmatrix}$ are such that $4A+3X=5B$ then $X=?$

   (a) $\begin{bmatrix}4 & -5\\ -6 & 2\end{bmatrix}$ (b) $\begin{bmatrix}4 & 5\\ -6 & -2\end{bmatrix}$ (c) $\begin{bmatrix}-4 & 5\\ 6 & -2\end{bmatrix}$ (d) none of these

4. If $(A-2B)=\begin{bmatrix}1 & -2\\ 3 & 0\end{bmatrix}$ and $(2A-3B)=\begin{bmatrix}-2 & 2\\ 3 & -3\end{bmatrix}$ then $B=?$

   (a) $\begin{bmatrix}6 & -4\\ -3 & 3\end{bmatrix}$ (b) $\begin{bmatrix}-4 & 6\\ -3 & -3\end{bmatrix}$ (c) $\begin{bmatrix}4 & -6\\ 3 & -3\end{bmatrix}$ (d) none of these

5. If $(2A-B)=\begin{bmatrix}6 & -6 & 0\\ -4 & 2 & 1\end{bmatrix}$ and $(2B+A)=\begin{bmatrix}3 & 2 & 5\\ -2 & 1 & -7\end{bmatrix}$ then $A=?$

   (a) $\begin{bmatrix}-3 & 2 & 1\\ 2 & 1 & -1\end{bmatrix}$ (b) $\begin{bmatrix}3 & 2 & -1\\ 2 & -1 & 1\end{bmatrix}$

   (c) $\begin{bmatrix}3 & -2 & 1\\ -2 & 1 & -1\end{bmatrix}$ (d) none of these

6. If $2\begin{bmatrix}3 & 4\\ 5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\ 10 & 5\end{bmatrix}$ then

   (a) $(x=-2,\ y=8)$ (b) $(x=2,\ y=-8)$

   (c) $(x=3,\ y=-6)$ (d) $(x=-3,\ y=6)$

7. If $\begin{bmatrix}x-y & 2x-y\\ 2x+z & 3z+w\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 5 & 13\end{bmatrix}$ then

   (a) $z=3,\ w=4$ (b) $z=4,\ w=3$

   (c) $z=1,\ w=2$ (d) $z=2,\ w=-1$

8. If $\begin{bmatrix} x & y \\ 3y & x \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ then

(a) $x = 1, y = 2$ (b) $x = 2, y = 1$ (c) $x = 1, y = 1$ (d) none of these

9. If the matrix $A = \begin{bmatrix} 3-2x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular then $x = ?$

(a) 0 (b) 1 (c) –1 (d) –2

10. If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ then $(A_\alpha)^2 = ?$

(a) $\begin{bmatrix} \cos^2\alpha & \sin^2\alpha \\ -\sin^2\alpha & \cos^2\alpha \end{bmatrix}$ (b) $\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$

(c) $\begin{bmatrix} 2\cos\alpha & 2\sin\alpha \\ -\sin\alpha & 2\cos\alpha \end{bmatrix}$ (d) none of these

11. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ be such that $A + A' = I$, then $\alpha = ?$

(a) π (b) $\frac{\pi}{3}$ (c) π (d) $\frac{2\pi}{3}$

12. If $A = \begin{bmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{bmatrix}$ is singular then $k = ?$

(a) $\frac{16}{3}$ (b) $\frac{34}{5}$ (c) $\frac{33}{2}$ (d) none of these

13. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then adj $A = ?$

(a) $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ (b) $\begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$ (c) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (d) $\begin{bmatrix} -d & -b \\ c & a \end{bmatrix}$

14. If $A = \begin{bmatrix} 2x & 0 \\ x & x \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ then $x = ?$

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) –2

15. If A and B are square matrices of the same order then $(A + B)(A - B) = ?$

(a) $(A^2 - B^2)$ (b) $A^2 + AB - BA - B^2$

(c) $A^2 - AB + BA - B^2$ (d) none of these

16. If A and B are square matrices of the same order then $(A + B)^2 = ?$

(a) $A^2 + 2AB + B^2$ (b) $A^2 + AB + BA + B^2$

(c) $A^2 + 2BA + B^2$ (d) none of these

17. If A and B are square matrices of the same order then $(A - B)^2 = ?$

(a) $A^2 - 2AB + B^2$ (b) $A^2 - AB - BA + B^2$

(c) $A^2 - 2BA + B^2$ (d) none of these

18. If A and B are symmetric matrices of the same order then $(AB - BA)$ is always

(a) a symmetric matrix (b) a skew-symmetric matrix

(c) a zero matrix (d) an identity matrix

19. Matrices A and B are inverses of each other only when

(a) $AB = BA$ (b) $AB = BA = O$ (c) $AB = O$, $BA = I$ (d) $AB = BA = I$

20. For square matrices A and B of the same order, we have adj $(AB) = ?$

(a) (adj A)(adj B) (b) (adj B)(adj A)

(c) $|AB|$ (d) none of these

21. If A is a 3-rowed square matrix and $|A| = 4$ then adj (adj A) = ?

(a) $4A$ (b) $16A$ (c) $64A$ (d) none of these

22. If A is a 3-rowed square matrix and $|A| = 5$ then $|\text{adj } A| = ?$

(a) 5 (b) 25 (c) 125 (d) none of these

23. For any two matrices $A$and B,

(a) $AB = BA$ is always true

(b) $AB = BA$ is never true

(c) sometimes $AB = BA$ and sometimes $AB \neq BA$

(d) whenever AB exists, then BA exists

24. If $A \cdot \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ then $A = ?$

(a) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (d) none of these

25. If A is an invertible square matrix then $|A^{-1}| = ?$

(a) $|A|$ (b) $\frac{1}{|A|}$ (c) 1 (d) 0

26. If A and B are invertible matrices of the same order then $(AB)^{-1} = ?$

(a) $(A^{-1} \times B^{-1})$ (b) $(A \times B^{-1})$ (c) $(A^{-1} \times B)$ (d) $(B^{-1} \times A^{-1})$

27. If A and B are two nonzero square matrices of the same order such that $AB = 0$ then

(a) $|A| = 0$ or $|B| = 0$ (b) $|A| = 0$ and $|B| = 0$

(c) $|A| \neq 0$ and $|B| \neq 0$ (d) none of these

28. If A is a square matrix such that $|A| \neq 0$ and $A^2 - A + 2I = O$ then $A^{-1} = ?$

(a) $(I - A)$ (b) $(I + A)$ (c) $\frac{1}{2}(I - A)$ (d) $\frac{1}{2}(I + A)$

29. If $A = \begin{bmatrix} 1 & \lambda & 2 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ is not invertible then $\lambda = ?$

(a) 2 (b) 1 (c) –1 (d) 0

30. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then $A^{-1} = ?$

(a) A (b) $-A$ (c) adj A (d) $-$adj A

31. The matrix $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is

(a) idempotent (b) orthogonal
(c) nilpotent (d) none of these

32. The matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is

(a) nonsingular (b) idempotent
(c) nilpotent (d) orthogonal

33. If A is singular then $A(\text{adj } A) = ?$

(a) a unit matrix (b) a null matrix
(c) a symmetric matrix (d) none of these

34. For any 2-rowed square matrix A, if $A \cdot (\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$ then the value of $|A|$ is

(a) 0 (b) 8 (c) 64 (d) 4

35. If $A = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix}$ then $|A^{-1}| = ?$

(a) –5 (b) $\frac{-1}{5}$ (c) $\frac{1}{25}$ (d) 25

36. If $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$ and $A^2 + xI = yA$ then the values of x and y are

(a) $x = 6, y = 6$ (b) $x = 8, y = 8$ (c) $x = 5, y = 8$ (d) $x = 6, y = 8$

37. If matrices A and B anticommute then

(a) $AB = BA$ (b) $AB = -BA$ (c) $(AB) = (BA)^{-1}$ (d) none of these

38. If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ then adj $A = ?$

(a) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$ (d) none of these

39. If $A = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$ and B is a square matrix of order 2 such that $AB = I$ then $B = ?$

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$ (d) none of these

40. If A and B are invertible square matrices of the same order then $(AB)^{-1} = ?$

(a) AB^{-1} (b) $A^{-1}B$ (c) $A^{-1}B^{-1}$ (d) $B^{-1}A^{-1}$

41. If $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, then $A^{-1} = ?$

(a) $\begin{bmatrix} \frac{3}{7} & \frac{-1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{3} & \frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$ (d) none of these

42. If $|A| = 3$ and $A^{-1} = \begin{bmatrix} 3 & -1 \\ \frac{-5}{3} & \frac{2}{3} \end{bmatrix}$ then adj $A = ?$

(a) $\begin{bmatrix} 9 & 3 \\ -5 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} -9 & 3 \\ 5 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 9 & -3 \\ 5 & -2 \end{bmatrix}$

43. If A is an invertible matrix and $A^{-1} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ then $A = ?$

(a) $\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{6} \end{bmatrix}$ (c) $\begin{bmatrix} -3 & 2 \\ \frac{5}{2} & \frac{-3}{2} \end{bmatrix}$ (d) none of these

44. If $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ and $f(x) = 2x^2 - 4x + 5$ then $f(A) = ?$

(a) $\begin{bmatrix} 19 & -32 \\ -16 & 51 \end{bmatrix}$ (b) $\begin{bmatrix} 19 & -16 \\ -32 & 51 \end{bmatrix}$

(c) $\begin{bmatrix} 19 & -11 \\ -27 & 51 \end{bmatrix}$ (d) none of these

45. If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ then $A^2 - 4A = ?$

(a) I (b) $5I$ (c) $3I$ (d) 0

46. If A is a 2-rowed square matrix and $|A| = 6$ then $A \cdot \text{adj } A = ?$

(a) $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$ (d) none of these

47. If A is an invertible square matrix and k is a non-negative real number then $(kA)^{-1} = ?$

(a) $k \cdot A^{-1}$ (b) $\frac{1}{k} \cdot A^{-1}$ (c) $-k \cdot A^{-1}$ (d) none of these

48. If $A = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 0 & -2 \\ -2 & -1 & 2 \end{bmatrix}$ then $A^{-1} = ?$

(a) $\begin{bmatrix} 2 & 9 & -8 \\ -2 & 8 & 7 \\ -1 & 5 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 9 & -8 \\ 2 & 8 & 7 \\ -1 & -5 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & -9 & -8 \\ 2 & 8 & 7 \\ -1 & -5 & -4 \end{bmatrix}$ (d) none of these

49. If A is a square matrix then $(A + A')$ is

(a) a null matrix (b) an identity matrix

(c) a symmetric matrix (d) a skew-symmetric matrix

50. If A is a square matrix then $(A - A')$ is

(a) a null matrix (b) an identity matrix

(c) a symmetric matrix (d) a skew-symmetric matrix

51. If A is a 3-rowed square matrix and $|3A| = k|A|$ then $k = ?$

(a) 3 (b) 9 (c) 27 (d) 1

52. Which one of the following is a scalar matrix?

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix}$ (d) none of these

53. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = (A^2 + B^2)$ then

(a) $a = 2,\ b = -3$ (b) $a = -2,\ b = 3$

(c) $a = 1,\ b = 4$ (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (b)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (b)	**7.** (a)	**8** (c)	**9.** (b)	**10.** (b)
11. (b)	**12.** (c)	**13.** (c)	**14.** (c)	**15.** (c)	**16.** (b)	**17.** (b)	**18.** (b)	**19.** (d)	**20.** (b)
21. (a)	**22.** (b)	**23.** (c)	**24.** (c)	**25.** (b)	**26.** (d)	**27.** (b)	**28.** (c)	**29.** (b)	**30.** (c)
31. (c)	**32.** (b)	**33.** (b)	**34.** (b)	**35.** (b)	**36.** (b)	**37.** (b)	**38.** (a)	**39.** (c)	**40.** (d)
41. (b)	**42.** (b)	**43.** (c)	**44.** (b)	**45.** (b)	**46.** (a)	**47.** (b)	**48.** (c)	**49.** (c)	**50.** (d)
51. (c)	**52.** (c)	**53.** (c)							

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

1. $2A=(A+B)+(A-B)$ and $2B=(A+B)-(A-B)$.

2. $2A=\begin{bmatrix}5 & 6\\ -7 & 10\end{bmatrix}-\begin{bmatrix}3 & -2\\ 5 & 6\end{bmatrix}=\begin{bmatrix}2 & 8\\ -12 & 4\end{bmatrix}$.

3. $4A+3X=5B \Rightarrow 3X=(5B-4A) \Rightarrow X=\frac{1}{3}(5B-4A)$.

4. $B=(2A-3B)-2(A-2B)$.

5. $5A=2(2A-B)+(2B+A)$. Then, $A=\frac{1}{5}(5A)$.

6. $[2x+1=5 \Rightarrow x=2]$ and $[8+y=0 \Rightarrow y=-8]$.

7. $(x-y=-1$ and $2x-y=0) \Rightarrow (x=1, y=2)$
$(2x+z=5 \Rightarrow z=3)$ and $(3z+w=13 \Rightarrow w=4)$.

8. Solve $x+2y=3$ and $3y+2x=5$.

9. A is singular $\Leftrightarrow |A|=0$.

12. A is singular $\Leftrightarrow |A|=0$.

14. Use $AA^{-1}=I$.

15. Using distributive law, we have
$(A+B)\cdot(A-B)=A(A-B)+B(A-B)=(A^2-AB+BA-B^2)$.

16. $(A+B)^2=(A+B)\cdot(A+B)=A(A+B)+B(A+B)=(A^2+AB+BA+B^2)$.

17. $(A-B)^2=(A-B)\cdot(A-B)=A(A-B)-B(A-B)=(A^2-AB-BA+B^2)$.

18. Given $A'=A$ and $B'=B$.

$\therefore$ $(AB-BA)'=(AB)'-(BA)'=(B'A'-A'B')=(BA-AB)=-(AB-BA)$

$\therefore$ $(AB-BA)$ is skew symmetric.

19. A and B are inverses of each other only when $AB=BA=I$.

20. $\text{adj}(AB)=(\text{adj}\,B)(\text{adj}\,A)$.

21. $\text{adj}(\text{adj}\,A)=|A|^{(n-2)}=|A|^{(3-2)}\cdot A=|A|\cdot A=4A$.

22. $|\text{adj}\,A|=|A|^{(n-1)}=|A|^2=5^2=25$.

24. Let $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}3 & 2\\ 1 & -1\end{bmatrix}=\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}$. Find a, b, c and d.

25. $AA^{-1} = I \Rightarrow |AA^{-1}| = |I| = 1$

$$\Rightarrow |A|\cdot|A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}.$$

26. $(AB)^{-1} = B^{-1}A^{-1}$.

27. $[AB = 0 \text{ and } A \neq 0, B \neq 0] \Rightarrow |A| = 0 \text{ and } |B| = 0.$

28. $2I = (A - A^2) \Rightarrow 2A^{-1} = A^{-1}A - A^{-1}AA = I - IA = (I - A)$

$\therefore \quad \Rightarrow A^{-1} = \frac{1}{2}(I - A).$

29. A is not invertible $\Leftrightarrow |A| = 0$.

30. $|A| = 1 \Rightarrow A^{-1} = \frac{1}{|A|}\text{adj } A = (\text{adj } A).$

31. $A^2 = O \Rightarrow A$ is nilpotent.

32. $A^2 = A \Rightarrow A$ is idempotent.

33. Given $|A| = 0$. So, $A(\text{adj } A) = |A|\cdot I = 0\cdot I = 0.$

$\therefore \quad A(\text{adj } A)$ is a null matrix.

34. $A\cdot\text{adj } A = |A|\cdot I = 8\times\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = 8I \rightarrow |A| = 8.$

35. $AA^{-1} = I \Rightarrow |AA^{-1}| = |I| \Rightarrow |A|\cdot|A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}.$

$$|A| = \begin{vmatrix}-2 & 3\\1 & 1\end{vmatrix} = (-2 - 3) = (-5) \Rightarrow |A^{-1}| = \frac{-1}{5}.$$

36. $\begin{bmatrix}3 & 1\\7 & 5\end{bmatrix}\begin{bmatrix}3 & 1\\7 & 5\end{bmatrix} + x\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = y\begin{bmatrix}3 & 1\\7 & 5\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}16 & 8\\56 & 32\end{bmatrix} + \begin{bmatrix}x & 0\\0 & x\end{bmatrix} = \begin{bmatrix}3y & y\\7y & 5y\end{bmatrix} \Rightarrow \begin{bmatrix}16 + x & 8\\56 & 32 + x\end{bmatrix} = \begin{bmatrix}3y & y\\7y & 5y\end{bmatrix}.$$

$\therefore \quad y = 8 \text{ and } (16 + x = 3y = 3\times 8 = 24) \Rightarrow x = 8.$

37. A and B anticommute $\Leftrightarrow AB = -BA$.

39. $AB = I \Rightarrow B = A^{-1}$.

40. $(AB)^{-1} = B^{-1}A^{-1}$.

41. $|A| = \begin{vmatrix}2 & -1\\1 & 3\end{vmatrix} = (6 + 1) = 7 \neq 0$

$M_{11} = 3, M_{12} = 1, M_{21} = -1 \text{ and } M_{22} = 2$

$\therefore \quad C_{11} = 3, C_{12} = -1, C_{21} = 1 \text{ and } C_{22} = 2$

$$\Rightarrow \text{Adj } A = \begin{bmatrix}3 & -1\\1 & 2\end{bmatrix}' = \begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{7}\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix} = \begin{bmatrix}\frac{3}{7} & \frac{1}{7}\\-\frac{1}{7} & \frac{2}{7}\end{bmatrix}.$$

42. $A^{-1} = \frac{1}{|A|} \cdot \text{adj } A \rightarrow \text{adj } A = |A| \cdot A^{-1} = 3A^{-1} = \begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix}$.

43. $A = (A^{-1})^{-1}$. So, find the inverse of A^{-1}.

44. $f(A) = 2A^2 - 4A + 5I$.

45. $A^2 - 4A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 5I$.

46. $A \cdot (\text{adj } A) = |A| \cdot I = 6 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$.

47. $(kA)^{-1} = \frac{1}{k} \cdot A^{-1}$ is true.

48. $A^{-1} = \frac{1}{|A|} \cdot \text{adj } A$.

49. A is a square matrix $\Rightarrow$ $(A + A')$ is symmetric.

50. A is a square matrix $\Rightarrow$ $(A - A')$ is skew-symmetric.

51. $|3A| = (3 \times 3 \times 3)|A| = 27 \cdot |A|$.

52. A scalar matrix is a square matrix each of whose non-diagonal elements is 0 and all diagonal elements are equal.

53. $(A + B)^2 = (A^2 + B^2) \Leftrightarrow A^2 + B^2 + AB + BA = (A^2 + B^2) \Leftrightarrow AB = -BA$.

$AB = -BA \Leftrightarrow \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix} = \begin{bmatrix} -a-2 & a+1 \\ -b+2 & b-1 \end{bmatrix}$

Now, $(a + 1 = 2$ and $b - 1 = 3) \Rightarrow (a = 1$ and $b = 4)$.

9. CONTINUITY AND DIFFERENTIABILITY

FUNCTIONS

Real Functions

Let R be the set of all real numbers, and let X and *Y be any two nonempty subsets of R. Then, a rule f which associates to each $x \in X$, a unique real number $f(x) \in Y$ is called a real function from X to Y and we write, $f : X \to Y$.*

$f(x)$ is called the *image* of x or the *value of the function at x*. The sets X and Y are respectively known as the *domain* and the *codomain* of f.

Also, the set $\{f(x) : x \in X\}$ is called the *range* of f.

Clearly, range $(f) \subseteq Y$.

However, if range $(f) = Y$, we say that f is an *onto* function; otherwise f is said to be an *into* function.

If two or more than two elements in X have the same image in Y then f is said to be a *many-one* function.

On the other hand, if different elements in X have different images in Y, we say that f is *one-one.*

Clearly, f is one-one $\Leftrightarrow$ $[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$.

A one-one onto function is called a *one-to-one correspondence.*

REMARK 1 Sometimes a function is described only by a formula and the domain of the function is not explicitly stated. In such cases, the domain of the function is the set of all those real numbers for which the formula is meaningful.

REMARK 2 Usually the domain of a real function is an *interval.* For any two real numbers a and b, where $a < b$, we define

(i) closed interval $[a, b] = \{x \in R : a \leq x \leq b\}$
(ii) open interval $]a, b[= \{x \in R : a < x < b\}$
(iii) right-half open interval $[a, b[= \{x \in R : a \leq x < b\}$
(iv) left-half open interval $]a, b] = \{x \in R : a < x \leq b\}$
(v) $]a, \infty[= \{x \in R : x > a\}$ (vi) $[a, \infty[= \{x \in R : x \geq a\}$
(vii) $]-\infty, a[= \{x \in R : x < a\}$ (viii) $]-\infty, a] = \{x \in R : x \leq a\}$

Sometimes, we write, $R =]-\infty, \infty[$.

Some Important Functions

1. CONSTANT FUNCTION *Let c be a fixed real number. Then, the function defined by $f(x) = c$ for all $x \in R$ is called a constant function c.*

Clearly, dom $(f) = R$ and range $(f) = \{c\}$.

2. IDENTITY FUNCTION *The function defined by $f(x) = x$ for all $x \in R$ is called the identity function.*

Clearly, its domain is R and its range is R.

3. MODULUS FUNCTION *The function defined by*

$$f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$$

is called the modulus function.

Since the modulus of every real number is a unique non-negative real number, so dom $f(x) = R$.

Since $|x|$ is either 0 or a positive real number, we have

range $(f) = \{|x| : x \in R\}$ = set of non-negative real numbers.

4. RECIPROCAL FUNCTION *The function defined by $f(x) = \frac{1}{x}$ is called the reciprocal function.*

Clearly, $\frac{1}{x}$ is not defined when $x = 0$.

$\therefore$ dom $(f) = R - \{0\}$.

Also, $y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}$.

Clearly, x is defined for all real numbers y except when $y = 0$.

$\therefore$ range $(f) = R - \{0\}$.

5. SIGNUM FUNCTION *This function is defined by* $f(x) = \begin{cases} \frac{|x|}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0. \end{cases}$

Thus, we have $f(x) = \begin{cases} 1, & \text{when } x > 0 \\ 0, & \text{when } x = 0 \\ -1, & \text{when } x < 0. \end{cases}$

Clearly, its domain is R and range $= \{-1, 0, 1\}$.

6. SQUARE-ROOT FUNCTION Let $f(x) = +\sqrt{x}$.

We know that the negative real numbers do not have real square roots. So, $f(x)$ is not defined when x is a negative real number.

$\therefore$ dom (f) = set of all non-negative real numbers $= [0, \infty[$.

Clearly, range $(f) = \{+\sqrt{x} : x \in [0, \infty[\} = [0, \infty[$.

7. STEP FUNCTION OR THE GREATEST INTEGER FUNCTION

If $x \in R$ then $[x]$ is defined as the greatest integer not exceeding x.

For example, we have

$[2.01] = 2$; $[2.9] = 2$; $[-1.3] = -2$; $[3] = 3$ and $[-1] = -1$, etc.

Now, if we consider $f(x) = [x]$ then clearly for each $x \in R$, $[x]$ is defined.

So dom $(f) = R$.

By definition, $[x]$ is an integer.

So, range $(f) = \{[x] : x \in R\}$ = set of all integers.

EXAMPLE *Find a set of all real numbers x such that* $[x] = 2$.

SOLUTION Clearly, for all x such that $2 \le x < 3$, we have $f(x) = [x] = 2$.

$\therefore$ required set $= \{x \in R : 2 \le x < 3\} = [2, 3[$.

8. SMALLEST INTEGER FUNCTION (OR CEILING FUNCTION) *For any real number x, we define* $\lceil x \rceil$ *as the smallest integer greater than or equal to x.*

For example,

$$\lceil 6.3 \rceil = 7,\ \lceil 7.01 \rceil = 8,\ \lceil -6.1 \rceil = -6,\ \lceil -2.9 \rceil = -2,\ \lceil -3 \rceil = -3,\ \lceil 5 \rceil = 5.$$

The function $f : R \in R : f(x) = \lceil x \rceil$, $x \in R$ is called the smallest integer function or ceiling function.

Clearly, domain $(f) = R$ and range $(f) = I$.

9. POLYNOMIAL FUNCTION *A function of the form*

$$p(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n,$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ *are real numbers,* $a_0 \ne 0$ *and n is a non-negative integer, is called a polynomial function of degree n.*

Polynomials of degree 1, 2, 3 and 4 are respectively called *linear, quadratic, cubic* and *biquadratic* polynomials.

Thus, (i) $f(x) = ax + b$, $a \ne 0$, is a linear polynomial.

(ii) $f(x) = ax^2 + bx + c$, $a \ne 0$, is a quadratic polynomial.

(iii) $f(x) = ax^3 + bx^2 + cx + d$, $a \ne 0$, is a cubic polynomial.

10. RATIONAL FUNCTION *A function of the form* $f(x) = \frac{p(x)}{q(x)}$, *where* $p(x)$ *and* $q(x)$ *are polynomials and* $q(x) \ne 0$, *is called a rational function.*

Thus, $f(x) = \left(\frac{x^2+1}{x^3-2x+5}\right)$ is a rational function, where $x^3 - 2x + 5 \ne 0$.

Graphs

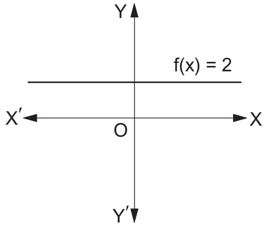
Graph of a Function

For a given function $f(x)$, *the aggregate of the points* $\{x, f(x)\}$ *is called the graph or the curve representing the function.*

In practice, we plot some of the points and join them freehand to obtain the graph.

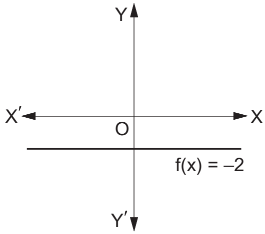
EXAMPLE 1 *Draw the graphs of the constant functions*
(i) $f(x) = 2$ *(ii)* $f(x) = 0$ *(iii)* $f(x) = -2$

SOLUTION (i) When $f(x) = 2$ for all $x \in R$, some of the points on the graph may be taken as $(0, 2), (-1, 2), (1, 2), (-2, 2)$, $(2, 2)$, etc. Joining these points, we obtain a line $y = 2$, drawn parallel to the x-axis at a distance of 2 units from it, as the required graph.



(ii) The graph of the function $f(x) = 0$ is the line $y = 0$, i.e., the x-axis.

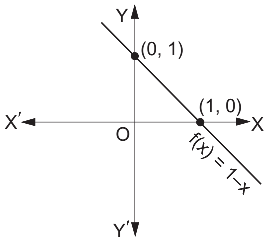
(iii) The graph of the function $f(x) = -2$ is the line $y = -2$, drawn parallel to the x-axis at a distance of 2 units below the x-axis.



EXAMPLE 2 *Draw the graphs of the linear functions*
(i) $f(x) = 1 - x$ *(ii)* $f(x) = 2x + 1$

SOLUTION (i) When $f(x) = 1 - x$, some of the points on the graph are $(0, 1)$, $(1, 0), (2, -1), (-1, 2)$, etc.

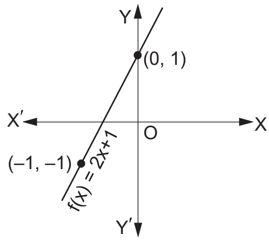
Joining these points, we get a line as the graph of the function.



(ii) Let $f(x) = 2x + 1$.

Some of the points on the graph are $(0, 1), (1, 3), (-1, -1), (2, 5)$, etc.

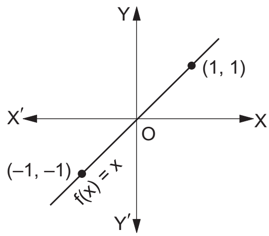
Joining these points, we obtain a line as the graph.



REMARK The graph of a linear function is a straight line.

EXAMPLE 3 *Draw the graph of the identity function* $f(x) = x$.

SOLUTION $f(x) = x$ is clearly a linear function whose graph must be a line. Plotting the points $(0, 0), (1, 1), (-1, -1)$, etc., and joining them, we get the required graph.

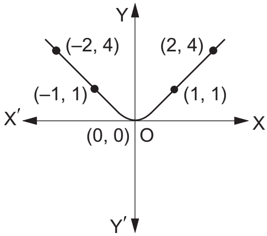


EXAMPLE 4 *Draw the graphs of the polynomial functions*

(i) $f(x) = x^2$ *(ii)* $f(x) = 1 - x^2$

(iii) $f(x) = x^3 - x$

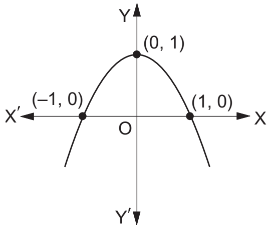
SOLUTION (i) The function $f(x) = x^2$ is a quadratic function. Some of the points on the graph are (0, 0), (1, 1), (−1, 1), (−2, 4), (2, 4), (−3, 9), (3, 9), etc. Joining these points, we get a parabola as the graph.



(ii) $f(x) = 1 - x^2$ is also a quadratic function. Some of the points on the graph are

(0, 1), (1, 0), (−1, 0), (2, −3), (−2, −3), (−3, −8), (3, −8), etc.

Joining these points, we obtain a parabola as its graph.

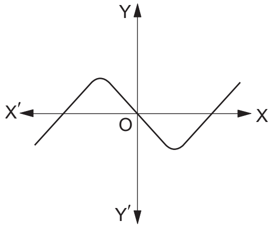


(iii) Let $f(x) = x^3 - x$.

This is a cubic function.

Some of the points on the graph are (0, 0), (−1, 0), (1, 0), (−0.5, 0.375), (0.5, −0.375), (2, 6), (−2, −6), etc.

Joining these points, we obtain the required graph.



REMARK It may be observed here that whenever (x, y) is a point on the graph then

$(-x, -y)$ is also a point on the graph. So, *the graph is symmetrical about the origin.*

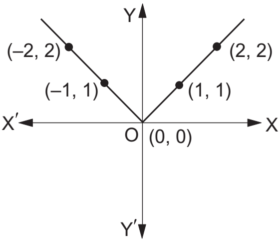
This is an important property possessed by the graph of an odd function.

EXAMPLE 5 *Draw the graph of the modulus function* $f(x) = |x|$.

SOLUTION $f(x) = |x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$

Some of the points on the graph are (0, 0), (−1, 1), (−2, 2), (1, 1), (2, 2), etc.

Joining these points, we get the required graph.

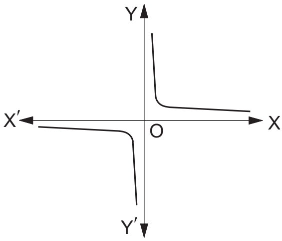


REMARK It may be observed here that whenever (x, y) is a point on the graph, then $(-x, y)$ is also a point of it. Thus, *the graph is symmetrical about the y-axis*. This is an important property possessed by the graphs of even functions.

EXAMPLE 6 *Draw the graph of the reciprocal function* $f(x) = \frac{1}{x}$.

SOLUTION Clearly, $f(x) = \frac{1}{x}$ is not defined at $x = 0$. Some points on the graph are $(1, 1)$, $(-1, -1)$, $(1/2, 2)$, $(2, 1/2)$, $(-1/2, -2)$, $(-2, -1/2)$, $(1/3, 3)$, $(-1/3, -3)$, $(3, 1/3)$, etc.

Joining these points, we get the required graph. Since $f(x) = \frac{1}{x}$ is an odd function, it is symmetrical about the origin.



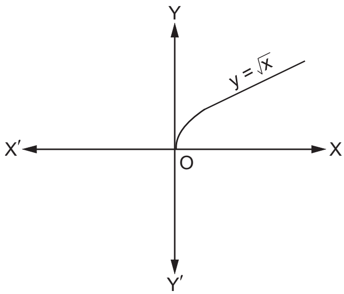
EXAMPLE 7 *Draw the graph of the square-root function* $f(x) = \sqrt{x}$.

SOLUTION Let f be a real-valued function which associates to each non-negative real number x, its non-negative square root.

Then, $f : R_0^+ \to R_0^+ : f(x) = \sqrt{x}$, is called the square-root function.

Domain $f(x) = R_0^+$, and range $(f) = R_0^+$.

Some of the points on the graph are (0, 0), (1, 1), (2, 1.4), (3, 1.7), (4, 2), (5, 2.2), etc. Joining these points, we get the required graph.



EXAMPLE 8 *Draw the graph of the rational function* $f(x) = \frac{x^2 - 1}{x - 1}$.

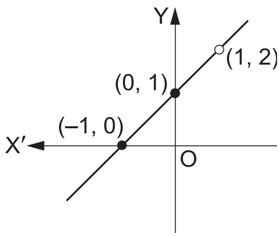
SOLUTION Let $f(x) = \frac{x^2 - 1}{x - 1}$.

Now, $f(1) = \frac{0}{0}$, which is meaningless.

So, the function is not defined at $x = 1$.

Also, when $x \neq 1$, we have

$$f(x) = \left(\frac{x^2 - 1}{x - 1}\right) = (x + 1).$$



This being a linear function, its graph is a straight line.

Some of the points on the graph are (0, 1), (–1, 0), (2, 3), (3, 4), (–2, –1), (–3, –2), etc.

Joining these points, we obtain the required graph. Clearly, the point (1, 2) does not lie on the graph. So, *it is a broken graph,* and we shall say that the given function is discontinuous at $x = 1$.

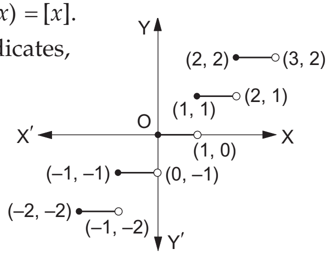
EXAMPLE 9 *Draw the graph of the step function* $f(x) = [x]$.

SOLUTION As the definition of the function indicates,

for all x such that $-2 \le x < -1$,
we have $f(x) = -2$;

for all x such that $-1 \le x < 0$,
we have $f(x) = -1$;

for all x such that $0 \le x < 1$,
we have $f(x) = 0$;



for all x such that $1 \le x < 2$, we have $f(x) = 1$, and so on,

$$\text{i.e., } f(x) = \begin{cases} -2 \text{ when } x \in [-2, -1[\\ -1 \text{ when } x \in [-1, 0[\\ 0 \text{ when } x \in [0, 1[\\ 1 \text{ when } x \in [1, 2[\\ \text{and so on.} \end{cases}$$

Clearly, the function jumps at the points (–1, –2), (0, –1), (1, 0), (2, 1), etc.

In other words, *the given function is discontinuous at each integral value of x.*

EXAMPLE 10 *Draw the graph of the smallest integer function* $f(x) = \lceil x \rceil$.

SOLUTION As the definition of the function suggests,

for all x such that $-3 < x \le -2$, we have $f(x) = -2$;

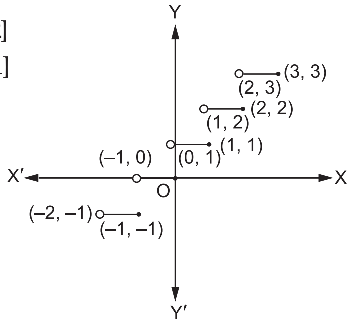
for all x such that $-2 < x \le -1$, we have $f(x) = -1$;

for all x such that $-1 < x \le 0$, we have $f(x) = 0$;

for all x such that $0 < x \le 1$, we have $f(x) = 1$;

and so on.

$$\text{i.e., } f(x) = \begin{cases} -2 \text{ when } x \in]-3, -2] \\ -1 \text{ when } x \in]-2, -1] \\ 0 \text{ when } x \in]-1, 0] \\ 1 \text{ when } x \in]0, 1] \\ 2 \text{ when } x \in]1, 2] \\ 3 \text{ when } x \in]2, 3] \\ \text{and so on.} \end{cases}$$



Plotting these points, we can get the required graph. The function jumps at the points (–2, –1), (–1, 0), (0, 1), (1, 2), etc., or is *discontinuous at each integral value of x.*

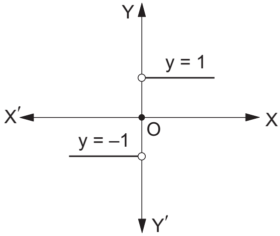
EXAMPLE 11 *Draw the graph of the signum function* $f(x) = \begin{cases} \dfrac{|x|}{x} & when \quad x \neq 0 \\ 0 & when \quad x = 0. \end{cases}$

SOLUTION Clearly, (0, 0) is a point on the graph. Now, when $x > 0$, we have $|x| = x$, and so in this case, we have, $f(x) = 1$, i.e., $f(x) = 1$ for all values of $x > 0$.

And, when $x < 0$, we have $|x| = -x$ and therefore,

$f(x) = -1$ for all values of $x < 0$.

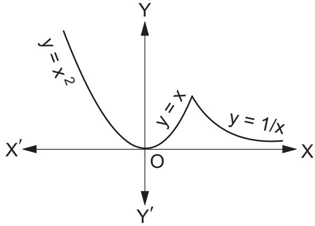
Hence the graph may be drawn, as shown in the adjoining figure.



Clearly, *the function is broken* (i.e., *it is discontinuous*) *at each of the points* $x = -1$, 0 and 1.

EXAMPLE 12 *Draw the graph of the function* $f(x) = \begin{cases} x^2, & when \quad x < 0 \\ x, & when \quad 0 \leq x \leq 1 \\ 1/x, & when \quad 1 \leq x < \infty. \end{cases}$

SOLUTION Here, the graph consists of three parts. Some of the points of the graph are (–3, 9), (–2, 4), (–1, 1), (0, 0), $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{3}{4}, \frac{3}{4}\right)$, (1, 1), $\left(2, \frac{1}{2}\right)$, $\left(3, \frac{1}{3}\right)$, etc.



And, the graph may now be drawn, as shown in the adjoining figure.

EXAMPLE 13 *Draw the graph of the function* $f(x) = |x| + |x - 1|$.

SOLUTION Let us consider the following cases.

Case I *When* $x < 0$

In this case, $(x - 1) < 0$.

$\therefore \quad |x| = -x$ and $|x - 1| = -(x - 1) = 1 - x$.

Consequently, $|x| + |x - 1| = -x + 1 - x = 1 - 2x$.

Case II *When* $0 \leq x \leq 1$

In this case, $|x| = x$ and $|x - 1| = -(x - 1) = 1 - x$.

$\therefore \quad |x| + |x - 1| = x + 1 - x = 1$.

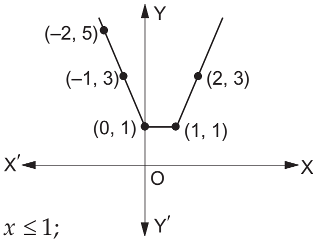
Case III *When* $x > 1$

In this case, $|x| = x$ and $|x - 1| = x - 1$.

$\therefore \quad |x| + |x - 1| = x + (x - 1) = (2x - 1)$.

Thus, we may define the above function as

$$f(x) = \begin{cases} 1-2x, & \text{when } x < 0 \\ 1, & \text{when } 0 \le x \le 1 \\ 2x-1, & \text{when } x > 1. \end{cases}$$



So, we have

(i) a linear function $1 - 2x$ when $x < 0$;

(ii) a constant function 1 when $0 \le x \le 1$;

(iii) a linear function $2x - 1$ when $x > 1$.

The corresponding points on these parts of the graph are $(-1, 3), (-2, 5), (0, 1), (1, 1), (2, 3), (3, 5)$, etc.

Joining these points, we obtain the graph as shown.

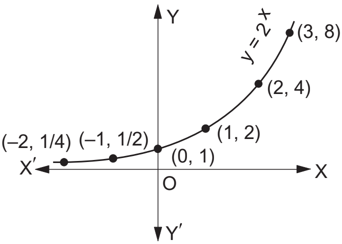
EXAMPLE 14 *Draw the graph of the exponential function:*

(i) $f(x) = 2^x$ *(ii)* $f(x) = \left(\frac{1}{3}\right)^x$

SOLUTION (i) Let $f(x) = 2^x$.

Some of the points on the graph are

$$(0, 1), (1, 2), (2, 4), (3, 8), \left(-1, \frac{1}{2}\right), \left(-2, \frac{1}{4}\right), \left(-3, \frac{1}{8}\right), \text{etc.}$$

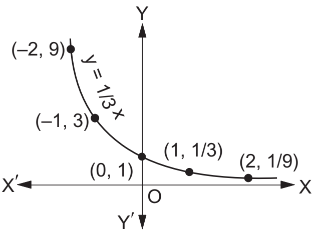


And so the graph takes the form, shown in the adjoining figure.

It may be observed here that the *given function is strictly increasing*. Also, as the value of x decreases, the corresponding value of the function decreases, and therefore, on the left-hand side of the y-axis, the curve comes closer and closer to the x-axis.

REMARK This is the case of the exponential function a^x, where $a > 1$.

(ii) Let $f(x) = \left(\frac{1}{3}\right)^x = \frac{1}{3^x}$.



Some of the points on the graph are $(-2, 9)$, $(-1, 3), (0, 1), \left(1, \frac{1}{3}\right), \left(2, \frac{1}{9}\right)$, etc.

Joining these points, we obtain the graph as shown. It follows from the graph that *the given function is strictly decreasing.*

On the RHS of the y-axis, the curve comes closer and closer to the x-axis.

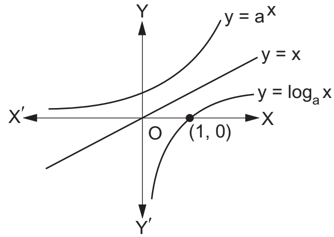
REMARK This is the case of the exponential function a^x, where $0 < a < 1$.

EXAMPLE 15 *Draw the graphs of the logarithmic functions*

(i) $\log_a x$, *when* $a > 1$ *(ii)* $\log_a x$, *when* $0 < a < 1$.

SOLUTION (i) We know that when $a > 1$, the function a^x is strictly increasing, i.e., different values of x give different values of a^x. Also, the range of this function is R.

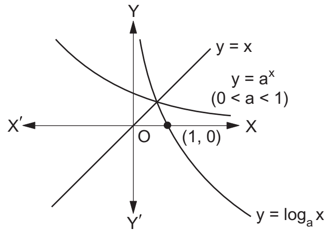
So, the function $f(x) = a^x$ is one-one onto and therefore invertible.



Its graph is of the form shown in the adjoining figure. The graph of the function $g(x) = \log_a x$ is the reflection of the graph of $f(x) = a^x$ in the line $y = x$.

It may be noted that the graph passes through (1, 0).

(ii) We know that when $0 < a < 1$, the function a^x is strictly decreasing, i.e., different values of x give different values of a^x. So, the function is one-one. Also, its range is R. So, it is onto. Thus, the function a^x is invertible.

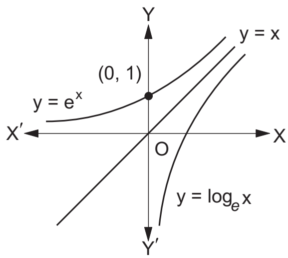


Its inverse is the log function reflected by the line $y = x$, as shown. The graph clearly passes through (1, 0).

EXAMPLE 16 *On the same scale draw the graphs of* e^x *and* $\log_e x$.

SOLUTION Since e lies between 2 and 3, it follows that $e > 1$. So, it is a particular case of a^x, where $a > 1$. The function e^x is strictly increasing. Also, its range is R. So, the function is one-one onto and therefore invertible.

The graph passes through the point (0, 1) and comes closer and closer to the x-axis and the values of x decrease. Thus, the graph of e^x may be drawn as shown in the figure. Its reflection in the line $y = x$ gives the graph of $\log_e x$.

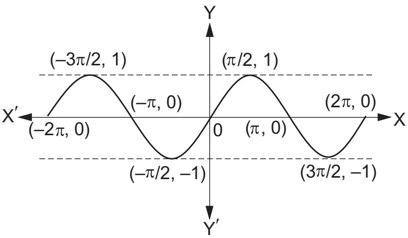


Graphs of Trigonometric Functions

1. GRAPH OF A SINE FUNCTION We know that a sine function is periodic with period 2π. So, we have to sketch the graph in the interval $[0, 2\pi]$ and then we may complete the graph by repeating it over intervals, each of length 2π. We first draw it in the interval $[0, \pi/2]$. We know that it is strictly increasing in this interval. Also, we may use the table, given below.

x	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = 0.707$	$\frac{\sqrt{3}}{2} = 0.87$	1

Thus, we may draw the graph in the interval $\left[0, \frac{\pi}{2}\right]$.



Now, we know that $\sin(\pi - x) = \sin x$. So, we may get some other values of the function in the interval $[\pi/2, \pi]$ Moreover, the function is strictly decreasing in this interval. Thus, we may draw it in the interval $[\pi/2, \pi]$. Finally, we draw it in $(\pi, 2\pi)$, using the fact that $\sin(\pi + x) = -\sin x$.

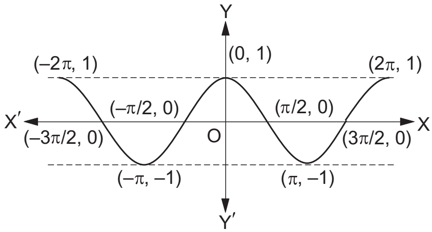
The graph may be completed now by making repetitions over each interval of length 2π.

2. GRAPH OF A COSINE FUNCTION We know that a cosine function is periodic with period 2π. By making use of the table given below, we may first draw it in the interval $[0, \pi/2]$, keeping in view that it is strictly decreasing in this interval.

x	$0°$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos x$	1	$\frac{\sqrt{3}}{2} = 0.87$	$\frac{1}{\sqrt{2}} = 0.707$	$\frac{1}{2}$	0

Now, $\cos(\pi - x) = -\cos x$, so, we may draw the graph in the interval $[\pi/2, \pi]$. It is strictly decreasing in this interval also. Further, making use of $\cos(\pi + x) = \cos x$, we may draw the graph from π to 2π, as shown in the figure.

The graph may now be completed by making repetitions over each interval of length 2π.



3. GRAPH OF A TANGENT FUNCTION We know that a tangent function is a periodic function with period π. Therefore, it is enough to draw the graph over an interval of length π. The complete graph then consists of infinitely many repetitions of the same to the left as well as to the right.

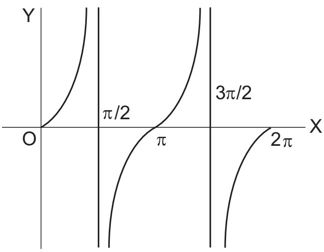
Since $\tan(-x) = -\tan x$, therefore, if $(x, \tan x)$ is any point on the graph then $(-x, -\tan x)$ is also a point on the graph. Thus, we can say that the graph of $y = \tan x$ is symmetrical in opposite quadrants.

Some of the values of the function are given below:

x	$0°$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	0	$\frac{\sqrt{3}}{3} = 0.58$	1	$\sqrt{3} = 1.73$	undefined

x	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\tan x$	$-\sqrt{3} = -1.73$	-1	$-\frac{\sqrt{3}}{3} = -0.58$	0

The graph may thus be drawn as shown below.



Similarly, the graphs of cosec x, sec x and cot x may be drawn.

Continuity

CONTINUITY AT A POINT *A real function $f(x)$ is said to be continuous at a point a of its domain if $\lim_{x \to a} f(x)$ exists and equals $f(a)$.*

Thus, $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = f(a).$$

If $f(x)$ is not continuous at a point, it is said to be *discontinuous* at that point.

REMARK $f(x)$ is discontinuous at $x = a$ in each of the following cases:

(i) $f(a)$ is not defined

(ii) $\lim_{x \to a} f(x)$ does not exist

(iii) $\lim_{x \to a} f(x) \neq f(a)$ {*removable discontinuity*}

SOLVED EXAMPLES

EXAMPLE 1 *Show that $f(x) = x^3$ is continuous at $x = 2$.*

SOLUTION We have $f(2) = 2^3 = 8$;

$$\lim_{x \to 2+} f(x) = \lim_{h \to 0} (2+h)^3 = \lim_{h \to 0} (8 + h^3 + 12h + 6h^2) = 8;$$

$$\lim_{x \to 2-} f(x) = \lim_{h \to 0} (2-h)^3 = \lim_{h \to 0} (8 - h^3 - 12h + 6h^2) = 8.$$

$$\therefore \quad \lim_{x \to 2+} f(x) = \lim_{x \to 2-} f(x) = f(2).$$

Hence, $f(x)$ is continuous at $x = 2$.

EXAMPLE 2 *Show that $f(x) = [x]$ is not continuous at $x = n$, where n is an integer.*

SOLUTION We have $f(n) = [n] = n$;

$$\lim_{x \to n+} f(x) = \lim_{h \to 0} f(n+h) = \lim_{h \to 0} [n+h] = n \quad \{\because \ [n+h] = n\};$$

$$\lim_{x \to n-} f(x) = \lim_{h \to 0} f(n-h) = \lim_{h \to 0} [n-h] = (n-1) \ \{\because \ [n-h] = (n-1)\}.$$

Thus, $\lim_{x \to n+} f(x) \neq \lim_{x \to n-} f(x)$ and therefore, $\lim_{x \to n} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = n$.

EXAMPLE 3 *Show that the function* $f(x) = \begin{cases} x, & \text{if } x \text{ is an integer;} \\ 0, & \text{if } x \text{ is not an integer} \end{cases}$

is discontinuous at each integral value of x.

SOLUTION Let $x = n$, where n is an integer. Then, $f(n) = n$;

$$\lim_{x \to n+} f(x) = \lim_{h \to 0} f(n+h) = 0$$
$$[\because \ (n+h) \text{ is not an integer} \Rightarrow f(n+h) = 0];$$

and $$\lim_{x \to n-} f(x) = \lim_{h \to 0} f(n-h) = 0$$
$$[\because \ (n-h) \text{ is not an integer} \Rightarrow f(n-h) = 0].$$

$\therefore$ $$\lim_{x \to n+} f(x) = \lim_{x \to n-} f(x) = 0.$$

So, $$\lim_{x \to n} f(x) = 0 \neq f(n).$$

Hence, $f(x)$ is discontinuous at $x = n$.

EXAMPLE 4 *Discuss the continuity of the function $f(x)$ at $x = 0$, if*

$$f(x) = \begin{cases} 2x - 1, & x < 0 \\ 2x + 1, & x \geq 0. \end{cases}$$ **[CBSE 2002]**

SOLUTION Clearly, $f(0) = (2 \times 0 + 1) = 1.$

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$
$$= \lim_{h \to 0} [2(0+h) + 1] = \lim_{h \to 0} (2h+1) = 1.$$
$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$$
$$= \lim_{h \to 0} [2(0-h) - 1)] = \lim_{h \to 0} (-2h-1) = -1.$$

Thus, $\lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$ and therefore, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE 5 *Show that the function* $f(x) = \begin{cases} 3x - 2, & \textit{when } x \leq 0 \\ x + 1, & \textit{when } x > 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION We have, $f(0) = (3 \times 0 - 2) = -2.$

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$
$$= \lim_{h \to 0} (h+1) = 1.$$
$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$$
$$= \lim_{h \to 0} [3(-h) - 2] = \lim_{h \to 0} (-3h - 2) = -2.$$

$\therefore$ $\lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$ and therefore, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE 6 *Show that the function* $f(x) = \begin{cases} \dfrac{x}{|x|}, & when \;\; x \neq 0 \\ 1, & when \;\; x = 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION It is being given that $f(0) = 1$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{h}{|h|} = \lim_{h \to 0} \frac{h}{h} = 1.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{-h}{|h|} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

$\therefore \quad \lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$.

So, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE 7 *Examine the continuity of the function*

$$f(x) = \begin{cases} \dfrac{|\sin x|}{x}, & x \neq 0 \\ 1, & x = 0 \;\; at \;\; x = 0. \end{cases}$$ **[CBSE 2004]**

SOLUTION We have $f(0) = 1$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{|\sin(0+h)|}{(0+h)} = \lim_{h \to 0} \frac{|\sin h|}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{|\sin(-h)|}{-h} = \lim_{h \to 0} \frac{|-\sin h|}{-h} = \lim_{h \to 0} \frac{\sin h}{-h} = -1.$$

$\therefore \quad \lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$. So, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE 8 *Show that the function* $f(x) = 2x - |x|$ *is continuous at* $x = 0$.
[CBSE 2002]

SOLUTION We have, $f(0) = (2 \times 0) - |0| = 0$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} (2h - |h|) = \lim_{h \to 0} (2h - h) = \lim_{h \to 0} h = 0.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \{2(-h) - |-h|\} = \lim_{h \to 0} (-2h - h) = \lim_{h \to 0} (-3h) = 0.$$

Thus, $\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = 0$ and therefore, $\lim_{x \to 0} f(x) = 0$.

$\therefore \quad \lim_{x \to 0} f(x) = f(0) = 0.$

Hence, $f(x)$ is continuous at $x = 0$.

EXAMPLE 9 *Let* $f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \ge 3. \end{cases}$

Show that $f(x)$ is continuous at $x = -3$ but discontinuous at $x = 3$.

SOLUTION We have $f(-3) = |-3| + 3 = (3 + 3) = 6$.

$$\lim_{x \to (-3)+} f(x) = \lim_{h \to 0} f(-3 + h)$$
$$= \lim_{h \to 0} \{-2(-3 + h)\} = \lim_{h \to 0} (6 - 2h) = 6.$$

$$\lim_{x \to (-3)-} f(x) = \lim_{h \to 0} f(-3 - h)$$
$$= \lim_{h \to 0} (|-3 - h| + 3) = \lim_{h \to 0} \{|-(3 + h)| + 3\}$$
$$= \lim_{h \to 0} \{(3 + h) + 3\} = 6.$$

$\therefore \quad \lim_{x \to (-3)+} f(x) = \lim_{x \to (-3)-} f(x) = 6.$ So, $\lim_{x \to (-3)} f(x) = 6.$

Thus, $\lim_{x \to (-3)} f(x) = f(-3) = 6.$

Hence, $f(x)$ is continuous at $x = -3$.

Now, $f(3) = (6 \times 3 + 2) = 20.$

$$\lim_{x \to 3-} f(x) = \lim_{h \to 0} f(3 - h)$$
$$= \lim_{h \to 0} \{-2(3 - h)\} = \lim_{h \to 0} (-6 + 2h) = -6.$$

Thus, $f(3) \ne \lim_{x \to 3-} f(x)$.

Hence, $f(x)$ is discontinuous at $x = 3$.

EXAMPLE 10 *Find the value of k for which*

$$f(x) = \begin{cases} kx + 5, & \text{when } x \le 2 \\ x - 1, & \text{when } x > 2 \end{cases}$$

is continuous at $x = 2$. **[CBSE 2002]**

SOLUTION We have, $f(2) = (k \times 2 + 5) = (2k + 5).$

$$\lim_{x \to 2+} f(x) = \lim_{h \to 0} f(2 + h)$$

$$= \lim_{h \to 0} \{(2+h) - 1\} = \lim_{h \to 0} (1+h) = 1.$$

$$\lim_{x \to 2-} f(x) = \lim_{h \to 0} f(2-h)$$

$$= \lim_{h \to 0} \{k(2-h)+5\} = \lim_{h \to 0} \{(2k+5) - kh\} = (2k+5).$$

Now, $\lim_{x \to 2} f(x)$ exists only when $2k+5 = 1$, i.e., when $k = -2$.

When $k = -2$, we have $\lim_{x \to 2} f(x) = f(2) = 1$.

Hence, $f(x)$ is continuous at $x = 2$ when $k = -2$.

EXAMPLE 11 *If the following function $f(x)$ is continuous at $x = 0$, find the value of k:*

$$f(x) = \begin{cases} \dfrac{1 - \cos 2x}{2x^2}, & x \neq 0 \\ k, & x = 0. \end{cases}$$ **[CBSE 2008]**

SOLUTION We have, $f(0) = k$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{(1 - \cos 2h)}{2h^2} = \lim_{h \to 0} \frac{2\sin^2 h}{2h^2} = \lim_{h \to 0} \left(\frac{\sin h}{h}\right)^2$$

$$= \left\{\lim_{h \to 0} \left(\frac{\sin h}{h}\right)\right\}^2 = (1)^2 = 1.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \left\{\frac{1 - \cos 2(-h)}{2(-h)^2}\right\} = \lim_{h \to 0} \frac{\{1 - \cos(-2h)\}}{2h^2}$$

$$= \lim_{h \to 0} \frac{(1 - \cos 2h)}{2h^2} = \lim_{h \to 0} \frac{2\sin^2 h}{2h^2} = \lim_{h \to 0} \left(\frac{\sin h}{h}\right)^2$$

$$= \left\{\lim_{h \to 0} \frac{\sin h}{h}\right\}^2 = 1^2 = 1.$$

Since $f(x)$ is continuous at $x = 0$, we must have

$$f(0) = \lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) \Rightarrow k = 1.$$

EXAMPLE 12 *For what value of k is the following function continuous at $x = 0$?*

$$f(x) = \begin{cases} \dfrac{1 - \cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$$ **[CBSE 2014C]**

SOLUTION We have, $f(0) = k$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{(1-\cos 4h)}{8h^2} = \lim_{h \to 0} \frac{2\sin^2 2h}{8h^2} = \lim_{h \to 0} \left(\frac{\sin 2h}{2h}\right)^2$$

$$= 1^2 = 1.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{\{1-\cos 4(-h)\}}{8(-h)^2} = \lim_{h \to 0} \frac{\{1-\cos(-4h)\}}{8h^2} = \lim_{h \to 0} \frac{(1-\cos 4h)}{8h^2}$$

$[\because \cos(-\theta) = \cos\theta]$

$$= \lim_{h \to 0} \frac{2\sin^2 2h}{8h^2} = \lim_{h \to 0} \left(\frac{\sin 2h}{2h}\right)^2$$

$$= 1^2 = 1.$$

$$\therefore \quad \lim_{x \to 0} f(x) = 1.$$

For continuity at $x = 0$, we must have

$$f(0) = \lim_{x \to 0} f(x) \Rightarrow k = 1 \quad \left[\because f(0) = k \text{ and } \lim_{x \to 0} f(x) = 1\right].$$

EXAMPLE 13 *Given that*

$$f(x) = \begin{cases} \dfrac{(1-\cos 4x)}{x^2}, & \text{if } x < 0 \\ a, & \text{if } x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{if } x > 0. \end{cases}$$

If $f(x)$ is continuous at $x = 0$, find the value of a. **[CBSE 2006C, '12C, '13C]**

SOLUTION We have, $f(0) = a$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \left\{\frac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4}\right\}$$

$$= \lim_{h \to 0} \left\{\frac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4} \times \frac{\sqrt{16+\sqrt{h}}+4}{\sqrt{16+\sqrt{h}}+4}\right\}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}\cdot\{\sqrt{16+\sqrt{h}}+4\}}{\{(16+\sqrt{h})-16\}} = \lim_{h \to 0} \frac{\sqrt{h}\cdot\{\sqrt{16+\sqrt{h}}+4\}}{\sqrt{h}}$$

$$= \lim_{h \to 0} \{\sqrt{16+\sqrt{h}}+4\} = (4+4) = 8.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \frac{\{1 - \cos 4(-h)\}}{(-h)^2} = \lim_{h \to 0} \frac{\{1 - \cos(-4h)\}}{h^2}$$

$$= \lim_{h \to 0} \frac{(1 - \cos 4h)}{h^2} = \lim_{h \to 0} \frac{2\sin^2 2h}{h^2}$$

$$= (2 \times 4) \cdot \lim_{h \to 0} \frac{(\sin^2 2h)}{(2h)^2} = 8 \cdot \lim_{h \to 0} \left(\frac{\sin 2h}{2h}\right)^2$$

$$= (8 \times 1^2) = 8.$$

$$\therefore \quad \lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = 8 \Rightarrow \lim_{x \to 0} f(x) = 8.$$

But, by continuity of f at $x = 0$, we have

$$\lim_{x \to 0} f(x) = f(0) \Rightarrow a = 8 \qquad [\because f(0) = a \text{ and } \lim_{x \to 0} f(x) = 8].$$

Hence, $a = 8$.

EXAMPLE 14 *If the following function $f(x)$ is continuous at $x = 0$, find the values of a, b and c.*

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{if } x < 0 \\ c, & \text{if } x = 0 \\ \dfrac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}}, & \text{if } x > 0. \end{cases}$$

[CBSE 2008]

SOLUTION We have, $f(0) = c$.

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{\sin(a+1)(-h) + \sin(-h)}{(-h)} = \lim_{h \to 0} \frac{-\{\sin(a+1)h + \sin h\}}{-h}$$

$$= \lim_{h \to 0} \frac{\sin(a+1)h + \sin h}{h} = \lim_{h \to 0} \frac{2\sin\left(\frac{a}{2}+1\right)h \cdot \cos\frac{ah}{2}}{h}$$

$$= 2 \cdot \lim_{h \to 0} \left\{ \frac{\sin\left(\frac{a}{2}+1\right)h}{\left(\frac{a}{2}+1\right)h} \cdot \left(\frac{a}{2}+1\right) \cdot \cos\frac{ah}{2} \right\}$$

$$= 2\left(\frac{a}{2}+1\right)\cdot \lim_{h\to 0} \frac{\sin\left(\frac{a}{2}+1\right)h}{\left(\frac{a}{2}+1\right)h}\cdot \lim_{h\to 0} \cos\frac{ah}{2}$$

$$= (a+2)\times 1\times 1 = (a+2).$$

$$\lim_{x\to 0+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0}\left\{\frac{[\sqrt{h+bh^2}-\sqrt{h}]}{bh^{3/2}}\times\frac{[\sqrt{h+bh^2}+\sqrt{h}]}{[\sqrt{h+bh^2}+\sqrt{h}]}\right\}$$

$$= \lim_{h\to 0}\frac{(h+bh^2-h)}{bh^{3/2}(\sqrt{h+bh^2}+\sqrt{h})} = \lim_{h\to 0}\frac{bh^2}{bh^2(\sqrt{1+bh}+1)}$$

$$= \lim_{h\to 0}\frac{1}{(\sqrt{1+bh}+1)} = \frac{1}{2}\cdot$$

Since $f(x)$ is continuous at $x = 0$, we have

$$f(0) = \lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) \Rightarrow c = \frac{1}{2} \text{ and } a+2 = \frac{1}{2}\cdot$$

$\therefore \quad c = \dfrac{1}{2}$ and $a = \dfrac{-3}{2}\cdot$

EXAMPLE 15 *If the function* $f(x) = \begin{cases} 3ax+b, & \text{for } x>1 \\ 11, & \text{for } x=1 \\ 5ax-2b, & \text{for } x<1 \end{cases}$

is continuous at $x=1$, *find the values of a and b.* **[CBSE 2012C]**

SOLUTION We have, $f(1) = 11.$

$$\lim_{x\to 1+} f(x) = \lim_{h\to 0} f(1+h)$$

$$= \lim_{h\to 0}\{3a(1+h)+b\} = \lim_{h\to 0}\{(3a+b)+3ah\}$$

$$= (3a+b).$$

$$\lim_{x\to 1-} f(x) = \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to 0}\{5a(1-h)-2b\} = \lim_{h\to 0}\{(5a-2b)-5ah\}$$

$$= (5a-2b).$$

Since $f(x)$ is continuous at $x=1$, we have

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1-} f(x) = f(1).$$

$\therefore \quad 3a+b = 5a-2b = 11.$

On solving $(3a+b=11)$ and $(5a-2b=11)$, we get $a=3,\ b=2.$

Hence, $a=3,\ b=2.$

EXAMPLE 16 *For what value of k is the function*

$$f(x)=\begin{cases}k(x^2-2x), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$$

(i) continuous at $x=0$? *(ii) continuous at* $x=1$?

(iii) continuous at $x=-1$?

SOLUTION (i) We have $f(0)=k(0-2\times 0)=0.$

$$\lim_{x\to 0+} f(x)=\lim_{h\to 0} f(0+h)$$

$$=\lim_{h\to 0}\{4(0+h)+1\}=\lim_{h\to 0}(4h+1)=1.$$

$$\lim_{x\to 0-} f(x)=\lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0} k\{(-h)^2-2(-h)\}=\lim_{h\to 0} k(h^2+2h)=0.$$

Thus, $\lim\limits_{x\to 0+} f(x)\ne \lim\limits_{x\to 0-} f(x)$, and thus $\lim\limits_{x\to 0} f(x)$ does not exist.

So, $f(x)$ is not continuous at $x=0$ for any value of k.

(ii) $f(1)=(4\times 1+1)=5.$

$$\lim_{x\to 1+} f(x)=\lim_{h\to 0} f(1+h)$$

$$=\lim_{h\to 0}\{4(1+h)+1\}=\lim_{h\to 0}(5+4h)=5.$$

$$\lim_{x\to 1-} f(x)=\lim_{h\to 0} f(1-h)$$

$$=\lim_{h\to 0}\{4(1-h)+1\}=\lim_{h\to 0}(5-4h)=5.$$

Thus, $\lim\limits_{x\to 1+} f(x)=\lim\limits_{x\to 1-} f(x)=\lim\limits_{x\to 1} f(x)=5.$

Hence, $f(x)$ is continuous at $x=1$ for every real value of k.

(iii) $f(-1)=k\{(-1)^2-2\times(-1)\}=3k.$

$$\lim_{x\to(-1)+} f(x)=\lim_{h\to 0} f(-1+h)$$

$$=\lim_{h\to 0} k\{(-1+h)^2-2(-1+h)\}$$

$$=\lim_{h\to 0} k\{1+h^2-2h+2-2h\}$$

$$=\lim_{h\to 0} k(3+h^2-4h)=3k.$$

$$\lim_{x\to(-1)-} f(x)=\lim_{h\to 0} f(-1-h)$$

$$=\lim_{h\to 0} k\{(-1-h)^2-2(-1-h)\}=\lim_{h\to 0} k\{1+h^2+2h+2+2h\}$$

$$=\lim_{h\to 0} k(3+4h+h^2)=3k.$$

Thus, $\lim_{x\to(-1)+} f(x) = \lim_{x\to(-1)-} f(x) = f(-1) = 3k.$

Hence, $f(x)$ is continuous at $x = -1$ for each real value of k.

EXAMPLE 17 *Show that the function* $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$

is discontinuous at $x = 0$.

Redefine the function in such a way that it becomes continuous at $x = a$.

SOLUTION Clearly, $f(0) = 1$.

Also, $\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{\sin^2 ax}{x^2} = a^2 \cdot \lim_{ax\to 0} \left(\dfrac{\sin ax}{ax}\right)^2 = a^2.$

Since $\lim_{x\to 0} f(x) \neq f(0)$, $f(x)$ is discontinuous at $x = 0$.

However, it becomes continuous if $f(0) = a^2$.

So, the desired function is $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & \text{when } x \neq 0 \\ a^2, & \text{when } x = 0. \end{cases}$

EXAMPLE 18 *Is the function* $f(x) = \dfrac{(3x + 4\tan x)}{x}$ *continuous at* $x = 0$*? If not, how may the function be defined to make it continuous at this point?*

SOLUTION Since $f(x)$ is not defined at $x = 0$, it cannot be continuous at $x = 0$.

However, $\lim_{x\to 0} f(x) = \lim_{x\to 0} \left(\dfrac{3x + 4\tan x}{x}\right) = \lim_{x\to 0} \left[3 + 4 \cdot \dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x}\right]$

$$= 3 + 4 \cdot \lim_{x\to 0} \left\{\frac{\sin x}{x}\right\} \cdot \left\{\lim_{x\to 0} \frac{1}{\cos x}\right\} = 7.$$

So, in order to make $f(x)$ continuous at $x = 0$, we define it as

$$f(x) = \begin{cases} \dfrac{(3x + 4\tan x)}{x}, & \text{when } x \neq 0 \\ 7, & \text{when } x = 0. \end{cases}$$

EXAMPLE 19 *Show that the function* $f(x) = \begin{cases} \left(\dfrac{e^{1/x} - 1}{e^{1/x} + 1}\right), & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION Clearly, $f(0) = 0$.

Now, $\lim\limits_{x \to 0} f(x) = \lim\limits_{h \to 0} f(0+h) = \lim\limits_{h \to 0} f(h) = \lim\limits_{h \to 0} \left(\dfrac{e^{1/h} - 1}{e^{1/h} + 1} \right)$

$$= \lim_{h \to 0} \frac{e^{1/h}\left(1 - \dfrac{1}{e^{1/h}}\right)}{e^{1/h}\left(1 + \dfrac{1}{e^{1/h}}\right)} = \lim_{h \to 0} \frac{\left(1 - \dfrac{1}{e^{1/h}}\right)}{\left(1 + \dfrac{1}{e^{1/h}}\right)} = 1.$$

And, $\lim\limits_{x \to 0} f(x) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} f(-h) = \lim\limits_{h \to 0} \left(\dfrac{e^{-1/h} - 1}{e^{-1/h} + 1} \right)$

$$= \lim_{h \to 0} \frac{\left(\dfrac{1}{e^{1/h}} - 1\right)}{\left(\dfrac{1}{e^{1/h}} + 1\right)} = -1.$$

Thus, $\lim\limits_{x \to 0+} f(x) \neq \lim\limits_{x \to 0-} f(x)$, and therefore, $\lim\limits_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXERCISE 9A

1. Show that $f(x) = x^2$ is continuous at $x = 2$.
2. Show that $f(x) = (x^2 + 3x + 4)$ is continuous at $x = 1$.

Prove that

3. $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & \text{when } x \neq 3; \\ 5, & \text{when } x = 3 \end{cases}$ is continuous at $x = 3$.

4. $f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5}, & \text{when } x \neq 5; \\ 10, & \text{when } x = 5 \end{cases}$ is continuous at $x = 5$.

5. $f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & \text{when } x \neq 0; \\ 1, & \text{when } x = 0 \end{cases}$ is discontinuous at $x = 0$.

6. $f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2}, & \text{when } x \neq 0; \\ 1, & \text{when } x = 0 \end{cases}$ is discontinuous at $x = 0$. **[CBSE 2000]**

7. $f(x) = \begin{cases} 2 - x, & \text{when } x < 2; \\ 2 + x, & \text{when } x \geq 2 \end{cases}$ is discontinuous at $x = 2$.

8. $f(x) = \begin{cases} 3 - x, & \text{when } x \leq 0; \\ x^2, & \text{when } x > 0 \end{cases}$ is discontinuous at $x = 0$.

9. $f(x) = \begin{cases} 5x - 4, & \text{when } 0 < x \le 1; \\ 4x^2 - 3x, & \text{when } 1 < x < 2 \end{cases}$ is continuous at $x = 1$. **[CBSE 2001C]**

10. $f(x) = \begin{cases} x - 1, & \text{when } 1 \le x < 2; \\ 2x - 3, & \text{when } 2 \le x \le 3 \end{cases}$ is continuous at $x = 2$.

11. $f(x) = \begin{cases} \cos x, & \text{when } x \ge 0; \\ -\cos x, & \text{when } x < 0 \end{cases}$ is discontinuous at $x = 0$.

12. $f(x) = \begin{cases} \dfrac{|x - a|}{x - a}, & \text{when } x \ne a; \\ 1, & \text{when } x = a \end{cases}$ is discontinuous at $x = a$.

13. $f(x) = \begin{cases} \dfrac{1}{2}(x - |x|), & \text{when } x \ne 0; \\ 2, & \text{when } x = 0 \end{cases}$ is discontinuous at $x = 0$.

14. $f(x) = \begin{cases} \sin \dfrac{1}{x}, & \text{when } x \ne 0; \\ 0, & \text{when } x = 0 \end{cases}$ is discontinuous at $x = 0$.

15. $f(x) = \begin{cases} 2x, & \text{when } x < 2; \\ 2, & \text{when } x = 2; \\ x^2, & \text{when } x > 2 \end{cases}$ is discontinuous at $x = 2$.

16. $f(x) = \begin{cases} -x, & \text{when } x < 0; \\ 1, & \text{when } x = 0; \\ x, & \text{when } x > 0 \end{cases}$ is discontinuous at $x = 0$.

17. Find the value of k for which

$$f(x) = \begin{cases} \dfrac{\sin 2x}{5x}, & \text{when } x \ne 0; \\ k, & \text{when } x = 0 \end{cases} \text{ is continuous at } x = 0.$$

18. Find the value of λ for which

$$f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x + 1}, & \text{when } x \ne -1; \\ \lambda, & \text{when } x = -1 \end{cases} \text{ is continuous at } x = -1.$$

19. For what value of k is the following function continuous at $x = 2$?

$$f(x) = \begin{cases} 2x + 1, & x < 2 \\ k, & x = 2 \\ 3x - 1, & x > 2 \end{cases}$$

[CBSE 2008]

20. For what value of k is the function

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & \text{when } x \ne 3; \\ k, & \text{when } x = 3 \end{cases} \text{ is continuous at } x = 3?$$

21. Find the value of k for which the function

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2}; \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$$ is continuous at $x = \dfrac{\pi}{2}$. **[CBSE 2012C]**

22. Show that the function: $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

23. Show that: $f(x) = \begin{cases} x+1, & \text{if } x \geq 1; \\ x^2+1, & \text{if } x < 1 \end{cases}$ is continuous at $x = 1$. **[CBSE 2007]**

24. Show that: $f(x) = \begin{cases} x^3-3, & \text{if } x \leq 2; \\ x^2+1, & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$. **[CBSE 2007]**

25. Find the values of a and b such that the following function is continuous:

$$f(x) = \begin{cases} 5, \text{ when } x \leq \\ ax + b, \text{ when } 2 < x < 10 \\ 21, \text{ when } x \geq 10. \end{cases}$$ **[CBSE 2011]**

26. Find the values of a for which the function f, defined as

$$f(x) = \begin{cases} a \sin \dfrac{\pi}{2}(x+1), & x \leq 0 \\ \dfrac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$ is continuous at $x = 0$. **[CBSE 2011]**

27. Prove that the function f given by $f(x) = |x-3|$, $x \in R$ is continuous but not differentiable at $x = 3$. **[CBSE 2012C]**

ANSWERS (EXERCISE 9A)

17. $k = \dfrac{2}{5}$ **18.** $\lambda = -4$ **19.** $k = 5$ **20.** $k = 6$ **21.** $k = 6$ **25.** $a = 2, b = 1$ **26.** $a = \dfrac{1}{2}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 9A)

12. $\lim\limits_{x \to a+} f(x) = \lim\limits_{h \to 0} \dfrac{|a+h-a|}{(a+h-a)} = \lim\limits_{h \to 0} \dfrac{|h|}{h} = \lim\limits_{h \to 0} \dfrac{h}{h} = 1.$

$\lim\limits_{x \to a-} f(x) = \lim\limits_{h \to 0} \dfrac{|a-h-a|}{(a-h-a)} = \lim\limits_{h \to 0} \dfrac{|-h|}{-h} = \lim\limits_{h \to 0} \dfrac{h}{-h} = -1.$

13. $\lim\limits_{x \to 0+} f(x) = \lim\limits_{h \to 0} \dfrac{1}{2}(h - |h|) = 0$

and $\lim\limits_{x \to 0-} f(x) = \lim\limits_{h \to 0} \dfrac{1}{2}(-h - |-h|) = -\lim\limits_{h \to 0} h = 0.$

14. $\lim\limits_{x \to 0} \sin \dfrac{1}{x}$ does not exist, since as $x \to 0$, the value of $\sin(1/x)$ oscillates between -1 and 1, and it does not approach a definite number.

21. $f\left(\frac{\pi}{2}-0\right)=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)}=\lim_{h\to 0}\frac{k\sin h}{2h}=\frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h}=\left(\frac{k}{2}\times 1\right)=\frac{k}{2}\cdot$

$f\left(\frac{\pi}{2}+0\right)=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}=\lim_{h\to 0}\frac{-k\sin h}{-2h}=\frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h}=\left(\frac{k}{2}\times 1\right)=\frac{k}{2}\cdot$

$\therefore\ \frac{k}{2}=3 \Rightarrow k=6.$

22. $f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}(-h)^2\sin\left(\frac{-1}{h}\right)=\lim_{h\to 0} -h^2\sin\frac{1}{h}=(0\times \text{a finite quantity})=0.$

$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} h^2\sin\frac{1}{h}=(0\times\text{a finite quantity})=0.$

$\therefore\ f(0-0)=f(0+0)=f(0).$

Hence, $f(x)$ is continuous at $x=0$.

Continuous Functions

CONTINUITY IN AN INTERVAL *A function $f(x)$ is said to be continuous in an open interval $]a, b[$ if it is continuous at each point of $]a, b[$.*

If $f(x)$ is defined on a closed interval $[a, b]$, we say that

(i) f is continuous at a if $\lim_{x\to a+} f(x)=f(a)$;

(ii) f is continuous at b if $\lim_{x\to b-} f(x)=f(b)$;

(iii) f is continuous in $[a, b]$ if it is continuous at a, at b and at each point of $]a, b[$.

CONTINUOUS FUNCTIONS *A function $f(x)$ is said to be continuous if it is continuous at each point of its domain.*

Algebra of Continuous Functions

THEOREM 1 *Every constant function is continuous.*

PROOF Let $f(x)=c$, where c is constant.

Clearly, the domain of a constant function is R.

Let a be an arbitrary real number.

Then, $f(a)=c$ and $\lim_{x\to a} f(x)=\lim_{x\to a} c=c$ $[\because\ f(x)=c]$.

$\therefore\ \lim_{x\to a} f(x)=f(a).$

Thus, $f(x)$ is continuous at $x=a$ for all $a\in R$.

Hence, $f(x)$ is continuous.

THEOREM 2 *Show that the identity function is continuous.*

PROOF Let $f(x) = x$ for all $x \in R$.

Let a be an arbitrary real number. Then, $f(a) = a$.

And, $\lim_{x \to a} f(x) = \lim_{x \to a} x = a \qquad [\because \; f(x) = x]$.

$\therefore \qquad \lim_{x \to a} f(x) = f(a) = a.$

This shows that $f(x)$ is continuous at $x = a$ for all $a \in R$.

Hence, the identity function is continuous.

THEOREM 3 *If f and g be continuous functions then*

(i) $f + g$ *is continuous* *(ii)* $f - g$ *is continuous*

(iii) cf *is continuous* *(iv)* fg *is continuous*

(v) $\left(\frac{f}{g}\right)$ *is continuous at those points where* $g(x) \neq 0$

PROOF (i) Let $\text{dom}(f) = D_1$ and $\text{dom}(g) = D_2$. Then, $\text{dom}(f+g) = D_1 \cap D_2$.

Let $a \in D_1 \cap D_2$. Then, $a \in D_1$ and $a \in D_2$.

By continuity of f and g, we have

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} g(x) = g(a).$$

$$\therefore \quad \lim_{x \to a} (f+g)(x) = \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$
$$= f(a) + g(a) = (f+g)(a).$$

This shows that $(f+g)$ is continuous at a for all $a \in D_1 \cap D_2$.

Hence, $(f+g)$ is continuous.

Similarly, (ii) may be proved.

(iii) Let $\text{dom}(f) = D_1$. Then, $\text{dom}(cf) = D_1$. Let $a \in D_1$.

Then, by the continuity of f, we have $\lim_{x \to a} f(x) = f(a)$.

$$\therefore \quad \lim_{x \to a} (cf)(x) = \lim_{x \to a} c \cdot f(x) = c \cdot \lim_{x \to a} f(x) = c \cdot f(a) = (cf)(a).$$

This shows that (cf) is continuous at a for all $a \in D_1$.

Hence, cf is continuous.

(iv) Let $\text{dom}(f) = D_1$ and $\text{dom}(g) = D_2$. Then, $\text{dom}(fg) = D_1 \cap D_2$.

Let $a \in D_1 \cap D_2$. Then, $a \in D_1$ and $a \in D_2$.

By continuity of f and g, we have

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} g(x) = g(a).$$

$$\therefore \quad \lim_{x \to a} (fg)(x) = \lim_{x \to a} [f(x) \cdot g(x)]$$
$$= \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = f(a) \cdot g(a) = (fg)(a).$$

This shows that fg is continuous.

(v) Let dom$(f) = D_1$ and dom$(g) = D_2$.

Then, dom $\left(\frac{f}{g}\right) = [(D_1 \cap D_2) - \{x : g(x) = 0\}] = D$ (say).

Let $a \in D$. Then $a \in D_1$, $a \in D_2$ and $g(a) \neq 0$.

By continuity of f and g, we have

$\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$.

$$\therefore \lim_{x \to a} \left(\frac{f}{g}\right)(x) = \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a), \text{ since } g(a) \neq 0.$$

Thus, $\left(\frac{f}{g}\right)$ is continuous at a for all $a \in D$. Hence, $\left(\frac{f}{g}\right)$ is continuous.

THEOREM 4 *Every polynomial function is continuous.*

PROOF Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ be a polynomial. We shall prove the theorem by induction on n.

When $n = 0$ then $f(x) = a_0$, which being a constant function is continuous.

When $n = 1$ then $f(x) = a_0 + a_1 x$.

Clearly, $f(x)$ is the sum of a constant function and a multiple of the identity function. It, being the sum of two continuous functions, is continuous.

Let every polynomial of degree at most n be continuous.

Consider a general polynomial of degree $(n + 1)$, namely,

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + a_{n+1} x^{n+1}.$$

This can be written as $a_0 + x(a_1 + a_2 x + \ldots + a_n x^{n-1} + a_{n+1} x^n)$.

This is the sum of a constant function a_0 (which is continuous) and the product of the identity function x (which is continuous) and the polynomial function $a_1 + a_2 x + \ldots + a_{n+1} x^n$ of degree at most n (which we assumed to be continuous).

Therefore, it is continuous.

Thus, the continuity of a polynomial of degree n implies the continuity of a polynomial of degree $(n + 1)$.

Hence, by the principle of induction, every polynomial function is continuous.

THEOREM 5 *Every rational function is continuous.*

PROOF Let $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

Then, dom$[p(x)] = R$ and dom$[q(x)] = R$.

$$\therefore \text{dom}[f(x)] = \text{dom}\left\{\frac{p(x)}{q(x)}\right\} = R \cap R - \{x : q(x) = 0\} = R - \{x : q(x) = 0\}.$$

Let a be an arbitrary element of dom $f(x)$.

Then, $a \in R$ and $q(a) \neq 0$.

But, every polynomial function being continuous, we have

$$\lim_{x \to a} p(x) = p(a) \text{ and } \lim_{x \to a} q(x) = q(a).$$

$$\therefore \quad \lim_{x \to a} f(x) = \lim_{x \to a} = \frac{\lim\limits_{x \to a} p(x)}{\lim\limits_{x \to a} q(x)} = \frac{p(a)}{q(a)}, \text{ where } q(a) \neq 0.$$

This shows that $f(x)$ is continuous at $x = a$ for all $a \in$ dom $f(x)$.

Hence, every rational function is continuous.

THEOREM 6 *Let f and g be real functions such that $f \circ g$ is defined.*
If g is continuous at a and f is continuous at $g(a)$, show that $f \circ g$ is continuous at a.

PROOF Since $f \circ g$ is defined, we have range $(g) \subseteq \text{dom}(f)$.

Since g is continuous at a, we have $\lim\limits_{x \to a} g(x) = g(a)$. ... (i)

Also, f being continuous at $g(a)$, we have $\lim\limits_{y \to g(a)} f(y) = f\{g(a)\}$. ... (ii)

$$\therefore \quad \lim_{x \to a} (f \circ g)(x) = \lim_{x \to a} f\{g(x)\} = \lim_{g(x) \to g(a)} f\{g(x)\}$$

$$[\because \quad x \to a \Rightarrow g(x) \to g(a) \text{ from (i)}]$$

$$= \lim_{y \to g(a)} f(y) = f\{g(a)\} \qquad \text{[using (ii)]}$$

$$= (f \circ g)(a).$$

Thus, $(f \circ g)$ is continuous at a.

THEOREM 7 *The composite of two continuous functions is continuous.*

PROOF Let f and g be continuous functions such that $g \circ f$ is defined.
Then, range $(f) \subseteq \text{dom}(g)$. Let $a \in \text{dom}(f)$.

Then, $a \in \text{dom}(f) \Rightarrow f(a) \in \text{range}(f)$

$\Rightarrow f(a) \in \text{dom}(g) \qquad [\because \text{ range}(f) \subseteq \text{dom}(g)]$.

Thus, f is continuous at a, and g is continuous at $f(a)$.

Consequently, $\lim\limits_{x \to a} f(x) = f(a)$. ... (i)

And, $\lim\limits_{y \to f(a)} g(y) = g[f(a)]$. ... (ii)

$$\therefore \quad \lim_{x \to a} (g \circ f)(x) = \lim_{x \to a} g[f(x)]$$

$$= \lim_{f(x) \to f(a)} g[f(x)] \; [\because x \to a \Rightarrow f(x) \to f(a) \text{ from (i)}]$$

$$= \lim_{y \to f(a)} g(y) = g[f(a)] \qquad \text{[using (ii)]}$$

$$= (g \circ f)(a).$$

This shows that $g \circ f$ is continuous at a for all $a \in \text{dom}(f)$.

Hence $g \circ f$ is continuous.

An important result

For continuity of f at a, it is sufficient to show that $\lim_{h \to 0} f(a+h) = f(a)$, since f is continuous $\Leftrightarrow \lim_{x \to a} f(x) = f(a)$

$$\Leftrightarrow \lim_{(a+h) \to a} f(a+h) = f(a) \quad [\text{putting } x = (a+h)]$$

$$\Leftrightarrow \lim_{h \to 0} f(a+h) = f(a).$$

THEOREM 8 *The exponential function is continuous.*

PROOF Let $f(x) = e^x$. Then, clearly, $\text{dom}(f) = R$.

Let a be an arbitrary real number. Then,

$$\lim_{x \to a} e^x = \lim_{h \to 0} e^{a+h} = \lim_{h \to 0} e^a \cdot e^h = e^a \cdot \lim_{h \to 0} e^h = e^a \times 1 = e^a.$$

Thus, $f(x) = e^x$ is continuous at $x = a$ for all $a \in R$.

Hence, the exponential function is continuous.

THEOREM 9 *(i) The sine function is continuous.*
(ii) The cosine function is continuous.
(iii) The tangent function is continuous.

PROOF (i) Let $f(x) = \sin x$. Then, clearly $\text{dom}(f) = R$.

Let a be an arbitrary real number. Then,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \sin x = \lim_{h \to 0} \sin(a+h) = \lim_{h \to 0} [\sin a \cos h + \cos a \sin h]$$
$$= \sin a \cdot \lim_{h \to 0} \cos h + \cos a \cdot \lim_{h \to 0} \sin h$$
$$= (\sin a \times 1 + \cos a \times 0) = \sin a = f(a).$$

Thus, $\lim_{x \to a} f(x) = f(a)$ for all $a \in R$.

$\therefore$ $f(x) = \sin x$ is continuous at a for all $a \in R$.

Hence, $\sin x$ is continuous.

(ii) Let $f(x) = \cos x$. Clearly, $\text{dom}(f) = R$.

Let a be an arbitrary real number. Then,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \cos x = \lim_{h \to 0} \cos(a+h) = \lim_{h \to 0} [\cos a \cos h - \sin a \sin h]$$
$$= \cos a \cdot \lim_{h \to 0} \cos h - \sin a \cdot \lim_{h \to 0} \sin h = (\cos a \times 1 - \sin a \times 0)$$
$$= \cos a = f(a).$$

Thus, $\lim_{x \to a} f(x) = f(a)$ for all $a \in R$.

$\therefore$ $f(x) = \cos x$ is continuous at a for all $a \in R$.

Hence, $\cos x$ is continuous.

(iii) We have $\tan x = \dfrac{\sin x}{\cos x}$ and $\text{dom}(\tan x) = R - \{(2n+1)\frac{\pi}{2} : n \in I\}$.

Let $a \in R - \{(2n+1)\frac{\pi}{2} : n \in I\}$. Then,

$$\lim_{x \to a} \tan x = \lim_{x \to a} \frac{\sin x}{\cos x} = \frac{\lim_{h \to 0} \sin (a+h)}{\lim_{h \to 0} \cos (a+h)}$$

$$= \frac{\lim_{h \to 0} (\sin a \cos h + \cos a \sin h)}{\lim_{h \to 0} (\cos a \cos h - \sin a \sin h)}$$

$$= \left(\frac{\sin a \times \cos 0 + \cos a \times \sin 0}{\cos a \times \cos 0 - \sin a \times \sin 0} \right) = \left(\frac{\sin a \times 1 + \cos a \times 0}{\cos a \times 1 - \sin a \times 0} \right)$$

$$= \tan a.$$

Thus, $\tan x$ is continuous at a for all $a \in R - \left\{ (2n+1)\frac{\pi}{2} : n \in I \right\}$.

Hence, $\tan x$ is continuous.

THEOREM 10 *The logarithmic function is continuous.*

PROOF Let $f(x) = \log x$. Then, $\text{dom}(f) =$ set of positive real numbers.

Let a be any positive real number. Then,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \log x$$

$$= \lim \left[\log \left(a \cdot \frac{x}{a} \right) \right] = \lim_{x \to a} \left[\log a + \log \frac{x}{a} \right]$$

$$= \log a + \lim_{x \to a} \log \frac{x}{a} = \log a = f(a) \left[\because \lim_{x \to 0} \log \frac{x}{a} = \log 1 = 0 \right].$$

Thus, $f(x) = \log x$ is continuous at $a \in R^+$.

Hence, $f(x) = \log x$ is continuous.

THEOREM 11 $\sin |x|$ *is continuous.*

PROOF Let $f(x) = |x|$ and $g(x) = \sin x$. Then,

$$(g \text{ o } f)(x) = g\{f(x)\} = g(|x|) = \sin |x|.$$

Now, f and g being continuous, it follows that their composite $(g \text{ o } f)$ is continuous.

Hence, $\sin |x|$ is continuous.

SOLVED EXAMPLES

EXAMPLE 1 *Let* $f(x) = \begin{cases} x & \text{if } x \geq 1 \\ x^2 & \text{if } x < 1. \end{cases}$ *Is f a continuous function? Why?*

SOLUTION Let a be any real number. Then, three possibilities arise.

Case I *When* $a > 1$

In this case, $f(a) = a$.

Also, $\lim_{x \to a+} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} (a+h) = a.$

And, $\lim_{x \to a-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} (a-h) = a$

$[\because \ a > 1 \text{ and } h \text{ is very small} \Rightarrow (a-h) > 1].$

$\therefore \lim_{x \to a+} f(x) = \lim_{a \to a-} f(x) = a = f(a).$

So, $f(x)$ is continuous at each $a > 1$.

Case II *When* $a < 1$

In this case, $f(a) = a^2$.

$$\lim_{x \to a+} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} (a+h)^2 = a^2$$

$[\because \ a < 1 \text{ and } h \text{ is very small} \Rightarrow (a+h) < 1].$

And, $\lim_{x \to a-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} (a-h)^2 = a^2$

$[\because \ a < 1 \text{ and } h \text{ is very small} \Rightarrow (a-h) < 1].$

$\therefore \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = a^2 = f(a).$

Case III *When* $a = 1$

In this case, $f(1) = 1$.

$$\lim_{x \to 1+} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} (1+h) = 1 \quad [\because \ (1+h) > 1].$$

$$\lim_{x \to 1-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1-h)^2 = 1 \quad [\because \ (1-h) < 1].$$

$\therefore \lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x) = 1.$

So, $\lim_{x \to 1} f(x) = 1 = f(1).$

So, f is continuous at $a = 1$.

Thus, from all the above three cases, it follows that $f(x)$ is continuous at $x = a$ for all $a \in R$.

Hence, $f(x)$ is continuous.

EXAMPLE 2 *Prove that* $f(x) = |x|$ *is a continuous function.*

SOLUTION Let $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$

Clearly, $\text{dom}(f) = R$.

Let a be any real number. Then, the following cases arise.

Case I *When* $a < 0$

In this case, $f(a) = -a$.

$$\lim_{x \to a+} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \{-(a+h)\} = -a$$

$[\because \ a < 0 \text{ and } h \text{ is very small and positive} \Rightarrow a+h < 0].$

And, $\lim_{x \to a-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \{-(a-h)\} = -a.$

$[\because \ a < 0 \text{ and } h \text{ is very small and positive} \Rightarrow a-h < 0].$

So, $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = -a = f(a).$

Thus, $f(x)$ is continuous at each $a < 0$.

Case II *When* $a > 0$

In this case, $f(a) = a$.

$$\lim_{x \to a+} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} (a+h) = a \quad [\because \ (a+h) > 0].$$

$$\lim_{x \to a-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} (a-h) = a \quad [\because \ (a-h) > 0].$$

$$\therefore \ \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = f(a).$$

So, $f(x)$ is continuous at each $a > 0$.

Case III *When* $a = 0$

Clearly, $f(0) = 0$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h = 0.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \{-(-h)\}$$

$$= \lim_{h \to 0} h = 0.$$

$$\therefore \ \lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = 0 = f(0).$$

So, $f(x)$ is continuous at $x = 0$.

Thus, from all the above cases, it follows that $f(x) = |x|$ is continuous at a for all $a \in R$.

Hence, $f(x) = |x|$ is continuous.

EXAMPLE 3 *Discuss the continuity of the function* $f(x) = \begin{cases} 2x - 1, & \text{if} \ \ x < 0 \\ 2x + 1, & \text{if} \ \ x \geq 0. \end{cases}$

[CBSE 2002]

SOLUTION When $x < 0$, we have $f(x) = 2x - 1$, which being a polynomial function, is continuous at each point where $x < 0$.

Also, when $x > 0$, we have $f(x) = 2x + 1$, which being a polynomial function, is continuous at each point where $x > 0$.

Let us consider the point $x = 0$.

$$f(0) = (2 \times 0 + 1) = 1.$$

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (2h+1) = 1.$$

And, $\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} [(2(-h) - 1)] = -1.$

$$\therefore \quad \lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x).$$

So, $\lim_{x \to 0} f(x)$ does not exist, and therefore $f(x)$ is not continuous at $x = 0$.

Thus, the given function is continuous at each point except at $x = 0$, where it is discontinuous.

EXAMPLE 4 *Discuss the continuity of the function* $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if} \ \ x < 0 \\ (x + 1), & \text{if} \ \ x \geq 0. \end{cases}$

SOLUTION We know that $\sin x$ as well as the identity function x is continuous.

So, the quotient function, $\frac{\sin x}{x}$ is continuous at each $x < 0$.

Also, when $x > 0$, we have $f(x) = (x + 1)$, which being a polynomial function, is continuous.

Let us consider the point $x = 0$.

Clearly, $f(0) = (0 + 1) = 1$.

We have $f(0) = (0 + 1) = 1$;

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} (h + 1) = 1$$

and, $\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \frac{\sin(-h)}{-h} = \lim_{h \to 0} \left(\frac{-\sin h}{-h} \right) = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

$\therefore$ $\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = 1.$

So, $\lim_{x \to 0} f(x) = 1 = f(0).$

Thus, $f(x)$ is continuous at $x = 0$ also.

Hence, $f(x)$ is continuous at all points.

EXAMPLE 5 *Discuss the continuity of the function* $f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$

SOLUTION We know that the identity function x is continuous and the modulus function $|x|$ is continuous.

So, the quotient function $\frac{x}{|x|}$ is continuous at each $x \neq 0$.

It has already been proved that $f(x)$ is discontinuous at $x = 0$.

Hence, the given function is continuous at each point, except at $x = 0$.

EXAMPLE 6 *Locate the point of discontinuity of the function* $f(x) = \begin{cases} \frac{x^4 - 16}{x - 2}, & \text{if } x \neq 2 \\ 16, & \text{if } x = 2. \end{cases}$

SOLUTION The function $f(x) = \frac{x^4 - 16}{x - 2}$ being a rational function, is continuous at all points of its domain, i.e., for all real numbers except 2.

Now, consider the given function at $x = 2$.

We have $f(2) = 16$.

$$\lim_{x \to 2} f(x) = \lim_{h \to 0} f(2 + h) = \lim_{h \to 0} \left\{ \frac{(2 + h)^4 - 16}{2 + h - 2} \right\} = \lim_{2+h \to 2} \left\{ \frac{(2 + h)^4 - 2^4}{(2 + h) - 2} \right\}$$

$$= 4 \times 2^{4-1} = 32 \qquad \left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1} \right].$$

Thus, $\lim_{x \to 2+} f(x) \neq f(2)$, and therefore $\lim_{x \to 2} f(x) \neq f(2)$.

Hence, $f(x)$ is discontinuous at $x = 2$.

EXAMPLE 7 *Determine the value of k so that the function*

$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2; \\ 3, & \text{if } x > 2 \end{cases} \quad \textit{is continuous.}$$ **[CBSE 1990C, '91]**

SOLUTION Since a polynomial function is continuous and a constant function is continuous, the given function is continuous for all $x < 2$ and for all $x > 2$.

So, consider the point $x = 2$. We have $f(2) = 4k$.

$$\lim_{x \to 2+} f(x) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} 3 = 3.$$

And, $\lim_{x \to 2-} f(x) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} k(2-h)^2 = 4k.$

$\therefore$ for continuity, we must have $4k = 3$ or $k = \frac{3}{4}$.

EXAMPLE 8 *Let* $f(x) = \begin{cases} 1, & \text{if } x \leq 3 \\ ax+b, & \text{if } 3 < x < 5 \\ 7, & \text{if } 5 \leq x. \end{cases}$

Find the values of a and b so that $f(x)$ is continuous.

SOLUTION We know that a constant function is continuous, and a polynomial function is continuous.

So, the given function is continuous for all $x < 3$; for all $x > 5$ and for all x lying in $]3, 5[$.

Now, consider the point $x = 3$. We have $f(3) = 1$.

$$\lim_{x \to 3+} f(x) = \lim_{h \to 0} f(3+h) = \lim_{h \to 0} [a(3+h)+b] = (3a+b).$$

And, $\lim_{x \to 3-} f(x) = \lim_{h \to 0} (3-h) = \lim_{h \to 0} 1 = 1.$

Since $f(x)$ is given to be continuous, it must be continuous at $x = 3$ also.

So, we must have $f(3) = \lim_{x \to 3-} f(x) = \lim_{x \to 3+} f(x)$, i.e., $3a + b = 1$. … (i)

Again, consider the point $x = 5$. We have $f(5) = 7$.

$$\lim_{x \to 5+} f(x) = \lim_{h \to 0} f(5+h) = \lim_{h \to 0} 7 = 7.$$

And, $\lim_{x \to 5-} f(x) = \lim_{h \to 0} f(5-h) = \lim_{h \to 0} [a(5-h)+b] = (5a+b).$

Now, by continuity of $f(x)$ at $x = 5$, we have

$$f(5) = \lim_{x \to 5+} f(x) = \lim_{x \to 5-} f(x) \quad \text{or} \quad 5a + b = 7. \qquad \text{… (ii)}$$

Solving (i) and (ii), we get $a = 3$ and $b = -8$.

EXAMPLE 9 *Show that the function $f(x) = \sqrt{x^4 + 3}$ is continuous at each point.*

SOLUTION Let $g(x) = x^4 + 3$ and $h(y) = \sqrt{y}$.

Since every polynomial function is continuous at each point where $x = a$, it follows that $g(x)$ is continuous at $x = a$.

Clearly, $g(a)$ is positive. Moreover, $h[g(a)] = \sqrt{g(a)} = \lim_{y \to g(a)} \sqrt{y}$.

$\therefore$ $h(y)$ is continuous at $g(a)$.

Since the composite of continuous functions is continuous, it follows that $(h \text{ o } g)(x)$ is continuous.

But, $(h \text{ o } g)(x) = h[g(x)] = h(x^4 + 3) = \sqrt{x^4 + 3} = f(x)$.

Hence, $f(x)$ is continuous.

EXAMPLE 10 *Show that the function* $f(x) = |\sin x + \cos x|$ *is continuous at* $x = \pi$.

SOLUTION Let $g(x) = \sin x + \cos x$ and $k(y) = |y|$.

We shall first show that g is continuous at $x = \pi$ and k is continuous at $y = g(\pi)$.

Now, $\lim_{x \to \pi} g(x) = \lim_{x \to \pi} (\sin x + \cos x) = \sin \pi + \cos \pi = -1$.

Also, $g(\pi) = (\sin \pi + \cos \pi) = -1$. $\therefore$ $\lim_{x \to \pi} g(x) = g(\pi)$.

So, g is continuous at $x = \pi$. Now, $g(\pi) = -1$.

$\therefore$ $k[g(\pi)] = k(-1) = |-1| = 1$.

Now, $\lim_{y \to (-1)+} k(y) = \lim_{y \to (-1)+} |y| = \lim_{h \to 0} |-1 + h| = 1$.

And, $\lim_{y \to (-1)-} k(y) = \lim_{y \to (-1)-} |y| = \lim_{h \to 0} |-1 - h| = 1$.

$\therefore$ $k[g(\pi)] = \lim_{y \to g(\pi)} k(y)$. This shows that k is continuous at $g(\pi)$.

Consequently, $(k \text{ o } g)$ is continuous at $x = \pi$.

But, $(k \text{ o } g)(x) = k[g(x)] = k(\sin x + \cos x) = |\sin x + \cos x| = f(x)$.

Hence, $f(x)$ is continuous at $x = \pi$.

EXERCISE 9B

1. Show that the function $f(x) = \begin{cases} (7x + 5), & \text{when } x \geq 0; \\ (5 - 3x), & \text{when } x < 0 \end{cases}$ is a continuous function.

2. Show that the function $f(x) = \begin{cases} \sin x, & \text{if } x < 0; \\ x, & \text{if } x \geq 0 \end{cases}$ is continuous.

3. Show that the function $f(x) = \begin{cases} \dfrac{x^n - 1}{x - 1}, & \text{when } x \neq 1; \\ n, & \text{when } x = 1 \end{cases}$ is continuous.

4. Show that $\sec x$ is a continuous function.

5. Show that $\cos |x|$ is a continuous function.

6. Show that the function $f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{when } x \neq 0;\\ 2, & \text{when } x=0\end{cases}$

is continuous at each point except 0.

7. Discuss the continuity of $f(x)=[x]$.

8. Show that $f(x)=\begin{cases}(2x-1), & \text{if } x<2;\\ \dfrac{3x}{2}, & \text{if } x \geq 2\end{cases}$ is continuous.

9. Show that $f(x)=\begin{cases}x, & \text{if } x \neq 0;\\ 1, & \text{if } x=0\end{cases}$ is continuous at each point except 0.

10. Locate the point of discontinuity of the function

$$f(x)=\begin{cases}(x^3-x^2+2x-2), & \text{if } x \neq 1;\\ 4, & \text{if } x=1.\end{cases}$$

11. Discuss the continuity of the function $f(x)=|x|+|x-1|$ in the interval $[-1, 2]$.

ANSWERS (EXERCISE 9B)

7. discontinuous at $x=n$, where n is an integer

10. discontinuous at $x=1$ **11.** continuous

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 9B)

1. Since a polynomial function is continuous, the given function is continuous at each $x>0$ as well as at each $x<0$. Test its continuity at $x=0$.

2. Let $a \in R$.

Case I When $a>0$ then, $\lim\limits_{x \to a+} f(x) = \lim\limits_{x \to a-} f(x) = f(a) = a$.

Case II When $a<0$ then, $\lim\limits_{x \to a+} f(x) = \lim\limits_{x \to a-} f(x) = f(a) = \sin a$.

Case III When $a=0$ then, $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0-} f(x) = f(0) = 0$.

7. $f(x)$ is discontinuous at each integral value of x.

8. Take three cases: (i) $a>2$ (ii) $a<2$ (iii) $a=2$.

10. Test at $x=1$.

11. The given function may be expressed as

$$f(x)=\begin{cases}(1-2x), & \text{when } -1 \leq x < 0;\\ 1, & \text{when } 0 \leq x \leq 1;\\ (2x-1), & \text{when } 1 < x \leq 2.\end{cases}$$

Since a polynomial function as well as a constant function is continuous, it follows that $f(x)$ is continuous when $-1<x<0$; $0<x<1$ and $1<x<2$.

So, test the continuity at $-1, 0, 1$ and 2.

At $x=-1$, $\lim\limits_{x \to (-1)+} f(x) = f(-1)$, so it is continuous at $x=-1$.

At $x = 0$, $\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = f(0) = 1$, so it is continuous at $x = 0$.

At $x = 1$, $\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x) = f(1) = 1$.

At $x = 2$, $\lim_{x \to 2-} f(x) = f(2) = 3$, so it is continuous at $x = 2$.

Differentiability

Let $f(x)$ be a real function and a be any real number. Then, we define

(i) Right-hand derivative $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, if it exists, is called the right-hand derivative of $f(x)$ at $x = a$, and it is denoted by $Rf'(a)$.

(ii) Left-hand derivative $\lim_{h \to 0} \dfrac{f(a-h) - f(a)}{-h}$, if it exists, is called the left-hand derivative of $f(x)$ at $x = a$, and it is denoted by $Lf'(a)$.

DIFFERENTIABILITY *A function $f(x)$ is said to be differentiable at $x = a$, if $Rf'(a) = Lf'(a)$.*

The common value of $Rf'(a)$ and $Lf'(a)$ is denoted by $f'(a)$ and it is known as the *derivative* of $f(x)$ at $x = a$.

If, however, $Rf'(a) \neq Lf'(a)$, we say that $f(x)$ is not differentiable at $x = a$.

REMARK In each case, h is taken as positive and very small.

SOLVED EXAMPLES

EXAMPLE 1 *Show that $f(x) = x^2$ is differentiable at $x = 1$ and find $f'(1)$.*

SOLUTION
$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - (1)^2}{h}$$
$$= \lim_{h \to 0} \left(\frac{1 + h^2 + 2h - 1}{h} \right) = \lim_{h \to 0} (h + 2) = 2.$$

And,
$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{(1-h)^2 - (1)^2}{-h}$$
$$= \lim_{h \to 0} \left(\frac{1 + h^2 - 2h - 1}{-h} \right) = \lim_{h \to 0} (-h + 2) = 2.$$

$\therefore$ $Rf'(1) = Lf'(1) = 2.$

This shows that $f(x)$ is differentiable at $x = 1$ and $f'(1) = 2$.

EXAMPLE 2 *Show that $f(x) = [x]$ is not differentiable at $x = 1$.*

SOLUTION We have
$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[1+h] - [1]}{h} = 0$$
$\{\because\ [1+h] = 1 \text{ and } [1] = 1\}$,

and $$Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{[1-h]-[1]}{-h} = \infty$$

$\{\because \ [1-h] = 0 \text{ and } [1] = 1\}$.

Thus, $Rf'(1) \neq Lf'(1)$.

Hence, $f(x) = [x]$ is not differentiable at $x = 1$.

EXAMPLE 3 *(i) Show that $f(x) = x^{4/3}$ is differentiable at $x = 0$, and hence find $f'(0)$.*

(ii) Show that $g(x) = x^{3/2}$ is not differentiable at $x = 0$.

SOLUTION (i) We have $$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^{4/3}-0}{h} = \lim_{h\to 0} \frac{h^{4/3}}{h} = \lim_{h\to 0} h^{1/3} = 0.$$

And, $$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-0}{-h}$$

$$= \lim_{h\to 0} \frac{(-h)^{4/3}-0}{(-h)} = \lim_{h\to 0} \frac{(-h)^{4/3}}{(-h)} = \lim_{h\to 0} (-h)^{1/3} = 0.$$

Thus, $Rf'(0) = Lf'(0) = 0$.

Hence, $f(x) = x^{4/3}$ is differentiable at $x = 0$ and $f'(0) = 0$.

(ii) Consider $g(x) = x^{3/2}$.

Now, $$Rg'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{g(h)-g(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^{3/2}-0}{h} = \lim_{h\to 0} \frac{h^{3/2}}{h} = \lim_{h\to 0} h^{1/2} = 0.$$

And, $$Lg'(0) = \lim_{h\to 0} \frac{g(0-h)-g(0)}{-h} = \lim_{h\to 0} \frac{g(-h)-g(0)}{-h}$$

$$= \lim_{h\to 0} \frac{(-h)^{3/2}-0}{(-h)} = \lim_{h\to 0} \frac{(-h)^{3/2}}{(-h)}$$

$= \lim_{h\to 0} (-h)^{1/2}$, which is imaginary.

Thus, $Lg'(0)$ does not exist.

Hence, $g(x) = x^{3/2}$ is not differentiable at $x = 0$.

EXAMPLE 4 *Show that the function* $f(x) = \begin{cases} 1+x, & \text{if } x \le 2; \\ 5-x, & \text{if } x > 2 \end{cases}$ *is not differentiable at* $x = 2$.

SOLUTION $$Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \left[\frac{5-(2+h)-3}{h}\right]$$

$$= \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0} (-1) = -1.$$

And, $Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$

$$= \lim_{h \to 0} \left[\frac{1 + (2-h) - 3}{-h} \right] = \lim_{h \to 0} \frac{-h}{-h} = \lim_{h \to 0} 1 = 1.$$

Thus, $Rf'(2) \neq Lf'(2)$.

Hence, $f(x)$ is not differentiable at $x = 2$.

EXAMPLE 5 *Let* $f(x) = \begin{cases} (1 + \sin x), & when \;\; 0 \le x < \frac{\pi}{2} \\ 1, & when \;\; x < 0. \end{cases}$

Show that $f'(0)$ *does not exist.*

SOLUTION We have $Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{(1 + \sin h) - 1}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

And, $Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \to 0} \frac{(1-1)}{-h} = 0.$

Thus, $Rf'(0) \neq Lf'(0)$. Hence, $f'(0)$ does not exist.

EXAMPLE 6 *Let* $f(x) = mx + c$ *and* $f(0) = f'(0) = 1$. *Find* $f(2)$.

SOLUTION Clearly, $f(0) = (m \times 0 + c) = c = 1$ (given).

Also, $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{(mh + c) - 1}{h} = \lim_{h \to 0} \frac{mh + 1 - 1}{h} \qquad [\because \;\; c = 1]$$

$$= \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m.$$

Thus, $f'(0) = 1 \Rightarrow m = 1$.

So, $f(x) = 1 \cdot x + 1$, i.e., $f(x) = (x + 1)$. Hence, $f(2) = (2 + 1) = 3$.

Relation between Continuity and Differentiability

THEOREM *Every differentiable function is continuous. But, every continuous function need not be differentiable.*

PROOF Let $f(x)$ be a differentiable function and let a be any real number in its domain. Then,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a). \qquad \ldots \text{(i)}$$

Now, $\lim_{h \to 0} [f(a+h) - f(a)] = \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \times h \right]$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \times \lim_{h \to 0} h$$

$$= f'(a) \times 0 = 0 \qquad \text{[using (i)]}.$$

Thus, $\lim_{h \to 0} [f(a+h) - f(a)] = 0$ or $\lim_{h \to 0} f(a+h) = f(a)$.

This shows that $f(x)$ is continuous at a for all a.

Hence, every differentiable function is continuous.

In order to show that a continuous function need not be differentiable, it is sufficient to give an example of a function which is continuous but not differentiable. Consider $f(x) = |x|$ at $x = 0$.

Clearly, $f(0) = 0$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} |h| = \lim_{h \to 0} h = 0.$$

And, $$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} |-h| = \lim_{h \to 0} h = 0.$$

Thus, $$\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = f(0).$$

So, $f(x) = |x|$ is continuous at $x = 0$.

But, $$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h| - 0}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

And, $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{|-h| - 0}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1.$$

Thus, $Rf'(0) \neq Lf'(0)$.

This shows that $f(x) = |x|$ is not differentiable at $x = 0$. Thus, $f(x) = |x|$ is continuous but not differentiable at $x = 0$. Hence, a continuous function need not be differentiable.

EXAMPLE 7 *Show that the function* $$f(x) = \begin{cases} x \sin(1/x), & \text{when } x \neq 0; \\ 0, & \text{when } x = 0 \end{cases}$$

is continuous but not differentiable at $x = 0$.

SOLUTION We have already discussed the above function for continuity at $x = 0$.

Now, $$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h}$$

$$= \lim_{h \to 0} \sin(1/h), \text{ which does not exist.}$$

Hence, $f(x)$ is not differentiable at $x = 0$.

EXAMPLE 8 *Show that* $f(x) = |x - 2|$ *is continuous but not differentiable at* $x = 2$.

SOLUTION We have $f(2) = |2 - 2| = 0$.

$$\lim_{x \to 2+} f(x) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} |2 + h - 2| = \lim_{h \to 0} |h| = \lim_{h \to 0} h = 0.$$

$$\lim_{x \to 2-} f(x) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} |2 - h - 2| = \lim_{h \to 0} |-h| = \lim_{h \to 0} h = 0.$$

$$\therefore \lim_{x \to 2+} f(x) = \lim_{x \to 2-} f(x) = f(2) = 0.$$

So, $f(x)$ is continuous at $x = 2$.

But, $Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{|2+h-2|-0}{h}$

$= \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$

And, $Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0} \frac{|2-h-2|-0}{-h}$

$= \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1.$

Thus, $Rf'(2) \neq Lf'(2)$.

This shows that $f(x)$ is not differentiable at $x = 2$.

EXERCISE 9C

1. Show that $f(x) = x^3$ is continuous as well as differentiable at $x = 3$.
2. Show that $f(x) = (x-1)^{1/3}$ is not differentiable at $x = 1$.
3. Show that a constant function is always differentiable.
4. Show that $f(x) = |x-5|$ is continuous but not differentiable at $x = 5$.
5. Let $f(x) = \begin{cases} (2-x), & \text{when } x \geq 1 \\ x, & \text{when } 0 \leq x \leq 1. \end{cases}$

   Show that $f(x)$ is continuous but not differentiable at $x = 1$.
6. Show that $f(x) = [x]$ is neither continuous nor derivable at $x = 2$.
7. Show that the function $f(x) = \begin{cases} (1-x), & \text{when } x < 1; \\ (x^2-1), & \text{when } x \geq 1 \end{cases}$

   is continuous but not differentiable at $x = 1$.
8. Let $f(x) = \begin{cases} (2+x), & \text{if } x \geq 0 \\ (2-x), & \text{if } x < 0. \end{cases}$ Show that $f(x)$ is not derivable at $x = 0$.
9. If $f(x) = |x|$, show that $f'(2) = 1$.
10. Find the values of a and b so that the function

    $f(x) = \begin{cases} (x^2 + 3x + a), & \text{when } x \leq 1; \\ (bx + 2), & \text{when } x > 1 \end{cases}$ is differentiable at each $x \in R$.

ANSWERS (EXERCISE 9C)

10. $a = 3, b = 5$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 9C)

9. Show that $Rf'(2) = Lf'(2) = 1$.
10. $Rf'(1) = Lf'(1) = 5$.

10. DIFFERENTIATION

1. Derivatives of Some Functions

In class 11, we have studied about the derivatives of algebraic and trigonometric functions. We derived the following results.

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$

(ii) $\frac{d}{dx}(\sin x) = \cos x$

(iii) $\frac{d}{dx}(\cos x) = -\sin x$

(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$

(v) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(vi) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(vii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

In algebra of derivatives, we have established the following rules.

Some Rules of Differentiation

(i) $\frac{d}{dx}(u+v) = \left(\frac{du}{dx} + \frac{dv}{dx}\right)$

(ii) $\frac{d}{dx}(u-v) = \left(\frac{du}{dx} - \frac{dv}{dx}\right)$

(iii) $\frac{d}{dx}(uv) = \left(u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}\right)$

(iv) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\left(v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}\right)}{v^2}$

DERIVATIVES OF COMPOSITE FUNCTIONS (CHAIN RULE)

(i) Let $y = f(t)$ and $t = g(x)$. Then, $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$.

(ii) Let $y = f(t)$, $t = g(u)$ and $u = h(x)$. Then, $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}\right)$.

This rule may be extended further on more variables.

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate each of the following w.r.t. x:*

(i) $\sin x^3$ (ii) $\cos^3 x$ (iii) $\tan \sqrt{x}$

SOLUTION (i) Let $y = \sin x^3$.

Putting $x^3 = t$, we get

$y = \sin t$ and $t = x^3$

$$\Rightarrow \frac{dy}{dt} = \cos t \text{ and } \frac{dt}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$$

$$= (\cos t \cdot 3x^2) = 3x^2 \cos t = 3x^2 \cos x^3.$$

Hence, $\frac{d}{dx}(\sin x^3) = 3x^2 \cos x^3$.

(ii) Let $y = \cos^3 x = (\cos x)^3$.

Putting $\cos x = t$, we get

$y = t^3$ and $t = \cos x$

$$\Rightarrow \frac{dy}{dt} = 3t^2 \text{ and } \frac{dt}{dx} = -\sin x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$$

$$= (-3t^2 \sin x) = (-3 \sin x)t^2 = (-3 \sin x \cos^2 x).$$

Hence, $\frac{d}{dx}(\cos^3 x) = -3 \sin x \cos^2 x$.

(iii) Let $y = \tan \sqrt{x}$.

Putting $\sqrt{x} = t$, we get

$y = \tan t$ and $t = \sqrt{x}$

$$\Rightarrow \frac{dy}{dt} = \sec^2 t \text{ and } \frac{dt}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$$

$$= \left(\sec^2 t \cdot \frac{1}{2\sqrt{x}}\right) = \frac{\sec^2 \sqrt{x}}{2\sqrt{x}} \quad [\because t = \sqrt{x}].$$

Hence, $\frac{d}{dx}(\tan \sqrt{x}) = \frac{\sec^2 \sqrt{x}}{2\sqrt{x}}$.

EXAMPLE 2 *Differentiate each of the following w.r.t. x:*

(i) $(ax + b)^m$ (ii) $(2x + 3)^5$ (iii) $\sqrt{ax^2 + 2bx + c}$

SOLUTION (i) Let $y = (ax + b)^m$.

Putting $(ax + b) = t$, we get

$$y = t^m \text{ and } t = (ax + b)$$

$$\Rightarrow \frac{dy}{dt} = mt^{m-1} \text{ and } \frac{dt}{dx} = a$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = (mt^{m-1} \times a) = mat^{m-1} = ma(ax + b)^{m-1}.$$

$$\therefore \frac{d}{dx}(ax + b)^m = ma(ax + b)^{m-1}.$$

(ii) Let $y = (2x + 3)^5$.

Putting $(2x + 3) = t$, we get

$$y = t^5 \text{ and } t = 2x + 3$$

$$\Rightarrow \frac{dy}{dt} = 5t^4 \text{ and } \frac{dt}{dx} = 2$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = 10t^4 = 10(2x + 3)^4.$$

(iii) Let $y = \sqrt{ax^2 + 2bx + c}$.

Putting $(ax^2 + 2bx + c) = t$, we get

$$y = \sqrt{t} \text{ and } t = (ax^2 + 2bx + c)$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx} = (2ax + 2b) = 2(ax + b)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{1}{2\sqrt{t}} \times 2(ax + b)$$

$$= \frac{(ax + b)}{\sqrt{t}} = \frac{(ax + b)}{\sqrt{ax^2 + 2bx + c}}.$$

EXAMPLE 3 *Differentiate* $\sin 3x \cos 5x$ *w.r.t.* x.

SOLUTION Let $y = \sin 3x \cos 5x = \frac{1}{2}\{2\cos 5x \sin 3x\}$

$$= \frac{1}{2}\{\sin (5x + 3x) - \sin (5x - 3x)\}$$

$$= \frac{1}{2}\cdot(\sin 8x - \sin 2x) = \frac{1}{2}\sin 8x - \frac{1}{2}\sin 2x.$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\cdot\frac{d}{dx}(\sin 8x) - \frac{1}{2}\cdot\frac{d}{dx}(\sin 2x)$$

$$= \left(\frac{1}{2}\cdot 8\cos 8x - \frac{1}{2} \times 2\cos 2x\right) = (4\cos 8x - \cos 2x).$$

EXAMPLE 4 *Differentiate* $\sin 2x \sin 4x$ *w.r.t.* x.

SOLUTION Let $y = \sin 2x \sin 4x = \frac{1}{2}(2\sin 4x \sin 2x)$

$$= \frac{1}{2}\{\cos(4x - 2x) - \cos(4x + 2x)\}$$

$$= \frac{1}{2}[\cos 2x - \cos 6x].$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2} \cdot \frac{d}{dx}(\cos 2x) - \frac{1}{2} \cdot \frac{d}{dx}(\cos 6x)$$

$$= \frac{1}{2} \cdot (-2\sin 2x) - \frac{1}{2} \cdot (-6\sin 6x)$$

$$= (3\sin 6x - \sin 2x).$$

EXAMPLE 5 *Differentiate* $\sqrt{\frac{1-\tan x}{1+\tan x}}$ *w.r.t. x.*

SOLUTION Let $y = \sqrt{\frac{1-\tan x}{1+\tan x}}$.

Putting $\frac{(1-\tan x)}{(1+\tan x)} = t$, we get

$$y = \sqrt{t} \text{ and } t = \frac{(1-\tan x)}{(1+\tan x)}.$$

$$\therefore \quad \frac{dy}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}.$$

$$\text{And, } \frac{dt}{dx} = \frac{(1+\tan x) \cdot \frac{d}{dx}(1-\tan x) - (1-\tan x) \cdot \frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)(-\sec^2 x) - (1-\tan x)(\sec^2 x)}{(1+\tan x)^2} = \frac{-2\sec^2 x}{(1+\tan x)^2}.$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{1}{2\sqrt{t}} \times \frac{-2\sec^2 x}{(1+\tan x)^2}$$

$$= \frac{-\sec^2 x}{(1+\tan x)^2} \times \frac{\sqrt{1+\tan x}}{\sqrt{1-\tan x}}$$

$$= \frac{-\sec^2 x}{(1+\tan x)^{3/2}(1-\tan x)^{1/2}}.$$

EXAMPLE 6 *If* $y = \frac{1}{\sqrt{a^2 - x^2}}$, *find* $\frac{dy}{dx}$.

SOLUTION Putting $(a^2 - x^2) = t$, we get

$$y = \frac{1}{\sqrt{t}} = t^{-1/2} \text{ and } t = (a^2 - x^2)$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{1}{2}t^{-3/2} = \frac{-1}{2t^{3/2}} \text{ and } \frac{dt}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{-1}{2t^{3/2}} \times (-2x) = \frac{x}{t^{3/2}} = \frac{x}{(a^2 - x^2)^{3/2}}.$$

EXAMPLE 7 *If* $y = \cos^2 x^2$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $y = (\cos x^2)^2$.

Putting $x^2 = t$ and $\cos t = u$, we get

$$y = u^2,\ u = \cos t \text{ and } t = x^2$$

$$\Rightarrow \frac{dy}{du} = 2u,\ \frac{du}{dt} = -\sin t \text{ and } \frac{dt}{dx} = 2x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= (2u) \times (-\sin t) \times 2x$$

$$= -4xu \sin t = -4x \cos t \sin t = -4x \cos x^2 \sin x^2.$$

$$\therefore \frac{dy}{dx} = -4x \cos x^2 \sin x^2.$$

EXAMPLE 8 *If* $y = \sin(\cos x^2)$, *find* $\frac{dy}{dx}$.

SOLUTION Putting $x^2 = t$ and $\cos x^2 = \cos t = u$, we get

$$y = \sin u,\ u = \cos t \text{ and } t = x^2$$

$$\Rightarrow \frac{dy}{du} = \cos u,\ \frac{du}{dt} = -\sin t \text{ and } \frac{dt}{dx} = 2x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \{\cos u \times (-\sin t) \times 2x\} = -2x \sin t \cos u$$

$$= -2x \sin t \cos(\cos t) = -2x \sin x^2 \cos(\cos x^2).$$

$$\therefore \frac{dy}{dx} = -2x \sin x^2 \cos(\cos x^2).$$

EXAMPLE 9 *If* $y = \sin(\sqrt{\sin x + \cos x})$, *find* $\frac{dy}{dx}$.

SOLUTION Putting $(\sin x + \cos x) = t$ and $\sqrt{t} = u$, we get

$$y = \sin u,\ u = \sqrt{t} \text{ and } t = (\sin x + \cos x)$$

$$\Rightarrow \frac{dy}{du} = \cos u,\ \frac{du}{dt} = \frac{1}{2} t^{-1/2} = \frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx} = (\cos x - \sin x)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \left\{\cos u \times \frac{1}{2\sqrt{t}} \times (\cos x - \sin x)\right\} = \frac{\cos\sqrt{t}}{2\sqrt{t}} \cdot (\cos x - \sin x)$$

$$= \frac{\cos(\sqrt{\sin x + \cos x})}{2\sqrt{\sin x + \cos x}} \cdot (\cos x - \sin x).$$

EXAMPLE 10 *If* $y = \sin[\sqrt{\sin\sqrt{x}}]$, *find* $\frac{dy}{dx}$.

SOLUTION Putting $\sqrt{x} = t$, $\sin\sqrt{x} = \sin t = u$ and $\sqrt{\sin\sqrt{x}} = \sqrt{u} = v$, we get

$$y = \sin v,\ v = \sqrt{u},\ u = \sin t \text{ and } t = \sqrt{x}.$$

$$\therefore \quad \frac{dy}{dv} = \cos v;\ \frac{dv}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}};\ \frac{du}{dt} = \cos t \text{ and } \frac{dt}{dx} = \frac{1}{2\sqrt{x}}.$$

$$\text{So,} \quad \frac{dy}{dx} = \left(\frac{dy}{dv}\times\frac{dv}{du}\times\frac{du}{dt}\times\frac{dt}{dx}\right) = \left[\cos v\cdot\frac{1}{2\sqrt{u}}\cdot\cos t\cdot\frac{1}{2\sqrt{x}}\right]$$

$$= \left[\cos\sqrt{u}\cdot\frac{1}{2\sqrt{u}}\cos t\cdot\frac{1}{2\sqrt{x}}\right] \qquad [\because\ v = \sqrt{u}]$$

$$= \frac{1}{4}\cos(\sqrt{\sin t})\cdot\frac{1}{\sqrt{\sin t}}\cdot\cos\sqrt{x}\cdot\frac{1}{\sqrt{x}} \qquad [\because\ u = \sin t]$$

$$= \frac{1}{4}\cos(\sqrt{\sin\sqrt{x}})\cdot\frac{1}{\sqrt{\sin\sqrt{x}}}\cdot\cos\sqrt{x}\cdot\frac{1}{\sqrt{x}} \qquad [\because\ t = \sqrt{x}]$$

$$= \frac{\cos(\sqrt{\sin\sqrt{x}})}{4\sqrt{x}\sqrt{\sin\sqrt{x}}}\cdot\cos\sqrt{x}.$$

EXAMPLE 11 *If* $y = \frac{5x}{\sqrt[3]{1-x^2}} + \sin^2(2x+3)$, *find* $\frac{dy}{dx}$. **[CBSE 2000C]**

SOLUTION We have

$$y = 5x(1-x^2)^{-1/3} + \sin^2(2x+3)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\{5x(1-x^2)^{-1/3}\} + \frac{d}{dx}\{\sin^2(2x+3)\}$$

$$= \left\{5x\cdot\left(\frac{-1}{3}\right)(1-x^2)^{-4/3}\cdot(-2x) + (1-x^2)^{-1/3}-5\right\}$$

$$+ \{2\sin(2x+3)\cos(2x+3)\cdot 2\}$$

$$= \frac{10x^2}{3(1-x^2)^{4/3}} + \frac{5}{(1-x^2)^{1/3}} + 2\sin(4x+6)$$

$$= \frac{10x^2 + 15(1-x^2)}{3(1-x^2)^{4/3}} + 2\sin(4x+6)$$

$$= \frac{(15-5x^2)}{3(1-x^2)^{4/3}} + 2\sin(4x+6).$$

EXERCISE 10A

Differentiate each of the following w.r.t. x:

1. $\sin 4x$
2. $\cos 5x$
3. $\tan 3x$
4. $\cos x^3$
5. $\cot^2 x$
6. $\tan^3 x$

7. $\cot \sqrt{x}$
8. $\sqrt{\tan x}$
9. $(5 + 7x)^6$
10. $(3 - 4x)^5$
11. $(2x^2 - 3x + 4)^5$
12. $(ax^2 + bx + c)^6$
13. $\dfrac{1}{(x^2 - 3x + 5)^3}$
14. $\sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}$
15. $\sqrt{\dfrac{1 + \sin x}{1 - \sin x}}$
16. $\cos^2 x^3$
17. $\sec^3(x^2 + 1)$
18. $\sqrt{\cos 3x}$
19. $\sqrt[3]{\sin 2x}$
20. $\sqrt{1 + \cot x}$
21. $\operatorname{cosec}^3 \dfrac{1}{x^2}$
22. $\sqrt{\sin x^3}$
23. $\sqrt{x \sin x}$
24. $\sqrt{\cot \sqrt{x}}$
25. $\cot^3 x^2$
26. $\cos(\sin \sqrt{ax + b})$
27. $\sqrt{\operatorname{cosec}(x^3 + 1)}$
28. $\sin 5x \cos 3x$
29. $\sin 2x \sin x$
30. $\cos 4x \cos 2x$

Find $\dfrac{dy}{dx}$***, when:***

31. $y = \sin\left(\dfrac{1 + x^2}{1 - x^2}\right)$
32. $y = \dfrac{(\sin x + x^2)}{\cot 2x}$
33. If $y = \dfrac{(\cos x - \sin x)}{(\cos x + \sin x)}$, prove that $\dfrac{dy}{dx} + y^2 + 1 = 0.$
34. If $y = \dfrac{(\cos x + \sin x)}{(\cos x - \sin x)}$, prove that $\dfrac{dy}{dx} = \sec^2\left(x + \dfrac{\pi}{4}\right)$.

ANSWERS (EXERCISE 10A)

1. $4\cos 4x$
2. $-5\sin 5x$
3. $3\sec^2 3x$
4. $-3x^2 \sin x^3$
5. $-2\cot x \operatorname{cosec}^2 x$
6. $3\tan^2 x \sec^2 x$
7. $\dfrac{-1}{2\sqrt{x}} \operatorname{cosec}^2 \sqrt{x}$
8. $\dfrac{\sec^2 x}{2\sqrt{\tan x}}$
9. $42(5 + 7x)^5$
10. $-20(3 - 4x)^4$
11. $5(2x^2 - 3x + 4)^4(4x - 3)$
12. $6(ax^2 + bx + c)(2ax + b)$
13. $\dfrac{-3(2x - 3)}{(x^2 - 3x + 5)^4}$
14. $\dfrac{-2a^2 x}{(a^2 + x^2)^{3/2}(a^2 - x^2)^{1/2}}$
15. $\sec x(\sec x + \tan x)$
16. $-3x^2 \sin(2x^3)$
17. $6x \sec^3(x^2 + 1)\tan(x^2 + 1)$
18. $\dfrac{-3}{2} \cdot \dfrac{\sin 3x}{\sqrt{\cos 3x}}$
19. $\dfrac{2}{3} \cdot \dfrac{\cos 2x}{(\sin 2x)^{2/3}}$
20. $-\dfrac{1}{2} \cdot \dfrac{\operatorname{cosec}^2 x}{\sqrt{1 + \cot x}}$
21. $\dfrac{6}{x^3} \operatorname{cosec}^3 \dfrac{1}{x^2} \cot \dfrac{1}{x^2}$
22. $\dfrac{3}{2} x^2 \cdot \dfrac{\cos x^3}{\sqrt{\sin x^3}}$
23. $\dfrac{(x\cos x + \sin x)}{2\sqrt{x \sin x}}$

24. $\dfrac{-\text{cosec}^2\sqrt{x}}{4\sqrt{x}\,\sqrt{\cot\sqrt{x}}}$

25. $-6x\cot^2 x^2\,\text{cosec}^2 x^2$

26. $\dfrac{-a\cos\sqrt{ax+b}}{2\sqrt{ax+b}}\cdot\sin\{\sin\sqrt{ax+b}\}$

27. $-\dfrac{3}{2}x^2\sqrt{\text{cosec}\,(x^3+1)}\cdot\cot\,(x^3+1)$

28. $(4\cos 8x+\cos 2x)$

29. $\left(\dfrac{3}{2}\sin\,3x-\dfrac{1}{2}\sin\,x\right)$

30. $-(3\sin\,6x+\sin\,2x)$

31. $\dfrac{4x}{(1-x^2)^2}\cdot\cos\left(\dfrac{1+x^2}{1-x^2}\right)$

32. $2(\sin\,x+x^2)\sec^2 2x+(\cos\,x+2x)\tan\,2x$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10A)

15. $y=\dfrac{\sqrt{1+\sin x}}{\sqrt{1-\sin x}}\times\dfrac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}=\dfrac{(1+\sin x)}{\cos x}=(\sec x+\tan x)$

$\Rightarrow\ \dfrac{dy}{dx}=(\sec x\tan x+\sec^2 x)=\sec x(\sec x+\tan x).$

32. $y=(\sin x+x^2)\tan 2x$

$\Rightarrow\ \dfrac{dy}{dx}=(\sin x+x^2)\cdot\dfrac{d}{dx}(\tan 2x)+\tan 2x\cdot\dfrac{d}{dx}(\sin x+x^2).$

33. $y=\left(\dfrac{1-\tan x}{1+\tan x}\right)$ [on dividing num. and denom. by cos x]

$\Rightarrow\ y=\tan\left(\dfrac{\pi}{4}-x\right)$

$\Rightarrow\ \dfrac{dy}{dx}=-\sec^2\left(\dfrac{\pi}{4}-x\right)$

$\Rightarrow\ \dfrac{dy}{dx}+y^2+1=-\sec^2\left(\dfrac{\pi}{4}-x\right)+\tan^2\left(\dfrac{\pi}{4}-x\right)+1=0.$

34. $y=\left(\dfrac{1+\tan x}{1-\tan x}\right)$

$\Rightarrow\ y=\tan\left(\dfrac{\pi}{4}+x\right)$

$\Rightarrow\ \dfrac{dy}{dx}=\sec^2\left(\dfrac{\pi}{4}+x\right).$

2. Derivatives of Exponential and Logarithmic Functions

EXPONENTIAL FUNCTION

Let a be a real number such that $a>1$.

Then, $f(x)=a^x$ is called an ***exponential function***.

Its domain = R and range = R^+.

When $a = e$, we have the exponential function, $f(x) = e^x$.

This is called ***natural exponential function***.

LOGARITHMIC FUNCTION

Let a be a real number such that $a > 1$.

If $a^x = b$, we define, $\log_a x = b$.

We say that log of x to the base a is equal to b.

$\log_{10} x$ is called ***common logarithm of x.***

$\log_e x$ is called ***natural logarithm of x*** and we denote it simply by $\log x$.

It is easy to verify the following results:

(i) $\log (xy) = (\log x) + (\log y)$ (ii) $\log \left(\frac{x}{y}\right) = (\log x) - (\log y)$

(iii) $\log (x^n) = n \log x$ (iv) $\log_a x = \frac{1}{\log_x a}$

(v) $\log_x x = 1$ and $\log 1 = 0$

Derivatives of Exponential and Logarithmic Functions

We have (i) $\frac{d}{dx}(e^x) = e^x$ (ii) $\frac{d}{dx}(\log x) = \frac{1}{x}$.

Derivative of a^x

Let $y = a^x$. Then, $\log y = x \log a$. ...

(i)

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log a \Rightarrow \frac{dy}{dx} = y(\log a)$$

$$\Rightarrow \frac{dy}{dx} = a^x (\log a).$$

Hence, $\frac{d}{dx}(a^x) = a^x (\log a)$.

SUMMARY

(i) $\frac{d}{dx}(e^x) = e^x$ (ii) $\frac{d}{dx}(a^x) = a^x (\log a)$

(iii) $\frac{d}{dx}(\log x) = \frac{1}{x}$ (iv) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate each of the following w.r.t. x:*

(i) e^{x^2} (ii) e^{-3x} (iii) $e^{\cos x}$

SOLUTION (i) Let $y = e^{x^2}$.

Putting $x^2 = t$, we get

$y = e^t$ and $t = x^2$

$\Rightarrow \frac{dy}{dt} = e^t$ and $\frac{dt}{dx} = 2x$

$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = (e^t \times 2x) = (e^{x^2} \times 2x) = 2x e^{x^2}$.

Hence, $\frac{d}{dx}(e^{x^2}) = 2x e^{x^2}$.

(ii) Let $y = e^{-3x}$.

Putting $-3x = t$, we get

$y = e^t$ and $t = -3x$

$\Rightarrow \frac{dy}{dt} = e^t$ and $\frac{dt}{dx} = -3$

$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = -3e^t = -3e^{-3x}$.

Hence, $\frac{d}{dx}(e^{-3x}) = -3e^{-3x}$.

(iii) Let $y = e^{\cos x}$.

Putting $\cos x = t$, we get

$y = e^t$ and $t = \cos x$

$\Rightarrow \frac{dy}{dt} = e^t$ and $\frac{dt}{dx} = -\sin x$

$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = (-e^t \sin x) = (-e^{\cos x} \sin x)$.

Hence, $\frac{d}{dx}(e^{\cos x}) = -e^{\cos x} \sin x$.

EXAMPLE 2 *Differentiate each of the following w.r.t. x:*

(i) $\sin(\log x),\ x > 0$ *(ii)* $\log(\log x),\ x > 1$

SOLUTION (i) Let $y = \sin(\log x)$.

Putting $\log x = t$, we get

$y = \sin t$ and $t = \log x$

$\Rightarrow \frac{dy}{dt} = \cos t$ and $\frac{dt}{dx} = \frac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \left(\cos t \times \frac{1}{x}\right) = \cos(\log x) \times \frac{1}{x} = \frac{\cos(\log x)}{x}.$$

Hence, $\frac{d}{dx}\{\sin(\log x)\} = \frac{\cos(\log x)}{x}$.

(ii) Let $y = \log(\log x)$.

Putting $\log x = t$, we get

$$y = \log t \text{ and } t = \log x$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{t} \text{ and } \frac{dt}{dx} = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \left(\frac{1}{t} \times \frac{1}{x}\right) = \left(\frac{1}{\log x} \times \frac{1}{x}\right) = \frac{1}{(x \log x)}.$$

$$\therefore \frac{d}{dx}\{\log(\log x)\} = \frac{1}{(x \log x)}.$$

EXAMPLE 3 *If* $y = e^{\sqrt{\cot x}}$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $y = e^{\sqrt{\cot x}}$.

Putting $\cot x = t$ and $\sqrt{\cot x} = \sqrt{t} = u$, we get

$$y = e^u, \quad u = \sqrt{t} \text{ and } t = \cot x$$

$$\Rightarrow \frac{dy}{du} = e^u, \quad \frac{du}{dt} = \frac{1}{2} t^{-1/2} = \frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx} = -\operatorname{cosec}^2 x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \left\{e^u \cdot \frac{1}{2\sqrt{t}} \cdot (-\operatorname{cosec}^2 x)\right\} = e^{\sqrt{\cot x}} \cdot \frac{1}{2\sqrt{\cot x}} \cdot (-\operatorname{cosec}^2 x)$$

$$= \frac{(-\operatorname{cosec}^2 x)\, e^{\sqrt{\cot x}}}{2\sqrt{\cot x}}.$$

EXAMPLE 4 *If* $y = \log \tan \frac{x}{2}$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $y = \log \tan \frac{x}{2}$.

Putting $\frac{x}{2} = t$ and $\tan \frac{x}{2} = \tan t = u$, we get

$$y = \log u, \quad u = \tan t \text{ and } t = \frac{x}{2}$$

$$\Rightarrow \frac{dy}{du} = \frac{1}{u}, \quad \frac{du}{dt} = \sec^2 t \text{ and } \frac{dt}{dx} = \frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \left(\frac{1}{u} \times \sec^2 t \times \frac{1}{2}\right) = \left(\frac{1}{2\tan t} \times \sec^2 t\right) = \frac{1}{\sin 2t} = \frac{1}{\sin x} = \operatorname{cosec} x.$$

Hence, $\frac{dy}{dx} = \operatorname{cosec} x$.

EXAMPLE 5 *If* $y = \frac{1}{\log \cos x}$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $y = (\log \cos x)^{-1}$.

Putting $\cos x = t$ and $\log \cos x = \log t = u$, we get

$$y = u^{-1} = \frac{1}{u},\ u = \log t \text{ and } t = \cos x$$

$$\Rightarrow \frac{dy}{du} = \frac{-1}{u^2},\ \frac{du}{dt} = \frac{1}{t} \text{ and } \frac{dt}{dx} = -\sin x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \left\{\frac{-1}{u^2} \times \frac{1}{t} \times (-\sin x)\right\} = \left\{\frac{1}{(\log \cos x)^2} \cdot \frac{1}{\cos x} \cdot \sin x\right\}$$

$$= \frac{\tan x}{(\log \cos x)^2}.$$

EXAMPLE 6 *If* $y = \sqrt{e^{\sqrt{x}}}$, *find* $\frac{dy}{dx}$.

SOLUTION Putting $\sqrt{x} = t$, $e^{\sqrt{x}} = e^t = u$, we get

$$y = \sqrt{u},\ u = e^t \text{ and } t = \sqrt{x}$$

$$\Rightarrow \frac{dy}{du} = \frac{1}{2} u^{-1/2} = \frac{1}{2\sqrt{u}},\ \frac{du}{dt} = e^t \text{ and } \frac{dt}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \left(\frac{1}{2\sqrt{u}} \times e^t \times \frac{1}{2\sqrt{x}}\right) = \left\{\frac{1}{2\sqrt{u}} \times u \times \frac{1}{2\sqrt{x}}\right\} = \frac{\sqrt{u}}{4\sqrt{x}} = \frac{e^{\frac{1}{2}t}}{4\sqrt{x}}$$

$$= \frac{e^{\frac{1}{2}\sqrt{x}}}{4\sqrt{x}}.$$

EXAMPLE 7 *If* $y = \log \log \log x^3$, *find* $\frac{dy}{dx}$.

SOLUTION Let $x^3 = t$, $\log x^3 = \log t = u$ and $\log \log x^3 = \log u = v$.

Then, $y = \log v$, $v = \log u$, $u = \log t$ and $t = x^3$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \left(\frac{1}{v} \times \frac{1}{u} \times \frac{1}{t} \times 3x^2\right) = \frac{3x^2}{tuv}$$

$$= \frac{3x^2}{x^3(\log t)(\log u)} = \frac{3}{x(\log t)(\log \log t)}$$

$$= \frac{3}{x(\log x^3)(\log \log x^3)} = \frac{1}{x(\log x)(\log \log x^3)} \cdot$$

EXAMPLE 8 *If* $y = \sqrt{\log\left\{\sin\left(\frac{x^2}{3} - 1\right)\right\}}$*, find* $\frac{dy}{dx}\cdot$

SOLUTION Put $\left(\frac{x^2}{3} - 1\right) = t$, $\sin\left(\frac{x^2}{3} - 1\right) = \sin t = u$

and $\log\left\{\sin\left(\frac{x^2}{3} - 1\right)\right\} = \log u = v.$

Then, $y = \sqrt{v}$, where $v = \log u$, $u = \sin t$ and $t = \left(\frac{x^2}{3} - 1\right)\cdot$

$\therefore \quad \frac{dy}{dv} = \frac{1}{2}v^{-1/2} = \frac{1}{2\sqrt{v}}$; $\frac{dv}{du} = \frac{1}{u}$; $\frac{du}{dt} = \cos t$ and $\frac{dt}{dx} = \frac{2x}{3}\cdot$

So, $$\frac{dy}{dx} = \left(\frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \left(\frac{1}{2\sqrt{v}} \cdot \frac{1}{u} \cdot \cos t \cdot \frac{2x}{3}\right)$$

$$= \frac{x}{3} \cdot \frac{\cos t}{u \cdot \sqrt{\log u}} = \frac{x}{3} \cdot \frac{\cos t}{\sin t\sqrt{\log \sin t}}$$

[$\because$ $v = \log u$ and $u = \sin t$]

$$= \frac{x \cot t}{3\sqrt{\log \sin t}} = \frac{x \cot\left(\frac{x^2}{3} - 1\right)}{3 \cdot \sqrt{\log \sin\left(\frac{x^2}{3} - 1\right)}} \qquad \left[\because \; t = \left(\frac{x^2}{3} - 1\right)\right].$$

EXAMPLE 9 *If* $y = e^x \log(\sin 2x)$*, find* $\frac{dy}{dx}\cdot$

SOLUTION We have

$$\frac{dy}{dx} = \frac{d}{dx}\{e^x \log(\sin 2x)\}$$

$$= e^x \cdot \frac{d}{dx}\{\log(\sin 2x)\} + \log(\sin 2x) \cdot \frac{d}{dx}(e^x)$$

$$= e^x \cdot \left\{\frac{1}{\sin 2x} \cdot \cos 2x \cdot 2\right\} + \log(\sin 2x) \cdot e^x$$

$$= 2e^x \cot 2x + e^x \log(\sin 2x)$$

$$= e^x\{2\cot 2x + \log(\sin 2x)\}.$$

EXAMPLE 10 *If* $y = e^{ax}\cos(bx + c)$, *find* $\frac{dy}{dx}$.

SOLUTION We have

$$\frac{dy}{dx} = e^{ax}\cdot\frac{d}{dx}\{\cos(bx + c)\} + \cos(bx + c)\cdot\frac{d}{dx}(e^{ax})$$

$$= e^{ax}\cdot\{-b\sin(bx + c)\} + \cos(bx + c)\cdot ae^{ax}$$

$$= e^{ax}\cdot\{a\cos(bx + c) - b\sin(bx + c)\}.$$

EXAMPLE 11 *Differentiate* $\log\sqrt{\frac{1 + \cos^2 x}{(1 - e^{2x})}}$ *w.r.t. x.* **[CBSE 2003C]**

SOLUTION Let $y = \log\sqrt{\frac{1+\cos^2 x}{(1-e^{2x})}} = \log\left(\frac{1+\cos^2 x}{1-e^{2x}}\right)^{1/2} = \frac{1}{2}\log\left(\frac{1+\cos^2 x}{1-e^{2x}}\right)$.

$$\therefore\ \ y = \frac{1}{2}\log(1+\cos^2 x) - \frac{1}{2}\log(1-e^{2x})$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{(1+\cos^2 x)}(2\cos x)(-\sin x) - \frac{1}{2}\cdot\frac{1}{(1-e^{2x})}\cdot(-2e^{2x})$$

$$= \left\{\frac{-\sin x\cos x}{(1+\cos^2 x)} + \frac{e^{2x}}{(1-e^{2x})}\right\}.$$

Hence, $\frac{d}{dx}\left\{\log\sqrt{\frac{1+\cos^2 x}{(1-e^{2x})}}\right\} = \left\{\frac{-\sin x\cos x}{(1+\cos^2 x)} + \frac{e^{2x}}{(1-e^{2x})}\right\}$.

EXAMPLE 12 *If* $y = \log\sqrt{\frac{1 + \sin^2 x}{1 - \sin x}}$, *find* $\frac{dy}{dx}$. **[CBSE 2003C]**

SOLUTION We have

$$y = \frac{1}{2}\{\log(1+\sin^2 x) - \log(1-\sin x)\}. \qquad \text{... (i)}$$

On differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}\cdot\left[\frac{d}{dx}\{\log(1+\sin^2 x)\} - \frac{d}{dx}\{\log(1-\sin x)\}\right]$$

$$= \frac{1}{2}\cdot\left\{\frac{2\sin x\cos x}{(1+\sin^2 x)} - \frac{(-\cos x)}{(1-\sin x)}\right\}$$

$$= \frac{1}{2}\cdot\left\{\frac{\sin 2x}{(1+\sin^2 x)} + \frac{\cos x}{(1-\sin x)}\right\}.$$

EXAMPLE 13 *If* $y = \log\sin(e^x + 5x + 8)$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $y = \log\sin(e^x + 5x + 8)$. ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sin(e^x + 5x + 8)} \cdot \cos(e^x + 5x + 8) \cdot \frac{d}{dx}(e^x + 5x + 8)$$

$$= \{\cot(e^x + 5x + 8)\}(e^x + 5) = (e^x + 5) \cdot \cot(e^x + 5x + 8).$$

EXAMPLE 14 *If* $y = \sqrt{x^2 + 1} - \log\left\{\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}}\right\}$, *find* $\frac{dy}{dx}$. **[CBSE 2008]**

SOLUTION We have

$$y = \sqrt{x^2 + 1} - \log\left\{\frac{1 + \sqrt{x^2 + 1}}{x}\right\}$$

$$\Rightarrow \quad y = \sqrt{x^2 + 1} - \log\{1 + \sqrt{x^2 + 1}\} + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x - \frac{1}{\{1 + \sqrt{x^2 + 1}\}} \cdot \left\{\frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x\right\} + \frac{1}{x}$$

$$= \frac{x}{\sqrt{x^2 + 1}} - \frac{1}{\{1 + \sqrt{x^2 + 1}\}} \cdot \frac{x}{\sqrt{x^2 + 1}} + \frac{1}{x}$$

$$= \frac{x\{1 + \sqrt{x^2 + 1}\} - x}{(\sqrt{x^2 + 1})\{1 + \sqrt{x^2 + 1}\}} + \frac{1}{x} = \frac{x\sqrt{x^2 + 1}}{(\sqrt{x^2 + 1})\{1 + \sqrt{x^2 + 1}\}} + \frac{1}{x}$$

$$= \frac{x}{\{1 + \sqrt{x^2 + 1}\}} + \frac{1}{x} = \frac{(x^2 + 1) + \sqrt{x^2 + 1}}{x\{1 + \sqrt{x^2 + 1}\}}$$

$$= \frac{(\sqrt{x^2 + 1})\{(\sqrt{x^2 + 1}) + 1\}}{x\{1 + \sqrt{x^2 + 1}\}} = \frac{\sqrt{x^2 + 1}}{x}.$$

EXERCISE 10B

Differentiate each of the following w.r.t. x:

1. (i) e^{4x} (ii) e^{-5x} (iii) e^{x^3}
2. (i) $e^{2/x}$ (ii) $e^{\sqrt{x}}$ (iii) $e^{-2\sqrt{x}}$
3. (i) $e^{\cot x}$ (ii) $e^{-\sin 2x}$ (iii) $e^{\sqrt{\sin x}}$
4. (i) $\tan(\log x)$ (ii) $\log \sec x$ (iii) $\log \sin \frac{x}{2}$
5. (i) $\log_3 x$ (ii) 2^{-x} (iii) 3^{x+2}
6. (i) $\log\left(x + \frac{1}{x}\right)$ (ii) $\log \sin 3x$ (iii) $\log(x + \sqrt{1 + x^2})$ **[CBSE 2003]**
7. $e^{\sqrt{x}} \log x$
8. $\log \sin \sqrt{x^2 + 1}$ **[CBSE 2003]**
9. $e^{2x} \sin 3x$
10. $e^{3x} \cos 2x$
11. $e^{-5x} \cot 4x$
12. $e^x \log(\sin 2x)$
13. $\log(\operatorname{cosec} x - \cot x)$
14. $\log\left(\sec \frac{x}{2} + \tan \frac{x}{2}\right)$

15. $\sqrt{\dfrac{1+e^x}{1-e^x}}$

16. $\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

17. $xe^{\sqrt{\sin\, x}}$

18. $e^{\sin x} \sin(e^x)$

19. $e^{\sqrt{1-x^2}} \tan x$

20. $\dfrac{e^x}{(1+\cos x)}$

21. $x^3 e^x \cos x$

22. $e^{x\cos x}$

ANSWERS (EXERCISE 10B)

1. (i) $4e^{4x}$ (ii) $-5e^{-5x}$ (iii) $(3x^2)e^{x^3}$

2. (i) $-\dfrac{2}{x^2} \cdot e^{2/x}$ (ii) $\dfrac{1}{2\sqrt{x}} e^{\sqrt{x}}$ (iii) $\dfrac{-1}{\sqrt{x}} e^{-2\sqrt{x}}$

3. (i) $-e^{\cot x}(\text{cosec}^2 x)$ (ii) $(-2\cos 2x)e^{-\sin 2x}$ (iii) $\dfrac{\cos x}{2\sqrt{\sin x}} \cdot e^{\sqrt{\sin x}}$

4. (i) $\dfrac{\sec^2(\log x)}{x}$ (ii) $\tan x$ (iii) $\dfrac{1}{2}\cot\dfrac{x}{2}$

5. (i) $\dfrac{1}{x(\log 3)}$ (ii) $-2^{-x}\log 2$ (iii) $(9 \times 3^x \log 3)$

6. (i) $\dfrac{(x^2-1)}{x(x^2+1)}$ (ii) $3\cot 3x$ (iii) $\dfrac{1}{\sqrt{1+x^2}}$

7. $\dfrac{e^{\sqrt{x}}(2+\sqrt{x}\log x)}{2x}$

8. $\dfrac{x}{\sqrt{x^2+1}}\cot\sqrt{x^2+1}$

9. $e^{2x}(3\cos 3x + 2\sin 3x)$

10. $e^{3x}(3\cos 2x - 2\sin 2x)$

11. $-e^{-5x}(5\cot 4x + 4\text{cosec}^2 4x)$

12. $e^x\{2\cot 2x + \log\sin 2x\}$

13. $\text{cosec}\, x$

14. $\dfrac{1}{2}\sec\dfrac{x}{2}$

15. $\dfrac{e^x}{(1-e^x)\sqrt{1-e^{2x}}}$

16. $\dfrac{-4}{(e^x - e^{-x})^2}$

17. $e^{\sqrt{\sin x}} \cdot \left(\dfrac{x\cos x}{2\sqrt{\sin x}} + 1\right)$

18. $e^{\sin x}(e^x \cos e^x + \cos x \sin e^x)$

19. $e^{\sqrt{1-x^2}}\left\{\sec^2 x - \dfrac{x\tan x}{\sqrt{1-x^2}}\right\}$

20. $\dfrac{e^x(1+\cos x+\sin x)}{(1+\cos x)^2}$

21. $e^x x^2(x\cos x - x\sin x + 3\cos x)$

22. $e^{x\cos x}(\cos x - x\sin x)$

---

3. Derivatives of Inverse Trigonometric Functions

In the table given below, we mention the domain and range of various inverse trigonometric functions.

Function	Domain	Range
(i) $\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(ii) $\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
(iii) $\tan^{-1}x$	R	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(iv) $\cot^{-1}x$	R	$]0, \pi[$
(v) $\sec^{-1}x$	$R -]-1, 1[$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
(vi) $\text{cosec}^{-1}x$	$R -]-1, 1[$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Derivatives of Inverse Trigonometric Functions

EXAMPLE 1 *Prove that* $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$, *where* $x \in]-1, 1[$.

SOLUTION Let $y = \sin^{-1}x$, where $x \in]-1, 1[$ and $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ · Then,

$$y = \sin^{-1}x \Rightarrow x = \sin y$$

$$\Rightarrow \frac{dx}{dy} = \cos y \geq 0 \text{ since } y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

$$\Rightarrow \frac{dx}{dy} = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

Hence, $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$.

EXAMPLE 2 *Prove that* $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$, *where* $x \in]-1, 1[$.

SOLUTION Let $y = \cos^{-1}x$, where $x \in]-1, 1[$ and $y \in \left]0, \frac{\pi}{2}\right[$ · Then,

$$y = \cos^{-1}x \Rightarrow x = \cos y$$

$$\Rightarrow \frac{dx}{dy} = -\sin y, \text{ where } \sin y > 0, \text{ since } y \in \left]0, \frac{\pi}{2}\right[$$

$$\Rightarrow \frac{dx}{dy} = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}.$$

Hence, $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}\cdot$

EXAMPLE 3 *Prove that* $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{(1+x^2)}$, *where* $x \in R$.

SOLUTION Let $y = \tan^{-1}x$, where $x \in R$ and $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\cdot$ Then,

$$x = \tan y$$

$$\Rightarrow \frac{dx}{dy} = \sec^2 y = (1+\tan^2 y) = (1+x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(1+x^2)}\cdot$$

Hence, $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{(1+x^2)}\cdot$

EXAMPLE 4 *Prove that* $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{(1+x^2)}$, *where* $x \in R$.

SOLUTION Let $y = \cot^{-1}x$, where $x \in R$ and $y \in]0, \pi[$. Then,

$$x = \cot y$$

$$\Rightarrow \frac{dx}{dy} = -\text{cosec}^2 y = -(1+\cot^2 y) = -(1+x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{(1+x^2)}\cdot$$

Hence, $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{(1+x^2)}\cdot$

EXAMPLE 5 *Prove that* $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$, *where* $x \in R - [-1, 1]$.

SOLUTION Let $y = \sec^{-1}x$, where $x \in R - [-1, 1]$ and $y \in]0, \pi[- \left\{\frac{\pi}{2}\right\}$. Then,

$$x = \sec y$$

$$\Rightarrow \frac{dx}{dy} = \sec y \tan y > 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y \cdot \sqrt{\sec^2 y - 1}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}\cdot$$

Hence, $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}\cdot$

EXAMPLE 6 *Prove that* $\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$, *where* $x \in R - [-1, 1]$.

SOLUTION Let $y = \text{cosec}^{-1}x$, where $x \in R - [-1, 1]$ and $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[- \{0\}$. Then,

$$x = \text{cosec}\, y$$

$$\Rightarrow \frac{dx}{dy} = -\text{cosec}\, y \cot y, \text{ where cosec}\, y \cot y > 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\text{cosec}\, y \cot y} = \frac{-1}{(\text{cosec}\, y)\sqrt{\text{cosec}^2 y - 1}} = \frac{-1}{|x|\sqrt{x^2-1}}.$$

Hence, $\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$.

SUMMARY

(i) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ (ii) $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{(1+x^2)}$ (iv) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{(1+x^2)}$

(v) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$ (vi) $\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{|x| \cdot \sqrt{x^2-1}}$

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate the following w.r.t. x:*
(i) $\sin^{-1}2x$ *(ii)* $\tan^{-1}\sqrt{x}$ *(iii)* $\cos^{-1}(\cot x)$

SOLUTION (i) Let $y = \sin^{-1}2x$.

Putting $2x = t$, we get $y = \sin^{-1}t$ and $t = 2x$.

Now, $y = \sin^{-1}t \Rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}$.

And, $t = 2x \Rightarrow \frac{dt}{dx} = 2$.

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{2}{\sqrt{1-t^2}} = \frac{2}{\sqrt{1-4x^2}} \qquad [\because t = 2x].$$

Hence, $\frac{d}{dx}(\sin^{-1}2x) = \frac{2}{\sqrt{1-4x^2}}$.

(ii) Let $y = \tan^{-1}\sqrt{x}$.

Putting $\sqrt{x} = t$, we get $y = \tan^{-1}t$ and $t = \sqrt{x}$.

Now, $y = \tan^{-1}t \Rightarrow \dfrac{dy}{dt} = \dfrac{1}{(1+t^2)}$.

And, $t = \sqrt{x} \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

$\therefore \dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right) = \dfrac{1}{(1+t^2)} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}(1+x)}$ $[\because t = \sqrt{x}]$.

Hence, $\dfrac{d}{dx}(\tan^{-1}\sqrt{x}) = \dfrac{1}{2\sqrt{x}(1+x)}$.

(iii) Let $y = \cos^{-1}(\cot x)$.

Putting $\cot x = t$, we get $y = \cos^{-1}t$ and $t = \cot x$.

Now, $y = \cos^{-1}t \Rightarrow \dfrac{dy}{dt} = \dfrac{-1}{\sqrt{1-t^2}}$.

And, $t = \cot x \Rightarrow \dfrac{dt}{dx} = -\text{cosec}^2 x$.

$\therefore \dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right) = \dfrac{\text{cosec}^2 x}{\sqrt{1-t^2}} = \dfrac{\text{cosec}^2 x}{\sqrt{1-\cot^2 x}}$ $[\because t = \cot x]$.

Hence, $\dfrac{d}{dx}\{\cos^{-1}(\cot x)\} = \dfrac{\text{cosec}^2 x}{\sqrt{1-\cot^2 x}}$.

EXAMPLE 2 *Differentiate the following w.r.t. x:*

(i) $\sec(\tan^{-1}x)$ *(ii)* $\sin(\tan^{-1}x)$ *(iii)* $\cot(\cos^{-1}x)$

SOLUTION (i) Let $y = \sec(\tan^{-1}x)$.

Putting $\tan^{-1}x = t$, we get $y = \sec t$ and $t = \tan^{-1}x$.

Now, $y = \sec t \Rightarrow \dfrac{dy}{dt} = \sec t \tan t$.

And, $t = \tan^{-1}x \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{(1+x^2)}$.

$\therefore \dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right) = \dfrac{\sec t \tan t}{(1+x^2)} = \dfrac{(\sqrt{1+\tan^2 t})(\tan t)}{(1+x^2)}$

$= \dfrac{(\sqrt{1+x^2})x}{(1+x^2)} = \dfrac{x}{\sqrt{1+x^2}}$ $[\because t = \tan^{-1}x \Rightarrow \tan t = x]$

.

Hence, $\dfrac{d}{dx}\{\sec(\tan^{-1}x)\} = \dfrac{x}{\sqrt{1+x^2}}$.

(ii) Let $y = \sin(\tan^{-1}x)$.

Putting $\tan^{-1}x = t$, we get $y = \sin t$ and $t = \tan^{-1}x$.

Now, $y = \sin t \Rightarrow \dfrac{dy}{dt} = \cos t$.

And, $t = \tan^{-1} x \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{(1+x^2)}$.

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \cos t \cdot \frac{1}{(1+x^2)} = \frac{1}{(1+x^2)^{3/2}}$$

$$\left[\because \ \tan t = x \Rightarrow \cos t = \frac{1}{\sqrt{1+x^2}}\right].$$

Hence, $\dfrac{d}{dx}\{\sin(\tan^{-1} x)\} = \dfrac{1}{(1+x^2)^{3/2}}$.

(iii) Let $y = \cot(\cos^{-1} x)$.

Putting $\cos^{-1} x = t$, we get $y = \cot t$ and $t = \cos^{-1} x$.

Now, $y = \cot t \Rightarrow \dfrac{dy}{dt} = -\text{cosec}^2 t$.

And, $t = \cos^{-1} x \Rightarrow \dfrac{dt}{dx} = \dfrac{-1}{\sqrt{1-x^2}}$.

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{\text{cosec}^2 t}{\sqrt{1-x^2}} = \frac{1}{(1-x^2)^{3/2}}$$

$$\left[\because \ \cos t = x \Rightarrow \text{cosec}^2 t = \frac{1}{(1-x^2)}\right].$$

Hence, $\dfrac{d}{dx}\{\cot(\cos^{-1} x)\} = \dfrac{1}{(1-x^2)^{3/2}}$.

EXAMPLE 3 *If* $y = \sin(\tan^{-1} 2x)$, *prove that* $\dfrac{dy}{dx} = \dfrac{2}{(1+4x^2)^{3/2}}$.

SOLUTION Putting $\tan^{-1} 2x = t$, we get $y = \sin t$ and $t = \tan^{-1} 2x$.

Now, $y = \sin t \Rightarrow \dfrac{dy}{dt} = \cos t$.

And, $t = \tan^{-1} 2x \Rightarrow \dfrac{dt}{dx} = \left\{\dfrac{1}{(1+4x^2)} \times 2\right\} = \dfrac{2}{(1+4x^2)}$.

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \left\{\cos t \times \frac{2}{(1+4x^2)}\right\}. \qquad \text{... (i)}$$

Now, $t = \tan^{-1} 2x \Rightarrow \tan t = 2x$

$$\Rightarrow \sec t = \sqrt{1+\tan^2 t} = \sqrt{1+4x^2}$$

$$\Rightarrow \cos t = \frac{1}{\sec t} = \frac{1}{\sqrt{1+4x^2}}. \qquad \text{... (ii)}$$

Putting the value of $\cos t$ from (ii) in (i), we get

$$\frac{dy}{dx} = \left\{\frac{1}{\sqrt{1+4x^2}} \times \frac{2}{(1+4x^2)}\right\} = \frac{2}{(1+4x^2)^{3/2}}.$$

EXAMPLE 4 *Differentiate* $\sqrt{\cot^{-1}\sqrt{x}}$ *w.r.t. x.*

SOLUTION Let $y = \sqrt{\cot^{-1}\sqrt{x}}$.

Putting $\sqrt{x} = t$ and $\cot^{-1}\sqrt{x} = \cot^{-1}t = u$, we get

$$y = \sqrt{u}, \text{ where } u = \cot^{-1}t \text{ and } t = \sqrt{x}.$$

Now, $y = \sqrt{u} \Rightarrow \dfrac{dy}{du} = \dfrac{1}{2}u^{-1/2} = \dfrac{1}{2\sqrt{u}}$;

$$u = \cot^{-1}t \Rightarrow \frac{du}{dt} = \frac{-1}{(1+t^2)}.$$

And, $t = \sqrt{x} \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right) = \frac{-1}{4\sqrt{u}(1+t^2)\sqrt{x}}$$

$$= \frac{-1}{4(\sqrt{\cot^{-1}t})(1+t^2)\sqrt{x}} \qquad [\because \ u = \cot^{-1}t]$$

$$= \frac{-1}{4(\sqrt{\cot^{-1}\sqrt{x}})(1+x)\sqrt{x}} \qquad [\because \ t = \sqrt{x}].$$

EXAMPLE 5 *Differentiate* $e^{\tan^{-1}\sqrt{x}}$ *w.r.t. x.*

SOLUTION Let $y = e^{\tan^{-1}\sqrt{x}}$.

Putting $\sqrt{x} = t$ and $\tan^{-1}\sqrt{x} = \tan^{-1}t = u$, we get

$$y = e^u, \text{ where } u = \tan^{-1}t \text{ and } t = \sqrt{x}.$$

Now, $y = e^u \Rightarrow \dfrac{dy}{du} = e^u$;

$$u = \tan^{-1}t \Rightarrow \frac{du}{dt} = \frac{1}{(1+t^2)}.$$

And, $t = \sqrt{x} \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right) = e^u \cdot \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}}$$

$$= e^{\tan^{-1}t} \cdot \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}} \qquad [\because \ u = \tan^{-1}t]$$

$$= \frac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)} \qquad [\because \ t = \sqrt{x}].$$

Hence, $\dfrac{dy}{dx} = \dfrac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)}$.

EXAMPLE 6 *If* $y = \dfrac{x \sin^{-1} x}{\sqrt{1-x^2}}$, *find* $\dfrac{dy}{dx}$.

SOLUTION $y = \dfrac{x \sin^{-1} x}{\sqrt{1-x^2}}$ ... (i)

$\Rightarrow \; y\sqrt{1-x^2} = x \sin^{-1} x.$ ... (ii)

On differentiating both sides of (ii) w.r.t. x, we get

$$y \cdot \frac{d}{dx}(\sqrt{1-x^2}) + (\sqrt{1-x^2})\frac{dy}{dx} = x \cdot \frac{d}{dx}(\sin^{-1}x) + \sin^{-1}x \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \; y \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) + (\sqrt{1-x^2})\frac{dy}{dx} = x \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x \cdot 1$$

$$\Rightarrow \; \frac{-x\,y}{\sqrt{1-x^2}} + (\sqrt{1-x^2})\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$$

$$\Rightarrow \; \frac{-x^2 \sin^{-1}x}{(1-x^2)} + (\sqrt{1-x^2})\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x \qquad \text{[using (i)]}$$

$$\Rightarrow \; -x^2 \sin^{-1}x + (1-x^2)^{3/2}\frac{dy}{dx} = x(\sqrt{1-x^2}) + (1-x^2)\sin^{-1}x$$

$$\Rightarrow \; (1-x^2)^{3/2}\frac{dy}{dx} = x(\sqrt{1-x^2}) + \sin^{-1}x$$

$$\Rightarrow \; \frac{dy}{dx} = \frac{x(\sqrt{1-x^2}) + \sin^{-1}x}{(1-x^2)^{3/2}}.$$

EXAMPLE 7 *Find* $\dfrac{d}{dx}[\sqrt{1-x^2}\sin^{-1}x - x]$.

SOLUTION We have

$$\frac{d}{dx}[(\sqrt{1-x^2})\sin^{-1}x - x]$$

$$= \frac{d}{dx}[(\sqrt{1-x^2}\sin^{-1}x] - \frac{d}{dx}(x)$$

$$= (\sqrt{1-x^2}) \cdot \frac{d}{dx}(\sin^{-1}x) + (\sin^{-1}x) \cdot \frac{d}{dx}(\sqrt{1-x^2}) - 1$$

$$= (\sqrt{1-x^2}) \cdot \frac{1}{(\sqrt{1-x^2})} + (\sin^{-1}x) \cdot \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) - 1$$

$$= \left\{1 - \frac{x\sin^{-1}x}{\sqrt{1-x^2}} - 1\right\} = \frac{-x\sin^{-1}x}{\sqrt{1-x^2}}.$$

EXAMPLE 8 *Show that* $\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]=\sqrt{a^2-x^2}.$ **[CBSE 2004C]**

SOLUTION We have

$$\frac{d}{dx}\left[\frac{x}{2}\cdot\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot\sin^{-1}\frac{x}{a}\right]$$

$$=\frac{d}{dx}\left[\frac{x}{2}\cdot\sqrt{a^2-x^2}\right]+\frac{a^2}{2}\cdot\frac{d}{dx}\left[\sin^{-1}\frac{x}{a}\right]$$

$$=\frac{x}{2}\cdot\frac{d}{dx}(\sqrt{a^2-x^2})+(\sqrt{a^2-x^2})\cdot\frac{d}{dx}\left(\frac{x}{2}\right)+\frac{a^2}{2}\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}$$

$$=\frac{x}{2}\cdot\frac{1}{2}(a^2-x^2)^{-1/2}\cdot(-2x)+(\sqrt{a^2-x^2})\cdot\frac{1}{2}+\frac{a^2}{2\sqrt{a^2-x^2}}$$

$$=\frac{-x^2}{2\sqrt{a^2-x^2}}+\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2\sqrt{a^2-x^2}}$$

$$=\frac{-x^2+(a^2-x^2)+a^2}{2\sqrt{a^2-x^2}}=\frac{(a^2-x^2)}{\sqrt{(a^2-x^2)}}=\sqrt{(a^2-x^2)}.$$

Hence, $\frac{d}{dx}\left[\frac{x}{2}\cdot\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot\sin^{-1}\frac{x}{a}\right]=\sqrt{a^2-x^2}.$

EXERCISE 10C

Differentiate each of the following w.r.t. x:

1. $\cos^{-1}2x$
2. $\tan^{-1}x^2$
3. $\sec^{-1}\sqrt{x}$
4. $\sin^{-1}\frac{x}{a}$
5. $\tan^{-1}(\log x)$
6. $\cot^{-1}(e^x)$
7. $\log(\tan^{-1}x)$
8. $\cot^{-1}x^3$
9. $\sin^{-1}(\cos x)$
10. $(1+x^2)\tan^{-1}x$
11. $\tan^{-1}(\cot x)$
12. $\log(\sin^{-1}x^4)$
13. $(\cot^{-1}x^2)^3$
14. $\tan^{-1}(\cos\sqrt{x})$
15. $\tan(\sin^{-1}x)$
16. $e^{\tan^{-1}\sqrt{x}}$
17. $\sqrt{\sin^{-1}x^2}$
18. If $y=\sin^{-1}(\cos x)+\cos^{-1}(\sin x)$, prove that $\frac{dy}{dx}=-2.$
19. Prove that $\frac{d}{dx}\{2x\tan^{-1}x-\log(1+x^2)\}=2\tan^{-1}x.$

ANSWERS (EXERCISE 10C)

1. $\frac{-2}{\sqrt{1-4x^2}}$ **2.** $\frac{2x}{(1+x^4)}$ **3.** $\frac{1}{2x\sqrt{x-1}}$ **4.** $\frac{1}{\sqrt{a^2-x^2}}$

5. $\dfrac{1}{x\{1+(\log x)^2\}}$ **6.** $\dfrac{-e^x}{(1+e^{2x})}$ **7.** $\dfrac{1}{(1+x^2)\tan^{-1}x}$ **8.** $\dfrac{-3x^2}{(1+x^6)}$

9. -1 **10.** $(1+2x\tan^{-1}x)$ **11.** -1 **12.** $\dfrac{4x^3}{(\sin^{-1}x^4)\sqrt{1-x^8}}$

13. $\dfrac{-6x(\cot^{-1}x^2)^2}{(1+x^4)}$ **14.** $\dfrac{-\sin\sqrt{x}}{(2\sqrt{x})(1+\cos^2\sqrt{x})}$ **15.** $\dfrac{1}{(1-x^2)^{3/2}}$

16. $\dfrac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)}$ **17.** $\dfrac{x}{(\sqrt{1-x^4})\sqrt{\sin^{-1}x^2}}$

4. Differentiation by Trigonometrical Transformations

SOME USEFUL RESULTS

(i) $(1-\cos x) = 2\sin^2\left(\dfrac{x}{2}\right)$ **(ii)** $(1+\cos x) = 2\cos^2\left(\dfrac{x}{2}\right)$

(iii) $\sin 3x = (3\sin x - 4\sin^3 x)$ **(iv)** $\cos 3x = (4\cos^3 x - 3\cos x)$

(v) $\sin x = \dfrac{2\tan(x/2)}{1+\tan^2(x/2)}$ **(vi)** $\cos x = \dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$

(vii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left\{\dfrac{x-y}{1+xy}\right\}$ **(viii)** $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left\{\dfrac{x+y}{1-xy}\right\}$

SOME USEFUL SUBSTITUTIONS

Suppose we are given $\sin^{-1}f(x)$, $\cos^{-1}f(x)$, $\tan^{-1}f(x)$, etc.

Rule 1. If $f(x)=\sqrt{a^2-x^2}$, put $x=a\sin\theta$ or $x=a\cos\theta$.

Rule 2. If $f(x)=\sqrt{a^2+x^2}$, put $x=a\tan\theta$ or $x=a\cot\theta$.

Rule 3. If $f(x)=\sqrt{x^2-a^2}$, put $x=a\sec\theta$ or $x=a\operatorname{cosec}\theta$.

Rule 4. If $f(x)=\sqrt{a-x}$, put $x=a\cos 2\theta$.

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate each of the following w.r.t. x:*

(i) $\tan^{-1}\left(\dfrac{1-\cos x}{\sin x}\right)$ *(ii)* $\tan^{-1}\left(\dfrac{\cos x-\sin x}{\cos x+\sin x}\right)$

SOLUTION (i) Let $y=\tan^{-1}\left(\dfrac{1-\cos x}{\sin x}\right)=\tan^{-1}\left\{\dfrac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}\right\}$

$$=\tan^{-1}\left\{\tan\frac{x}{2}\right\}=\frac{x}{2}.$$

$\therefore \quad y = \frac{x}{2}\cdot$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}\cdot$

(ii) Let $y = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right)$

[on dividing num. and denom. by $\cos x$]

$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - x\right)\right\} = \left(\frac{\pi}{4} - x\right)\cdot$

$\therefore \quad y = \left(\frac{\pi}{4} - x\right)\cdot$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - x\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{d}{dx}(x) = (0 - 1) = -1.$

EXAMPLE 2 *Differentiate w.r.t. x:*

(i) $\tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$ *(ii)* $\tan^{-1}(\sec x + \tan x)$

SOLUTION (i) Let $y = \tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right) = \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)}\right\}$

$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right\}$

$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\} = \left(\frac{\pi}{4} - \frac{x}{2}\right)\cdot$

$\therefore \quad y = \left(\frac{\pi}{4} - \frac{x}{2}\right)\cdot$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{d}{dx}\left(\frac{x}{2}\right) = \left(0 - \frac{1}{2}\right) = \frac{-1}{2}\cdot$

(ii) Let $y = \tan^{-1}(\sec x + \tan x)$

$= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) = \tan^{-1}\left(\frac{1 + \sin x}{\cos x}\right)$

$= \tan^{-1}\left\{\frac{1 - \cos\left(\frac{\pi}{2} + x\right)}{\sin\left(\frac{\pi}{2} + x\right)}\right\}$

$\left\{\because \cos\left(\frac{\pi}{2} + x\right) = -\sin x;\ \sin\left(\frac{\pi}{2} + x\right) = \cos x\right\}$

$$= \tan^{-1}\left\{\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \left(\frac{\pi}{4}+\frac{x}{2}\right).$$

$\therefore \quad y = \left(\frac{\pi}{4}+\frac{x}{2}\right).$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{d}{dx}\left(\frac{x}{2}\right) = \left(0+\frac{1}{2}\right) = \frac{1}{2}.$

EXAMPLE 3 *Differentiate w.r.t. x:*

(i) $\tan^{-1}\left\{\sqrt{\frac{1+\cos x}{1-\cos x}}\right\}$ *(ii)* $\tan^{-1}\left\{\sqrt{\frac{1+\sin x}{1-\sin x}}\right\}$ **[CBSE 2003C, '04C]**

SOLUTION (i) Let $y = \tan^{-1}\left\{\sqrt{\frac{1+\cos x}{1-\cos x}}\right\} = \tan^{-1}\left\{\sqrt{\frac{2\cos^2(x/2)}{2\sin^2(x/2)}}\right\}$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right\} = \left(\frac{\pi}{2}-\frac{x}{2}\right).$$

$\therefore \quad y = \left(\frac{\pi}{2}-\frac{x}{2}\right).$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}\left(\frac{x}{2}\right) = \left(0-\frac{1}{2}\right) = \frac{-1}{2}.$

(ii) Let $y = \tan^{-1}\left\{\sqrt{\frac{1+\sin x}{1-\sin x}}\right\} = \tan^{-1}\left\{\frac{1-\cos\left(\frac{\pi}{2}+x\right)}{1+\cos\left(\frac{\pi}{2}+x\right)}\right\}^{\frac{1}{2}}$

$$= \tan^{-1}\left\{\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\}^{\frac{1}{2}} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \left(\frac{\pi}{4}+\frac{x}{2}\right).$$

$\therefore \quad y = \left(\frac{\pi}{4}+\frac{x}{2}\right).$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{d}{dx}\left(\frac{x}{2}\right) = \left(0+\frac{1}{2}\right) = \frac{1}{2}.$

EXAMPLE 4 *Differentiate* $\cos^{-1}\left\{\sqrt{\frac{1+\cos x}{2}}\right\}$ *w.r.t. x.*

SOLUTION Let $y = \cos^{-1}\left\{\sqrt{\frac{1+\cos x}{2}}\right\} = \cos^{-1}\left\{\sqrt{\frac{2\cos^2(x/2)}{2}}\right\}$

$= \cos^{-1}\{\cos(x/2)\} = \frac{x}{2}\cdot$

$\therefore \quad y = \frac{x}{2}\cdot$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}\cdot$

EXAMPLE 5 *If* $y = \cot^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}$, *find* $\frac{dy}{dx}\cdot$ **[CBSE 2004C]**

SOLUTION We have

$$y = \cot^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}} = \cot^{-1}\sqrt{\frac{1+\cos\left(\frac{\pi}{2}+x\right)}{1-\cos\left(\frac{\pi}{2}+x\right)}}$$

$$= \cot^{-1}\sqrt{\frac{2\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}} = \cot^{-1}\left\{\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \left(\frac{\pi}{4}+\frac{x}{2}\right).$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{d}{dx}\left(\frac{x}{2}\right) = \left(0+\frac{1}{2}\right) = \frac{1}{2}\cdot$$

EXAMPLE 6 *If* $y = \cot^{-1}\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$ *and* $0 < x < \frac{\pi}{2}$, *find* $\frac{dy}{dx}\cdot$

[CBSE 2008]

SOLUTION We have

$$(1+\sin x) = \{\cos^2(x/2) + \sin^2(x/2) + 2\sin(x/2)\cos(x/2)\}$$
$$= \{\cos(x/2) + \sin(x/2)\}^2.$$
$$(1-\sin x) = \{\cos^2(x/2) + \sin^2(x/2) - 2\sin(x/2)\cos(x/2)\}$$
$$= \{\cos(x/2) - \sin(x/2)\}^2.$$

$$\therefore \quad \sqrt{1+\sin x} = \sqrt{\{\cos(x/2)+\sin(x/2)\}^2} = \cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)$$

and $\sqrt{1-\sin x} = \sqrt{\{\cos(x/2)-\sin(x/2)\}^2} = \cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\cdot$

$$\therefore \quad y = \cot^{-1}\left\{\frac{[\cos(x/2)+\sin(x/2)]+[\cos(x/2)-\sin(x/2)]}{[\cos(x/2)+\sin(x/2)]-[\cos(x/2)-\sin(x/2)]}\right\}$$

$$= \cot^{-1}\left\{\frac{2\cos(x/2)}{2\sin(x/2)}\right\} = \cot^{-1}\{\cot(x/2)\} = \frac{x}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}.$$

EXAMPLE 7 *If* $y = \tan^{-1} \dfrac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})}{(\sqrt{1+\sin x} - \sqrt{1-\sin x})}$, *find* $\dfrac{dy}{dx}$. **[CBSE 2004]**

SOLUTION We have

$$y = \tan^{-1}\left\{\frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \times \frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x}}{(\sqrt{1+\sin x} + \sqrt{1-\sin x}}\right\}$$

$$= \tan^{-1}\left\{\frac{(1+\sin x) + (1-\sin x) + 2\sqrt{1-\sin^2 x}}{(1+\sin x) - (1-\sin x)}\right\} = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \tan^{-1}\left\{\frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2)}\right\} = \tan^{-1}\left\{\cot\frac{x}{2}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right\}$$

$$= \left(\frac{\pi}{2} - \frac{x}{2}\right).$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - \frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}\left(\frac{x}{2}\right) = \left(0 - \frac{1}{2}\right) = -\frac{1}{2}.$$

EXAMPLE 8 *Differentiate w.r.t. x:*

(i) $\cot^{-1}\left(\dfrac{1}{x}\right)$ *(ii)* $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ *(iii)* $\cot^{-1}\left(\dfrac{1-x}{1+x}\right)$

SOLUTION (i) Let $y = \cot^{-1}\left(\dfrac{1}{x}\right)$.

Putting $x = \tan\theta$, we get

$$y = \cot^{-1}\left(\frac{1}{\tan\theta}\right) = \cot^{-1}(\cot\theta) = \theta = \tan^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{(1+x^2)}.$$

Hence, $\dfrac{d}{dx}\left\{\cot^{-1}\left(\dfrac{1}{x}\right)\right\} = \dfrac{1}{(1+x^2)}$.

(ii) Let $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$.

Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta = 2\tan^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{(1+x^2)}.$$

Hence, $\dfrac{d}{dx}\left\{\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)\right\} = \dfrac{2}{(1+x^2)}$.

(iii) Let $y = \cot^{-1}\left(\frac{1-x}{1+x}\right)$.

Putting $x = \tan\theta$, we get

$$y = \cot^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \cot^{-1}\left\{\tan\left(\frac{\pi}{4}-\theta\right)\right\}$$

$$= \cot^{-1}\left[\cot\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-\theta\right)\right\}\right] = \left(\frac{\pi}{4}+\theta\right) = \frac{\pi}{4}+\tan^{-1}x.$$

$\therefore \quad \frac{dy}{dx} = \frac{1}{(1+x^2)}$.

Hence, $\frac{d}{dx}\left\{\cot^{-1}\left(\frac{1-x}{1+x}\right)\right\} = \frac{1}{(1+x^2)}$.

EXAMPLE 9 *Differentiate w.r.t. x:*

(i) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ *(ii)* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *(iii)* $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

SOLUTION (i) Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

Putting $x = \tan\theta$, we get

$$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1}x.$$

$\therefore \quad \frac{dy}{dx} = \frac{2}{(1+x^2)}$.

Hence, $\frac{d}{dx}\left\{\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\} = \frac{2}{(1+x^2)}$.

(ii) Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Putting $x = \tan\theta$, we get

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x.$$

$\therefore \quad \frac{dy}{dx} = \frac{2}{(1+x^2)}$.

Hence, $\frac{d}{dx}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\} = \frac{2}{(1+x^2)}$.

(iii) Let $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$.

Putting $x = \cos\theta$, we get

$$y = \sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$$= \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1}x.$$

$\therefore \quad y = 2\cos^{-1}x.$

Hence, $\dfrac{dy}{dx} = 2\dfrac{d}{dx}(\cos^{-1}x) = \dfrac{-2}{\sqrt{1-x^2}}\cdot$

EXAMPLE 10 *Differentiate w.r.t. x:*

(i) $\cos^{-1}(4x^3 - 3x)$ *(ii)* $\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ *(iii)* $\sec^{-1}\left(\dfrac{x^2+1}{x^2-1}\right)$

SOLUTION (i) Let $y = \cos^{-1}(4x^3 - 3x)$.

Putting $x = \cos\theta$, we get

$$y = \cos^{-1}(4\cos^3\theta - 3\cos\theta) = \cos^{-1}(\cos 3\theta) = 3\theta.$$

$\therefore \quad y = 3\theta \Rightarrow y = 3\cos^{-1}x$

$$\Rightarrow \frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}\cdot$$

(ii) Let $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$.

Putting $x = \tan\theta$, we get

$$y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\cos 2\theta) = \sin^{-1}\left[\sin\left(\frac{\pi}{2} - 2\theta\right)\right]$$

$$= \left(\frac{\pi}{2} - 2\theta\right) = \left(\frac{\pi}{2} - 2\tan^{-1}x\right)\cdot$$

$\therefore \quad y = \left(\dfrac{\pi}{2} - 2\tan^{-1}x\right)\cdot$

Hence, $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - 2\tan^{-1}x\right) = \dfrac{d}{dx}\left(\dfrac{\pi}{2}\right) - 2\cdot\dfrac{d}{dx}(\tan^{-1}x)$

$$= \left\{0 - \frac{2}{(1+x^2)}\right\} = \frac{-2}{(1+x^2)}\cdot$$

(iii) Let $y = \sec^{-1}\left(\dfrac{x^2+1}{x^2-1}\right)$.

Putting $x = \cot\theta$, we get

$$y = \sec^{-1}\left(\frac{\cot^2\theta+1}{\cot^2\theta-1}\right) = \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$$

$$= \sec^{-1}(\sec 2\theta) = 2\theta = 2\cot^{-1}x.$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-2}{(1+x^2)}\cdot$

Hence, $\dfrac{d}{dx}\left\{\sec^{-1}\left(\dfrac{x^2+1}{x^2-1}\right)\right\} = \dfrac{-2}{(1+x^2)}\cdot$

EXAMPLE 11 *Differentiate* $\cot^{-1}\left(\frac{1-x}{1+x}\right)$ *w.r.t. x.* **[CBSE 2004]**

SOLUTION Let $y = \cot^{-1}\left(\frac{1-x}{1+x}\right)$.

Putting $x = \tan\theta$, we get

$$y = \cot^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \cot^{-1}\left\{\tan\left(\frac{\pi}{4}-\theta\right)\right\}$$

$$= \cot^{-1}\left[\cot\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-\theta\right)\right\}\right] = \cot^{-1}\left[\cot\left(\frac{\pi}{4}+\theta\right)\right]$$

$$= \left(\frac{\pi}{4}+\theta\right)$$

$$= \frac{\pi}{4} + \tan^{-1}x \qquad [\because \ x = \tan\theta \Rightarrow \theta = \tan^{-1}x].$$

$$\therefore \ y = \frac{\pi}{4} + \tan^{-1}x$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left\{\frac{\pi}{4} + \tan^{-1}x\right\} = \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{d}{dx}(\tan^{-1}x)$$

$$= \left\{0 + \frac{1}{(1+x^2)}\right\} = \frac{1}{(1+x^2)}.$$

Hence, $\frac{dy}{dx} = \frac{1}{(1+x^2)}$.

EXAMPLE 12 *Differentiate* $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$ *w.r.t. x.* **[CBSE 2004]**

SOLUTION Let $y = \tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$.

Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left\{\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right\} = \tan^{-1}\left\{\frac{\sec\theta - 1}{\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{\left(\frac{1}{\cos\theta}-1\right)}{\sin\theta}\cdot\cos\theta\right\} = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left\{\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right\} = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

$$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x.$$

$$\therefore \quad y = \frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\cdot\frac{d}{dx}(\tan^{-1}x) = \frac{1}{2(1+x^2)}\cdot$$

EXAMPLE 13 *If* $y = \tan^{-1}\left\{\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right\}$, *find* $\dfrac{dy}{dx}\cdot$ **[CBSE 2006]**

SOLUTION Putting $x^2 = \cos\theta$, we get

$$y = \tan^{-1}\left\{\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2(\theta/2)}-\sqrt{2\sin^2(\theta/2)}}{\sqrt{2\cos^2(\theta/2)}+\sqrt{2}\sin^2(\theta/2)}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2}\cos(\theta/2)-\sqrt{2}\sin(\theta/2)}{\sqrt{2}\cos(\theta/2)+\sqrt{2}\sin(\theta/2)}\right\} = \tan^{-1}\left\{\frac{\cos(\theta/2)-\sin(\theta/2)}{\cos(\theta/2)+\sin(\theta/2)}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}$$

$$= \left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2.$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2\right\} = \frac{d}{dx}\left(\frac{\pi}{4}\right)-\frac{1}{2}\cdot\frac{d}{dx}(\cos^{-1}x^2)$$

$$= \left\{0-\frac{1}{2}\cdot\frac{-1}{\sqrt{1-x^4}}\cdot 2x\right\} = \frac{x}{\sqrt{1-x^4}}\cdot$$

EXAMPLE 14 *If* $y = \sin^{-1}[x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}]$, *find* $\dfrac{dy}{dx}$. **[CBSE 2010]**

SOLUTION Putting $x = \sin\theta$ and $\sqrt{x} = \sin\phi$, we get

$$y = \sin^{-1}[\sin\theta\cos\phi - \sin\phi\cos\theta] = \sin^{-1}[\sin(\theta-\phi)]$$

$$= (\theta-\phi) = \sin^{-1}x - \sin^{-1}\sqrt{x}.$$

$$\therefore \quad y = \sin^{-1}x - \sin^{-1}\sqrt{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\{\sin^{-1}x - \sin^{-1}\sqrt{x}\}$$

$$= \frac{d}{dx}(\sin^{-1}x) - \frac{d}{dx}\{\sin^{-1}\sqrt{x}\}$$

$$= \left[\frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}\cdot\sqrt{1-x}}\right].$$

EXAMPLE 15 *Differentiate* $\tan^{-1}\left(\dfrac{x^{1/3}+a^{1/3}}{1-x^{1/3}a^{1/3}}\right)$ *w.r.t. x.*

SOLUTION Let $y=\tan^{-1}\left(\dfrac{x^{1/3}+a^{1/3}}{1-x^{1/3}a^{1/3}}\right)$.

Putting $x^{1/3}=\tan\theta$ and $a^{1/3}=\tan\phi$, we get

$$y=\tan^{-1}\left(\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}\right)=\tan^{-1}[\tan(\theta+\phi)]$$

$$=(\theta+\phi)=\tan^{-1}(x^{1/3})+\tan^{-1}(a^{1/3}).$$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}\{\tan^{-1}(x^{1/3})+\tan^{-1}(a^{1/3})\}$$

$$=\frac{d}{dx}\{\tan^{-1}(x^{1/3})\}+\frac{d}{dx}\{\tan^{-1}(a^{1/3})\}=\frac{1}{(1+x^{2/3})}\cdot\frac{1}{3}x^{-2/3}$$

$$=\frac{1}{3x^{2/3}(1+x^{2/3})}\qquad[\because\ \tan^{-1}(a^{1/3})=\text{constant}].$$

EXAMPLE 16 *Differentiate* $\tan^{-1}\left(\dfrac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$ *w.r.t. x.*

SOLUTION Let $y=\tan^{-1}\left(\dfrac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$.

Putting $a=r\sin\theta$ and $b=r\cos\theta$, we get

$$y=\tan^{-1}\left\{\frac{r(\sin\theta\cos x-\cos\theta\sin x)}{r(\cos\theta\cos x+\sin\theta\sin x)}\right\}$$

$$=\tan^{-1}\left\{\frac{\sin(\theta-x)}{\cos(\theta-x)}\right\}=\tan^{-1}\{\tan(\theta-x)\}$$

$$=\theta-x=\left(\tan^{-1}\frac{a}{b}-x\right)\quad\left[\because\ \frac{a}{b}=\tan\theta\Rightarrow\theta=\tan^{-1}\frac{a}{b}\right].$$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}\left(\tan^{-1}\frac{a}{b}-x\right)$$

$$=\frac{d}{dx}\left(\tan^{-1}\frac{a}{b}\right)-\frac{d}{dx}(x)=-1\qquad\left[\because\ \tan^{-1}\frac{a}{b}=\text{constant}\right].$$

EXAMPLE 17 *If* $y=\sin^{-1}\left\{\dfrac{\sqrt{1+x}-\sqrt{1-x}}{2}\right\}$, *find* $\dfrac{dy}{dx}$.

SOLUTION Putting $x=\cos 2\theta$, we get

$$y=\sin^{-1}\left\{\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{2}\right\}$$

$$=\sin^{-1}\left\{\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{2}\right\}=\sin^{-1}\left\{\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{2}\right\}$$

$$= \sin^{-1}\left\{\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right\} = \sin^{-1}\left\{\sin\frac{\pi}{4}\cos\theta - \cos\frac{\pi}{4}\sin\theta\right\}$$

$$= \sin^{-1}\left\{\sin\left(\frac{\pi}{4} - \theta\right)\right\}$$

$$= \left(\frac{\pi}{4} - \theta\right) = \left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}x\right)$$

$$\left[\because\ x = \cos 2\theta \Rightarrow 2\theta = \cos^{-1}x \Rightarrow \theta = \frac{1}{2}\cos^{-1}x\right]$$

$\therefore\ y = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x.$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}x\right\} = \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{1}{2}\frac{d}{dx}(\cos^{-1}x)$

$$= \left\{0 - \frac{1}{2}\cdot\frac{(-1)}{\sqrt{1-x^2}}\right\} = \frac{1}{2\sqrt{1-x^2}}.$$

EXAMPLE 18 *Differentiate each of the following w.r.t. x:*

(i) $\tan^{-1}(\sqrt{1+x^2} + x)$ *(ii)* $\tan^{-1}(\sqrt{1+x^2} - x)$

SOLUTION (i) Let $y = \tan^{-1}(\sqrt{1+x^2} + x)$.

Putting $x = \cot\theta$, we get

$$y = \tan^{-1}(\operatorname{cosec}\theta + \cot\theta) = \tan^{-1}\left(\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right)$$

$$= \tan^{-1}\left(\cot\frac{\theta}{2}\right) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right\}$$

$$= \frac{\pi}{2} - \frac{1}{2}\theta = \frac{\pi}{2} - \frac{1}{2}\cot^{-1}x.$$

$$\therefore\ \frac{dy}{dx} = -\frac{1}{2}\cdot\frac{-1}{(1+x^2)} = \frac{1}{2(1+x^2)}.$$

Hence, $\frac{d}{dx}\{\tan^{-1}(\sqrt{1+x^2} + x)\} = \frac{1}{2(1+x^2)}.$

(ii) Let $y = \tan^{-1}(\sqrt{1+x^2} - x)$.

Putting $x = \cot\theta$, we get

$$y = \tan^{-1}(\operatorname{cosec}\theta - \cot\theta) = \tan^{-1}\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left\{\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right\}$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\theta = \frac{1}{2}\cot^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{2(1+x^2)}\cdot$$

$$\text{Hence,} \quad \frac{d}{dx}\{\tan^{-1}(\sqrt{1+x^2}-x)\} = \frac{-1}{2(1+x^2)}\cdot$$

EXAMPLE 19 *Differentiate* $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+1}{x}\right)$ *w.r.t. x.*

SOLUTION Let $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+1}{x}\right)$.

Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{\sec\theta+1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left\{\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right\} = \tan^{-1}\left\{\cot\frac{\theta}{2}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\} = \left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \frac{\pi}{2}-\frac{1}{2}\tan^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2(1+x^2)}\cdot$$

$$\text{Hence,} \quad \frac{d}{dx}\left\{\tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)\right\} = \frac{-1}{2(1+x^2)}\cdot$$

EXAMPLE 20 *Differentiate* $\cot^{-1}(\sqrt{1+x^2}+x)$ *w.r.t. x.*

SOLUTION Let $y = \cot^{-1}(\sqrt{1+x^2}+x)$.

Putting $x = \cot\theta$, we get

$$y = \cot^{-1}(\operatorname{cosec}\theta + \cot\theta) = \cot^{-1}\left(\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$= \cot^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \cot^{-1}\left\{\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right\}$$

$$= \cot^{-1}\left(\cot\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\cot^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{2(1+x^2)}\cdot$$

$$\text{Hence,} \quad \frac{d}{dx}\{\cot^{-1}(\sqrt{1+x^2}+x)\} = \frac{-1}{2(1+x^2)}\cdot$$

EXAMPLE 21 *Differentiate* $\cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right)$ *w.r.t. x.*

SOLUTION Let $y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right) = \cos^{-1}\left\{\frac{\left(x - \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)}\right\} = \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right)$.

Putting $x = \tan\theta$, we get

$$y = \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right) = \cos^{-1}(-\cos 2\theta)$$

$$= \cos^{-1}\{\cos(\pi - 2\theta)\} = (\pi - 2\theta) = \pi - 2\tan^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\pi - 2\tan^{-1}x) = \frac{d}{dx}(\pi) - 2\cdot\frac{d}{dx}(\tan^{-1}x)$$

$$= \left(0 - \frac{2}{1 + x^2}\right) = \frac{-2}{(1 + x^2)}.$$

EXAMPLE 22 *If* $y = \tan^{-1}\left(\frac{\sqrt{a} - \sqrt{x}}{1 + \sqrt{ax}}\right)$, *find* $\frac{dy}{dx}$.

SOLUTION Putting $\sqrt{a} = \tan\alpha$ and $\sqrt{x} = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta}\right) = \tan^{-1}\{\tan(\alpha - \theta)\} = (\alpha - \theta).$$

$$\therefore \quad y = (\alpha - \theta) \Rightarrow y = \tan^{-1}\sqrt{a} - \tan^{-1}\sqrt{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}\sqrt{a}) - \frac{d}{dx}(\tan^{-1}\sqrt{x})$$

$$= \left\{0 - \frac{1}{(1 + x)}\cdot\frac{1}{2}x^{-1/2}\right\} = \frac{-1}{2\sqrt{x}(1 + x)}.$$

Hence, $\frac{dy}{dx} = \frac{-1}{2\sqrt{x}(1 + x)}$.

EXAMPLE 23 *If* $y = \sin^{-1}\left\{\frac{5x + 12\sqrt{1 - x^2}}{13}\right\}$, *find* $\frac{dy}{dx}$. **[CBSE 2004C, '05C, '08]**

SOLUTION We have

$$y = \sin^{-1}\left\{\frac{5}{13}\cdot x + \frac{12}{13}\cdot\sqrt{1 - x^2}\right\}.$$

Let $\frac{5}{13} = \sin\alpha$ and $x = \cos\theta$. Then,

$$\cos\alpha = \sqrt{1 - \frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

and $\sqrt{1 - x^2} = \sqrt{1 - \cos^2\theta} = \sqrt{\sin^2\theta} = \sin\theta$.

$$\therefore \quad y = \sin^{-1}\{\sin\alpha\cos\theta + \cos\alpha\sin\theta\}$$

$$= \sin^{-1}\{\sin(\alpha + \theta)\}$$

$$= \alpha + \theta = \sin^{-1}\frac{5}{13} + \cos^{-1}x.$$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}\left\{\sin^{-1}\frac{5}{13}+\cos^{-1}x\right\}=\frac{d}{dx}\left\{\sin^{-1}\frac{5}{13}\right\}+\frac{d}{dx}(\cos^{-1}x)$$

$$=\left\{0-\frac{1}{\sqrt{1-x^2}}\right\}=\frac{-1}{\sqrt{1-x^2}}.$$

Hence, $\dfrac{dy}{dx}=\dfrac{-1}{\sqrt{1-x^2}}$.

EXERCISE 10D

Differentiate each of the following w.r.t. x:

1. $\sin^{-1}\left\{\sqrt{\dfrac{1-\cos x}{2}}\right\}$
2. $\tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$
3. $\cot^{-1}\left(\dfrac{1+\cos x}{\sin x}\right)$
4. $\cot^{-1}\left(\sqrt{\dfrac{1+\cos x}{1-\cos x}}\right)$
5. $\tan^{-1}\left(\dfrac{\cos x+\sin x}{\cos x-\sin x}\right)$
6. $\cot^{-1}\left(\dfrac{\cos x-\sin x}{\cos x+\sin x}\right)$
7. $\cot^{-1}\left(\sqrt{\dfrac{1+\cos 3x}{1-\cos 3x}}\right)$
8. $\sec^{-1}\left(\dfrac{1+\tan^2 x}{1-\tan^2 x}\right)$
9. $\sin^{-1}\left(\dfrac{1-\tan^2 x}{1+\tan^2 x}\right)$
10. $\operatorname{cosec}^{-1}\left(\dfrac{1+\tan^2 x}{2\tan x}\right)$
11. $\cot^{-1}(\operatorname{cosec} x+\cot x)$
12. $\tan^{-1}(\cot x)+\cot^{-1}(\tan x)$
13. $\sin^{-1}\{\sqrt{1-x^2}\}$
14. $\sin^{-1}\left(\sqrt{\dfrac{1-x}{2}}\right)$
15. $\cos^{-1}\left\{\sqrt{\dfrac{1+x}{2}}\right\}$
16. $\cos^{-1}\{\sqrt{1-x^2}\}$
17. $\sin^{-1}\{2x\sqrt{1-x^2}\}$
18. $\sin^{-1}(3x-4x^3)$
19. $\sin^{-1}(1-2x^2)$
20. $\sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$
21. $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$
22. $\tan^{-1}\left(\dfrac{x}{1+\sqrt{1-x^2}}\right)$
23. $\cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$
24. $\sec^{-1}\left(\dfrac{1}{1-2x^2}\right)$
25. $\sin^{-1}\left\{\dfrac{1}{\sqrt{1+x^2}}\right\}$
26. $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$
27. $\cot^{-1}\left(\dfrac{1+x}{1-x}\right)$
28. $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

29. $\operatorname{cosec}^{-1}\left(\frac{1+x^2}{2x}\right)$

30. $\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

31. $\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

32. $\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

33. $\cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)$

34. $\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}$

35. $\sin^{-1}\{2ax\sqrt{1-a^2x^2}\}$

36. $\tan^{-1}\left\{\frac{\sqrt{1+a^2x^2}-1}{ax}\right\}$

37. $\sin^{-1}\left\{\frac{x^2}{\sqrt{x^4+a^4}}\right\}$

38. $\tan^{-1}\left(\frac{e^{2x}+1}{e^{2x}-1}\right)$

39. $\cos^{-1}(2x)+2\cos^{-1}\sqrt{1-4x^2}$

40. $\tan^{-1}\left(\frac{a-x}{1+ax}\right)$

41. $\tan^{-1}\left\{\frac{\sqrt{x}-x}{1+x^{3/2}}\right\}$

42. $\tan^{-1}\left(\frac{\sqrt{a}+\sqrt{x}}{1-\sqrt{ax}}\right)$

43. $\tan^{-1}\left(\frac{3-2x}{1+6x}\right)$

44. $\tan^{-1}\left(\frac{5x}{1-6x^2}\right)$

45. $\tan^{-1}\left(\frac{2x}{1+15x^2}\right)$

46. If $y=\tan^{-1}\left(\frac{ax-b}{bx+a}\right)$, prove that $\frac{dy}{dx}=\frac{1}{(1+x^2)}$.

47. If $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$, show that $\frac{dy}{dx}=\frac{4}{(1+x^2)}$.

48. If $y=\sec^{-1}\left(\frac{x+1}{x-1}\right)+\sin^{-1}\left(\frac{x-1}{x+1}\right)$, show that $\frac{dy}{dx}=0$.

49. If $y=\sin\left\{2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right\}$, show that $\frac{dy}{dx}=\frac{-x}{\sqrt{1-x^2}}$.

50. If $y=\tan^{-1}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\}$, prove that $\frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}$. **[CBSE 2003]**

51. Differentiate $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$ w.r.t. x.

ANSWERS (EXERCISE 10D)

1. $\frac{1}{2}$ **2.** $\frac{1}{2}$ **3.** $\frac{1}{2}$ **4.** $\frac{1}{2}$ **5.** 1 **6.** 1 **7.** $\frac{3}{2}$ **8.** 2

9. -2 **10.** 2 **11.** $\frac{1}{2}$ **12.** -2 **13.** $\frac{-1}{\sqrt{1-x^2}}$ **14.** $\frac{-1}{2\sqrt{1-x^2}}$

15. $\frac{-1}{2\sqrt{1-x^2}}$ **16.** $\frac{1}{\sqrt{1-x^2}}$ **17.** $\frac{2}{\sqrt{1-x^2}}$ **18.** $\frac{3}{\sqrt{1-x^2}}$

19. $\dfrac{-2}{\sqrt{1-x^2}}$ **20.** $\dfrac{1}{\sqrt{1-x^2}}$ **21.** $\dfrac{1}{\sqrt{1-x^2}}$ **22.** $\dfrac{1}{2\sqrt{1-x^2}}$

23. $\dfrac{1}{\sqrt{1-x^2}}$ **24.** $\dfrac{2}{\sqrt{1-x^2}}$ **25.** $\dfrac{1}{(1+x^2)}$ **26.** $\dfrac{1}{(1+x^2)}$

27. $\dfrac{-1}{(1+x^2)}$ **28.** $\dfrac{3}{(1+x^2)}$ **29.** $\dfrac{2}{(1+x^2)}$ **30.** $\dfrac{2}{(1+x^2)}$

31. $\dfrac{-1}{(1+x^2)}$ **32.** $\dfrac{-2}{(1+x^2)}$ **33.** $\dfrac{2nx^{n-1}}{(1+x^{2n})}$ **34.** $\dfrac{1}{\sqrt{a^2-x^2}}$

35. $\dfrac{2a}{\sqrt{1-a^2x^2}}$ **36.** $\dfrac{a}{2(1+a^2x^2)}$ **37.** $\dfrac{2a^2x}{(a^4+x^4)}$ **38.** $\dfrac{-2e^{2x}}{(1+e^{4x})}$

39. $\dfrac{2}{\sqrt{1-4x^2}}$ **40.** $\dfrac{-1}{(1+x^2)}$ **41.** $\dfrac{1}{2\sqrt{x}(1+x)}-\dfrac{1}{(1+x^2)}$

42. $\dfrac{1}{2\sqrt{x}(1+x)}$ **43.** $\dfrac{-2}{(1+4x^2)}$ **44.** $\left(\dfrac{3}{1+9x^2}+\dfrac{2}{1+4x^2}\right)$

45. $\dfrac{5}{(1+25x^2)}-\dfrac{3}{(1+9x^2)}$ **51.** $\dfrac{2^{x+1}(\log 2)}{(1+4^x)}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10D)

(i) Put $x=\cos\theta$ in Q. 13, 14, 15.

(ii) Put $x=\sin\theta$ in Q. 16, 17, 18, 19, 20, 21, 22, 23, 24.

(iii) Put $x=\tan\theta$ in Q. 25, 26, 27, 28, 29, 30.

(iv) Put $x=\cot\theta$ in Q. 31, 32.

33. Put $x^n=\tan\theta$. 34. Put $x=a\sin\theta$. 35. Put $ax=\sin\theta$.

36. Put $ax=\tan\theta$. 37. Put $x^2=a^2\tan\theta$. 38. Put $e^{2x}=\cot\theta$.

39. Put $2x=\cos\theta$. 40. $y=\tan^{-1}a-\tan^{-1}x$. 41. $y=\tan^{-1}\sqrt{x}-\tan^{-1}x$.

42. $y=\tan^{-1}\sqrt{a}+\tan^{-1}\sqrt{x}$. 43. $y=\tan^{-1}3-\tan^{-1}2x$.

44. $y=\tan^{-1}3x+\tan^{-1}2x$. 45. $y=\tan^{-1}5x-\tan^{-1}3x$.

46. $y=\tan^{-1}\left[\dfrac{x-(b/a)}{(b/a)x+1}\right]=\left[\tan^{-1}x-\tan^{-1}\dfrac{b}{a}\right]$.

48. $\sec^{-1}\theta=\cos^{-1}\dfrac{1}{\theta}$ and $\cos^{-1}\theta+\sin^{-1}\theta=\dfrac{\pi}{2}$.

51. Let $y=\sin^{-1}\left\{\dfrac{2^x\cdot 2}{1+(2^x)^2}\right\}$. Puttting $2^x=\tan\theta$, we get

$$y=\sin^{-1}\left\{\frac{2\tan\theta}{1+\tan^2\theta}\right\}=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}2^x.$$

$$\therefore\ \frac{dy}{dx}=2\cdot\frac{d}{dx}(\tan^{-1}2^x)=2\cdot\frac{1}{\{1+(2^x)^2\}}\cdot\frac{d}{dx}(2^x)=\frac{2}{(1+4^x)}\cdot 2^x(\log 2)=\frac{2^{x+1}(\log 2)}{(1+4^x)}.$$

5. Differentiation of Implicit Function

Let $f(x, y) = a$ be a function of x and y defined in such a manner that y is not expressible directly in terms of x. Then, $f(x, y) = a$ is called an implicit function of x and y. In differentiating such a function, we differentiate both sides of the equation termwise, keeping in mind that

$$\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}\ ;\ \frac{d}{dx}(y^3) = 3y^2 \cdot \frac{dy}{dx}, \text{ and so on.}$$

SOLVED EXAMPLES

EXAMPLE 1 *If* $x^3 + y^3 = 3axy$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $x^3 + y^3 = 3axy$. ... (i)

Differentiating both sides of (i) w.r.t. x, we get

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3a \cdot \left\{x \cdot \frac{dy}{dx} + y \cdot 1\right\}$$

$$\Rightarrow\ 3(y^2 - ax) \cdot \frac{dy}{dx} = 3(ay - x^2)$$

$$\Rightarrow\ \frac{dy}{dx} = \left(\frac{ay - x^2}{y^2 - ax}\right).$$

EXAMPLE 2 *If* $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. ... (i)

Differentiating both sides of (i) w.r.t. x, we get

$$2ax + 2h\left(x \cdot \frac{dy}{dx} + y \cdot 1\right) + 2by \cdot \frac{dy}{dx} + 2g + 2f \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow\ (2ax + 2hy + 2g) + (2hx + 2by + 2f) \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow\ \frac{dy}{dx} = -\left(\frac{ax + hy + g}{hx + by + f}\right).$$

EXAMPLE 3 *If* $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$, *prove that* $\frac{dy}{dx} = \sqrt{\frac{1 - y^2}{1 - x^2}}$.

SOLUTION Given: $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$. ... (i)

Putting $x = \sin\theta$ and $y = \sin\phi$, it becomes

$$\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$$

$$\Rightarrow\ \frac{\cos\theta + \cos\phi}{\sin\theta - \sin\phi} = a$$

$$\Rightarrow \frac{2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)}{2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)} = a$$

$$\Rightarrow \cot\left(\frac{\theta-\phi}{2}\right) = a \Rightarrow \theta-\phi = 2\cot^{-1} a$$

$$\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a. \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} = 0.$$

Hence, $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

EXAMPLE 4 *If* $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$, *prove that* $\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$.

SOLUTION We have $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$. ... (i)

Putting $x = \sin\theta$ and $y = \cos\phi$ in (i), we get

$$\sin\theta\cos\phi + \cos\theta\sin\phi = 1$$

$$\Rightarrow \sin(\theta+\phi) = 1$$

$$\Rightarrow (\theta+\phi) = \sin^{-1}(1)$$

$$\Rightarrow \sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}. \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} = 0.$$

$$\therefore \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}.$$

EXAMPLE 5 *If* $x\sqrt{1+y} + y\sqrt{1+x} = 0$, *prove that* $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$.

SOLUTION $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$$

$$\Rightarrow x^2(1+y) = y^2(1+x) \qquad \text{[on squaring both sides]}$$

$$\Rightarrow (x-y)(x+y+xy) = 0$$

$$\Rightarrow x+y+xy = 0 \qquad [\because\ x = y \text{ does not satisfy the given equation}]$$

$$\Rightarrow y = \frac{-x}{1+x}.$$

$$\therefore\ \frac{dy}{dx} = -\left\{\frac{(1+x)\cdot 1 - x\cdot 1}{(1+x)^2}\right\} = \frac{-1}{(1+x)^2} \qquad \text{[using the quotient rule].}$$

EXAMPLE 6 *If* $\sin y = x \sin(a+y)$, *prove that* $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$. **[CBSE 2012]**

SOLUTION $\sin y = x\sin(a+y)$

$$\Rightarrow\ x = \frac{\sin y}{\sin(a+y)}. \qquad \text{... (i)}$$

On differentiating both sides of (i) w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)} \quad \text{[using the quotient rule]}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)} = \frac{\sin a}{\sin^2(a+y)}.$$

Hence, $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

EXAMPLE 7 *If* $y\cdot\sqrt{x^2+1} = \log\{\sqrt{x^2+1} - x\}$, *show that*

$$(x^2+1)\frac{dy}{dx} + xy + 1 = 0. \qquad \textbf{[CBSE 2006]}$$

SOLUTION Given: $y\cdot\sqrt{x^2+1} = \log\{\sqrt{x^2+1} - x\}$. ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$y\cdot\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x + (\sqrt{x^2+1})\cdot\frac{dy}{dx}$$

$$= \frac{1}{\{\sqrt{x^2+1} - x\}}\cdot\left\{\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x - 1\right\}$$

$$\Rightarrow\ \frac{xy}{\sqrt{x^2+1}} + (\sqrt{x^2+1})\frac{dy}{dx} = \frac{1}{\{\sqrt{x^2+1} - x\}}\cdot\left\{\frac{x}{\sqrt{x^2+1}} - 1\right\}$$

$$\Rightarrow\ \frac{xy}{\sqrt{x^2+1}} + (\sqrt{x^2+1})\frac{dy}{dx} = \frac{1}{\{\sqrt{x^2+1} - x\}}\cdot\frac{\{x - \sqrt{x^2+1}\}}{\sqrt{x^2+1}}$$

$$\Rightarrow\ xy + (x^2+1)\frac{dy}{dx} = -1$$

$$\Rightarrow\ (x^2+1)\frac{dy}{dx} + xy + 1 = 0.$$

EXAMPLE 8 *If* $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, *show that* $2x\dfrac{dy}{dx} + y = 2\sqrt{x}$. **[CBSE 2006]**

SOLUTION $y = \sqrt{x} + \dfrac{1}{\sqrt{x}} \ \Rightarrow\ \sqrt{x}\,y = x + 1$ ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$\sqrt{x}\cdot\frac{dy}{dx}+y\cdot\frac{1}{2}x^{-1/2}=1$$

$$\Rightarrow\ \sqrt{x}\,\frac{dy}{dx}+\frac{y}{2\sqrt{x}}=1$$

$$\Rightarrow\ 2x\,\frac{dy}{dx}+y=2\sqrt{x}.$$

EXAMPLE 9 *If* $\cos(x+y)=y\sin x$, *find* $\frac{dy}{dx}\cdot$

SOLUTION Given: $\cos(x+y)=y\sin x.$ ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$-\sin(x+y)\cdot\frac{d}{dx}(x+y)=y\cos x+\sin x\cdot\frac{dy}{dx}$$

$$\Rightarrow\ -\sin(x+y)\cdot\left(1+\frac{dy}{dx}\right)=y\cos x+\sin x\cdot\frac{dy}{dx}$$

$$\Rightarrow\ \{\sin(x+y)+\sin x\}\cdot\frac{dy}{dx}=-\{\sin(x+y)+y\cos x\}$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{-\{\sin(x+y)+y\cos x\}}{\{\sin(x+y)+\sin x\}}\cdot$$

EXAMPLE 10 *Find* $\frac{dy}{dx}$ *when* $\sin(xy)+\frac{x}{y}=x^2-y.$

SOLUTION Given: $\sin(xy)+\frac{x}{y}=x^2-y.$

Differentiating both sides w.r.t. x, we get

$$\cos(xy)\cdot\frac{d}{dx}(xy)+x\cdot\left(-\frac{1}{y^2}\right)\frac{dy}{dx}+\frac{1}{y}\cdot 1=2x-\frac{dy}{dx}$$

$$\Rightarrow\ \cos(xy)\cdot\left[x\cdot\frac{dy}{dx}+y\cdot 1\right]-\frac{x}{y^2}\cdot\frac{dy}{dx}+\frac{1}{y}=2x-\frac{dy}{dx}$$

$$\Rightarrow\ \left[x\cos(xy)-\frac{x}{y^2}+1\right]\cdot\frac{dy}{dx}=2x-\frac{1}{y}-y\cos(xy)$$

$$\Rightarrow\ \{xy^2\cos(xy)-x+y^2\}\cdot\frac{dy}{dx}=2xy^2-y-y^3\cos(xy).$$

Hence, $\frac{dy}{dx}=\left\{\frac{2xy^2-y-y^3\cos(xy)}{xy^2\cos(xy)-x+y^2}\right\}.$

EXAMPLE 11 *If* $e^x+e^y=e^{x+y}$, *prove that* $\frac{dy}{dx}=-e^{y-x}.$

SOLUTION Given: $e^x+e^y=e^{x+y}.$ ... (i)

On dividing throughout by e^{x+y}, we get

$$e^{-y} + e^{-x} = 1. \quad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$e^{-y} \cdot \left(\frac{-dy}{dx}\right) + e^{-x}(-1) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-e^{-x}}{e^{-y}} = -e^{(y-x)}.$$

EXAMPLE 12 *If* $\tan^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = a$, *show that* $\frac{dy}{dx} = \frac{x(1 - \tan a)}{y(1 + \tan a)}$.

SOLUTION $\tan^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = a \Rightarrow \frac{(x^2 - y^2)}{(x^2 + y^2)} = \tan a$

$$\therefore \ (x^2 - y^2) = (x^2 + y^2)\tan a. \quad \text{... (i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$2x - 2y \cdot \frac{dy}{dx} = 2x \tan a + 2y \cdot \frac{dy}{dx} \cdot \tan a$$

$$\Rightarrow y(1 + \tan a)\frac{dy}{dx} = x(1 - \tan a)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x(1 - \tan a)}{y(1 + \tan a)}.$$

EXERCISE 10E

Find $\frac{dy}{dx}$, ***when:***

1. $x^2 + y^2 = 4$
2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
3. $\sqrt{x} + \sqrt{y} = \sqrt{a}$
4. $x^{2/3} + y^{2/3} = a^{2/3}$
5. $xy = c^2$
6. $x^2 + y^2 - 3xy = 1$
7. $xy^2 - x^2y - 5 = 0$
8. $(x^2 + y^2)^2 = xy$
9. $x^2 + y^2 = \log(xy)$
10. $x^n + y^n = a^n$
11. $x \sin 2y = y \cos 2x$
12. $\sin^2 x + 2\cos y + xy = 0$
13. $y \sec x + \tan x + x^2y = 0$
14. $\cot(xy) + xy = y$
15. $y \tan x - y^2 \cos x + 2x = 0$
16. $e^x \log y = \sin^{-1}x + \sin^{-1}y$
17. $xy \log(x + y) = 1$
18. $\tan(x + y) + \tan(x - y) = 1$
19. $\log\sqrt{x^2 + y^2} = \tan^{-1}\frac{y}{x}$
20. If $y = x \sin y$, prove that $\left(x \cdot \frac{dy}{dx}\right) = \frac{y}{(1 - x\cos y)}$.
21. If $xy = \tan(xy)$, show that $\frac{dy}{dx} = \frac{-y}{x}$.

22. If $y\log x = (x - y)$, prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(1 + \log x)^2}$.

23. If $\cos y = x\cos(y + a)$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(y + a)}{\sin a}$.

24. If $\cos^{-1}\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right) = \tan^{-1}a$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

ANSWERS (EXERCISE 10E)

1. $\dfrac{-x}{y}$ **2.** $\dfrac{-b^2x}{a^2y}$ **3.** $-\sqrt{\dfrac{y}{x}}$ **4.** $\dfrac{-y^{1/3}}{x^{1/3}}$ **5.** $\dfrac{-c^2}{x^2}$

6. $\dfrac{(2x - 3y)}{(3x - 2y)}$ **7.** $\dfrac{(y^2 - 2xy)}{(x^2 - 2xy)}$ **8.** $\dfrac{(y - 4xy^2 - 4x^3)}{(4y^3 + 4x^2y - x)}$ **9.** $\dfrac{y(1 - 2x^2)}{x(2y^2 - 1)}$

10. $\dfrac{-x^{n-1}}{y^{n-1}}$ **11.** $\dfrac{(2y\sin 2x + \sin 2y)}{(\cos 2x - 2x\cos 2y)}$ **12.** $\dfrac{(y + \sin 2x)}{(2\sin y - x)}$

13. $\dfrac{-(y\sec x\tan x + \sec^2 x + 2xy)}{(x^2 + \sec x)}$ **14.** $\dfrac{-y\cot^2(xy)}{\{1 + x\cot^2(xy)\}}$

15. $\dfrac{(y\sec^2 x + y^2\sin x + 2)}{(2y\cos x - \tan x)}$ **16.** $y\cdot\sqrt{\dfrac{1 - y^2}{1 - x^2}}\cdot\left\{\dfrac{1 - e^x\log y\cdot\sqrt{1 - x^2}}{(e^x\sqrt{1 - y^2}) - y}\right\}$

17. $\dfrac{-(x + y + x^2y)}{x^2\{y + (x + y)\log(x + y)\}}$ **18.** $\dfrac{\sec^2(x + y) + \sec^2(x - y)}{\sec^2(x - y) - \sec^2(x + y)}$ **19.** $\dfrac{x + y}{x - y}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10E)

8. $x^4 + y^4 + 2x^2y^2 = xy$

$\Rightarrow 4x^3 + 4y^3\cdot\dfrac{dy}{dx} + 4x^2y\cdot\dfrac{dy}{dx} + 4xy^2 = x\dfrac{dy}{dx} + y$

$\Rightarrow (4y^3 + 4x^2y - x)\dfrac{dy}{dx} = (y - 4xy^2 - 4x^3).$

9. $x^2 + y^2 = \log x + \log y$

$\Rightarrow 2x + 2y\dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{1}{y}\cdot\dfrac{dy}{dx}.$

10. $x^n + y^n = a^n \Rightarrow nx^{n-1} + ny^{n-1}\dfrac{dy}{dx} = 0.$

14. $\cot(xy) + xy = y$

$\Rightarrow -\operatorname{cosec}^2(xy)\cdot\left\{x\dfrac{dy}{dx} + y\right\} + \left(x\dfrac{dy}{dx} + y\right) = \dfrac{dy}{dx}$

$$\Rightarrow \{\text{cosec}^2(xy) - 1\} \cdot x\frac{dy}{dx} + \frac{dy}{dx} = \{1 - \text{cosec}^2(xy)\}y$$

$$\Rightarrow \{x\cot^2(xy) + 1\} \cdot \frac{dy}{dx} = -y\cot^2(xy).$$

17. $y\log(x+y) = \dfrac{1}{x}$

$$\Rightarrow y \cdot \frac{1}{(x+y)} \cdot \left(1 + \frac{dy}{dx}\right) + \log(x+y) \cdot \frac{dy}{dx} = \frac{-1}{x^2}$$

$$\Rightarrow \left\{\frac{y}{(x+y)} + \log(x+y)\right\} \cdot \frac{dy}{dx} = -\left(\frac{1}{x^2} + \frac{y}{x+y}\right).$$

19. $\dfrac{1}{2}\log(x^2+y^2) = \tan^{-1}\left(\dfrac{y}{x}\right)$

$$\Rightarrow \log(x^2+y^2) = 2\tan^{-1}\frac{y}{x} \Rightarrow (x^2+y^2) = e^{2\tan^{-1} y/x}. \quad \text{... (i)}$$

On differentiating (i) w.r.t. x, we get

$$2x + 2y \cdot \frac{dy}{dx} = e^{2\tan^{-1} y/x} \cdot \frac{2}{\left(1 + \frac{y^2}{x^2}\right)} \cdot \frac{\left(x\frac{dy}{dx} - y\right)}{x^2}$$

$$\Rightarrow x + y\frac{dy}{dx} = (x^2+y^2) \cdot \frac{x^2}{(x^2+y^2)} \cdot \frac{\left(x\frac{dy}{dx} - y\right)}{x^2} \quad \text{[using (i)]}$$

$$\Rightarrow x + y\frac{dy}{dx} = x\frac{dy}{dx} - y$$

$$\Rightarrow (x-y)\frac{dy}{dx} = (x+y) \Rightarrow \frac{dy}{dx} = \frac{(x+y)}{(x-y)}.$$

24. $\dfrac{x^2-y^2}{x^2+y^2} = \cos\{\tan^{-1} a\} = \text{constant}$

$$\Rightarrow \frac{d}{dx}\left(\frac{x^2-y^2}{x^2+y^2}\right) = 0$$

$$\Rightarrow \frac{(x^2+y^2) \cdot \frac{d}{dx}(x^2-y^2) - (x^2-y^2) \cdot \frac{d}{dx}(x^2+y^2)}{(x^2+y^2)^2} = 0$$

$$\Rightarrow (x^2+y^2)\left[2x - 2y\frac{dy}{dx}\right] - (x^2-y^2)\left(2x + 2y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow x\{(x^2+y^2) - (x^2-y^2)\} = y\{(x^2-y^2) + (x^2+y^2)\}\frac{dy}{dx}.$$

Hence, $\dfrac{dy}{dx} = \dfrac{y}{x}$.

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6. Differentiation Using Logarithms

When the given function is a power of some expression or a product of expressions, then we take the logarithm on both sides and differentiate termwise, as shown below.

SOLVED EXAMPLES

EXAMPLE 1 *If* $y=\sqrt{\dfrac{(x-3)(x^2+4)}{(3x^2+4x+5)}}$, *find* $\dfrac{dy}{dx}$. **[CBSE 2006]**

SOLUTION Given: $y=\sqrt{\dfrac{(x-3)(x^2+4)}{(3x^2+4x+5)}}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y=\frac{1}{2}\{\log(x-3)+\log(x^2+4)-\log(3x^2+4x+5)\}.$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\cdot\left\{\frac{1}{(x-3)}+\frac{2x}{(x^2+4)}-\frac{(6x+4)}{(3x^2+4x+5)}\right\}$$

$$\Rightarrow\ \frac{dy}{dx}=\left(\frac{1}{2}y\right)\cdot\left\{\frac{1}{(x-3)}+\frac{2x}{(x^2+4)}-\frac{(6x+4)}{(3x^2+4x+5)}\right\}$$

$$=\frac{1}{2}\cdot\sqrt{\frac{(x-3)(x^2+4)}{(3x^2+4x+5)}}\cdot\left\{\frac{1}{(x-3)}+\frac{2x}{(x^2+4)}-\frac{(6x+4)}{(3x^2+4x+5)}\right\}.$$

EXAMPLE 2 *If* $y=\dfrac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}$, *find* $\dfrac{dy}{dx}$.

SOLUTION Given: $y=\dfrac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y=\frac{1}{2}\log x+\frac{3}{2}\log(x+4)-\frac{4}{3}\log(4x-3).$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\cdot\frac{1}{x}+\frac{3}{2}\cdot\frac{1}{(x+4)}-\frac{4}{3}\cdot\frac{4}{(4x-3)}$$

$$\Rightarrow\ \frac{dy}{dx}=y\left[\frac{1}{2x}+\frac{3}{2(x+4)}-\frac{16}{3(4x-3)}\right]$$

$$=\frac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}\cdot\left[\frac{1}{2x}+\frac{3}{2(x+4)}-\frac{16}{3(4x-3)}\right].$$

EXAMPLE 3 *Differentiate* $(x+1)^2(x+2)^3(x+3)^4$ *w.r.t.* x.

SOLUTION Let $y=(x+1)^2(x+2)^3(x+3)^4$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y=2\log(x+1)+3\log(x+2)+4\log(x+3). \quad \text{... (ii)}$$

Differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{2}{(x+1)}+\frac{3}{(x+2)}+\frac{4}{(x+3)}$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \left[\frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)} \right]$$

$$= (x+1)^2 (x+2)^3 (x+3)^4 \cdot \left[\frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)} \right].$$

EXAMPLE 4 *Differentiate* $\sqrt{(x-1)(x-2)(x-3)(x-4)}$ *w.r.t. x.*

SOLUTION Let $y = \sqrt{(x-1)(x-2)(x-3)(x-4)}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = \frac{1}{2} \{\log (x-1) + \log (x-2) + \log (x-3) + \log (x-4)\}. \quad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} \cdot \left\{ \frac{1}{(x-1)} + \frac{1}{(x-2)} + \frac{1}{(x-3)} + \frac{1}{(x-4)} \right\}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{2} \right) \cdot \left\{ \frac{1}{(x-1)} + \frac{1}{(x-2)} + \frac{1}{(x-3)} + \frac{1}{(x-4)} \right\}$$

$$= \frac{1}{2} \cdot \sqrt{(x-1)(x-2)(x-3)(x-4)}$$

$$\cdot \left\{ \frac{1}{(x-1)} + \frac{1}{(x-2)} + \frac{1}{(x-3)} + \frac{1}{(x-4)} \right\}.$$

EXAMPLE 5 *Differentiate* $(e^x \cos^3 x \sin^2 x)$ *w.r.t. x.*

SOLUTION Let $y = e^x \cos^3 x \sin^2 x$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = x + 3 \log \cos x + 2 \log \sin x. \quad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 + \frac{3}{\cos x} \cdot (-\sin x) + \frac{2}{\sin x} \cdot \cos x$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \{1 - 3 \tan x + 2 \cot x\}$$

$$= (e^x \cos^3 x \sin^2 x)(1 - 3 \tan x + 2 \cot x).$$

EXAMPLE 6 *Differentiate* $(\tan x \tan 2x \tan 3x \tan 4x)$ *w.r.t. x.*

SOLUTION Let $y = \tan x \tan 2x \tan 3x \tan 4x$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = \log \tan x + \log \tan 2x + \log \tan 3x + \log \tan 4x. \quad \text{... (ii)}$$

On diffferentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \left\{ \frac{\sec^2 x}{\tan x} + \frac{2\sec^2 2x}{\tan 2x} + \frac{3\sec^2 3x}{\tan 3x} + \frac{4\sec^2 4x}{\tan 4x} \right\}$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \left[\frac{1}{\sin x \cos x} + \frac{2}{\sin 2x \cos 2x} + \frac{3}{\sin 3x \cos 3x} + \frac{4}{\sin 4x \cos 4x} \right]$$

$$= y \cdot \left[\frac{2}{\sin 2x} + \frac{4}{\sin 4x} + \frac{6}{\sin 6x} + \frac{8}{\sin 8x} \right]$$

$$= [2 \tan x \tan 2x \tan 3x \tan 4x] \times [\text{cosec } 2x + 2 \text{cosec } 4x + 3 \text{cosec } 6x + 4 \text{cosec } 8x].$$

EXAMPLE 7 *If* $y = (2x+3)^{(3x-5)}$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $y = (2x+3)^{(3x-5)}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = (3x - 5) \log (2x + 3). \quad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = (3x-5) \cdot \frac{d}{dx} \{\log (2x+3)\} + \log (2x+3) \cdot \frac{d}{dx} (3x - 5)$$

$$= (3x-5) \cdot \frac{1}{(2x+3)} \cdot 2 + \log (2x+3) \cdot 3$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \left\{ \frac{(6x-10)}{(2x+3)} + 3 \log (2x+3) \right\}$$

$$\Rightarrow \frac{dy}{dx} = (2x+3)^{(3x-5)} \cdot \left\{ \frac{(6x-10)}{(2x+3)} + 3 \log (2x+3) \right\}.$$

EXAMPLE 8 *If* $y = \frac{5^x}{x^5}$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $y = \frac{5^x}{x^5}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = \log (5^x) - \log (x^5)$$

$$\Rightarrow \log y = x \log 5 - 5 \log x$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = (\log 5) \cdot 1 - \frac{5}{x} \qquad \text{[differentiating both sides w.r.t. } x\text{]}$$

$$\Rightarrow \frac{dy}{dx} = y \left(\log 5 - \frac{5}{x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{5^x}{x^5} \left(\log 5 - \frac{5}{x} \right).$$

EXAMPLE 9 *Differentiate* x^x *w.r.t.* x.

SOLUTION Let $y = x^x$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = x \log x. \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x)$$

$$= \left(x \cdot \frac{1}{x} + \log x \cdot 1 \right) = (1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^x (1 + \log x).$$

EXAMPLE 10 *Differentiate* $(\sin x)^x$ *w.r.t. x.*

SOLUTION Let $y = (\sin x)^x$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = x \log (\sin x). \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx}\{\log (\sin x)\} + \log (\sin x) \cdot \frac{d}{dx}(x)$$

$$= x \cdot \frac{1}{\sin x} \cdot \cos x + \log (\sin x) \cdot 1$$

$$= x \cot x + \log (\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y \cdot [x \cot x + \log (\sin x)]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^x [x \cot x + \log (\sin x)].$$

EXAMPLE 11 *Differentiate* $x^{\sin^{-1} x}$ *w.r.t. x.* **[CBSE 2000]**

SOLUTION Let $y = x^{\sin^{-1} x}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = (\sin^{-1} x)(\log x). \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = (\sin^{-1} x) \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(\sin^{-1} x)$$

$$= (\sin^{-1} x) \frac{1}{x} + (\log x) \cdot \frac{1}{\sqrt{1 - x^2}}$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \left[\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1 - x^2}} \right]$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin^{-1} x} \cdot \left\{ \frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1 - x^2}} \right\}.$$

EXAMPLE 12 *Differentiate* $(\sin x)^{\log x}$ *w.r.t. x.*

SOLUTION Let $y = (\sin x)^{\log x}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = (\log x)(\log \sin x). \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = (\log x) \cdot \frac{d}{dx}(\log \sin x) + (\log \sin x) \cdot \frac{d}{dx}(\log x)$$

$$= (\log x) \cdot \frac{1}{\sin x} \cdot \cos x + (\log \sin x) \cdot \frac{1}{x}$$

$$= (\log x) \cot x + \frac{(\log \sin x)}{x}$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \left[(\log x) \cot x + \frac{(\log \sin x)}{x} \right]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x} \cdot \left[(\log x) \cot x + \frac{(\log \sin x)}{x} \right].$$

EXAMPLE 13 *Differentiate* $x^x \sin^{-1}\sqrt{x}$ *w.r.t. x.* **[CBSE 2005C]**

SOLUTION Let $y = x^x \sin^{-1}\sqrt{x}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = x \log x + \log (\sin^{-1}\sqrt{x}). \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(x \log x) + \frac{d}{dx}\{\log (\sin^{-1}\sqrt{x})\}$$

$$= \left(x \cdot \frac{1}{x} + \log x \cdot 1 \right) + \frac{1}{\sin^{-1}\sqrt{x}} \cdot \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2} x^{-1/2}$$

$$= \left\{ (1 + \log x) + \frac{1}{2(\sqrt{x - x^2}) \sin^{-1}\sqrt{x}} \right\}$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \left\{ (1 + \log x) + \frac{1}{2(\sqrt{x - x^2}) \sin^{-1}\sqrt{x}} \right\}$$

$$= (x^x \sin^{-1}\sqrt{x}) \left\{ (1 + \log x) + \frac{1}{2(\sqrt{x - x^2}) \sin^{-1}\sqrt{x}} \right\}.$$

$$= \left\{ (x^x \sin^{-1}\sqrt{x})(1 + \log x) + \frac{x^x}{2(\sqrt{x - x^2})} \right\}.$$

EXAMPLE 14 *Differentiate* $\dfrac{e^{x^2} \tan^{-1} x}{\sqrt{1 + x^2}}$ *w.r.t. x.*

SOLUTION Let $y = \dfrac{e^{x^2} \tan^{-1} x}{\sqrt{1 + x^2}}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = x^2 + \log(\tan^{-1}x) - \frac{1}{2}\log(1+x^2). \quad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2x + \frac{1}{\tan^{-1}x}\cdot\frac{1}{(1+x^2)} - \frac{1}{2}\cdot\frac{2x}{(1+x^2)}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left[2x + \frac{1}{(1+x^2)\tan^{-1}x} - \frac{x}{(1+x^2)}\right]$$

$$= \frac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}\cdot\left[2x + \frac{1}{(1+x^2)\tan^{-1}x} - \frac{x}{(1+x^2)}\right].$$

EXAMPLE 15 *Differentiate* $\left\{x^{\tan x} + \sqrt{\frac{x^2+1}{x}}\right\}$ *w.r.t. x.* **[CBSE 2006C]**

SOLUTION Let $y = u + v$, where $u = x^{\tan x}$ and $v = \sqrt{\frac{x^2+1}{x}}$.

Now, $u = x^{\tan x}$

$$\Rightarrow \quad \log u = (\tan x)(\log x)$$

$$\Rightarrow \quad \frac{1}{u}\cdot\frac{du}{dx} = (\tan x)\cdot\frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}(\tan x)$$

[differentiating w.r.t. x]

$$= (\tan x)\cdot\frac{1}{x} + (\log x)\sec^2 x$$

$$\Rightarrow \quad \frac{du}{dx} = u\cdot\left[\frac{\tan x}{x} + (\log x)\sec^2 x\right]$$

$$\Rightarrow \quad \frac{du}{dx} = x^{\tan x}\cdot\left\{\frac{\tan x}{x} + (\log x)\sec^2 x\right\}. \quad \text{... (i)}$$

And, $v = \sqrt{\frac{x^2+1}{x}}$

$$\Rightarrow \quad \log v = \frac{1}{2}\cdot\{\log(x^2+1) - \log x\}$$

$$\Rightarrow \quad \frac{1}{v}\cdot\frac{dv}{dx} = \frac{1}{2}\cdot\left\{\frac{2x}{(x^2+1)} - \frac{1}{x}\right\} \quad \text{[differentiating w.r.t. } x\text{]}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{v}{2}\cdot\left\{\frac{2x^2 - (x^2+1)}{x(x^2+1)}\right\}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{2}\sqrt{\frac{x^2+1}{x}}\cdot\left\{\frac{(x^2-1)}{x(x^2+1)}\right\}. \quad \text{... (ii)}$$

$$\therefore \quad y = u + v$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x^{\tan x} \cdot \left\{ \frac{\tan x}{x} + (\log x)\sec^2 x \right\} + \frac{1}{2} \cdot \sqrt{\frac{x^2+1}{x}} \cdot \left\{ \frac{(x^2-1)}{x(x^2+1)} \right\}.$$

EXAMPLE 16 *If* $y = (x)^{\cos x} + (\cos x)^{\sin x}$, *find* $\dfrac{dy}{dx}\cdot$ **[CBSE 2005C, '06, '09]**

SOLUTION Let $y = u + v$, where $u = (x)^{\cos x}$ and $v = (\cos x)^{\sin x}$.

Now, $u = (x)^{\cos x}$

$$\Rightarrow \log u = (\cos x)(\log x)$$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = (\cos x) \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(\cos x)$$

[on differentiating w.r.t. x]

$$= (\cos x) \cdot \frac{1}{x} + (\log x)(-\sin x)$$

$$\Rightarrow \frac{du}{dx} = u \cdot \left\{ \frac{\cos x}{x} - (\log x)(\sin x) \right\}$$

$$\Rightarrow \frac{du}{dx} = (x)^{\cos x} \left\{ \frac{\cos x}{x} - (\log x)(\sin x) \right\} \cdot \qquad \text{... (i)}$$

And, $v = (\cos x)^{\sin x}$

$$\Rightarrow \log v = (\sin x) \log (\cos x)$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = (\sin x) \cdot \frac{d}{dx}\{\log (\cos x)\} + \log (\cos x) \cdot \frac{d}{dx}(\sin x)$$

[on differentiating w.r.t. x]

$$\Rightarrow \frac{dv}{dx} = v \cdot \left\{ (\sin x) \cdot \frac{(-\sin x)}{\cos x} + \log (\cos x) \cdot \cos x \right\}$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^{\sin x} \cdot \{-\sin x \tan x + \cos x \cdot \log (\cos x)\}. \qquad \text{... (ii)}$$

$$\therefore \quad y = (u + v)$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (x)^{\cos x} \cdot \left\{ \frac{\cos x}{x} - (\log x) \sin x \right\}$$

$$+ (\cos x)^{\sin x} \cdot \{-\sin x \tan x + \cos x \cdot \log (\cos x)\}.$$

EXAMPLE 17 *If* $y = (\sin x)^{\tan x} + (\cos x)^{\sec x}$, *find* $\dfrac{dy}{dx}\cdot$

SOLUTION Let $y = u + v$, where $u = (\sin x)^{\tan x}$ and $v = (\cos x)^{\sec x}$.

Now, $u = (\sin x)^{\tan x}$

$$\Rightarrow \log u = (\tan x)(\log \sin x)$$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = (\tan x)\cdot\frac{d}{dx}(\log \sin x) + (\log \sin x)\cdot\frac{d}{dx}(\tan x)$$

[on differentiating w.r.t. x]

$$= (\tan x)\cdot\frac{1}{\sin x}\cdot\cos x + (\log \sin x)\cdot\sec^2 x$$

$$\Rightarrow \frac{du}{dx} = u[1 + (\log \sin x)\sec^2 x]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^{\tan x}\cdot\{1 + (\log \sin x)\sec^2 x\}. \quad \text{... (i)}$$

And, $v = (\cos x)^{\sec x}$

$$\Rightarrow \log v = (\sec x)\cdot\log(\cos x)$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = (\sec x)\cdot\frac{d}{dx}\{\log(\cos x)\} + \log(\cos x)\cdot\frac{d}{dx}(\sec x)$$

[on differentiating w.r.t. x]

$$= (\sec x)\cdot\frac{1}{\cos x}\cdot(-\sin x) + \log(\cos x)\cdot\sec x \tan x$$

$$= (\sec x \tan x)\,[\log(\cos x) - 1]$$

$$\Rightarrow \frac{dv}{dx} = v\cdot(\sec x \tan x)\,[\log(\cos x) - 1]$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^{\sec x}\cdot(\sec x \tan x)\,[\log(\cos x) - 1]. \quad \text{... (ii)}$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\tan x}\cdot\{1 + (\log \sin x)\sec^2 x\}$$

$$+ (\cos x)^{\sec x}\cdot(\sec x \tan x)\,[\log(\cos x) - 1] \quad \text{[from (i) and (ii)].}$$

EXAMPLE 18 *If* $x^y = y^x$, *find* $\frac{dy}{dx}\cdot$

SOLUTION Given: $x^y = y^x$

$$\Rightarrow y \log x = x \log y. \quad \text{... (i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$y\cdot\frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}(y) = x\cdot\frac{d}{dx}(\log y) + (\log y)\cdot\frac{d}{dx}(x)$$

$$\Rightarrow y\cdot\frac{1}{x} + (\log x)\cdot\frac{dy}{dx} = x\cdot\frac{1}{y}\cdot\frac{dy}{dx} + (\log y)\cdot 1$$

$$\Rightarrow \left(\log x - \frac{x}{y}\right)\frac{dy}{dx} = \left(\log y - \frac{y}{x}\right)$$

$$\Rightarrow \frac{(y \log x - x)}{y}\cdot\frac{dy}{dx} = \frac{(x \log y - y)}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(x \log y - y)}{x(y \log x - x)}.$$

EXAMPLE 19 *If* $x^y \cdot y^x = 1$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $x^y \cdot y^x = 1$

$\Rightarrow (y \log x) + (x \log y) = 0.$ ... (i) $[\because \log 1 = 0]$

On differentiating both sides of (i) w.r.t. x, we get

$$y \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(y) + x \cdot \frac{d}{dx}(\log y) + (\log y) \cdot \frac{d}{dx}(x) = 0$$

$$\Rightarrow y \cdot \frac{1}{x} + (\log x) \cdot \frac{dy}{dx} + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + (\log y) \cdot 1 = 0$$

$$\Rightarrow \left(\log x + \frac{x}{y}\right) \cdot \frac{dy}{dx} = -\left(\log y + \frac{y}{x}\right)$$

$$\Rightarrow \frac{(y \log x + x)}{y} \cdot \frac{dy}{dx} = \frac{-(x \log y + y)}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y(x \log y + y)}{x(y \log x + x)}.$$

EXAMPLE 20 *If* $x^y = e^{x-y}$, *prove that* $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$. **[CBSE 2010C]**

SOLUTION We have

$$x^y = e^{x-y} \Rightarrow y \log x = (x - y)$$

$$\Rightarrow (1 + \log x)y = x$$

$$\Rightarrow y = \frac{x}{(1 + \log x)}. \quad \text{... (i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1 + \log x) \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(1 + \log x)}{(1 + \log x)^2}$$

$$= \frac{(1 + \log x) \cdot 1 - x \cdot \frac{1}{x}}{(1 + \log x)^2} = \frac{(1 + \log x - 1)}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}.$$

EXAMPLE 21 *If* $x^y + y^x = a^b$, *find* $\frac{dy}{dx}$.

SOLUTION Let $u = x^y$ and $v = y^x$.

Then, $u + v = a^b \Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0.$ ... (i) $[\because a^b = \text{constant}]$

Now, $u = x^y \Rightarrow \log u = y \log x$ [taking log on both sides]

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx} \quad \text{[on differentiation]}$$

$$\Rightarrow \frac{du}{dx} = u\left[\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right]$$

$$\Rightarrow \frac{du}{dx} = x^y\left[\frac{y + x\log x \cdot \frac{dy}{dx}}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{y-1}\left[y + x\log x \cdot \frac{dy}{dx}\right].$$

And, $v = y^x \Rightarrow \log v = x\log y$ [taking log on both sides]

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = x\cdot\frac{1}{y}\cdot\frac{dy}{dx} + \log y \quad \text{[on differentiation]}$$

$$\Rightarrow \frac{dv}{dx} = v\cdot\left[\frac{x}{y}\cdot\frac{dy}{dx} + \log y\right] \Rightarrow \frac{dv}{dx} = y^x\left\{\frac{x\cdot\frac{dy}{dx} + y\log y}{y}\right\}$$

$$\Rightarrow \frac{dv}{dx} = y^{(x-1)}\cdot\left\{x\cdot\frac{dy}{dx} + y\log y\right\}.$$

Using (i), we get $\frac{du}{dx} + \frac{dv}{dx} = 0$

$$\Rightarrow x^{(y-1)}\left\{y + x\log x\cdot\frac{dy}{dx}\right\} + y^{(x-1)}\cdot\left\{x\frac{dy}{dx} + y\log y\right\} = 0$$

$$\Rightarrow \{x^y(\log x) + x\cdot y^{(x-1)}\}\cdot\frac{dy}{dx} = -\{y\cdot x^{(y-1)} + y^x(\log y)\}.$$

$$\therefore \frac{dy}{dx} = \frac{-\{y\cdot x^{(y-1)} + y^x(\log y)\}}{\{x^y(\log x) + x\,y^{(x-1)}\}}.$$

EXAMPLE 22 *If* $x^x + x^y + y^x = a^b$, *find* $\frac{dy}{dx}$.

SOLUTION Let $u = x^x$, $v = x^y$ and $w = y^x$. Then,

$$u + v + w = a^b$$

$$\Rightarrow \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0. \quad \text{... (i)} \quad [\because a^b = \text{constant}]$$

Now, $u = x^x$

$$\Rightarrow \log u = x\log x$$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x\cdot\frac{d}{dx}(\log x) + \log x\cdot\frac{d}{dx}(x)$$

[on differentiating both sides w.r.t. x]

$$\Rightarrow \frac{du}{dx} = u\cdot\left\{x\cdot\frac{1}{x} + (\log x)\cdot 1\right\}$$

$$\Rightarrow \frac{du}{dx} = x^x(1 + \log x). \quad \text{... (ii)}$$

And, $v = x^y$

$\Rightarrow \log v = y \log x$

$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = y \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(y)$

[on differentiating both sides w.r.t. x]

$\Rightarrow \frac{dv}{dx} = v \cdot \left[y \cdot \frac{1}{x} + (\log x) \cdot \frac{dy}{dx} \right]$

$\Rightarrow \frac{dv}{dx} = x^y \left\{ \frac{y}{x} + (\log x) \frac{dy}{dx} \right\}$. ... (iii)

And, $w = y^x$

$\Rightarrow \log w = x \log y$

$\Rightarrow \frac{1}{w} \cdot \frac{dw}{dx} = x \cdot \frac{d}{dx}(\log y) + (\log y) \cdot \frac{d}{dx}(x)$

[on differentiating both sides w.r.t. x]

$\Rightarrow \frac{dw}{dx} = w \cdot \left\{ x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + (\log y) \cdot 1 \right\}$

$\Rightarrow \frac{dw}{dx} = y^x \cdot \left\{ \frac{x}{y} \cdot \frac{dy}{dx} + (\log y) \right\}$. ... (iv)

Using (ii), (iii) and (iv) in (i), we get

$$x^x(1 + \log x) + x^y \left\{ \frac{y}{x} + (\log x) \frac{dy}{dx} \right\} + y^x \cdot \left\{ \frac{x}{y} \cdot \frac{dy}{dx} + (\log y) \right\} = 0$$

$\Rightarrow \{x^x(1 + \log x) + y \cdot x^{(y-1)} + y^x(\log y)\} + \{x^y(\log x) + xy^{(x-1)}\} \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-\{x^x(1 + \log x) + y \cdot x^{(y-1)} + y^x(\log y)\}}{\{x^y(\log x) + xy^{(x-1)}\}}$.

EXAMPLE 23 *If* $y = x^{(x^x)}$, *find* $\frac{dy}{dx}$.

SOLUTION Let $x^x = u$. Then, $y = x^u$.

$\therefore \; x \log x = \log u$ and $\log y = u \log x$.

Now, $\log u = x \log x$

$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x)$ [on differentiating w.r.t. x]

$\Rightarrow \frac{du}{dx} = u \cdot \left[x \cdot \frac{1}{x} + \log x \cdot 1 \right]$

$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$. ... (i)

And, $\log y = u \log x$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = u \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(u)$

$= u \cdot \frac{1}{x} + (\log x) \cdot \frac{du}{dx}$

$$\Rightarrow \quad \frac{dy}{dx} = y \cdot \left[\frac{u}{x} + (\log x) \cdot \frac{du}{dx} \right]$$

$$= x^{(x^x)} \cdot \left[\frac{x^x}{x} + (\log x)\{x^x (1 + \log x)\} \right]$$

$$= x^{(x^x)} \cdot [x^{(x-1)} + x^x (\log x) + x^x (\log x)^2].$$

EXAMPLE 24 *If* $y = (\log x)^{\cos x} + \dfrac{x^2+1}{x^2-1}$, *find* $\dfrac{dy}{dx}$. **[CBSE 2008C]**

SOLUTION Let $(\log x)^{\cos x} = u$ and $\dfrac{x^2+1}{x^2-1} = v$. Then,

$$u = (\log x)^{\cos x} \Rightarrow \log u = \cos x \cdot \log (\log x). \qquad \text{... (i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \cos x \cdot \frac{d}{dx} \{\log (\log x)\} + \log (\log x) \cdot \frac{d}{dx} (\cos x)$$

$$= \cos x \frac{1}{\log x} \cdot \frac{1}{x} - (\sin x) \cdot \log (\log x).$$

$$\therefore \quad \frac{du}{dx} = u \cdot \left\{ \frac{\cos x}{x \log x} - (\sin x) \cdot \log (\log x) \right\}$$

$$\Rightarrow \quad \frac{du}{dx} = (\log x)^{\cos x} \cdot \left\{ \frac{\cos x}{x \log x} - (\sin x) \cdot \log (\log x) \right\}.$$

And, $v = \dfrac{(x^2+1)}{(x^2-1)} \Rightarrow \dfrac{dv}{dx} = \dfrac{(x^2-1) \cdot \frac{d}{dx}(x^2+1) - (x^2+1) \cdot \frac{d}{dx}(x^2-1)}{(x^2-1)^2}$

$$\Rightarrow \frac{dv}{dx} = \frac{(x^2-1) \cdot 2x - (x^2+1) \cdot 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

$$\therefore \quad y = u + v$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= (\log x)^{\cos x} \cdot \left\{ \frac{\cos x}{x \log x} - (\sin x) \cdot \log (\log x) \right\} - \frac{4x}{(x^2-1)^2}.$$

EXAMPLE 25 *Differentiate* $(\log x)^x + x^{\log x}$ *w.r.t. x.* **[CBSE 2013]**

SOLUTION Let $y = (\log x)^x + x^{\log x}$.

Let $(\log x)^x = u$ and $x^{\log x} = v$. Then,

$$y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}. \qquad \text{... (i)}$$

Now, $u = (\log x)^x \Rightarrow \log u = x \log (\log x)$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log (\log x) \cdot 1$$

$$\Rightarrow \frac{du}{dx} = u \cdot \left\{ \frac{1}{\log x} + \log(\log x) \right\}$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x \cdot \left\{ \frac{1}{\log x} + \log(\log x) \right\}. \quad \ldots \text{(ii)}$$

$$v = x^{\log x} \Rightarrow \log v = (\log x)(\log x) = (\log x)^2$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = 2\log x \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dv}{dx} = v\left(\frac{2\log x}{x} \right)$$

$$\Rightarrow \frac{dv}{dx} = x^{\log x} \cdot \frac{2\log x}{x} = x^{\log x - 1} \cdot 2\log x. \quad \ldots \text{(iii)}$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (ii) and (iii) in (i), we get

$$\frac{dy}{dx} = (\log x)^{x-1} + (\log x)^x \times \log(\log x) + x^{\log x - 1} \cdot 2\log x.$$

EXAMPLE 26 *If* $y = (\sin x)^x + \sin^{-1}\sqrt{x}$, *find* $\frac{dy}{dx}$. **[CBSE 2013C]**

SOLUTION Let $(\sin x)^x = u$ and $\sin^{-1}\sqrt{x} = v$. Then,

$$y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}. \quad \ldots \text{(i)}$$

Now, $u = (\sin x)^x \Rightarrow \log u = x \log(\sin x)$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{1}{\sin x} \cdot \cos x + \log(\sin x) \cdot 1$$

$$\Rightarrow \frac{du}{dx} = u\{x \cot x + \log(\sin x)\}$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \cdot \{x \cot x + \log(\sin x)\}. \quad \ldots \text{(ii)}$$

$$v = \sin^{-1}\sqrt{x} \Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{(1-x)}} \cdot \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}\sqrt{(1-x)}} = \frac{1}{2\sqrt{x - x^2}}. \quad \ldots \text{(iii)}$$

Using (ii) and (iii) in (i), we get

$$\frac{dy}{dx} = (\sin x)^x \{x \cot x + \log(\sin x)\} + \frac{1}{2\sqrt{x - x^2}}.$$

EXAMPLE 27 *If* $(\tan^{-1}x)^y + y^{\cot x} = 1$, *then find* $\frac{dy}{dx}$. **[CBSE 2014C]**

SOLUTION Let $(\tan^{-1}x)^y = u$ and $y^{\cot x} = v$. Then,

$$u + v = 1 \Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0. \quad \ldots \text{(i)}$$

Now, $u = (\tan^{-1}x)^y$

$\Rightarrow \log u = y \log(\tan^{-1}x)$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = y\cdot\frac{1}{\tan^{-1}x}\cdot\frac{1}{(1+x^2)} + \log(\tan^{-1}x)\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = u\cdot\left\{\frac{y}{(1+x^2)\tan^{-1}x} + \log(\tan^{-1}x)\cdot\frac{dy}{dx}\right\}$$

$$\Rightarrow \frac{du}{dx} = (\tan^{-1}x)^y\cdot\left\{\frac{y}{(1+x^2)\tan^{-1}x} + \log(\tan^{-1}x)\cdot\frac{dy}{dx}\right\}$$

$$\Rightarrow \frac{du}{dx} = \frac{y(\tan^{-1}x)^{y-1}}{(1+x^2)} + (\tan^{-1}x)^y\log(\tan^{-1}x)\cdot\frac{dy}{dx}. \quad \ldots \text{(ii)}$$

Again, $v = y^{\cot x}$

$$\Rightarrow \log v = (\cot x)\log y$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = (\cot x)\cdot\frac{1}{y}\cdot\frac{dy}{dx} + (\log y)(-\operatorname{cosec}^2 x)$$

$$\Rightarrow \frac{dv}{dx} = v\left\{\frac{\cot x}{y}\cdot\frac{dy}{dx} - (\operatorname{cosec}^2 x)(\log y)\right\}$$

$$\Rightarrow \frac{dv}{dx} = y^{\cot x}\left\{\frac{\cot x}{y}\cdot\frac{dy}{dx} - (\operatorname{cosec}^2 x)(\log y)\right\}$$

$$\Rightarrow \frac{dv}{dx} = (\cot x)y^{(\cot x)-1}\cdot\frac{dy}{dx} - y^{\cot x}(\operatorname{cosec}^2 x)(\log y). \quad \ldots \text{(iii)}$$

Using (ii) and (iii) in (i), we get

$$\frac{y(\tan^{-1}x)^{y-1}}{(1+x^2)} + (\tan^{-1}x)^y\log(\tan^{-1}x)\frac{dy}{dx} + (\cot x)y^{(\cot x)-1}\cdot\frac{dy}{dx} - y^{\cot x}(\operatorname{cosec}^2 x)(\log y) = 0$$

$$\Rightarrow \{(\tan^{-1}x)^y\log(\tan^{-1}x) + (\cot x)y^{(\cot x)-1}\}\frac{dy}{dx} = \left\{y^{\cot x}(\operatorname{cosec}^2 x)(\log y) - \frac{y(\tan^{-1}x)^{y-1}}{(1+x^2)}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^{\cot x}(\operatorname{cosec}^2 x)(\log y) - \dfrac{y(\tan^{-1}x)^{y-1}}{(1+x^2)}}{(\tan^{-1}x)^y\log(\tan^{-1}x) + (\cot x)y^{(\cot x)-1}}.$$

EXERCISE 10F

Find $\frac{dy}{dx}$, when:

1. $y = x^{1/x}$
2. $y = x^{\sqrt{x}}$
3. $y = (\log x)^x$
4. $y = x^{\sin x}$
5. $y = x^{(\cos^{-1}x)}$
6. $y = (\tan x)^{1/x}$

7. $y = (\sin x)^{\cos x}$ 8. $y = (\log x)^{\sin x}$ 9. $y = (\cos x)^{\log x}$

10. $y = (\tan x)^{\sin x}$ 11. $y = (\cos x)^{\cos x}$ 12. $y = (\tan x)^{\cot x}$

13. $y = x^{\sin 2x}$ 14. $y = (\sin^{-1} x)^x$ 15. $y = \sin (x^x)$

16. $y = (3x + 5)^{(2x - 3)}$ 17. $y = (x+1)^3 (x+2)^4 (x+3)^5$

18. $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ 19. $y = (2-x)^3 (3+2x)^5$

20. $y = \cos x \cos 2x \cos 3x$ 21. $y = \dfrac{x^5 \sqrt{x+4}}{(2x+3)^2}$

22. $y = \dfrac{(x+1)^2 \sqrt{x-1}}{(x+4)^3 \cdot e^x}$ 23. $y = \dfrac{\sqrt{x}(3x+5)^2}{\sqrt{x+1}}$

24. $y = \dfrac{x^2 \sqrt{1+x}}{(1+x^2)^{3/2}}$ 25. $y = \sqrt{(x-2)(2x-3)(3x-4)}$

26. $y = \sin 2x \sin 3x \sin 4x$ 27. $y = \dfrac{x^3 \sin x}{e^x}$

28. $y = \dfrac{e^x \log x}{x^2}$ 29. $y = \dfrac{x \cos^{-1} x}{\sqrt{1-x^2}}$

30. $y = (1+x)(1+x^2)(1+x^4)(1+x^6)$ 31. $y = x^x - 2^{\sin x}$

32. $y = (\log x)^x + x^{\log x}$ 33. $y = x^{\sin x} + (\sin x)^{\cos x}$

34. $y = (x \cos x)^x + (x \sin x)^{1/x}$ 35. $y = (\sin x)^x + \sin^{-1} \sqrt{x}$

36. $y = x^{x \cos x} + \left(\dfrac{x^2+1}{x^2-1}\right)$ **[CBSE 2011]** 37. $y = e^x \sin^3 x \cos^4 x$

38. $y = 2^x \cdot e^{3x} \sin 4x$ 39. $y = x^x \cdot e^{(2x+5)}$

40. $y = (2x+3)^5 (3x-5)^7 (5x-1)^3$ 41. $(\cos x)^y = (\cos y)^x$ **[CBSE 2012]**

42. $(\tan x)^y = (\tan y)^x$ 43. $y = (\log x)^x + (x)^{\log x}$ **[CBSE 2004C]**

44. If $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, prove that $(1-x^2)\dfrac{dy}{dx} = (xy+1)$.

45. If $y = \sqrt{x+y}$, prove that $\dfrac{dy}{dx} = \dfrac{1}{(2y-1)}$.

46. If $x^a y^b = (x+y)^{(a+b)}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

47. If $(x^x + y^x) = 1$, show that $\dfrac{dy}{dx} = -\left\{\dfrac{x^x(1+\log x) + y^x(\log y)}{xy^{x-1}}\right\}$

48. If $y = e^{\sin x} + (\tan x)^x$, prove that

$$\frac{dy}{dx} = e^{\sin x} \cos x + (\tan x)^x [2x \operatorname{cosec} 2x + \log \tan x].$$ **[CBSE 2003]**

49. If $y = \log (x + \sqrt{1+x^2})$, prove that

$$\frac{dy}{dx} = \frac{1}{\log (x + \sqrt{1+x^2})} \cdot \frac{1}{\sqrt{1+x^2}}.$$ **[CBSE 2003]**

50. If $y = \log \sin \sqrt{x^2+1}$, prove that $\dfrac{dy}{dx} = \dfrac{x \cot \sqrt{x^2+1}}{\sqrt{x^2+1}}\cdot$ **[CBSE 2003]**

51. If $y = \log \sqrt{\dfrac{1-\cos x}{1+\cos x}}$, show that $\dfrac{dy}{dx} = \operatorname{cosec} x.$ **[CBSE 2003]**

52. If $y = \log \tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$, show that $\dfrac{dy}{dx} = \sec x.$ **[CBSE 2002]**

53. If $y = \sqrt{\dfrac{1-\sin 2x}{1+\sin 2x}}$, show that $\dfrac{dy}{dx} + \sec^2\left(\dfrac{\pi}{4}-x\right) = 0.$ **[CBSE 2002]**

54. If $y = \log \sqrt{\dfrac{1+\cos^2 x}{1-e^{2x}}}$, show that $\dfrac{dy}{dx} = \dfrac{e^{2x}}{(1-e^{2x})} - \dfrac{\sin x \cos x}{(1+\cos^2 x)}\cdot$ **[CBSE 2003C]**

55. If $y = (x)^{\cos x} + (\sin x)^{\tan x}$, prove that

$$\frac{dy}{dx} = x^{\cos x}\left\{\frac{\cos x}{x} - (\sin x)\log x\right\} + (\sin x)^{\tan x}\cdot\{1 + (\log \sin x)\sec^2 x\}.$$

[CBSE 2004C, '09]

56. If $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$, prove that

$$\frac{dy}{dx} = (\sin x)^{\cos x}\cdot[\cot x \cos x - \sin x(\log \sin x)] + (\cos x)^{\sin x}\cdot[\cos x(\log \cos x) - \sin x \tan x].$$

57. If $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$, prove that

$$\frac{dy}{dx} = (\tan x)^{\cot x}.\ \operatorname{cosec}^2 x(1 - \log \tan x) + (\cot x)^{\tan x}\cdot \sec^2 x[\log(\cot x) - 1].$$

58. If $y = x^{\cos x} + (\cos x)^x$, prove that

$$\frac{dy}{dx} = x^{\cos x}\cdot\left\{\frac{\cos x}{x} - (\sin x)\log x\right\} + (\cos x)^x[(\log \cos x) - x\tan x].$$

59. If $y = x^{\log x} + (\log x)^x$, prove that

$$\frac{dy}{dx} = x^{(\log x)}\left\{\frac{2\log x}{x}\right\} + (\log x)^x\cdot\left\{\frac{1}{\log x} + \log(\log x)\right\}\cdot$$

60. If $y = x^{(x^2-3)} + (x-3)^{x^2}$, find $\dfrac{dy}{dx}\cdot$

61. If $f(x) = \left(\dfrac{3+x}{1+x}\right)^{2+3x}$, find $f'(0)$. **[CBSE 2005C]**

62. If $y = (\sin x)^x + \sin^{-1}\sqrt{x}$, find $\dfrac{dy}{dx}\cdot$ **[CBSE 2009]**

63. If $(x^2+y^2)^2 = xy$, find $\dfrac{dy}{dx}\cdot$ **[CBSE 2009]**

64. If $y = x^{\cot x} + \dfrac{2x^2-3}{x^2+x+2}$, find $\dfrac{dy}{dx}\cdot$ **[CBSE 2012C]**

65. If $y = \tan^{-1}\dfrac{a}{x} + \log\sqrt{\dfrac{x-a}{x+a}}$, prove that $\dfrac{dy}{dx} = \dfrac{2a^3}{(x^4-a^4)}\cdot$ **[CBSE 2014C]**

66. If $x^m y^n = (x+y)^{m+n}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x}\cdot$ **[CBSE 2014]**

ANSWERS (EXERCISE 10F)

1. $\dfrac{x^{1/x}(1-\log x)}{x^2}$ **2.** $\dfrac{x^{\sqrt{x}}(2+\log x)}{2\sqrt{x}}$ **3.** $(\log x)^x\left\{\dfrac{1}{\log x}+\log(\log x)\right\}$

4. $x^{\sin x}\left[\dfrac{\sin x}{x}+(\cos x)\log x\right]$ **5.** $x^{\cos^{-1}x}\left[\dfrac{\cos^{-1}x}{x}-\dfrac{\log x}{\sqrt{1-x^2}}\right]$

6. $(\tan x)^{1/x}\left\{\dfrac{2x\operatorname{cosec}2x-\log(\tan x)}{x^2}\right\}$

7. $(\sin x)^{\cos x}\cdot\{\cot x\cos x-\sin x\log(\sin x)\}$

8. $(\log x)^{\sin x}\cdot\left\{\dfrac{\sin x}{x\log x}+\cos x\cdot\log(\log x)\right\}$

9. $(\cos x)^{\log x}\cdot\left\{\dfrac{\log(\cos x)}{x}-(\tan x)(\log x)\right\}$

10. $(\tan x)^{\sin x}\cdot\{\cos x\cdot\log(\tan x)+\sec x\}$

11. $(\cos x)^{\cos x}\cdot\{(-\sin x)(1+\log\cos x)\}$

12. $(\tan x)^{\cot x}\cdot\{\operatorname{cosec}^2x(1-\log\tan x)\}$

13. $x^{\sin 2x}\cdot\left\{\dfrac{\sin 2x}{x}+(2\cos 2x)\log x\right\}$

14. $(\sin^{-1}x)^x\cdot\left\{\log(\sin^{-1}x)+\dfrac{x}{(\sin^{-1}x)\sqrt{1-x^2}}\right\}$ **15.** $[\cos(x^x)]\,x^x(1+\log x)$

16. $(3x+5)^{(2x-3)}\cdot\left\{\dfrac{3(2x-3)}{(3x+5)}+2\log(3x+5)\right\}$

17. $(x+1)^3(x+2)^4(x+3)^5\cdot\left\{\dfrac{3}{(x+1)}+\dfrac{4}{(x+2)}+\dfrac{5}{(x+3)}\right\}$

18. $\dfrac{1}{2}\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\cdot\left[\dfrac{1}{(x-1)}+\dfrac{1}{(x-2)}-\dfrac{1}{x-3}-\dfrac{1}{(x-4)}-\dfrac{1}{(x-5)}\right]$

19. $(2-x)^3(3+2x)^5\cdot\left[\dfrac{10}{(3+2x)}-\dfrac{3}{(2-x)}\right]$

20. $-(\cos x\cos 2x\cos 3x)\cdot[\tan x+2\tan 2x+3\tan 3x]$

21. $\dfrac{x^5\sqrt{x+4}}{(2x+3)^2}\cdot\left\{\dfrac{5}{x}+\dfrac{1}{2(x+4)}-\dfrac{4}{(2x+3)}\right\}$

22. $\dfrac{(x+1)^2\sqrt{x-1}}{(x+4)^3\cdot e^x}\cdot\left\{\dfrac{2}{(x+1)}+\dfrac{1}{2(x-1)}-\dfrac{3}{(x+4)}-1\right\}$

23. $\dfrac{\sqrt{x}(3x+5)^2}{\sqrt{x+1}}\cdot\left\{\dfrac{1}{2x}+\dfrac{6}{(3x+5)}-\dfrac{1}{2(x+1)}\right\}$

24. $\dfrac{x^2\sqrt{1+x}}{(1+x^2)^{3/2}}\cdot\left\{\dfrac{2}{x}+\dfrac{1}{2(1+x)}-\dfrac{3x}{(1+x^2)}\right\}$

25. $\dfrac{1}{2}\sqrt{(x-2)(2x-3)(3x-4)}\cdot\left\{\dfrac{1}{(x-2)}+\dfrac{2}{(2x-3)}+\dfrac{3}{(3x-4)}\right\}$

26. $(\sin 2x \sin 3x \sin 4x)(2\cot 2x + 3\cot 3x + 4\cot 4x)$

27. $\dfrac{x^3 \sin x}{e^x} \cdot \left(\dfrac{3}{x} + \cot x - 1\right)$

28. $\dfrac{e^x \log x}{x^2} \cdot \left\{1 + \dfrac{1}{(x \log x)} - \dfrac{2}{x}\right\}$

29. $\dfrac{x \cos^{-1} x}{\sqrt{1-x^2}} \cdot \left\{\dfrac{1}{x} - \dfrac{1}{(\cos^{-1} x)\sqrt{1-x^2}} + \dfrac{x}{(1-x^2)}\right\}$

30. $(1+x)(1+x^2)(1+x^4)(1+x^6) \cdot \left\{\dfrac{1}{(1+x)} + \dfrac{2x}{(1+x^2)} + \dfrac{4x^3}{(1+x^4)} + \dfrac{6x^5}{(1+x^6)}\right\}$

31. $x^x(1+\log x) + (\log 2) \cdot 2^{\sin x}(\cos x)$

32. $(\log x)^x \cdot \left\{\dfrac{1}{\log x} + \log(\log x)\right\} + e^{(\log x)^2} \cdot \dfrac{2\log x}{x}$

33. $x^{\sin x} \cdot \left\{\dfrac{\sin x}{x} + \cos x \cdot \log x\right\} + (\sin x)^{\cos x} \cdot \{\cos x \cot x - \sin x \log(\sin x)\}$

34. $(x\cos x)^x\{1 - x\tan x + \log x + \log \cos x\}$

$+ (x\sin x)^{1/x} \cdot \left\{\dfrac{1}{x^2} + \dfrac{\cot x}{x} - \dfrac{\log x}{x^2} - \dfrac{\log \sin x}{x^2}\right\}$

35. $(\sin x)^x \cdot \{x\cot x + \log \sin x\} + \dfrac{1}{2\sqrt{x-x^2}}$

36. $x^{x\cos x} \cdot \{\cos x - x \sin x(\log x) + \cos x(\log x)\} - \dfrac{4x}{(x^2-1)^2}$

37. $e^x \sin^3 x \cos^4 x(1 + 3\cot x - 4\tan x)$

38. $2^x \cdot e^{3x} \sin 4x \cdot \{(\log 2) + 3 + 4\cot 4x\}$

39. $x^x \cdot e^{(2x+5)} \cdot (3 + \log x)$

40. $(2x+3)^5(3x-5)^7(5x-1)^3 \cdot \left[\dfrac{10}{(2x+3)} + \dfrac{21}{(3x-5)} + \dfrac{15}{(5x-1)}\right]$

41. $\dfrac{(\log \cos y + y\tan x)}{(\log \cos x + x\tan y)}$

42. $\dfrac{\log \tan y - 2y \operatorname{cosec} 2x}{\log \tan x - 2x \operatorname{cosec} 2y}$

43. $(\log x)^x\left\{\log(\log x) + \dfrac{1}{\log x}\right\} + (x)^{\log x} \cdot \dfrac{2}{x}\log x$

60. $x^{(x^2-3)} \cdot \left\{\dfrac{(x^2-3)}{x} + (2x)\log x\right\} + (x-3)^{x^2} \cdot \left\{\dfrac{x^2}{(x-3)} + 2x\log(x-3)\right\}$

61. $(27\log 3) - 12$

62. $(\sin x)^x[x\cot x + \log(\sin x)] + \dfrac{1}{2\sqrt{x-x^2}}$

63. $\dfrac{y(3x^2-y^2)}{x(x^2-3y^2)}$

64. $x^{\cot x}\left(\dfrac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x\right) + \dfrac{2x^2+14x+3}{(x^2+x+2)^2}$.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10F)

15. Let $x^x = t$. Then,

$$\log t = x\log x \Rightarrow \frac{1}{t}\cdot\frac{dt}{dx} = x\cdot\frac{1}{x} + \log x \cdot 1$$

$$\Rightarrow \frac{dt}{dx} = t[1 + \log x] = x^x(1 + \log x).$$

$\therefore\ y = \sin t \Rightarrow \dfrac{dy}{dt} = \cos t = \cos x^x.$

$\therefore\ \dfrac{dy}{dx} = \dfrac{(dy/dt)}{(dx/dt)} = [\cos(x^x)]\, x^x(1 + \log x).$

18. $\log y = \dfrac{1}{2}\cdot\{\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\}$

$\Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{1}{2}\cdot\left\{\dfrac{1}{(x-1)} + \dfrac{1}{(x-2)} - \dfrac{1}{(x-3)} - \dfrac{1}{(x-4)} - \dfrac{1}{(x-5)}\right\}.$

20. $\log y = \log\cos x + \log\cos 2x + \log\cos 3x$

$\Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{-\sin x}{\cos x} - \dfrac{2\sin 2x}{\cos 2x} - \dfrac{3\sin 3x}{\cos 3x}$

$\Rightarrow \dfrac{dy}{dx} = y\cdot\{-\tan x - 2\tan 2x - 3\tan 3x\}.$

21. $\log y = 5\log x + \dfrac{1}{2}\log(x+4) - 2\log(2x+3)$

$\Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \left\{\dfrac{5}{x} + \dfrac{1}{2(x+4)} - \dfrac{4}{(2x+3)}\right\}.$

22. $\log y = 2\log(x+1) + \dfrac{1}{2}\log(x-1) - 3\log(x+4) - x$

$\Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \left\{\dfrac{2}{(x+1)} + \dfrac{1}{2(x-1)} - \dfrac{3}{(x+4)} - 1\right\}.$

25. $\log y = \dfrac{1}{2}\cdot\{\log(x-2) + \log(2x-3) + \log(3x-4)\}$

$\Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{1}{2}\left\{\dfrac{1}{(x-2)} + \dfrac{2}{(2x-3)} + \dfrac{3}{(3x-4)}\right\}.$

26. $\log y = \log\sin 2x + \log\sin 3x + \log\sin 4x$

$\Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \left\{\dfrac{2\cos 2x}{\sin 2x} + \dfrac{3\cos 3x}{\sin 3x} + \dfrac{4\cos 4x}{\sin 4x}\right\}.$

29. $\log y = \log x + \log\cos^{-1}x - \dfrac{1}{2}\log(1-x^2)$

$\Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \left\{\dfrac{1}{x} - \dfrac{1}{\cos^{-1}x}\cdot\dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{2}\times\dfrac{(-2x)}{(1-x^2)}\right\}$

$\Rightarrow \dfrac{dy}{dx} = y\cdot\left\{\dfrac{1}{x} - \dfrac{1}{(\cos^{-1}x)(\sqrt{1-x^2})} + \dfrac{x}{(1-x^2)}\right\}.$

30. $\log y = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^6)$

$\Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \left\{\dfrac{1}{(1+x)} + \dfrac{2x}{(1+x^2)} + \dfrac{4x^3}{(1+x^4)} + \dfrac{6x^5}{(1+x^6)}\right\}.$

31. Let $y = u - v$, where $u = x^x$ and $v = 2^{\sin x}$.

Then, $\dfrac{dy}{dx} = \dfrac{du}{dx} - \dfrac{dv}{dx}\cdot$

Now, $u = x^x \Rightarrow \log u = x\log x$

$\Rightarrow \dfrac{1}{u}\cdot\dfrac{du}{dx} = \left(x\cdot\dfrac{1}{x} + \log x\cdot 1\right)$

$\Rightarrow \dfrac{du}{dx} = u(1 + \log x) = x^x(1 + \log x).$

And, $v = 2^{\sin x} \Rightarrow \log v = (\sin x)(\log 2)$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = (\log 2)(\cos x)$$

$$\Rightarrow \frac{dv}{dx} = v(\log 2)(\cos x) = 2^{\sin x}(\log 2)(\cos x).$$

32. Let $y = u + v$, where $u = (\log x)^x$ and $v = x^{\log x}$.

Then, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}\cdot$

Now, $u = (\log x)^x \Rightarrow \log u = x \log(\log x)$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x\cdot\frac{1}{\log x}\cdot\frac{1}{x} + \log(\log x)\cdot 1$$

$$\Rightarrow \frac{du}{dx} = u\cdot\left\{\frac{1}{\log x} + \log(\log x)\right\}$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x\cdot\left\{\frac{1}{\log x} + \log(\log x)\right\}.$$

And, $v = x^{\log x} \Rightarrow \log v = (\log x)(\log x) = (\log x)^2$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = \frac{2(\log x)}{x}$$

$$\Rightarrow \frac{dv}{dx} = v\cdot\frac{2(\log x)}{x} = e^{(\log x)^2}\cdot\frac{2(\log x)}{x}.$$

34. Let $y = u + v$, where $u = (x\cos x)^x$ and $v = (x\sin x)^{1/x}$.

Then, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}\cdot$

Now, $u = (x\cos x)^x \Rightarrow \log u = x\log(x\cos x)$

$$\Rightarrow \log u = x\cdot\{\log x + \log\cos x\}$$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x\cdot\left\{\frac{1}{x} - \frac{\sin x}{\cos x}\right\} + \{\log x + \log\cos x\}\cdot 1$$

$$\Rightarrow \frac{du}{dx} = u\{(1 - x\tan x) + (\log x + \log\cos x)\}$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\{(1 - x\tan x + \log x + \log\cos x)\}.$$

And, $v = (x\sin x)^{1/x} \Rightarrow \log v = \frac{1}{x}\log(x\sin x)$

$$\Rightarrow \log v = \frac{1}{x}\cdot\{\log x + \log\sin x\}$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = \frac{1}{x}\cdot\left(\frac{1}{x} + \frac{\cos x}{\sin x}\right) + (\log x + \log\sin x)\left(\frac{-1}{x^2}\right)$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = \left\{\frac{1}{x^2} + \frac{1}{x}\cot x - \frac{\log x}{x^2} - \frac{\log\sin x}{x^2}\right\}$$

$$\Rightarrow \frac{dv}{dx} = v\cdot\left\{\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log x}{x^2} - \frac{\log\sin x}{x^2}\right\}$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{1/x}\cdot\left\{\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log x}{x^2} - \frac{\log\sin x}{x^2}\right\}.$$

35. Let $y = u + v$, where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$.

Now, $u = (\sin x)^x \Rightarrow \log u = x\log\sin x$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x\cdot\frac{\cos x}{\sin x} + \log\sin x\cdot 1$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \cdot \{x \cot x + \log \sin x\}.$$

And, $v = \sin^{-1}\sqrt{x} \Rightarrow \dfrac{dv}{dx} = \dfrac{1}{\sqrt{1-x}} \cdot \dfrac{1}{2} x^{-1/2} = \dfrac{1}{2\sqrt{x - x^2}}$.

$\therefore \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$.

36. Let $u = x^{x \cos x}$ and $v = \left(\dfrac{x^2+1}{x^2-1}\right)$. Then,

$$\log u = (x \cos x) \log x$$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = (x \cos x) \cdot \frac{1}{x} + (\log x)(-x \sin x + \cos x)$$

$$\Rightarrow \frac{du}{dx} = u \cdot \{\cos x - x \sin x (\log x) + \cos x (\log x)\}$$

$$\Rightarrow \frac{du}{dx} = x^{x \cos x} \cdot \{\cos x - x \sin x (\log x) + \cos x (\log x)\}.$$

And, $\dfrac{dv}{dx} = \dfrac{(x^2-1) \cdot 2x - (x^2+1) \cdot 2x}{(x^2-1)^2} = \dfrac{-4x}{(x^2-1)^2}$.

37. $\log y = x + 3 \log \sin x + 4 \log \cos x$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \left\{1 + \frac{3 \cos x}{\sin x} - \frac{4}{\cos x} \cdot \sin x\right\}.$$

38. $\log y = x \log 2 + 3x + \log \sin 4x$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = (\log 2) + 3 + \frac{4 \cos 4x}{\sin 4x}.$$

39. $\log y = x \log x + (2x + 5)$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \left(x \cdot \frac{1}{x} + \log x\right) + 2 = (3 + \log x).$$

40. $\log y = 5 \log (2x + 3) + 7 \log (3x - 5) + 3 \log (5x - 1)$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \left\{\frac{5}{(2x+3)} \cdot 2 + \frac{7}{(3x-5)} \cdot 3 + \frac{3}{(5x-1)} \cdot 5\right\}.$$

41. $(\cos x)^y = (\cos y)^x$

$$\Rightarrow y \log \cos x = x \log \cos y$$

$$\Rightarrow y \cdot \left(\frac{-\sin x}{\cos x}\right) + (\log \cos x) \frac{dy}{dx} = x \cdot \left(\frac{-\sin y}{\cos y}\right) \cdot \frac{dy}{dx} + (\log \cos y) \cdot 1$$

$$\Rightarrow (\log \cos x + x \tan y) \frac{dy}{dx} = (\log \cos y + y \tan x).$$

44. $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}} \Rightarrow y(\sqrt{1-x^2}) = \sin^{-1} x$

$$\Rightarrow y \cdot \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) + \sqrt{1-x^2} \cdot \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{-xy}{\sqrt{1-x^2}} + \sqrt{1-x^2} \cdot \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow -xy + (1-x^2) \frac{dy}{dx} = 1 \Rightarrow (1-x^2) \frac{dy}{dx} = (xy + 1).$$

45. $y^2 = (x + y) \Rightarrow 2y\dfrac{dy}{dx} = 1 + \dfrac{dy}{dx}$

$$\Rightarrow (2y - 1)\frac{dy}{dx} = 1.$$

46. $x^a y^b = (x + y)^{(a+b)} \Rightarrow a \log x + b \log y = (a + b) \log (x + y).$

$$\therefore \frac{a}{x} + \frac{b}{y} \cdot \frac{dy}{dx} = (a + b) \cdot \frac{1}{(x + y)}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \left(\frac{a + b}{x + y} - \frac{b}{y}\right)\frac{dy}{dx} = \left\{\frac{a}{x} - \frac{a + b}{x + y}\right\}.$$

60. Let $u = x^{(x^2 - 3)}$ and $v = (x - 3)^{x^2}$. Then, $y = u + v$.

Now, $u = x^{(x^2 - 3)}$

$$\Rightarrow \log u = (x^2 - 3) \log x$$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = (x^2 - 3) \cdot \frac{1}{x} + (2x) \log x$$

$$\Rightarrow \frac{du}{dx} = u \cdot \left[\frac{(x^2 - 3)}{x} + (2x) \log x\right] = x^{(x^2 - 3)} \cdot \left\{\frac{(x^2 - 3)}{x} + (2x) \log x\right\}.$$

And, $v = (x - 3)^{x^2}$

$$\Rightarrow \log v = x^2 \log (x - 3)$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = x^2 \cdot \frac{1}{(x - 3)} + 2x \log (x - 3)$$

$$\Rightarrow \frac{dv}{dx} = v \cdot \left\{\frac{x^2}{(x - 3)} + 2x \log (x - 3)\right\} = (x - 3)^{x^2}\left\{\frac{x^2}{(x - 3)} + 2x \log (x - 3)\right\}.$$

61. $y = \left(\dfrac{3 + x}{1 + x}\right)^{2+3x} \Rightarrow \log y = (2 + 3x)[\log (3 + x) - \log (1 + x)]$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = (2 + 3x)\left[\frac{1}{3 + x} - \frac{1}{1 + x}\right] + [\log (3 + x) - \log (1 + x)] \times 3.$$

$$\therefore f'(x) = \frac{dy}{dx} = \left(\frac{3 + x}{1 + x}\right)^{2+3x} \cdot \left\{\frac{(2 + 3x)(-2)}{(3 + x)(1 + x)} + 3 \log\left(\frac{3 + x}{1 + x}\right)\right\}.$$

Hence, $f'(0) = 3^2 \cdot \left(\dfrac{-4}{3} + 3 \log 3\right) = -12 + 27 \log 3.$

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7. Derivatives of an Infinite Series

If we take out a single term from an infinite series, it remains unaffected. We utilize this result in finding the derivative of an infinite series.

SOLVED EXAMPLES

EXAMPLE 1 *If* $y = x^{x^{x \ldots \infty}}$, *prove that* $\dfrac{dy}{dx} = \dfrac{y^2}{x(1 - y \log x)}$.

SOLUTION We know that an infinite series is not affected by the exclusion of a single term.

So, we may write the given function as $y = x^y$.

Now, $y = x^y \Rightarrow \log y = y \log x$. ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{(1 - y\log x)}{y}\cdot\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{dx} = \left\{\frac{y}{x}\times\frac{y}{(1 - y\log x)}\right\} \Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}.$$

EXAMPLE 2 *If* $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots \text{to } \infty}}}$, *prove that* $\frac{dy}{dx} = \frac{\cos x}{(2y - 1)}$.

SOLUTION We may write the given series as

$$y = \sqrt{\sin x + y} \Rightarrow y^2 = (\sin x + y). \quad \ldots \text{(i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$2y\cdot\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow (2y - 1)\cdot\frac{dy}{dx} = \cos x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(2y - 1)}.$$

EXAMPLE 3 *If* $y = e^{x + e^{x + e^{x + \ldots \text{ to } \infty}}}$, *prove that* $\frac{dy}{dx} = \frac{y}{(1 - y)}$.

SOLUTION We may write the given series as

$$y = e^{x + y} \Rightarrow \log y = (x + y). \quad \ldots \text{(i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y} - 1\right)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{(1 - y)}{y}\cdot\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{(1 - y)}.$$

EXAMPLE 4 *If* $y = (\sqrt{x})^{(\sqrt{x})^{(\sqrt{x})\ldots\infty}}$, *prove that* $x\left(\frac{dy}{dx}\right) = \frac{y^2}{(2 - y\log x)}$.

SOLUTION We may write the given series as

$$y = (\sqrt{x})^y \Rightarrow y = x^{y/2}$$

$$\Rightarrow \log y = \frac{y}{2} \cdot \log x. \qquad \text{... (i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{y}{2} \cdot \frac{1}{x} + \frac{1}{2} \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y} - \frac{1}{2} \log x\right) \cdot \frac{dy}{dx} = \frac{y}{2x}$$

$$\Rightarrow \frac{(2 - y \log x)}{2y} \cdot \frac{dy}{dx} = \frac{y}{2x}$$

$$\Rightarrow x \cdot \frac{dy}{dx} = \frac{y^2}{(2 - y \log x)}.$$

EXERCISE 10G

1. If $y = (\sin x)^{(\sin x)^{(\sin x) \ldots \infty}}$, prove that $\dfrac{dy}{dx} = \dfrac{y^2 \cot x}{(1 - y \log \sin x)}$.

2. If $y = (\cos x)^{(\cos x)^{(\cos x) \ldots \infty}}$, prove that $\dfrac{dy}{dx} = \dfrac{-y^2 \tan x}{(1 - y \log \cos x)}$.

3. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots \infty}}}$, prove that $\dfrac{dy}{dx} = \dfrac{1}{(2y - 1)}$.

4. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \infty}}}$, prove that $\dfrac{dy}{dx} = \dfrac{\sin x}{(1 - 2y)}$.

5. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots \infty}}}$, prove that $\dfrac{dy}{dx} = \dfrac{\sec^2 x}{(2y - 1)}$.

6. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots \infty}}}$, show that $(2y - 1) \cdot \dfrac{dy}{dx} = \dfrac{1}{x}$.

7. If $y = a^{x^{a^{x \ldots \infty}}}$, prove that $\dfrac{dy}{dx} = \dfrac{y^2 (\log y)}{x [1 - y (\log x)(\log y)]}$.

8. If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots \infty}}}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{(2y - x)}$.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10G)

1. $y = (\sin x)^y$. 3. $y = \sqrt{x + y} \Rightarrow y^2 = x + y$.

7. $y = a^{(x^y)} \Rightarrow \log y = x^y (\log a) \Rightarrow \log (\log y) = (y \log x) + \log (\log a)$.

8. $y = x + \dfrac{1}{y}$.

8. Derivative of One Function With Respect To Another Function

Let $f(x)$ and $g(x)$ be two functions of x. In order to find the derivative of $f(x)$ with respect to $g(x)$, we put $u = f(x)$ and $v = g(x)$. Now, find $\frac{du}{dv} = \frac{(du/dx)}{(dv/dx)}$, which is the required derivative.

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate* e^x *w.r.t.* $\sqrt{x}$.

SOLUTION Let $u = e^x$ and $v = \sqrt{x}$.

Then, $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

$$\therefore \quad \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{e^x}{(1/2\sqrt{x})} = 2e^x\sqrt{x}.$$

EXAMPLE 2 *Differentiate* $\sin^2 x$ *w.r.t.* $e^{\cos x}$.

SOLUTION Let $u = \sin^2 x$ and $v = e^{\cos x}$. Then,

$$\frac{du}{dx} = 2\sin x\cos x$$

and $\frac{dv}{dx} = e^{\cos x}\cdot(-\sin x) = (-\sin x)\cdot e^{\cos x}$.

$$\therefore \quad \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{2\sin x\cos x}{(-\sin x)e^{\cos x}} = \frac{-2\cos x}{e^{\cos x}}.$$

EXAMPLE 3 *Differentiate* $\sin^{-1}x$ *w.r.t.* $\tan^{-1}x$.

SOLUTION Let $u = \sin^{-1}x$ and $v = \tan^{-1}x$.

Then, $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ and $\frac{dv}{dx} = \frac{1}{(1+x^2)}$.

$$\therefore \quad \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{1}{\sqrt{1-x^2}}\times(1+x^2) = \frac{(1+x^2)}{\sqrt{1-x^2}}.$$

EXAMPLE 4 *Differentiate* $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *w.r.t.* $\tan^{-1}x$.

SOLUTION Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \tan^{-1}x$

Now, $v = \tan^{-1}x \Rightarrow x = \tan v$.

Putting $x = \tan v$, we get

$$u = \tan^{-1}\left\{\frac{\sqrt{1+\tan^2 v}-1}{\tan v}\right\} = \tan^{-1}\left(\frac{\sec v - 1}{\tan v}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos v}{\sin v}\right) = \tan^{-1}\left\{\frac{2\sin^2(v/2)}{2\sin(v/2)\cos(v/2)}\right\}$$

$$= \tan^{-1}\left\{\tan\frac{v}{2}\right\} = \frac{v}{2}.$$

$$\therefore \quad u = \frac{v}{2} \Rightarrow \frac{du}{dv} = \frac{1}{2}.$$

EXAMPLE 5 *Differentiate* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *w.r.t.* $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

SOLUTION Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

Putting $x = \tan\theta$, we get

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta,$$

$$v = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta.$$

$$\therefore \quad u = v \Rightarrow \frac{du}{dv} = 1.$$

EXAMPLE 6 *Differentiate* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *w.r.t.* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$. **[CBSE 2004C]**

SOLUTION Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Putting $x = \tan\theta$, we get

$$u = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta,$$

$$v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta.$$

$$\therefore \quad u = v \Rightarrow \frac{du}{dv} = 1.$$

EXAMPLE 7 *Differentiate* $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ *w.r.t.* $\cos^{-1}x^2$.

SOLUTION Let $u = \tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ and $v = \cos^{-1}x^2$.

Then, $\cos^{-1}x^2 = v \Rightarrow x^2 = \cos v$.

Putting $x^2 = \cos v$, we get

$$u = \tan^{-1}\left\{\frac{\sqrt{1+\cos v}-\sqrt{1-\cos v}}{\sqrt{1+\cos v}+\sqrt{1-\cos v}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2(v/2)}-\sqrt{2\sin^2(v/2)}}{\sqrt{2\cos^2(v/2)}+\sqrt{2\sin^2(v/2)}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos(v/2) - \sin(v/2)}{\cos(v/2) + \sin(v/2)}\right\} = \tan^{-1}\left\{\frac{1 - \tan(v/2)}{1 + \tan(v/2)}\right\}$$

[dividing num. and denom. by $\cos(v/2)$]

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - \frac{v}{2}\right)\right\} = \left(\frac{\pi}{4} - \frac{v}{2}\right).$$

$$\therefore \quad u = \left(\frac{\pi}{4} - \frac{v}{2}\right) \Rightarrow \frac{du}{dv} = \frac{-1}{2}.$$

EXERCISE 10H

1. Differentiate x^6 with respect to $(1/\sqrt{x})$.
2. Differentiate $\log x$ with respect to $\cot x$.
3. Differentiate $e^{\sin x}$ with respect to $\cos x$.
4. Differentiate $\tan^{-1}\sqrt{\dfrac{1-x^2}{1+x^2}}$ with respect to $\cos^{-1}x^2$.
5. Differentiate $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$.
6. Differentiate $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ with respect to $\cos^{-1}(2x^2-1)$.
7. Differentiate $\sin^3 x$ with respect to $\cos^3 x$.
8. Differentiate $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ with respect to $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$.
9. Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$.
10. Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$ w.r.t. $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$. **[CBSE 2014]**

ANSWERS (EXERCISE 10H)

1. $-12x^{13/2}$ **2.** $\dfrac{-\sin^{-1}x}{x}$ **3.** $-e^{\sin x}\cot x$ **4.** $\dfrac{1}{2}$ **5.** 1
6. $-\dfrac{1}{2}$ **7.** $-\tan x$ **8.** $\dfrac{2}{3}$ **9.** $\dfrac{1}{4}$ **10.** $-\dfrac{1}{2}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10H)

4. Put $x^2 = \cos v$. 8. Put $x = \tan\theta$. 10. Put $x = \cos\theta$.

9. Derivatives of Parametric Functions

Sometimes x and y are given as functions of a variable t. Then, t is called a *parameter*.

Let $x = f(t)$ and $y = g(t)$. Then,

$$\frac{dx}{dt} = f'(t) \text{ and } \frac{dy}{dt} = g'(t).$$

$$\therefore \quad \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{g'(t)}{f'(t)}, \text{ where } f'(t) \neq 0.$$

SUMMARY

Let $x = f(t)$ and $y = g(t)$. Then,

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{g'(t)}{f'(t)}, \text{ where } f'(t) \neq 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find* $\frac{dy}{dx}$, *when* $x = a(t + \sin t)$ *and* $y = a(1 - \cos t)$.

SOLUTION We have:

$$x = a(t + \sin t) \Rightarrow \frac{dx}{dt} = a(1 + \cos t);$$

$$y = a(1 - \cos t) \Rightarrow \frac{dy}{dt} = a \sin t.$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{a \sin t}{a(1 + \cos t)} = \frac{2a \sin(t/2) \cos(t/2)}{2a \cos^2(t/2)} = \tan\frac{t}{2}\cdot$$

EXAMPLE 2 *If* $x = a[\cos\theta + \log \tan(\theta/2)]$ *and* $y = a \sin\theta$, *find* $\frac{dy}{dx}$ *at* $\theta = (\pi/4)$.

[CBSE 2008]

SOLUTION We have

$$x = a\{\cos\theta + \log \tan(\theta/2)\}$$

$$\Rightarrow \frac{dx}{d\theta} = a\left\{-\sin\theta + \frac{\sec^2(\theta/2)}{2\tan(\theta/2)}\right\} = a\left\{-\sin\theta + \frac{1}{2\sin(\theta/2)\cos(\theta/2)}\right\}$$

$$= a\left\{-\sin\theta + \frac{1}{\sin\theta}\right\} = \frac{a(1 - \sin^2\theta)}{\sin\theta} = \frac{a\cos^2\theta}{\sin\theta}\cdot$$

And, $y = a \sin\theta \Rightarrow \frac{dy}{d\theta} = a\cos\theta.$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{d\theta} \times \frac{d\theta}{dx}\right) = \left(a\cos\theta \cdot \frac{\sin\theta}{a\cos^2\theta}\right) = \tan\theta.$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{\theta = \pi/4} = \tan\frac{\pi}{4} = 1.$$

EXAMPLE 3 *If* $x = a\sin 2t(1+\cos 2t)$ *and* $y = b\cos 2t(1-\cos 2t)$, *show that* $\left(\frac{dy}{dx}\right)_{at\ t=\frac{\pi}{4}} = \frac{b}{a}\cdot$ **[CBSE 2006]**

SOLUTION We have

$$x = a\sin 2t(1+\cos 2t)$$

$$\Rightarrow \frac{dx}{dt} = a\cdot[\sin 2t(-2\sin 2t) + (1+\cos 2t)(2\cos 2t)]$$

$$= (2a)\cdot[-\sin^2 2t + \cos 2t + \cos^2 2t]$$

$$= (2a)[\cos 4t + \cos 2t] \qquad \{\because\ (\cos^2 2t - \sin^2 2t) = \cos 4t\}.$$

And, $y = b\cos 2t(1-\cos 2t)$

$$\Rightarrow \frac{dy}{dt} = b[\cos 2t(2\sin 2t) + (1-\cos 2t)(-2\sin 2t)]$$

$$= (2b)[\sin 2t\cos 2t - \sin 2t + \sin 2t\cos 2t]$$

$$= (2b)[2\sin 2t\cos 2t - \sin 2t] = (2b)[\sin 4t - \sin 2t].$$

$$\therefore\ \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{(2b)(\sin 4t - \sin 2t)}{(2a)(\cos 4t + \cos 2t)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(t=\frac{\pi}{4}\right)} = \frac{b}{a}\cdot\frac{\left\{\sin\left(4\times\frac{\pi}{4}\right) - \sin\left(2\times\frac{\pi}{4}\right)\right\}}{\left\{\cos\left(4\times\frac{\pi}{4}\right) + \cos\left(2\times\frac{\pi}{4}\right)\right\}}$$

$$= \frac{b}{a}\cdot\frac{\left(\sin\pi - \sin\frac{\pi}{2}\right)}{\left(\cos\pi + \cos\frac{\pi}{2}\right)} = \frac{b}{a}\cdot\frac{(0-1)}{(-1+0)} = \frac{b}{a}\cdot$$

EXAMPLE 4 *If* $x = a\left(\frac{1+t^2}{1-t^2}\right)$ *and* $y = \frac{2t}{(1-t^2)}$, *find* $\frac{dy}{dx}\cdot$ **[CBSE 2005]**

SOLUTION We have

$$x = a\left[-1 + \frac{2}{(1-t^2)}\right] = a[-1 + 2(1-t^2)^{-1}]$$

$$\Rightarrow \frac{dx}{dt} = a[0 + 2(-1)(1-t^2)^{-2}(-2t)] = a\times\frac{4t}{(1-t^2)^2} = \frac{4at}{(1-t^2)^2}\cdot$$

And, $y = \frac{2t}{(1-t^2)}$

$$\Rightarrow \frac{dy}{dt} = \frac{(1-t^2)\cdot 2 - 2t(-2t)}{(1-t^2)^2} = \frac{2(1+t^2)}{(1-t^2)^2}\cdot$$

$$\therefore\ \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \left\{\frac{2(1+t^2)}{(1-t^2)^2}\times\frac{(1-t^2)^2}{4at}\right\} = \frac{(1+t^2)}{2at}\cdot$$

EXAMPLE 5 *If* $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, *find* $\frac{d^2y}{dx^2}$ *at* $t = \frac{\pi}{3}\cdot$

[CBSE 2005C]

SOLUTION We have

$$x = 3\sin t - \sin 3t$$

$$\Rightarrow \frac{dx}{dt} = 3\cos t - 3\cos 3t. \quad \text{... (i)}$$

And, $y = 3\cos t - \cos 3t$

$$\Rightarrow \frac{dy}{dt} = -3\sin t + 3\sin 3t. \quad \text{... (ii)}$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{-3\sin t + 3\sin 3t}{3\cos t - 3\cos 3t} = \frac{\sin 3t - \sin t}{\cos t - \cos 3t}$$

$$= \frac{2\cos 2t \sin t}{2\sin 2t \sin t} = \cot 2t.$$

$$\therefore \frac{d^2y}{dx^2} = -2\text{cosec}^2 2t \cdot \frac{dt}{dx} = \frac{-2\text{cosec}^2 2t}{(dx/dt)} = \frac{-2\text{cosec}^2 2t}{3(\cos t - \cos 3t)}\cdot$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{\left(t=\frac{\pi}{3}\right)} = \frac{-2\text{cosec}^2(2\pi/3)}{3\cos\frac{\pi}{3} - \cos\pi} = -2\times\left(\frac{2}{\sqrt{3}}\right)^2\cdot\frac{1}{3\left(\frac{1}{2}+1\right)}$$

$$= \left(-2\times\frac{4}{3}\times\frac{2}{9}\right) = \frac{-16}{27}\cdot$$

EXAMPLE 6 *If* $x = \sqrt{a^{\sin^{-1}t}}$ *and* $y = \sqrt{a^{\cos^{-1}t}}$, *show that* $\frac{dy}{dx} = \frac{-y}{x}\cdot$

SOLUTION We have

$$x^2 = a^{\sin^{-1}t} \text{ and } y^2 = a^{\cos^{-1}t}$$

$$\Rightarrow 2x\frac{dx}{dt} = a^{\sin^{-1}t}\cdot\frac{1}{\sqrt{1-t^2}} \text{ and } 2y\frac{dy}{dt} = a^{\cos^{-1}t}\cdot\frac{(-1)}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{2y}{2x}\cdot\frac{(dy/dt)}{(dx/dt)} = \frac{-a^{\cos^{-1}t}}{\sqrt{1-t^2}}\times\frac{\sqrt{1-t^2}}{a^{\sin^{-1}t}}$$

$$\Rightarrow \frac{y}{x}\cdot\frac{dy}{dx} = -\frac{a^{\cos^{-1}t}}{a^{\sin^{-1}t}} = -\frac{y^2}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x} \quad \text{[on dividing both sides by } \frac{y}{x}\text{]}.$$

Hence, $\frac{dy}{dx} = \frac{-y}{x}\cdot$

EXAMPLE 7 *If* $a > 0$, $x = \left(t+\frac{1}{t}\right)^a$ *and* $y = a^{\left(t+\frac{1}{t}\right)}$, *find* $\frac{dy}{dx}\cdot$

SOLUTION We have

$$x = \left(t+\frac{1}{t}\right)^a \Rightarrow \frac{dx}{dt} = a\left(t+\frac{1}{t}\right)^{(a-1)}\cdot\frac{d}{dt}\left(t+\frac{1}{t}\right) = a\left(t+\frac{1}{t}\right)^{(a-1)}\cdot\left(1-\frac{1}{t^2}\right)\cdot$$

$$\text{And, } y = a^{\left(t+\frac{1}{t}\right)} \Rightarrow \frac{dy}{dt} = a^{\left(t+\frac{1}{t}\right)} \log a \cdot \frac{d}{dt}\left(t + \frac{1}{t}\right)$$

$$= a^{\left(t+\frac{1}{t}\right)} \log a \cdot \left(1 - \frac{1}{t^2}\right).$$

$$\therefore \quad \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{a^{\left(t+\frac{1}{t}\right)} \log a \cdot \left(1 - \frac{1}{t^2}\right)}{a\left(t + \frac{1}{t}\right)^{(a-1)}\left(1 - \frac{1}{t^2}\right)} = \frac{a^{\left(t+\frac{1}{t}-1\right)} \cdot \log a}{\left(t + \frac{1}{t}\right)^{(a-1)}}.$$

EXERCISE 10I

Find $\frac{dy}{dx}$***, when***

1. $x = at^2,\ \ y = 2at$
2. $x = a\cos\theta,\ \ y = b\sin\theta$
3. $x = a\cos^2\theta,\ y = b\sin^2\theta$
4. $x = a\cos^3\theta,\ y = a\sin^3\theta$
5. $x = a(1 - \cos\theta),\ y = a(\theta + \sin\theta)$
6. $x = a\log t,\ \ y = b\sin t$
7. $x = (\log t + \cos t),\ \ y = (e^t + \sin t)$
8. $x = \cos\theta + \cos 2\theta,\ \ y = \sin\theta + \sin 2\theta$
9. $x = \sqrt{\sin 2\theta},\ \ y = \sqrt{\cos 2\theta}$
10. $x = e^{\theta}(\sin\theta + \cos\theta),\ \ y = e^{\theta}(\sin\theta - \cos\theta)$
11. $x = a(\cos\theta + \theta\sin\theta),\ \ y = a(\sin\theta - \theta\cos\theta)$ **[CBSE 2003C]**
12. $x = \frac{3at}{(1+t^2)},\ \ y = \frac{3at^2}{(1+t^2)}$
13. $x = \frac{1-t^2}{1+t^2},\ \ y = \frac{2t}{1+t^2}$
14. $x = \cos^{-1}\frac{1}{\sqrt{1+t^2}},\ \ y = \sin^{-1}\frac{t}{\sqrt{1+t^2}}$
15. If $x = 3\cos t - 2\cos^3 t,\ \ y = 3\sin t - 2\sin^3 t,$ show that $\frac{dy}{dx} = \cot t.$
16. If $x = \frac{1+\log t}{t^2}$ and $y = \frac{3+2\log t}{t},$ show that $\frac{dy}{dx} = t.$
17. If $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta),$ find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}\cdot$ **[CBSE 2003C]**
18. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta,$ show that $\frac{dy}{dx} = \tan\frac{3\theta}{2}\cdot$ **[CBSE 2013C]**
19. If $x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ \ y = \frac{\cos^3 t}{\sqrt{\cos 2t}},$ find $\frac{dy}{dx}\cdot$
20. If $x = (2\cos\theta - \cos 2\theta)$ and $y = (2\sin\theta - \sin 2\theta),$ find $\left(\frac{d^2y}{dx^2}\right)_{\theta = \frac{\pi}{2}}\cdot$ **[CBSE 2005C]**
21. If $x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta),$ find $\frac{d^2y}{dx^2}\cdot$ **[CBSE 20011]**

ANSWERS (EXERCISE 10I)

1. $\frac{1}{t}$ **2.** $\frac{-b\cot\theta}{a}$ **3.** $\frac{-b}{a}$ **4.** $-\tan\theta$

5. $\cot\frac{\theta}{2}$ **6.** $\frac{bt\cos t}{a}$ **7.** $\frac{t(e^t+\cos t)}{(1-t\sin t)}$ **8.** $-\left(\frac{\cos\theta+2\cos 2\theta}{\sin\theta+2\sin 2\theta}\right)$

9. $-(\tan 2\theta)^{3/2}$ **10.** $\tan\theta$ **11.** $\tan\theta$ **12.** $\frac{2t}{(1-t^2)}$

13. $\frac{(t^2-1)}{2t}$ **14.** 1 **17.** 1 **19.** $-\cot 3t$ **20.** $\frac{-3}{2}$ **21.** $\frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10I)

14. Put $t=\tan\theta$.

19. $\log x = 3\log\sin t - \frac{1}{2}\log\cos 2t$ and $\log y = 3\log\cos t - \frac{1}{2}\log\cos 2t$

$$\Rightarrow \frac{1}{x}\cdot\frac{dx}{dt} = \frac{3}{\sin t}\cdot\cos t - \frac{1}{2}\cdot\frac{1}{\cos 2t}(-2\sin 2t),\ \frac{dy}{dt} = y\cdot(-3\tan t+\tan 2t)$$

$$\Rightarrow \frac{dx}{dt} = x\cdot[3\cot t+\tan 2t],\ \frac{dy}{dt} = y\cdot[-3\tan t+\tan 2t]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{y}{x}\cdot\frac{(-3\tan t+\tan 2t)}{(3\cot t+\tan 2t)} = \frac{\cot^3 t\cdot\left(-3\tan t+\frac{2\tan t}{1-\tan^2 t}\right)}{\left(\frac{3}{\tan t}+\frac{2\tan t}{1-\tan^2 t}\right)}$$

$$= \frac{-(1-3\tan^2 t)}{(3\tan t-\tan^3 t)} = \frac{-1}{\tan 3t} = -\cot 3t.$$

20. $\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \tan\frac{3\theta}{2}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{3}{2}\sec^2\frac{3\theta}{2}\cdot\frac{d\theta}{dx} = \frac{3}{2}\sec^2\frac{3\theta}{2}\cdot\frac{1}{2(\sin 2\theta-\sin\theta)}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}} = \frac{3}{4}\sec^2\frac{3\pi}{4}\cdot\frac{1}{\left(\sin\pi-\sin\frac{\pi}{2}\right)} = \frac{3}{4}\times 2\times\frac{1}{(0-1)} = \frac{-3}{2}.$$

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10. Second-Order Derivatives

Let $y=f(x)$ be a differentiable function of x whose second-order derivative exists. We denote the second-order derivative of y w.r.t. x by $\frac{d^2y}{dx^2}$ or y_2.

SOLVED EXAMPLES

EXAMPLE 1 *Find the second-order derivative of:*

(i) x^{10} *(ii)* $\log x$ *(iii)* $\tan^{-1} x$

SOLUTION (i) Let $y = x^{10}$. Then,

$$\frac{dy}{dx} = 10x^9.$$

$$\therefore \frac{d^2y}{dx^2} = (10 \times 9)x^8 = 90x^8.$$

Hence, $\frac{d^2y}{dx^2}(x^{10}) = 90x^8$.

(ii) Let $y = \log x$. Then,

$$\frac{dy}{dx} = \frac{1}{x} = x^{-1}.$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(x^{-1}) = (-1)x^{(-1-1)} = -x^{-2} = \frac{-1}{x^2}\cdot$$

Hence, $\frac{d^2}{dx^2}(\log x) = \frac{-1}{x^2}\cdot$

(iii) Let $y = \tan^{-1} x$. Then,

$$\frac{dy}{dx} = \frac{1}{(1+x^2)} = (1+x^2)^{-1}.$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(1+x^2)^{-1} = (-1)(1+x^2)^{-2}\cdot(2x) = \frac{-2x}{(1+x^2)^2}\cdot$$

Hence, $\frac{d^2}{dx^2}\{\tan^{-1} x\} = \frac{-2x}{(1+x^2)^2}\cdot$

EXAMPLE 2 *If* $y = (\tan x + \sec x)$, *prove that* $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}\cdot$

SOLUTION Given that $y = (\tan x + \sec x)$.

$$\therefore \frac{dy}{dx} = \sec^2 x + \sec x \tan x = \left(\frac{1}{\cos^2 x} + \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\right)$$

$$= \left(\frac{1+\sin x}{\cos^2 x}\right) = \left(\frac{1+\sin x}{1-\sin^2 x}\right) = \frac{1}{(1-\sin x)}\cdot$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{1}{(1-\sin x)}\right\} = \frac{d}{dx}(1-\sin x)^{-1}$$

$$= (-1)(1-\sin x)^{-2}(-\cos x) = \frac{\cos x}{(1-\sin x)^2}\cdot$$

Hence, $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}\cdot$

EXAMPLE 3 *If* $y = e^{4x} \sin 3x$, *find* $\dfrac{d^2y}{dx^2}$.

SOLUTION Let $y = e^{4x} \sin 3x$. Then,

$$\frac{dy}{dx} = 3e^{4x} \cos 3x + 4e^{4x} \sin 3x = e^{4x}(3\cos 3x + 4\sin 3x).$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\{e^{4x}(3\cos 3x + 4\sin 3x)\}$$

$$= e^{4x}(-9\sin 3x + 12\cos 3x) + 4e^{4x}(3\cos 3x + 4\sin 3x)$$

$$= e^{4x}(7\sin 3x + 24\cos 3x).$$

EXAMPLE 4 *If* $y = (x^4 + \cot x)$, *find* $\dfrac{d^2y}{dx^2}$.

SOLUTION We have

$$y = (x^4 + \cot x)$$

$$\Rightarrow \frac{dy}{dx} = 4x^3 - \operatorname{cosec}^2 x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(4x^3 - \operatorname{cosec}^2 x)$$

$$= 4 \cdot \frac{d}{dx}(x^3) - \frac{d}{dx}(\operatorname{cosec}^2 x)$$

$$= (4 \times 3x^2) - 2\operatorname{cosec} x(-\operatorname{cosec} x \cot x)$$

$$= (12x^2 + 2\operatorname{cosec}^2 x \cot x).$$

EXAMPLE 5 *Find the second derivative of* $\log(\log x)$ *w.r.t.* x.

SOLUTION Let $y = \log(\log x)$. Then,

$$\frac{dy}{dx} = \frac{1}{(\log x)} \cdot \frac{1}{x} = \frac{1}{(x \log x)} = (x \log x)^{-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(x \log x)^{-1}$$

$$= (-1)(x \log x)^{-2} \cdot \frac{d}{dx}(x \log x)$$

$$= \frac{-1}{(x \log x)^2} \cdot \left(x \cdot \frac{1}{x} + \log x \cdot 1\right) = \frac{-(1 + \log x)}{(x \log x)^2}.$$

EXAMPLE 6 *If* $y = \sin(\log x)$, *find* $\dfrac{d^2y}{dx^2}$.

SOLUTION We have

$$y = \sin(\log x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\{\sin(\log x)\} = \cos(\log x) \cdot \frac{1}{x} = \frac{\cos(\log x)}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{\cos(\log x)}{x}\right\}$$

$$= \frac{x \cdot \frac{d}{dx}\{\cos(\log x)\} - \cos(\log x) \cdot \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\left\{-\sin(\log x) \cdot \frac{1}{x}\right\} - \cos(\log x) \cdot 1}{x^2}$$

$$= \frac{-\{\sin(\log x) + \cos(\log x)\}}{x^2}.$$

EXAMPLE 7 *If* $e^y(x+1) = 1$, *prove that* $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

SOLUTION We have

$$e^y(x+1) = 1 \Rightarrow e^y = \frac{1}{(x+1)} \quad \text{... (i)}$$

$$\Rightarrow y = \log\left\{\frac{1}{(x+1)}\right\} = \log 1 - \log(x+1)$$

$$\Rightarrow y = -\log(x+1). \quad \text{... (ii)}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{(x+1)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} = \left(\frac{dy}{dx}\right)^2.$$

Hence, $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

EXAMPLE 8 *If* $y = A\cos nx + B\sin nx$, *prove that* $\frac{d^2y}{dx^2} + n^2y = 0$.

SOLUTION We have

$$y = A\cos nx + B\sin nx$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(A\cos nx) + \frac{d}{dx}(B\sin nx)$$

$$= -An\sin nx + Bn\cos nx$$

$$= n(B\cos nx - A\sin nx)$$

$$\Rightarrow \frac{d^2y}{dx^2} = n \cdot \frac{d}{dx}(B\cos nx - A\sin nx)$$

$$= n \cdot \left\{B \cdot \frac{d}{dx}(\cos nx) - A \cdot \frac{d}{dx}(\sin nx)\right\}$$

$$= n \cdot \{-Bn\sin nx - An\cos nx\}$$

$$= -n^2(A\cos nx + B\sin nx) = -n^2y$$

$$\Rightarrow \frac{d^2y}{dx^2} + n^2y = 0.$$

EXAMPLE 9 *If* $y = e^x(\sin x + \cos x)$, *prove that* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$. **[CBSE 2002, '09]**

SOLUTION We have

$$y = e^x(\sin x + \cos x)$$

$$\Rightarrow \frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\sin x + \cos x) + (\sin x + \cos x) \cdot \frac{d}{dx}(e^x)$$

$$= e^x(\cos x - \sin x) + (\sin x + \cos x) \cdot e^x = 2e^x \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \cdot \frac{d}{dx}(e^x \cos x)$$

$$= 2 \cdot \left\{ e^x \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(e^x) \right\}$$

$$= 2 \cdot \{e^x(-\sin x) + (\cos x)e^x\} = 2e^x(\cos x - \sin x).$$

$$\therefore \left(\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y \right)$$

$$= 2e^x(\cos x - \sin x) - 4e^x \cos x + 2e^x(\sin x + \cos x) = 0.$$

Hence, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

EXAMPLE 10 *If* $y = 3e^{2x} + 2e^{3x}$, *prove that* $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

SOLUTION We have

$$y = 3e^{2x} + 2e^{3x} \quad \text{... (i)}$$

$$\Rightarrow \frac{dy}{dx} = 3 \cdot \frac{d}{dx}(e^{2x}) + 2 \cdot \frac{d}{dx}(e^{3x}) \text{ [on differentiating (i) w.r.t. } x\text{]}$$

$$= (3 \times 2e^{2x}) + (2 \times 3e^{3x}) = (6e^{2x} + 6e^{3x})$$

$$\Rightarrow \frac{dy}{dx} = 6(e^{2x} + e^{3x}). \quad \text{... (ii)}$$

On differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 6 \cdot \left\{ \frac{d}{dx}(e^{2x}) + \frac{d}{dx}(e^{3x}) \right\}$$

$$= 6 \cdot (2e^{2x} + 3e^{3x}).$$

$$\therefore \left(\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y \right)$$

$$= 6(2e^{2x} + 3e^{3x}) - 30(e^{2x} + e^{3x}) + (18e^{2x} + 12e^{3x})$$

$$= (12 - 30 + 18)e^{2x} + (18 - 30 + 12)e^{3x} = 0.$$

Hence, $\left(\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y \right) = 0$.

EXAMPLE 11 *If* $y = \sin^{-1} x$, *prove that* $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0.$ **[CBSE 2012]**

SOLUTION Given: $y = \sin^{-1}x.$ ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \; y_1^2 = \frac{1}{(1-x^2)}$$

$$\Rightarrow \; (1-x^2)y_1^2 = 1. \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$(1-x^2)\cdot 2y_1y_2 + y_1^2(-2x) = 0$$

$$\Rightarrow \; (1-x^2)y_2 - xy_1 = 0.$$

Hence, $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0.$

EXAMPLE 12 *If* $y = (\tan^{-1}x)^2$, *prove that* $(1+x^2)^2y_2 + 2x(1+x^2)y_1 = 2.$ **[CBSE 2012]**

SOLUTION Given: $y = (\tan^{-1}x)^2.$ ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$y_1 = 2\tan^{-1}x \cdot \frac{1}{(1+x^2)}$$

$$\Rightarrow \; (1+x^2)y_1 = 2\tan^{-1}x$$

$$\Rightarrow \; (1+x^2)^2y_1^2 = 4(\tan^{-1}x)^2 \qquad \text{[on squaring both sides]}$$

$$\Rightarrow \; (1+x^2)^2y_1^2 - 4y = 0. \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$(1+x^2)^2\cdot 2y_1y_2 + y_1^2\cdot 2(1+x^2)\cdot 2x - 4y_1 = 0$$

$$\Rightarrow \; (1+x^2)^2y_2 + 2x(1+x^2)y_1 - 2 = 0.$$

Hence, $(1+x^2)^2y_2 + 2x(1+x^2)y_1 = 2.$

EXAMPLE 13 *If* $x = a(\theta + \sin\theta)$ *and* $y = a(1-\cos\theta)$, *find* $\dfrac{d^2y}{dx^2}$ *at* $\theta = \dfrac{\pi}{2}\cdot$

[CBSE 2005, '09]

SOLUTION We have

$$x = a(\theta + \sin\theta) \text{ and } y = a(1-\cos\theta)$$

$$\Rightarrow \; \frac{dx}{d\theta} = a(1+\cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta$$

$$\Rightarrow \; \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)}$$

$$= \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\sin\theta}{(1+\cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = \tan\frac{\theta}{2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{d\theta}{dx} = \left(\frac{1}{2}\sec^2\frac{\theta}{2}\right)\times\frac{1}{a(1+\cos\theta)}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}} = \frac{1}{2}\sec^2\frac{\pi}{4}\cdot\frac{1}{a\left(1+\cos\frac{\pi}{2}\right)} = \frac{1}{a}\cdot$$

EXAMPLE 14 *If* $x = (2\cos\theta - \cos 2\theta)$ *and* $y = (2\sin\theta - \sin 2\theta)$, *find* $\frac{d^2y}{dx^2}$ *at* $\theta = \frac{\pi}{2}\cdot$

[CBSE 2005C]

SOLUTION We have

$$x = (2\cos\theta - \cos 2\theta) \text{ and } y = (2\sin\theta - \sin 2\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = (-2\sin\theta + 2\sin 2\theta) \text{ and } \frac{dy}{d\theta} = (2\cos\theta - 2\cos 2\theta)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{(2\cos\theta - 2\cos 2\theta)}{(-2\sin\theta + 2\sin 2\theta)} = \frac{(\cos\theta - \cos 2\theta)}{(\sin 2\theta - \sin\theta)}$$

$$= \frac{2\sin\left(\frac{3\theta}{2}\right)\sin\frac{\theta}{2}}{2\cos\left(\frac{3\theta}{2}\right)\sin\frac{\theta}{2}} = \tan\frac{3\theta}{2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\tan\frac{3\theta}{2}\right) = \frac{3}{2}\sec^2\frac{3\theta}{2}\cdot\frac{d\theta}{dx} = \frac{3}{2}\sec^2\frac{3\theta}{2}\cdot\frac{1}{2(\sin 2\theta - \sin\theta)}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}} = \frac{3}{2}\cdot\sec^2\left(\frac{3\pi}{4}\right)\cdot\frac{1}{\left(2\sin\pi - \sin\frac{\pi}{2}\right)}$$

$$= \frac{-3}{4}\sec^2\frac{\pi}{4} = \frac{-3}{4}\times(\sqrt{2})^2 = \frac{-3}{2}$$

$$\left[\because \sec\frac{3\pi}{4} = \sec\left(\pi - \frac{\pi}{4}\right) = -\sec\frac{\pi}{4}\right]\cdot$$

EXERCISE 10J

1. Find the second derivative of:
   (i) x^{11} (ii) 5^x (iii) $\tan x$ (iv) $\cos^{-1}x$
2. Find the second derivative of:
   (i) $x \sin x$ (ii) $e^{2x}\cos 3x$ (iii) $x^3 \log x$
3. If $y = x + \tan x$, show that $\cos^2 x\cdot\frac{d^2y}{dx^2} - 2y + 2x = 0.$
4. If $y = 2\sin x + 3\cos x$, show that $y + \frac{d^2y}{dx^2} = 0.$

5. If $y = 3\cos(\log x) + 4\sin(\log x)$, prove that $x^2 y_2 + x y_1 + y = 0$. **[CBSE 2009, '12]**

6. If $y = e^{-x} \cos x$, show that $\frac{d^2y}{dx^2} = 2e^{-x} \sin x$.

7. If $y = \sec x - \tan x$, show that $(\cos x) \frac{d^2y}{dx^2} = y^2$.

8. If $y = (\operatorname{cosec} x + \cot x)$, prove that $(\sin x) \frac{d^2y}{dx^2} - y^2 = 0$.

9. If $y = \tan^{-1} x$, show that $(1 + x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0$.

10. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + (\tan x) \frac{dy}{dx} + y \cos^2 x = 0$.

11. If $y = a\cos(\log x) + b\sin(\log x)$, prove that $x^2 y_2 + x y_1 + y = 0$. **[CBSE 2009]**

12. Find the second derivative of $e^{3x} \sin 4x$.

13. Find the second derivative of $\sin 3x \cos 5x$.

    HINT: $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$.

14. If $y = e^{\tan x}$, prove that $(\cos^2 x) \frac{d^2y}{dx^2} - (1 + \sin 2x) \cdot \frac{dy}{dx} = 0$. **[CBSE 2009C]**

15. If $y = \frac{\log x}{x}$, show that $\frac{d^2y}{dx^2} = \frac{(2\log x - 3)}{x^3}$.

16. If $y = e^{ax} \cos bx$, show that $\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2) y = 0$.

17. If $y = e^{a \cos^{-1} x}$, $-1 \le x \le 1$, show that $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$.

18. If $x = at^2$ and $y = 2at$, find $\frac{d^2y}{dx^2}$ at $t = 2$.

19. If $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, find $\frac{d^2y}{dx^2}$ at $\theta = \pi$.

20. If $y = \sin(\log x)$, prove that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$. **[CBSE 2007]**

21. If $y = \frac{\sin^{-1} x}{\sqrt{1 - x^2}}$, show that $(1 - x^2) \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} - y = 0$. **[CBSE 2009]**

22. If $y = e^x \sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$. **[CBSE 2009C]**

23. If $x = a\left(\cos\theta + \log \tan \frac{\theta}{2}\right)$ and $y = a \sin\theta$, show that the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$ is $\frac{4}{a}$. **[CBSE 2009, '13]**

24. If $x = \cos t + \log \tan \frac{t}{2}$, $y = \sin t$ then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. **[CBSE 2012C]**

25. If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$ **[CBSE 2014]**

26. If $y = (\cot^{-1} x)^2$, then show that $(x^2+1)^2 \frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$ **[CBSE 2010C]**

27. If $y = \{x + \sqrt{x^2+1}\}^m$, then show that $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0.$ **[CBSE 2013C]**

28. If $y = \log\,[x + \sqrt{x^2+a^2}]$, then prove that $(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$ **[CBSE 2013C]**

29. If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$, show that $\frac{d^2y}{dx^2} = \frac{1}{a}\left(\frac{\sec^3\theta}{\theta}\right)$. **[CBSE 2011C, 12C]**

30. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, show that

$$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0.$$ **[CBSE 2014]**

ANSWERS (EXERCISE 10J)

1. (i) $110x^9$ (ii) $5^x(\log 5)^2$ (iii) $2\sec^2 x\tan x$ (iv) $\frac{-x}{(1-x^2)^{3/2}}$

2. (i) $-x\sin x + 2\cos x$ (ii) $-e^{2x}(5\cos 3x + 12\sin 3x)$ (iii) $(5x + 6x\log x)$

12. $e^{3x}(24\cos 4x - 7\sin 4x)$ **13.** $(2\sin 2x - 32\sin 8x)$

18. $\frac{-1}{16a}$ **19.** $\frac{-1}{4a}$ **24.** $\frac{-1}{\sqrt{2}}$, $2\sqrt{2}$

11. APPLICATIONS OF DERIVATIVES

1. Derivative as a Rate Measure

Rate of Change of Quantities

Let $y = f(x)$. Then, $\frac{dy}{dx}$ denotes the rate of change of y w.r.t. x and its value at $x = a$ is denoted by $\left[\frac{dy}{dx}\right]_{x=a}$.

If $x = f(t)$, $y = g(t)$ then by chain rule, we have

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \left(\frac{dy}{dt} \cdot \frac{dt}{dx}\right).$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the rate of change of the area of a circle with respect to its radius r when* $r = 6$ cm.

SOLUTION Let A be the area of a circle of radius r. Then,

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

$$\Rightarrow \left[\frac{dA}{dr}\right]_{r=6\text{ cm}} = (2\pi \times 6)\text{cm}^2/\text{cm} = (12\pi)\text{cm}^2/\text{cm}.$$

Hence, the area is changing at the rate of (12π) cm^2/cm.

EXAMPLE 2 *A stone is dropped into a quiet lake and the waves move in circles. If the radius of a circular wave increases at the rate of* 4 cm/sec, *find the rate of increase in its area at the instant when its radius is* 10 cm.

SOLUTION At any instant t, let the radius of the circle be r cm and its area be A cm^2. Then,

$$\frac{dr}{dt} = 4\text{ cm/sec} \quad \text{(given)} \qquad \text{... (i)}$$

Now, $A = \pi r^2 \Rightarrow \frac{dA}{dt} = \left(\frac{dA}{dr} \cdot \frac{dr}{dt}\right)$

$$= \frac{d}{dr}(\pi r^2) \cdot 4 \quad \left[\because\ A = \pi r^2 \text{ and } \frac{dr}{dt} = 4\right]$$

$$= (2\pi r \times 4)\text{ cm}^2/\text{sec} = (8\pi r)\text{ cm}^2/\text{sec}$$

$$\Rightarrow \quad \left[\frac{dA}{dt}\right]_{r=10} = (8\pi \times 10)\ \text{cm}^2/\text{sec} = (80\pi)\ \text{cm}^2/\text{sec}.$$

Hence, the area of the circle is increasing at the rate of $(80\pi)\ \text{cm}^2/\text{sec}$ at the instant when $r = 10$ cm.

EXAMPLE 3 *A spherical soap bubble is expanding so that its radius is increasing at the rate of* 0.02 cm/sec. *At what rate is the surface area increasing when its radius is* 5 cm? *(Take* π = 3.14.)

SOLUTION A soap bubble is in the form of a sphere. At an instant *t*, let its radius be *r* and surface area *S*. Then,

$$\frac{dr}{dt} = 0.02\ \text{cm/sec} \quad \text{(given)} \qquad \text{... (i)}$$

Now, $S = 4\pi r^2$

$$\Rightarrow \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} = (8 \times 3.14 \times r \times 0.02)\ \text{cm}^2/\text{sec}$$

$$\Rightarrow \left[\frac{dS}{dt}\right]_{r=5} = (8 \times 3.14 \times 5 \times 0.02)\text{cm}^2/\text{sec} = 2.512\ \text{cm}^2/\text{sec}.$$

Hence, the surface area of the bubble is increasing at the rate of $2.512\ \text{cm}^2/\text{sec}$ at the instance when its radius is 5 cm.

EXAMPLE 4 *The volume of a spherical balloon is increasing at the rate of* $20\ \text{cm}^3/\text{sec}$. *Find the rate of change of its surface area at the instant when its radius is* 8 cm.

SOLUTION At any instant *t*, let *r* be the radius, *V* the volume and *S* the surface area of the balloon. Then,

$$\frac{dV}{dt} = 20\ \text{cm}^3/\text{sec} \quad \text{(given)} \qquad \text{... (i)}$$

Now, $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$$\Rightarrow 20 = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \cdot \frac{dr}{dt}$$

$$\Rightarrow 20 = \frac{4}{3}\pi \times 3r^2 \times \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{5}{\pi r^2} \qquad \text{... (ii)}$$

$$\therefore \quad S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$$

$$= \frac{d}{dr}(4\pi r^2) \cdot \frac{5}{\pi r^2}$$

$$= \left(8\pi r \times \frac{5}{\pi r^2}\right) = \frac{40}{r}$$

$$\Rightarrow \left[\frac{dS}{dt}\right]_{r=8\text{ cm}} = \left(\frac{40}{8}\right)\text{ cm}^2/\text{sec} = 5\text{ cm}^2/\text{sec}.$$

Hence, the rate of change of surface area at the instant when $r = 8$ cm is $5\text{ cm}^2/\text{sec}$.

EXAMPLE 5 *The surface area of a spherical balloon is increasing at* $2\text{ cm}^2/\text{sec}$. *At what rate is the volume of the bubble increasing when the radius of the bubble is* 6 cm? **[CBSE 2005]**

SOLUTION At any instant t, let r be the radius, V the volume and S the surface area of the balloon. Then,

$$\frac{dS}{dt} = 2\text{ cm}^2/\text{sec} \quad \text{(given)} \qquad \text{... (i)}$$

$$\text{Now, } S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = \frac{dS}{dr}\cdot\frac{dr}{dt}$$

$$\Rightarrow 2 = \frac{d}{dr}(4\pi r^2)\cdot\frac{dr}{dt}$$

$$\Rightarrow 8\pi r\cdot\frac{dr}{dt} = 2$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r} \qquad \text{... (ii)}$$

$$\therefore\ V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\cdot\frac{dr}{dt} = 4\pi r^2\cdot\frac{dr}{dt}$$

$$= \left(4\pi r^2\cdot\frac{1}{4\pi r}\right) = r \quad \text{[using (ii)]}$$

$$\Rightarrow \left[\frac{dV}{dt}\right]_{r=6\text{ cm}} = 6\text{ cm}^3/\text{sec}.$$

EXAMPLE 6 *The volume of a cube is increasing at the rate of* $7\text{ cm}^3/\text{sec}$. *How fast is its surface area increasing at the instant when the length of an edge of the cube is* 12 cm? **[CBSE 2006C]**

SOLUTION At any instant t, let the length of each edge of the cube be x, V be its volume and S be its surface area. Then,

$$\frac{dV}{dt} = 7\text{ cm}^3/\text{sec} \quad \text{(given)} \qquad \text{... (i)}$$

$$\text{Now, } V = x^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dx}\cdot\frac{dx}{dt}$$

$$\Rightarrow 7 = \frac{d}{dx}(x^3)\cdot\frac{dx}{dt} \qquad [\because\ V = x^3]$$

$$\Rightarrow 3x^2\cdot\frac{dx}{dt} = 7$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{7}{3x^2} \qquad \text{... (ii)}$$

$$\therefore \quad S = 6x^2 \Rightarrow \frac{dS}{dt} = \frac{dS}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{d}{dx}(6x^2) \cdot \frac{7}{3x^2}$$

$$= \left(12x \times \frac{7}{3x^2}\right) = \frac{28}{x}$$

$$\Rightarrow \left[\frac{dS}{dt}\right]_{x=12} = \left(\frac{28}{12}\right) \text{cm}^2/\text{sec} = 2\frac{1}{3} \text{ cm}^2/\text{sec}.$$

Hence, the surface area of the cube is increasing at the rate of $2\frac{1}{3}$ cm^2/sec at the instant when its edge is 12 cm.

EXAMPLE 7 *The length x of a rectangle is decreasing at the rate of* 5 cm/minute *and the width y is increasing at the rate of* 4 cm/minute. *When x* = 8 cm *and y* = 6 cm, *find the rate of change of (i) the perimeter, and (ii) the area of the rectangle.*

SOLUTION Given that $\frac{dx}{dt} = -5$ cm/minute and $\frac{dy}{dt} = 4$ cm/minute.

(i) Let P be the perimeter of the rectangle at any instant. Then,

$$P = 2(x+y) \Rightarrow \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-5+4) \text{ cm/minute}$$

$$= -2 \text{ cm/minute.}$$

Hence, the perimeter of the rectangle is decreasing at the rate of 2 cm/minute.

(ii) Let A be the area of the rectangle at any instant. Then,

$$A = x \cdot y \Rightarrow \frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$$

$$= [(-5) \times 6 + 8 \times 4] \text{ cm}^2/\text{minute}$$

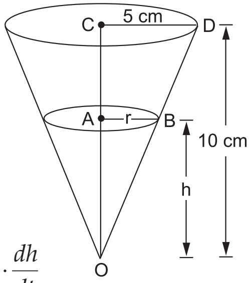
$$= 2 \text{ cm}^2/\text{minute.}$$

Hence, the area of the rectangle is increasing at the rate of 2 cm^2/minute.

EXAMPLE 8 *Water is leaking from a conical funnel at the rate of* 5 cm^3/sec. *If the radius of the base of the funnel is* 5 cm *and its altitude is* 10 cm, *find the rate at which the water level is dropping when it is* 2.5 cm *from the top.*

[CBSE 2004]

SOLUTION At any instant t, let r be the radius of the water level, h the height of the water level and V the volume of the water in the conical funnel. Then,



$$\frac{dV}{dt} = -5 \quad \text{(given)} \qquad \text{... (i)}$$

From similar Δs OAB and OCD, we have

$$\frac{AB}{OA} = \frac{CD}{OC} \Rightarrow \frac{r}{h} = \frac{5}{10} = \frac{1}{2} \Rightarrow r = \frac{1}{2}h.$$

Now, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times \left(\frac{1}{2}h\right)^2 \times h = \frac{1}{12}\pi h^3.$

$$\therefore \quad \frac{dV}{dt} = \left(\frac{dV}{dh} \times \frac{dh}{dt}\right) = \frac{d}{dh}\left(\frac{1}{12}\pi h^3\right) \cdot \frac{dh}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

$$\Rightarrow \quad -5 = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{-20}{\pi h^2} \qquad \text{... (i)}$$

$$\Rightarrow \quad \left(\frac{dh}{dt}\right)_{h=7.5\text{ cm}} = \frac{-20}{\pi \times (7.5)^2} \qquad [\because \quad h = (10 - 2.5) \text{ cm} = 7.5 \text{ cm}]$$

$$= \frac{-16}{45\pi} \text{ cm/sec.}$$

Hence, the rate of change of water level at $h = 7.5$ cm is $\frac{-16}{45\pi}$ cm/sec.

EXAMPLE 9 *Sand is pouring from a pipe at the rate of* 12 cm^3/sec. *The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing, when the height is* 4 cm? **[CBSE 2011]**

SOLUTION At any instant t, let r be the radius, h the height and V the volume of the cone.

Then, $h = \frac{r}{6} \Rightarrow r = 6h.$

$$\therefore \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (6h)^2 h = 12\pi h^3.$$

Now, $V = 12\pi h^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$$\Rightarrow \quad 12 = \frac{d}{dh}(12\pi h^3) \times \frac{dh}{dt}$$

$$\Rightarrow \quad 12 = 36\pi h^2 \times \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{1}{3\pi h^2}$$

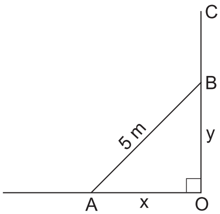
$$\Rightarrow \quad \left(\frac{dh}{dt}\right)_{h=4\text{ cm}} = \frac{1}{(3 \times \pi \times 4 \times 4)} \text{ cm/sec}$$

$$= \frac{1}{48\pi} \text{ cm/sec.}$$

Hence, the rate of increase of the height of the sand cone at the instant when $h = 4$ cm is $\frac{1}{48\pi}$ cm/sec.

EXAMPLE 10 *A* 5-m *long ladder is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall at the rate of* 2 m/sec. *How fast is its height on the wall decreasing when the foot of the ladder is* 4 m *away from the wall?*

SOLUTION Let OC be the wall. At a certain instant t, let AB be the position of the ladder such that $OA = x$ and $OB = y$.



Length of the ladder $AB = 5$ m.

Given that $\frac{dx}{dt} = 2$ m/sec.

From right $\triangle AOB$, we have

$$x^2 + y^2 = 25$$

$$\Rightarrow \quad 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \quad 2x \cdot 2 + 2y \cdot \frac{dy}{dt} = 0 \qquad \left[\because \quad \frac{dx}{dt} = 2\right]$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-2x}{y} \qquad \text{... (i)}$$

Now, $x = 4 \Rightarrow y = \sqrt{5^2 - 4^2} = 3$.

Putting $x = 4$, $y = 3$ in (i) we get $\frac{dy}{dt} = \frac{-8}{3}$.

Hence, the required rate of decrease in the height of the ladder on the wall is (8/3) m/sec.

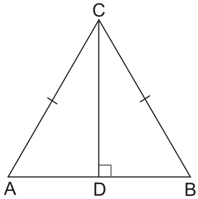
EXAMPLE 11 *The two equal sides of an isosceles triangle with fixed base b* cm *are decreasing at the rate of* 3 cm/sec. *How fast is the area decreasing when each of the equal sides is equal to the base?* **[CBSE 2006C]**

SOLUTION At a certain instant t, let $\triangle ABC$ be an isosceles triangle in which $AB = AC = x$ cm and $BC = b$ cm.

Then, $\frac{dx}{dt} = 3$ cm/sec (given) ... (i)

Draw $AD \perp BC$. Then, $BD = DC = \frac{b}{2}$.

$$\therefore \quad AD = \sqrt{x^2 - \frac{b^2}{4}} = \frac{\sqrt{4x^2 - b^2}}{2}$$



$$\therefore \quad A = \frac{1}{2} b \cdot \frac{\sqrt{4x^2 - b^2}}{2} = \frac{b\sqrt{4x^2 - b^2}}{4}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$$

$$= \left(\frac{dA}{dx} \times 3\right) \qquad \left[\because \quad \frac{dx}{dt} = 3 \text{ cm/sec}\right]$$

$$= \frac{d}{dx}\left\{\frac{b\sqrt{4x^2 - b^2}}{4}\right\} \times 3$$

$$= \frac{3b}{4} \cdot \frac{1}{2}(4x^2 - b^2)^{-1/2} \cdot (8x) = \frac{3bx}{\sqrt{4x^2 - b^2}}.$$

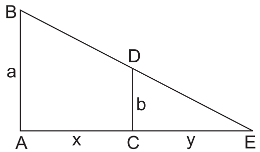
$$\therefore \quad \left[\frac{dA}{dt}\right]_{x=b} = \frac{3b^2}{\sqrt{4b^2 - b^2}} = \frac{3b^2}{\sqrt{3}\,b} = \sqrt{3}\,b.$$

EXAMPLE 12 *A point source of light along a straight road is at a height of 'a' metres. A boy 'b' metres in height is walking along the road. How fast is his shadow increasing if he is walking away from the light at the rate of 'c' metres per minute?* **[CBSE 2006C]**

SOLUTION Let AB be the lamp post, the lamp being at B. Then, $AB = a$ metres. At any instant t, let CD be the position of the boy and CE be his shadow. Then, $CD = b$ metres.

Let $AC = x$ metres and $CE = y$ metres.

Given that $\frac{dx}{dt} = c$ metres/min.



Clearly, $\triangle BAE \sim \triangle DCE$.

$$\therefore \quad \frac{AB}{CD} = \frac{AE}{CE}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{x + y}{y}$$

$$\Rightarrow \quad (a - b)y = bx$$

$$\Rightarrow \quad (a - b)\frac{dy}{dt} = b \cdot \frac{dx}{dt} = bc \qquad \left[\because \quad \frac{dx}{dt} = c\right]$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{bc}{(a - b)} \text{ m/min.}$$

Hence, the shadow is increasing at the rate of $\frac{bc}{(a - b)}$ m/min.

EXAMPLE 13 *A man* 160 cm *tall, walks away from a source of light situated at the top of a pole* 6 m *high, at the rate of* 1.1 m/s. *How fast is the length of his shadow increasing when he is* 1 m *away from the pole?*

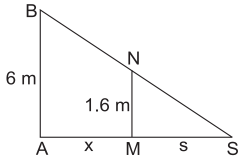
SOLUTION Let AB be the lamp post, the lamp being at B.

Then, $AB = 6$ m.

At any instance t, let MN be the position of the man and MS be his shadow. Then, $MN = 1.6$ m.

Let $AM = x$ metres and $MS = s$ metres.

Given that $\frac{dx}{dt} = 1.1$ m/s.



Clearly, $\triangle SAB \sim \triangle SMN$.

$$\therefore \quad \frac{AS}{MS} = \frac{AB}{MN} = \frac{6}{1.6} = \frac{15}{4}$$

$$\Rightarrow \quad \frac{x+s}{s} = \frac{15}{4} \quad \Rightarrow \quad x = \frac{11}{4}s, \text{ where } MS = s.$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{11}{4} \cdot \frac{ds}{dt} \quad \Rightarrow \quad 1.1 = \frac{11}{4} \cdot \frac{ds}{dt}$$

$$\Rightarrow \quad \frac{ds}{dt} = \left(\frac{1.1 \times 4}{11}\right) = 0.4 \text{ m/s}.$$

Hence, the length of the shadow is increasing at the rate of 0.4 m/s.

EXAMPLE 14 *A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.* **[CBSE 2002]**

SOLUTION Let the required point be (x, y).

Given: $\frac{dy}{dt} = 8\frac{dx}{dt}$ ... (i)

Given curve is $6y = x^3 + 2$ ... (ii)

On differentiating both sides of (ii) w.r.t. t, we get

$$6\frac{dy}{dt} = 3x^2 \frac{dx}{dt}$$

$$\Rightarrow \quad 6\left(8\frac{dx}{dt}\right) = 3x^2 \frac{dx}{dt} \quad \text{[using (i)]}$$

$$\Rightarrow \quad 3x^2 = 48 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4.$$

Putting $x = 4$ in (ii), we get $y = 11$.

Putting $x = -4$ in (ii), we get $y = \frac{-62}{6} = \frac{-31}{3}$.

Hence, the required points are (4, 11) and $\left(-4, \frac{-31}{3}\right)$.

EXAMPLE 15 *Find the points on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate.*

SOLUTION Let the required point be (x, y).

Given: $\frac{dy}{dt} = \frac{dx}{dt}$ ... (i)

Given curve is $y^2 = 8x$ ... (ii)

On differentiating both sides of (ii) w.r.t. t, we get

$$2y\frac{dy}{dt} = 8\frac{dx}{dt}$$

$$\Rightarrow\ 2y \cdot \frac{dx}{dt} = 8\frac{dx}{dt} \qquad \text{[using (i)]}$$

$$\Rightarrow\ 2y = 8 \ \Rightarrow\ y = 4.$$

Putting $y = 4$ in (ii), we get $8x = 16$ and therefore, $x = 2$.

Hence, the required point is (2, 4).

EXAMPLE 16 *At what points of the ellipse* $16x^2 + 9y^2 = 400$ *does the ordinate decrease at the same rate at which the abscissa increases?*

SOLUTION Let the required point be (x, y). Then,

$$\frac{dy}{dt} = -\frac{dx}{dt} \qquad \text{... (i)}$$

Given curve is $16x^2 + 9y^2 = 400$ ... (ii)

On differentiating both sides of (ii) w.r.t. t, we get

$$32x \cdot \frac{dx}{dt} + 18y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow\ 32x \cdot \frac{dx}{dt} + 18y \cdot \left(-\frac{dx}{dt}\right) = 0 \qquad \text{[using (i)]}$$

$$\Rightarrow\ (32x - 18y) = 0 \ \Rightarrow\ y = \frac{16}{9}x.$$

Putting $y = \frac{16}{9}x$ in (ii), we get

$$16x^2 + 9 \times \left(\frac{16x}{9}\right)^2 = 400 \ \Rightarrow\ x^2 = 9 \ \Rightarrow\ x = \pm 3.$$

Now, $x = 3 \ \Rightarrow\ y = \left(\frac{16}{9} \times 3\right) = \frac{16}{3}$,

$$x = -3 \ \Rightarrow\ y = \left[\frac{16}{9} \times (-3)\right] = \frac{-16}{3}.$$

Hence, the required points are $\left(3, \frac{16}{3}\right)$ and $\left(-3, \frac{-16}{3}\right)$.

MARGINAL COST AND MARGINAL REVENUE

Marginal Cost: Let C be the total cost of producing and marketing x units of a product. Then, the marginal cost (MC) is defined as, $\boldsymbol{MC = \frac{dC}{dx}}$.

Marginal Revenue: The rate of change of total revenue with respect to the quantity sold is called the marginal revenue (MR) and therefore, $\boldsymbol{MR = \frac{dR}{dx}}$.

EXAMPLE 17 *The total cost C(x) of producing x items in a firm is given by*

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 6000.$$

Find the marginal cost when 4 units are produced.

SOLUTION Given: $C(x) = 0.005x^3 - 0.02x^2 + 30x + 6000$

$$\Rightarrow \; MC = \frac{dC}{dx}$$

$$= \frac{d}{dx}(0.005x^3 - 0.02x^2 + 30x + 6000)$$

$$= \{(0.005 \times 3x^2) - (0.02 \times 2x) + 30\}$$

$$\Rightarrow \; [MC]_{x=4} = \{(0.005 \times 3 \times 4^2) - (0.02 \times 2 \times 4) + 30\}$$

$$= (0.24 - 0.16 + 30) = 30.08.$$

Hence, the required marginal cost is ₹ 30.08.

EXAMPLE 18 *The total revenue received from the sale of x units of a product is given by*

$$R(x) = 3x^2 + 40x + 10.$$

Find the marginal revenue when $x = 5$.

SOLUTION Given: $R(x) = 3x^2 + 40x + 10$

$$\Rightarrow \quad MR = \frac{dR}{dx}$$

$$= \frac{d}{dx}(3x^2 + 40x + 10) = 6x + 40$$

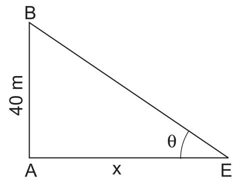
$$\Rightarrow \quad [MR]_{x=5} = (6 \times 5 + 40) = 70.$$

Hence, the required marginal revenue is ₹ 70.

EXERCISE 11A

1. The side of a square is increasing at the rate of 0.2 cm/s. Find the rate of increase of the perimeter of the square.
2. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?
3. The radius of a circle is increasing uniformly at the rate of 0.3 centimetre per second. At what rate is the area increasing when the radius is 10 cm? (Take $\pi = 3.14$.)
4. The side of a square sheet of metal is increasing at 3 centimetres per minute. At what rate is the area increasing when the side is 10 cm long?
5. The radius of a circular soap bubble is increasing at the rate of 0.2 cm/s. Find the rate of increase of its surface area when the radius is 7 cm.

6. The radius of an air bubble is increasing at the rate of 0.5 centimetre per second. At what rate is the volume of the bubble increasing when the radius is 1 centimetre?
7. The volume of a spherical balloon is increasing at the rate of 25 cubic centimetres per second. Find the rate of change of its surface at the instant when its radius is 5 cm.
8. A balloon which always remains spherical is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm.
9. The bottom of a rectangular swimming tank is 25 m by 40 m. Water is pumped into the tank at the rate of 500 cubic metres per minute. Find the rate at which the level of water in the tank is rising.
10. A stone is dropped into a quiet lake and waves move in circles at a speed of 3.5 cm per second. At the instant when the radius of the circular wave is 7.5 cm, how fast is the enclosed area increasing? (Take $\pi = 22/7$.)
11. A 2-m tall man walks at a uniform speed of 5 km per hour away from a 6-metre-high lamp post. Find the rate at which the length of his shadow increases.
12. An inverted cone has a depth of 40 cm and a base of radius 5 cm. Water is poured into it at a rate of 1.5 cubic centimetres per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.
13. Sand is pouring from a pipe at the rate of 18 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is one-sixth of the radius of the base. How fast is the height of the sand cone increasing when its height is 3 cm?
14. Water is dripping through a tiny hole at the vertex in the bottom of a conical funnel at a uniform rate of 4 cm^3/s. When the slant height of the water is 3 cm, find the rate of decrease of the slant height of the water, given that the vertical angle of the funnel is 120°.
15. Oil is leaking at the rate of 16 mL/s from a vertically kept cylindrical drum containing oil. If the radius of the drum is 7 cm and its height is 60 cm, find the rate at which the level of the oil is changing when the oil level is 18 cm.
16. A 13-m long ladder is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the ladder is 5 m away from the wall?
17. A man is moving away from a 40-m high tower at a speed of 2 m/s. Find the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of 30 metres from the foot of the tower. Assume that the eye level of the man is 1.6 m from the ground.



18. Find an angle x which increases twice as fast as its sine.

19. The radius of a balloon is increasing at the rate of 10 cm/s. At what rate is the surface area of the balloon increasing when the radius is 15 cm?

20. An edge of a variable cube is increasing at the rate of 5 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

21. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. Find the rate at which the area is increasing when the side is 10 cm. **[CBSE 2014C]**

ANSWERS (EXERCISE 11A)

1. $\frac{dP}{dt} = 0.8 \text{ cm/s}$ **2.** $\frac{dC}{dt} = 4.4 \text{ cm/s}$ **3.** $\frac{dA}{dt} = 18.84 \text{ cm}^2/\text{s}$

4. $\frac{dA}{dt} = 60 \text{ cm}^2/\text{min}$ **5.** $\frac{dS}{dt} = 35.2 \text{ cm}^2/\text{s}$ **6.** $\frac{dV}{dt} = 6.28 \text{ cm}^3/\text{s}$

7. $\frac{dS}{dt} = 10 \text{ cm}^2/\text{s}$ **8.** $\frac{dr}{dt} = 0.32 \text{ cm/s}$ **9.** $\frac{dh}{dt} = 0.5 \text{ m/min}$

10. $165 \text{ cm}^2/\text{s}$ **11.** 2.5 km/h **12.** $\frac{1}{10\pi}$ cm/s

13. $\frac{1}{18\pi}$ cm/s **14.** $\frac{32}{27\pi}$ cm/s **15.** $\frac{16}{49\pi}$ cm/s

16. $\frac{5}{6}$ m/s **17.** 0.032 radian/second **18.** $\frac{\pi}{3}$

19. $1200\pi \text{ cm}^2/\text{s}$ **20.** $1500 \text{ cm}^3/\text{s}$ **21.** $10\sqrt{3} \text{ cm}^2/\text{s}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11A)

5. $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = (8\pi r) \cdot \frac{dr}{dt} = (8\pi r)(0.2)$

$$\Rightarrow \left[\frac{dS}{dt}\right]_{r=7} = (1.6\pi \times 7) \text{ cm}^2/\text{s}.$$

6. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = (4\pi r^2) \cdot \frac{dr}{dt} = (4\pi r^2) \times 0.5.$

7. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = (4\pi r^2) \cdot \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{25}{4\pi r^2}.$

$$S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} = \left(8\pi r \cdot \frac{25}{4\pi r^2}\right) = \frac{50}{r}$$

$$\Rightarrow \left[\frac{dS}{dt}\right]_{r=5} = \left(\frac{50}{5}\right) \text{ cm}^2/\text{s}.$$

8. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$

$$\Rightarrow \left[\frac{dr}{dt}\right]_{r=15} = \frac{225}{\pi \times (15)^2} = \frac{1}{\pi} \text{ cm/s}.$$

9. $V = (25 \times 40 \times h) \Rightarrow \frac{dV}{dt} = \left(1000 \times \frac{dh}{dt}\right)$

$$\Rightarrow 500 = \left(1000 \times \frac{dh}{dt}\right) \Rightarrow \frac{dh}{dt} = 0.5 \text{ m/min.}$$

10. $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} = (2\pi r \times 3.5) = 7\pi r$

$$\Rightarrow \left[\frac{dA}{dt}\right]_{r=7.5} = (7\pi \times 7.5) \text{ cm}^2/\text{s} = 165 \text{ cm}^2/\text{s}.$$

13. $h = \frac{r}{6} \Rightarrow r = 6h$. So, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3$.

$\therefore \frac{dV}{dt} = 36\pi h^2 \cdot \frac{dh}{dt} \Rightarrow 18 = 36\pi h^2 \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{18}{36\pi h^2} = \frac{1}{2\pi h^2}$

$$\Rightarrow \left[\frac{dh}{dt}\right]_{h=3} = \frac{1}{(2\pi \times 9)} \text{ cm/s.}$$

14. $V = \frac{1}{3}\pi(l \sin 60°)^2(l \cos 60°) = \frac{\pi}{8} l^3$.

$\therefore \frac{dV}{dt} = \frac{3\pi l^2}{8} \cdot \frac{dl}{dt} \Rightarrow \frac{3\pi l^2}{8} \cdot \frac{dl}{dt} = -4 \Rightarrow \frac{dl}{dt} = \frac{-32}{3\pi l^2} \Rightarrow \left[\frac{dl}{dt}\right]_{l=3} = \frac{-32}{27\pi}$ cm/s.

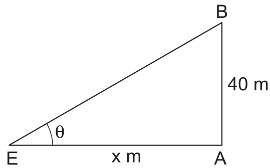
15. $V = (\pi \times 7 \times 7 \times h) \Rightarrow \frac{dV}{dt} = 49\pi \cdot \frac{dh}{dt} \Rightarrow 49\pi \cdot \frac{dh}{dt} = 16 \Rightarrow \frac{dh}{dt} = \frac{16}{49\pi}$ cm/s.

17. Let at any time t, the man be at a distance of x metres from tower AB and let θ be the angle of elevation at that time.

Then, $x = 40 \cot \theta \Rightarrow \frac{dx}{dt} = -40 \operatorname{cosec}^2\theta \cdot \frac{d\theta}{dt}$.

$\therefore \frac{d\theta}{dt} = \frac{-2\sin^2\theta}{40} = -\frac{1}{20} \cdot \left(\frac{AB}{BE}\right)^2 = -\frac{1}{20} \cdot \frac{(40)^2}{(40)^2 + x^2}$.

Find its value when $x = 30$ m.



18. $\frac{dx}{dt} = 2 \cdot \frac{d}{dt}(\sin x) = 2\cos x \cdot \frac{dx}{dt}$. So, $\cos x = \frac{1}{2}$ or $x = \frac{\pi}{3}$.

19. $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} = (8\pi \times 15 \times 10) \text{ cm}^2/\text{s}$.

20. $V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt} = [3 \times (10)^2 \times 5] \text{ cm}^3/\text{s}$.

21. $A = \frac{\sqrt{3}}{4}a^2 \Rightarrow \frac{dA}{da} = \frac{\sqrt{3}}{2}a \Rightarrow \frac{dA}{dt} \cdot \frac{dt}{da} = \frac{\sqrt{3}}{2}a$

$$\Rightarrow \frac{dA}{dt} \times \frac{1}{2} = \frac{\sqrt{3}}{2}a \Rightarrow \frac{dA}{dt} = \sqrt{3}a \Rightarrow \left[\frac{dA}{dt}\right]_{a=10} = \sqrt{3} \times 10 \text{ cm}^2.$$

2. Errors and Approximation

Let $y = f(x)$. Then, $\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = f'(x)$.

$$\therefore \frac{f(x + \delta x) - f(x)}{\delta x} = f'(x) + \in, \text{ where } \in \to 0 \text{ when } \delta x \to 0$$

$\Rightarrow\ f(x+\delta x)-f(x)=f'(x)\cdot\delta x+\in\cdot\delta x$

$\Rightarrow\ f(x+\delta x)-f(x)=f'(x)\cdot\delta x$ (approximately)

$\Rightarrow\ \delta y=f'(x)\cdot\delta x$ $[\because\ f(x+\delta x)-f(x)=\delta y]$.

Thus, if δx is an error in x then the corresponding error in y is δy. These small values δx and δy are called ***differentials.***

(i) ***Absolute Error***: δx is called an absolute error in x.

(ii) ***Relative Error***: $\frac{\delta x}{x}$ is called the relative error.

(iii) ***Percentage Error***: $\left(\frac{\delta x}{x}\times 100\right)$ is called the percentage error.

SOLVED EXAMPLES

EXAMPLE 1 *Using differentials, find the approximate value of* $(82)^{1/4}$ *up to three places of decimal.* **[CBSE 2005]**

SOLUTION Let $f(x)=x^{1/4}$. Then, $f'(x)=\frac{1}{4x^{3/4}}\cdot$

Now, $\{f(x+\delta x)-f(x)\}=f'(x)\cdot\delta x$

$$\Rightarrow\ \{f(x+\delta x)-f(x)\}=\frac{1}{4x^{3/4}}\cdot\delta x \qquad \text{... (i)}$$

We may write, $82=(81+1)$.

Putting $x=81$ and $\delta x=1$ in (i), we get

$$f(81+1)-f(81)=\frac{1}{4\times(81)^{3/4}}\cdot 1$$

$$\Rightarrow\ f(82)-f(81)=\frac{1}{(4\times 3^3)}=\frac{1}{108}$$

$$\Rightarrow\ f(82)=\left\{f(81)+\frac{1}{108}\right\}=\left\{(81)^{1/4}+\frac{1}{108}\right\}=3+0.009=3.009.$$

EXAMPLE 2 *Find the approximate value of the cube root of* 127.

SOLUTION Let $f(x)=x^{1/3}$. Then, $f'(x)=\frac{1}{3x^{2/3}}\cdot$

Now, $\{f(x+\delta x)-f(x)\}=f'(x)\cdot\delta x$

$$\Rightarrow\ \{f(x+\delta x)-f(x)\}=\frac{1}{3x^{2/3}}\cdot\delta x \qquad \text{... (i)}$$

We may write, $127=(125+2)$.

Putting $x=125$ and $\delta x=2$ in (i), we get

$$f(125+2)-f(125)=\frac{1}{3\times(125)^{2/3}}\times 2$$

$$\Rightarrow\ f(127) - f(125) = \frac{2}{75}$$

$$\Rightarrow\ f(127) = f(125) + \frac{2}{75} = \left\{(125)^{1/3} + \frac{2}{75}\right\} = \left(5 + \frac{2}{75}\right) = \frac{377}{75}$$

$$\Rightarrow\ \sqrt[3]{127} = \frac{377}{75} = 5.026.$$

EXAMPLE 3 *Using differentials find the approximate value of the square root of* 26. **[CBSE 2000]**

SOLUTION Let $f(x) = \sqrt{x}$. Then, $f'(x) = \frac{1}{2\sqrt{x}}\cdot$

Now, $\{f(x+\delta x) - f(x)\} = f'(x)\cdot\delta x$

$$\Rightarrow\ \{f(x+\delta x) - f(x)\} = \frac{1}{2\sqrt{x}}\cdot\delta x \qquad \text{... (i)}$$

We may write, $26 = (25+1)$.

Putting $x = 25$ and $\delta x = 1$ in (i), we get

$$f(26) - f(25) = \frac{1}{2\sqrt{25}}\times 1$$

$$\Rightarrow\ f(26) = f(25) + \frac{1}{10} = \left(\sqrt{25} + \frac{1}{10}\right) = \left(5 + \frac{1}{10}\right) = (5+0.1) = 5.1$$

$$\Rightarrow\ \sqrt{26} = 5.1.$$

Hence, $\sqrt{26} = 5.1$.

EXAMPLE 4 *Using differentials find the approximate value of* $\sqrt{0.037}$. **[CBSE 2005C]**

SOLUTION Let $f(x) = \sqrt{x}$. Then, $f'(x) = \frac{1}{2\sqrt{x}}\cdot$

Now, $\{f(x+\delta x) - f(x)\} = f'(x)\cdot\delta x$

$$\Rightarrow\ \{f(x+\delta x) - f(x)\} = \frac{1}{2\sqrt{x}}\cdot\delta x \qquad \text{... (i)}$$

We may write, $0.037 = (0.04 - 0.003)$.

Putting $x = 0.04$ and $\delta x = -0.003$ in (i), we get

$$f(0.04 - 0.003) - f(0.04) = \frac{1}{2\sqrt{0.04}}\times(-0.003)$$

$$\Rightarrow\ f(0.037) = f(0.04) - \frac{0.003}{0.4}$$

$$\Rightarrow\ \sqrt{0.037} = \left(\sqrt{0.04} - \frac{3}{400}\right) = \left(0.2 - \frac{3}{400}\right) = \frac{77}{400}$$

$$\Rightarrow\ \sqrt{0.037} = 0.1925.$$

EXAMPLE 5 *Find the approximate value of* tan 46°, *it is being given that* 1° = 0.01745 *radian.*

SOLUTION Let $f(x) = \tan x$. Then, $f'(x) = \sec^2 x$.

Now, $f(x+\delta x) - f(x) = f'(x) \cdot \delta x$

$\Rightarrow\ f(x+\delta x) - f(x) = \sec^2 x \cdot \delta x$ ... (i)

Putting $x = 45°,\ \ \delta x = 1° = 0.01745$ in (i), we get

$$f(46°) - f(45°) = (\sec^2 45°) \times 0.01745$$

$$\Rightarrow\ \tan 46° - \tan 45° = (\sec^2 45°) \times 0.01745$$

$$\Rightarrow\ \tan 46° = \tan 45° + (\sec^2 45°) \times 0.01745$$

$$= (1 + 2 \times 0.01745) = 1.03490.$$

Hence, $\tan 46° = 1.03490$.

EXAMPLE 6 *Find the approximate value of* $\log_{10} 10.1$, *it being given that* $\log_{10} e = 0.4343$.

SOLUTION Let $f(x) = \log_{10} x$. Then, $f'(x) = \frac{1}{x}(\log_{10} e)$.

Now, $f(x+\delta x) - f(x) = f'(x) \cdot \delta x$

$\Rightarrow\ f(x+\delta x) - f(x) = \frac{1}{x}(\log_{10} e) \cdot \delta x$

$\Rightarrow\ f(x+\delta x) - f(x) = \frac{0.4343}{x} \cdot \delta x$ ... (i)

Putting $x = 10$ and $\delta x = 0.1$ in (i), we get

$$f(10.1) - f(10) = \frac{0.4343}{10} \times 0.1$$

$$\Rightarrow\ \log_{10}(10.1) - \log_{10} 10 = 0.004343$$

$$\Rightarrow\ \log_{10}(10.1) = (\log_{10} 10) + 0.004343$$

$$= (1 + 0.004343) = 1.004343.$$

Hence, $\log_{10}(10.1) = 1.004343$.

EXAMPLE 7 *If* $f(x) = 3x^2 + 15x + 5$, *then find the approximate value of* $f(3.02)$, *using differentials.* **[CBSE 2008C, '14]**

SOLUTION $f(x) = 3x^2 + 15x + 5\ \Rightarrow\ f'(x) = 6x + 15$

Now, $f(x+\delta x) - f(x) = f'(x) \cdot \delta x$

$\Rightarrow\ f(x+\delta x) - f(x) = (6x + 15) \cdot \delta x$ ... (i)

Putting $x = 3$ and $\delta x = 0.02$ in (i), we get

$$f(3.02) - f(3) = (6 \times 3 + 15) \times (0.02)$$

$$\Rightarrow\ f(3.02) - 77 = 0.66 \qquad [\because\ f(3) = 3 \times 3^2 + 15 \times 3 + 5 = 77]$$

$$\Rightarrow\ f(3.02) = 77 + 0.66 = 77.66.$$

Use of the Formula: $\delta y = \frac{dy}{dx} \cdot \delta x$

EXAMPLE 8 *If the radius of a circle increases from* 5 cm *to* 5.1 cm, *find the increase in area.*

SOLUTION Area of a circle of radius r is given by $A = \pi r^2$.

Now, $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

$$\Rightarrow \left[\frac{dA}{dr}\right]_{r=5} = (2\pi \times 5) \text{ cm}^2 = 10\pi \text{ cm}^2.$$

Also, $\delta r = (5.1 - 5)$ cm $= 0.1$ cm.

$\therefore \quad \delta A = \frac{dA}{dr} \cdot \delta r = (10\pi \times 0.1) \text{ cm}^2 = \pi \text{ cm}^2.$

Hence, the increase in area is π cm^2.

EXAMPLE 9 *If* $y = (x^4 - 12)$ *and if* x *changes from* 2 *to* 1.99, *what is the approximate change in* y? **[CBSE 2002]**

SOLUTION $y = (x^4 - 12) \Rightarrow \frac{dy}{dx} = 4x^3$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=2} = (4 \times 2^3) = 32.$$

Let $\delta x = (1.99 - 2) = -0.01$.

$\therefore \quad \delta y = \frac{dy}{dx} \cdot \delta x = 32 \times (-0.01) = -0.32.$

EXAMPLE 10 *The time* T *of oscillation of a simple pendulum of length* l *is given by* $T = 2\pi\sqrt{\frac{l}{g}}$ $\cdot$ *Find the percentage error in* T, *corresponding to an error of* 2% *in the value of* l.

SOLUTION $T = 2\pi\sqrt{\frac{l}{g}} \Rightarrow \log T = \log 2 + \log \pi + \frac{1}{2}\log l - \frac{1}{2}\log g$

$$\Rightarrow \frac{1}{T} \cdot \frac{dT}{dl} = \frac{1}{2l} \Rightarrow \frac{1}{T} \cdot \frac{dT}{dl} \cdot \delta l = \frac{1}{2l}\,\delta l$$

$$\Rightarrow \frac{1}{T} \cdot \delta T = \frac{1}{2l} \cdot \delta l \qquad \left[\because \ \delta T = \frac{dT}{dl} \cdot \delta l\right]$$

$$\Rightarrow \left(\frac{\delta T}{T} \times 100\right) = \frac{1}{2} \cdot \left(\frac{\delta l}{l} \times 100\right) = \frac{1}{2} \times 2 = 1.$$

$\therefore$ percentage error in $T = 1\%$.

EXAMPLE 11 *If the error committed in measuring the radius of a circle be* 0.01%, *find the corresponding error in calculating the area.*

SOLUTION $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r \Rightarrow \frac{dA}{dr} \cdot \delta r = 2\pi r \cdot \delta r$

$$\Rightarrow \delta A = 2\pi r^2 \cdot \frac{\delta r}{r} \quad [\because \frac{dA}{dr} \cdot \delta r = \delta A]$$

$$\Rightarrow \frac{\delta A}{A} = 2 \cdot \frac{\delta r}{r} \quad [\because A = \pi r^2]$$

$$\Rightarrow \left(\frac{\delta A}{A} \times 100\right) = 2 \cdot \left(\frac{\delta r}{r} \times 100\right)$$

$\Rightarrow$ percentage error in area = $(2 \times 0.01) = 0.02\%$.

Hence, the percentage error in area = 0.02%.

EXAMPLE 12 *If in a triangle ABC, the side c and the angle C remain constant while the remaining elements are changed slightly, show that* $\frac{da}{\cos A} + \frac{db}{\cos B} = 0.$

SOLUTION As given, we have $\frac{c}{\sin C} = k$ (constant).

$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = k \Rightarrow a = k \sin A$ and $b = k \sin B$.

$\therefore \quad da = k \cos A \cdot dA$ and $db = k \cos B \cdot dB$

or $\frac{da}{\cos A} + \frac{db}{\cos B} = k(dA + dB) = kd(A + B) = kd(\pi - C) = 0.$

Hence, $\frac{da}{\cos A} + \frac{db}{\cos B} = 0.$

EXAMPLE 13 *The area S of a triangle is calculated by measuring the sides b and c, and ∠A. If there be an error δA in the measurement of ∠A, show that the relative error in area is given by*

$$\frac{\delta S}{S} = \cot A \cdot \delta A.$$

SOLUTION $S = \frac{1}{2} bc \sin A \Rightarrow \frac{dS}{dA} = \frac{1}{2} bc \cos A \Rightarrow \frac{dS}{dA} \cdot \delta A = \frac{1}{2} bc \cos A \cdot \delta A$

$$\Rightarrow \delta S = \frac{1}{2} bc \cos A \cdot \delta A \Rightarrow \delta S = \frac{1}{2} bc \sin A \cdot (\cot A) \cdot \delta A$$

$$\Rightarrow \delta S = S \cdot (\cot A) \cdot \delta A \Rightarrow \frac{\delta S}{S} = (\cot A) \cdot \delta A.$$

Hence, $\frac{\delta S}{S} = (\cot A) \cdot \delta A.$

EXERCISE 11B

Using differentials, find the approximate values of:

1. $\sqrt{37}$ **[CBSE 2000]**
2. $\sqrt[3]{29}$

3. $\sqrt[3]{27}$

4. $\sqrt{0.24}$ **[CBSE 2001]**

5. $\sqrt{49.5}$ **[CBSE 2012]**

6. $\sqrt[4]{15}$

7. $\dfrac{1}{(2.002)^2}$

8. $\log_e 10.02$, given that $\log_e 10 = 2.3026$

9. $\log_{10}(4.04)$, it being given that $\log_{10} 4 = 0.6021$ and $\log_{10} e = 0.4343$

10. $\cos 61°$, it being given that $\sin 60° = 0.86603$ and $1° = 0.01745$ radian

11. If $y = \sin x$ and x changes from $\dfrac{\pi}{2}$ to $\dfrac{22}{14}$, what is the approximate change in y?

12. A circular metal plate expands under heating so that its radius increases by 2%. Find the approximate increase in the area of the plate, if the radius of the plate before heating is 10 cm.

13. If the length of a simple pendulum is decreased by 2%, find the percentage decrease in its period T, where $T = 2\pi\sqrt{\dfrac{l}{g}}\cdot$

14. The pressure p and the volume V of a gas are connected by the relation, $pV^{1/4} = k$, where k is a constant. Find the percentage increase in the pressure, corresponding to a diminution of 0.5% in the volume.

15. The radius of a sphere shrinks from 10 cm to 9.8 cm. Find approximately the decrease in (i) volume, and (ii) surface area.

16. If there is an error of 0.1% in the measurement of the radius of a sphere, find approximately the percentage error in the calculation of the volume of the sphere.

17. Show that the relative error in the volume of a sphere, due to an error in measuring the diameter, is three times the relative error in the diameter.

ANSWERS (EXERCISE 11B)

1. 6.08 **2.** 3.074 **3.** 5.027 **4.** 0.49

5. 7.0357 **6.** $\dfrac{63}{32}$ **7.** 0.2495 **8.** 2.3046

9. 0.606443 **10.** 0.4849 **11.** No change **12.** $\dfrac{2\pi}{5}\text{ cm}^2$

13. 1% **14.** 0.125% **15.** (i) $80\pi\text{ cm}^3$ (ii) $16\pi\text{ cm}^2$

16. 0.3%

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11B)

7. Let $f(x) = \frac{1}{x^2}\cdot$ Then $f(x+\delta x) - f(x) = \frac{-2}{x^3}\cdot \delta x.$

Take $x = 2$ and $\delta x = 0.002.$

12. $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r = 20\pi$, when $r = 10.$

$\therefore \quad \delta A = \frac{dA}{dr}\cdot \delta r = \left(20\pi \times \frac{2}{100}\right) = \frac{2\pi}{5}\cdot$

13. $\log T = \log\left(\frac{2\pi}{\sqrt{g}}\right) + \frac{1}{2}\log l \Rightarrow \frac{1}{T}\cdot\frac{dT}{dl} = \frac{1}{2l}\cdot$

$\therefore \quad \delta T = \frac{T}{2l}\cdot \delta l \Rightarrow \left(\frac{\delta T}{T}\times 100\right) = \left(\frac{1}{2}\cdot\frac{\delta l}{l}\times 100\right) = \left\{\frac{1}{2}\times\left(\frac{-2}{100}\right)\times 100\right\} = -1.$

14. $pV^{1/4} = k \Rightarrow \log p + \frac{1}{4}\log V = \log k$

$$\Rightarrow \frac{1}{p}\cdot\frac{dp}{dV} + \frac{1}{4V} = 0 \Rightarrow \frac{1}{p}\cdot\frac{dp}{dV}\cdot\delta V = -\frac{1}{4V}\cdot\delta V$$

$$\Rightarrow \frac{1}{p}\cdot\delta p = -\frac{1}{4V}\cdot\delta V \Rightarrow \left(\frac{\delta p}{p}\times 100\right)$$

$$\Rightarrow \left(\frac{\delta p}{p}\times 100\right) = -\frac{1}{4}\left(\frac{\delta V}{V}\times 100\right) = \left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right) = \frac{1}{8}\cdot$$

3. Rolle's and Lagrange's Theorems

Rolle's Theorem

Let f be a real-valued function, defined in the closed interval $[a, b]$ *such that*

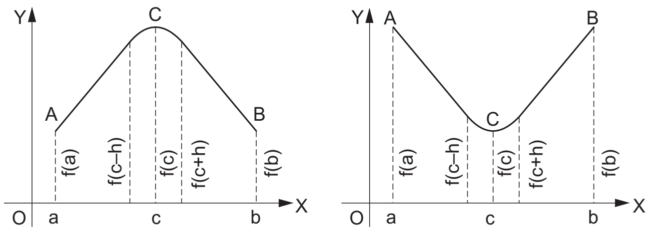
(i) f *is continuous on* $[a, b]$; *(ii)* f *is differentiable on* $]a, b[$;

(iii) $f(a) = f(b)$.

Then, there exists a real number c in the open interval $]a, b[$ *such that* $f'(c) = 0$.

PROOF If $f(x) = c$, where c is a constant then $f'(x) = 0$ for all $x \in]a, b[$.
So, in this case, the theorem follows.

Now, consider the case, when f is not a constant function. Since $f(a) = f(b)$, the function should either increase or decrease, when x takes values slightly greater than a.



Let $f(x)$ be increasing at $x > a$.

Since $f(b) = f(a)$, $f(x)$ must cease to increase and begin to decrease at some point $x = c \in]a, \; b[$.

Clearly, at such a point, the function has a maximum value. Thus, for a very small and positive value of h, we have

$$f(c+h) - f(c) < 0 \text{ and } f(c-h) - f(c) < 0.$$

$$\therefore \quad \frac{f(c+h) - f(c)}{h} < 0 \text{ and } \frac{f(c-h) - f(c)}{-h} > 0.$$

So, $$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \le 0 \text{ and } \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} \ge 0.$$

But, if $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \neq \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$ then

$Rf'(c) \neq Lf'(c)$ and this would imply that $f(x)$ is not differentiable at $x = c$, contradicting the given hypothesis.

$$\therefore \quad \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = 0.$$

Hence, $f'(c) = 0$, where $a < c < b$.

Similarly, the theorem can be proved for the case when $f(x)$ is initially decreasing for values of $x > a$, and it ceases to decrease at $x = c$.

Hence, there exists $c \in]a, b[$ such that $f'(c) = 0$.

REMARK Rolle's theorem is applicable to a function only when all the three properties above listed are satisfied by it. If even a single property is violated, we say that the theorem is not applicable on such a function.

GEOMETRICAL SIGNIFICANCE OF ROLLE'S THEOREM Let f be a real function defined on $[a, b]$ and let Rolle's theorem be applicable on it. Then, f being continuous on $[a, b]$, it follows that we can draw a graph of $f(x)$ between the values $x = a$ and $x = b$.

Also, $f(x)$ being differentiable in $]a, \; b[$, it follows that the graph of $f(x)$ has a tangent at each point of $]a, \; b[$.

Now, the existence of a real number $c \in]a, b[$ such that $f'(c) = 0$ shows that the tangent to the curve at $x = c$ has a slope 0, i.e., it is parallel to the x-axis.

SOLVED EXAMPLES

EXAMPLE 1 *Verify Rolle's theorem for the function* $f(x) = x^3 - 6x^2 + 11x - 6$ *in the interval* $[1, 3]$.

SOLUTION Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[1, 3]$.

(ii) $f'(x) = 3x^2 - 12x + 11$, which clearly exists for all values of $x \in [1, 3]$.

So, $f(x)$ is differentiable on the open interval $]1, 3[$.

(iii) $f(1) = (1^3 - 6 \times 1^2 + 11 \times 1 - 6) = 0$

and $f(3) = (3^3 - 6 \times 3^2 + 11 \times 3 - 6) = 0$.

$\therefore \quad f(1) = f(3)$.

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist some $c \in]1, 3[$ such that $f'(c) = 0$.

Now, $\quad f'(c) = 0 \Rightarrow 3c^2 - 12c + 11 = 0$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 132}}{6} \Rightarrow c = \left(2 \pm \frac{1}{\sqrt{3}}\right).$$

Clearly, both the values of c lie in the interval $]1, 3[$.

Hence, Rolle's theorem is verified.

EXAMPLE 2 *Verify Rolle's theorem for the function* $f(x) = x(x-1)^2$ *in the interval* $[0, 1]$.

SOLUTION We have $\quad f(x) = x^3 - 2x^2 + x$.

We observe here that

(i) $f(x)$ being a polynomial function, is continuous on $[0, 1]$.

(ii) $f'(x) = (3x^2 - 4x + 1)$, which clearly exists for all values of $x \in [0, 1]$.

So, $f(x)$ is differentiable on the interval $]0, 1[$.

(iii) $f(0) = 0$ and $f(1) = 0$.

$\therefore \; f(0) = f(1)$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist a real number $c \in]0, 1[$ such that $f'(c) = 0$.

Now, $\; f'(c) = 0 \Rightarrow 3c^2 - 4c + 1 = 0 \Rightarrow (c-1)(3c-1) = 0$

$$\Rightarrow c = 1 \text{ or } c = \frac{1}{3}.$$

Out of these two values, clearly $\frac{1}{3} \in]0, 1[$.

Thus, $\; c = \frac{1}{3} \in]0, 1[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is satisfied.

EXAMPLE 3 *Verify Rolle's theorem for the function* $f(x) = (x-a)^m (x-b)^n$ *in the interval* $[a, b]$, *where m and n are positive integers.*

SOLUTION Let $f(x) = (x-a)^m (x-b)^n$, where $a \le x \le b$.

On expanding $(x-a)^m$ and $(x-b)^n$ by the binomial theorem and then taking the product, we find that $f(x)$ is a polynomial of degree $(m+n)$. But, a polynomial function being continuous everywhere, it follows that $f(x)$ is continuous in $[a, b]$.

Also, $f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m (x-b)^{n-1}$
$= (x-a)^{m-1}(x-b)^{n-1} \cdot [m(x-b) + n(x-a)]$,

which clearly exists for all $x \in]a, b[$.

$\therefore$ $f(x)$ is differentiable on $]a, b[$.

Also, $f(a) = f(b) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied. Consequently, there must exist some $c \in]a, b[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow (c-a)^{m-1}(c-b)^{n-1}[m(c-b) + n(c-a)] = 0$

$\Leftrightarrow (c-a) = 0$ or $(c-b) = 0$ or $m(c-b) + n(c-a) = 0$

$\Leftrightarrow c = a$ or $c = b$ or $c = \left(\frac{mb+na}{m+n}\right)$.

Clearly, $c = \left(\frac{mb+na}{m+n}\right) \in]a, b[$ such that $f'(c) = 0$

$\left\{\because \left(\frac{mb+na}{m+n}\right) \text{ divides }]a, b[\text{ in the ratio } m : n\right\}$.

Hence, Rolle's theorem is verified.

EXAMPLE 4 *Verify Rolle's theorem for each of the following functions:*

(i) $f(x) = \sin 2x$ *in* $\left[0, \frac{\pi}{2}\right]$

(ii) $f(x) = (\sin x + \cos x)$ *in* $\left[0, \frac{\pi}{2}\right]$ **[CBSE 2006]**

(iii) $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ *in* $\left[0, \frac{\pi}{2}\right]$

(iv) $f(x) = (\sin x - \sin 2x)$ *in* $[0, \pi]$

SOLUTION (i) Consider $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$.

Since the sine function is continuous at each $x \in R$, it follows that $f(x) = \sin 2x$ is continuous on $\left[0, \frac{\pi}{2}\right]$.

Also, $f'(x) = 2\cos 2x$, which clearly exists for all $x \in \left[0, \frac{\pi}{2}\right]$.

So, $f(x)$ is differentiable on $\left]0, \frac{\pi}{2}\right[\cdot$

Also, $f(0) = f\left(\frac{\pi}{2}\right) = 0.$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist a real number $c \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Now, $f'(c) = 0 \Leftrightarrow 2\cos 2c = 0 \Leftrightarrow \cos 2c = 0$

$$\Leftrightarrow 2c = \frac{\pi}{2}, \text{ i.e., } c = \frac{\pi}{4}\cdot$$

Thus, $c = \frac{\pi}{4} \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Hence, Rolle's theorem is verified.

(ii) Consider $f(x) = (\sin x + \cos x)$ in $\left[0, \frac{\pi}{2}\right]\cdot$

By the continuity of the sine function, the cosine function and the sum of continuous functions, it follows that $f(x)$ is continuous on $\left[0, \frac{\pi}{2}\right]\cdot$

Also, $f'(x) = (\cos x - \sin x)$, which clearly exists for all values of $x \in \left[0, \frac{\pi}{2}\right]\cdot$

So, $f(x)$ is differentiable on $\left]0, \frac{\pi}{2}\right[\cdot$ Also, $f(0) = f\left(\frac{\pi}{2}\right) = 1.$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist some $c \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Now, $f'(c) = 0 \Leftrightarrow \cos c - \sin c = 0 \Leftrightarrow \cos c = \sin c \Leftrightarrow c = \frac{\pi}{4}\cdot$

Thus, $c = \frac{\pi}{4} \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Hence, Rolle's theorem is verified.

(iii) Consider $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ in $\left[0, \frac{\pi}{2}\right]\cdot$

Since the cosine function is continuous everywhere, it follows that $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ is continuous on $\left[0, \frac{\pi}{2}\right]\cdot$

Also, $f'(x) = -2\sin\left(2x - \frac{\pi}{2}\right) = 2\cos 2x$, which clearly exists for all $x \in \left]0, \frac{\pi}{2}\right[\cdot$

$\therefore\;\; f(x)$ is differentiable on $\left]0, \frac{\pi}{2}\right[\cdot$

Further, $f(0) = \cos 2\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0.$

And, $\quad f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0.$

$\therefore \qquad f(0) = f\left(\frac{\pi}{2}\right) = 0.$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist $c \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Now, $\quad f'(c) = 0 \;\Leftrightarrow\; 2\cos 2c = 0 \;\Leftrightarrow\; 2c = \frac{\pi}{2} \;\Rightarrow\; c = \frac{\pi}{4}\cdot$

Thus, $\quad c = \frac{\pi}{4} \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Hence, Rolle's theorem is verified.

(iv) Consider $f(x) = (\sin x - \sin 2x)$ in $[0, \pi]$.

Since the sine function is continuous, it follows that $g(x) = \sin x$ and $h(x) = \sin 2x$ are both continuous and so their difference is also continuous.

Consequently, $f(x) = g(x) - h(x)$ is differentiable on $[0, \pi]$.

Also, $f'(x) = (\cos x - 2\cos 2x)$, which clearly exists for all $x \in [0, \pi]$.

$\therefore\;\; f(x)$ is differentiable on $]0, \pi[$.

And, $f(0) = f(\pi) = 0.$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist a real number $c \in]0, \pi[$ such that $f'(c) = 0.$

$$\begin{aligned}
\text{Now, } f'(c) = 0 &\Leftrightarrow \cos c - 2\cos 2c = 0\\
&\Leftrightarrow \cos c - 2(2\cos^2 c - 1) = 0\\
&\Leftrightarrow 4\cos^2 c - \cos c - 2 = 0\\
&\Leftrightarrow \cos c = \frac{1 \pm \sqrt{33}}{8} = 0.8431 \;\text{ or }\; -0.5931\\
&\Leftrightarrow \cos c = 0.8431 \;\text{ or }\; \cos(180° - c) = 0.5931.\\
&\Leftrightarrow c = 32°\cdot 32' \;\text{ or }\; c = 126°23'.
\end{aligned}$$

Thus, $c \in]0, \pi[$ such that $f'(c) = 0.$

Hence, Rolle's theorem is satisfied.

EXAMPLE 5 *Verify Rolle's theorem for each of the following functions:*

(i) $f(x) = \sin^2 x$ *in* $0 \le x \le \pi$

(ii) $f(x) = e^x \cos x$ *in* $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(iii) $f(x) = \frac{\sin x}{e^x}$ *in* $0 \le x \le \pi$

SOLUTION (i) Consider $f(x) = \sin^2 x$ in $[0, \pi]$.

Let $g(x) = \sin x$ and $h(x) = x^2$.

Then, the sine function being continuous for all $x \in R$ and every polynomial function being continuous for all $x \in R$, it follows that g and h are both continuous for all $x \in R$.

But the composite of continuous functions is continuous.

$\therefore \quad (h \circ g)(x) = f(x) = \sin^2 x$ is continuous on $[0, \pi]$.

Also, $f'(x) = 2\sin x \cos x = \sin 2x$, which clearly exists for all $x \in]0, \pi[$.

So, $f(x)$ is differentiable on $]0, \pi[$.

Also, $f(0) = f(\pi) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist $c \in]0, \pi[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow \sin 2c = 0 \Leftrightarrow 2c = \pi \Leftrightarrow c = \frac{\pi}{2}\cdot$

Thus, $c = \frac{\pi}{2} \in]0, \pi[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

(ii) Consider $f(x) = e^x \cos x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\cdot$

Let $g(x) = e^x$ and $h(x) = \cos x$.

Then, the exponential function as well as the cosine function being continuous, it follows that $g(x) \cdot h(x) = f(x)$ is continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\cdot$

Also, $f'(x) = -e^x \sin x + e^x \cos x = e^x(\cos x - \sin x)$, which clearly exists for all $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\cdot$

So, $f(x)$ is differentiable on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\cdot$

Also, $f\left(-\frac{\pi}{2}\right) = e^{-\pi/2} \cos\left(-\frac{\pi}{2}\right) = 0$. And, $f\left(\frac{\pi}{2}\right) = e^{\pi/2} \cos\frac{\pi}{2} = 0$.

$\therefore \quad f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 0.$

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist some $c \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow e^c(\cos c - \sin c) = 0 \Leftrightarrow \cos c - \sin c = 0$

$$\Leftrightarrow \sin c = \cos c \Leftrightarrow c = \frac{\pi}{4}\cdot$$

Thus, $c = \frac{\pi}{4} \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

(iii) Consider $f(x) = \dfrac{\sin x}{e^x}$ in $[0, \pi]$.

Since $e^x \neq 0$ for any $x \in [0, \pi]$ and $f(x)$ is the quotient of two continuous functions, it follows that $f(x)$ is continuous on $[0, \pi]$.

Also, $f'(x) = \dfrac{e^x \cos x - e^x \sin x}{e^{2x}} = \dfrac{(\cos x - \sin x)}{e^x}$, which clearly exists for all $x \in]0, \pi[$.

So, $f(x)$ is differentiable on $]0, \pi[$. Also, $f(0) = f(\pi) = 0$.

So, all the conditions of Rolle's theorem are satisfied.

$\therefore$ there must exist $c \in]0, \pi[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow \dfrac{(\cos c - \sin c)}{e^c} = 0$

$$\Leftrightarrow \cos c - \sin c = 0 \Leftrightarrow \sin c = \cos c \Leftrightarrow c = \frac{\pi}{4}\cdot$$

Thus, $c = \frac{\pi}{4} \in]0, \pi[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

EXAMPLE 6 *Verify Rolle's theorem for the function* $f(x) = x(x+3)e^{-(x/2)}$ *in* $[-3, 0]$.

SOLUTION Since a polynomial function as well as an exponential function is continuous and the product of two continuous functions is continuous, it follows that $f(x)$ is continuous on the given interval $[-3, 0]$.

Now, $f'(x) = (2x+3)e^{-(x/2)} - \frac{1}{2}e^{-(x/2)}(x^2 + 3x)$

$$= e^{-(x/2)}\left(\frac{x + 6 - x^2}{2}\right),$$

which is clearly finite for all values of x in $]-3, 0[$.

So, $f(x)$ is differentiable on $]-3, 0[$. Also, $f(-3) = f(0) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist $c \in]-3, 0[$ such that $f'(c) = 0$.

But, $f'(c) = 0 \Leftrightarrow e^{-(c/2)}\left(\dfrac{c + 6 - c^2}{2}\right) = 0 \Leftrightarrow c + 6 - c^2 = 0$

$$\Leftrightarrow (3 - c)(c + 2) = 0 \Leftrightarrow c = 3 \text{ or } c = -2.$$

Thus, $c = -2 \in]-3, 0[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

EXAMPLE 7 *Verify Rolle's theorem for the function* $f(x) = \{\log(x^2 + 2) - \log 3\}$ *on* $[-1, 1]$.

SOLUTION Clearly, $f(x) = \{\log(x^2 + 2) - \log 3\}$ has a unique and definite value for each $x \in [-1, 1]$. So, at each point of $[-1, 1]$, the limit of the function is equal to the value of the function.

So, $f(x)$ is continuous on $[-1, 1]$.

Also, $f'(x) = \dfrac{2x}{(x^2 + 2)}$, which is clearly finite for each $x \in [-1, 1]$.

So, $f(x)$ is differentiable on $]-1, 1[$.

Also, $f(-1) = f(1) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist $c \in]-1, 1[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow \dfrac{2c}{(c^2 + 2)} = 0 \Leftrightarrow 2c = 0$, i.e., $c = 0$.

Thus, $c = 0 \in]-1, 1[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

EXAMPLE 8 *Verify Rolle's theorem for the following functions:*

(i) $f(x) = \sqrt{4 - x^2}$ *in* $[-2, 2]$

(ii) $f(x) = \log\left(\dfrac{x^2 + ab}{(a + b)x}\right]$ *in* $[a, b]$, *where* $0 < a < b$

SOLUTION (i) Consider $f(x) = \sqrt{4 - x^2}$ in $[-2, 2]$.

Clearly, $f(x) = \sqrt{4 - x^2}$ has a unique and definite value for each $x \in [-2, 2]$. So, at each point of $[-2, 2]$, the limit is equal to the value of the function.

$\therefore$ $f(x)$ is continuous on $[-2, 2]$.

Also, $f'(x) = \dfrac{-x}{\sqrt{4 - x^2}}$, which clearly exists for each $x \in [-2, 2]$.

So, $f(x)$ is differentiable on $]-2, 2[$.

Also, $f(-2) = f(2) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist $c \in]-2, 2[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow \dfrac{-c}{\sqrt{4 - c^2}} = 0 \Leftrightarrow c = 0$.

Thus, $c = 0 \Leftrightarrow]-2, 2[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

(ii) Consider

$$f(x) = \log\left[\frac{x^2 + ab}{(a+b)x}\right] = \log(x^2 + ab) - \log(a+b) - \log x.$$

Clearly, $f(x)$ has a unique and definite value for each $x \in [a, b]$, where $0 < a < x < b$.

So, the value of the function is equal to the limit of the function at each point of $[a, b]$.

$\therefore$ $f(x)$ is continuous on $[a, b]$.

Also, $f'(x) = \left[\frac{2x}{(x^2 + ab)} - \frac{1}{x}\right]$, which is clearly definite and finite for all values of x in $]a, b[$.

So, $f(x)$ is differentiable on $]a, b[$. Moreover, $f(a) = f(b) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist $c \in]a, b[$ such that $f'(c) = 0$.

But, $f'(c) = 0 \Leftrightarrow \frac{2c}{(c^2 + ab)} - \frac{1}{c} = 0$

$$\Leftrightarrow \frac{(c^2 - ab)}{c(c^2 + ab)} = 0 \Leftrightarrow c^2 - ab = 0 \Leftrightarrow c = \sqrt{ab}.$$

Now, c being the geometric mean of a and b, it follows that $a < c < b$.

Thus, $c \in]a, b[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

EXAMPLE 9 *Verify Rolle's theorem for the function* $f(x) = 2x^3 + x^2 - 4x - 2$.

SOLUTION Since a polynomial function is everywhere continuous and differentiable, the given function is continuous as well as differentiable on every interval.

To identify the interval, we solve the equation $f(x) = 0$.

Now, $f(x) = 0 \Leftrightarrow 2x^3 + x^2 - 4x - 2 = 0$

$$\Leftrightarrow (x^2 - 2)(2x + 1) = 0 \Leftrightarrow x^2 = 2 \text{ or } x = -\frac{1}{2}$$

$$\Leftrightarrow x = \sqrt{2} \text{ or } x = -\sqrt{2} \text{ or } x = -\frac{1}{2}\cdot$$

So, we consider the given function in $[-\sqrt{2}, \sqrt{2}]$.

Clearly, $f(-\sqrt{2}) = f(\sqrt{2}) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist $c \in]-\sqrt{2}, \sqrt{2}[$ such that $f'(c) = 0$.

But, $f'(x) = (6x^2 + 2x - 4)$.

$\therefore$ $f'(c) = 0 \Leftrightarrow 6c^2 + 2c - 4 = 0 \Leftrightarrow 2(3c - 2)(c + 1) = 0$

$$\Leftrightarrow c = 2/3 \text{ or } c = -1.$$

Clearly, both these points lie in $[-\sqrt{2}, \sqrt{2}]$.

Hence, Rolle's theorem is verified.

EXAMPLE 10 *Discuss the applicability of Rolle's theorem to the functions:*
(i) $f(x) = x^2$ *in* [1, 2] *(ii)* $f(x) = x^{2/3}$ *in* [–1, 1] **[CBSE 1999]**

SOLUTION (i) Consider $f(x) = x^2$ in [1, 2].

Since a polynomial function is continuous and differentiable everywhere, the first two conditions of Rolle's theorem are satisfied.

But, $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$. So, $f(1) \neq f(2)$.

And, therefore, the condition $f(a) = f(b)$ is violated.

Hence, Rolle's theorem is not applicable for $f(x) = x^2$ in [1, 2].

(ii) Consider $f(x) = x^{2/3}$ in [–1, 1].

We have

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^{2/3} - 0}{h} = \lim_{h\to 0} \frac{1}{h^{1/3}} = \infty.$$

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{(-h)^{2/3} - 0}{-h} = \lim_{h\to 0} \frac{-1}{(h)^{1/3}} = -\infty.$$

$\therefore$ $Rf'(0) \neq Lf'(0)$, i.e., $f'(0)$ does not exist.

Thus, $f(x)$ is not differentiable at $x = 0 \in]-1, 1[$.

So, the condition of differentiability at each point of the given interval is not satisfied.

Hence, Rolle's theorem is not applicable to $f(x) = x^{2/3}$ in [–1, 1].

EXAMPLE 11 *Discuss the applicability of Rolle's theorem on:*
(i) $f(x) = |x|$ *in* [–1, 1] *(ii)* $f(x) = \tan x$ *in* [0, π]

SOLUTION (i) Consider $f(x) = |x|$ in [–1, 1].

We may express it as $f(x) = \begin{cases} -x & \text{when } -1 \leq x < 0 \\ x & \text{when } 0 \leq x \leq 1 \end{cases}$

Clearly, $f(-1) = f(1) = 1$.

But, $Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$

And,

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1.$$

$\therefore$ $Rf'(0) \neq Lf'(0)$.

This shows that $f(x)$ is not differentiable at $x = 0$.

Thus, the condition of differentiability at each point of the given interval is not satisfied.

(ii) Consider $f(x) = \tan x$ in [0, π].

Clearly, $f(0) = f(\pi) = 0$. But, $\tan \frac{\pi}{2}$ does not have a definite value.

So, $f(x) = \tan x$ is not continuous at $x = \frac{\pi}{2}\cdot$

Hence, the condition of continuity at each point of the given interval is violated.

EXAMPLE 12 *Discuss the applicability of Rolle's theorem on the function*

$$f(x) = \begin{cases} (x^2 + 1), & \text{when } 0 \le x \le 1 \\ (3 - x), & \text{when } 1 < x \le 2. \end{cases}$$

SOLUTION The given function has been defined in [0, 2].

And, $f(0) = f(2) = 1$.

Now, we consider the differentiability of $f(x)$ at $x = 1$.

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{[3-(1+h)]-2}{h} = \lim_{h\to 0} \frac{-h}{h} = -1.$$

$$Lf'(1) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0} \frac{(1-h)^2+1-2}{-h} = \lim_{h\to 0} (2-h) = 2.$$

Thus, $Rf'(1) \ne Lf'(1)$.

So, $f(x)$ is not differentiable at $x = 1 \in]0, 2[$.

Thus, the condition of differentiability at each point of the given interval is not satisfied.

EXAMPLE 13 *If Rolle's theorem holds for the function* $f(x) = x^3 + bx^2 + ax + 5$ *on* $[1, 3]$ *with* $c = \left(2 + \frac{1}{\sqrt{3}}\right)$, *find the values of a and b.*

SOLUTION We have, $f'(x) = 3x^2 + 2bx + a$.

$$\therefore \quad f'(c) = 0 \Leftrightarrow 3c^2 + 2bc + a = 0$$

$$\Leftrightarrow c = \frac{-2b \pm \sqrt{4b^2 - 12a}}{6} = \frac{-b \pm \sqrt{b^2 - 3a}}{3}\cdot$$

$$\text{Now, } c = \left(2 + \frac{1}{\sqrt{3}}\right) \Leftrightarrow \frac{-b + \sqrt{b^2 - 3a}}{3} = \left(2 + \frac{1}{\sqrt{3}}\right)$$

$$\Leftrightarrow \frac{-b}{3} = 2 \text{ and } \frac{\sqrt{b^2 - 3a}}{3} = \frac{1}{\sqrt{3}}$$

$$\Leftrightarrow b = -6 \text{ and } b^2 - 3a = 3 \Leftrightarrow b = -6 \text{ and } a = 11.$$

Hence, $a = 11$ and $b = -6$.

EXAMPLE 14 *At what points on the curve* $y = (\cos x - 1)$ *in* $[0, 2\pi]$, *is the tangent parallel to the x-axis?*

SOLUTION Consider $f(x) = (\cos x - 1)$ in $[0, 2\pi]$.

Now, the cosine function being continuous and the constant function being continuous, it follows that their difference $(\cos x - 1)$ is continuous. So, $f(x)$ is continuous on $[0, 2\pi]$.

$f'(x) = -\sin x$, which exists for all $x \in]0, 2\pi[$.

$\therefore \quad f(x)$ is differentiable on $]0, 2\pi[$.

Also, $f(0) = f(2\pi) = 0$.

So, all the conditions of Rolle's theorem are satisfied.

So, there must exist some $c \in]0, 2\pi[$ such that $f'(c) = 0$.

Now, $\quad f'(c) = 0 \Leftrightarrow -\sin c = 0 \Leftrightarrow \sin c = 0$

$\Leftrightarrow c = 0$ or $c = \pi$ or $c = 2\pi$.

Thus, the tangent to the curve is parallel to the x-axis at each of the points $x = 0$, $x = \pi$ and $x = 2\pi$.

Now, $x = 0$ and $y = \cos x - 1 \Rightarrow y = 0$;

$x = \pi$ and $y = \cos x - 1 \Rightarrow y = -2$;

$x = 2\pi$ and $y = \cos x - 1 \Rightarrow y = 0$.

$\therefore$ the required points are $(0, 0)$, $(\pi, -2)$ and $(2\pi, 0)$.

EXERCISE 11C

Verify Rolle's theorem for each of the following functions:

1. $f(x) = x^2$ on $[-1, 1]$
2. $f(x) = x^2 - x - 12$ in $[-3, 4]$ **[CBSE 2001]**
3. $f(x) = x^2 - 5x + 6$ in $[2, 3]$ **[CBSE 2002C]**
4. $f(x) = x^2 - 3x - 18$ in $[-3, 6]$
5. $f(x) = x^2 - 4x + 3$ in $[1, 3]$ **[CBSE 2007]**
6. $f(x) = x(x-4)^2$ in $[0, 4]$
7. $f(x) = x^3 - 7x^2 + 16x - 12$ in $[2, 3]$
8. $f(x) = x^3 + 3x^2 - 24x - 80$ in $[-4, 5]$
9. $f(x) = (x-1)(x-2)(x-3)$ in $[1, 3]$
10. $f(x) = (x-1)(x-2)^2$ in $[1, 2]$ **[CBSE 2006]**
11. $f(x) = (x-2)^4(x-3)^3$ in $[2, 3]$
12. $f(x) = \sqrt{1-x^2}$ in $[-1, 1]$
13. $f(x) = \cos x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
14. $f(x) = \cos 2x$ in $[0, \pi]$
15. $f(x) = \sin 3x$ in $[0, \pi]$
16. $f(x) = \sin x + \cos x$ in $\left[0, \frac{\pi}{2}\right]$
17. $f(x) = e^{-x} \sin x$ in $[0, \pi]$
18. $f(x) = e^{-x}(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$
19. $f(x) = \sin x - \sin 2x$ in $[0, 2\pi]$
20. $f(x) = x(x+2)e^x$ in $[-2, 0]$
21. Show that $f(x) = x(x-5)^2$ satisfies Rolle's theorem on $[0, 5]$ and that the value of c is $(5/3)$.

Discuss the applicability of Rolle's theorem, when:

22. $f(x) = (x-1)(2x-3)$, where $1 \le x \le 3$
23. $f(x) = x^{1/2}$ on $[-1, 1]$
24. $f(x) = 2 + (x-1)^{2/3}$ on $[0, 2]$

25. $f(x) = \cos \frac{1}{x}$ on $[-1, 1]$

26. $f(x) = [x]$ on $[-1, 1]$, where $[x]$ denotes the greatest integer not exceeding x

27. Using Rolle's theorem, find the point on the curve $y = x(x-4)$, $x \in [0, 4]$, where the tangent is parallel to the x-axis. **[CBSE 2000]**

ANSWERS (EXERCISE 11C)

22. Not applicable, since $f(1) \neq f(3)$

23. Not applicable, since $f'(0)$ does not exist

24. Not applicable, since $f'(1)$ does not exist

25. Not applicable, since $f(x)$ is discontinuous at $x = 0$

26. Not applicable, since $f(x)$ is discontinuous at $x = 0$

27. $(2, -4)$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11C)

9. $f(x) = (x^3 - 6x^2 + 11x - 6)$.

22. $f(1) \neq f(3)$.

23. $f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(0)$ does not exist, where $0 \in [-1, 1]$.

24. $f'(1)$ does not exist.

25. $f(x)$ is discontinuous at $x = 0$.

26. $f(x)$ is not continuous at $x = 0$.

Lagrange's Mean-value Theorem

Let f be a real function such that

(i) $f(x)$ is continuous on $[a, b]$, *(ii) $f(x)$ is differentiable on $]a, b[$.*

Then, there exists a real number $c \in]a, b[$ such that $f'(c) = \frac{f(b) - f(a)}{(b-a)}$.

PROOF Let $F(x) = f(x) + Ax$, where A is a constant to be chosen in such a way that $F(a) = F(b)$.

Now, $F(a) = F(b) \Leftrightarrow f(a) + Aa = f(b) + Ab \Leftrightarrow A = \frac{f(b) - f(a)}{(a-b)}$.

$$\therefore \quad F(x) = f(x) + \frac{f(b) - f(a)}{(a-b)} \cdot x \qquad \text{... (i)}$$

Now, $f(x)$ is continuous on $[a, b]$, and $\left\{\frac{f(b) - f(a)}{(a-b)} \cdot x\right\}$ being a polynomial function, is continuous on $[a, b]$.

And, the sum of two continuous functions being continuous, it follows from (i) that $F(x)$ is continuous.

Also, $f(x)$ is differentiable on $]a, b[$, and since the polynomial function $\frac{f(b)-f(a)}{(a-b)}x$ is differentiable on $]a, b[$, it follows that $F(x)$ is differentiable on $]a, b[$.

Also, $F(a) = F(b)$.

Thus, all the conditions of Rolle's theorem are satisfied by $F(x)$.

So, there must exist some $c \in]a, b[$ such that $F'(c) = 0$.

Now, $F(x) = f(x) + \frac{f(b)-f(a)}{(a-b)} \cdot x \Rightarrow F'(x) = f'(x) + \frac{f(b)-f(a)}{(a-b)}$.

$$\therefore \quad F'(c) = 0 \Leftrightarrow f'(c) + \frac{f(b)-f(a)}{(a-b)} = 0 \Leftrightarrow f'(c) = \frac{f(b)-f(a)}{(b-a)}.$$

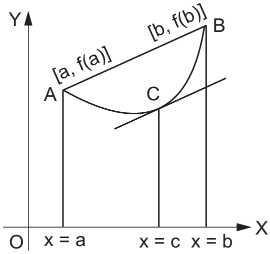
GEOMETRICAL SIGNIFICANCE OF THE MEAN-VALUE THEOREM Let $y = f(x)$ be a given function defined on $[a, b]$, which is continuous on $[a, b]$ and differentiable on $]a, b[$.

Then, by Lagrange's mean-value theorem, there exists some $c \in]a, b[$ such that

$$f'(c) = \frac{f(b)-f(a)}{(b-a)} \qquad \ldots \text{(i)}$$

Now, if we draw the curve $y = f(x)$ and take the points $A[a, f(a)]$ and $B[b, f(b)]$ on the curve then

$$\text{slope of chord } AB = \frac{f(b)-f(a)}{(b-a)} \qquad \ldots \text{(ii)}$$



Thus, from (i) and (ii), we have

$f'(c) =$ slope of chord AB.

This shows that *the tangent to the curve $y = f(x)$ at the point $x = c$ is parallel to the chord AB.*

PHYSICAL SIGNIFICANCE OF THE MEAN-VALUE THEOREM Let a particle be moving in a straight line and let $f(a)$ and $f(b)$ be its positions from the starting point, at the times a and b respectively.

Then, average speed of the particle $= \frac{f(b)-f(a)}{(b-a)}$.

We also know that $f'(c)$ denotes the instantaneous speed of the particle at time c.

Now, by the mean-value theorem, $f'(c) = \frac{f(b)-f(a)}{(b-a)}$.

So, *the mean-value theorem says that at some time c between a and b, the instantaneous speed of the particle would be equal to the average speed.*

SOLVED EXAMPLES

EXAMPLE 1 *Verify Lagrange's mean-value theorem for the following functions:*

(i) $f(x) = x(2-x)$ *in* $[0, 1]$

(ii) $f(x) = x(x+4)^2$ *in* $[0, 4]$

(iii) $f(x) = x + \frac{1}{x}$ *in* $[1, 3]$

(iv) $f(x) = (x-1)(x-2)(x-3)$ *in* $[0, 4]$

(v) $f(x) = \sqrt{x^2 - 4}$ **[CBSE 2002]**

SOLUTION (i) Consider $f(x) = x(2-x)$ in $[0, 1]$.

The given function is $f(x) = 2x - x^2$.

It, being a polynomial function, is continuous on $[0, 1]$.

Also, $f'(x) = 2 - 2x$, which exists for all x in $[0, 1]$.

So, $f(x)$ is differentiable on $]0, 1[$.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in]0, 1[$ such that

$$f'(c) = \frac{f(1) - f(0)}{(1-0)} = 1.$$

Now, $f'(c) = 1 \Leftrightarrow 2 - 2c = 1 \Leftrightarrow c = \frac{1}{2} \in]0, 1[$.

Thus, $c = \frac{1}{2} \in]0, 1[$ such that $f'(c) = \frac{f(1) - f(0)}{1 - 0}$.

Hence, Lagrange's mean-value theorem is verified.

(ii) Consider $f(x) = x(x+4)^2$ in $[0, 4]$.

The given function is $f(x) = x^3 + 8x^2 + 16x$.

It, being a polynomial function, is continuous on $[0, 4]$.

Also, $f'(x) = (3x^2 + 16x + 16)$, which exists for all $x \in]0, 4[$.

So, $f(x)$ is differentiable on $]0, 4[$.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in]0, 4[$ such that

$$f'(c) = \frac{f(4) - f(0)}{(4-0)} = \left(\frac{256 - 0}{4}\right) = 64.$$

Now, $f'(c) = 64 \Leftrightarrow 3c^2 + 16c + 16 = 64$

$$\Leftrightarrow 3c^2 + 16c - 48 = 0 \Leftrightarrow c = \frac{-8 \pm 4\sqrt{13}}{3}.$$

Clearly, $c = \frac{-8 + 4\sqrt{13}}{3} = 2.13 \in]0, 4[$.

Hence, Lagrange's mean-value theorem is verified.

(iii) Consider $f(x) = x + \frac{1}{x}$ in [1, 3]. The given function is

$$f(x) = \frac{x^2+1}{x}.$$

Since it is a rational function such that $x \neq 0$, it is continuous in [1, 3].

Also, $f'(x) = \left(1 - \frac{1}{x^2}\right) = \left(\frac{x^2-1}{x^2}\right)$, which clearly exists for all values of x in]1, 3[.

So, $f(x)$ is differentiable on]1, 3[.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in]1, 3[$ such that

$$f'(c) = \frac{f(3)-f(1)}{(3-1)} = \frac{\left(\frac{10}{3}-2\right)}{2} = \frac{2}{3}.$$

Now, $\quad f'(c) = \frac{2}{3} \Leftrightarrow \frac{c^2-1}{c^2} = \frac{2}{3} \Leftrightarrow c = \sqrt{3} \in]1, 3[.$

Thus, $c = \sqrt{3} \in]1, 3[$ such that $f'(c) = \frac{f(3)-f(1)}{(3-1)}$.

Hence, Lagrange's mean-value theorem is verified.

(iv) Consider $f(x) = (x-1)(x-2)(x-3)$ in [0, 4].

The given function is $f(x) = x^3 - 6x^2 + 11x - 6$.

Being a polynomial function, it is continuous on [0, 4].

Also, $f'(x) = (3x^2 - 12x + 11)$, which exists for all $x \in]0, 4[$.

So, $f(x)$ is differentiable on]0, 4[.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there exists some $c \in]0, 4[$ such that

$$f'(c) = \frac{f(4)-f(0)}{(4-0)} = \frac{6-(-6)}{4} = 3.$$

Now, $f'(c) = 3 \Leftrightarrow 3c^2 - 12c + 11 = 3 \Leftrightarrow 3c^2 - 12c + 8 = 0$

$$\Leftrightarrow c = \frac{12 \pm \sqrt{144-96}}{6} = \frac{6 \pm 2\sqrt{3}}{3}.$$

$$\Leftrightarrow c = 3.155 \text{ or } c = 0.845 \quad [\because \sqrt{3} = 1.732].$$

Clearly, both these values of c lie in]0, 4[.

Hence, Lagrange's mean-value theorem is verified.

(v) Consider $f(x) = \sqrt{x^2-4}$ in [2, 4].

Clearly, $f(x)$ has a definite and unique value for each $x \in [2, 4]$. So, at every point [2, 4], the value of $f(x)$ is equal to the limit of $f(x)$.

So, $f(x)$ is continuous on [2, 4].

Also, $f'(x) = \dfrac{x}{\sqrt{x^2-4}}$, which exists for all $x \in]2, 4[$.

So, $f(x)$ is differentiable on]2, 4[.

So, there must exist some $c \in]2, 4[$ such that

$$f'(c) = \frac{f(4)-f(2)}{(4-2)} = \frac{\sqrt{12}-0}{2} = \sqrt{3}.$$

Now,

$$f'(c) = \sqrt{3} \Leftrightarrow \frac{c}{\sqrt{c^2-4}} = \sqrt{3} \Leftrightarrow c^2 = 3(c^2-4) \Leftrightarrow c = \pm\sqrt{6}.$$

Clearly, $c = \sqrt{6} \in]2, 4[$ such that $f'(c) = \dfrac{f(4)-f(2)}{(4-2)}$.

Hence, Lagrange's mean-value theorem is verified.

EXAMPLE 2 *Verify the hypothesis and conclusion of Lagrange's mean-value theorem for the function* $f(x) = \dfrac{1}{(4x-1)}$, $1 \le x \le 4$.

SOLUTION Clearly, for each $x \in [1, 4]$, $f(x)$ has a definite and unique value.

So, $f(x)$ is continuous on [1, 4].

Also, $f'(x) = \dfrac{-4}{(4x-1)^2}$, which exists for all $x \in]1, 4[$.

So, $f(x)$ is differentiable on]1, 4[.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in]1, 4[$ such that

$$f'(c) = \frac{f(4)-f(1)}{(4-1)} = \frac{1}{3}\left(\frac{1}{15}-\frac{1}{3}\right) = \frac{-4}{45}.$$

Now, $f'(c) = \dfrac{-4}{45} \Leftrightarrow \dfrac{-4}{(4c-1)^2} = \dfrac{-4}{45} \Leftrightarrow (4c-1)^2 = 45$

$$\Leftrightarrow (4c-1) = \pm 3\sqrt{5} \Leftrightarrow c = \left(\frac{1 \pm 3\sqrt{5}}{4}\right).$$

Clearly, $c = \dfrac{1+3\sqrt{5}}{4} = \left(\dfrac{1+3\times 2.23}{4}\right) = 1.92 \in]1, 4[$.

Hence, Lagrange's mean-value theorem is verified.

EXAMPLE 3 *Find 'c' of the mean-value theorem for the functions:*

(i) $f(x) = 2x^2 - 10x + 29$ *in* [2, 7]

(ii) $f(x) = x(x-1)(x-2)$ *in* $\left[0, \dfrac{1}{2}\right]$

SOLUTION (i) $f(x) = 2x^2 - 10x + 29 \Leftrightarrow f'(x) = 4x - 10.$

Now, $f'(c) = \frac{f(7) - f(2)}{(7-2)} \Leftrightarrow (4c - 10) = \frac{(57 - 17)}{5}$

$\Leftrightarrow 4c = 18 \Leftrightarrow c = 4.5 \in]2, 7[.$

$\therefore \; c = 4.5.$

(ii) The given function is $f(x) = x^3 - 3x^2 + 2x.$

$\therefore \; f'(x) = 3x^2 - 6x + 2.$

Now,

$$f'(c) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\left(\frac{1}{2} - 0\right)} \Leftrightarrow 3c^2 - 6c + 2 = \frac{\left(\frac{3}{8} - 0\right)}{\left(\frac{1}{2}\right)}$$

$$\Leftrightarrow 12c^2 - 24c + 5 = 0$$

$$\Leftrightarrow c = \frac{24 \pm \sqrt{576 - 240}}{24} = \frac{6 \pm \sqrt{21}}{6}.$$

Clearly, $c = \frac{6 - \sqrt{21}}{6} = \frac{6 - 4.58}{6} = 0.24 \in \left[0, \frac{1}{2}\right].$

Hence, $c = 0.24.$

EXAMPLE 4 *Using Lagrange's mean-value theorem, find a point on the curve* $y = \sqrt{x-2}$*, defined in the interval* $[2, 3]$*, where the tangent is parallel to the chord joining the end points of the curve.*

SOLUTION Let $f(x) = \sqrt{x-2}.$

Clearly, for each $x \in [2, 3]$, the function $f(x)$ has a definite and unique value.

So, $f(x)$ is continuous on $[2, 3]$.

Also, $f'(x) = \frac{1}{2\sqrt{x-2}}$, which exists for all x in $]2, 3[$.

So, $f(x)$ is differentiable on $]2, 3[$.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in]2, 3[$ such that $f'(c) = \frac{f(3) - f(2)}{(3-2)} = 1.$

Now, $f'(c) = 1 \Leftrightarrow \frac{1}{2\sqrt{c-2}} = 1 \Leftrightarrow c - 2 = \frac{1}{4}$, i.e., $c = \frac{9}{4} \in]2, 3[.$

Now, $x = \frac{9}{4}$ and $y = \sqrt{x-2} \Leftrightarrow y = \sqrt{\left(\frac{9}{4} - 2\right)} = \frac{1}{2}.$

$\therefore$ the tangent to the given curve at the point $\left(\frac{9}{4}, \frac{1}{2}\right)$ is parallel to the chord joining the end points of the curve.

EXAMPLE 5 *Find a point on the parabola* $y=(x-3)^2$, *where the tangent is parallel to the chord joining* $(3, 0)$ *and* $(4, 1)$. **[CBSE 2000C]**

SOLUTION Let us apply Lagrange's mean-value theorem for the function $f(x)=(x-3)^2$ in the interval $[3, 4]$.

Now, $f(x)$ being a polynomial function, it is continuous on $[3, 4]$.

Also, $f'(x)=2(x-3)$, which exists for all $x \in]3, 4[$.

So, $f(x)$ is differentiable on $]3, 4[$.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist a point $c \in]3, 4[$ such that

$$f'(c)=\frac{f(4)-f(3)}{(4-3)}=1.$$

Now, $f'(c)=1 \Leftrightarrow 2(c-3)=1 \Leftrightarrow c=\frac{7}{2} \in]3, 4[$.

Now, $x=\frac{7}{2}$ and $y=(x-3)^2 \Leftrightarrow y=\frac{1}{4}\cdot$

Thus, at the point $\left(\frac{7}{2}, \frac{1}{4}\right)$ on the given curve the tangent is parallel to the chord joining (3, 0) and (4, 1).

EXERCISE 11D

Verify Lagrange's mean-value theorem for each of the following functions:

1. $f(x)=x^2+2x+3$ on $[4, 6]$ **[CBSE 2006]**
2. $f(x)=x^2+x-1$ on $[0, 4]$ **[CBSE 2002]**
3. $f(x)=2x^2-3x+1$ on $[1, 3]$
4. $f(x)=x^3+x^2-6x$ on $[-1, 4]$
5. $f(x)=(x-4)(x-6)(x-8)$ on $[4, 10]$
6. $f(x)=e^x$ on $[0, 1]$
7. $f(x)=x^{2/3}$ on $[0, 1]$
8. $f(x)=\log x$ on $[1, e]$
9. $f(x)=\tan^{-1}x$ on $[0, 1]$
10. $f(x)=\sin x$ on $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$
11. $f(x)=(\sin x+\cos x)$ on $\left[0, \frac{\pi}{2}\right]$
12. Show that Lagrange's mean-value theorem is not applicable to $f(x)=|x|$ on $[-1, 1]$.
13. Show that Lagrange's mean-value theorem is not applicable to $f(x)=\frac{1}{x}$ on $[-1, 1]$.

14. Find 'c' of Lagrange's mean-value theorem for

(i) $f(x) = (x^3 - 3x^2 + 2x)$ on $\left[0, \frac{1}{2}\right]$

(ii) $f(x) = \sqrt{25 - x^2}$ on $[1, 5]$

(iii) $f(x) = \sqrt{x + 2}$ on $[4, 6]$

15. Using Lagrange's mean-value theorem, find a point on the curve $y = x^2$, where the tangent is parallel to the line joining the points (1, 1) and (2, 4). **[CBSE 2000C]**

16. Find a point on the curve $y = x^3$, where the tangent to the curve is parallel to the chord joining the points (1, 1) and (3, 27).

17. Find the points on the curve $y = x^3 - 3x$, where the tangent to the curve is parallel to the chord joining (1, –2) and (2, 2).

18. If $f(x) = x(1 - \log x)$, where $x > 0$, show that

$$(a - b)\log c = b(1 - \log b) - a(1 - \log a), \text{ where } 0 < a < c < b.$$

ANSWERS (EXERCISE 11D)

14. (i) $1 - \sqrt{\frac{7}{12}}$ (ii) $\sqrt{15}$ (iii) 4.964 **15.** $\left(\frac{3}{2}, \frac{9}{4}\right)$

16. $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$ **17.** $\left(\sqrt{\frac{7}{3}}, \frac{-2}{3}\sqrt{\frac{7}{3}}\right), \left(-\sqrt{\frac{7}{3}}, \frac{2}{3}\sqrt{\frac{7}{3}}\right)$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11D)

6. $c = \log(e - 1)$. 8. $2 < e < 3$. 12. $f'(0)$ does not exist.

13. $f(x)$ is discontinuous at $x = 0$.

18. $f(x)$ is continuous for all $x > 0$ and $f'(x) = -\log x$, which exists for all $x > 0$.

$$\therefore \; f'(c) = \frac{f(b) - f(a)}{(b - a)} \Rightarrow -\log c = \frac{b(1 - \log b) - a(1 - \log a)}{(b - a)}.$$

4. Maxima and Minima

ABSOLUTE MAXIMUM VALUE OF A FUNCTION *A function $f(x)$ is said to have the greatest value or absolute maximum value at a point 'a' in its domain if $f(x) \le f(a)$ for all x in the domain of $f(x)$.*

In this case, 'a' is called the *point of maximum.*

ABSOLUTE MINIMUM VALUE OF A FUNCTION *A function $f(x)$ is said to have the smallest value or absolute minimum value at a point 'a' in its domain if $f(a) \le f(x)$ for all x in the domain of $f(x)$.*

The point a in this case, is called the *point of minimum.*

Examples (i) Consider $f(x) = \sin x$ on the interval $[0, \pi]$.

Clearly, the greatest value of the function, i.e., the absolute maximum, is $f(\pi/2) = 1$.

Also, the smallest value of the function, i.e., the absolute minimum, is $f(0) = f(\pi) = 0$.

(ii) Consider $f(x) = x^2$ for all $x \in R$.

It is clear that the smallest value of the function, i.e., the absolute minimum, is 0.

But, we can say nothing about the greatest value of the function. In other words, we can say that the absolute maximum does not exist.

(iii) Consider $f(x) = x^3$ for all $x \in R$.

Here, we observe that as the value of x increases, the value of the function increases, and as x decreases, the value of $f(x)$ decreases.

But, the function has neither a greatest value nor a least value.

Thus, the function has neither an absolute maximum nor an absolute minimum.

(iv) Consider $f(x) = -(x-1)^2 + 5$ for all $x \in R$.

For this function, we obtain the absolute maximum when its negative part is 0, i.e, when $(x-1)^2$ is 0. So, the greatest value of the function is 5.

Clearly, it does not have a least value.

EXAMPLE 1 *Without using the derivative, find the maximum or minimum values, if any, of the function $f(x) = 4x^2 - 4x + 7$ for all $x \in R$.*

SOLUTION We have $f(x) = 4x^2 - 4x + 7 = (2x-1)^2 + 6$.

Clearly, $(2x-1)^2$ is non-negative for all $x \in R$.

The least value of $(2x-1)^2$ is 0.

So, the least value of the function is 6.

Clearly, this happens when $2x - 1 = 0$, i.e., $x = (1/2)$.

Thus, $x = (1/2)$ is a point of absolute minimum. However, $f(x)$ does not have an absolute maximum.

EXAMPLE 2 *Find the maximum or minimum values, if any, of the following functions, without using the derivatives:*

(i) $f(x) = |x+2|$ *(ii)* $f(x) = -|x+1| + 3$

(iii) $f(x) = |\sin(4x+3)|$ *(iv)* $f(x) = \sin(\sin x)$

SOLUTION (i) Clearly, $|x+2|$ is non-negative for all $x \in R$.

The least value of $|x+2|$ is 0.

So, the minimum value of the function is 0.

Clearly, $f(x) = |x+2|$ does not have a maximum value.

(ii) The value of $f(x) = -|x+1| + 3$ is maximum when its negative part has a value 0. Thus, the maximum value of $f(x)$ is 3.

Clearly, there is no minimum value of $f(x)$.

(iii) The maximum value of $|\sin\theta|$ is 1, so the maximum value of $|\sin(4x+3)|$ is 1. Also, the value of $|\sin(4x+3)|$ is non-negative. So, the minimum value of $|\sin(4x+3)|$ is 0.

(iv) $-1 \le \sin x \le 1$

$\Rightarrow \sin(-1) \le \sin(\sin x) \le \sin 1$

$\{\because \sin x \text{ is increasing on } [-1, 1]\}$

$\Rightarrow -\sin 1 \le \sin(\sin x) \le \sin 1.$

This shows that the maximum value of $f(x)$ is $\sin 1$ and the minimum value is $-\sin 1$.

Local Maxima and Local Minima

Here, our main interest lies in finding the maximum and minimum values of a function in a small interval rather than considering it for the whole of the domain. So, we shall deal with the topics of local maxima and local minima.

LOCAL MAXIMUM VALUE OF A FUNCTION *We say that c is a point of local maximum of a function $f(x)$ if there is an open interval I containing c such that $f(x) < f(c)$ for all $x \in I$.*

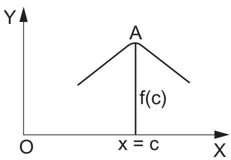
Also, in this case, $f(c)$ is called a *local maximum value* of $f(x)$.

LOCAL MINIMUM VALUE OF A FUNCTION *We say that c is a point of local minimum of a function $f(x)$ if there is an open interval I containing c such that $f(c) < f(x)$ for all $x \in I$.*

BEHAVIOUR OF $f'(x)$ AT LOCAL MAXIMA AND LOCAL MINIMA

Case I *When c is a point of local maxima*

In this case, $f(x)$ is increasing for values of x slightly less than c; it ceases to increase at $x = c$ and then decreases.



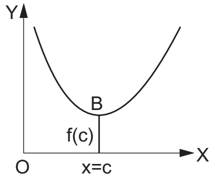
$\therefore$ for small positive values of h, we have

$f'(x) > 0$ on $[c-h, c]$

and, $f'(x) < 0$ on $[c, c+h]$.

Thus, $f'(x)$ changes sign from positive to negative as x increases through c.

Case II *When c is a point of local minima*



In this case $f(x)$ is decreasing for values of x slightly less than c; it ceases to decrease at $x = c$ and then increases for values of x slightly greater than c.

$\therefore$ for small positive values of h, we have

$f'(x) < 0$ on $[c-h, c]$

and, $f'(x) > 0$ on $[c, c+h]$.

Thus, $f'(x)$ changes sign from negative to positive as x increases through c.

EXTREMUM VALUES *The maximum or minimum value of a function is called an extreme or extremum value of the function.*

THEOREM *If $f'(x)$ exists on an interval $[a, b]$ and $f(x)$ has a maximum or minimum value at $x = c \in]a, b[$ then $f'(c) = 0$.*

PROOF The proof is beyond the scope of this book.

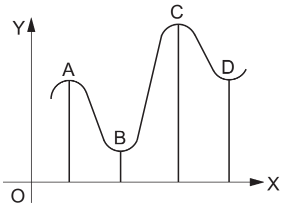
STATIONARY POINTS *The points at which $f'(c) = 0$ are known as stationary points or turning points.*

REMARKS (i) A function can have at the most one absolute maximum and at the most one absolute minimum.

(ii) A function may have more than one local maximum and more than one local minimum.

(iii) A local minimum may be greater than a local maximum.

(iv) Local maximums and local minimums occur alternately.



In the given figure, A and C are the points of local maxima, while B and D are the points of local minima.

Clearly, the local minimum at D is greater than the local maximum at A.

Tests for Finding Local Extremum Values

We must remember that

(i) c is a point of local maximum if $f'(c) = 0$ and $f'(x)$ changes sign from positive to negative as x increases through c

(ii) c is a point of local minimum if $f'(c) = 0$ and $f'(x)$ changes sign from negative to positive as x increases through c.

FIRST-DERIVATIVE TEST

Working Rule for Finding Extremum Values

In order to find the extremum values of a given function $f(x)$ by using the first-derivative test, we proceed according to the following steps.

1. Find $f'(x)$.
2. Solve $f'(x) = 0$. Let its roots be $a, b, c,$ etc. Then, these are the candidates for maxima or minima.
3. We test the function at each one of these values. Let us take $x = c$.
4. Determine the sign of $f'(x)$ for values of x slightly less than c and for values of x slightly greater than c.

CONCLUSION (i) If $f'(x)$ changes sign from positive to negative as x increases through c then $x = c$ is a point of maximum.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c then $x = c$ is a point of minimum.

(iii) If $f'(x)$ does not change sign as x increases through c, we say that x is neither a point of maximum nor a point of minimum.
In this case, $x = c$ is called a *point of inflexion.*

Similarly, we may deal with the other roots of $f'(x) = 0$ and examine them for maxima or minima.

SOLVED EXAMPLES

EXAMPLE 1 *Find the local maxima or local minima, if any, of*

(i) $f(x) = \dfrac{1}{(x^2+2)}$ *(ii)* $f(x) = (x^3 - 3x)$

In each case, find the local maximum or the local minimum values, as the case may be.

SOLUTION (i) $f(x) = \dfrac{1}{(x^2+2)} \Rightarrow f'(x) = \dfrac{-2x}{(x^2+2)^2}\cdot$

For a local maxima or minima, we have $f'(x) = 0$.

$\therefore\ f'(x) = 0 \Rightarrow \dfrac{-2x}{(x^2+2)^2} = 0 \Rightarrow x = 0.$

Now, when x is slightly less than 0, i.e., when x is negative, then $f'(x) = \dfrac{-2x}{(x^2+2)^2}$ is positive.

When x is slightly greater than 0 then $f'(x) = \dfrac{-2x}{(x^2+2)^2} < 0.$

Thus, $f'(x)$ changes sign from positive to negative.

So, $x = 0$ is a point of local maximum.

And, local maximum value is $f(0) = \dfrac{1}{(0^2+2)} = \dfrac{1}{2}\cdot$

(ii) $f(x) = (x^3 - 3x) \Rightarrow f'(x) = (3x^2 - 3).$

Now, for a maximum or a minimum, we have $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow 3x^2 - 3 = 0$

$\Rightarrow 3(x-1)(x+1) = 0 \Rightarrow x = 1$ or $x = -1.$

Thus, $x = 1$ and $x = -1$ are the candidates for local maxima or local minima.

Consider $x = 1$.

We know that $f'(x) = 3(x^2 - 1)$.

When x is slightly less than 1 then x^2 is slightly less than 1 and therefore, $3(x^2 - 1)$ is negative.

Again, when x is slightly more than 1 then x^2 is slightly more than 1 and therefore, $3(x^2 - 1)$ is positive.

Thus, $f'(x)$ changes values from negative to positive as x increases through 1.

So, $x = 1$ is a point of local minimum.

Local minimum value $= f(1) = (1^3 - 3 \times 1) = -2$.

Consider $x = -1$.

When x is slightly less than -1 then x^2 is slightly more than 1. So, $3(x^2 - 1)$ is positive.

Again, when x is slightly more than -1 then x^2 is slightly less than 1. So, $3(x^2 - 1)$ is negative.

Thus, $f'(x)$ changes sign from positive to negative as x increases through -1.

So, $x = -1$ is a point of local maximum.

Local maximum value $= f(-1) = (-1)^3 - 3 \times (-1) = 2$.

EXAMPLE 2 *Find the local maxima or local minima of* $f(x) = x^3 - 6x^2 + 9x + 15$. *Also, find the local maximum or local minimum values as the case may be.*

SOLUTION Here, $f(x) = x^3 - 6x^2 + 9x + 15 \Rightarrow f'(x) = 3x^2 - 12x + 9$.

For a local maxima or minima, we must have $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0$

$\Rightarrow 3(x-3)(x-1) = 0 \Rightarrow x = 3$ or $x = 1$.

Case I *When* $x = 3$

In this case, when x is slightly less than 3 then $f'(x) = 3(x-3)(x-1)$ is negative and when x is slightly more than 3 then $f'(x)$ is positive.

Thus, $f'(x)$ changes from negative to positive as x increases through 3.

So, $x = 3$ is a point of local minimum.

Local minimum value $= f(3) = 15$.

Case II *When* $x = 1$

In this case, when x is slightly less than 1 then $f'(x) = 3(x-3)(x-1)$ is positive and when x is slightly more than 1 then $f'(x)$ is negative.

Thus, $f'(x)$ changes sign from positive to negative as x increases through 1.

So, $x = 1$ is a point of local maximum.

Local maximum value $= f(1) = 19$.

EXAMPLE 3 *Show that* $(x^5 - 5x^4 + 5x^3 - 1)$ *has a local maximum when* $x = 1$, *a local minimum when* $x = 3$, *and neither when* $x = 0$.

SOLUTION Let $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Then,

$$f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3) = 5x^2(x - 3)(x - 1).$$

For a local maxima or minima, we have $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow 5x^2(x - 3)(x - 1) = 0 \Rightarrow x = 0$ or $x = 3$ or $x = 1$.

Case I *When* $x = 1$

In this case, when x is slightly less than 1 then $f'(x) = 5x^2(x - 3)(x - 1)$ is positive and when x is slightly more than 1 then $f'(x)$ is negative.

Thus, $f'(x)$ changes sign from positive to negative.

So, $x = 1$ is a point of local maximum.

Case II *When* $x = 3$

Clearly, when x is slightly less than 3 then $f'(x) = 5x^2(x - 3)(x - 1)$ is negative and when x is slightly greater than 3 then $f'(x)$ is positive.

Thus, $f'(x)$ changes sign from negative to positive.

So, $x = 3$ is a point of local minimum.

Case III *When* $x = 0$

Clearly, when x is slightly less than 0 then $f'(x) = 5x^2(x - 3)(x - 1)$ is positive and also when x is slightly more than 0 then $f'(x)$ is positive.

Thus, $f'(x)$ does not change sign as it passes through 0.

So, it is neither a point of local maximum nor a point of local minimum, i.e., $x = 0$ is a point of inflexion.

EXAMPLE 4 *Find the points of local maxima and local minima as well as the corresponding local maximum and local minimum values for the function*

$$f(x) = (x - 1)^3(x + 1)^2.$$

SOLUTION $f(x) = (x-1)^3(x+1)^2 \Rightarrow f'(x) = 2(x-1)^3(x+1) + 3(x-1)^2(x+1)^2$

$$= (x-1)^2(x+1)(5x+1).$$

$\therefore\ f'(x) = 0 \Rightarrow (x-1)^2(x+1)(5x+1) = 0$

$$\Rightarrow x = 1 \text{ or } x = -1 \text{ or } x = -\frac{1}{5}\cdot$$

Consider $x = 1$.

When x is slightly less than 1 then clearly $f'(x)$ is +ve. When x is slightly greater than 1, then clearly $f'(x)$ is +ve. Thus, $f'(x)$ does not change sign as x passes through 1.

So, $x = 1$ is neither a point of maximum nor a point of minimum.

Consider $x = -1$.

When x is slightly less than -1 then $f'(x) = (+)(-)(-) = +$ve.

When x is slightly greater than -1 then $f'(x) = (+)(+)(-) = -$ve.

Thus, $f'(x)$ changes sign from +ve to –ve as x passes through –1.

So, $x = -1$ is a point of local maximum.

Local maximum value $= f(-1) = (-2)^3(-1+1)^2 = 0$.

Consider $x = -\frac{1}{5}\cdot$

When x is slightly less than $-\frac{1}{5}$ then $f'(x) = (+)(+)(-) = -$ve.

When x is slightly greater than $-\frac{1}{5}$ then $f'(x) = (+)(+)(+) = +$ve.

Thus, $f'(x)$ changes sign from –ve to +ve as x passes through $-\frac{1}{5}\cdot$

So, $x = -\frac{1}{5}$ is a point of local minimum.

Local minimum value $= f\left(-\frac{1}{5}\right) = \frac{-3456}{3125}\cdot$

SECOND-DERIVATIVE TEST

Working Rule for Finding Extremum Values

1. Find $f'(x)$.
2. Solve $f'(x) = 0$.
   Each value of x so obtained is a candidate for maximum or minimum.
   Let $x = c$ be one of its points.
3. Find $f''(x)$ and put $x = c$ to get $f''(c)$.
   Now, if $f''(c) < 0$ then $x = c$ is a point of local maximum;
   if $f''(c) > 0$ then $x = c$ is a point of local minimum;
   if $f''(c) = 0$ then use the first-derivative test.

EXAMPLE 5 *Find all the points of local maxima and minima and the corresponding maximum and minimum values of the function*

$$f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105.$$

SOLUTION $f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105.$

$\therefore\ f'(x) = -3x^3 - 24x^2 - 45x = -3x(x^2 + 8x + 15).$

Now, $f'(x) = 0 \Rightarrow -3x(x^2 + 8x + 15) = 0 \Rightarrow -3x(x+5)(x+3) = 0$

$$\Rightarrow x = 0 \text{ or } x = -5 \text{ or } x = -3.$$

Thus, $x = 0$, $x = -5$ and $x = -3$ are the candidates for local maxima or minima.

Moreover, $f''(x) = (-9x^2 - 48x - 45)$.

Case I *When* $x = 0$

We have $f''(0) = -45 < 0$.

So, $x = 0$ is a point of local maximum.

And, local maximum value at $x = 0$ is $f(0) = 105$.

Case II *When* $x = -5$

We have $f''(-5) = -30 < 0$.

So, $x = -5$ is a point of local maximum.

And, local maximum value at $x = -5$ is $f(-5) = \frac{295}{4}\cdot$

Case III *When* $x = -3$

We have $f''(-3) = 18 > 0$.

So, $x = -3$ is a point of local minimum.

Local minimum value at $x = -3$ is $f(-3) = \frac{231}{4}\cdot$

EXAMPLE 6 *Find the local maxima and local minima of the functions:*

(i) $f(x) = \sin 2x$, *where* $0 < x < \pi$

(ii) $f(x) = (\sin 2x - x)$, *where* $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

SOLUTION (i) $f(x) = \sin 2x \Rightarrow f'(x) = 2\cos 2x.$

$\therefore\ f'(x) = 0 \Rightarrow 2\cos 2x = 0, \text{ i.e., } \cos 2x = 0$

$$\Rightarrow 2x = \frac{\pi}{2} \text{ or } 2x = \frac{3\pi}{2} \Rightarrow x = \frac{\pi}{4} \text{ or } x = \frac{3\pi}{4}\cdot$$

Thus, $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ are the candidates for local maxima or minima.

Moreover, $f'(x) = 2\cos 2x \Rightarrow f''(x) = -4\sin 2x.$

Case I *When* $x = (\pi/4)$

We have $f''\left(\frac{\pi}{4}\right) = -4\sin\frac{\pi}{2} = -4 < 0.$

$\therefore \quad x = \frac{\pi}{4}$ is a point of local maximum.

Local maximum value $= f\left(\frac{\pi}{4}\right) = 1.$

Case II *When* $x = (3\pi/4)$

We have $f''(3\pi/4) = -4\sin\frac{3\pi}{2} = 4 > 0.$

$\therefore \quad x = \frac{3\pi}{4}$ is a point of local minimum.

Local minimum value $= f\left(\frac{3\pi}{4}\right) = -1.$

(ii) $f(x) = (\sin 2x - x) \Rightarrow f'(x) = (2\cos 2x - 1)$ and $f''(x) = -4\sin 2x.$

$\therefore \quad f'(x) = 0 \Rightarrow (2\cos 2x - 1) = 0 \Rightarrow \cos 2x = \frac{1}{2}$

$\Rightarrow \quad 2x = -\frac{\pi}{3}$ or $2x = \frac{\pi}{3} \Rightarrow x = \frac{-\pi}{6}$ or $x = \frac{\pi}{6}\cdot$

Thus, $x = \frac{-\pi}{6}$ and $x = \frac{\pi}{6}$ are the candidates for local maxima or local minima.

Case I *When* $x = -(\pi/6)$

We have $f''\left(-\frac{\pi}{6}\right) = 4\sin\frac{\pi}{3} = 2\sqrt{3} > 0.$

So, $x = \frac{-\pi}{6}$ is a point of local minimum.

The local minimum value

$$= f\left(-\frac{\pi}{6}\right) = \left(-\sin\frac{\pi}{3} - \frac{\pi}{6}\right) = -\left(\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right)\cdot$$

Case II *When* $x = \frac{\pi}{6}$

We have $f''\left(\frac{\pi}{6}\right) = -4\sin\left(\frac{\pi}{3}\right) = -2\sqrt{3} < 0.$

$\therefore \quad x = \frac{\pi}{6}$ is a point of local maximum.

And, the local maximum value

$$= f\left(\frac{\pi}{6}\right) = \left(\sin\frac{\pi}{3} - \frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)\cdot$$

EXAMPLE 7 *Find the local maxima and local minima of the functions:*

(i) $f(x) = (\sin x - \cos x)$, *where* $0 < x < \frac{\pi}{2}$

(ii) $f(x) = (2\cos x + x)$, *where* $0 < x < \pi$

SOLUTION (i) $f(x) = (\sin x - \cos x)$.

$\therefore \quad f'(x) = (\cos x + \sin x)$ and $f''(x) = (-\sin x + \cos x)$.

Now, $f'(x) = 0 \Rightarrow \cos x + \sin x = 0$

$$\Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}.$$

Thus, $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ are the candidates for local maxima or local minima.

Case I *When* $x = (3\pi/4)$

$$f''\left(\frac{3\pi}{4}\right) = \left(-\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \frac{-2}{\sqrt{2}}$$

$$= -\sqrt{2} < 0.$$

So, $x = (3\pi/4)$ is a point of local maximum.

Local maximum value $= f\left(\frac{3\pi}{4}\right) = \sqrt{2}$.

Case II *When* $x = (7\pi/4)$

$$f''\left(\frac{7\pi}{4}\right) = f''\left(-\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}\right) = \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right)$$

$$= \sqrt{2} > 0.$$

So, $x = (7\pi/4)$ is a point of local minimum.

Local minimum value $= f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$

$$= \left(-\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) = -\sqrt{2}.$$

(ii) $f(x) = (2\cos x + x)$.

$\therefore \quad f'(x) = (-2\sin x + 1)$ and $f''(x) = -2\cos x$.

Now, $f'(x) = 0 \Rightarrow -2\sin x + 1 = 0$

$$\Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6} \text{ or } x = \frac{5\pi}{6} \text{ in }]0, \pi[.$$

Thus, $x = (\pi/6)$ and $x = (5\pi/6)$ are the candidates for local maxima or local minima.

Case I *When* $x = (\pi/6)$

We have $f''\left(\frac{\pi}{6}\right) = -2\cos\frac{\pi}{6} = \left(-2 \times \frac{\sqrt{3}}{2}\right) = -\sqrt{3} < 0.$

$\therefore \quad x = (\pi/6)$ is a point of local maximum.

Local maximum value $= f\left(\frac{\pi}{6}\right) = \left(\sqrt{3} + \frac{\pi}{6}\right)$.

Case II *When* $x = (5\pi/6)$

We have $f''\left(\frac{5\pi}{6}\right) = -2\cos\frac{5\pi}{6} = 2\cos\frac{\pi}{6} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3} > 0.$

So, $x = (5\pi/6)$ is a point of local minimum.

Local minimum value $= f\left(\frac{5\pi}{6}\right) = \left(\frac{5\pi}{6} - \sqrt{3}\right)\cdot$

EXAMPLE 8 *Find the points of local maxima or local minima of the function*

$$f(x) = (\sin^4 x + \cos^4 x) \;\; in \;\; 0 < x < \frac{\pi}{2}\cdot$$

SOLUTION $f(x) = \sin^4 x + \cos^4 x$

$\Rightarrow f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$

$= -4\sin x \cos x(\cos^2 x - \sin^2 x) = -2\sin 2x \cos 2x = -\sin 4x.$

And, $f''(x) = -4\cos 4x.$

Now, $f'(x) = 0 \Rightarrow -\sin 4x = 0 \Rightarrow 4x = \pi$, i.e., $x = \frac{\pi}{4}\cdot$

$\therefore$ $x = (\pi/4)$ is a point of local maximum or local minimum.

Now, $f''\left(\frac{\pi}{4}\right) = -4\cos \pi = 4 > 0.$

$\therefore$ $x = (\pi/4)$ is a point of local minimum.

Local minimum value $= f\left(\frac{\pi}{4}\right) = \frac{1}{2}\cdot$

EXAMPLE 9 *Find the local maxima and local minima, and the corresponding local maximum and local minimum values of the following functions:*

(i) $f(x) = x\sqrt{1-x}$, *where* $x > 0$

(ii) $f(x) = \frac{x}{(x-1)(x-4)}$, *where* $1 < x < 4$

SOLUTION (i) $f(x) = x\sqrt{1-x}.$

$\therefore$ $f'(x) = \frac{(2-3x)}{2\sqrt{1-x}}$ and $f''(x) = \frac{(3x-4)}{4(1-x)^{3/2}}\cdot$

Now, $f'(x) = 0 \Rightarrow (2-3x) = 0 \Rightarrow x = \frac{2}{3}$;

and, $f''\left(\frac{2}{3}\right) = \frac{\left(3 \times \frac{2}{3} - 4\right)}{4\left(1 - \frac{2}{3}\right)^{3/2}} = \frac{-3^{3/2}}{2} < 0.$

$\therefore$ $x = \frac{2}{3}$ is a point of local maximum.

Local maximum value $= f\left(\frac{2}{3}\right) = \frac{2}{3\sqrt{3}}\cdot$

(ii) $f(x) = \frac{x}{(x-1)(x-4)} = \frac{x}{(x^2 - 5x + 4)}\cdot$

$\therefore$ $f'(x) = \frac{(4 - x^2)}{(x^2 - 5x + 4)^2}\cdot$

And,

$$f''(x) = \frac{(x^2 - 5x + 4)^2(-2x) - (4 - x^2)2(x^2 - 5x + 4)(2x - 5)}{(x^2 - 5x + 4)^4}\cdot$$

$$f'(x) = 0 \Rightarrow (4 - x^2) = 0 \Rightarrow x = 2 \quad [\because 1 < x < 4].$$

$$f''(2) = \frac{-16}{16} = -1 < 0.$$

$\therefore$ $x = 2$ is a point of local maximum.

Local maximum value $= f(2) = -1$.

EXAMPLE 10 *Prove that the maximum value of* $\left(\frac{1}{x}\right)^x$ *is* $e^{1/e}$.

SOLUTION Let $y = \left(\frac{1}{x}\right)^x = x^{-x}$. Then, $\log y = -x \log x$.

$$\therefore \quad \frac{1}{y} \cdot \frac{dy}{dx} = -(1 + \log x) \quad \text{or} \quad \frac{dy}{dx} = -y(1 + \log x).$$

And, $\frac{d^2y}{dx^2} = -y \cdot \frac{1}{x} - (1 + \log x) \cdot \frac{dy}{dx} = -\left(\frac{1}{x}\right)^{x+1} - (1 + \log x) \cdot \frac{dy}{dx}$.

Now, $\frac{dy}{dx} = 0 \Rightarrow 1 + \log x = 0$

$$\Rightarrow \log x = -1 = -\log e = \log(1/e) \Rightarrow x = (1/e).$$

Also, $\frac{d^2y}{dx^2}$ at $x = (1/e)$ is $-e^{1+(1/e)} < 0$ $\quad \left[\because \frac{dy}{dx} = 0\right]$.

So, $x = (1/e)$ is a point of local maximum.

Local maximum value = {value of y when $x = (1/e)$} $= e^{1/e}$.

EXAMPLE 11 *Find the point on the parabola* $y^2 = 2x$ *which is closest to the point* (1, 4).

SOLUTION Let $A(x, y)$ be the required point which is closest to the point $B(1, 4)$. Then, the distance AB should be minimum. And, therefore AB^2 should be minimum.

Now, $AB^2 = (x-1)^2 + (y-4)^2 = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2 = \frac{(y^4 - 32y + 68)}{4}$.

Let $f(y) = \frac{y^4 - 32y + 68}{4}$.

Then, $f'(y) = \frac{4y^3 - 32}{4} = (y^3 - 8)$. And, $f''(y) = 3y^2$.

Now, $f'(y) = 0 \Rightarrow y^3 - 8 = 0 \Rightarrow y = 2$. Also, $f''(2) = 3 \times 4 = 12 > 0$.

So, $y = 2$ is a point of minimum.

Now, $y = 2 \Rightarrow x = \frac{y^2}{2} = \frac{4}{2} = 2$. So, the required point is (2, 2).

EXAMPLE 12 *Show that none of the following functions has a maxima or minima.*

(i) e^x *(ii)* $\log x$ *(iii)* $x^3 + x^2 + x + 1$

SOLUTION (i) $f(x) = e^x \Rightarrow f'(x) = e^x$.

For a maxima or a minima, we must have $f'(x) = 0$.

But, $f'(x) = 0 \Rightarrow e^x = 0$, which is not possible.

So, $f(x) = e^x$ does not have a maxima or a minima.

(ii) $f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}\cdot$

For a maxima or a minima, we must have $f'(x) = 0$.

But, $f'(x) = 0 \Rightarrow \frac{1}{x} = 0$, which is not possible for any value of x.

$\therefore$ $f(x) = \log x$ does not have a maxima or a minima.

(iii) $f(x) = x^3 + x^2 + x + 1 \Rightarrow f'(x) = 3x^2 + 2x + 1$.

For a maxima or a minima, we must have $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow 3x^2 + 2x + 1 = 0$

$$\Rightarrow x = \frac{-2 \pm \sqrt{4-12}}{6}, \text{ which are imaginary.}$$

Thus, $f'(x) \neq 0$ for any real value of x.

Hence, $f(x)$ does not have a maxima or a minima.

EXAMPLE 13 *Show that* $\sin^p\theta \cos^q\theta$ *attains a maximum when* $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$.

SOLUTION Let $y = \sin^p\theta \cos^q\theta$.

Then, $$\frac{dy}{d\theta} = p\sin^{p-1}\theta\cos^{q+1}\theta - q\cos^{q-1}\theta\sin^{p+1}\theta$$
$$= (\sin^{p-1}\theta\cos^{q-1}\theta)(p\cos^2\theta - q\sin^2\theta).$$

Now, for maxima or minima, we have $\frac{dy}{d\theta} = 0$.

But $\frac{dy}{d\theta} = 0 \Rightarrow \sin^{p-1}\theta = 0$ or $\cos^{q-1}\theta = 0$ or $p\cos^2\theta - q\sin^2\theta = 0$

$$\Rightarrow \theta = 0 \text{ or } \theta = \frac{\pi}{2} \text{ or } \theta = \tan^{-1}\sqrt{\frac{p}{q}}\cdot$$

Moreover, we may write

$$\frac{dy}{d\theta} = \frac{y(p\cos^2\theta - q\sin^2\theta)}{\sin\theta\cos\theta} = y(p\cot\theta - q\tan\theta).$$

$$\therefore \quad \frac{d^2y}{d\theta^2} = y(-p\,\text{cosec}^2\theta - q\sec^2\theta) + (p\cot\theta - q\tan\theta)\cdot\frac{dy}{d\theta}\cdot$$

Thus, at $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$, we have $\frac{dy}{d\theta} = 0$ and therefore,

$$\left[\frac{d^2y}{d\theta^2} \text{ at } \theta = \tan^{-1}\sqrt{\frac{p}{q}}\right] = \sin^p\theta\cos^q\theta(-p\,\text{cosec}^2\theta - q\sec^2\theta) < 0.$$

Hence, y is maximum when $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$.

MAXIMA AND MINIMA ON A CLOSED INTERVAL Let $f(x)$ be a given function defined on $[a, b]$. Let the local minimum value of $f(x)$ be m, and let the local maximum value of $f(x)$ be M.

Then, *minimum value of* $f(x)$ *on* $[a, b]$ *is the smallest of* m, $f(a)$ *and* $f(b)$.

The maximum value of $f(x)$ *on* $[a, b]$ *is the greatest of* M, $f(a)$ *and* $f(b)$.

EXAMPLE 14 *Find the maximum and minimum values of*

$$(3x^4 - 8x^3 + 12x^2 - 48x + 25) \quad on \quad [0, 3].$$

SOLUTION Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$.

Then, $f'(x) = 12x^3 - 24x^2 + 24x - 48$

and $f''(x) = 36x^2 - 48x + 24$.

Now, $f'(x) = 0 \Rightarrow 12(x^3 - 2x^2 + 2x - 4) = 0$

$\Rightarrow 12(x-2)(x^2+2) = 0$

$\Rightarrow x = 2$ [neglecting the imaginary values].

And, $f''(2) = 72 > 0$.

So, $x = 2$ is a point of local minima.

Now, $f(2) = -39$; $f(0) = 25$ and $f(3) = 16$.

$\therefore$ minimum value of $f(x)$ on $[0, 3]$ is -39 at $x = 2$.

Maximum value of $f(x)$ on $[0, 3]$ is 25 at $x = 0$.

EXAMPLE 15 *Find the maximum and minimum values of* $(x + \sin 2x)$ *on* $[0, 2\pi]$.

SOLUTION Let $f(x) = (x + \sin 2x)$.

Then, $f'(x) = (1 + 2\cos 2x)$ and $f''(x) = -4\sin 2x$.

Now, $f'(x) = 0 \Rightarrow \cos 2x = -\frac{1}{2}$, where $x \in [0, 2\pi]$

$$\Rightarrow 2x = \frac{2\pi}{3} \text{ or } 2x = \frac{4\pi}{3} \Rightarrow x = (\pi/3) \text{ or } x = (2\pi/3).$$

$$f''\left(\frac{\pi}{3}\right) = -4\sin\left(\frac{2\pi}{3}\right) = -4\sin\frac{\pi}{3} = -2\sqrt{3} < 0.$$

$\therefore$ $x = (\pi/3)$ is a point of local maxima.

Now, $f\left(\frac{\pi}{3}\right) = \frac{2\pi + 3\sqrt{3}}{6}$, $f(0) = 0$ and $f(2\pi) = 2\pi$.

$\therefore$ maximum value of $f(x)$ is 2π at $x = 2\pi$.

Again, we consider the point $x = (2\pi/3)$.

$$f''\left(\frac{2\pi}{3}\right) = -4\sin\left(2 \times \frac{2\pi}{3}\right) = -4\sin\frac{4\pi}{3} = 4\sin\frac{\pi}{3} = 2\sqrt{3} > 0.$$

So, $x = \frac{2\pi}{3}$ is a point of local minima.

Now, $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \sin\frac{\pi}{3} = \frac{4\pi - 3\sqrt{3}}{6}$;

$f(0) = 0$ and $f(2\pi) = 2\pi + \sin 4\pi = 2\pi$.

$\therefore$ the minimum value of $f(x)$ is 0 at $x = 0$.

EXAMPLE 16 *Show that* $\sin x(1 + \cos x)$, $x \in [0, \pi]$ *is maximum at* $x = (\pi/3)$.

SOLUTION Let $f(x) = \sin x(1 + \cos x)$.

Then, $f'(x) = -\sin^2 x + \cos x(1 + \cos x)$

$= 2\cos^2 x + \cos x - 1 = (2\cos x - 1)(\cos x + 1).$

And, $f''(x) = (-4\cos x \sin x - \sin x) = -\sin x(1 + 4\cos x).$

Now, $f'(x) = 0 \Rightarrow (2\cos x - 1)(\cos x + 1) = 0$

$\Rightarrow \cos x = \frac{1}{2}$ or $\cos x = -1 \Rightarrow x = \frac{\pi}{3}$ or $x = \pi.$

But, $f''\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3}\left(1 + 4\cos\frac{\pi}{3}\right) = \frac{-3\sqrt{3}}{2} < 0.$

$\therefore$ $x = (\pi/3)$ is a point of local maximum.

Now, $f(0) = 0$, $f(\pi) = 0$ and $f\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{4}\cdot$

$\therefore$ the maximum value of $f(x)$ is $\frac{3\sqrt{3}}{4}$ at $x = \frac{\pi}{3}\cdot$

EXERCISE 11E

Find the maximum or minimum values, if any, without using derivatives, of the functions:

1. $(5x-1)^2 + 4$
2. $-(x-3)^2 + 9$
3. $-|x+4| + 6$
4. $\sin 2x + 5$
5. $|\sin 4x + 3|$

Find the points of local maxima or local minima and the corresponding local maximum and minimum values of each of the following functions:

6. $f(x) = (x-3)^4$
7. $f(x) = x^2$
8. $f(x) = 2x^3 - 21x^2 + 36x - 20$
9. $f(x) = x^3 - 6x^2 + 9x + 15$
10. $f(x) = x^4 - 62x^2 + 120x + 9$
11. $f(x) = -x^3 + 12x^2 - 5$
12. $f(x) = (x-1)(x+2)^2$
13. $f(x) = -(x-1)^3(x+1)^2$
14. $f(x) = \frac{x}{2} + \frac{2}{x}, x > 0$
15. Find the maximum and minimum values of $2x^3 - 24x + 107$ on the interval $[-3, 3]$.
16. Find the maximum and minimum values of $3x^4 - 8x^3 + 12x^2 - 48x + 1$ on the interval $[1, 4]$.
17. Find the maximum and minimum values of

$$f(x) = \left(\sin x + \frac{1}{2}\cos x\right) \text{ in } 0 \le x \le \frac{\pi}{2}\cdot$$

18. Show that the maximum value of $x^{1/x}$ is $e^{1/e}$.
19. Show that $\left(x + \frac{1}{x}\right)$ has a maximum and minimum, but the maximum value is less than the minimum value.

20. Find the maximum profit that a company can make, if the profit function is given by $p(x) = 41 + 24x - 18x^2$.

21. An enemy jet is flying along the curve $y = (x^2 + 2)$. A soldier is placed at the point $(3, 2)$. Find the nearest point between the soldier and the jet.

22. Find the maximum and minimum values of

$$f(x) = (-x + 2\sin x) \text{ on } [0, 2\pi].$$

ANSWERS (EXERCISE 11E)

1. min. value = 4 **2.** max. value = 9 **3.** max. value = 6

4. max. value = 4, min. value = 6 **5.** max. value = 4, min. value = 2

6. local max. value is 0 at $x = 3$ **7.** local min. value is 0 at $x = 0$

8. local max. value is –3 at $x = 1$ and local min. value is – 128 at $x = 6$

9. local max. value is 19 at $x = 1$ and local min. value is 15 at $x = 3$

10. local max. value is 68 at $x = 1$ and local min. values are –1647 at $x = -6$ and –316 at $x = 5$

11. local max. value is 251 at $x = 8$ and local min. value is –5 at $x = 0$

12. local max. value is 0 at $x = -2$ and local min. value is –4 at $x = 0$

13. local max. value is 0 at each of the points $x = 1$ and $x = -1$ and local min. value is $\frac{-3456}{3125}$ at $x = -\frac{1}{5}$

14. local min. value is 2 at $x = 2$

15. max. value is 139 at $x = -2$ and min. value is 89 at $x = 3$

16. max. value is 257 at $x = 4$ and min. value is –63 at $x = 2$

17. max. value is $\frac{3}{4}$ at $x = \frac{\pi}{6}$ and min. value is $\frac{1}{2}$ at $x = \frac{\pi}{2}$

20. 49 **21.** (1, 3)

22. max. value is $\left(-\frac{\pi}{3} + \sqrt{3}\right)$ at $x = \frac{\pi}{3}$ and min. value is $\left(\frac{5\pi}{3} + \sqrt{3}\right)$ at $x = \frac{5\pi}{3}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11E)

4. $-1 \le \sin 2x \le 1$.
5. $-1 \le \sin 4x \le 1 \Rightarrow (-1 + 3) \le (\sin 4x + 3) \le (1 + 3)$.
6. Use the first-derivative test.
21. Find a point $(x, x^2 + 2)$ nearest to (3, 2).

SOME MORE SOLVED EXAMPLES

EXAMPLE 1 *Amongst all pairs of positive numbers with sum 24, find those whose product is maximum.*

SOLUTION Let the numbers be x and $(24-x)$.

Let $P = x(24-x) = (24x - x^2)$.

Then, $\frac{dP}{dx} = (24-2x)$ and $\frac{d^2P}{dx^2} = -2$.

Now, $\frac{dP}{dx} = 0 \Rightarrow (24-2x) = 0 \Rightarrow x = 12$.

Thus, $\left\{\frac{d^2P}{dx^2}\right\}_{x=12} = -2 < 0$.

$\therefore$ $x = 12$ is a point of maximum.

Hence, the required numbers are 12 and 12.

EXAMPLE 2 *Amongst all pairs of positive numbers with product 256, find those whose sum is the least.*

SOLUTION Let the required numbers be x and $\frac{256}{x}\cdot$

Let $S = \left(x + \frac{256}{x}\right)\cdot$ Then, $\frac{dS}{dx} = \left(1 - \frac{256}{x^2}\right)$ and $\frac{d^2S}{dx^2} = \frac{512}{x^3}\cdot$

For a maxima or minima, we have $\frac{dS}{dx} = 0$.

Now, $\frac{dS}{dx} = 0 \Rightarrow \left(1 - \frac{256}{x^2}\right) = 0 \Rightarrow x^2 = 256$ or $x = 16$.

Also, $\left[\frac{d^2S}{dx^2}\right]_{x=16} = \left(\frac{512}{16 \times 16 \times 16}\right) = \frac{1}{8} > 0$.

So, $x = 16$ is a point of minimum.

Hence, the required numbers are 16 and 16.

EXAMPLE 3 *Find two positive numbers x and y such that $(x+y) = 60$ and xy^3 is maximum.*

SOLUTION Let $(x+y) = 60$ and let $P = xy^3$.

Then, $P = (60-y)y^3$ $\quad [\because x = (60-y)]$.

$$\therefore \quad \frac{dP}{dy} = 3y^2(60-y) + y^3(-1) = (180y^2 - 4y^3) = 4y^2(45-y)$$

and, $\frac{d^2P}{dy^2} = (360y - 12y^2) = 12y(30-y)$.

Now, $\frac{dP}{dy} = 0 \Rightarrow 4y^2(45-y) = 0 \Rightarrow y = 0$ or $y = 45$.

Neglecting $y = 0$, we are left with $y = 45$.

Now, $\left[\dfrac{d^2P}{dy^2}\right]_{y=45} = (12 \times 45)(30 - 45) = -8100 < 0.$

So, $y = 45$ is a point of maximum.

Hence, the numbers are 45 and 15.

EXAMPLE 4 *Find two numbers whose sum is 16 and the sum of whose cubes is minimum.*

SOLUTION Let the numbers be x and $(16 - x)$.

Let $S = x^3 + (16 - x)^3$.

Then, $\dfrac{dS}{dx} = 3x^2 - 3(16 - x)^2$ and $\dfrac{d^2S}{dx^2} = 6x + 6(16 - x) = 96.$

Now, $\dfrac{dS}{dx} = 0 \Rightarrow 3x^2 - 3(16 - x)^2 = 0 \Rightarrow x = 8.$

And, $\left[\dfrac{d^2S}{dx^2}\right]_{x=8} = 96 > 0.$

$\therefore$ $x = 8$ is a point of minimum.

Hence, the required numbers are 8 and 8.

EXAMPLE 5 *Show that of all the rectangles with a given perimeter, the square has the largest area.*

SOLUTION Let a be the fixed perimeter.

Consider a rectangle with sides x and y and perimeter a.

Let A be the area of the rectangle.

Then, $2x + 2y = a$ ... (i)

And, $A = xy = x\left(\dfrac{a - 2x}{2}\right)$ [using (i)].

$\therefore$ $\dfrac{dA}{dx} = \left(\dfrac{1}{2}a - 2x\right) = \dfrac{1}{2}(a - 4x)$ and $\dfrac{d^2A}{dx^2} = -2.$

Now, $\dfrac{dA}{dx} = 0 \Rightarrow \dfrac{1}{2}(a - 4x) = 0 \Rightarrow x = \dfrac{a}{4}\cdot$

Also, $\left[\dfrac{d^2A}{dx^2}\right]_{x=(a/4)} = -2 < 0.$

$\therefore$ $x = (a/4)$ is a point of maximum.

Also, from (i), we have $y = \dfrac{1}{2}(a - 2x) = \dfrac{1}{2}\left(a - \dfrac{a}{2}\right) = \dfrac{1}{4}a$, when $x = \dfrac{a}{4}\cdot$

Thus, $x = y$ [each $= (a/4)$].

Hence, the rectangle is a square.

EXAMPLE 6 *Show that of all the rectangles of a given area, the square has the smallest perimeter.* **[CBSE 2011]**

SOLUTION Let A be the fixed area.

Consider a rectangle with sides x and y, and area A.

Let P be the perimeter of this rectangle.

Then, $A = xy \Rightarrow y = \frac{A}{x}$ … (i)

And, $P = 2x + 2y = 2x + \frac{2A}{x}$ [using (i)].

Now, $P = \left(2x + \frac{2A}{x}\right) \Rightarrow \frac{dP}{dx} = \left(2 - \frac{2A}{x^2}\right)$ and $\frac{d^2P}{dx^2} = \frac{4A}{x^3}$.

Now, $\frac{dP}{dx} = 0 \Rightarrow 2 - \frac{2A}{x^2} = 0 \Rightarrow x = \sqrt{A}$

and, $\left[\frac{d^2P}{dx^2}\right]_{x=\sqrt{A}} = \frac{4A}{A^{3/2}} = \frac{4}{\sqrt{A}} > 0.$

So, $x = \sqrt{A}$ is a point of minimum.

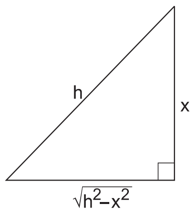
Moreover, $x = \sqrt{A} \Rightarrow y = \frac{A}{x} = \frac{A}{\sqrt{A}} = \sqrt{A} = x.$

So, when the perimeter is smallest, the rectangle is a square.

EXAMPLE 7 *Prove that the area of a right-angled triangle of a given hypotenuse is maximum when the triangle is isosceles.* **[CBSE 2012C]**

SOLUTION Let h be the hypotenuse of the triangle and x be its altitude.

Then, the base of the triangle $= \sqrt{h^2 - x^2}$.



Now, $A = \frac{1}{2}x\sqrt{h^2 - x^2}$

$\therefore \quad \frac{dA}{dx} = \frac{1}{2}\left\{x \cdot \frac{1}{2}(h^2 - x^2)^{-1/2}(-2x) + \sqrt{h^2 - x^2}\right\}$

$= \frac{1}{2}\left(\frac{h^2 - 2x^2}{\sqrt{h^2 - x^2}}\right)$

and, $\frac{d^2A}{dx^2} = \frac{(\sqrt{h^2 - x^2})(-4x) - (h^2 - 2x^2) \cdot \frac{1}{2}(h^2 - x^2)^{-1/2}(-2x)}{2(h^2 - x^2)}$

$= \frac{(2x^3 - 3xh^2)}{2(h^2 - x^2)^{3/2}}$.

Now, $\frac{dA}{dx} = 0 \Rightarrow (h^2 - 2x^2) = 0 \Rightarrow x = \frac{h}{\sqrt{2}}$.

$\therefore \left[\frac{d^2A}{dx^2}\right]_{x=\frac{h}{\sqrt{2}}} = -2 < 0.$

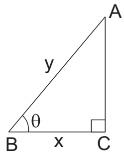
Thus, A is maximum at $x = (h/\sqrt{2})$.

$\therefore$ base $= \sqrt{h^2 - x^2} = \sqrt{\left(h^2 - \frac{h^2}{2}\right)} = \frac{h}{\sqrt{2}} =$ altitude.

Hence, the triangle is isosceles.

EXAMPLE 8 *If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is* $(\pi/3)$. **[CBSE 2009]**

SOLUTION Let us consider a right-angled triangle with base $= x$ and hypotenuse $= y$. Let $x + y = k$, where k is a constant. Let θ be the angle between the base and the hypotenuse. Let A be the area of the triangle. Then,



$$A = \frac{1}{2} \times BC \times AC = \frac{1}{2} x\sqrt{y^2 - x^2}.$$

$$\therefore \quad A^2 = \frac{x^2(y^2 - x^2)}{4} = \frac{x^2[(k-x)^2 - x^2]}{4} \quad [\because \; y = (k - x)]$$

$$\text{or } A^2 = \frac{k^2x^2 - 2kx^3}{4} \qquad \ldots \text{(i)}$$

On differentiating (i), we get $\quad 2A \cdot \frac{dA}{dx} = \frac{2k^2x - 6kx^2}{4}$ ... (ii)

$$\text{or} \quad \frac{dA}{dx} = \frac{k^2x - 3kx^2}{4A}.$$

Now, $\quad \frac{dA}{dx} = 0 \Rightarrow (k^2x - 3kx^2) = 0 \Rightarrow x = \frac{k}{3}$ [neglecting $x = 0$].

Now, differentiating (ii), we get

$$2\left(\frac{dA}{dx}\right)^2 + 2A \cdot \frac{d^2A}{dx^2} = \frac{2k^2 - 12kx}{4} \qquad \ldots \text{(iii)}$$

Putting $\frac{dA}{dx} = 0$ and $x = \frac{k}{3}$ in (iii), we get $\frac{d^2A}{dx^2} = \frac{-k^2}{4A} < 0.$

Thus, A is maximum when $x = (k/3)$.

Now, $\quad x = \frac{k}{3} \Rightarrow y = \left(k - \frac{k}{3}\right) = \frac{2k}{3}.$

$$\therefore \quad \frac{x}{y} = \cos\theta \Rightarrow \cos\theta = \frac{(k/3)}{(2k/3)} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

Hence, the area is maximum when $\theta = \frac{\pi}{3}$.

EXAMPLE 9 *Two sides of a triangle are given. Find the angle between them such that the area is maximum.*

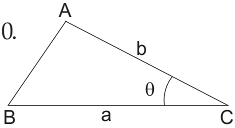
SOLUTION Let a and b be the lengths of the given sides and let θ be the angle between them. Let A be the area of the triangle.

Then, $A = \frac{1}{2} ab \sin\theta.$

$\therefore \quad \frac{dA}{d\theta} = \frac{1}{2}ab\cos\theta$ and $\frac{d^2A}{d\theta^2} = -\frac{1}{2}ab\sin\theta.$

Now, $\frac{dA}{d\theta} = 0 \Rightarrow \frac{1}{2}ab\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}\cdot$

And, $\left[\frac{d^2A}{d\theta^2}\right]_{\theta=(\pi/2)} = -\frac{1}{2}ab\sin\frac{\pi}{2} = -\frac{1}{2}ab < 0.$



$\therefore \quad \theta = \frac{\pi}{2}$ is a point of maximum.

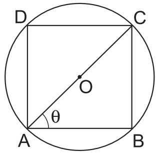
Hence, the area of the triangle is maximum when the angle between the given sides is $(\pi/2)$.

EXAMPLE 10 *Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.* **[CBSE 2008, '13C]**

SOLUTION Let $ABCD$ be a rectangle inscribed in a given circle with centre O and radius r.

Let $\angle CAB = \theta$.

Then, $AC = 2r$, $AB = 2r\cos\theta$ and $BC = 2r\sin\theta$.



Let A be the area of rectangle $ABCD$.

Then, $A = AB \times BC = 4r^2 \sin\theta\cos\theta = 2r^2\sin 2\theta$.

Thus, $A = 2r^2\sin 2\theta$, where r is constant.

$\therefore \quad \frac{dA}{d\theta} = 4r^2\cos 2\theta$ and $\frac{d^2A}{d\theta^2} = -8r^2\sin 2\theta.$

Now, $\frac{dA}{d\theta} = 0 \Rightarrow 4r^2\cos 2\theta = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}$, i.e., $\theta = \frac{\pi}{4}\cdot$

And, $\left[\frac{d^2A}{d\theta^2}\right]_{\theta=(\pi/4)} = -8r^2\sin\frac{\pi}{2} = -8r^2 < 0.$

$\therefore \quad \theta = (\pi/4)$ is a point of maximum.

Thus, area is maximum when $\theta = (\pi/4)$.

In this case, $AB = 2r\cos\frac{\pi}{4} = r\sqrt{2}$

and, $BC = 2r\sin\frac{\pi}{4} = r\sqrt{2}.$

Thus, $AB = BC$ and therefore, $ABCD$ is a square.

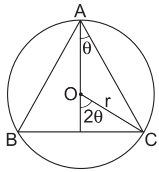
EXAMPLE 11 *Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.*

SOLUTION Let ABC be a triangle inscribed in a given circle with centre O and radius r.

For maximum area, the vertex A should be at a maximum distance from the base BC.

Therefore, A must lie on the diameter, perpendicular to BC. Thus, $AD \perp BC$.

So, triangle ABC must be isosceles.



Let $\angle CAD = \theta$.

Now, $BC = 2CD = 2OC \sin 2\theta = 2r \sin 2\theta$

and, $AD = (OA + OD) = (r + r\cos 2\theta)$.

Let A be the area of the triangle.

Then, $A = \frac{1}{2} BC \times AD = r^2 \sin 2\theta (1 + \cos 2\theta)$.

$$\therefore \quad \frac{dA}{d\theta} = r^2[\sin 2\theta(-2\sin 2\theta) + (1 + \cos 2\theta) \cdot 2\cos 2\theta]$$

$$= r^2[2(\cos^2 2\theta - \sin^2 2\theta) + 2\cos 2\theta] = 2r^2[\cos 4\theta + \cos 2\theta].$$

And, $$\frac{d^2A}{d\theta^2} = 2r^2[-4\sin 4\theta - 2\sin 2\theta] = -4r^2(2\sin 4\theta + \sin 2\theta).$$

Now, $$\frac{dA}{d\theta} = 0 \Rightarrow \cos 4\theta + \cos 2\theta = 0$$

$$\Rightarrow \cos 4\theta = -\cos 2\theta = \cos(\pi - 2\theta)$$

$$\Rightarrow 4\theta = \pi - 2\theta \Rightarrow \theta = \frac{\pi}{6}\cdot$$

And, $$\left[\frac{d^2A}{d\theta^2}\right]_{\theta = (\pi/6)} = -4r^2\left(2\sin\frac{2\pi}{3} + \sin\frac{\pi}{3}\right) = -6r^2\sqrt{3} < 0.$$

$\therefore \; \theta = \frac{\pi}{6}$ is a point of maximum.

So, in this case, each angle of the triangle is $(\pi/3)$.

Hence, ABC is an equilateral triangle.

EXAMPLE 12 *The combined resistance R of two resistors R_1 and R_2 where $R_1, R_2 > 0$ is given by* $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}\cdot$

If $R_1 + R_2 = C$ (constant), show that the maximum resistance R is obtained by choosing $R_1 = R_2$.

SOLUTION We have $R = \frac{R_1 R_2}{(R_1 + R_2)}$ and $R_1 + R_2 = C$.

$$\therefore \quad R = \frac{R_1(C - R_1)}{C} \qquad [\because \; R_2 = (C - R_1)].$$

So, $$\frac{dR}{dR_1} = \frac{C - 2R_1}{C} \text{ and } \frac{d^2R}{dR_1^{\,2}} = -\frac{2}{C}\cdot$$

For a maxima or minima, we have $\frac{dR}{dR_1} = 0.$

Now, $\frac{dR}{dR_1} = 0 \Rightarrow \frac{C - 2R_1}{C} = 0$ or $R_1 = \frac{C}{2}\cdot$

And, $\left[\frac{d^2R}{dR_1^2}\right]_{R_1 = (C/2)} = \frac{-2}{C} < 0.$

Thus, $R_1 = \frac{C}{2}$ is a point of maximum.

When, $R_1 = \frac{C}{2}$, we have $R_2 = \left(C - \frac{C}{2}\right) = \frac{C}{2}\cdot$ Hence, $R_1 = R_2$.

EXAMPLE 13 *A beam of length l is supported at one end. If W is the uniform load per unit length, the bending moment M at a distance x from the end is given by* $M = \left(\frac{lx}{2} - \frac{Wx^2}{2}\right)\cdot$ *Find the point on the beam at which the bending moment has the maximum value.*

SOLUTION $M = \left(\frac{lx}{2} - \frac{Wx^2}{2}\right) \Rightarrow \frac{dM}{dx} = \left(\frac{l}{2} - Wx\right)$ and $\frac{d^2M}{dx^2} = -W.$

For a maxima or minima, we have $\frac{dM}{dx} = 0.$

Now, $\frac{dM}{dx} = 0 \Rightarrow \frac{l}{2} - Wx = 0 \Rightarrow x = \frac{l}{2W}\cdot$

Also, $\frac{d^2M}{dx^2} = -W < 0$ for all values of x.

$\therefore \quad x = \frac{l}{2W}$ is a point of maxima.

So, the required point is at a distance of $(l/2W)$ from the supporting end.

EXAMPLE 14 *A wire of length* 25 m *is to be cut into two pieces. One of the wires is to be made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area of the square and the circle is minimum?* **[CBSE 1996]**

SOLUTION Let x metres and $(25 - x)$ metres be the required lengths.

Let a be the side of the square formed and r be the radius of the circle formed. Then,

$$4a = x \text{ and } 2\pi r = 25 - x \text{ or } a = \frac{x}{4} \text{ and } r = \frac{25 - x}{2\pi}\cdot$$

$\therefore$ area of the square $= \left(\frac{x^2}{16}\right)$ sq metres.

Area of the circle $= \pi\left(\frac{25 - x}{2\pi}\right)^2 = \frac{(25 - x)^2}{4\pi}\cdot$

Now, the combined area $A = \frac{x^2}{16} + \frac{(25-x)^2}{4\pi}\cdot$

$\therefore \quad \frac{dA}{dx} = \frac{x}{8} - \frac{(25-x)}{2\pi} = \frac{(\pi+4)x - 100}{8\pi}$, and $\frac{d^2A}{dx^2} = \frac{(\pi+4)}{8\pi}\cdot$

For a maxima or minima, we have $\frac{dA}{dx} = 0$.

Now, $\frac{dA}{dx} = 0 \Rightarrow (\pi+4)x - 100 = 0 \Rightarrow x = \frac{100}{(\pi+4)}\cdot$

And, $\frac{d^2A}{dx^2} = \frac{(\pi+4)}{8\pi} > 0$ for all values of x.

$\therefore \quad x = \frac{100}{(\pi+4)}$ is a point of minimum.

$\therefore$ lengths of the pieces are $\left(\frac{100}{\pi+4}\right)$ metres and $\left(\frac{25\pi}{\pi+4}\right)$ metres.

Some Examples Related to Volume and Area of Solids

EXAMPLE 15 *Show that a cylinder of a given volume which is open at the top has minimum total surface area, provided its height is equal to the radius of its base.* **[CBSE 2009C, '14]**

SOLUTION Let r be the radius and h the height of the cylinder of given volume V.

Then, $V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}$ … (i)

Let the total surface area be S. Then,

$$S = \pi r^2 + 2\pi rh = \left(\pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2}\right) \quad \text{[using (i)]}.$$

$$\therefore \quad S = \left(\pi r^2 + \frac{2V}{r}\right).$$

So, $\frac{dS}{dr} = \left(2\pi r - \frac{2V}{r^2}\right)$ and $\frac{d^2S}{dr^2} = \left(2\pi + \frac{4V}{r^3}\right)$.

For a maxima or minima, we have $\frac{dS}{dr} = 0$.

Now, $\frac{dS}{dr} = 0 \Rightarrow 2\pi r - \frac{2V}{r^2} = 0 \Rightarrow 2\pi r^3 = 2V = 2\pi r^2 h \Rightarrow r = h$.

$$\left(\frac{d^2S}{dr^2}\right)_{r=h} = \left(2\pi + \frac{4V}{h^3}\right) > 0.$$

Thus, $r = h$ is a point of minimum.

This shows that the total surface area of the cylinder is minimum when $h = r$, i.e., when the height of the cylinder is equal to the radius of the base.

EXAMPLE 16 *Show that the height of a cylinder which is open at the top, having a given surface and the greatest volume, is equal to the radius of its base.*

[CBSE 2004, 2010]

SOLUTION Let r be the radius, h the height and S the given surface area of an open cylinder. Then, $S = (\pi r^2 + 2\pi rh)$ or $h = \left(\frac{S - \pi r^2}{2\pi r}\right)$ ... (i)

Let V be the volume of the cylinder.

Then, $V = \pi r^2 h = \pi r^2 \cdot \left(\frac{S - \pi r^2}{2\pi r}\right) = \left(\frac{rS - \pi r^3}{2}\right)$ [using (i)].

$\therefore \quad \frac{dV}{dr} = \left(\frac{S - 3\pi r^2}{2}\right)$ and $\frac{d^2V}{dr^2} = -3\pi r.$

Now, $\frac{dV}{dr} = 0 \Rightarrow S - 3\pi r^2 = 0 \Rightarrow \pi r^2 + 2\pi rh - 3\pi r^2 = 0 \Rightarrow h = r.$

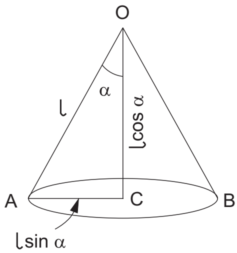
And, $\left(\frac{d^2V}{dr^2}\right)_{r=h} = -3\pi h < 0.$ Thus, $r = h$ is a point of maximum.

So, V is maximum when $r = h$, i.e., when the height of the cylinder is equal to the radius of its base.

EXAMPLE 17 *Show that the semivertical angle of a cone of maximum volume and of given slant height is* $\cos^{-1}\frac{1}{\sqrt{3}}$.

[CBSE 2008C, '14]

SOLUTION Let α be the semivertical angle of a cone of given slant height l. Then, height of the cone $= l\cos\alpha$, radius of the base $= l\sin\alpha$.



Let V be the volume of the cone. Then,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(l^2\sin^2\alpha)(l\cos\alpha)$$

or $\quad V = \frac{1}{3}\pi l^3 \sin^2\alpha\cos\alpha.$

$$\therefore \frac{dV}{d\alpha} = \frac{1}{3}\pi l^3(-\sin^3\alpha + 2\sin\alpha\cos^2\alpha) = \frac{1}{3}\pi l^3\sin\alpha(2\cos^2\alpha - \sin^2\alpha).$$

$$\text{And, } \frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi l^3\cos\alpha(2\cos^2\alpha - \sin^2\alpha) + \frac{1}{3}\pi l^3\sin\alpha(-4\sin\alpha\cos\alpha - 2\sin\alpha\cos\alpha)$$

$$= \frac{1}{3}\pi l^3\cos^3\alpha(2 - 7\tan^2\alpha).$$

Now, for a maxima or minima, we have $\frac{dV}{d\alpha} = 0.$

Now, $\frac{dV}{d\alpha} = 0 \Rightarrow \frac{1}{3}\pi l^3\sin\alpha(2\cos^2\alpha - \sin^2\alpha) = 0$

$\Rightarrow \sin\alpha = 0$ or $2\cos^2\alpha - \sin^2\alpha = 0$

$$\Rightarrow \alpha = 0 \text{ or } \alpha = \tan^{-1}\sqrt{2}.$$

Neglecting $\alpha = 0$, we have $\alpha = \tan^{-1}\sqrt{2}$, i.e., $\tan\alpha = \sqrt{2}$.

Now, $\tan\alpha = \sqrt{2} \Rightarrow \sec^2\alpha = 3 \Rightarrow \cos\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\frac{1}{\sqrt{3}}$

$$\therefore \quad \left[\frac{d^2V}{d\alpha^2}\right]_{\alpha=\tan^{-1}\sqrt{2}} = \frac{1}{3}\pi l^3 \cdot \left(\frac{1}{\sqrt{3}}\right)^3 (2-14) = -\frac{4\pi l^3}{3\sqrt{3}} < 0.$$

Thus, V is maximum when $\alpha = \cos^{-1}\frac{1}{\sqrt{3}}$, i.e., when the semivertical angle of the cone is $\cos^{-1}\frac{1}{\sqrt{3}}\cdot$

EXAMPLE 18 *Show that the semivertical angle of a right circular cone of given surface area and maximum volume is* $\sin^{-1}(1/3)$.

SOLUTION Let r be the radius, l the slant height and h the height of the cone. Let S denote the surface area and V the volume of the cone.

Then, $S = (\pi r^2 + \pi rl) = \text{constant}.$

$$\therefore \quad l = \left(\frac{S}{\pi r} - r\right) \quad \ldots \text{(i)}$$

Now, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{l^2 - r^2}.$

$$\therefore \quad V^2 = \frac{1}{9}\pi^2 r^4 (l^2 - r^2) = \frac{1}{9}\pi^2 r^4 \cdot \left\{\left(\frac{S}{\pi r} - r\right)^2 - r^2\right\} \quad \text{[using (i)]}$$

$$= \frac{1}{9}S(Sr^2 - 2\pi r^4).$$

Thus, $V^2 = \left(\frac{1}{9}S^2 r^2 - \frac{2\pi}{9}Sr^4\right)$

$$\therefore \ 2V \cdot \frac{dV}{dr} = \left(\frac{2}{9}S^2 r - \frac{8\pi S}{9}r^3\right) = \frac{2rS}{9}(S - 4\pi r^2) \quad \ldots \text{(ii)}$$

$\therefore \frac{dV}{dr} = 0 \Rightarrow r = 0$ or $(S - 4\pi r^2) = 0 \Rightarrow r^2 = \frac{S}{4\pi}$ (neglecting $r = 0$).

On differentiating (ii), we get $2\left(\frac{dV}{dr}\right)^2 + 2V \cdot \frac{d^2V}{dr^2} = \frac{1}{9}S(2S - 24\pi r^2).$

Putting $\frac{dV}{dr} = 0$ and $r^2 = \frac{S}{4\pi}$, we get

$$2V \cdot \frac{d^2V}{dr^2} = \frac{1}{9}S(2S - 6S) = -\frac{4}{9}S^2 < 0.$$

$\therefore$ when the volume is maximum, we have

$$r^2 = \frac{S}{4\pi} = \frac{(\pi r^2 + \pi rl)}{4\pi} \Rightarrow l = 3r.$$

Now, if α is the semivertical angle of the cone then

$$\frac{r}{l} = \sin\alpha \Rightarrow \frac{r}{3r} = \sin\alpha \Rightarrow \sin\alpha = \frac{1}{3} \Rightarrow \alpha = \sin^{-1}\left(\frac{1}{3}\right).$$

Hence, the semivertical angle of a right cone of a given surface and maximum volume is $\sin^{-1}(1/3)$.

EXAMPLE 19 *Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.* **[CBSE 2005]**

SOLUTION Let V be the fixed volume of a closed cuboid with length a, breadth a and height h.

Let S be its surface area.

Then, $V = (a \times a \times h)$ or $h = \dfrac{V}{a^2}$ ... (i)

Now, $S = 2(a^2 + ah + ah) = 2(a^2 + 2ah) = 2\left(a^2 + \dfrac{2V}{a}\right)$ [using (i)]

i.e., $S = 2\left(a^2 + \dfrac{2V}{a}\right)$. $\therefore \dfrac{dS}{da} = 2\left(2a - \dfrac{2V}{a^2}\right)$ and $\dfrac{d^2S}{da^2} = \left(4 + \dfrac{8V}{a^3}\right)$.

Now, $\dfrac{dS}{da} = 0 \Rightarrow V = a^3 \Rightarrow a \times a \times h = a^3 \Rightarrow h = a.$

Now, when $h = a$, we have $V = a^3$.

$$\therefore \left[\frac{d^2S}{da^2}\right]_{h=a} = \left(4 + \frac{8a^3}{a^3}\right) = 12 > 0.$$

So, S is minimum when length $= a$, breadth $= a$ and height $= a$, i.e., when it is a cube.

EXAMPLE 20 *Show that the height of a closed cylinder of given surface and maximum volume is equal to the diameter of its base.*

SOLUTION Let S be the fixed surface area of a closed cylinder of radius r and height h. Let V be its volume.

Then, $S = (2\pi r^2 + 2\pi rh) \Rightarrow h = \left(\dfrac{S - 2\pi r^2}{2\pi r}\right)$ ... (i)

And, $V = \pi r^2 h = \pi r^2\left(\dfrac{S - 2\pi r^2}{2\pi r}\right) = \left(\dfrac{rS - 2\pi r^3}{2}\right)$ [using (i)].

$$\Rightarrow \frac{dV}{dr} = \left(\frac{S - 6\pi r^2}{2}\right) \text{ and } \frac{d^2V}{dr^2} = -6\pi r.$$

Now, $\dfrac{dV}{dr} = 0 \Rightarrow (S - 6\pi r^2) = 0 \Rightarrow S = 6\pi r^2$

$$\Rightarrow (2\pi r^2 + 2\pi rh) = 6\pi r^2 \Rightarrow h = 2r, \text{ i.e., } r = \frac{1}{2}h.$$

And, $\left[\dfrac{d^2V}{dr^2}\right]_{r=(h/2)} = -6\pi \times \dfrac{1}{2}h = -3\pi h < 0.$

So, V is maximum when $h = 2r$.

EXAMPLE 21 *A closed right circular cylinder has a volume of 2156 cubic units. What should be the radius of its base so that its total surface area may be maximum?*

SOLUTION Let r be the radius and h the height of the cylinder.

Then, $\pi r^2 h = 2156 \Rightarrow h = \left(\frac{2156}{\pi r^2}\right)\cdot$ Let S be its total surface area.

Then, $S = (2\pi r^2 + 2\pi rh) = \left(2\pi r^2 + 2\pi r \times \frac{2156}{\pi r^2}\right) = \left(2\pi r^2 + \frac{4312}{r}\right)\cdot$

$\therefore \quad \frac{dS}{dr} = \left(4\pi r - \frac{4312}{r^2}\right)$ and $\frac{d^2S}{dr^2} = \left(4\pi + \frac{8624}{r^3}\right)\cdot$

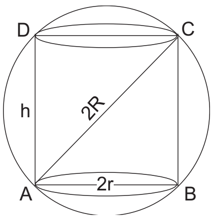
Now, $\frac{dS}{dr} = 0 \Rightarrow \left(4\pi r - \frac{4312}{r^2}\right) = 0$

$$\Rightarrow 4\pi r^3 = 4312 \Rightarrow r = \left(\frac{4312}{4\pi}\right)^{1/3} = \left(\frac{4312 \times 7}{4 \times 22}\right)^{1/3} = 7.$$

And, $\left[\frac{d^2S}{dr^2}\right]_{r=7} = \left(4\pi + \frac{8624}{343}\right) > 0. \quad \therefore \quad S$ is maximum when $r = 7$.

EXAMPLE 22 *Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is* $\frac{2R}{\sqrt{3}}\cdot$ *Find the volume of the largest cylinder inscribed in a sphere of radius R.* **[CBSE 2006, '09C, '12C, '13C]**

SOLUTION Let r be the radius and h the height of the inscribed cylinder $ABCD$. Let V be its volume.



Then, $V = \pi r^2 h$ … (i)

Clearly, $AC = 2R$.

Also, $AC^2 = AB^2 + BC^2$

$\Rightarrow (2R)^2 = (2r)^2 + h^2$

$\Rightarrow r^2 = \frac{1}{4}(4R^2 - h^2)$ … (ii)

Using (ii) in (i), we get

$$V = \frac{\pi h}{4}(4R^2 - h^2) \Rightarrow \frac{dV}{dh} = \left(\pi R^2 - \frac{3}{4}\pi h^2\right) \text{ and } \frac{d^2V}{dh^2} = -\frac{3}{2}\pi h.$$

For a maxima or minima, we have $(dV/dh) = 0$.

Now, $\frac{dV}{dh} = 0 \Rightarrow \pi R^2 - \frac{3}{4}\pi h^2 = 0 \Rightarrow h = \frac{2R}{\sqrt{3}}\cdot$

$$\therefore \quad \left[\frac{d^2V}{dh^2}\right]_{h=(2R/\sqrt{3})} = -\frac{3}{2}\pi \times \frac{2R}{\sqrt{3}} = -\pi R\sqrt{3} < 0.$$

So, V is maximum when $h = \frac{2R}{\sqrt{3}}\cdot$

Hence, the height of the cylinder of maximum volume is $\frac{2R}{\sqrt{3}}$.

Largest volume of the cylinder $= \pi \times \frac{1}{4}\left[4R^2 - \frac{4R^2}{3}\right] \times \frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$.

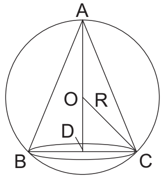
EXAMPLE 23 *Show that the cone of greatest volume which can be inscribed in a given sphere is such that three times its altitude is twice the diameter of the sphere. Find the volume of the largest cone inscribed in a sphere of radius R.* **[CBSE 2010C]**

SOLUTION Let R be the radius of the given sphere with centre O, and let V be the volume of the inscribed cone, h be its height and r be the radius of the base.

In the given figure, we have $OD = AD - AO = (h - R)$.

$\therefore \quad R^2 = (h-R)^2 + r^2$ or $r^2 = h(2R - h)$ ... (i)

Now, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^2(2R - h)$ [using (i)].



$\therefore \quad \frac{dV}{dh} = \frac{1}{3}\pi h(4R - 3h)$, and $\frac{d^2V}{dh^2} = \left(\frac{4}{3}\pi R - 2\pi h\right)$.

For a maxima or minima, we have $\frac{dV}{dh} = 0$.

Now, $\frac{dV}{dh} = 0 \Rightarrow \frac{1}{3}\pi h(4R - 3h) = 0$

$\Rightarrow h = 0$ or $(4R - 3h) = 0 \Rightarrow h = \frac{4}{3}R$ [$\because h \neq 0$].

And, $\left[\frac{d^2V}{dh^2}\right]_{h=(4/3)R} = -\frac{4\pi R}{3} < 0$.

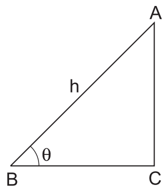
So, V is maximum when $h = \frac{4}{3}R$, i.e., when $3h = 2(2R)$

i.e., 3 times the height = 2 times the diameter.

Volume of the largest cone $= \frac{1}{3}\pi \times \frac{16R^2}{9} \times \left(2R - \frac{4R}{3}\right) = \frac{32\pi R^3}{81}$.

EXAMPLE 24 *Prove that the perimeter of a right-angled triangle of given hypotenuse is maximum when the triangle is isosceles.*

SOLUTION Let h be the hypotenuse and θ be the angle between the hypotenuse and the base.



Then, base $= h\cos\theta$ and altitude $= h\sin\theta$.

Let P be the perimeter.

Then, $P = h + h\cos\theta + h\sin\theta$.

$\therefore \quad \frac{dP}{d\theta} = -h\sin\theta + h\cos\theta = h(\cos\theta - \sin\theta)$.

And, $\frac{d^2P}{d\theta^2} = h(-\sin\theta - \cos\theta) = -h(\sin\theta + \cos\theta)$.

Now, $\frac{dP}{d\theta} = 0 \Rightarrow h(\cos\theta - \sin\theta) = 0$

$\Rightarrow \cos\theta = \sin\theta$, i.e., $\tan\theta = 1 \Rightarrow \theta = (\pi/4)$.

Also, $\left[\frac{d^2P}{d\theta^2}\right]_{\theta=(\pi/4)} = -h\left[\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right] = -h\sqrt{2} < 0$.

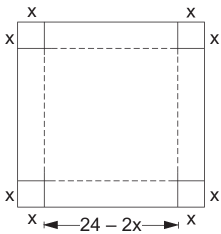
$\therefore$ P is maximum when $\theta = (\pi/4)$.

In this case, base $= h\cos\frac{\pi}{4} = \frac{h}{\sqrt{2}}$ and altitude $= h\sin\frac{\pi}{4} = \frac{h}{\sqrt{2}}$.

$\therefore$ base = altitude, and hence the triangle is isosceles.

EXAMPLE 25 *An open box is to be made out of a piece of cardboard measuring* (24 cm × 24 cm) *by cutting off equal squares from the corners and turning up the sides. Find the height of the box when it has maximum volume.* **[CBSE 2004]**

SOLUTION Let the length of the side of each square cut off from the corners be x cm.



Then, height of the box = x cm.

$\therefore$ $V = (24-2x)^2 \times x = 4x^3 - 96x^2 + 576x$

$\Rightarrow \frac{dV}{dx} = 12(x^2 - 16x + 48)$

and, $\frac{d^2V}{dx^2} = 24(x-8)$.

Now, $\frac{dV}{dx} = 0 \Rightarrow x^2 - 16x + 48 = 0$, i.e., $(x-12)(x-4) = 0$

$\Rightarrow x = 4 \quad [\because x \neq 12]$

$\left[\frac{d^2V}{dx^2}\right]_{x=4} = -96 < 0$. $\quad\therefore$ V is maximum at $x = 4$.

Hence, the volume of the box is maximum when its height is 4 cm.

EXERCISE 11F

1. Find two positive numbers whose product is 49 and the sum is minimum.
2. Find two positive numbers whose sum is 16 and the sum of whose squares is minimum.
3. Divide 15 into two parts such that the square of one number multiplied with the cube of the other number is maximum.

4. Divide 8 into two positive parts such that the sum of the square of one and the cube of the other is minimum.
5. Divide a into two parts such that the product of the pth power of one part and the qth power of the second part may be maximum.
6. The rate of working of an engine is given by

$$R = 15v + \frac{6000}{v}, \text{ where } 0 < v < 30$$

   and v is the speed of the engine. Show that R is the least when $v = 20$.
7. Find the dimensions of the rectangle of area 96 cm^2 whose perimeter is the least. Also, find the perimeter of the rectangle.
8. Prove that the largest rectangle with a given perimeter is a square.
9. Given the perimeter of a rectangle, show that its diagonal is minimum when it is a square.
10. Show that a rectangle of maximum perimeter which can be inscribed in a circle of radius a is a square of side $\sqrt{2}\,a$.
11. The sum of the perimeters of a square and a circle is given. Show that the sum of their areas is least when the side of the square is equal to the diameter of the circle.
12. Show that the right triangle of maximum area that can be inscribed in a circle is an isosceles triangle.
13. Prove that the perimeter of a right-angled triangle of given hypotenuse is maximum when the triangle is isosceles.
14. The perimeter of a triangle is 8 cm. If one of the sides of the triangle be 3 cm, what will be the other two sides for maximum area of the triangle?
15. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 metres. Find the dimensions of the window to admit maximum light through it. **[CBSE 2000, '02, '06]**
16. A square piece of tin of side 12 cm is to be made into a box without a lid by cutting a square from each corner and folding up the flaps to form the sides. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find this maximum volume.
17. An open box with a square base is to be made out of a given cardboard of area c^2 (square) units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ (cubic) units. **[CBSE 2001]**
18. A cylindrical can is to be made to hold 1 litre of oil. Find the dimensions which will minimize the cost of the metal to make the can.
19. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base. **[CBSE 2007]**

20. Find the radius of a closed right circular cylinder of volume 100 cm^3 which has the minimum total surface area.

21. Show that the height of a closed cylinder of given volume and the least surface area is equal to its diameter.

22. Prove that the volume of the largest cone that can be inscribed in a sphere is $\frac{8}{27}$ of the volume of the sphere. **[CBSE 2004, '04C, '05C]**

23. Which fraction exceeds its pth power by the greatest number possible?

24. Find the point on the curve $y^2 = 4x$ which is nearest to the point $(2, -8)$.

25. A right circular cylinder is inscribed in a cone. Show that the curved surface area of the cylinder is maximum when the diameter of the cylinder is equal to the radius of the base of the cone. **[CBSE 2006C]**

26. Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.

27. A rectangle is inscribed in a semicircle of radius r with one of its sides on the diameter of the semicircle. Find the dimensions of the rectangle so that its area is maximum. Find also this area. **[CBSE 2004]**

28. Two sides of a triangle have lengths a and b and the angle between them is θ. What value of θ will maximize the area of the triangle? **[CBSE 2002C]**

29. Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is (500π) cm^3. **[CBSE 2004, 05C]**

30. A square tank of capacity 250 cubic metres has to be dug out. The cost of the land is ₹ 50 per square metre. The cost of digging increases with the depth and for the whole tank, it is ₹ $(400 \times h^2)$, where h metres is the depth of the tank. What should be the dimensions of the tank so that the cost is minimum? **[CBSE 2003]**

31. A square piece of tin of side 18 cm is to be made into a box without the top, by cutting a square piece from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find the maximum volume of the box. **[CBSE 2003]**

32. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when the depth of the tank is half of its width. **[CBSE 2003, '07C]**

33. A wire of length 36 cm is cut into two pieces. One of the pieces is turned in the form of a square and the other in the form of an equilateral triangle. Find the length of each piece so that the sum of the areas of the two be minimum. **[CBSE 2005]**

34. Find the largest possible area of a right-angled triangle whose hypotenuse is 5 cm. **[CBSE 2006C]**

ANSWERS (EXERCISE 11F)

1. 7, 7 **2.** 8, 8 **3.** 6, 9 **4.** 6, 2 **5.** $\frac{ap}{p+q}, \frac{aq}{p+q}$

7. length $= 4\sqrt{6}$ cm, breadth $= 4\sqrt{6}$ cm, perimeter $= 16\sqrt{6}$ cm

14. $b = c = 2.5$ cm **15.** breadth $= \frac{20}{(\pi+4)}$ m, height $= \frac{10}{(\pi+4)}$ m

16. 2 cm, 128 cm^3 **18.** $r = \left(\frac{500}{\pi}\right)^{1/3}$ cm, $h = \frac{1000}{\pi^{1/3}(500)^{2/3}}$ cm **20.** $\left(\frac{50}{\pi}\right)^{1/3}$ cm

23. $\left(\frac{1}{p}\right)^{1/(p-1)}$ **24.** $(4, -4)$

27. length $= r\sqrt{2}$, breadth $= \frac{r\sqrt{2}}{2}$, area $= r^2$ sq units **28.** $\frac{\pi}{2}$

30. Side = 10 cm, depth = 2.5 m **31.** 3 cm, 432 cm^3

33. $\frac{144\sqrt{3}}{4\sqrt{3}+9}$ cm, $\frac{324}{4\sqrt{3}+9}$ cm **34.** $\frac{25}{4}$ sq cm

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11F)

7. Let P be the fixed perimeter and x, y be the sides.

Then, $96 = xy \Rightarrow y = \frac{96}{x}$.

$P = 2(x+y) \Rightarrow P = 2\left(x + \frac{96}{x}\right)$.

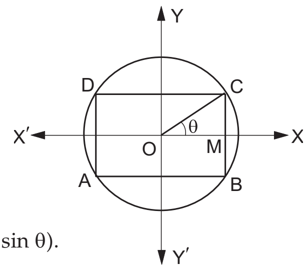
$P = 2(x+y) \Rightarrow y = \frac{1}{2}(P - 2x)$. $\quad \therefore A = xy \Rightarrow A = \frac{1}{2}x(P-2x)$.

9. $P = 2(x+y) \Rightarrow y = \frac{1}{2}(P-2x)$.

$\therefore \quad z = D^2 = x^2 + y^2 \Rightarrow z = x^2 + \frac{1}{4}(P-2x)^2$.

Now, D is minimum when z is minimum.

10. Let $ABCD$ be the rectangle inscribed in a circle of radius a and with centre O. Join OC. Let $\angle COX = \theta$. Then, the coordinates of C are $(a\cos\theta, a\sin\theta)$.



$\therefore \quad OM = a\cos\theta$ and $MC = a\sin\theta$.

$BC = 2MC = 2a$ and $CD = 2OM = 2a\cos\theta$.

$\therefore \quad P = 2a(\cos\theta + \sin\theta)$.

So, $\frac{dP}{d\theta} = 2a(-\sin\theta + \cos\theta)$ and $\frac{d^2P}{d\theta^2} = -2a(\cos\theta + \sin\theta)$.

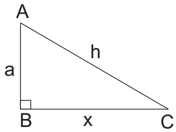
$$\frac{dP}{d\theta} = 0 \Rightarrow \theta = \frac{\pi}{4} \text{ and at this value of } \theta, \frac{d^2P}{d\theta} < 0.$$

So, P is maximum at $\theta = (\pi/4)$.
Then, $BC = \sqrt{2}\, a = CD$.

11. Let $S = 4x + 2\pi y$. Then, $y = \frac{(S - 4x)}{2\pi}$.

Let $A = x^2 + \pi y^2$. Then, $A = x^2 + \frac{(S-4x)^2}{4\pi}$.

13. Let ABC be the right triangle with given hypotenuse h.
Let, base $BC = x$ and altitude $BA = a$.



Then, $h^2 = x^2 + a^2 \Rightarrow a = \sqrt{h^2 - x^2}$.

$\therefore \quad P = a + x + h \Rightarrow P = \sqrt{(h^2 - x^2)} + x + h.$

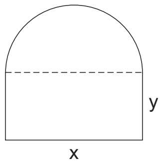
So, $\frac{dP}{dx} = \frac{-x}{\sqrt{h^2 - x^2}} + 1$ and $\frac{d^2P}{dx^2} = \frac{-h^2}{(h^2 - x^2)^{3/2}}$.

Now, $\frac{dP}{dx} = 0 \Rightarrow x = \frac{h}{\sqrt{2}}$.

And for $x = \frac{h}{\sqrt{2}}, \frac{d^2P}{dx^2} = \frac{-2^{3/2}}{h} < 0.$

$\therefore$ P is maximum when $x = \frac{h}{\sqrt{2}}$ and $a = \frac{h}{\sqrt{2}}$, i.e., when $x = a$.

15. Let the length and breadth of the rectangle be x and y metres respectively. Then, radius of the semicircle $= (x/2)$ metres.
So, the perimeter of the window is given by



$$10 = x + 2y + \frac{\pi x}{2} \Rightarrow y = 5 - \left(\frac{2+\pi}{4}\right)x.$$

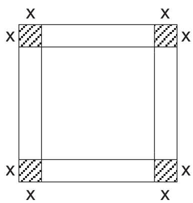
$\therefore \quad A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 \Rightarrow A = x\left[5 - \frac{(2+\pi)}{4}\cdot x\right] + \frac{\pi x^2}{8}$

$\therefore \quad \frac{dA}{dx} = \left(5 - x - \frac{\pi x}{4}\right)$ and $\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0.$

Now, $\frac{dA}{dx} = 0 \Rightarrow x = \frac{20}{(\pi + 4)}$.

$\therefore$ A is maximum when $x = \frac{20}{(\pi+4)}$ and $y = \frac{10}{(\pi+4)}$.

16. Let the length cut off from each corner be x cm. Then, length of the box $= (12 - 2x)$ cm = breadth and height $= x$ cm.



$\therefore \quad V = (12 - 2x)^2 \times x \Rightarrow V = 4x^3 - 48x^2 + 144x.$

$\therefore \quad \frac{dV}{dx} = 12(x-6)(x-2)$ and $\frac{d^2V}{dx^2} = 24(x-4)$.

Then, $\frac{dV}{dx} = 0 \Rightarrow x = 2$ [$\because$ $x = 6$ is not possible].

Also at $x = 2$, $\frac{d^2V}{dx^2} < 0.$

$\therefore$ V is maximum when $x = 2$ and maximum $V = [(12-4)^2 \times 2]\ \text{cm}^3$.

17. Let each side of the base be a and height be h. Then,

$$c^2 = (a^2 + 4ah) \Rightarrow h = \frac{(c^2 - a^2)}{4a}.$$

So, $V = a^2 \times h \Rightarrow V = \dfrac{(c^2 a - a^3)}{4}.$

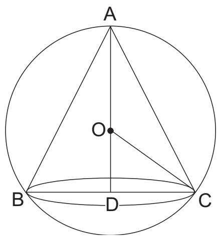
$$\therefore \quad \frac{dV}{da} = \frac{(c^2 - 3a^2)}{4} \text{ and } \frac{d^2V}{da^2} = -\frac{3}{2}a < 0.$$

Thus, V is maximum when $a^2 = \dfrac{c^2}{3}$ and $h = \dfrac{c}{2\sqrt{3}}.$

18. $\pi r^2 h = 1000 \Rightarrow h = \dfrac{1000}{\pi r^2}$ $\quad [\because V = 1 \text{ litre} = 1000 \text{ cm}^3].$

$$\therefore \quad S = (2\pi r^2 + 2\pi rh) \Rightarrow S = 2\left(\pi r^2 + \frac{1000}{r}\right).$$

22. Let the radius of the given sphere be a. Let V be the volume of the inscribed cone, h be its height and r be its radius.



Then, $V = \dfrac{1}{3}\pi r^2 h.$

Now, $OD = (AD - OA) = (h - a).$

$OC^2 = OD^2 + DC^2 \Rightarrow a^2 = (h-a)^2 + r^2$

$\Rightarrow r^2 = h(2a - h).$

$$\therefore \quad V = \frac{1}{3}\pi h^2(2a - h) = \left[\frac{2\pi a h^2}{3} - \frac{1}{3}\pi h^3\right].$$

$$\therefore \quad \frac{dV}{dh} = \left(\frac{4\pi a h}{3} - \pi h^2\right) \text{ and } \frac{d^2V}{dh^2} = \left(\frac{4\pi a}{3} - 2\pi h\right).$$

So, $\dfrac{dV}{dh} = 0 \Rightarrow h = \dfrac{4a}{3}$ and $\left[\dfrac{d^2V}{dh^2} \text{ at } h = \dfrac{4a}{3}\right] = -\dfrac{4\pi a}{3} < 0.$

$\therefore$ V is maximum when $h = \dfrac{4a}{3}$ and $r^2 = \dfrac{8a^2}{9}.$

Maximum volume $= \dfrac{1}{3}\pi \times \dfrac{8a^2}{9} \times \dfrac{4a}{3} = \dfrac{8}{27} \times \left(\dfrac{4}{3}\pi a^3\right).$

23. Let the required fraction be x. Let $y = (x - x^p)$. Then,

$$\frac{dy}{dx} = 0 \Rightarrow x = \left(\frac{1}{p}\right)^{1/(p-1)}. \qquad \frac{d^2y}{dx^2} \text{ at this point } = -p(p-1)\left(\frac{1}{p}\right)^{\left(\frac{p-2}{p-1}\right)} < 0.$$

$\therefore$ y is maximum when $x = \left(\dfrac{1}{p}\right)^{1/(p-1)}.$

24. Let the required point be (x, y). Then,

$$D = (x-2)^2 + (y+8)^2 \Rightarrow D = x^2 + y^2 - 4x + 16y + 68$$

$$\Rightarrow D = \frac{y^4}{16} + 16y + 68 \qquad \left[\because x = \frac{y^2}{4}\right]$$

$$\Rightarrow \frac{dD}{dy} = \left(\frac{4y^3}{16} + 16\right) \text{ and } \frac{d^2D}{dy^2} = \frac{3}{4}y^2.$$

Now, $\frac{dD}{dy} = 0 \Rightarrow y^3 = -64 \Rightarrow y = -4.$

$$\left[\frac{d^2D}{dy^2}\right]_{y=-4} = \frac{3}{4} \times 16 = 12 > 0.$$

$\therefore$ D is minimum when $y = -4$.

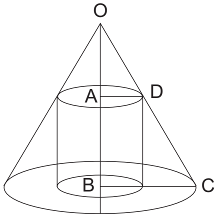
At this value, we have $x = 4$.

Hence, the required point is $(4, -4)$.

25. Let r_1 be the radius of the cone and h_1 be its height and let r be the radius and h the height of the inscribed cylinder.

Clearly, $\triangle OAD$ and $\triangle OBC$ are similar.

$$\therefore \quad \frac{r}{r_1} = \frac{h_1 - h}{h_1} \Rightarrow r = \frac{r_1}{h_1}(h_1 - h) \qquad \ldots \text{(i)}$$



Let S be the curved surface area of the cylinder.

Then, $S = 2\pi rh \Rightarrow S = \frac{2\pi r_1}{h_1}(h_1 - h)h$

$$\Rightarrow \frac{dS}{dh} = \frac{2\pi r_1}{h_1}(h_1 - 2h) \text{ and } \frac{d^2S}{dh^2} = \frac{-4\pi r_1}{h_1} < 0.$$

Now, $\frac{dS}{dh} = 0 \Leftrightarrow h_1 = 2h$. Also, $\frac{d^2S}{dh^2} < 0$.

So, S is maximum when $h_1 = 2h$.

$$\therefore \quad \frac{r}{r_1} = \frac{2h - h}{2h} = \frac{1}{2} \Leftrightarrow r_1 = 2r = \text{diameter of the cylinder.}$$

26. Let V be the volume, x the side of the square base and h the height of the cuboid.

$$\text{Then,} \quad V = x^2h \Rightarrow h = \frac{V}{x^2} \qquad \ldots \text{(i)}$$

$$\therefore \quad S = 2(x^2 + xh + xh) \Rightarrow S = 2x^2 + 4xh.$$

$$\therefore \quad S = 2x^2 + \frac{4V}{x} \quad \text{[using (i)]}$$

$$\Rightarrow \quad \frac{dS}{dx} = 4x - \frac{4V}{x^2} \text{ and } \frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3}.$$

Now, $\frac{dS}{dx} = 0 \Rightarrow x^3 = V$.

For this value of x, $\frac{d^2S}{dx^2} = 4 + \frac{8V}{V} = 4 + 8 = 12 > 0$.

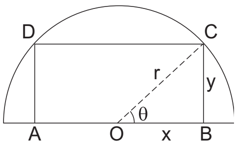
$\therefore$ surface area is minimum when $x^3 = V$.

Also, $h = \frac{V}{x^2} = \frac{x^3}{x^2} = x$.

$\therefore$ S is minimum when the cuboid is a cube.

27. Let $ABCD$ be the rectangle of length $2x$ and breadth y, inscribed in a semicircle of radius r and centre O. Let $\angle BOC = \theta$.

Then, $\frac{y}{r} = \sin\theta$ and $\frac{x}{r} = \cos\theta$.



$\therefore$ area of the rectangle is given by

$$A = 2xy = r^2 \sin 2\theta$$

$$\Rightarrow \frac{dA}{d\theta} = 2r^2 \cos 2\theta \text{ and } \frac{d^2A}{d\theta^2} = -4r^2 \sin 2\theta.$$

Now, $\dfrac{dA}{d\theta} = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \dfrac{\pi}{2} \Rightarrow \theta = \dfrac{\pi}{4}\cdot$

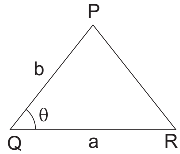
For this value of θ, we have $\dfrac{d^2A}{d\theta^2} = -4r^2 < 0.$

$\therefore$ area is maximum when $2x = 2r\cos\dfrac{\pi}{4} = r\sqrt{2}$ and $y = r\sin\dfrac{\pi}{4} = \dfrac{r\sqrt{2}}{2}\cdot$

Maximum area $= 2xy = \dfrac{r\sqrt{2} \times r\sqrt{2}}{2} = r^2$ sq units.

28. Let A be the area of $\triangle PQR$.

Then, $A = \dfrac{1}{2}ab\sin\theta$



$\Rightarrow \dfrac{dA}{d\theta} = \dfrac{1}{2}ab\cos\theta$

$\Rightarrow \dfrac{d^2A}{d\theta^2} = -\dfrac{1}{2}ab\sin\theta < 0.$

For maxima or minima, we have

$$\frac{dA}{d\theta} = 0 \Leftrightarrow \frac{1}{2}ab\cos\theta = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2}\cdot$$

$\therefore$ A is maximum, when $\theta = \dfrac{\pi}{2}\cdot$

30. Let the side of the square tank be x metres.

Then, total cost is given by $C = 50x^2 + 400h^2$ ... (i)

Also, $x^2h = 250 \Rightarrow h = \dfrac{250}{x^2}$ ... (ii)

$\therefore$ $C = 50x^2 + \dfrac{400 \times 62500}{x^4}$

$\Rightarrow \dfrac{dC}{dx} = (100x - 10^8 \cdot x^{-5})$ and $\dfrac{d^2C}{dx^2} = \left(100 + \dfrac{10^8 \times 5}{x^6}\right)\cdot$

Now, $\dfrac{dC}{dx} = 0 \Leftrightarrow x = 10$ and $\left[\dfrac{d^2C}{dx^2}\right]_{(x=10)} = 600 > 0.$

$\therefore$ cost is minimum when $x = 10$ and $h = \left(\dfrac{250}{100}\right)$ m = 2.5 m.

Hence, for minimum cost, the tank must have a square base of side 10 m and depth 2.5 m.

31. Let the side of the square piece cut from each corner of the given square plate of side 18 cm be x cm.

Then, the dimensions of the open box are $(18 - 2x)$ cm, $(18 - 2x)$ cm and x cm.

$\therefore$ $V = (18 - 2x)^2 \times x$

$\Rightarrow$ $V = 4x^3 - 72x^2 + 324x$ ... (i)

$\Rightarrow \dfrac{dV}{dx} = 12(x^2 - 12x + 27)$ and $\dfrac{d^2V}{dx^2} = 12(2x - 12).$

Now, $\frac{dV}{dx} = 0 \Leftrightarrow x^2 - 12x + 27 = 0 \Leftrightarrow (x-3)(x-9) = 0 \Leftrightarrow x = 3 \quad [\because \; x \neq 9]$

and $\left[\frac{d^2V}{dx^2}\right]_{(x=3)} = 12(2 \times 3 - 12) = -72 < 0.$

$\therefore$ V is maximum at $x = 3$ cm, and

maximum volume $= [4 \times 3^3 - 72 \times 3^2 + 324 \times 3]\text{ cm}^3 = 432\text{ cm}^3.$

32. Let x be the side of the square base and h be the depth.

Then, $V = x^2 h.$

Let the cost per square metre be ₹ p. Then,

$$C = (x^2 + 4xh)\,p \Rightarrow C = \left(x^2 + 4x \times \frac{V}{x^2}\right)p \Rightarrow C = \left(x^2 + \frac{4V}{x}\right)p.$$

$\therefore \frac{dC}{dx} = \left(2x - \frac{4V}{x^2}\right)p$ and $\frac{d^2C}{dx^2} = \left(2 + \frac{8V}{x^3}\right)p.$

Now, $\frac{dC}{dx} = 0 \Leftrightarrow \left(2x - \frac{4V}{x^2}\right) = 0 \Leftrightarrow x = (2V)^{1/3}$

And, $\left[\frac{d^2C}{dx^2}\right]_{x=(2V)^{1/3}} = p\left(2 + \frac{8V}{2V}\right) = 6p > 0.$

$\therefore$ C is minimum, when $x = (2V)^{1/3}$.

Then, $h = \frac{V}{x^2} = \frac{1}{2}(2V)^{1/3} = \frac{1}{2}x.$

Hence, for minimum cost, the depth of the tank should be equal to half of the side of its square base.

33. Let the perimeter of the square be x cm.

Then the perimeter of the triangle is $(36 - x)$ cm.

$\therefore$ side of the square $= \frac{x}{4}$ cm.

And, side of the triangle $= \frac{1}{3}(36 - x)$ cm.

$$\therefore A = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(12 - \frac{x}{3}\right)^2 = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right)$$

$$\Rightarrow A = \left(\frac{\sqrt{3}}{36} + \frac{1}{16}\right)x^2 - 2\sqrt{3}x + 36\sqrt{3}$$

$\Rightarrow \frac{dA}{dx} = \frac{(4\sqrt{3} + 9)}{144} \times 2x - 2\sqrt{3}$ and $\frac{d^2A}{dx^2} = \frac{4\sqrt{3} + 9}{72} > 0$

$\therefore \frac{dA}{dx} = 0 \Leftrightarrow x = \frac{144\sqrt{3}}{(4\sqrt{3} + 9)}$ cm.

34. Consider a right-angled $\triangle ABC$ in which hyp. $AC = 5$ cm and let $\angle BAC = \theta$.

Then, $AB = 5\cos\theta$ and $BC = 5\sin\theta$.

$\therefore$ area, $A = \frac{1}{2}AB \times BC = \frac{1}{2} \times 5\cos\theta \times 5\sin\theta$

$\Rightarrow A = \frac{25}{4}\sin 2\theta$

$$\Rightarrow \frac{dA}{d\theta} = \frac{25}{2}\cos 2\theta \text{ and } \frac{d^2A}{d\theta^2} = -25\sin 2\theta.$$

Now, $\frac{dA}{d\theta} = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$.

$$\left.\frac{d^2A}{d\theta^2}\right]_{\theta=\frac{\pi}{4}} = -25\sin\left(\frac{\pi}{2}\right) = -25 < 0.$$

$\therefore \theta = \frac{\pi}{4}$ is a point of maxima.

Maximum area $= \frac{25}{4} \times \sin\left(2 \times \frac{\pi}{4}\right) = \frac{25}{4}$ sq cm.

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5. Increasing and Decreasing Functions

INCREASING FUNCTION *A function f(x) defined on]a, b[is said to be increasing if*

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2) \text{ for all } x_1, x_2 \in]a, b[$$

or $$x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2) \text{ for all } x_1, x_2 \in]a, b[.$$

STRICTLY INCREASING FUNCTION *A function f(x) defined on]a, b[is said to be strictly increasing if*

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all } x_1, x_2 \in]a, b[$$

or $$x_1 > x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in]a, b[.$$

DECREASING FUNCTION *A function f(x) defined on]a, b[is said to be decreasing if*

$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2) \text{ for all } x_1, x_2 \in]a, b[$$

or $$x_1 > x_2 \Rightarrow f(x_1) \le f(x_2) \text{ for all } x_1, x_2 \in]a, b[.$$

STRICTLY DECREASING FUNCTION *A function f(x) defined on]a, b[is said to be strictly decreasing if*

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in]a, b[$$

or $$x_1 > x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all } x_1, x_2 \in]a, b[.$$

EXAMPLE 1 *Show that $f(x) = 3x + 5$ is a strictly increasing function on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 < x_2$. Then,

$$x_1 < x_2 \Rightarrow 3x_1 < 3x_2$$
$$\Rightarrow 3x_1 + 5 < 3x_2 + 5$$
$$\Rightarrow f(x_1) < f(x_2).$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in R$.

Hence, $f(x)$ is strictly increasing on R.

EXAMPLE 2 *Show that the function $f(x) = e^x$ is strictly increasing on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 > x_2$. Then,

$$x_1 > x_2 \Rightarrow e^{x_1} > e^{x_2} \quad [\because e > 1 \text{ and } x_1 > x_2 \Rightarrow e^{x_1} > e^{x_2}]$$

$$\Rightarrow f(x_1) > f(x_2).$$

Thus, $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in R$.

Hence, $f(x)$ is strictly increasing on R.

EXAMPLE 3 *Show that $f(x) = e^{-x}$ is a strictly decreasing function on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 > x_2$. Then,

$$x_1 > x_2 \Rightarrow e^{x_1} > e^{x_2} \quad [\because e > 1 \text{ and } x_1 > x_2 \Rightarrow e^{x_1} > e^{x_2}]$$

$$\Rightarrow \frac{1}{e^{x_1}} < \frac{1}{e^{x_2}}$$

$$\Rightarrow e^{-x_1} < e^{-x_2}$$

Thus, $x_1 > x_2 \Rightarrow e^{-x_1} < e^{-x_2}$ for all $x_1, x_2 \in R$.

Hence, $f(x) = e^{-x}$ is strictly decreasing on R.

EXAMPLE 4 *If a is a real number greater than 1, show that the function $f(x) = a^x$ is strictly increasing on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 > x_2$. Then,

$$x_1 > x_2 \Rightarrow a^{x_1} > a^{x_2} \quad [\because a > 1 \text{ and } x_1 > x_2 \Rightarrow a^{x_1} > a^{x_2}]$$

$$\Rightarrow f(x_1) > f(x_2).$$

Thus, $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in R$.

Hence, $f(x)$ is strictly increasing on R.

EXAMPLE 5 *If a is a real number such that $0 < a < 1$, show that the function $f(x) = a^x$ is strictly decreasing on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 < x_2$. Then,

$$x_1 < x_2 \Rightarrow a^{x_1} > a^{x_2} \quad [\because 0 < a < 1 \text{ and } x_1 < x_2 \Rightarrow a^{x_1} > a^{x_2}]$$

$$\Rightarrow f(x_1) > f(x_2).$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in R$.

Hence, $f(x)$ is strictly decreasing on R.

THEOREM 1 *Let $f(x)$ be a function continuous on $[a, b]$ and differentiable on $]a, b[$. Then,*

$$f'(x) > 0 \text{ for all } x \in]a, b[\Rightarrow f(x) \text{ is increasing on }]a, b[.$$

PROOF Let $f'(x) > 0$ for all $x \in]a, b[$.

Let $x_1, x_2 \in]a, b[$ such that $x_1 < x_2$.

Then, clearly $f(x)$ is continuous on $[x_1, x_2]$ and differentiable on $]x_1, x_2[$.

So, by the mean-value theorem, there exists a real number $c \in]x_1, x_2[$ such that $$f'(c) = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)}.$$

Since $f'(x) > 0$ for all $x \in]a, b[$, in particular, $f'(c) > 0$.

Now, $$f'(c) > 0 \Rightarrow \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} > 0$$
$$\Rightarrow f(x_2) - f(x_1) > 0 \quad [\because \; x_2 - x_1 > 0 \text{ when } x_1 < x_2]$$
$$\Rightarrow f(x_2) > f(x_1) \Rightarrow f(x_1) < f(x_2).$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

Hence, $f(x)$ is an increasing function on $]a, b[$.

THEOREM 2 *Let $f(x)$ be a function, continuous on $[a, b]$ and differentiable on $]a, b[$. Then, $f'(x) < 0$ for all $x \in]a, b[\Rightarrow f(x)$ is decreasing on $]a, b[$.*

PROOF Let it be given that $f'(x) < 0$ for all $x \in]a, b[$.

Let $x_1, x_2 \in]a, b[$ such that $x_1 < x_2$.

Then, $f(x)$ is continuous on $[x_1, x_2]$ and differentiable on $]x_1, x_2[$.

So, by the mean-value theorem, there exists a real number $c \in]x_1, x_2[$ such that $$f'(c) = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)}.$$

Since, $f'(x) < 0$ for all $x \in]a, b[$, in particular, $f'(c) < 0$.

Now, $$f'(c) < 0 \Rightarrow \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} < 0$$
$$\Rightarrow f(x_2) - f(x_1) < 0 \quad [\because \; x_2 - x_1 > 0 \text{ when } x_1 < x_2]$$
$$\Rightarrow f(x_2) < f(x_1) \Rightarrow f(x_1) > f(x_2).$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

This shows that $f(x)$ is decreasing on $]a, b[$.

SUMMARY

(i) If $f'(x) > 0$ for all $x \in]a, b[$ then $f(x)$ is increasing on $]a, b[$.

(ii) If $f'(x) < 0$ for all $x \in]a, b[$ then $f(x)$ is decreasing on $]a, b[$.

COROLLARY 1 *If $f'(x) > 0$ for all $x \in]a, b[$, except for a finite number of points where $f'(x) = 0$ then the function is increasing on $]a, b[$.*

PROOF Let $f'(x) > 0$ for all $x \in]a, b[$ except at one point c, where $f'(c) = 0$.

Then, $f'(x) > 0$ for all $x \in]a, c[$ and $f'(x) > 0$ for all $x \in]c, b[$.

Thus, $f(x)$ is increasing on $]a, c[$ as well as on $]c, b[$.

Hence, $f(x)$ is increasing on $]a, b[$.

The result may now be extended to the case when $f'(x) = 0$ at a finite number of points.

COROLLARY 2 *If $f'(x) < 0$ for all $x \in]a, b[$, except for a finite number of points where $f'(x) = 0$, then $f(x)$ is decreasing on $]a, b[$.*

PROOF It is similar to that of Corollary 1.

Two Important Results

I. (i) If $f'(x) \geq 0$ for all $x \in]a, b[$ then $f(x)$ is increasing on $[a, b]$.

(ii) If $f'(x) > 0$ for all $x \in]a, b[$ then $f(x)$ is strictly increasing on $[a, b]$.

II. (i) If $f'(x) \leq 0$ for all $x \in]a, b[$ then $f(x)$ is decreasing on $[a, b]$.

(ii) If $f'(x) < 0$ for all $x \in]a, b[$ then $f(x)$ is strictly decreasing on $[a, b]$.

SOLVED EXAMPLES

EXAMPLE 1 *Show that the function $f(x) = (x^3 - 6x^2 + 12x - 18)$ is an increasing function on R.* **[CBSE 2002C]**

SOLUTION $f(x) = (x^3 - 6x^2 + 12x - 18)$

$\Rightarrow$ $f'(x) = 3x^2 - 12x + 12$

$= 3(x^2 - 4x + 4) = 3(x-2)^2 \geq 0$ for all $x \in R$.

Thus, $f'(x) \geq 0$ for all $x \in R$.

Hence, $f(x)$ is an increasing function on R.

EXAMPLE 2 *Show that the function $f(x) = e^x$ is strictly increasing on R.*

SOLUTION $f(x) = e^x \Rightarrow f'(x) = e^x$.

Case I *When $x > 0$*

In this case, $e^x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) > 1$

$\Rightarrow$ $e^x > 0$ for all $x > 0$

$\Rightarrow$ $f'(x) > 0$ for all $x > 0$.

Case II *When $x = 0$*

In this case, $e^x = e^0 = 1 > 0$

$\Rightarrow$ $f'(x) > 0$ when $x = 0$.

Case III *When $x < 0$*

Let $x = -y$, where y is positive.

$$\text{Then,} \quad e^x = e^{-y} = \frac{1}{e^y} = \frac{1}{(\text{a +ve quantity})} > 0.$$

So, $f'(x) > 0$ when $x < 0$.

Thus, from all the cases, we have $f'(x) > 0$ for all $x \in R$.

Hence, $f(x)$ is strictly increasing for all $x \in R$.

EXAMPLE 3 *Show that $f(x) = e^{1/x}$ is a strictly decreasing function for all $x > 0$.*

SOLUTION $f(x) = e^{1/x} \Rightarrow f'(x) = -\frac{1}{x^2} e^{1/x} < 0$ for all $x > 0$

[$\because$ $x^2 > 0$ and $e^{1/x} > 0$ when $x > 0$].

Thus, $f'(x) < 0$ for all $x > 0$.

Hence, $f(x)$ is strictly decreasing for all $x > 0$.

EXAMPLE 4 *Show that $f(x) = (x-1)e^x + 1$ is a strictly increasing function for all $x > 0$.*

SOLUTION $f(x) = (x-1)e^x + 1 \Rightarrow f'(x) = (x-1)e^x + e^x$

$\Rightarrow f'(x) = xe^x > 0$ for all $x > 0$

[$\because$ $x > 0$ and $e^x > 0$ when $x > 0$].

Thus, $f'(x) > 0$ for all $x > 0$.

Hence, $f(x)$ is a strictly increasing function for all $x > 0$.

EXAMPLE 5 *Show that the function $f(x) = (x - \sin x)$ is increasing for all $x \in R$.*

SOLUTION $f(x) = (x - \sin x)$

$\Rightarrow$ $f'(x) = (1 - \cos x) \geq 0$ for all $x \in R$ [$\because$ $-1 \leq \cos x \leq 1$]

$\Rightarrow$ $f'(x) \geq 0$ for all $x \in R$.

Hence, $f(x) = (x - \sin x)$ is increasing for all $x \in R$.

EXAMPLE 6 *Show that the function $f(x) = \cos^2 x$ is strictly decreasing on $\left]0, \frac{\pi}{2}\right[$.*

SOLUTION $f(x) = \cos^2 x$

$\Rightarrow$ $f'(x) = -2\cos x \sin x = -(\sin 2x) < 0$ in $\left]0, \frac{\pi}{2}\right[$

[$\because$ $\sin 2x > 0$ in the 1st quadrant]

$\Rightarrow$ $f'(x) < 0$ for all $x \in \left]0, \frac{\pi}{2}\right[$.

Hence, $f(x) = \cos^2 x$ is strictly decreasing on $\left]0, \frac{\pi}{2}\right[$.

EXAMPLE 7 *Show that $f(x) = \log \sin x$ is (a) strictly increasing on $\left]0, \frac{\pi}{2}\right[$ and (b) strictly decreasing on $\left]\frac{\pi}{2}, \pi\right[$.*

SOLUTION $f(x) = \log \sin x$

$$\Rightarrow \quad f'(x) = \frac{1}{\sin x} \cdot \cos x \Rightarrow f'(x) = \cot x \qquad \ldots \text{(i)}$$

(a) We know that for each $x \in \left]0, \frac{\pi}{2}\right[$,

$$f'(x) = \cot x > 0.$$

$\therefore \quad f(x)$ is strictly increasing in $\left]0, \frac{\pi}{2}\right[$.

(b) We know that for each $x \in \left]\frac{\pi}{2}, \pi\right[$,

$$f'(x) = \cot x < 0.$$

$\therefore \quad f(x)$ is strictly decreasing in $\left]\frac{\pi}{2}, \pi\right[$.

EXAMPLE 8 *Show that the function $f(x) = \sin x$ is*

(a) strictly increasing on $\left]0, \frac{\pi}{2}\right[$

(b) strictly decreasing on $\left]\frac{\pi}{2}, \pi\right[$

(c) neither increasing nor decreasing on $]0, \pi[$

SOLUTION $f(x) = \sin x \;\Rightarrow\; f'(x) = \cos x.$

(a) We know that for each $x \in \left]0, \frac{\pi}{2}\right[$, $\cos x > 0$.

$\therefore \quad f'(x) > 0$ for all $x \in \left]0, \frac{\pi}{2}\right[$.

Hence, $f(x)$ is increasing on $\left]0, \frac{\pi}{2}\right[$.

(b) We know that for each $x \in \left]\frac{\pi}{2}, \pi\right[$, $\cos x < 0$.

$\therefore \quad f'(x) < 0$ for all $x \in \left]\frac{\pi}{2}, \pi\right[$.

Hence, $f(x)$ is decreasing on $\left]\frac{\pi}{2}, \pi\right[$.

(c) It follows from the above two results that $f(x) = \sin x$ is increasing on $\left]0, \frac{\pi}{2}\right[$ and decreasing on $\left]\frac{\pi}{2}, \pi\right[$.

So, it is neither increasing nor decreasing on $]0, \pi[$.

EXAMPLE 9 *Show that the function $f(x) = (x^2 - x + 1)$ is neither increasing nor decreasing on* $]0, 1[$.

SOLUTION $f(x) = (x^2 - x + 1) \;\Rightarrow\; f'(x) = (2x - 1).$

Now, $f'(x) > 0 \;\Leftrightarrow\; (2x-1) > 0 \;\Leftrightarrow\; x > \frac{1}{2}\cdot$

$\therefore\;$ $f(x)$ is increasing when $x \in \left]\frac{1}{2}, 1\right[\cdot$

And, $f'(x) < 0 \;\Leftrightarrow\; (2x-1) < 0 \;\Leftrightarrow\; x < \frac{1}{2}\cdot$

$\therefore\;$ $f(x)$ is decreasing when $x \in \left]0, \frac{1}{2}\right[\cdot$

Hence, the given function is neither increasing nor decreasing on]0, 1[.

EXAMPLE 10 *Prove that the function $f(x) = 10^x$ is strictly increasing on R.*

SOLUTION $f(x) = 10^x \;\Rightarrow\; f'(x) = (10^x)(\log 10) > 0$ for all $x \in R$.

Hence, $f(x) = 10^x$ is strictly increasing on R.

EXAMPLE 11 *Show that $f(x) = \tan^{-1}(\cos x + \sin x)$ is a strictly increasing function on the interval $\left(0, \frac{\pi}{4}\right)\cdot$* **[CBSE 2007]**

SOLUTION $f(x) = \tan^{-1}(\cos x + \sin x)$

$$\Rightarrow\; f'(x) = \frac{1}{1+(\cos x + \sin x)^2}\cdot\frac{d}{dx}(\cos x + \sin x)$$

$$= \frac{(-\sin x + \cos x)}{(1+\cos^2 x + \sin^2 x + 2\sin x\,\cos x)}$$

$$= \frac{(\cos x - \sin x)}{(2+\sin 2x)}$$

Now, when $0 < x < \frac{\pi}{4}$, we have $\cos x > \sin x$ and $\sin 2x > 0$.

$\therefore\;$ $(\cos x - \sin x) > 0$ and $(2 + \sin 2x) > 0$.

$\therefore\;$ $f'(x) > 0$ for all x when $0 < x < \frac{\pi}{4}\cdot$

Hence, $f(x)$ is strictly increasing in $\left(0, \frac{\pi}{4}\right)\cdot$

EXAMPLE 12 *Find the intervals on which the function $f(x) = 10 - 6x - 2x^2$ is (a) strictly increasing (b) strictly decreasing.*

SOLUTION $f(x) = 10 - 6x - 2x^2$

$$\Rightarrow\; f'(x) = -6 - 4x = -2(3+2x) = -4\left(x + \frac{3}{2}\right) \qquad \ldots\text{(i)}$$

(a) $f(x)$ is strictly increasing

$\Leftrightarrow\; f'(x) > 0$

$\Leftrightarrow\; -4\left(x+\frac{3}{2}\right) > 0$ [from (i)]

$$\Leftrightarrow \left(x+\frac{3}{2}\right)<0 \Leftrightarrow x<\frac{-3}{2}.$$

$\therefore$ $f(x)$ is strictly increasing on the interval $\left]-\infty, \frac{-3}{2}\right[$.

(b) $f(x)$ is strictly decreasing

$$\Leftrightarrow f'(x)<0$$

$$\Leftrightarrow -4\left(x+\frac{3}{2}\right)<0 \qquad \text{[from (i)]}$$

$$\Leftrightarrow \left(x+\frac{3}{2}\right)>0$$

$$\Leftrightarrow x>\frac{-3}{2}.$$

$\therefore$ $f(x)$ is strictly decreasing on the interval $\left]\frac{-3}{2}, \infty\right[$.

Hence, $f(x)$ is strictly increasing on the interval $\left]-\infty, \frac{-3}{2}\right[$ and strictly decreasing on $\left]\frac{-3}{2}, \infty\right[$.

EXAMPLE 13 *Find the intervals on which the function* $f(x)=-2x^3-9x^2-12x+1$ *is (a) strictly increasing (b) strictly decreasing.*

SOLUTION $f(x)=-2x^3-9x^2-12x+1$

$$\Rightarrow f'(x)=-6x^2-18x-12=-6(x^2+3x+2)=-6(x+2)(x+1) \quad \ldots\text{(i)}$$

(a) $f(x)$ is strictly increasing

$$\Leftrightarrow f'(x)>0$$

$$\Leftrightarrow -6(x+2)(x+1)>0 \qquad \text{[from (i)]}$$

$$\Leftrightarrow (x+2)(x+1)<0$$

$$\Leftrightarrow -2<x<-1$$

$$\Leftrightarrow x\in\,]-2,-1[.$$

$\therefore$ $f(x)$ is strictly increasing on the interval $]-2,-1[$.

(b) $f(x)$ is strictly decreasing

$$\Leftrightarrow f'(x)<0$$

$$\Leftrightarrow -6(x+2)(x+1)<0 \qquad \text{[from (i)]}$$

$$\Leftrightarrow (x+2)(x+1)>0$$

$$\Leftrightarrow [(x+2)>0 \text{ and } (x+1)>0] \text{ or } [(x+2)<0 \text{ and } (x+1)<0]$$

$$\Leftrightarrow (x>-2 \text{ and } x>-1) \text{ or } (x<-2 \text{ and } x<-1)$$

$$\Leftrightarrow (x>-1) \text{ or } (x<-2)$$

$$\Leftrightarrow x\in\,]-1,\infty[\text{ or } x\in\,]-\infty,-2[$$

$$\Leftrightarrow x\in\,]-\infty,-2[\,\cup\,]-1,\infty[.$$

$\therefore$ $f(x)$ is strictly decreasing on $]-\infty,-2[\,\cup\,]-1,\infty[$.

EXAMPLE 14 *Find the intervals on which the function* $f(x) = 2x^3 - 15x^2 + 36x + 6$ *is (a) increasing (b) decreasing.* **[CBSE 2004C, '05, '09C]**

SOLUTION $f(x) = 2x^3 - 15x^2 + 36x + 6$

$\Rightarrow f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-3)(x-2)$ ...(i)

(a) $f(x)$ is increasing

$\Leftrightarrow f'(x) \geq 0$

$\Leftrightarrow 6(x-3)(x-2) \geq 0$ [from (i)]

$\Leftrightarrow (x-3)(x-2) \geq 0$

$\Leftrightarrow [(x-3) \geq 0 \text{ and } (x-2) \geq 0]$

or $[(x-3) \leq 0 \text{ and } (x-2) \leq 0]$

$\Leftrightarrow [x \geq 3 \text{ and } x \geq 2] \text{ or } [x \leq 3 \text{ and } x \leq 2]$

$\Leftrightarrow [x \geq 3] \text{ or } [x \leq 2]$

$\Leftrightarrow x \in [3, \infty[\text{ or } x \in]-\infty, 2]$

$\Leftrightarrow x \in]-\infty, 2] \cup [3, \infty[.$

$\therefore f(x)$ is increasing on $]-\infty, 2] \cup [3, \infty[$.

(b) $f(x)$ is decreasing

$\Leftrightarrow f'(x) \leq 0$

$\Leftrightarrow 6(x-3)(x-2) \leq 0$ [from (i)]

$\Leftrightarrow (x-3)(x-2) \leq 0$

$\Leftrightarrow 2 \leq x \leq 3$

$\Leftrightarrow x \in [2, 3].$

$\therefore f(x)$ is decreasing on $[2, 3]$.

Hence, $f(x)$ is increasing on $]-\infty, 2] \cup [3, \infty[$ and decreasing on $[2, 3]$.

EXAMPLE 15 *Find the intervals on which the function* $f(x) = x^3 + 2x^2 - 1$ *is (a) increasing (b) decreasing.*

SOLUTION $f(x) = x^3 + 2x^2 - 1$

$\Rightarrow f'(x) = 3x^2 + 4x = 3x\left(x + \frac{4}{3}\right)$ ...(i)

(a) $f(x)$ is increasing

$\Leftrightarrow f'(x) \geq 0$

$\Leftrightarrow 3x\left(x + \frac{4}{3}\right) \geq 0$ [from (i)]

$\Leftrightarrow x\left(x + \frac{4}{3}\right) \geq 0$

$\Leftrightarrow [x \geq 0 \text{ and } \left(x + \frac{4}{3}\right) \geq 0] \text{ or } [x \leq 0 \text{ and } \left(x + \frac{4}{3}\right) \leq 0]$

$\Leftrightarrow [x \geq 0 \text{ and } x \geq -\frac{4}{3}] \text{ or } [x \leq 0 \text{ and } x \leq -\frac{4}{3}]$

$\Leftrightarrow (x \geq 0) \text{ or } \left(x \geq -\frac{4}{3}\right)$

$$\Leftrightarrow \quad x \in [0, \infty[\text{ or } x \in]-\infty, -\frac{4}{3}]$$

$$\Leftrightarrow \quad x \in \left]-\infty, -\frac{4}{3}\right] \cup [0, \infty[.$$

$\therefore$ $f(x)$ is increasing on $\left]-\infty, -\frac{4}{3}\right] \cup [0, \infty[$.

(b) $f(x)$ is decreasing

$$\Leftrightarrow \quad f'(x) \le 0$$

$$\Leftrightarrow \quad 3x\left(x + \frac{4}{3}\right) \le 0 \qquad \text{[from (i)]}$$

$$\Leftrightarrow \quad x\left(x + \frac{4}{3}\right) \le 0$$

$$\Leftrightarrow \quad -\frac{4}{3} \le x \le 0$$

$$\Leftrightarrow \quad x \in \left[-\frac{4}{3}, 0\right].$$

$\therefore$ $f(x)$ is decreasing on $\left[-\frac{4}{3}, 0\right]$.

Hence, $f(x)$ is increasing on $\left]-\infty, -\frac{4}{3}\right] \cup [0, \infty[$ and decreasing on $\left[-\frac{4}{3}, 0\right]$.

EXAMPLE 16 *Find the intervals on which the function* $f(x) = x^3 + 3x^2 - 105x + 25$ *is (a) increasing (b) decreasing.*

SOLUTION $f(x) = x^3 + 3x^2 - 105x + 25$

$$\Rightarrow \quad f'(x) = 3x^2 + 6x - 105 = 3(x^2 + 2x - 35) = 3(x+7)(x-5) \quad \ldots \text{(i)}$$

(a) $f(x)$ is increasing

$$\Leftrightarrow \quad f'(x) \ge 0$$

$$\Leftrightarrow \quad 3(x+7)(x-5) \ge 0 \qquad \text{[from (i)]}$$

$$\Leftrightarrow \quad (x+7)(x-5) \ge 0$$

$$\Leftrightarrow \quad [(x+7) \ge 0 \text{ and } (x-5) \ge 0] \text{ or } [(x+7) \le 0 \text{ and } (x-5) \le 0]$$

$$\Leftrightarrow \quad [x \ge -7 \text{ and } x \ge 5] \text{ or } [x \le -7 \text{ and } x \le 5]$$

$$\Leftrightarrow \quad (x \ge 5) \text{ or } (x \le -7)$$

$$\Leftrightarrow \quad x \in [5, \infty[\text{ or } x \in]-\infty, -7]$$

$$\Leftrightarrow \quad x \in]-\infty, -7] \cup [5, \infty[.$$

$\therefore$ $f(x)$ is increasing on $]-\infty, -7] \cup [5, \infty[$.

(b) $f(x)$ is decreasing

$$\Leftrightarrow \quad f'(x) \le 0$$

$$\Leftrightarrow \quad 3(x+7)(x-5) \le 0 \qquad \text{[from (i)]}$$

$$\Leftrightarrow \quad (x+7)(x-5) \le 0$$

$$\Leftrightarrow \quad -7 \le x \le 5$$

$$\Leftrightarrow \quad x \in [-7, 5].$$

$\therefore\ f(x)$ is decreasing on $[-7, 5]$.

Hence, $f(x)$ is increasing on $]-\infty, -7] \cup [5, \infty[$ and decreasing on $[-7, 5]$.

EXAMPLE 17 *Find the intervals on which the function* $f(x) = 5 + 36x + 3x^2 - 2x^3$ *is (a) increasing (b) decreasing.*

SOLUTION $f(x) = 5 + 36x + 3x^2 - 2x^3$

$\Rightarrow\ f'(x) = 36 + 6x - 6x^2 = -6(x^2 - x - 6) = -6(x+2)(x-3)$ ...(i)

(a) $f(x)$ is increasing

$\Leftrightarrow\ f'(x) \geq 0$

$\Leftrightarrow\ -6(x+2)(x-3) \geq 0$ [from (i)]

$\Leftrightarrow\ (x+2)(x-3) \leq 0$

$\Leftrightarrow\ -2 \leq x \leq 3$

$\Leftrightarrow\ x \in [-2, 3]$.

$\therefore\ f(x)$ is increasing on $[-2, 3]$.

(b) $f(x)$ is decreasing

$\Leftrightarrow\ f'(x) \leq 0$

$\Leftrightarrow\ -6(x+2)(x-3) \leq 0$ [from (i)]

$\Leftrightarrow\ (x+2)(x-3) \leq 0$

$\Leftrightarrow\ [(x+2) \geq 0 \text{ and } (x-3) \geq 0]$ or $[(x+2) \leq 0 \text{ and } (x-3) \leq 0]$

$\Leftrightarrow\ [x \geq -2 \text{ and } x \geq 3]$ or $[x \leq -2 \text{ and } x \leq 3]$

$\Leftrightarrow\ (x \geq 3)$ or $(x \leq -2)$

$\Leftrightarrow\ x \in [3, \infty[$ or $x \in]-\infty, -2]$

$\Leftrightarrow\ x \in]-\infty, -2] \cup [3, \infty[$.

$\therefore\ f(x)$ is decreasing on $]-\infty, -2] \cup [3, \infty[$.

Hence, $f(x)$ is increasing on $[-2, 3]$ and decreasing on $]-\infty, -2] \cup [3, \infty[$.

EXAMPLE 18 *Find the intervals on which the function* $f(x) = (x+1)^3 (x-3)^3$ *is (a) increasing (b) decreasing.* **[CBSE 2001C]**

SOLUTION $f(x) = (x+1)^3 (x-3)^3$

$$\Rightarrow\ f'(x) = (x+1)^3 \cdot \frac{d}{dx}(x-3)^3 + (x-3)^3 \cdot \frac{d}{dx}(x+1)^3$$

$$= (x+1)^3 \cdot 3(x-3)^2 + (x-3)^3 \cdot 3(x+1)^2$$

$$= 3(x+1)^2 (x-3)^2 [(x+1) + (x-3)]$$

$$= 6(x+1)^2 (x-3)^2 (x-1) \quad \text{...(i)}$$

(a) $f(x)$ is increasing

$\Leftrightarrow\ f'(x) \geq 0$

$\Leftrightarrow\ 6(x+1)^2 (x-3)^2 (x-1) \geq 0$ [from (i)]

$\Leftrightarrow\ (x-1) \geq 0$

$\Leftrightarrow\ x \geq 1$

$\Leftrightarrow\ x \in [1, \infty[$.

$\therefore\ f(x)$ is increasing on $[1, \infty[$.

(b) $f(x)$ is decreasing

$$\Leftrightarrow \quad f'(x) \le 0$$

$$\Leftrightarrow \quad 6(x+1)^2(x-3)^2(x-1) \le 0 \qquad \text{[from (i)]}$$

$$\Leftrightarrow \quad (x-1) \le 0 \quad \Leftrightarrow \quad x \le 1$$

$$\Leftrightarrow \quad x \in]-\infty, 1].$$

$\therefore$ $f(x)$ is decreasing on $]-\infty, 1]$.

Hence, $f(x)$ is increasing on $[1, \infty[$ and decreasing on $]-\infty, 1]$.

EXAMPLE 19 *Find the intervals on which the function* $f(x) = \dfrac{4x^2+1}{x}$, $(x \ne 0)$ *is (a) increasing (b) decreasing.* **[CBSE 2004]**

SOLUTION

$$f(x) = \frac{4x^2+1}{x},\ x \ne 0$$

$$\Rightarrow \quad f(x) = \left(4x + \frac{1}{x}\right),\ x \ne 0$$

$$\Rightarrow \quad f'(x) = \left(4 - \frac{1}{x^2}\right)$$

$$\Rightarrow \quad f'(x) = \frac{(4x^2-1)}{x^2} \qquad \ldots\text{(i)}$$

(a) $f(x)$ is increasing

$$\Leftrightarrow \quad f'(x) \ge 0 \Leftrightarrow \frac{(4x^2-1)}{x^2} \ge 0 \qquad \text{[from (i)]}$$

$$\Leftrightarrow \quad (4x^2-1) \ge 0 \qquad [\because x^2 > 0]$$

$$\Leftrightarrow \quad (2x-1)(2x+1) \ge 0 \Leftrightarrow 2\left(x - \frac{1}{2}\right) \cdot 2\left(x + \frac{1}{2}\right) \ge 0$$

$$\Leftrightarrow \quad \left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right) \ge 0$$

$$\Leftrightarrow \quad \left[\left(x - \frac{1}{2}\right) \ge 0 \text{ and } \left(x + \frac{1}{2}\right) \ge 0\right]$$

$$\text{or} \quad \left[\left(x - \frac{1}{2}\right) \le 0 \text{ and } \left(x + \frac{1}{2}\right) \le 0\right]$$

$$\Leftrightarrow \quad \left[x \ge \frac{1}{2} \text{ and } x \ge -\frac{1}{2}\right] \text{ or } \left[x \le \frac{1}{2} \text{ and } x \le -\frac{1}{2}\right]$$

$$\Leftrightarrow \quad \left(x \ge \frac{1}{2}\right) \text{ or } \left(x \le -\frac{1}{2}\right)$$

$$\Leftrightarrow \quad x \in \left[\frac{1}{2}, \infty\right[\text{ or } x \in \left]-\infty, -\frac{1}{2}\right]$$

$$\Leftrightarrow \quad x \in \left]-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right[.$$

$$\therefore \quad f(x) \text{ is increasing on } \left]-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right[.$$

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \le 0 \Leftrightarrow \dfrac{(4x^2-1)}{x^2} \le 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (4x^2-1) \le 0 \qquad [\because x^2 > 0]$

$\Leftrightarrow \quad (2x-1)(2x+1) \le 0 \Leftrightarrow 2\left(x-\frac{1}{2}\right)\cdot 2\left(x+\frac{1}{2}\right) \le 0$

$\Leftrightarrow \quad \left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right) \le 0$

$\Leftrightarrow \quad -\frac{1}{2} \le x \le \frac{1}{2}$

$\Leftrightarrow \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$

$\therefore \; f(x)$ is decreasing on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Hence, $f(x)$ is increasing on $\left]-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right[$ and decreasing on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

EXAMPLE 20 *Find the intervals on which the function* $f(x) = \dfrac{x}{(x^2+1)}$ *is*

(a) increasing *(b) decreasing.* **[CBSE 2003]**

SOLUTION

$$f(x) = \frac{x}{(x^2+1)}$$

$$\Rightarrow \quad f'(x) = \frac{(x^2+1)\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \frac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2}$$

$$\Rightarrow \quad f'(x) = \frac{(1-x^2)}{(x^2+1)^2} \qquad \ldots\text{(i)}$$

(a) $f(x)$ is increasing

$\Leftrightarrow \quad f'(x) \ge 0$

$\Leftrightarrow \quad \dfrac{(1-x^2)}{(x^2+1)^2} \ge 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (1-x^2) \ge 0 \qquad [\because (x^2+1)^2 > 0]$

$\Leftrightarrow \quad -(1-x^2) \le 0 \Leftrightarrow (x^2-1) \le 0$

$\Leftrightarrow \quad (x-1)(x+1) \le 0$

$\Leftrightarrow \quad -1 \le x \le 1$

$\Leftrightarrow \quad x \in [-1, 1].$

$\therefore \; f(x)$ is increasing on $[-1, 1]$.

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \le 0$

$\Leftrightarrow \quad \dfrac{(1-x^2)}{(x^2+1)^2} \le 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (1-x^2) \le 0 \qquad [\because\ (x^2+1)^2 > 0]$

$\Leftrightarrow \quad -(1-x^2) \ge 0$

$\Leftrightarrow \quad (x^2-1) \le 0 \quad \Leftrightarrow \quad (x-1)(x+1) \ge 0$

$\Leftrightarrow \quad [(x-1) \ge 0 \text{ and } (x+1) \ge 0] \text{ or } [(x-1) \le 0 \text{ and } (x+1) \le 0]$

$\Leftrightarrow \quad [x \ge 1 \text{ and } x \ge -1] \text{ or } [x \le 1 \text{ and } x \le -1]$

$\Leftrightarrow \quad (x \ge 1) \text{ or } (x \le -1)$

$\Leftrightarrow \quad x \in [1, \infty[\text{ or } x \in]-\infty, -1]$

$\Leftrightarrow \quad x \in]-\infty, -1] \cup [1, \infty[.$

$\therefore\ f(x)$ is decreasing on $]-\infty, -1] \cup [1, \infty[$.

Hence, $f(x)$ is increasing on $[-1, 1]$ and decreasing on $]-\infty, -1] \cup [1, \infty[$.

EXAMPLE 21 *Find the intervals on which the function $f(x) = (x+2)\, e^{-x}$ is*

(a) increasing (b) decreasing. **[CBSE 2000C]**

SOLUTION $f(x) = (x+2)\, e^{-x}$

$\Rightarrow\ f'(x) = -(x+2)e^{-x} + e^{-x} \cdot 1$

$\Rightarrow\ f'(x) = -(x+1)e^{-x} \qquad \ldots$(i)

(a) $f(x)$ is increasing

$\Leftrightarrow \quad f'(x) \ge 0$

$\Leftrightarrow \quad -(x+1)e^{-x} \ge 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (x+1)e^{-x} \le 0$

$\Leftrightarrow \quad (x+1) \le 0 \qquad [\because\ e^{-x} > 0]$

$\Leftrightarrow \quad x \le -1 \quad \Leftrightarrow \quad x \in]-\infty, -1].$

$\therefore\ f(x)$ is increasing on $]-\infty, -1]$.

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \le 0$

$\Leftrightarrow \quad -(x+1)e^{-x} \le 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (x+1)e^{-x} \ge 0$

$\Leftrightarrow \quad (x+1) \ge 0 \qquad [\because\ e^{-x} > 0]$

$\Leftrightarrow \quad x \ge -1$

$\Leftrightarrow \quad x \in [-1, \infty[.$

$\therefore\ f(x)$ is decreasing on $[-1, \infty[$.

Hence, $f(x)$ is increasing on $]-\infty, -1]$ and decreasing on $[-1, \infty[$.

EXAMPLE 22 *Find the intervals on which the function $f(x) = \log(1+x) - \dfrac{x}{(1+x)}$ is*

(a) increasing (b) decreasing. **[CBSE 2000C]**

SOLUTION $f(x) = \log(1+x) - \dfrac{x}{(1+x)}$

$$\Rightarrow\ f'(x)=\left\{\frac{1}{(1+x)}-\frac{(1+x)\cdot 1-x\cdot 1}{(1+x)^2}\right\}$$

$$=\left\{\frac{1}{(1+x)}-\frac{1}{(1+x)^2}\right\}=\frac{(1+x-1)}{(1+x)^2}$$

$$\Rightarrow\ f'(x)=\frac{x}{(1+x)^2} \qquad \ldots(\text{i})$$

(a) $f(x)$ is increasing

$\Leftrightarrow\ f'(x)\geq 0$

$\Leftrightarrow\ \dfrac{x}{(1+x)^2}\geq 0$ [from (i)]

$\Leftrightarrow\ x\geq 0$ [$\because\ (1+x)^2\geq 0$]

$\Leftrightarrow\ x\in[0,\infty[$.

(b) $f(x)$ is decreasing

$\Leftrightarrow\ f'(x)\leq 0$

$\Leftrightarrow\ \dfrac{x}{(1+x)^2}\leq 0$ [from (i)]

$\Leftrightarrow\ x\leq 0$

$\Leftrightarrow\ x\in]-\infty, 0]$.

Hence, $f(x)$ is increasing on $[0,\infty[$ and decreasing on $]-\infty, 0]$.

EXAMPLE 23 *Find the intervals on which the function* $f(x)=(\sin x-\cos x)$, $0<x<2\pi$, *is (a) increasing (b) decreasing.* **[CBSE 2000C, '09, '11]**

SOLUTION $f(x)=(\sin x-\cos x),\ 0<x<2\pi$

$$\Rightarrow\ f'(x)=(\cos x+\sin x)=\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)$$

$$\Rightarrow\ f'(x)=\sqrt{2}\left(\sin\frac{\pi}{4}\cos x+\cos\frac{\pi}{4}\sin x\right)$$

$$\Rightarrow\ f'(x)=\sqrt{2}\sin\left(\frac{\pi}{4}+x\right) \qquad \ldots(\text{i})$$

Now, $0<x<2\pi\ \Rightarrow\ \dfrac{\pi}{4}<\left(\dfrac{\pi}{4}+x\right)<\left(\dfrac{\pi}{4}+2\pi\right)$

$$\Rightarrow\ \frac{\pi}{4}<\left(\frac{\pi}{4}+x\right)<\frac{9\pi}{4} \qquad \ldots(\text{ii})$$

(a) $f(x)$ is increasing

$\Leftrightarrow\ f'(x)\geq 0$

$\Leftrightarrow\ \sqrt{2}\sin\left(\dfrac{\pi}{4}+x\right)\geq 0$ [from (i)]

$\Leftrightarrow\ \sin\left(\dfrac{\pi}{4}+x\right)\geq 0$

$$\Leftrightarrow \quad \left\{\frac{\pi}{4} < \left(\frac{\pi}{4} + x\right) \le \pi\right\} \text{ or } \left\{2\pi \le \left(\frac{\pi}{4} + x\right) < \frac{9\pi}{4}\right\}$$

$$\Leftrightarrow \quad \left\{0 < x \le \frac{3\pi}{4}\right\} \text{ or } \left\{\frac{7\pi}{4} \le x < 2\pi\right\}$$

$$\Leftrightarrow \quad x \in \left]0, \frac{3\pi}{4}\right] \text{ or } x \in \left[\frac{7\pi}{4}, 2\pi\right[$$

$$\Leftrightarrow \quad x \in \left]0, \frac{3\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 2\pi\right[.$$

$\therefore f(x)$ is increasing on $\left]0, \frac{3\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 2\pi\right[$.

(b) $f(x)$ is decreasing

$$\Leftrightarrow \quad f'(x) \le 0$$

$$\Leftrightarrow \quad \sqrt{2}\sin\left(\frac{\pi}{4} + x\right) \le 0 \qquad \text{[from (i)]}$$

$$\Leftrightarrow \quad \sin\left(\frac{\pi}{4} + x\right) \le 0$$

$$\Leftrightarrow \quad \pi \le \left(\frac{\pi}{4} + x\right) \le 2\pi$$

$$\Leftrightarrow \quad \frac{3\pi}{4} \le x \le \frac{7\pi}{4}$$

$$\Leftrightarrow \quad x \in \left[\frac{3\pi}{4}, \frac{7\pi}{4}\right].$$

$\therefore f(x)$ is decreasing on $\left[\frac{3\pi}{4}, \frac{7\pi}{4}\right]$.

EXAMPLE 24 *Separate the interval* $\left[0, \frac{\pi}{2}\right]$ *into subintervals in which* $f(x) = (\sin^4 x + \cos^4 x)$ *is (a) increasing (b) decreasing.* **[CBSE 2000C]**

SOLUTION $f(x) = (\sin^4 x + \cos^4 x)$

$$\Rightarrow f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$$

$$= -4\sin x \cos x(\cos^2 x - \sin^2 x)$$

$$= -2\sin 2x \cos 2x = -\sin 4x \qquad \ldots\text{(i)}$$

Also, $0 \le x \le \frac{\pi}{2} \Leftrightarrow 0 \le 4x \le 2\pi$.

(a) $f(x)$ is increasing

$$\Leftrightarrow f'(x) \ge 0$$

$$\Leftrightarrow -\sin 4x \ge 0 \qquad \text{[from (i)]}$$

$$\Leftrightarrow \sin 4x \le 0$$

$$\Leftrightarrow \pi \le 4x \le 2\pi$$

$$\Leftrightarrow \frac{\pi}{4} \le x \le \frac{\pi}{2}$$

$\Leftrightarrow \; x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right].$

$\therefore \; f(x)$ is increasing on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right].$

(b) $f(x)$ is decreasing

$\Leftrightarrow \; f'(x) \le 0$ [from (i)]

$\Leftrightarrow \; -\sin 4x \le 0$

$\Leftrightarrow \; \sin 4x \ge 0$

$\Leftrightarrow \; 0 \le 4x \le \pi$

$\Leftrightarrow \; 0 \le x \le \frac{\pi}{4}$

$\Leftrightarrow \; x \in \left[0, \frac{\pi}{4}\right].$

$\therefore \; f(x)$ is decreasing on $\left[0, \frac{\pi}{4}\right].$

Hence, $f(x)$ is increasing on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and decreasing on $\left[0, \frac{\pi}{4}\right].$

EXAMPLE 25 *Separate* $\left[0, \frac{\pi}{2}\right]$ *into subintervals in which* $f(x) = \sin 3x$ *is*

(a) increasing (b) decreasing.

SOLUTION $f(x) = \sin 3x \;\Rightarrow\; f'(x) = 3\cos 3x$ ...(i)

Also, $0 \le x \le \frac{\pi}{2} \;\Leftrightarrow\; 0 \le 3x \le \frac{3\pi}{2}$ ...(ii)

(a) $f(x)$ is increasing

$\Leftrightarrow \; f'(x) \ge 0$

$\Leftrightarrow \; 3\cos 3x \ge 0 \;\Leftrightarrow\; \cos 3x \ge 0$ [from (i)]

$\Leftrightarrow \; 0 \le 3x \le \frac{\pi}{2}$

$\Leftrightarrow \; 0 \le x \le \frac{\pi}{6}$

$\Leftrightarrow \; x \in \left[0, \frac{\pi}{6}\right].$

$\therefore \; f(x)$ is increasing on $\left[0, \frac{\pi}{6}\right].$

(b) $f(x)$ is decreasing

$\Leftrightarrow \; f'(x) \le 0$

$\Leftrightarrow \; 3\cos 3x \le 0 \;\Leftrightarrow\; \cos 3x \ge 0$ [from (i)]

$\Leftrightarrow \; \frac{\pi}{2} \le 3x \le \frac{3\pi}{2}$

$\Leftrightarrow \; \frac{\pi}{6} \le x \le \frac{\pi}{2}$

$\Leftrightarrow \; x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right].$

$\therefore \; f(x)$ is decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Hence, $f(x)$ is increasing on $\left[0, \frac{\pi}{6}\right]$ and decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

EXAMPLE 26 *Prove that* $\tan x > x$ *for all* $x \in \left[0, \frac{\pi}{2}\right]$.

SOLUTION Let c be an arbitrary real number such that $c \in \left]0, \frac{\pi}{2}\right[$.

Let $f(x) = \tan x - x$ for all $x \in [0, c]$.

$\therefore \; f'(x) = \sec^2 x - 1 = \tan^2 x > 0$ for all $x \in]0, c[$.

Thus, $f(x)$ is increasing on $]0, c[$.

Now, $x > 0 \;\Rightarrow\; f(x) > f(0)$

$\Rightarrow\; f(x) > 0$

$\Rightarrow\; \tan x - x > 0$

$\Rightarrow\; \tan x > x$.

EXAMPLE 27 *Find the values of* x *for which the function* $f(x) = x^x$, $x > 0$ *is*

(a) increasing (b) decreasing.

SOLUTION $f(x) = x^x$

$\Rightarrow\; f(x) = e^{x \log x}$

$\Rightarrow\; f'(x) = e^{x \log x} \cdot \frac{d}{dx}(x \log x)$

$\Rightarrow\; f'(x) = x^x(1 + \log_e x)$ ...(i)

(a) $f(x)$ is increasing

$\Leftrightarrow\; f'(x) \geq 0$

$\Leftrightarrow\; x^x(1 + \log_e x) \geq 0$ [from (i)]

$\Leftrightarrow\; (1 + \log_e x) \geq 0$ [$\because\; x^x > 0$ when $x > 0$]

$\Leftrightarrow\; \log_e x \geq -1$

$\Leftrightarrow\; x \geq e^{-1}$

$\Leftrightarrow\; x \in \left[\frac{1}{e}, \infty\right[$.

$\therefore \; f(x)$ is increasing on $\left[\frac{1}{e}, \infty\right[$.

(b) $f(x)$ is decreasing

$\Leftrightarrow\; f'(x) \leq 0$

$\Leftrightarrow\; x^x(1 + \log_e x) \leq 0$

$\Leftrightarrow\; (1 + \log_e x) \leq 0$ [$\because\; x^x > 0$]

$\Leftrightarrow\; \log_e x \leq -1$

$\Leftrightarrow\; x \leq e^{-1}$

$\Leftrightarrow\; 0 \leq x \leq \frac{1}{e}$

$\Leftrightarrow \; x \in \left[0, \frac{1}{e}\right]$.

$\therefore \; f(x)$ is decreasing on $\left[0, \frac{1}{e}\right]$.

Hence, $f(x)$ is increasing on $\left[\frac{1}{e}, \infty\right]$ and decreasing on $\left[0, \frac{1}{e}\right]$.

EXAMPLE 28 *Find the intervals on which the function* $f(x) = (x^4 - 2x^2)$ *is increasing or decreasing.*

SOLUTION $f(x) = x^4 - 2x^2 \Rightarrow f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$.

Now, $f'(x) = 0 \Rightarrow x = -1$ or $x = 0$ or $x = 1$.

These points divide the whole real line into four disjoint open intervals, namely, $]-\infty, -1[$, $]-1, 0[$, $]0, 1[$ and $]1, \infty[$.

Case I *When* $x \in]-\infty, -1[$

In this case, $x < -1$. So, $f'(x) = (-)(-)(-) = -$ ve.

So, $f'(x) < 0$ for all $x \in]-\infty, -1[$.

$\therefore \; f(x)$ is decreasing for all $x \in]-\infty, -1[$.

Case II *When* $x \in]-1, 0[$

In this case, $-1 < x < 0$.

So, $f'(x) = (-)(-)(+) = +$ ve.

$\therefore \; f'(x) > 0$ for all $x \in]-1, 0[$.

$\therefore \; f(x)$ is increasing on $]-1, 0[$.

Case III *When* $x \in]0, 1[$

In this case, $0 < x < 1$.

So, $f'(x) = (+)(-)(+) = -$ ve.

Thus, $f'(x) < 0$ for all $x \in]0, 1[$.

$\therefore \; f(x)$ is decreasing on $]0, 1[$.

Case IV *When* $x \in]1, \infty[$

In this case, $1 < x < \infty$.

So, $f'(x) = (+)(+)(+) = +$ ve.

Thus, $f'(x) > 0$ for all $x \in]1, \infty[$.

$\therefore \; f(x)$ is increasing on $]1, \infty[$.

From all the four cases, we conclude that

$f(x)$ is increasing on $]-1, 0[\cup]1, \infty[$

and, $f(x)$ is decreasing on $]-\infty, -1[\cup]0, 1[$.

EXAMPLE 29 *Prove that* $\frac{x}{(1+x)} < \log(1+x) < x$ *for* $x > 0$.

SOLUTION Let $f(x) = \log(1+x) - \frac{x}{(1+x)}$.

Then, $f'(x) = \left[\frac{1}{1+x} - \frac{1}{(1+x)^2}\right] = \frac{x}{(1+x)^2} > 0$ for $x > 0$.

Since $f'(x) > 0$ for all $x > 0$ and $f'(0) = 0$, it follows that $f(x)$ is increasing on $[0, \infty[$.

Now, $x > 0 \Rightarrow f(x) > f(0) \Rightarrow f(x) > 0$ $\quad [\because f(0) = 0]$

$$\Rightarrow \left[\log(1+x) - \frac{x}{(1+x)}\right] > 0 \Rightarrow \log(1+x) > \frac{x}{(1+x)} \quad \ldots \text{(i)}$$

Again, let $g(x) = [x - \log(1+x)]$.

Then, $g'(x) = \left[1 - \frac{1}{(1+x)}\right] = \frac{x}{(1+x)} > 0$ for $x > 0$.

Now, $g'(x) > 0$ for all $x > 0$ and $g'(0) = 0$.

$\therefore \quad g(x)$ is strictly increasing on $[0, \infty[$.

Also, $g(0) = 0$.

Now, $x > 0 \Rightarrow g(x) > g(0) \Rightarrow g(x) > 0$ $\quad [\because g(0) = 0]$

$$\Rightarrow [x - \log(1+x)] > 0 \Rightarrow x > \log(1+x) \quad \ldots \text{(ii)}$$

From (i) and (ii), we get $\frac{x}{(1+x)} < \log(1+x) < x$ for $x > 0$.

EXERCISE 11G

1. Show that the function $f(x) = 5x - 2$ is a strictly increasing function on R.
2. Show that the function $f(x) = -2x + 7$ is a strictly decreasing function on R.
3. Prove that $f(x) = ax + b$, where a and b are constants and $a > 0$, is a strictly increasing function on R.
4. Prove that the function $f(x) = e^{2x}$ is strictly increasing on R.
5. Show that the function $f(x) = x^2$ is
   (a) strictly increasing on $[0, \infty[$
   (b) strictly decreasing on $]-\infty, 0[$
   (c) neither strictly increasing nor strictly decreasing on R
6. Show that the function $f(x) = |x|$ is
   (a) strictly increasing on $]0, \infty[$
   (b) strictly decreasing on $]-\infty, 0[$
7. Prove that the function $f(x) = \log_e x$ is strictly increasing on $]0, \infty[$.
8. Prove that the function $f(x) = \log_a x$ is strictly increasing on $]0, \infty[$ when $a > 1$ and strictly decreasing on $]0, \infty[$ when $0 < a < 1$.
9. Prove that $f(x) = 3^x$ is strictly increasing on R.
10. Show that $f(x) = x^3 - 15x^2 + 75x - 50$ is increasing on R.
11. Show that $f(x) = \left(x - \frac{1}{x}\right)$ is increasing for all $x \in R$, where $x \neq 0$.
12. Show that $f(x) = \left(\frac{3}{x} + 5\right)$ is decreasing for all $x \in R$, where $x \neq 0$.

13. Show that $f(x) = \frac{1}{(1+x^2)}$ is increasing for all $x \le 0$.

14. Show that $f(x) = \left(x^3 + \frac{1}{x^3}\right)$ is decreasing on $]-1, 1[$. **[CBSE 2009]**

15. Show that $f(x) = \frac{x}{\sin x}$ is increasing on $\left]0, \frac{\pi}{2}\right[$.

16. Prove that the function $f(x) = \log(1+x) - \frac{2x}{(x+2)}$ is increasing for all $x > -1$. **[CBSE 2000C]**

17. Let I be an interval disjoint from $]-1, 1[$. Prove that the function $f(x) = \left(x + \frac{1}{x}\right)$ is strictly increasing on I.

18. Show that $f(x) = \frac{(x-2)}{(x+1)}$ is increasing for all $x \in R$, except at $x = -1$.

19. Find the intervals on which the function $f(x) = (2x^2 - 3x)$ is (a) strictly increasing (b) strictly decreasing.

20. Find the intervals on which the function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is (a) strictly increasing (b) strictly decreasing. **[CBSE 2005C]**

21. Find the intervals on which the function $f(x) = 6 - 9x - x^2$ is (a) strictly increasing (b) strictly decreasing.

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

22. $f(x) = \left(x^4 - \frac{x^3}{3}\right)$

23. $f(x) = x^3 - 12x^2 + 36x + 17$ **[CBSE 2006, '09C]**

24. $f(x) = (x^3 - 6x^2 + 9x + 10)$ **[CBSE 2000, '04C]**

25. $f(x) = (6 + 12x + 3x^2 - 2x^3)$

26. $f(x) = 2x^3 - 24x + 5$

27. $f(x) = (x-1)(x-2)^2$ **[CBSE 2009C]**

28. $f(x) = (x^4 - 4x^3 + 4x^2 + 15)$

29. $f(x) = 2x^3 + 9x^2 + 12x + 15$ **[CBSE 2008C]**

30. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ **[CBSE 2012C]**

31. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is (a) strictly increasing (b) strictly decreasing. **[CBSE 2014]**

32. $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is (a) strictly increasing (b) strictly decreasing. **[CBSE 2014C]**

ANSWERS (EXERCISE 11G)

19. (a) $f(x)$ is strictly increasing on $\left]\frac{3}{4}, \infty\right[\cdot$

(b) $f(x)$ is strictly decreasing on $\left]-\infty, \frac{3}{4}\right[\cdot$

20. (a) $f(x)$ is strictly increasing on $]-\infty, -2[\cup]3, \infty[$.

(b) $f(x)$ is strictly decreasing on $]-2, 3[$.

21. (a) $f(x)$ is strictly increasing on $\left]-\infty, -\frac{9}{2}\right[\cdot$

(b) $f(x)$ is strictly decreasing on $\left]-\frac{9}{2}, \infty\right[\cdot$

22. (a) $f(x)$ is increasing on $\left[\frac{1}{4}, \infty\right[\cdot$

(b) $f(x)$ is decreasing on $\left]-\infty, \frac{1}{4}\right]\cdot$

23. (a) $f(x)$ is increasing on $]-\infty, -2[\cup]6, \infty[$.

(b) $f(x)$ is decreasing on $]2, 6[$.

24. (a) $f(x)$ is increasing on $]-\infty, 1] \cup [3, \infty[$.

(b) $f(x)$ is decreasing on $[-1, 3]$.

25. (a) $f(x)$ is increasing on $[-1, 2]$.

(b) $f(x)$ is decreasing on $]-\infty, -1] \cup [2, \infty[$.

26. (a) $f(x)$ is increasing on $]-\infty, -2] \cup [1, \infty[$.

(b) $f(x)$ is decreasing on $[-2, 2]$.

27. (a) $f(x)$ is increasing on $\left]-\infty, \frac{4}{3}\right] \cup [2, \infty[$.

(b) $f(x)$ is decreasing on $\left[\frac{4}{3}, 2\right]\cdot$

28. (a) $f(x)$ is increasing on $[0, 1] \cup [2, \infty[$.

(b) $f(x)$ is decreasing on $]-\infty, 0] \cup [1, 2]$.

29. (a) $f(x)$ is increasing on $]-\infty, -2] \cup [-1, \infty[$.

(b) $f(x)$ is decreasing on $[-2, -1]$.

30. (a) $f(x)$ is increasing on $[1, 2] \cup [3, \infty[$

(b) $f(x)$ is decreasing on $]-\infty, 1] \cup [2, 3]$

31. (a) Strictly increasing in $]-1, 0[\cup]2, \infty[$.

(b) Strictly decreasing in $]-\infty, -1[\cup]0, 2[$.

32. (a) Strictly increasing in $]-2, 1[\cup]3, \infty[$.

(b) Strictly decreasing in $]-\infty, -2[\cup]1, 3[$.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11G)

1. $x_1 > x_2 \Rightarrow 5x_1 > 5x_2$
$\Rightarrow (5x_1 - 2) > (5x_2 - 2)$
$\Rightarrow f(x_1) > f(x_2)$.

2. $x_1 > x_2 \Rightarrow -2x_1 < -2x_2$
$\Rightarrow -2x_1 + 7 < -2x_2 + 7$
$\Rightarrow f(x_1) < f(x_2)$.

5. (a) Let $x_1, x_2 \in [0, \infty[$ such that $x_1 > x_2 \cdot$ Then,

$$x_1 > x_2 \Rightarrow x_1^2 > x_1 x_2 \text{ and } x_1 x_2 > x_2^2$$

$$\Rightarrow x_1^2 > x_2^2 \Rightarrow f(x_1) > f(x_2).$$

(b) Let $x_1, x_2 \in]-\infty, 0]$ such that $x_1 > x_2$. Then,

$$x_1 > x_2 \Rightarrow x_1 x_2 < x_2^2 \qquad [\because x_2 < 0].$$

Also, $x_1 > x_2 \Rightarrow x_1^2 < x_1 x_2 \qquad [\because x_1 < 0]$.

$\therefore\ x_1^2 < x_1 x_2 < x_2^2$.

Thus, $x_1 > x_2 \Rightarrow x_1^2 < x_2^2$

$$\Rightarrow f(x_1) < f(x_2).$$

(c) Since $f(x) = x^2$ is strictly increasing on $[0, \infty[$ and strictly decreasing on $]-\infty, 0]$, it is neither strictly increasing nor strictly decreasing on the whole real line.

6. (a) Let $x_1, x_2 \in]0, \infty[$ such that $x_1 > x_2$. Then,

$$x_1 > x_2 \Rightarrow |x_1| > |x_2|$$
$$\Rightarrow f(x_1) > f(x_2).$$

$\therefore\ f(x) = |x|$ is strictly increasing on $]0, \infty[$.

(b) Let $x_1, x_2 \in]-\infty, 0[$ such that $x_1 > x_2$. Then,

$$x_1 > x_2 \Rightarrow |x_1| < |x_2|$$
$$\Rightarrow f(x_1) < f(x_2).$$

Thus, $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$.

$\therefore\ f(x) = |x|$ is strictly decreasing on $]-\infty, 0[$.

7. For all $x_1, x_2 \in]0, \infty[$, we have
$$x_1 \geq x_2 \Rightarrow \log_e x_1 \geq \log_e x_2 \Rightarrow f(x_1) \geq f(x_2).$$

8. $f(x) = \log_a x = \dfrac{\log_e x}{\log_e a}$.

Case I Let $x_1, x_2 \in]0, \infty[$ and let $a > 1$.

Then, $x_1 > x_2 \Rightarrow \log x_1 > \log x_2$

$$\Rightarrow \frac{\log x_1}{\log a} > \frac{\log x_2}{\log a}$$

$$\Rightarrow \log_a x_1 > \log_a x_2.$$

$\therefore\ f(x)$ is strictly increasing on $[0, \infty[$ when $a > 1$.

Case II Let $x_1, x_2 \in]0, \infty[$ and let $0 < a < 1$.

Then, $x_1 > x_2 \Rightarrow \log x_1 > \log x_2$

$$\Rightarrow \frac{\log x_1}{\log a} < \frac{\log x_2}{\log a} \quad \left[\because 0 < a < 1 \Rightarrow \log a < 0 \Rightarrow \frac{1}{\log a} < 0\right]$$

$\Rightarrow (\log_a x_1) < (\log_a x_2)$.

$\therefore$ $f(x)$ is strictly decreasing on $]0, \infty[$ when $0 < a < 1$.

9. $f(x) = 3^x \Rightarrow f'(x) = (3^x \log 3) > 0$ for all $x \in R$.

10. $f(x) = x^3 - 15x^2 + 75x - 50$

$\Rightarrow f'(x) = 3x^2 - 30x + 75 = 3(x^2 - 10x + 25) = 3(x-5)^2 \geq 0$.

11. $f(x) = \left(x - \frac{1}{x}\right) \Rightarrow f'(x) = \left(1 + \frac{1}{x^2}\right) > 0$ for all $x \in R$, where $x \neq 0$.

12. $f(x) = \left(\frac{3}{x} + 5\right) \Rightarrow f'(x) = \frac{-3}{x^2} < 0$ for all $x \in R$, where $x \neq 0$.

13. $f'(x) = -(1 + x^2)^{-2} \cdot 2x = \frac{-2x}{(1+x^2)^2} \geq 0$ when $x \leq 0$.

14. $f'(x) < 0 \Leftrightarrow \frac{3(x^6 - 1)}{x^4} < 0 \Leftrightarrow (x^6 - 1) < 0$

$\Leftrightarrow (x-1)(x+1)(x^4 + x^2 + 1) < 0$

$\Leftrightarrow (x-1)(x+1) < 0 \Leftrightarrow -1 < x < 1$.

15. $f'(x) > 0 \Leftrightarrow \frac{\sin x - x\cos x}{\sin^2 x} > 0 \Leftrightarrow (\sin x - x\cos x) > 0$

$\Leftrightarrow \tan x > x$, which is true on $\left]0, \frac{\pi}{2}\right[$.

16. Clearly, $\log(1+x)$ is defined only when $(1+x) > 0$, i.e., when $x > -1$.

Now, $f(x) = \log(1+x) - \frac{2x}{(x+2)}$

$$\Rightarrow f'(x) = \left\{\frac{1}{(1+x)} - \frac{[(x+2)\cdot 2 - 2x \cdot 1]}{(x+2)^2}\right\} = \frac{x^2}{(1+x)(x+2)^2} > 0 \quad [\because (1+x) > 0].$$

17. $f(x) = x + \frac{1}{x} \Rightarrow f'(x) = \left(1 - \frac{1}{x^2}\right) = \frac{(x^2 - 1)}{x^2}$.

Now, $f'(x) > 0 \Leftrightarrow (x^2 - 1) > 0 \Leftrightarrow (x-1)(x+1) > 0$

$\Leftrightarrow [(x-1) > 0 \text{ and } (x+1) > 0]$ or $[(x-1) < 0 \text{ and } (x+1) < 0]$

$\Leftrightarrow [x > 1 \text{ and } x > -1]$ or $[x < 1 \text{ and } x < -1]$

$\Leftrightarrow (x > 1)$ or $(x < -1)$

$\Leftrightarrow x \in]-\infty, -1[$ or $x \in]1, \infty[$.

$\Leftrightarrow x \in]-\infty, -1[\cup]1, \infty[$.

6. Tangents and Normals

Some Important Theorems

THEOREM 1 *Prove that the equation of a tangent to a curve $y = f(x)$ at a point $P(x_1, y_1)$ is given by* $\dfrac{y - y_1}{x - x_1} = \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}$,

where $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}$ *denotes the value of* $\dfrac{dy}{dx}$ *at* $x = x_1$ *and* $y = y_1$,

and the equation of the normal at $P(x_1, y_1)$ is given by

$$\frac{y - y_1}{x - x_1} = \frac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}.$$

PROOF We know that the slope of the tangent to the curve $y = f(x)$ at a point $P(x_1, y_1)$ is $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}$.

Thus, the tangent to the given curve at a point P is a line passing through (x_1, y_1) with a slope of $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}$.

So, the equation of the tangent at P is $\dfrac{y - y_1}{x - x_1} = \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}$.

Again, the normal to the given curve at P is a line perpendicular to the tangent at P.

So, the slope of the normal is $\dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}$.

Also, the normal at P passes through (x_1, y_1).

So, the equation of the normal at P is $\dfrac{y - y_1}{x - x_1} = \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}$.

THEOREM 2 *The tangent to a curve $y = f(x)$ at a point $P(x_1, y_1)$ is parallel to the x-axis if and only if* $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = 0$.

PROOF The tangent is parallel to the x-axis $\Leftrightarrow$ its slope is 0

$$\Leftrightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0.$$

THEOREM 3 *The tangent to a curve $y = f(x)$ at a point $P(x_1, y_1)$ is parallel to the y-axis if and only if* $\left(\dfrac{dx}{dy}\right)_{(x_1, y_1)} = 0$.

PROOF The tangent is parallel to the y-axis

$$\Leftrightarrow \text{ its slope is } 90° \quad \Leftrightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \tan 90° = \infty \quad \Leftrightarrow \quad \left(\frac{dx}{dy}\right)_{(x_1,\, y_1)} = 0.$$

REMARK If $x = f(t)$ and $y = g(t)$, where $f'(t) \neq 0$ then

(i) equation of the tangent at 't' is $\dfrac{y - g(t)}{x - f(t)} = \dfrac{g'(t)}{f'(t)}$

and, (ii) equation of the normal at 't' is $\dfrac{y - g(t)}{x - f(t)} = -\dfrac{f'(t)}{g'(t)}$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the equations of the tangent and the normal to the curve*

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5 \text{ at the point } (1,\ 3).$$

SOLUTION The equation of the given curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$.

$$\therefore \quad \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10.$$

So, $\left(\dfrac{dy}{dx}\right)_{(1,\ 3)} = (4 \times 1^3 - 18 \times 1^2 + 26 \times 1 - 10) = 2.$

$\therefore$ the required equation of the tangent is

$$\frac{y - 3}{x - 1} = 2 \quad \text{or} \quad 2x - y + 1 = 0.$$

And, the required equation of the normal is

$$\frac{y - 3}{x - 1} = \frac{-1}{2} \quad \text{or} \quad x + 2y - 7 = 0.$$

EXAMPLE 2 *Find the equations of the tangent and the normal to the curve* $y = x^2 + 4x + 1$ *at the point where* $x = 3$.

SOLUTION When $x = 3$, we have $y = (3^2 + 4 \times 3 + 1) = 22$.

So, the point of contact is $(3,\ 22)$.

Now, $y = x^2 + 4x + 1 \Rightarrow \dfrac{dy}{dx} = 2x + 4 \Rightarrow \left(\dfrac{dy}{dx}\right)_{(3,\ 22)} = (2 \times 3 + 4) = 10.$

Equation of the tangent is $\dfrac{y - 22}{x - 30} = 10 \quad \Rightarrow \quad 10x - y - 278 = 0.$

And, equation of the normal is $\dfrac{y - 22}{x - 3} = \dfrac{-1}{10} \quad \Rightarrow \quad x + 10y - 223 = 0.$

EXAMPLE 3 *Show that the equation of the tangent to the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (x_1,\ y_1) \text{ is } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

SOLUTION $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} + \dfrac{2y}{b^2} \cdot \dfrac{dy}{dx} = 0$ [on differentiating w.r.t. x]

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \frac{-b^2x_1}{a^2y_1}\cdot$$

So, the equation of the tangent at (x_1, y_1) is

$$\frac{y-y_1}{x-x_1} = \frac{-b^2x_1}{a^2y_1} \Rightarrow b^2xx_1 + a^2yy_1 = b^2x_1^2 + a^2y_1^2.$$

On dividing throughout by a^2b^2, we get $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2}$,

i.e., $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1 \left[\because (x_1, y_1) \text{ lies on } \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1\right]\cdot$

EXAMPLE 4 *Find the equation of the tangent to the curve $y = \sqrt{5x-3} - 2$, which is parallel to the line $4x - 2y + 3 = 0$.* **[CBSE 2005, '13C]**

SOLUTION The given line is $4x - 2y + 3 = 0$ or $y = 2x + \dfrac{3}{2}\cdot$

$\therefore$ slope of the given line = 2.

So, the slope of the tangent = 2.

Let the point of contact be (x_1, y_1).

Now, $y = \sqrt{5x-3} - 2 \Rightarrow \dfrac{dy}{dx} = \dfrac{5}{2\cdot\sqrt{5x-3}}$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \frac{5}{2\cdot\sqrt{5x_1-3}}\cdot$$

$$\therefore \frac{5}{2\cdot\sqrt{5x_1-3}} = 2 \Rightarrow \frac{25}{4(5x_1-3)} = 4, \text{ i.e., } x_1 = \frac{73}{80}\cdot$$

$$\therefore y_1 = \sqrt{5x_1-3} - 2 = \sqrt{\left(5 \times \frac{73}{80} - 3\right)} - 2 = \frac{-3}{4}\cdot$$

So, the point of contact is $\left(\dfrac{73}{80}, \dfrac{-3}{4}\right)\cdot$

Hence, the required equation of the tangent is $\dfrac{\left(y + \dfrac{3}{4}\right)}{\left(x - \dfrac{73}{80}\right)} = 2$,

i.e., $80x - 40y - 103 = 0$.

EXAMPLE 5 *Find the equations of the normals to the curve $3x^2 - y^2 = 8$, parallel to the line $x + 3y = 4$.*

SOLUTION The given line is $x + 3y = 4$ or $y = -\dfrac{1}{3}x + \dfrac{4}{3}\cdot$

$\therefore$ slope of the given line $= -\dfrac{1}{3}$

So, the slope of the required normal $= -\dfrac{1}{3}$ ... (i)

Let the point of contact be (x_1, y_1).

Now, $3x^2 - y^2 = 8 \Rightarrow 6x - 2y \cdot \frac{dy}{dx} = 0$ [on differentiating w.r.t. $\boldsymbol{x}$]

$$\Rightarrow \frac{dy}{dx} = \frac{3x}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \frac{3x_1}{y_1}.$$

$\therefore$ slope of the normal $= \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1,\, y_1)}} = \dfrac{-y_1}{3x_1}$ ... (ii)

Thus, from (i) and (ii) we get $\dfrac{-y_1}{3x_1} = -\dfrac{1}{3} \Rightarrow y_1 = x_1$.

Also, since (x_1, y_1) lies on the given curve, we have

$$3x_1^2 - y_1^2 = 8 \text{ or } 3x_1^2 - x_1^2 = 8 \qquad [\because \; y_1 = x_1]$$

or $x_1^2 = 4$ or $x_1 = \pm 2$.

$\therefore \; y_1 = \pm 2 \qquad [\because \; y_1 = x_1]$.

Thus, the points of contact are $(2, 2)$ and $(-2, -2)$.

So, the equation of the required normal at $(2, 2)$ is $\dfrac{y-2}{x-2} = \dfrac{-1}{3}$,

i.e., $x + 3y - 8 = 0$.

The equation of the required normal at $(-2, -2)$ is $\dfrac{y+2}{x+2} = \dfrac{-1}{3}$,

i.e., $x + 3y + 8 = 0$.

EXAMPLE 6 *Prove that the curve* $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ *touches the straight line* $\frac{x}{a} + \frac{y}{b} = 2$ *at the point* (a, b), *whatever be the value of* n. **[CBSE 2007C]**

SOLUTION The equation of the curve is $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$.

On differentiating w.r.t. x, we get $n \cdot \left(\frac{x}{a}\right)^{n-1} \cdot \frac{1}{a} + n \cdot \left(\frac{y}{b}\right)^{n-1} \cdot \frac{1}{b} \cdot \frac{dy}{dx} = 0$

$$\Rightarrow \frac{n\,x^{n-1}}{a^n} + \frac{n\,y^{n-1}}{b^n} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^n x^{n-1}}{a^n y^{n-1}}.$$

$$\therefore \left(\frac{dy}{dx}\right)_{(a,\, b)} = \frac{-b^n a^{n-1}}{a^n b^{n-1}} = -\frac{b}{a}.$$

So, the equation of the tangent at (a, b) is $\dfrac{y-b}{x-a} = \dfrac{-b}{a}$,

i.e., $ay + bx = 2ab$.

$\therefore \; \dfrac{x}{a} + \dfrac{y}{b} = 2$, which is independent of n.

EXAMPLE 7 *At what points will the tangent to the curve* $y = 2x^3 - 15x^2 + 36x - 21$ *be parallel to the x-axis? Also, find the equations of tangents to the curve at these points.* **[CBSE 2008]**

SOLUTION Let the required point be $P(x_1, y_1)$.

Then, $\left[\frac{dy}{dx}\right]_{(x_1, y_1)} = 0.$

Now, $y = 2x^3 - 15x^2 + 36x - 21$

$\Rightarrow \quad \frac{dy}{dx} = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$

$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{(x_1, y_1)} = 6(x_1^2 - 5x_1 + 6).$

$\therefore \quad \left[\frac{dy}{dx}\right]_{(x_1, y_1)} = 0 \quad \Rightarrow \quad 6(x_1^2 - 5x_1 + 6) = 0$

$\Rightarrow \quad x_1^2 - 5x_1 + 6 = 0$

$\Rightarrow \quad (x_1 - 2)(x_1 - 3) = 0 \quad \Rightarrow \quad x_1 = 2 \text{ or } x_1 = 3.$

Since $P(x_1, y_1)$ lies on the given curve, we have

$y_1 = 2x_1^3 - 15x_1^2 + 36x_1 - 21.$

$\therefore \quad x_1 = 2 \quad \Rightarrow \quad y_1 = (2\times 8 - 15\times 4 + 36\times 2 - 21) = 7.$

$x_1 = 3 \quad \Rightarrow \quad y_1 = (2\times 27 - 15\times 9 + 36\times 3 - 21) = 6.$

So, the required points are $P(2, 7)$ and $P'(3, 6)$.

The equations of tangents at these points are $y = 7$ and $y = 6$ respectively.

EXAMPLE 8 *Prove that all points of the curve $y^2 = 4a\left[x + a\sin\frac{x}{a}\right]$ at which the tangent is parallel to the axis of x, lie on a parabola.*

SOLUTION Let the required point be (x_1, y_1).

Then, we must have $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0.$

Now, $y^2 = 4a\left[x + a\sin\frac{x}{a}\right] \Rightarrow 2y\cdot\frac{dy}{dx} = 4a\left[1 + \cos\frac{x}{a}\right]$

$\Rightarrow \frac{dy}{dx} = \frac{2a\left[1 + \cos\frac{x}{a}\right]}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{2a\left[1 + \cos\frac{x_1}{a}\right]}{y_1}.$

$\therefore \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0 \Rightarrow \frac{2a\left[1 + \cos\frac{x_1}{a}\right]}{y_1} = 0 \Rightarrow \left[1 + \cos\frac{x_1}{a}\right] = 0$

$\Rightarrow \cos\frac{x_1}{a} = -1 \Rightarrow \sin\frac{x_1}{a} = \sqrt{1 - \cos^2\frac{x_1}{a}} = 0.$

But, (x_1, y_1) lies on the given curve.

$\therefore \quad y_1^2 = 4a\left[x_1 + a\sin\frac{x_1}{a}\right] \Rightarrow y_1^2 = 4ax_1 \qquad \left[\because \sin\frac{x_1}{a} = 0\right].$

This shows that (x_1, y_1) lies on the parabola $y^2 = 4ax$.

EXAMPLE 9 *Tangents are drawn from the origin to the curve* $y = \sin x$. *Prove that their points of contact lie on the curve* $x^2y^2 = (x^2 - y^2)$.

SOLUTION Let the point of contact be (x_1, y_1).

Now, $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \cos x_1$.

$\therefore$ the equation of the tangent at (x_1, y_1) is $\frac{y - y_1}{x - x_1} = \cos x_1$.

Since the tangent passes through the origin, we have $\frac{-y_1}{-x_1} = \cos x_1$,

i.e., $\frac{y_1}{x_1} = \cos x_1$ ... (i)

But, (x_1, y_1) lies on the curve $y = \sin x$.

$\therefore$ $y_1 = \sin x_1$ ... (ii)

Squaring (i) and (ii) and adding, we get

$$\frac{y_1^2}{x_1^2} + y_1^2 = 1 \Rightarrow y_1^2 + x_1^2 y_1^2 = x_1^2$$

$$\Rightarrow x_1^2 y_1^2 = (x_1^2 - y_1^2).$$

This shows that (x_1, y_1) lies on the curve $x^2y^2 = (x^2 - y^2)$.

EXAMPLE 10 *Determine the points on the curve* $2y = (3 - x^2)$ *at which the tangent is parallel to the line* $x + y = 0$.

SOLUTION Let the required point be (x_1, y_1).

Then, slope of the tangent at $(x_1 \ y_1)$ = slope of the given line.

$\therefore$ $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -1$ $\quad [\because x + y = 0 \Rightarrow y = -x]$.

Now, $2y = (3 - x^2) \Rightarrow 2 \cdot \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -x \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -x_1$.

$\therefore$ $-x_1 = -1 \Rightarrow x_1 = 1$.

Since (x_1, y_1) lies on the curve $2y = (3 - x^2)$, we have $2y_1 = (3 - x_1^2)$.

When $x_1 = 1$, we have $y_1 = \left(\frac{3 - 1^2}{2}\right) = 1$.

Hence, the required point is (1, 1).

EXAMPLE 11 *Find the points on the curve* $4x^2 + 9y^2 = 1$, *where the tangents are perpendicular to the line* $2y + x = 0$.

SOLUTION The equation of the given line is $y = -\frac{1}{2}x$.

$\therefore$ slope of this line $= -\frac{1}{2}$ ... (i)

Let the required point be (x_1, y_1).

Now, $4x^2 + 9y^2 = 1 \Rightarrow 8x + 18y \cdot \frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{-4x}{9y}\right) \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-4x_1}{9y_1}.$$

$\therefore$ slope of the tangent at $(x_1, y_1) = \frac{-4x_1}{9y_1}$ ... (ii)

From (i) and (ii), we have $\frac{-4x_1}{9y_1} \times \left(-\frac{1}{2}\right) = -1$ or $y_1 = -\frac{2}{9}x_1$ ... (iii)

Since (x_1, y_1) lies on the curve $4x^2 + 9y^2 = 1$, we have

$$4x_1^2 + 9y_1^2 = 1 \Rightarrow 4x_1^2 + 9 \times \frac{4}{81}x_1^2 = 1 \quad \text{[using (iii)].}$$

$\therefore \quad x_1^2 = \frac{9}{40}$ or $x_1 = \pm\frac{3}{2\sqrt{10}}$.

So, $y_1 = \pm\frac{2}{9} \times \frac{3}{2\sqrt{10}} = \pm\frac{1}{3\sqrt{10}}$.

Hence, the required points are

$$\left(\frac{3}{2\sqrt{10}}, \frac{1}{3\sqrt{10}}\right) \text{ and } \left(\frac{-3}{2\sqrt{10}}, \frac{-1}{3\sqrt{10}}\right).$$

EXAMPLE 12 *Find the coordinates of the points on the curve $y = x^2 + 3x + 4$, the tangents at which pass through the origin.*

SOLUTION Let the required point be (x_1, y_1).

Now, $y = x^2 + 3x + 4 \Rightarrow \frac{dy}{dx} = 2x + 3 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = (2x_1 + 3)$.

So, the equation of the tangent at (x_1, y_1) is $\frac{y - y_1}{x - x_1} = (2x_1 + 3)$.

Since the tangent passes through the origin, we have

$$\frac{-y_1}{-x_1} = (2x_1 + 3)$$

$$\Rightarrow y_1 = (2x_1^2 + 3x_1) \quad \text{... (i)}$$

But, (x_1, y_1) lies on the given curve. So, $y_1 = x_1^2 + 3x_1 + 4$ ... (ii)

From (i) and (ii), we get

$$2x_1^2 + 3x_1 = x_1^2 + 3x_1 + 4 \text{ or } x_1^2 = 4 \text{ or } x_1 = \pm 2.$$

Now, $x_1 = 2 \Rightarrow y_1 = (2^2 + 3 \times 2 + 4) = 14$.

And, $x_1 = -2 \Rightarrow y_1 = [(-2)^2 + 3 \times (-2) + 4] = 2$.

Hence, the required points are (2, 14) and (−2, 2).

EXAMPLE 13 *If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$.*

SOLUTION Let the point of contact be (x_1, y_1).

Then, the equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ ...(i)

This is identical with $x \cos \alpha + y \sin \alpha = p$ ... (ii)

Comparing coefficients, we have $\frac{(x_1/a^2)}{\cos \alpha} = \frac{(y_1/b^2)}{\sin \alpha} = \frac{1}{p}$

$$\Rightarrow \quad \frac{(x_1/a)}{a \cos \alpha} = \frac{(y_1/b)}{b \sin \alpha} = \frac{1}{p}.$$

$\therefore \; \frac{x_1}{a} = \frac{a \cos \alpha}{p}$ and $\frac{y_1}{b} = \frac{b \sin \alpha}{p}$.

Squaring and adding, we get $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}{p^2}$

$$\Rightarrow \quad \frac{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}{p^2} = 1 \quad \left[\because \; \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \right]$$

$$\Rightarrow \quad a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2.$$

EXAMPLE 14 *If* $x \cos \alpha + y \sin \alpha = p$ *touches the curve* $x^m y^n = a^{m+n}$, *prove that*

$$p^{m+n} \cdot m^m \cdot n^n = (m+n)^{m+n} \cos^m \alpha \, \sin^n \alpha.$$

SOLUTION Let the point of contact be (x_1, y_1).

The equation of the given curve is $x^m y^n = a^{m+n}$.

This, on differentiation, gives

$$mx^{m-1} y^n + x^m \cdot ny^{n-1} \cdot \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-my}{nx}.$$

$$\therefore \; \left(\frac{dy}{dx} \right)_{(x_1, y_1)} = \frac{-my_1}{nx_1}.$$

So, the equation of the tangent at (x_1, y_1) is $\frac{y - y_1}{x - x_1} = \frac{-my_1}{nx_1}$,

i.e., $my_1 x + nx_1 y = (m+n) \, x_1 y_1$ ... (i)

This is identical with $x \cos \alpha + y \sin \alpha = p$ ... (ii)

Comparing coefficients, we get $\frac{my_1}{\cos \alpha} = \frac{nx_1}{\sin \alpha} = \frac{(m+n) x_1 y_1}{p}$.

$\therefore \; x_1 = \frac{pm}{(m+n) \cos \alpha}$ and $y_1 = \frac{pn}{(m+n) \sin \alpha}$.

Since (x_1, y_1) lies on the given curve, $x_1^m y_1^n = a^{m+n}$.

$$\therefore \; \left\{ \frac{pm}{(m+n) \cos \alpha} \right\}^m \cdot \left\{ \frac{pn}{(m+n) \sin \alpha} \right\}^n = a^{m+n}$$

or $p^{m+n} \cdot m^m \cdot n^n = (m+n)^{m+n} \cdot a^{m+n} \cos^m \alpha \, \sin^n \alpha$.

EXAMPLE 15 *Find the equation of the normal to the curve* $y = 2\sin^2 3x$ *at* $x = \frac{\pi}{6}\cdot$

SOLUTION When $x = \frac{\pi}{6}$, we have $y = 2\sin^2\left(\frac{3\pi}{6}\right) = 2.$

So, the point of contact is $\left(\frac{\pi}{6}, 2\right)\cdot$

$$\text{Now,} \quad y = 2\sin^2 3x \Rightarrow \frac{dy}{dx} = (4\sin 3x) \times 3 \times (\cos 3x)$$

$$\Rightarrow \frac{dy}{dx} = 12\sin 3x \cos 3x = 6\sin 6x.$$

$$\therefore \left(\frac{dy}{dx}\right) \text{at} \left(x = \frac{\pi}{6}\right) = 6\sin\left(6 \times \frac{\pi}{6}\right) = 6\sin \pi = 0.$$

So, the tangent is parallel to the x-axis.

Thus, the normal is parallel to the y-axis and passes through the point $\left(\frac{\pi}{6}, 2\right)\cdot$

So, the equation of the normal is $x = \frac{\pi}{6}\cdot$

EXAMPLE 16 *Find the equations of the tangent and the normal to the curve* $y(x-2)(x-3) - x + 7 = 0$ *at the point where it cuts the x-axis.*

[CBSE 2010]

SOLUTION The curve cuts the x-axis, where $y = 0$.

Putting $y = 0$ in the given equation, we get $x = 7$.

$\therefore$ the point of contact is $(7, 0)$.

$$\text{Now,} \quad y(x-2)(x-3) - x + 7 = 0 \Rightarrow y(x^2 - 5x + 6) - x + 7 = 0$$

$$\Rightarrow (x^2 - 5x + 6) \cdot \frac{dy}{dx} + y(2x - 5) - 1 = 0 \Rightarrow \frac{dy}{dx} = \frac{1 + 5y - 2xy}{x^2 - 5x + 6}\cdot$$

$$\therefore \left(\frac{dy}{dx}\right)_{(7,\,0)} = \left(\frac{1 + 5 \times 0 - 2 \times 7 \times 0}{49 - 35 + 6}\right) = \frac{1}{20}\cdot$$

So, the equation of the tangent is $\frac{y - 0}{x - 7} = \frac{1}{20}$ or $x - 20y - 7 = 0$.

Also, the equation of the normal is $\frac{y - 0}{x - 7} = -20$ or $20x + y - 140 = 0$.

EXAMPLE 17 *Show that* $\frac{x}{a} + \frac{y}{b} = 1$ *touches the curve* $y = be^{-x/a}$ *at the point where the curve crosses the axis of* y. **[CBSE 2005C, '07C]**

SOLUTION The equation of the curve is $y = be^{-x/a}$.

It crosses the y-axis at the point where $x = 0$.

Putting $x = 0$ in the equation of the curve, we get $y = b$.

So, the point of contact is $(0, b)$.

$$\text{Now,} \quad y = be^{-x/a} \Rightarrow \frac{dy}{dx} = \frac{-be^{-x/a}}{a}\cdot$$

$\therefore \quad \left(\frac{dy}{dx}\right)_{(0,\, b)} = -\frac{b}{a}\cdot$

So, the equation of the tangent is

$$\frac{y-b}{x-0} = \frac{-b}{a} \Rightarrow bx + ay = ab \Rightarrow \frac{x}{a} + \frac{y}{b} = 1.$$

Hence, $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at $(0, b)$.

EXAMPLE 18 *Find the equations of the tangent and the normal at the point 't' on the curve* $x = a\sin^3 t,\ y = b\cos^3 t$. **[CBSE 2009C, '14]**

SOLUTION We have $\frac{dx}{dt} = 3a\sin^2 t\cos t$ and $\frac{dy}{dt} = -3b\cos^2 t\sin t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3b\cos^2 t\sin t}{3a\sin^2 t\cos t} = \frac{-b\cos t}{a\sin t}\cdot$$

$\therefore$ the equation of the tangent at the point 't' is

$$\frac{y - b\cos^3 t}{x - a\sin^3 t} = \frac{-b\cos t}{a\sin t}$$

$$\Rightarrow \quad ay\sin t + bx\cos t = ab\sin t\cos t$$

$$\Rightarrow \quad \frac{x}{a\sin t} + \frac{y}{b\cos t} = 1 \quad \text{[on dividing throughout by } ab\sin t\cos t\text{]}.$$

Also, the equation of the normal at the point 't' is

$$\frac{y - b\cos^3 t}{x - a\sin^3 t} = \frac{a\sin t}{b\cos t}$$

$$\Rightarrow \quad ax\sin t - by\cos t = (a^2\sin^4 t - b^2\cos^4 t).$$

EXAMPLE 19 *Find the equations of the tangent and normal to the curve* $x = a\sin 3t,\ y = \cos 2t$ *at* $t = \frac{\pi}{4}\cdot$ **[CBSE 2008]**

SOLUTION We have

$$x = \sin 3t \Rightarrow \frac{dx}{dt} = 3\cos 3t$$

and $y = \cos 2t \Rightarrow \frac{dy}{dt} = -2\sin 2t$.

$$\therefore \quad \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{-2\sin 2t}{3\cos 3t}$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \frac{-2\sin\left(2\times\frac{\pi}{4}\right)}{3\cos\left(3\times\frac{\pi}{4}\right)} = \frac{-2\sin(\pi/2)}{-3\cos(\pi/4)} = \frac{2\sqrt{2}}{3}\cdot$$

When $t = \frac{\pi}{4}$, we have $x = \sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

and $y = \cos\frac{2\pi}{4} = \cos\frac{\pi}{2} = 0.$

$\therefore$ point of contact is $P\left(\frac{1}{\sqrt{2}}, 0\right)$.

Equation of the tangent at the point P is given by

$$\frac{y-0}{x-\frac{1}{\sqrt{2}}} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \frac{\sqrt{2}y}{\sqrt{2}x-1} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow 3\sqrt{2}y = 4x - 2\sqrt{2}$$

$$\Rightarrow 2\sqrt{2}x - 3y - 2 = 0.$$

Equation of the normal at the point P is given by

$$\frac{y-0}{x-\frac{1}{\sqrt{2}}} = \frac{-3}{2\sqrt{2}} \Rightarrow \frac{\sqrt{2}y}{\sqrt{2}x-1} = \frac{-3}{2\sqrt{2}}$$

$$\Rightarrow 4y = -3\sqrt{2}x + 3 \Rightarrow 3\sqrt{2}x + 4y - 3 = 0.$$

EXAMPLE 20 *For the curve $y = 4x^3 - 2x^5$, find all the points on the curve at which the tangent passes through the origin.* **[CBSE 2013C]**

SOLUTION The equation of the given curve is

$$y = 4x^3 - 2x^5. \quad \ldots \text{(i)}$$

On differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 12x^2 - 10x^4.$$

Let the required point be $P(x, y)$.

The equation of tangent to the given curve at $P(x, y)$ is given by

$$Y - y = (12x^2 - 10x^4)(X - x). \quad \ldots \text{(ii)}$$

If it passes through (0, 0) then, we have

$$y = (12x^2 - 10x^4)x$$

$$\Rightarrow 4x^3 - 2x^5 = 12x^3 - 10x^5 \quad \text{[using (i)]}$$

$$\Rightarrow 8x^3 - 8x^5 = 0 \Rightarrow 8x^3(1 - x^2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 1 \text{ or } x = -1.$$

Putting $x = 0$ in (i), we get $y = 0$.

Putting $x = 1$ in (i), we get $y = 2$.

Putting $x = -1$ in (i), we get $y = -2$.

So, the required points are (0, 0), (1, 2) and (–1, –2).

EXERCISE 11H

1. Find the slope of the tangent to the curve

   (i) $y = (x^3 - x)$ at $x = 2$ (ii) $y = (2x^2 + 3\sin x)$ at $x = 0$

   (iii) $y = (\sin 2x + \cot x + 2)^2$ at $x = \frac{\pi}{2}$

Find the equations of the tangent and the normal to the given curve at the indicated point:

2. $y = x^3 - 2x + 7$ at $(1, 6)$
3. $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$
5. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\theta, b\tan\theta)$
6. $y = x^3$ at $P(1, 1)$ **[CBSE 2006C]**
7. $y^2 = 4ax$ at $(at^2, 2at)$
8. $y = \cot^2 x - 2\cot x + 2$ at $x = \frac{\pi}{4}$
9. $16x^2 + 9y^2 = 144$ at $(2, y_1)$, where $y_1 > 0$
10. $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point where $x = 1$
11. Find the equation of the tangent to the curve $\sqrt{x} + \sqrt{y} = a$ at $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$.
12. Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.
13. Find the equation of the tangent to the curve $y = (\sec^4 x - \tan^4 x)$ at $x = \frac{\pi}{3}$.
14. Find the equation of the normal to the curve $y = (\sin 2x + \cot x + 2)^2$ at $x = \frac{\pi}{2}$.
15. Show that the tangents to the curve $y = 2x^3 - 4$ at the points $x = 2$ and $x = -2$ are parallel. **[CBSE 2001C]**
16. Find the equation of the tangent to the curve $x^2 + 3y = 3$, which is parallel to the line $y - 4x + 5 = 0$. **[CBSE 2005]**
17. At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, is the tangent parallel to the y-axis? **[CBSE 2000, '01C, '02C]**
18. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ where the tangent is parallel to the x-axis. **[CBSE 2011]**
19. Prove that the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) are at right angles.
20. Find the points on the curve $y = x^2 + 3x + 4$ at which the tangent passes through the origin.
21. Find the point on the curve $y = x^3 - 11x + 5$ at which the equation of tangent is $y = x - 11$. **[CBSE 2012]**
22. Find the equations of the tangents to the curve $2x^2 + 3y^2 = 14$, parallel to the line $x + 3y = 4$.

23. Find the equation of the tangent to the curve $x^2 + 2y = 8$, which is perpendicular to the line $x - 2y + 1 = 0$.

24. Find the point on the curve $y = 2x^2 - 6x - 4$ at which the tangent is parallel to the x-axis.

25. Find a point on the parabola $y = (x - 3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1). **[CBSE 2005C]**

26. Show that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$. **[CBSE 2005, '07C]**

27. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other. **[CBSE 2002]**

28. Show that the curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ cut orthogonally.

29. Find the equation of tangent to the curve $x = (\theta + \sin\theta)$, $y = (1 + \cos\theta)$ at $\theta = \frac{\pi}{4}\cdot$ **[CBSE 2006C]**

30. Find the equation of the tangent at $t = \frac{\pi}{4}$ for the curve $x = \sin 3t$, $y = \cos 2t$.

ANSWERS (EXERCISE 11H)

1. (i) 11 (ii) 3 (iii) –12 **2.** $x - y + 5 = 0$, $x + y - 7 = 0$

3. $m^2x - my + a = 0$, $m^2x + m^3y - 2am^2 - a = 0$

4. $bx\cos\theta + ay\sin\theta = ab$, $ax\sec\theta - by\,\text{cosec}\,\theta = (a^2 - b^2)$

5. $bx\sec\theta - ay\tan\theta = ab$, $by\cot\theta + ax\cos\theta = (a^2 + b^2)$

6. $y = 3x - 2$, $x + 3y = 4$ **7.** $x - ty + at^2 = 0$, $tx + y = at^3 + 2at$

8. $y = 1$, $x = \frac{\pi}{4}$ **9.** $8x + 3\sqrt{5}y - 36 = 0$, $9\sqrt{5}x - 24y + 14\sqrt{5} = 0$

10. $2x - y + 1 = 0$, $x + 2y - 7 = 0$ **11.** $2(x + y) = a^2$

13. $3y - 48\sqrt{3}x + 16\sqrt{3}\pi - 21 = 0$ **14.** $24y - 2x + \pi - 96 = 0$

16. $4x - y + 13 = 0$ **17.** (–1, 2) and (3, 2)

18. (1, 2), (1, –2) **20.** (–2, 2), (2, 14)

21. (2, –9) **22.** $x + 3y = 7$, $x + 3y = -7$

23. $2x + y - 6 = 0$ **24.** $\left(\frac{3}{2}, \frac{-17}{2}\right)$ **25.** $\left(\frac{7}{2}, \frac{1}{4}\right)$

29. $y = (1 - \sqrt{2})x + \frac{(\sqrt{2} - 1)\pi}{4} + 2$ **30.** $4x - 3\sqrt{2}y - 2\sqrt{2} = 0$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11H)

15. $x = 2 \Rightarrow y = (2 \times 8 - 4) = 12.$
$x = -2 \Rightarrow y = 2 \times (-8) - 4 = -20.$

$$\frac{dy}{dx} = 6x^2 \Rightarrow \left(\frac{dy}{dx}\right)_{(2,\,12)} = 24 \text{ and } \left(\frac{dy}{dx}\right)_{(x=-2)} = 24 \Rightarrow m_1 = m_2.$$

16. $x^2 + 3y = 3 \Rightarrow 2x + 3\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{3}$.

$y - 4x + 5 = 0 \Rightarrow y = 4x + 5 \Rightarrow$ its slope $= 4$.

$\therefore$ the slope of the tangent $= 4$.

So, $\frac{-2x}{3} = 4 \Rightarrow x = -6$.

$x^2 + 3y = 3, x = -6 \Rightarrow y = -11$.

The equation of the tangent at $(-6, -11)$ with slope $= 4$ is $y + 11 = 4(x + 6)$.

17. $x^2 + y^2 - 2x - 4y + 1 = 0 \Rightarrow 2x\frac{dx}{dy} + 2y - 2\frac{dx}{dy} - 4 = 0 \Rightarrow \frac{dx}{dy} = \left(\frac{y-2}{1-x}\right)$.

Now, $\frac{dx}{dy} = 0 \Rightarrow \frac{y-2}{1-x} = 0 \Rightarrow y - 2 = 0 \Rightarrow y = 2$.

$x^2 + 4 - 2x - 8 + 1 = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0$

$\Rightarrow x = 3$ or $x = -1$.

$\therefore$ the required points are $(3, 2)$ and $(-1, 2)$.

18. $2a^2\frac{dy}{dx} = 3x^2 - 6ax$ and so, $\frac{dy}{dx} = 0 \Rightarrow 3x(x - 2a) = 0 \Rightarrow x = 0$ or $x = 2a$.

20. $y = x^2 + 3x + 4 \Rightarrow \frac{dy}{dx} = (2x + 3)$.

Let the required point be $P(a, b)$.

Then, $\left[\frac{dy}{dx}\right]_{(a,\,b)} = 2a + 3$.

$\therefore$ the tangent is $\frac{y-b}{x-a} = 2a + 3 \Rightarrow \frac{-b}{-a} = 2a + 3$ [$\because$ the tangent passes through (0, 0)].

$\therefore b = 2a^2 + 3a$ … (i)

Also, (a, b) lies on the given curve.

$\therefore b = a^2 + 3a + 4$ … (ii)

$\therefore 2a^2 + 3a = a^2 + 3a + 4 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2$.

Now, $(a = 2 \Rightarrow b = 14)$ and $(a = -2 \Rightarrow b = 2)$.

21. The given curve is $y = x^3 - 11x + 5$. … (i)

Let (a, b) be the point on the given curve, the tangent at which is $y = x - 11$.

$\therefore$ slope of the tangent $= 1$.

Now, $y = x^3 - 11x + 5 \Rightarrow \frac{dy}{dx} = (3x^2 - 11) \Rightarrow \left(\frac{dy}{dx}\right)_{(a,\,b)} = (3a^2 - 11)$.

$3a^2 - 11 = 1 \Rightarrow 3a^2 = 12 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2$.

Since the point (a, b) lies on the curve $y = x^3 - 11x + 5$, we have $b = a^3 - 11a + 5$.

Now, $a = 2 \Rightarrow b = (2^3 - 11 \times 2 + 5) = (13 - 22) = -9$.

And, $a = -2 \Rightarrow b = [(-2)^3 - 11 \times (-2) + 5] = (-8 + 27) = 19$.

Thus, the desired points can be (2, –9) and (–2, 19).

But, (2, –9) satisfies (i) while (–2, 19) does not satisfy it.

Hence, the required point is (2, –9).

22. The given line is $x + 3y = 4 \Rightarrow y = -\frac{1}{3}x + \frac{4}{3}\cdot$

Slope of tangent = slope of given line = $\frac{-1}{3}\cdot$

Let the point of contact be (x_1, y_1).

Now, $2x^2 + 3y^2 = 14 \Rightarrow 4x + 6y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-4x}{6y} = \frac{-2x}{3y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-2x}{3y_1}\cdot$

$\therefore \quad \frac{-2x_1}{3y_1} = \frac{-1}{3} \Rightarrow 2x_1 = y_1$

Also, (x_1, y_1) lies on the given curve.

$\therefore \quad 2x_1^2 + 3y_1^2 = 14 \Rightarrow 2x_1^2 + 3 \times 4x_1^2 = 14 \Rightarrow x_1^2 = 1 \Rightarrow x_1 = \pm 1.$

$(x_1 = 1 \Rightarrow y_1 = 2)$ and $(x_1 = -1 \Rightarrow y_1 = -2)$.

So, the points of contact are (1, 2) and (–1, –2).

The respective tangents are

$\frac{y-2}{x-1} = \frac{-1}{3}$ and $\frac{y+2}{x+1} = \frac{-1}{3}$

$\Rightarrow 3y - 6 = -x + 1$ and $3y + 6 = -x - 1 \Rightarrow 3y + x = 7$ and $3y + x = -7$.

25. Let the required point be $P(x_1, y_1)$.

Slope of the line joining $A(3, 0)$ and $B(4, 1)$ is $\frac{(1-0)}{(4-3)} = 1$. Now, $y = (x-3)^2$

$\Rightarrow \frac{dy}{dx} = 2(x-3) \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2(x_1 - 3) \Rightarrow 2(x_1 - 3) = 1 \Rightarrow x_1 = \frac{7}{2}\cdot$

$\therefore \quad y = (x-3)^2 \Rightarrow y_1 = (x_1 - 3)^2 = \left(\frac{7}{2} - 3\right)^2 = \frac{1}{4}\cdot$

Hence, the required point is $\left(\frac{7}{2}, \frac{1}{4}\right)\cdot$

26. Let the point of intersection of the given curves be (x_1, y_1).

Then, $y^2 = x \Leftrightarrow 2y\frac{dy}{dx} = 1 \Leftrightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{1}{2y_1} \Leftrightarrow m_1 = \frac{1}{2y_1}\cdot$

$xy = k \Leftrightarrow x\frac{dy}{dx} + y = 0 \Leftrightarrow \frac{dy}{dx} = \frac{-y}{x} \Leftrightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = m_2 = \frac{-y_1}{x_1}\cdot$

$\therefore \quad m_1 m_2 = -1 \Leftrightarrow \frac{1}{2y_1} \times \frac{-y_1}{x_1} = -1 \Leftrightarrow x_1 = \frac{1}{2}\cdot$

(x_1, y_1) lies on $y^2 = x \Leftrightarrow y_1{}^2 = x_1 \Leftrightarrow y_1{}^2 = \frac{1}{2}\cdot$

(x_1, y_1) lies on $xy = k \Leftrightarrow x_1 y_1 = k \Leftrightarrow x_1^2 y_1^2 = k^2$

$\Leftrightarrow k^2 = \left(\frac{1}{4} \times \frac{1}{2}\right) = \frac{1}{8}\cdot$

$\Leftrightarrow 8k^2 = 1.$

27. Solving the given equations, we get their points of intersection as (a, a) and $(-a, -a)$.

$xy = a^2 \Rightarrow \frac{dy}{dx} = \frac{-y}{x} \Rightarrow \left(\frac{dy}{dx}\right)_{(a, a)} = -1 = m_1.$

$x^2 + y^2 = 2a^2 \Rightarrow \frac{dy}{dx} = \frac{-x}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(a, a)} = -1 = m_2.$

$\therefore\quad \tan\phi = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = 0 \Rightarrow \phi = 0.$

So, the given curves touch each other.

28. $m_1 m_2 = -1.$

29. $x = (\theta + \sin\theta) \Rightarrow \dfrac{dx}{d\theta} = (1 + \cos\theta).$

$y = (1 + \cos\theta) \Rightarrow \dfrac{dy}{d\theta} = -\sin\theta.$

$\therefore\quad \dfrac{dy}{dx} = \dfrac{(dy/d\theta)}{(dx/d\theta)} = \dfrac{-\sin\theta}{(1 + \cos\theta)}.$

$$\left(\frac{dy}{dx}\right) \text{at } \theta = \frac{\pi}{4} = \frac{-\sin(\pi/4)}{(1 + \cos\pi/4)} = \frac{-1}{\sqrt{2}} \cdot \frac{1}{(1 + 1/\sqrt{2})}$$

$$= \frac{-1}{(\sqrt{2} + 1)} \times \frac{(\sqrt{2} - 1)}{(\sqrt{2} - 1)} = (1 - \sqrt{2}).$$

At $\theta = \dfrac{\pi}{4}$, we have $x = \left(\dfrac{\pi}{4} + \sin\dfrac{\pi}{4}\right) = \left(\dfrac{\pi}{4} + \dfrac{1}{\sqrt{2}}\right)$, and

$$y = \left(1 + \cos\frac{\pi}{4}\right) = \left(1 + \frac{1}{\sqrt{2}}\right).$$

Required equation of tangent is

$$\frac{y - \left(1 + \frac{1}{\sqrt{2}}\right)}{x - \left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)} = (1 - \sqrt{2}) \Rightarrow y = (1 - \sqrt{2})x + \frac{(\sqrt{2} - 1)\pi}{4} + 2.$$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. If $y = 2^x$ then $\dfrac{dy}{dx} = ?$

(a) $x(2^{x-1})$ (b) $\dfrac{2^x}{(\log 2)}$ (c) $2^x(\log 2)$ (d) none of these

2. If $y = \log_{10} x$ then $\dfrac{dy}{dx} = ?$

(a) $\dfrac{1}{x}$ (b) $\dfrac{1}{x}(\log 10)$ (c) $\dfrac{1}{x(\log 10)}$ (d) none of these

3. If $y = e^{1/x}$ then $\dfrac{dy}{dx} = ?$

(a) $\dfrac{1}{x} \cdot e^{(1/x - 1)}$ (b) $\dfrac{-e^{1/x}}{x^2}$ (c) $e^{1/x} \log x$ (d) none of these

4. If $y = x^x$ then $\dfrac{dy}{dx} = ?$

(a) $x^x \log x$ (b) $x^x(1 + \log x)$ (c) $x(1 + \log x)$ (d) none of these

5. If $y = x^{\sin x}$ then $\frac{dy}{dx} = ?$

(a) $(\sin x) \cdot x^{(\sin x - 1)}$ (b) $(\sin x \cos x) \cdot x^{(\sin x - 1)}$

(c) $x^{\sin x}\left\{\frac{\sin x + x \log x \cdot \cos x}{x}\right\}$ (d) none of these

6. If $y = x^{\sqrt{x}}$ then $\frac{dy}{dx} = ?$

(a) $\sqrt{x} \cdot x^{(\sqrt{x} - 1)}$ (b) $\frac{x^{\sqrt{x}} \log x}{2\sqrt{x}}$ (c) $x^{\sqrt{x}}\left\{\frac{2 + \log x}{2\sqrt{x}}\right\}$ (d) none of these

7. If $y = e^{\sin \sqrt{x}}$ then $\frac{dy}{dx} = ?$

(a) $e^{\sin \sqrt{x}} \cdot \cos \sqrt{x}$ (b) $\frac{e^{\sin \sqrt{x}} \cos \sqrt{x}}{2\sqrt{x}}$ (c) $\frac{e^{\sin \sqrt{x}}}{2\sqrt{x}}$ (d) none of these

8. If $y = (\tan x)^{\cot x}$ then $\frac{dy}{dx} = ?$

(a) $\cot x \cdot (\tan x)^{\cot x - 1} \cdot \sec^2 x$ (b) $-(\tan x)^{\cot x} \cdot \operatorname{cosec}^2 x$

(c) $(\tan x)^{\cot x} \cdot \operatorname{cosec}^2 x (1 - \log \tan x)$ (d) none of these

9. If $y = (\sin x)^{\log x}$ then $\frac{dy}{dx} = ?$

(a) $(\log x) \cdot (\sin x)^{(\log x - 1)} \cdot \cos x$

(b) $(\sin x)^{\log x} \cdot \left\{\frac{x \log x + \log \sin x}{x}\right\}$

(c) $(\sin x)^{\log x} \cdot \left\{\frac{(x \log x) \cot x + \log \sin x}{x}\right\}$

(d) none of these

10. If $y = \sin (x^x)$ then $\frac{dy}{dx} = ?$

(a) $x^x \cos (x^x)$ (b) $x^x \cos x^x (1 + \log x)$

(c) $x^x \cos x^x \log x$ (d) none of these

11. If $y = \sqrt{x \sin x}$ then $\frac{dy}{dx} = ?$

(a) $\frac{(x \cos x + \sin x)}{2\sqrt{x \sin x}}$ (b) $\frac{1}{2}(x \cos x + \sin x) \cdot \sqrt{x \sin x}$

(c) $\frac{1}{2\sqrt{x \sin x}}$ (d) none of these

12. If $e^{x+y} = xy$ then $\frac{dy}{dx} = ?$

(a) $\frac{x(1 - y)}{y(x - 1)}$ (b) $\frac{y(1 - x)}{x(y - 1)}$ (c) $\frac{(x - xy)}{(xy - y)}$ (d) none of these

13. If $(x+y) = \sin(x+y)$ then $\frac{dy}{dx} = ?$

(a) -1 (b) 1 (c) $\frac{1-\cos(x+y)}{\cos^2(x+y)}$ (d) none of these

14. If $\sqrt{x} + \sqrt{y} = \sqrt{a}$ then $\frac{dy}{dx} = ?$

(a) $\frac{-\sqrt{x}}{\sqrt{y}}$ (b) $-\frac{1}{2}\cdot\frac{\sqrt{y}}{\sqrt{x}}$ (c) $\frac{-\sqrt{y}}{\sqrt{x}}$ (d) none of these

15. If $x^y = y^x$ then $\frac{dy}{dx} = ?$

(a) $\frac{(y - x\log y)}{(x - y\log x)}$ (b) $\frac{y(y - x\log y)}{x(x - y\log x)}$ (c) $\frac{y(y + x\log y)}{x(x + y\log x)}$ (d) none of these

16. If $x^p y^q = (x+y)^{(p+q)}$ then $\frac{dy}{dx} = ?$

(a) $\frac{x}{y}$ (b) $\frac{y}{x}$ (c) $\frac{x^{p-1}}{y^{q-1}}$ (d) none of these

17. If $y = x^2 \sin\frac{1}{x}$ then $\frac{dy}{dx} = ?$

(a) $x\sin\frac{1}{x} - \cos\frac{1}{x}$ (b) $-\cos\frac{1}{x} + 2x\sin\frac{1}{x}$

(c) $-x\sin\frac{1}{x} + \cos\frac{1}{x}$ (d) none of these

18. If $y = \cos^2 x^3$ then $\frac{dy}{dx} = ?$

(a) $-3x^2\sin(2x^3)$ (b) $-3x^2\sin^2 x^3$ (c) $-3x^2\cos^2(2x^3)$ (d) none of these

19. If $y = \log(x + \sqrt{x^2 + a^2})$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{2(x + \sqrt{x^2 + a^2})}$ (b) $\frac{-1}{\sqrt{x^2 + a^2}}$

(c) $\frac{1}{\sqrt{x^2 + a^2}}$ (d) none of these

20. If $y = \log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{\sqrt{x}(1-x)}$ (b) $\frac{-1}{x(1-\sqrt{x})^2}$ (c) $\frac{-\sqrt{x}}{2(1-\sqrt{x})}$ (d) none of these

21. If $y = \log\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{2}{\sqrt{1+x^2}}$ (b) $\frac{2\sqrt{1+x^2}}{x^2}$ (c) $\frac{-2}{\sqrt{1+x^2}}$ (d) none of these

22. If $y = \sqrt{\frac{1+\sin x}{1-\sin x}}$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{2}\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)$ (b) $\frac{1}{2}\operatorname{cosec}^2\left(\frac{\pi}{4}-\frac{x}{2}\right)$

(c) $\frac{1}{2}\operatorname{cosec}\left(\frac{\pi}{4}-\frac{x}{2}\right)\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)$ (d) none of these

23. If $y = \sqrt{\frac{\sec x - 1}{\sec x + 1}}$ then $\frac{dy}{dx} = ?$

(a) $\sec^2 x$ (b) $\frac{1}{2}\sec^2\frac{x}{2}$ (c) $\frac{-1}{2}\operatorname{cosec}^2\frac{x}{2}$ (d) none of these

24. If $y = \sqrt{\frac{1+\tan x}{1-\tan x}}$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{2}\sec^2 x \cdot \tan\left(x+\frac{\pi}{4}\right)$ (b) $\frac{\sec^2\left(x+\frac{\pi}{4}\right)}{2\sqrt{\tan\left(x+\frac{\pi}{4}\right)}}$

(c) $\frac{\sec^2\left(\frac{x}{4}\right)}{\sqrt{\tan\left(x+\frac{\pi}{4}\right)}}$ (d) none of these

25. If $y = \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$ then $\frac{dy}{dx} = ?$

(a) 1 (b) –1 (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

26. If $y = \tan^{-1}\left\{\frac{\cos x + \sin x}{\cos x - \sin x}\right\}$ then $\frac{dy}{dx} = ?$

(a) 1 (b) –1 (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

27. If $y = \tan^{-1}\left\{\frac{\cos x}{1+\sin x}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) 1 (d) –1

28. If $y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$ then $\frac{dy}{dx} = ?$

(a) $\frac{-1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{(1+x^2)}$ (d) none of these

29. If $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{a}{b}$ (b) $\frac{-b}{a}$ (c) 1 (d) –1

30. If $y = \sin^{-1}(3x - 4x^3)$ then $\frac{dy}{dx} = ?$

(a) $\frac{3}{\sqrt{1-x^2}}$ (b) $\frac{-4}{\sqrt{1-x^2}}$ (c) $\frac{3}{\sqrt{1+x^2}}$ (d) none of these

31. If $y = \cos^{-1}(4x^3 - 3x)$ then $\frac{dy}{dx} = ?$

(a) $\frac{3}{\sqrt{1-x^2}}$ (b) $\frac{-3}{\sqrt{1-x^2}}$ (c) $\frac{4}{\sqrt{1-x^2}}$ (d) $\frac{-4}{(3x^2-1)}$

32. If $y = \tan^{-1}\left(\frac{\sqrt{a}+\sqrt{x}}{1-\sqrt{ax}}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{(1+x)}$ (b) $\frac{1}{\sqrt{x}(1+x)}$ (c) $\frac{2}{\sqrt{x}(1+x)}$ (d) $\frac{1}{2\sqrt{x}(1+x)}$

33. If $y = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{2}{(1+x^2)}$ (b) $\frac{-2}{(1+x^2)}$ (c) $\frac{2x}{(1+x^2)}$ (d) none of these

34. If $y = \tan^{-1}\left(\frac{1+x^2}{1-x^2}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{2x}{(1+x^4)}$ (b) $\frac{-2x}{(1+x^4)}$ (c) $\frac{x}{(1+x^4)}$ (d) none of these

35. If $y = \cos^{-1}x^3$ then $\frac{dy}{dx} = ?$

(a) $\frac{-1}{(1+x)}$ (b) $\frac{2}{\sqrt{(1+x)}}$ (c) $\frac{-1}{2\sqrt{x}(1+x)}$ (d) none of these

36. If $y = \cos^{-1}x^3$ then $\frac{dy}{dx} = ?$

(a) $\frac{-1}{\sqrt{1-x^6}}$ (b) $\frac{-3x^2}{\sqrt{1-x^6}}$ (c) $\frac{-3}{x^2\sqrt{1-x^6}}$ (d) none of these

37. If $y = \tan^{-1}(\sec x + \tan x)$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) 1 (d) none of these

38. If $y = \cot^{-1}\left(\frac{1-x}{1+x}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{-1}{(1+x^2)}$ (b) $\frac{1}{(1+x^2)}$ (c) $\frac{1}{(1+x^2)^{3/2}}$ (d) none of these

39. If $y = \sqrt{\frac{1+x}{1-x}}$ then $\frac{dy}{dx} = ?$

(a) $\frac{2}{(1-x)^2}$ (b) $\frac{x}{(1-x)^{3/2}}$

(c) $\frac{1}{(1-x)^{3/2} \cdot (1+x)^{1/2}}$ (d) none of these

40. If $y = \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{-2}{(1+x^2)}$ (b) $\frac{2}{(1+x^2)}$ (c) $\frac{-1}{(1-x^2)}$ (d) none of these

41. If $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{-2}{(1+x^2)}$ (b) $\frac{-2}{(1-x^2)}$ (c) $\frac{-2}{\sqrt{1-x^2}}$ (d) none of these

42. If $y = \tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{(1+x^2)}$ (b) $\frac{2}{(1+x^2)}$ (c) $\frac{1}{2(1+x^2)}$ (d) none of these

43. If $y = \sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}$ then $\frac{dy}{dx} = ?$

(a) $\frac{-1}{2\sqrt{1-x^2}}$ (b) $\frac{1}{2\sqrt{1-x^2}}$ (c) $\frac{1}{2(1+x^2)}$ (d) none of these

44. If $x = at^2$, $y = 2at$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{t}$ (b) $\frac{-1}{t^2}$ (c) $\frac{-2}{t}$ (d) none of these

45. If $x = a\sec\theta$, $y = b\tan\theta$ then $\frac{dy}{dx} = ?$

(a) $\frac{b}{a}\sec\theta$ (b) $\frac{b}{a}\text{cosec}\,\theta$ (c) $\frac{b}{a}\cot\theta$ (d) none of these

46. If $x = a\cos^2\theta$, $y = b\sin^2\theta$ then $\frac{dy}{dx} = ?$

(a) $\frac{-a}{b}$ (b) $\frac{a}{b}\cot\theta$ (c) $\frac{-b}{a}$ (d) none of these

47. If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$ then $\frac{dy}{dx} = ?$

(a) $\cot\theta$ (b) $\tan\theta$ (c) $a\cot\theta$ (d) $a\tan\theta$

48. If $y = x^{x^{x\ldots\infty}}$ then $\frac{dy}{dx} = ?$

(a) $\frac{y}{x(1-\log x)}$ (b) $\frac{y^2}{x(1-\log x)}$ (c) $\frac{y^2}{x(1-y\log x)}$ (d) none of these

49. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\infty}}}$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{(2y-1)}$ (b) $\frac{1}{(y^2-1)}$ (c) $\frac{2y}{(y^2-1)}$ (d) none of these

50. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\infty}}}$ then $\frac{dy}{dx} = ?$

(a) $\frac{\sin x}{(2y-1)}$ (b) $\frac{\cos x}{(y-1)}$ (c) $\frac{\cos x}{(2y-1)}$ (d) none of these

51. If $y = e^x + e^{x+\ldots\infty}$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{(1-y)}$ (b) $\frac{y}{(1-y)}$ (c) $\frac{y}{(y-1)}$ (d) none of these

52. The value of k for which $f(x) = \begin{cases} \frac{\sin 5x}{3x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$ is

(a) $\frac{1}{3}$ (b) 0 (c) $\frac{3}{5}$ (d) $\frac{5}{3}$

53. Let $f(x) = \begin{cases} x\sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{when } x = 0. \end{cases}$

Then, which of the following is the true statement?

(a) $f(x)$ is not defined at $x = 0$ (b) $\lim_{x\to 0} f(x)$ does not exist

(c) $f(x)$ is continuous at $x = 0$ (d) $f(x)$ is discontinuous at $x = 0$

54. The value of k for which $f(x) = \begin{cases} \dfrac{3x + 4\tan x}{x}, & \text{when } x \neq 0 \\ k, & \text{when } x = 0 \end{cases}$

is continuous at $x = 0$, is

(a) 7 (b) 4 (c) 3 (d) none of these

55. Let $f(x) = x^{3/2}$. Then, $f'(0) = ?$

(a) $\dfrac{3}{2}$ (b) $\dfrac{1}{2}$ (c) does not exist (d) none of these

56. The function $f(x) = |x| \ \forall\ x \in R$ is

(a) continuous but not differentiable at $x = 0$

(b) differentiable but not continuous at $x = 0$

(c) neither continuous nor differentiable at $x = 0$

(d) none of these

57. The function $f(x) = \begin{cases} 1 + x, & \text{when } x \leq 2 \\ 5 - x, & \text{when } x > 2 \end{cases}$ is

(a) continuous as well as differentiable at $x = 2$

(b) continuous but not differentiable at $x = 2$

(c) differentiable but not continuous at $x = 2$

(d) none of these

58. If the function $f(x) = \begin{cases} kx + 5, & \text{when } x \leq 2 \\ x - 1, & \text{when } x > 2 \end{cases}$ is continuous at $x = 2$ then $k = ?$

(a) 2 (b) –2 (c) 3 (d) –3

59. If the function $f(x) = \begin{cases} \dfrac{1 - \cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$ then $k = ?$

(a) 1 (b) 2 (c) $\dfrac{1}{2}$ (d) $\dfrac{-1}{2}$

60. If the function $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & \text{when } x \neq 0 \\ k, & \text{when } x = 0 \end{cases}$ is continuous at $x = 0$ then $k = ?$

(a) a (b) a^2 (c) –2 (d) –4

61. If the function $f(x) = \begin{cases} \dfrac{k\cos x}{(\pi - 2x)}, & \text{when } x \neq \dfrac{\pi}{2} \\ 3, & \text{when } x = \dfrac{\pi}{2} \end{cases}$

be continuous at $x = \dfrac{\pi}{2}$, then the value of k is

(a) 3 (b) –3 (c) –5 (d) 6

62. At $x = 2$, $f(x) = [x]$ is
   (a) continuous but not differentiable
   (b) differentiable but not continuous
   (c) continuous as well as differentiable
   (d) none of these

63. Let $f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x + 1^2}, & \text{when } x \neq -1 \\ k, & \text{when } x = -1. \end{cases}$

   If $f(x)$ is continuous at $x = -1$ then $k = ?$
   (a) 4 (b) –4 (c) –3 (d) 2

64. The function $f(x) = x^3 - 6x^2 + 15x - 12$ is
   (a) strictly decreasing on R
   (b) strictly increasing on R
   (c) increasing in $(-\infty, 2]$ and decreasing in $(2, \infty)$
   (d) none of these

65. The function $f(x) = 4 - 3x + 3x^2 - x^3$ is
   (a) decreasing on R (b) increasing on R
   (c) strictly decreasing on R (d) strictly increasing on R

66. The function $f(x) = 3x + \cos 3x$ is
   (a) increasing on R (b) decreasing on R
   (c) strictly increasing on R (d) strictly decreasing on R

67. The function $f(x) = x^3 - 6x^2 + 9x + 3$ is decreasing for
   (a) $1 < x < 3$ (b) $x > 1$ (c) $x < 1$ (d) $x < 1$ or $x > 3$

68. The function $f(x) = x^3 - 27x + 8$ is increasing when
   (a) $|x| < 3$ (b) $|x| > 3$ (c) $-3 < x < 3$ (d) none of these

69. $f(x) = \sin x$ is increasing in
   (a) $\left(\frac{\pi}{2}, \pi\right)$ (b) $\left(\pi, \frac{3\pi}{2}\right)$ (c) $(0, \pi)$ (d) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

70. $f(x) = \dfrac{2x}{\log x}$ is increasing in
   (a) $(0, 1)$ (b) $(1, e)$ (c) (e, ∞) (d) $(-\infty, e)$

71. $f(x) = (\sin x - \cos x)$ is decreasing in
   (a) $\left(0, \frac{3\pi}{4}\right)$ (b) $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$ (c) $\left(\frac{7\pi}{4}, 2\pi\right)$ (d) none of these

72. $f(x) = \dfrac{x}{\sin x}$ is
   (a) increasing in $(0, 1)$
   (b) decreasing in $(0, 1)$

(c) increasing in $\left(0, \frac{1}{2}\right)$ and decreasing in $\left(\frac{1}{2}, 1\right)$

(d) none of these

73. $f(x) = x^x$ is decreasing in the interval

(a) $(0, e)$ (b) $\left(0, \frac{1}{e}\right)$ (c) $(0, 1)$ (d) none of these

74. $f(x) = x^2 e^{-x}$ is increasing in

(a) $(-2, 0)$ (b) $(0, 2)$ (c) $(2, \infty)$ (d) $(-\infty, \infty)$

75. $f(x) = \sin x - kx$ is decreasing for all $x \in R$, when

(a) $k < 1$ (b) $k \le 1$ (c) $k > 1$ (d) $k \ge 1$

76. $f(x) = (x+1)^3 (x-3)^3$ is increasing in

(a) $(-\infty, 1)$ (b) $(-1, 3)$ (c) $(3, \infty)$ (d) $(1, \infty)$

77. $f(x) = [x(x-3)]^2$ is increasing in

(a) $(0, \infty)$ (b) $(-\infty, 0)$

(c) $(1, 3)$ (d) $\left(0, \frac{3}{2}\right) \cup (3, \infty)$

78. If $f(x) = kx^3 - 9x^2 + 9x + 3$ is increasing for every real number x, then

(a) $k > 3$ (b) $k \ge 3$ (c) $k < 3$ (d) $k \le 3$

79. $f(x) = \frac{x}{(x^2+1)}$ is increasing in

(a) $(-1, 1)$ (b) $(-1, \infty)$

(c) $(-\infty, -1) \cup (1, \infty)$ (d) none of these

80. The least value of k for which $f(x) = x^2 + kx + 1$ is increasing on (1, 2), is

(a) –2 (b) –1 (c) 1 (d) 2

81. $f(x) = |x|$ has

(a) minimum at $x = 0$

(b) maximum at $x = 0$

(c) neither a maximum nor a minimum at $x = 0$

(d) none of these

82. When x is positive, the minimum value of x^x is

(a) e^e (b) $e^{1/e}$ (c) $e^{-1/e}$ (d) $(1/e)$

83. The maximum value of $\left(\frac{\log x}{x}\right)$ is

(a) $\left(\frac{1}{e}\right)$ (b) $\frac{2}{e}$ (c) e (d) 1

84. $f(x) = \text{cosec}\, x$ in $(-\pi, 0)$ has a maxima at

(a) $x = 0$ (b) $x = \frac{-\pi}{4}$ (c) $x = \frac{-\pi}{3}$ (d) $x = \frac{-\pi}{2}$

85. If $x > 0$ and $xy = 1$, the minimum value of $(x + y)$ is

(a) –2 (b) 1 (c) 2 (d) none of these

86. The minimum value of $\left(x^2 + \frac{250}{x}\right)$ is

(a) 0 (b) 25 (c) 50 (d) 75

87. The minimum value of $f(x) = 3x^4 - 8x^3 - 48x + 25$ on $[0, 3]$ is

(a) 16 (b) 25 (c) –39 (d) none of these

88. The maximum value of $f(x) = (x-2)(x-3)^2$ is

(a) $\frac{7}{3}$ (b) 3 (c) $\frac{4}{27}$ (d) 0

89. The least value of $f(x) = (e^x + e^{-x})$ is

(a) –2 (b) 0 (c) 2 (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (c)	**2.** (c)	**3.** (b)	**4.** (b)	**5.** (c)	**6.** (c)	**7.** (b)	**8** (c)	**9.** (c)	**10.** (b)
11. (a)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (b)	**16.** (b)	**17.** (b)	**18.** (a)	**19.** (c)	**20.** (a)
21. (a)	**22.** (b)	**23.** (b)	**24.** (b)	**25.** (c)	**26.** (a)	**27.** (b)	**28.** (b)	**29.** (d)	**30.** (a)
31. (b)	**32.** (d)	**33.** (b)	**34.** (a)	**35.** (c)	**36.** (b)	**37.** (a)	**38.** (b)	**39.** (c)	**40.** (a)
41. (c)	**42.** (c)	**43.** (a)	**44.** (a)	**45.** (b)	**46.** (c)	**47.** (b)	**48.** (c)	**49.** (a)	**50.** (c)
51. (b)	**52.** (d)	**53.** (c)	**54.** (a)	**55.** (c)	**56.** (a)	**57.** (b)	**58.** (b)	**59.** (c)	**60.** (b)
61. (d)	**62.** (d)	**63.** (b)	**64.** (b)	**65.** (a)	**66.** (a)	**67.** (a)	**68.** (b)	**69.** (d)	**70.** (c)
71. (b)	**72.** (a)	**73.** (b)	**74.** (b)	**75.** (c)	**76.** (d)	**77.** (d)	**78.** (a)	**79.** (a)	**80.** (a)
81. (a)	**82.** (c)	**83.** (a)	**84.** (d)	**85.** (c)	**86.** (d)	**87.** (c)	**88.** (c)	**89.** (c)	

HINTS TO THE GIVEN OBJECTIVE QUESTIONS

1. $\frac{dy}{dx} = 2^x(\log 2)$ 2. $y = \frac{\log x}{\log 10}$ 3. $\frac{dy}{dx} = \frac{e^{1/x}}{-x^2}$

4. $\log y = x \log x$. Now differentiate w.r.t. x.

5. $\log y = \sin x(\log x)$ 6. $\log y = \sqrt{x}(\log x)$ 7. $\log y = \sin \sqrt{x}$

8. $\log y = \cot x \cdot \log(\tan x)$ 9. $\log y = (\log x) \cdot (\log \sin x)$

10. Let $x^x = z$. Then, $y = \sin z$. Then, $\frac{dy}{dx} = \left(\frac{dy}{dz} \times \frac{dz}{dx}\right)$.

11. $y^2 = x\sin x \Rightarrow 2y \cdot \frac{dy}{dx} = (x\cos + \sin x).$

12. $(x+y) = \log x + \log y \Rightarrow 1 + \frac{dy}{dx} = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx}.$

13. $(x+y) = \sin(x+y) \Rightarrow 1 + \frac{dy}{dx} = \cos(x+y) \cdot \left[1 + \frac{dy}{dx}\right].$

14. $\sqrt{x} + \sqrt{y} = \sqrt{a} \Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0.$

15. $x^y = y^x \Rightarrow y\log x = x\log y.$ Now, differentiate both sides w.r.t. x.

16. $p\log x + q\log y = (p+q)\log(x+y).$

17. $\frac{dy}{dx} = x^2 \cdot \left(\cos\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) + 2x\sin\frac{1}{x}.$

18. $y = (\cos x^3)^2 \Rightarrow \frac{dy}{dx} = 2(\cos x^3)(-\sin x^3)(3x^2) = -\,3x^2\sin(2x^3).$

19. $\frac{dy}{dx} = \frac{1}{(x + \sqrt{x^2 + a^2}} \cdot \left\{1 + \frac{1}{2\sqrt{x^2 + a^2}} \times 2x\right\} = \frac{1}{\sqrt{x^2 + a^2}}.$

20. $y = \log(1 + \sqrt{x}) - \log(1 - \sqrt{x})$

$$\Rightarrow \frac{dy}{dx} = \left\{\frac{1}{2\sqrt{x}(1+\sqrt{x})} + \frac{1}{2\sqrt{x}(1-\sqrt{x})}\right\} = \frac{1}{2\sqrt{x}} \cdot \frac{2}{(1-x)} = \frac{1}{\sqrt{x}(1-x)}.$$

21. $y = \log(\sqrt{1+x^2} + x) - \log(\sqrt{1+x^2} - x).$ Now, differentiate.

22. $y = \left\{\frac{1 + \cos\left(\frac{\pi}{2} - x\right)}{1 - \cos\left(\frac{\pi}{2} - x\right)}\right\}^{\frac{1}{2}} = \left\{\frac{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right\}^{\frac{1}{2}} = \cot\left(\frac{\pi}{4} - \frac{x}{2}\right).$

23. $y = \left(\frac{1 - \cos x}{1 + \cos x}\right)^{\frac{1}{2}} = \left\{\frac{2\sin^2(x/2)}{2\cos^2(x/2)}\right\}^{\frac{1}{2}} = \tan\frac{x}{2}.$

24. $y = \left\{\tan\left(x + \frac{\pi}{4}\right)\right\}^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left\{\tan\left(x + \frac{\pi}{4}\right)\right\}^{-\frac{1}{2}} \cdot \sec^2\left(x + \frac{\pi}{4}\right).$

25. $y = \tan^{-1}\left(\frac{1 - \cos x}{\sin x}\right) = \tan^{-1}\left\{\frac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}\right\} = \tan^{-1}\left\{\tan\frac{x}{2}\right\} = \frac{x}{2}.$

26. $y = \tan^{-1}\left\{\frac{\cos x + \sin x}{\cos x - \sin x}\right\} = \tan^{-1}\left\{\frac{1 + \tan x}{1 - \tan x}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4} + x\right)\right\} = \left(\frac{\pi}{4} + x\right).$

27. $y = \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\} = \left(\frac{\pi}{4} - \frac{x}{2}\right).$

28. $y = \tan^{-1}\left\{\dfrac{2\sin^2(x/2)}{2\cos^2(x/2)}\right\}^{\frac{1}{2}} = \tan^{-1}\left\{\tan\dfrac{x}{2}\right\} = \dfrac{x}{2}.$

29. $y = \tan^{-1}\left(\dfrac{\dfrac{a}{b} - \tan x}{1 + \dfrac{a}{b}\tan x}\right) = \tan^{-1}\left(\dfrac{\tan\theta - \tan x}{1 + \tan\theta\tan x}\right)$

$$= \tan^{-1}\tan(\theta - x) = \theta - x = \left(\tan^{-1}\frac{a}{b} - x\right).$$

$\therefore\ \dfrac{dy}{dx} = -1.$

30. Putting $x = \sin\theta$, we get $y = \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1}x.$

31. Putting $x = \cos\theta$, we get $y = \cos^{-1}(\cos 3\theta) = 3\theta = 3\cos^{-1}x.$

32. Put $\sqrt{a} = \tan\theta$ and $\sqrt{x} = \tan\phi$. Then,

$y = \tan^{1}\{\tan(\theta + \phi)\} = \theta + \phi = \tan^{-1}\sqrt{a} + \tan^{-1}\sqrt{x}.$

33. Putting $x = \cot\theta$, we get:

$y = \cos^{-1}\left(\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta = 2\cot^{-1}x.$

34. Putting $x^2 = \tan\theta$, we get:

$y = \tan^{-1}\left\{\tan\left(\dfrac{\pi}{4} + \theta\right)\right\} = \left(\dfrac{\pi}{4} + \theta\right) = \left(\dfrac{\pi}{4} + \tan^{-1}x^2\right).$

35. $y = \cot^{-1}\sqrt{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{(1+x)}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{-1}{2\sqrt{x}(1+x)}.$

36. $y = \cos^{-1}x^3 \Rightarrow \dfrac{dy}{dx} = \dfrac{-3x^2}{\sqrt{1 - x^6}}.$

37. $y = \tan^{-1}\left(\dfrac{1 + \sin x}{\cos x}\right) = \tan^{-1}\left[\dfrac{\{\cos(x/2) + \sin(x/2)\}^2}{\cos^2(x/2) - \sin^2(x/2)}\right] = \tan^{-1}\left\{\dfrac{\cos(x/2) + \sin(x/2)}{\cos(x/2) - \sin(x/2)}\right\}$

$= \tan^{-1}\left\{\dfrac{1 + \tan(x/2)}{1 - \tan(x/2)}\right\} = \tan^{-1}\left\{\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)\right\} = \left(\dfrac{\pi}{4} + \dfrac{x}{2}\right).$

38. Put $x = \tan\theta$. Then, $y = \cot^{-1}\cdot\tan\left(\dfrac{\pi}{4} - \theta\right) = \cot^{-1}\left[\cot\left\{\dfrac{\pi}{2} - \left(\dfrac{\pi}{4} - \theta\right)\right\}\right] = \left(\dfrac{\pi}{4} + \theta\right).$

$\therefore\ y = \dfrac{\pi}{4} + \tan^{-1}x.$

39. $\log y = \dfrac{1}{2}\{\log(1+x) - \log(1-x)\} \Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{1}{2}\left\{\dfrac{1}{(1+x)} + \dfrac{1}{(1-x)}\right\}.$

40. Put $x = \cot\theta$. Then, $y = \sec^{-1}\left(\dfrac{1 + \tan^2\theta}{1 - \tan^2\theta}\right) = \sec^{-1}\left(\dfrac{1}{\cos 2\theta}\right) = \sec^{-1}(\sec 2\theta)$

$$= 2\theta = 2\cot^{-1}x.$$

41. Put $x = \cos\theta$. Then, $y = \sec^{-1}\left(\dfrac{1}{2\cos^2\theta - 1}\right) = \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1}x.$

42. Put $x = \tan\theta$. Then, $y = \dfrac{1}{2}\tan^{-1}x.$

43. Put $x = \cos\theta$. Then, $y = \dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1}x.$

44. $\dfrac{dx}{dt} = 2at,\ \dfrac{dy}{dt} = 2a.$ So, $\dfrac{dy}{dx} = \dfrac{(dy/dt)}{(dx/dt)} = \dfrac{1}{t}\cdot$

45. $\dfrac{dx}{d\theta} = a\sec\theta\tan\theta,\ \dfrac{dy}{d\theta} = b\sec^2\theta\cdot \dfrac{dy}{dx} = \dfrac{(dy/d\theta)}{(dx/d\theta)} = \dfrac{b}{a}\operatorname{cosec}\theta.$

46. $\dfrac{dx}{d\theta} = -2a\cos\theta\sin\theta,\ \dfrac{dy}{d\theta} = 2b\sin\theta\cos\theta \Rightarrow \dfrac{dy}{dx} = \dfrac{(dy/d\theta)}{(dx/d\theta)} = \dfrac{-b}{a}\cdot$

47. $\dfrac{dy}{dx} = \dfrac{(dy/d\theta)}{(dx/d\theta)}\cdot$

48. $y = x^y \Rightarrow \log y = y\log x \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{y}{x} + (\log x)\dfrac{dy}{dx}\cdot$

49. $y = \sqrt{x+y} \Rightarrow y^2 = x + y \Rightarrow 2y\dfrac{dy}{dx} = 1 + \dfrac{dy}{dx}\cdot$

50. $y = \sqrt{\sin x + y} \Rightarrow y^2 = \sin x + y \Rightarrow 2y\dfrac{dy}{dx} = \cos x + \dfrac{dy}{dx}\cdot$

51. $y = e^{x+y} \Rightarrow x + y = \log y \Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{1}{y}\cdot\dfrac{dy}{dx}\cdot$

52. For continuity at $x = 0$, we must have $\lim\limits_{x\to 0} f(x) = f(0)$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin 5x}{5x}\times\frac{5}{3} = \frac{5}{3}\lim_{5x\to 0}\frac{\sin 5x}{5x} = \left(\frac{5}{3}\times 1\right) = \frac{5}{3}\cdot$$

$\therefore$ we must have, $f(0) = \dfrac{5}{3} \Leftrightarrow k = \dfrac{5}{3}\cdot$

53. $f(0) = 0.$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\frac{1}{x} = 0\times(\text{a finite quantity}) = 0.$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

54. $f(0) = k.$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{3x + 4\tan x}{x} = \lim_{x\to 0}\left\{3 + \frac{4\tan x}{x}\right\} = (3+4) = 7.$$

$\therefore$ $f(x)$ is continuous at $x = 0 \Leftrightarrow f(0) = 7 \Leftrightarrow k = 7.$

55. $f'(0) = \lim\limits_{h\to 0}\dfrac{f(0-h) - f(0)}{-h} = \lim\limits_{h\to 0}\dfrac{(h^{3/2} - 0)}{-h} = \lim\limits_{h\to 0}(-h)^{1/2}$, which does not exist, since $(-h)^{1/2}$ is imaginary.

56. $f(0+0) = \lim\limits_{h\to 0}|0+h| = \lim\limits_{h\to 0}|h| = 0,$

$f(0-0) = \lim\limits_{h\to 0}|0-h| = \lim\limits_{h\to 0}|-h| = \lim\limits_{h\to 0}|h| = 0$ and $f(0) = 0$

$\therefore\ f(x)$ is continuous at $x = 0$

$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-0}{h} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1$

$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-0}{-h} = \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1.$

$\therefore\ Rf'(0) \neq Lf'(0)$, which shows that $f(x)$ is not differentiable at $x = 0$.

57. $f(2+0) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} \{5-(2+h)\} = 3;$

$f(2-0) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} \{1+(2-h)\} = 3$ and $f(2) = 3.$

$\therefore\ f(x)$ is continuous at $x = 2$

$Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(3-h)-3}{h} = \lim_{h\to 0} \frac{-h}{h} = -1.$

$Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0} \frac{(3-h)-3}{-h} = \lim_{h\to 0} \frac{h}{h} = 1.$

$\therefore\ f(x)$ is not differentiable at $x = 2.$

58. $f(2) = \lim_{x\to 2} f(x).$

$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} (2+h-1) = 1$

$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} \{k(2-h)+5\} = 2k+5$

$\therefore\ 2k+5 = 1 \Rightarrow k = -2.$

Also, $f(2) = 2k+5 = 1$. Hence, $k = -2.$

59. $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{1-\cos 4h}{8h^2} = \lim_{h\to 0} \frac{2\sin^2 2h}{8h^2}$

$$= \frac{1}{2} \lim_{h\to 0} \left(\frac{\sin 2h}{2h}\right)^2 = \left(\frac{1}{2}\times 1^2\right) = \frac{1}{2}\cdot$$

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{1-\cos 4(-h)}{8(-h)^2} = \lim_{h\to 0} \frac{(1-\cos 4h)}{8h^2} = \frac{1}{2}\cdot$

$\therefore\ \lim_{x\to 0^+} f(x) = \frac{1}{2}$

For continuity, we must have $f(0) = \frac{1}{2}\cdot$

60. $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin^2 ax}{a^2x^2}\times a^2 = a^2\cdot \lim_{ax\to 0} \left(\frac{\sin ax}{ax}\right)^2 = a^2\times 1^2 = a^2.$

For continuity, we must have $f(0) = a^2.$

61. $f\left(\frac{\pi}{2}-0\right) = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)} = \lim_{h\to 0} \frac{k\sin h}{2h} = \frac{k}{2}\lim_{h\to 0} \frac{\sin h}{h} = \left(\frac{k}{2}\times 1\right) = \frac{k}{2}\cdot$

$f\left(\frac{\pi}{2}+0\right) = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)} = \lim_{h\to 0} \frac{-k\sin h}{-2h} = \frac{k}{2}\lim_{h\to 0} \frac{\sin h}{h} = \frac{k}{2}$

$\therefore\ \frac{k}{2} = 3 \Rightarrow k = 6.$

62. $f(2) = [2] = 2.$

$f(2+0) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} [2+h] = 2.$

$f(2-0) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} [2-h] = 1.$

$\therefore$ $f(x)$ is not continuous at $x = 2$.

$$Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{[2+h]-[2]}{h} = \lim_{h\to 0} \frac{(2-2)}{h} = 0.$$

$$Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0} \frac{[2-h]-[2]}{-h} = \lim_{h\to 0} \frac{(1-2)}{-h} = \lim_{h\to 0} \frac{1}{h} = \infty$$

$\therefore$ $f(x)$ is not differentiable at $x = 2$.

63. $\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{(x-3)(x+1)}{(x+1)} = \lim_{x\to -1} (x-3) = -4.$

For continuity, we must have, $f(-1) = -4$.

64. $f'(x) = 3x^2 - 12x + 15 = 3(x^2 - 4x + 5) = 3[(x-2)^2 + 1] > 0.$

$\Rightarrow f'(x) > 0$ for all $x \in R \Rightarrow f(x)$ is strictly increasing on R.

65. $f'(x) = -3 + 6x - 3x^2 = -3(x^2 - 2x + 1) = -3(x-1)^2 \le 0.$

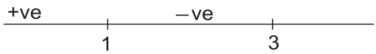
$\Rightarrow f'(x) \le 0$ for all $x \in R \Rightarrow f(x)$ is decreasing on R.

66. $f'(x) = 3 - 3\sin 3x = 3(1 - \sin 3x) \ge 0$ since $-1 \le \sin 3x \le 1$.

$\therefore$ $f'(x) \ge 0$ for all $x \in R \Rightarrow f(x)$ is increasing on R.

67. $f'(x) = 3x^2 - 12 + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$

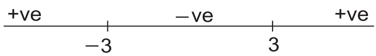
$f'(x) = 0 \Rightarrow x = 1$ or $x = 3$.



There are two factors in $f'(x)$, so we start with $+ve$ sign.

$\therefore$ $f(x)$ is decreasing for $1 < x < 3$.

68. $f'(x) = 3x^2 - 27 = 3(x^2 - 9) = 3(x+3)(x-3)$



$\therefore$ $f'(x) = 0 \Rightarrow x = -3$ or $x = 3$.

There are two factors in $f'(x)$, so we start with $+ve$ sign.

$\therefore$ $f(x)$ is increasing when $x < -3$ or $x > 3$, i.e., when $|x| > 3$.

69. $f'(x) = \cos x > 0$ in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \Rightarrow f(x)$ is increasing in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

70. $$f'(x) = \frac{(\log x)\cdot 2 - 2x \cdot \frac{1}{x}}{(\log x)^2} = \frac{2(\log x - 1)}{(\log x)^2}$$

$\therefore$ $f'(x) > 0 \Leftrightarrow \log x - 1 > 0 \Leftrightarrow \log x > 1 \Leftrightarrow \log x > \log e \Leftrightarrow x > e.$

$\therefore$ $f(x)$ is increasing in (e, ∞).

71. $$f'(x) = (\cos x + \sin x) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right) = \sqrt{2}\sin\left(\frac{\pi}{4} + x\right)$$

$\therefore$ $f'(x) < 0 \Rightarrow \sin\left(\frac{\pi}{4} + x\right) < 0 \Rightarrow \pi < \frac{\pi}{4} + x < 2\pi \Rightarrow \frac{3\pi}{4} < x < \frac{7\pi}{4}.$

$\therefore$ $f(x)$ is decreasing in $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$.

72. $f'(x) = \dfrac{\sin x - x\cos x}{\sin^2 x} = \dfrac{\cos x(\tan x - x)}{\sin^2 x}$.

$0 < x < 1 \Rightarrow \tan x > x$ and $\cos x > 0 \Rightarrow \cos x(\tan x - x) > 0$

$\Rightarrow f'(x) > 0$

$\therefore$ $f(x)$ is increasing in $(0, 1)$.

73. $f'(x) = x^x(1 + \log x)$

$f'(x) < 0 \Leftrightarrow (1 + \log x) < 0 \Rightarrow \log x < -1 = \log \dfrac{1}{e} \Rightarrow x > 0$ and $x < \dfrac{1}{e}$.

$\therefore$ $f(x)$ is decreasing in $\left(0, \dfrac{1}{e}\right)$.

74. $f'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x)$

$\therefore$ $f'(x) > 0 \Leftrightarrow x > 0$ and $(2 - x) > 0 \Leftrightarrow 0 < x < 2$.

$\therefore$ $f(x)$ is increasing in $(0, 2)$.

75. $f'(x) = (\cos x - k)$ and therefore,

$f(x)$ is decreasing $\Leftrightarrow f'(x) < 0 \Rightarrow \cos x - k < 0$

$\Rightarrow \cos x < k \Rightarrow k > \cos x \Rightarrow k > 1$.

76. $f(x) = (x + 1)^3(x - 3)^3$

$\therefore$ $f'(x) = 3(x + 1)^3(x - 3)^2 + 3(x + 1)^2(x - 3)^3$

$= 3(x + 1)^2(x - 3)^2[(x + 1) + (x - 3)] = 3(x + 1)^2(x - 3)^2(x - 1)$

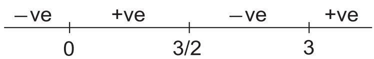
$\Rightarrow f'(x) > 0$ when $(x - 1) > 0$, i.e., when $x > 1$.

$\therefore$ $f(x)$ is increasing in $(1, \infty)$.

77. $f(x) = [x(x - 3)]^2 \Rightarrow f'(x) = 2x(x - 3)(2x - 3)$.

$\therefore$ $f'(x) = 0 \Rightarrow x = 0$ or $x = \dfrac{3}{2}$ or $x = 3$.

$f(x)$ is increasing when $0 < x < \dfrac{3}{2}$ or $x > 3$.



$\therefore$ $f(x)$ is increasing in $\left(0, \dfrac{3}{2}\right) \cup (3, \infty)$.

78. $f'(x) = 3kx^2 - 18x + 9 = 3(kx^2 - 6x + 3)$.

This is positive when $k > 0$ and $(36 - 12k) < 0 \Rightarrow k > 3$.

79. $f'(x) = \dfrac{(x^2 + 1)\cdot 1 - x \cdot 2x}{(x^2 + 1)^2} = \dfrac{(1 - x^2)}{(1 + x^2)^2}$.

$f'(x) > 0 \Leftrightarrow (1 - x^2) > 0 \Leftrightarrow x^2 < 1 \Leftrightarrow -1 < x < 1$.

$\therefore$ $f(x)$ is increasing in $(-1, 1)$.

80. $f'(x) = (2x + k)$.

$1 < x < 2 \Rightarrow 2 < 2x < 4 \Rightarrow 2 + k < 2x + k < 4 + k \Rightarrow 2 + k < f'(x) < 4 + k$.

$f(x)$ is increasing $\Leftrightarrow (2x + k) \geq 0 \Leftrightarrow 2 + k \geq 0 \Leftrightarrow k \geq -2$.

$\therefore$ least value of k is -2.

81. $f(x) = |x| \geq 0$ for all $x \in R$.

The least value of $|x|$ is 0 at $x = 0$.

$\therefore$ $f(x) = |x|$ has minima at $x = 0$.

82. $f(x) = x^2 \Rightarrow f'(x) = x^x(1 + \log x)$ and $f''(x) = x^x\left[\frac{1}{x} + (1 + \log x)^2\right]$.

$f'(x) = 0 \Rightarrow 1 + \log x = 0 \Rightarrow \log x = -1 = \log\left(\frac{1}{e}\right) \Rightarrow x = \left(\frac{1}{e}\right)$.

$[f''(x)]_{x = (1/e)} = e\left(\frac{1}{e}\right)^{1/e} > 0.$

$\therefore\ x = \frac{1}{e}$ is a point of minima.

Minimum value of x^x is $\left(\frac{1}{e}\right)^{1/e} = e^{-1/e}$.

83. $f(x) = \frac{\log x}{x} \Rightarrow f'(x) = \frac{x \cdot \frac{1}{x} - \log x \cdot 1}{x^2} = \frac{(1 - \log x)}{x^2}$

$$\Rightarrow f''(x) = \frac{x^2 \cdot \frac{1}{x} - (1 - \log x) \cdot 2x}{x^4} = \frac{(-3 + 2\log x)}{x^3}$$

$\therefore\ f'(x) = 0 \Rightarrow 1 - \log x = 0 \Rightarrow \log x = 1 = \log e \Rightarrow x = e.$

$f''(e) = \left(\frac{-3 + 2}{e^3}\right) = \frac{-1}{e^3} < 0.$

$\therefore\ x = e$ is a point of maxima.

Maximum value of $f(x)$ is $\frac{1}{e}(\log e) = \frac{1}{e}$.

84. $f(x) = \operatorname{cosec} x \Rightarrow f'(x) = -\operatorname{cosec} x \cot x$

$$\Rightarrow f''(x) = \operatorname{cosec}^3 x + \operatorname{cosec} x\,(\cot^2 x) = \operatorname{cosec} x\,(\operatorname{cosec}^2 x + \cot^2 x)$$

$$= \operatorname{cosec} x\,(2\operatorname{cosec}^2 x - 1)$$

$f'(x) = 0 \Rightarrow \cot x = 0 \Rightarrow x = \frac{-\pi}{2}$

$f''\left(\frac{-\pi}{2}\right) = \operatorname{cosec}\left(\frac{-\pi}{2}\right)\left[2\operatorname{cosec}^2\left(\frac{-\pi}{2}\right) - 1\right] = (-1)(2 - 1) = -1 < 0.$

$\therefore\ x = \frac{-\pi}{2}$ is a point of maxima.

85. $xy = 1 \Rightarrow y = \frac{1}{x}$.

Let $S = x + y = x + \frac{1}{x}$. Then, $\frac{ds}{dx} = \left(1 - \frac{1}{x^2}\right) = \frac{(x^2 - 1)}{x^2}$ and $\frac{d^2s}{dx^2} = \frac{2}{x^3}$.

$\frac{ds}{dx} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1.$

$\left.\frac{d^2s}{dx^2}\right]_{(x=-1)} = -2 < 0$ and $\left.\frac{d^2s}{dx^2}\right]_{(x=1)} = 2 > 0$

$\therefore\ S$ is minimum at $x = 1$ and minimum value of $S = (1 + 1) = 2.$

86. Let $f(x) = \left(x^2 + \frac{250}{x}\right)$. Then, $f'(x) = \left(2x - \frac{250}{x^2}\right)$ and $f''(x) = \left(2 + \frac{500}{x^3}\right)$

$f'(x) = 0 \Rightarrow 2x^3 - 250 = 0 \Rightarrow x^3 = 125 \Rightarrow x = 5.$

$f''(5) = \left(2 + \frac{500}{125}\right) = 6 > 0$

$\therefore$ $f(x)$ is minimum at $x = 5$ and minimum value $= \left(25 + \frac{250}{5}\right) = 75.$

87. $f'(x) = 12x^3 - 24x^2 + 24x - 48 = 12(x - 2)(x^2 + 2)$

$f''(x) = 36x^2 - 48x + 24 = 12(3x^2 - 4x + 2).$

$f'(x) = 0 \Rightarrow x = 2$ and $f''(2) = 12(3 \times 4 - 4 \times 2 + 2) = 72 > 0$

$\therefore$ $x = 2$ is a point of minima.

Minimum value $= \min\{f(0), f(2), f(3)\}$

$= \min\{25, -39, 16\} = -39.$

88. $f(x) = (x - 2)(x - 3)^2 \Rightarrow f'(x) = (x - 3)(3x - 7)$ and $f''(x) = (6x - 16).$

$f'(x) = 0 \Rightarrow x = 3$ or $x = \frac{7}{3}$

$f''(3) = 2 > 0$ and $f''\left(\frac{7}{3}\right) = -2 < 0.$

$\therefore$ $x = \frac{7}{3}$ is a point of maxima.

Maximum value $= \left(\frac{7}{3} - 2\right)\left(\frac{7}{3} - 3\right)^2 = \frac{4}{27}.$

89. $f'(x) = e^x - e^{-x}$ and $f''(x) = e^x + e^{-x}.$

$f'(x) = 0 \Rightarrow e^x - e^{-x} = 0 \Rightarrow e^x = e^{-x} \Rightarrow e^{2x} = e^0 \Rightarrow x = 0.$

$f''(0) = e^0 + \frac{1}{e^0} = (1 + 1) = 2 > 0.$

$\therefore$ $f(x)$ is minimum at $x = 0$ and minimum value of $f(x)$ is 2.

12. INDEFINITE INTEGRAL

INTEGRATION *It is the inverse process of differentiation.*

If the derivative of $F(x)$ is $f(x)$ then we say that the *antiderivative* or *integral* of $f(x)$ is $F(x)$ and we write,

$$\int f(x)\,dx = F(x).$$

Thus, $$\frac{d}{dx}[F(x)] = f(x) \;\Rightarrow\; \int f(x)\,dx = F(x).$$

Example Since $\frac{d}{dx}(\sin x) = \cos x$, we have $\int \cos x\,dx = \sin x$.

Moreover, if C is any constant then $\frac{d}{dx}(\sin x + C) = \cos x$.

So, in general, $\int \cos x\,dx = (\sin x + C)$.

Clearly, different values of C will give different integrals.

Thus, a given function may have an indefinite number of integrals. Because of this property, we call these integrals *indefinite integrals.*

Thus, $\frac{d}{dx}[F(x)] = f(x) \;\Rightarrow\; \int f(x)\,dx = F(x) + C$, where C is a constant, called the *constant of integration.* Any function to be integrated is known as an *integrand.*

The following two results are a direct consequence of the definition of an integral.

RESULT 1 $\int x^n dx = \frac{x^{(n+1)}}{(n+1)} + C$, *when* $n \neq -1$.

PROOF We have, $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{(n+1)} = x^n.$

$$\therefore \qquad \int x^n dx = \frac{x^{(n+1)}}{(n+1)} + C.$$

Thus, we have

(i) $\int x^6 dx = \frac{x^{(6+1)}}{(6+1)} + C = \frac{x^7}{7} + C.$

(ii) $\int x^{2/3}dx = \frac{x^{\left(\frac{2}{3}+1\right)}}{\left(\frac{2}{3}+1\right)} + C = \frac{3}{5}x^{5/3} + C.$

(iii) $\int x^{-3/4}dx = \frac{x^{\left(-\frac{3}{4}+1\right)}}{\left(-\frac{3}{4}+1\right)} = 4x^{1/4} + C.$

RESULT 2 $\int \frac{1}{x}\,dx = \log|x| + C,$ *where* $x \neq 0$.

PROOF Clearly, either $x > 0$ or $x < 0$.

Case I *When* $x > 0$

In this case, $|x| = x$.

$\therefore \quad \frac{d}{dx}[\log|x|] = \frac{d}{dx}(\log x) = \frac{1}{x}.$

So, we have, $\int \frac{1}{x}dx = \log|x| + C.$

Case II *When* $x < 0$

In this case $|x| = -x$.

$\therefore \quad \frac{d}{dx}[\log|x|] = \frac{d}{dx}[\log(-x)] = \frac{1}{(-x)}\cdot(-1) = \frac{1}{x}\cdot$

So, we have $\int \frac{1}{x}\,dx = \log|x| + C.$

Thus, from both the cases, we have $\int \frac{1}{x}\,dx = \log|x| + C.$

FORMULAE

On the basis of differentiation and the definition of integration, we have the following results.

1. $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1 \Rightarrow \int x^n dx = \frac{x^{n+1}}{(n+1)} + C$

2. $\frac{d}{dx}(\log|x|) = \frac{1}{x} \Rightarrow \int \frac{1}{x}\,dx = \log|x| + C$

3. $\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + C$

4. $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x \Rightarrow \int a^x dx = \frac{a^x}{\log a} + C$

5. $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x\,dx = \sin x + C$

6. $\frac{d}{dx}(-\cos x) = \sin x \Rightarrow \int \sin x\,dx = -\cos x + C$

7. $\dfrac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x\, dx = \tan x + C$

8. $\dfrac{d}{dx}(-\cot x) = \text{cosec}^2 x \Rightarrow \int \text{cosec}^2 x\, dx = -\cot x + C$

9. $\dfrac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x\, dx = \sec x + C$

10. $\dfrac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cot x \Rightarrow \int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$

11. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}} \Rightarrow \int \dfrac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$

12. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{(1+x^2)} \Rightarrow \int \dfrac{1}{(1+x^2)}\, dx = \tan^{-1} x + C$

13. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}} \Rightarrow \int \dfrac{1}{x\sqrt{x^2-1}}\, dx = \sec^{-1} x + C$

With the help of the above formulae, it is easy to evaluate the following integrals.

EXAMPLE 1 *Evaluate:*

(i) $\int x^9 dx$ *(ii)* $\int \sqrt[3]{x}\, dx$ *(iii)* $\int dx$

(iv) $\int \dfrac{1}{x^2}\, dx$ *(v)* $\int \dfrac{1}{x^{1/3}}\, dx$ *(vi)* $\int 5^x\, dx$

SOLUTION Using the standard formulae, we have

(i) $\int x^9 dx = \dfrac{x^{(9+1)}}{(9+1)} + C = \dfrac{x^{10}}{10} + C.$

(ii) $\int \sqrt[3]{x}\, dx = \int x^{1/3} dx = \dfrac{x^{\left(\frac{1}{3}+1\right)}}{\left(\frac{1}{3}+1\right)} + C = \dfrac{3}{4} x^{4/3} + C.$

(iii) $\int dx = \int x^0 dx = \dfrac{x^{(0+1)}}{(0+1)} + C = x + C.$

(iv) $\int \dfrac{1}{x^2}\, dx = \int x^{-2} dx = \dfrac{x^{(-2+1)}}{(-2+1)} + C = -\dfrac{1}{x} + C.$

(v) $\int \dfrac{1}{x^{1/3}} dx = \int x^{-1/3} dx = \dfrac{x^{\left(-\frac{1}{3}+1\right)}}{\left(-\frac{1}{3}+1\right)} + C = \dfrac{3}{2} x^{2/3} + C.$

(vi) $\int 5^x\, dx = \dfrac{5^x}{\log 5} + C.$

Some Standard Results on Integration

THEOREM 1 $\frac{d}{dx}\left\{\int f(x)\,dx\right\} = f(x).$

PROOF Let $\int f(x)\,dx = F(x).$ ... (i)

Then, $\frac{d}{dx}\{F(x)\} = f(x)$ [by def. of integral].

$\therefore \quad \frac{d}{dx}\left\{\int f(x)\,dx\right\} = f(x)$ [using (i)].

THEOREM 2 $\int k \cdot f(x)\; dx = k \cdot \int f(x)\; dx,$ *where k is a constant.*

PROOF Let $\int f(x)\; dx = F(x).$ ... (i)

Then, $\frac{d}{dx}\{F(x)\} = f(x).$ ... (ii)

$\therefore \quad \frac{d}{dx}\{k \cdot F(x)\} = k \cdot \frac{d}{dx}\{F(x)\} = k \cdot f(x)$ [using (ii)].

So, by the definition of an integral, we have

$\int \{k \cdot f(x)\}\; dx = k \cdot F(x) = k \cdot \int f(x)\; dx$ [using (i)].

EXAMPLE 2 *Evaluate:*

(i) $\int 3x^2 dx$ *(ii)* $\int 2^{(x+3)} dx$

SOLUTION (i) $\int 3x^2 dx = 3\int x^2 dx = 3 \cdot \frac{x^3}{3} + C = x^3 + C.$

(ii) $\int 2^{(x+3)} dx = \int 2^x \cdot 2^3 dx = 8\int 2^x\,dx = 8 \cdot \frac{2^x}{\log 2} + C = \frac{2^{(x+3)}}{\log 2} + C.$

THEOREM 3 *(i)* $\int \{f_1(x) + f_2(x)\}\; dx = \int f_1(x)\; dx + \int f_2(x)\; dx$

(ii) $\int \{f_1(x) - f_2(x)\}\; dx = \int f_1(x)\; dx - \int f_2(x)\; dx$

PROOF (i) Let $\int f_1(x)\; dx = F_1(x)$ and $\int f_2(x)\; dx = F_2(x).$ ... (i)

Then, $\frac{d}{dx}\{F_1(x)\} = f_1(x)$ and $\frac{d}{dx}\{F_2(x)\} = f_2(x).$ ... (ii)

Now, $\frac{d}{dx}\{F_1(x) + F_2(x)\} = \frac{d}{dx}\{F_1(x)\} + \frac{d}{dx}\{F_2(x)\}$

$= f_1(x) + f_2(x)$ [using (ii)].

$\therefore \quad \int \{f_1(x) + f_2(x)\}\; dx = F_1(x) + F_2(x)$

$= \int f_1(x)\; dx + \int f_2(x)\; dx$ [using (i)].

Similarly, (ii) can be proved.

REMARK In general, we have

$$\int \{k_1 \cdot f_1(x) \pm k_2 \cdot f_2(x) \pm \ldots \pm k_n \cdot f_n(x)\}\; dx$$

$$= k_1 \cdot \int f_1(x)\,dx \pm k_2 \cdot \int f_2(x)\,dx \pm \ldots \pm k_n \cdot \int f_n(x)\,dx.$$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\int\left(5x^3+2x^{-5}-7x+\frac{1}{\sqrt{x}}+\frac{5}{x}\right)dx$

(ii) $\int(3\sin x-4\cos x+5\sec^2x-2\operatorname{cosec}^2x)\,dx$

(iii) $\int(1-x)(2+3x)(5-4x)\,dx$

(iv) $\int\left(\frac{3x^4-5x^3+4x^2-x+2}{x^3}\right)dx$ (v) $\int\left(x^2+\frac{1}{x^2}\right)^3dx$

SOLUTION (i) $\int\left(5x^3+2x^{-5}-7x+\frac{1}{\sqrt{x}}+\frac{5}{x}\right)dx$

$$=5\int x^3dx+2\int x^{-5}dx-7\int x\,dx+\int x^{-1/2}dx+5\int\frac{1}{x}\,dx$$

$$=5\cdot\frac{x^4}{4}+2\cdot\frac{x^{-4}}{(-4)}-7\cdot\frac{x^2}{2}+\frac{x^{1/2}}{(1/2)}+5\log|x|+C$$

$$=\frac{5x^4}{4}-\frac{1}{2x^4}-\frac{7x^2}{2}+2\sqrt{x}+5\log|x|+C.$$

(ii) $\int(3\sin x-4\cos x+5\sec^2x-2\operatorname{cosec}^2x)\,dx$

$$=3\int\sin x\,dx-4\int\cos x\,dx+5\int\sec^2x\,dx-2\int\operatorname{cosec}^2x\,dx$$

$$=3(-\cos x)-4\sin x+5\tan x-2(-\cot x)+C$$

$$=(-3\cos x-4\sin x+5\tan x+2\cot x+C).$$

(iii) $\int(1-x)(2+3x)(5-4x)\,dx=\int(10-3x-19x^2+12x^3)\,dx$

$$=10\int dx-3\int x\,dx-19\int x^2dx+12\int x^3dx$$

$$=10x-3\cdot\frac{x^2}{2}-19\cdot\frac{x^3}{3}+12\cdot\frac{x^4}{4}+C$$

$$=10x-\frac{3x^2}{2}-\frac{19x^3}{3}+3x^4+C.$$

(iv) $\int\left(\frac{3x^4-5x^3+4x^2-x+2}{x^3}\right)dx=\int\left(3x-5+\frac{4}{x}-\frac{1}{x^2}+\frac{2}{x^3}\right)dx$

[dividing each term by x^3]

$$=3\int x\,dx-5\int dx+4\int\frac{1}{x}\,dx-\int x^{-2}dx+2\int x^{-3}dx$$

$$=3\cdot\frac{x^2}{2}-5x+4\log|x|-\left(-\frac{1}{x}\right)+2\left(\frac{x^{-2}}{-2}\right)+C$$

$$=\frac{3x^2}{2}-5x+4\log|x|+\frac{1}{x}-\frac{1}{x^2}+C.$$

(v) $\int\left(x^2+\frac{1}{x^2}\right)^3 dx = \int\left(x^6+\frac{1}{x^6}+3x^2+\frac{3}{x^2}\right)dx$

$$= \int x^6 dx + \int x^{-6} dx + 3\int x^2 dx + 3\int \frac{1}{x^2}\, dx$$

$$= \frac{x^7}{7} + \frac{x^{-5}}{(-5)} + 3\cdot\frac{x^3}{3} + 3\cdot\left(-\frac{1}{x}\right) + C$$

$$= \frac{x^7}{7} - \frac{1}{5x^5} + x^3 - \frac{3}{x} + C.$$

EXAMPLE 2 *Evaluate: (i)* $\int \frac{(x^3+4x^2-3x-2)}{(x+2)}\, dx$ *(ii)* $\int\left(\frac{x^4+1}{x^2+1}\right)dx$ **[CBSE 2011]**

SOLUTION (i) On dividing (x^3+4x^2-3x-2) by $(x+2)$, we get

$$\int \frac{(x^3+4x^2-3x-2)}{(x+2)}\, dx = \int\left\{x^2+2x-7+\frac{12}{x+2}\right\}dx$$

$$= \int x^2 dx + 2\int x\, dx - 7\int dx + 12\int \frac{1}{x+2}\, dx$$

$$= \frac{x^3}{3} + 2\cdot\frac{x^2}{2} - 7x + 12\log|x+2| + C$$

$$= \frac{x^3}{3} + x^2 - 7x + 12\log|x+2| + C.$$

(ii) On dividing (x^4+1) by (x^2+1), we get

$$\int\left(\frac{x^4+1}{x^2+1}\right)dx = \int\left[x^2-1+\frac{2}{(x^2+1)}\right]dx$$

$$= \int x^2 dx - \int dx + 2\int \frac{1}{x^2+1}\, dx$$

$$= \frac{x^3}{3} - x + 2\tan^{-1}x + C.$$

EXAMPLE 3 *Evaluate:*

(i) $\int \tan^2 x\, dx$ *(ii)* $\int \cot^2 x\, dx$ *(iii)* $\int \sin^2\frac{x}{2}\, dx$

SOLUTION (i) $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$

$$= \int \sec^2 x\, dx - \int dx = \tan x - x + C.$$

(ii) $\int \cot^2 x\, dx = \int (\operatorname{cosec}^2 x - 1)\, dx$

$$= \int \operatorname{cosec}^2 x\, dx - \int dx = -\cot x - x + C.$$

(iii) We know that $2\sin^2\frac{x}{2} = (1-\cos x)$.

$$\therefore \quad \int \sin^2 \frac{x}{2}\, dx = \frac{1}{2}\int (1 - \cos x)\, dx$$

$$= \frac{1}{2}\left[\int dx - \int \cos x\, dx\right] = \frac{1}{2}x - \frac{1}{2}\sin x + C.$$

EXAMPLE 4 *Evaluate* $\int \sqrt{1 - \sin 2x}\, dx.$ **[CBSE 2004]**

SOLUTION $\int \sqrt{1 - \sin 2x}\, dx = \int (\cos^2 x + \sin^2 x - 2\sin x \cos x)^{1/2}\, dx$

$$= \int \sqrt{(\cos x - \sin x)^2}\, dx$$

$$= \int (\cos x - \sin x)\, dx = \int \cos x\, dx - \int \sin x\, dx$$

$$= \sin x - (-\cos x) + C = \sin x + \cos x + C.$$

EXAMPLE 5 *Evaluate:*

(i) $\int \frac{dx}{1 + \sin x}$ **[CBSE 2002C]** *(ii)* $\int \left(\frac{\sin x}{1 + \sin x}\right) dx$

SOLUTION (i) $\int \frac{dx}{(1 + \sin x)} = \int \frac{1}{(1 + \sin x)} \times \frac{(1 - \sin x)}{(1 - \sin x)}\, dx$

$$= \int \frac{(1 - \sin x)}{(1 - \sin^2 x)}\, dx = \int \frac{(1 - \sin x)}{\cos^2 x}\, dx$$

$$= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx = \int (\sec^2 x - \sec x \tan x)\, dx$$

$$= \int \sec^2 x\, dx - \int \sec x \tan x\, dx = \tan x - \sec x + C.$$

(ii) $\int \left(\frac{\sin x}{1 + \sin x}\right) dx = \int \frac{(1 + \sin x) - 1}{(1 + \sin x)}\, dx$

$$= \int \left(1 - \frac{1}{1 + \sin x}\right) dx = \int dx - \int \frac{1}{(1 + \sin x)}\, dx$$

$$= \int dx - \int \frac{1}{(1 + \sin x)} \times \frac{(1 - \sin x)}{(1 - \sin x)}\, dx$$

$$= \int dx - \int \frac{(1 - \sin x)}{\cos^2 x}\, dx = \int dx - \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx$$

$$= \int dx - \int \sec^2 x\, dx + \int \sec x \tan x\, dx = x - \tan x + \sec x + C.$$

EXAMPLE 6 *Evaluate* $\int \frac{\sec x}{(\sec x + \tan x)}\, dx.$

SOLUTION $\int \frac{\sec x}{(\sec x + \tan x)}\, dx = \int \frac{\sec x}{(\sec x + \tan x)} \times \frac{(\sec x - \tan x)}{(\sec x - \tan x)}\, dx$

$$= \int \frac{(\sec^2 x - \sec x \tan x)}{(\sec^2 x - \tan^2 x)}\, dx$$

$$= \int (\sec^2 x - \sec x \tan x)\, dx$$

$$= \int \sec^2 x\, dx - \int \sec x \tan x\, dx = \tan x - \sec x + C.$$

EXAMPLE 7 *Evaluate:*

(i) $\int \left(\frac{4 - 5\cos x}{\sin^2 x} \right) dx$ (ii) $\int \left(\frac{1 - \cos 2x}{1 + \cos 2x} \right) dx$

(iii) $\int \frac{1}{\sin^2 x \cos^2 x}\, dx$ **[CBSE 2014]** (iv) $\int \frac{\cos 2x}{\cos^2 x \sin^2 x}\, dx$

SOLUTION (i) $\int \left(\frac{4 - 5\cos x}{\sin^2 x} \right) dx = \int \left(\frac{4}{\sin^2 x} - \frac{5\cos x}{\sin^2 x} \right) dx$

$$= \int (4\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x)\, dx$$

$$= 4\int \operatorname{cosec}^2 x\, dx - 5\int \operatorname{cosec} x \cot x\, dx$$

$$= 4(-\cot x) - 5(-\operatorname{cosec} x) + C$$

$$= -4\cot x + 5\operatorname{cosec} x + C.$$

(ii) $\int \left(\frac{1 - \cos 2x}{1 + \cos 2x} \right) dx = \int \frac{2\sin^2 x}{2\cos^2 x}\, dx = \int \tan^2 x\, dx$

$$= \int (\sec^2 x - 1)\, dx = \int \sec^2 x\, dx - \int dx$$

$$= \tan x - x + C.$$

(iii) $\int \frac{1}{\sin^2 x \cos^2 x}\, dx = \int \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \right) dx$

$$= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) dx$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx = \tan x - \cot x + C.$$

(iv) $\int \frac{\cos 2x}{\cos^2 x \sin^2 x}\, dx = \int \left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x \sin^2 x} \right) dx$

$$= \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx$$

$$= \int \operatorname{cosec}^2 x\, dx - \int \sec^2 x\, dx = -\cot x - \tan x + C.$$

EXAMPLE 8 *Evaluate* $\int \left(\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} \right) dx.$ **[CBSE 2013]**

SOLUTION $\int \left(\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} \right) dx = \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{(\cos x - \cos \alpha)}\, dx$

$$= 2\int \frac{(\cos^2 x - \cos^2 \alpha)}{(\cos x - \cos \alpha)}\, dx = 2\int (\cos x + \cos \alpha)\, dx$$

$$= 2\int \cos x\, dx + 2\cos \alpha \cdot \int dx = 2\sin x + 2x\cos \alpha + C.$$

EXAMPLE 9 *Evaluate* $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\} dx.$ **[CBSE 2003]**

SOLUTION $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\} dx = \int \tan^{-1}\left\{\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\right\} dx$

$$= \int \tan^{-1}(\tan x)\, dx = \int x\, dx = \frac{x^2}{2} + C.$$

EXAMPLE 10 *Evaluate* $\int \sin^{-1}(\cos x)\, dx.$

SOLUTION $\int \sin^{-1}(\cos x)\, dx = \int \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - x\right)\right\} dx$

$$= \int \left(\frac{\pi}{2} - x\right) dx = \frac{\pi}{2} \cdot \int dx - \int x\, dx = \frac{\pi x}{2} - \frac{x^2}{2} + C.$$

EXAMPLE 11 *Evaluate* $\int \tan^{-1}(\sec x + \tan x)\, dx.$

SOLUTION $\int \tan^{-1}(\sec x + \tan x)\, dx = \int \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) dx$

$$= \int \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right) dx = \int \tan^{-1}\left\{\frac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)}\right\} dx$$

$$= \int \tan^{-1}\left\{\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\} dx$$

$$= \int \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} dx = \int \left(\frac{\pi}{4}+\frac{x}{2}\right) dx$$

$$= \frac{\pi}{4} \cdot \int dx + \frac{1}{2}\int x\, dx = \frac{\pi x}{4} + \frac{1}{4}x^2 + C.$$

EXAMPLE 12 *Evaluate* $\int \left(\frac{1+\sin x}{1-\sin x}\right) dx.$ **[CBSE 2008C]**

SOLUTION $I = \int \frac{(1+\sin x)}{(1-\sin x)} \times \frac{(1+\sin x)}{(1+\sin x)}\, dx$

$$= \int \frac{(1+\sin x)^2}{(1-\sin^2 x)}\,dx = \int \frac{(1+\sin^2 x + 2\sin x)}{\cos^2 x}\,dx$$

$$= \int \left(\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} \right) dx = \int (\sec^2 x + \tan^2 x + 2\sec x \tan x)\,dx$$

$$= \int (2\sec^2 x - 1 + 2\sec x \tan x)\,dx$$

$$= 2\int \sec^2 x\,dx - \int dx + 2\int \sec x \tan x\,dx$$

$$= 2\tan x - x + 2\sec x + C.$$

EXAMPLE 13 *Evaluate* $\int \frac{(\sin^6 x + \cos^6 x)}{\sin^2 x \cos^2 x}\,dx.$ **[CBSE 2014]**

SOLUTION Using the formula, $(a^3 + b^3) = (a+b)^3 - 3ab(a+b)$, we get

$$I = \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x}\,dx$$

$$= \int \frac{(1 - 3\sin^2 x \cos^2 x)}{\sin^2 x \cos^2 x}\,dx = \int \left\{ \frac{1}{\sin^2 x \cos^2 x} - 3 \right\} dx$$

$$= \int \left\{ \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} - 3 \right\} dx = \int \left\{ \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} - 3 \right\} dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x - 3)\,dx = \tan x - \cot x - 3x + C.$$

EXERCISE 12

Very-Short-Answer Questions

Evaluate:

1. (i) $\int x^7 dx$ (ii) $\int x^{-7} dx$ (iii) $\int x^{-1} dx$

   (iv) $\int x^{5/3} dx$ (v) $\int x^{-5/4} dx$ (vi) $\int 2^x\,dx$

   (vii) $\int \sqrt[3]{x^2}\,dx$ (viii) $\int \frac{1}{\sqrt[4]{x^3}}\,dx$ (ix) $\int \frac{2}{x^2}\,dx$

2. (i) $\int \left(6x^5 - \frac{2}{x^4} - 7x + \frac{3}{x} - 5 + 4e^x + 7^x \right) dx$

   (ii) $\int \left(8 - x + 2x^3 - \frac{6}{x^3} + 2x^{-5} + 5x^{-1} \right) dx$ (iii) $\int \left(\frac{x}{a} + \frac{a}{x} + x^a + a^x + ax \right) dx$

3. (i) $\int (2-5x)(3+2x)(1-x)\,dx$ (ii) $\int \sqrt{x}(ax^2 + bx + c)\,dx$

   (iii) $\int \left(\sqrt{x} - \sqrt[3]{x^4} + \frac{7}{\sqrt[3]{x^2}} - 6e^x + 1 \right) dx$

4. (i) $\int \left(x^2 - \frac{1}{x^2}\right)^3 dx$ (ii) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$

(iii) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$ (iv) $\int \frac{(1+2x)^3}{x^4} dx$

(v) $\int \frac{(1+x)^3}{\sqrt{x}} dx$ (vi) $\int \frac{2x^2 + x - 2}{(x-2)} dx$

5. $\int \left[1 + \frac{1}{(1+x^2)} - \frac{2}{\sqrt{1-x^2}} + \frac{5}{x\sqrt{x^2-1}} + a^x\right] dx$

6. (i) $\int \left(\frac{x^2-1}{x^2+1}\right) dx$ (ii) $\int \left(\frac{x^6-1}{x^2+1}\right) dx$

(iii) $\int \left(\frac{x^4}{1+x^2}\right) dx$ (iv) $\int \left(\frac{x^2}{1+x^2}\right) dx$

7. $\int \left(9 \sin x - 7 \cos x - \frac{6}{\cos^2 x} + \frac{2}{\sin^2 x} + \cot^2 x\right) dx$

8. $\int \left(\frac{\cot x}{\sin x} - \tan^2 x - \frac{\tan x}{\cos x} + \frac{2}{\cos^2 x}\right) dx$

9. (i) $\int \sec x(\sec x + \tan x)\, dx$ **[CBSE 2011]**

(ii) $\int \operatorname{cosec} x(\operatorname{cosec} x - \cot x)\, dx$

10. (i) $\int (\tan x + \cot x)^2 dx$ (ii) $\int \left(\frac{1 + 2\sin x}{\cos^2 x}\right) dx$ (iii) $\int \left(\frac{3\cos x + 4}{\sin^2 x}\right) dx$

11. (i) $\int \frac{1}{(1-\cos x)} dx$ (ii) $\int \frac{1}{(1-\sin x)} dx$ **[CBSE 2002C]**

12. (i) $\int \frac{\tan x}{(\sec x + \tan x)} dx$ (ii) $\int \frac{\operatorname{cosec} x}{(\operatorname{cosec} x - \cot x)} dx$

13. (i) $\int \frac{\cos x}{1 + \cos x} dx$ (ii) $\int \frac{\sin x}{(1 - \sin x)} dx$

14. (i) $\int \sqrt{1 + \cos 2x}\, dx$ (ii) $\int \sqrt{1 - \cos 2x}\, dx$

15. (i) $\int \frac{1}{(1 + \cos 2x)} dx$ (ii) $\int \frac{1}{(1 - \cos 2x)} dx$

16. $\int \sqrt{1 + \sin 2x}\, dx$ 17. $\int \left(\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\right) dx$

18. $\int \tan^{-1}\left(\frac{\sin 2x}{1 + \cos 2x}\right) dx$ 19. $\int \cos^{-1}\left(\frac{1 - \tan^2 x}{1 + \tan^2 x}\right) dx$

20. $\int \cos^{-1}(\sin x)\,dx$

21. $\int \tan^{-1}\sqrt{\dfrac{1-\sin x}{1+\sin x}}\,dx$ **[CBSE 2003, '06C]**

22. $\int (3\cot x - 2\tan x)^2\,dx$

23. $\int (3\sin x + 4\operatorname{cosec} x)^2\,dx$

24. $\int \dfrac{dx}{(\sqrt{x+1}+\sqrt{x+2})}$ **[CBSE 2002]**

25. $\int \dfrac{dx}{(\sqrt{x+3}-\sqrt{x+2})}$ **[CBSE 2002]**

26. $\int \left(\dfrac{1+\cos x}{1-\cos x}\right)dx$

27. $\int \dfrac{(1+\tan x)}{(1-\tan x)}\,dx$

28. $\int \dfrac{\cos(x+a)}{\sin(x+b)}\,dx$ **[CBSE 2006C]**

29. $\int \dfrac{\sin(x-\alpha)}{\sin(x+\alpha)}\,dx$ **[CBSE 2013]**

30. $\int (1-x)\sqrt{x}\,dx$ **[CBSE 2012]**

31. $\int \dfrac{\sec^2 x}{\operatorname{cosec}^2 x}\,dx$ **[CBSE 2012C]**

32. $\int \left\{\dfrac{2-3\sin x}{\cos^2 x}\right\}dx$ **[CBSE 2011]**

ANSWERS (EXERCISE 12)

1. (i) $\dfrac{x^8}{8}+C$ (ii) $-\dfrac{1}{6x^6}+C$ (iii) $\log|x|+C$

(iv) $\dfrac{3}{8}x^{8/3}+C$ (v) $-4x^{-1/4}+C$ (vi) $\dfrac{2^x}{\log 2}+C$

(vii) $\dfrac{3}{5}x^{5/3}+C$ (viii) $4x^{1/4}+C$ (ix) $\dfrac{-2}{x}+C$

2. (i) $x^6+\dfrac{2}{3x^3}-\dfrac{7x^2}{2}+3\log|x|-5x+4e^x+\dfrac{7^x}{\log 7}+C$

(ii) $8x-\dfrac{x^2}{2}+\dfrac{x^4}{2}+\dfrac{3}{x^2}-\dfrac{1}{2x^4}+5\log|x|+C$

(iii) $\dfrac{x^2}{2a}+a\log|x|+\dfrac{x^{(a+1)}}{(a+1)}+\dfrac{a^x}{\log a}+\dfrac{ax^2}{2}+C$

3. (i) $6x-\dfrac{17x^2}{2}+\dfrac{x^3}{3}+\dfrac{5x^4}{2}+C$ (ii) $\dfrac{2ax^{7/2}}{7}+\dfrac{2bx^{5/2}}{5}+\dfrac{2cx^{3/2}}{3}+C$

(iii) $\dfrac{2}{3}x^{3/2}-\dfrac{3}{7}x^{7/3}+21x^{1/3}-6e^x+x+C$

4. (i) $\dfrac{x^7}{7}+\dfrac{1}{5x^5}-x^3-\dfrac{3}{x}+C$ (ii) $\dfrac{2}{3}x^{3/2}-2x^{1/2}+C$

(iii) $\dfrac{x^2}{2}+\log|x|+2x+C$ (iv) $-\dfrac{1}{3x^3}+8\log|x|-\dfrac{3}{x^2}-\dfrac{12}{x}+C$

(v) $2\sqrt{x}+\dfrac{2}{7}x^{7/2}+2x^{3/2}+\dfrac{6}{5}x^{5/2}+C$ (vi) $x^2+5x+8\log|x-2|+C$

5. $x+\tan^{-1}x-2\sin^{-1}x+5\sec^{-1}x+\dfrac{a^x}{\log a}+C$

6. (i) $x - 2\tan^{-1}x + C$ (ii) $\frac{x^5}{5} - \frac{x^3}{3} + x - 2\tan^{-1}x + C$

(iii) $\frac{x^3}{3} - x + \tan^{-1}x + C$ (iv) $x - \tan^{-1}x + C$

7. $-9\cos x - 7\sin x - 6\tan x - 3\cot x - x + C$

8. $-\operatorname{cosec} x + \tan x + x - \sec x + C$

9. (i) $\tan x + \sec x + C$ (ii) $-\cot x + \operatorname{cosec} x + C$

10. (i) $\tan x - \cot x + C$ (ii) $\tan x + 2\sec x + C$

(iii) $-3\operatorname{cosec} x - 4\cot x + C$

11. (i) $-\cot x - \operatorname{cosec} x + C$ (ii) $\tan x + \sec x + C$

12. (i) $\sec x - \tan x + x + C$ (ii) $-\cot x - \operatorname{cosec} x + C$

13. (i) $-\operatorname{cosec} x + \cot x + x + C$ (ii) $\sec x + \tan x - x + C$

14. (i) $\sqrt{2}\sin x + C$ (ii) $-\sqrt{2}\cos x + C$

15. (i) $\frac{1}{2}\tan x + C$ (ii) $-\frac{1}{2}\cot x + C$

16. $\sin x - \cos x + C$ **17.** $\sec x - \operatorname{cosec} x + C$

18. $\frac{x^2}{2} + C$ **19.** $x^2 + C$

20. $\left(\frac{\pi x}{2} - \frac{x^2}{2} + C\right)$ **21.** $\frac{\pi x}{4} - \frac{x^2}{4} + C$

22. $4\tan x - 9\cos x - 25x + C$ **23.** $\frac{57}{2}x - \frac{9}{4}\sin 2x - 16\cot x + C$

24. $\frac{2}{3}(x+2)^{3/2} - \frac{2}{3}(x+1)^{3/2} + C$ **25.** $\frac{2}{3}(x+3)^{3/2} + \frac{2}{3}(x+2)^{3/2} + C$

26. $-2\cot\frac{x}{2} - x + C$ **27.** $-\log|\cos x - \sin x| + C$

28. $\cos(a-b)\log|\sin(x+b)| - x\sin(a-b) + C$

29. $x\cos 2\alpha - \sin 2\alpha \cdot \log|\sin(x+\alpha)| + C$ **30.** $\frac{2}{15}x\sqrt{x}\,(5-3x) + C$

31. $\tan x - x + C$ **32.** $2\tan x - 3\sec x + C$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 12)

21. $I = \int \tan^{-1}\sqrt{\frac{1-\cos\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}}\,dx$

$$= \int \tan^{-1}\sqrt{\frac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}}\,dx = \int \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}dx$$

$$= \int \left(\frac{\pi}{4} - \frac{x}{2} \right) dx = \frac{\pi x}{4} - \frac{x^2}{4} + C.$$

22. $I = \int (9 \cot^2 x + 4 \tan^2 x - 12)\, dx$

$$= \int \{9(\operatorname{cosec}^2 x - 1) + 4(\sec^2 x - 1) - 12\}\, dx.$$

23. $I = \int (9 \sin^2 x + 16 \operatorname{cosec}^2 x + 24)\, dx$

$$= \int \left\{ 9\left(\frac{1 - \cos 2x}{2} \right) + 16 \operatorname{cosec}^2 x + 24 \right\} dx$$

$$= \int \left(\frac{57}{2} - \frac{9}{2} \cos 2x + 16 \operatorname{cosec}^2 x \right) dx.$$

24. $I = \int \dfrac{1}{(\sqrt{x+2} + \sqrt{x+1})} \times \dfrac{(\sqrt{x+2} - \sqrt{x+1})}{(\sqrt{x+2} - \sqrt{x+1})}\, dx$

$$= \int \sqrt{x+2}\, dx - \int \sqrt{x+1}\, dx = \frac{2}{3}(x+2)^{3/2} - \frac{2}{3}(x+1)^{3/2} + C.$$

26. $I = \int \dfrac{(1 + \cos x)}{(1 - \cos x)}\, dx = \int \dfrac{2\cos^2(x/2)}{2\sin^2(x/2)}\, dx$

$$= \int \cot^2(x/2)\, dx = \int \left(\operatorname{cosec}^2 \frac{x}{2} - 1 \right) dx$$

$$= \frac{-\cot(x/2)}{(1/2)} - x + C = -2 \cot \frac{x}{2} - x + C.$$

27. $I = \int \dfrac{\left(1 + \dfrac{\sin x}{\cos x}\right)}{\left(1 - \dfrac{\sin x}{\cos x}\right)}\, dx = \int \dfrac{(\cos x + \sin x)}{(\cos x - \sin x)}\, dx$

$$= -\int \frac{dt}{t}, \text{ where } (\cos x - \sin x) = t$$

$$= -\log|t| + C = -\log|\cos x - \sin x| + C.$$

28. $I = \int \dfrac{\cos(x + b + a - b)}{\sin(x + b)}\, dx$

$$= \int \frac{\cos(x+b)\cos(a-b) - \sin(x+b)\sin(a-b)}{\sin(x+b)}\, dx$$

$$= \cos(a-b) \int \cot(x+b)\, dx - \sin(a-b) \int dx.$$

29. $I = \int \dfrac{\sin(x + \alpha - 2\alpha)}{\sin(x + \alpha)}\, dx$

$$= \int \frac{\sin(x+\alpha)\cos 2\alpha - \cos(x+\alpha)\sin 2\alpha}{\sin(x+\alpha)}\, dx = \cos 2\alpha \int dx - \sin 2\alpha \cdot \int \frac{\cos(x+\alpha)}{\sin(x+\alpha)}\, dx$$

$$= x \cos 2\alpha - \sin 2\alpha \cdot \log|\sin(x+\alpha)| + C.$$

32. $I = \int \left\{ \dfrac{2}{\cos^2 x} - \dfrac{3 \sin x}{\cos^2 x} \right\} dx = \int (2 \sec^2 x - 3 \sec x \tan x)\, dx.$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. $\int x^6 dx = ?$

(a) $7x^7 + C$ (b) $\frac{x^7}{7} + C$ (c) $6x^5 + C$ (d) $6x^7 + C$

2. $\int x^{5/3} dx = ?$

(a) $\frac{3}{5}x^{2/3} + C$ (b) $\frac{8}{3}x^{8/3} + C$ (c) $\frac{3}{8}x^{8/3} + C$ (d) $\frac{5}{3}x^{8/3} + C$

3. $\int \frac{1}{x^3} dx = ?$

(a) $\frac{-3}{x^2} + C$ (b) $\frac{-1}{2x^2} + C$ (c) $\frac{-1}{3x^2} + C$ (d) $\frac{x^{-2}}{2} + C$

4. $\int \sqrt[3]{x}\, dx = ?$

(a) $\frac{3}{4}x^{3/4} + C$ (b) $\frac{4}{3}x^{3/4} + C$ (c) $\frac{3}{4}x^{4/3} + C$ (d) $\frac{4}{3}x^{4/3} + C$

5. $\int \frac{1}{\sqrt[3]{x}} dx = ?$

(a) $\frac{3}{2}x^{2/3} + C$ (b) $\frac{3}{2x^{2/3}} + C$ (c) $\frac{2}{3x^{2/3}} + C$ (d) $\frac{2}{3}x^{3/2} + C$

6. $\int \sqrt[3]{x^2}\, dx = ?$

(a) $\frac{5}{3}x^{5/3} + C$ (b) $\frac{3}{5}x^{5/3} + C$ (c) $\frac{5}{3}x^{3/5} + C$ (d) $\frac{3}{5}x^{3/5} + C$

7. $\int 3^x\, dx = ?$

(a) $3^x(\log 3) + C$ (b) $3^x + C$ (c) $\frac{3^x}{\log 3} + C$ (d) $\frac{\log 3}{3^x} + C$

8. $\int 2^{\log x}\, dx = ?$

(a) $\frac{2^{\log x + 1}}{(\log x + 1)} + C$ (b) $\frac{x^{(\log 2 + 1)}}{(\log 2 + 1)} + C$

(c) $\frac{2^{\log x}}{\log 2} + C$ (d) $\frac{2^{\log x}}{2} + C$

9. $\int \operatorname{cosec} x(\operatorname{cosec} x + \cot x)\, dx = ?$

(a) $\cot x - \operatorname{cosec} x + C$ (b) $-\cot x + \operatorname{cosec} x + C$

(c) $\cot x + \operatorname{cosec} x + C$ (d) $-\cot x - \operatorname{cosec} x + C$

10. $\int \frac{\sec x}{(\sec x + \tan x)} dx = ?$

(a) $\tan x + \sec x + C$ (b) $\tan x - \sec x + C$

(c) $-\tan x + \sec x + C$ (d) $-\tan x - \sec x + C$

11. $\int \frac{(1-\cos 2x)}{(1+\cos 2x)} dx = ?$

(a) $\tan x + x + C$ (b) $\tan x - x + C$

(c) $-\tan x + x + C$ (d) $-\tan x - x + C$

12. $\int \frac{1}{\sin^2 x \cos^2 x} dx = ?$

(a) $\tan x + \cot x + C$ (b) $-\tan x + \cot x + C$

(c) $\tan x - \cot x + C$ (d) none of these

13. $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx = ?$

(a) $-\cot x - \tan x + C$ (b) $-\cot x + \tan x + C$

(c) $\cot x - \tan x + C$ (d) $\cot x + \tan x + C$

14. $\int \frac{(\cos 2x - \cos 2\alpha)}{(\cos x - \cos \alpha)} dx = ?$

(a) $2\sin x + 2x\cos\alpha + C$ (b) $2\sin x - 2x\cos\alpha + C$

(c) $-2\sin x + 2x\cos\alpha + C$ (d) $-2\sin x - 2x\cos\alpha + C$

15. $\int \sqrt{1+\cos 2x}\, dx = ?$

(a) $\sqrt{2}\cos x + C$ (b) $\sqrt{2}\sin x + C$ (c) $-\sqrt{2}\cos x + C$ (d) $-\sqrt{2}\sin x + C$

16. $\int \sqrt{1+\sin 2x}\, dx = ?$

(a) $\sin x + \cos x + C$ (b) $-\sin x + \cos x + C$

(c) $\sin x - \cos x + C$ (d) $-\sin x - \cos x + C$

17. $\int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx = ?$

(a) $\cot x + \tan x + C$ (b) $-\cot x + \tan x + C$

(c) $\cot x - \tan x + C$ (d) $-\cot x - \tan x + C$

18. $\int \frac{dx}{(1-\cos 2x)} = ?$

(a) $\frac{1}{2}\cot x + C$ (b) $2\cot x + C$

(c) $-\frac{1}{2}\cot x + C$ (d) $-2\cot x + C$

19. $\int \frac{\sin 2x}{\sin x} dx = ?$

(a) $2\sin x + C$ (b) $\frac{1}{2}\sin x + C$ (c) $2\cos x + C$ (d) $\frac{1}{2}\cos x + C$

20. $\int \frac{(1-\sin x)}{\cos^2 x}\,dx = ?$

(a) $\tan x + \sec x + C$
(b) $\tan x - \sec x + C$
(c) $-\tan x + \sec x + C$
(d) $-\tan x - \sec x + C$

21. $\int \cot^2 x\,dx = ?$

(a) $-\cot x - x + C$
(b) $\cot x - x + C$
(c) $-\cot x + x + C$
(d) $\cot x + x + C$

22. $\int \sec x(\sec x + \tan x)\,dx = ?$

(a) $\tan x - \sec x + C$
(b) $-\tan x + \sec x + C$
(c) $\tan x + \sec x + C$
(d) $-\tan x - \sec x + C$

23. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\,dx = ?$

(a) $\tan x + x + C$
(b) $\tan x - x + C$
(c) $-\tan x + x + C$
(d) $-\tan x + x + C$

24. $\int \frac{\sin^2 x}{(1+\cos x)}\,dx = ?$

(a) $x + \sin x + C$
(b) $x - \sin x + C$
(c) $\sin x - x + C$
(d) $-\sin x - x + C$

25. $\int \frac{\cot x}{(\operatorname{cosec} x - \cot x)}\,dx = ?$

(a) $-\operatorname{cosec} x - \cot x - x + C$
(b) $\operatorname{cosec} x - \cot x - x + C$
(c) $-\operatorname{cosec} x + \cot x - x + C$
(d) $\operatorname{cosec} x + \cot x - x + C$

26. $\int \frac{\sin x}{(1+\sin x)}\,dx = ?$

(a) $\sec x + \tan x + x + C$
(b) $\sec x - \tan x + x + C$
(c) $-\sec x + \tan x + x + C$
(d) none of these

27. $\int \frac{(1+\sin x)}{(1-\sin x)}\,dx = ?$

(a) $2\tan x + 2\sec x + x + C$
(b) $2\tan x + 2\sec x - x + C$
(c) $\tan x + \sec x - x + C$
(d) none of these

28. $\int \frac{1}{(1+\cos x)}\,dx = ?$

(a) $-\cot x + \operatorname{cosec} x + C$
(b) $\cot x - \operatorname{cosec} x + C$
(c) $\cot x + \operatorname{cosec} x + C$
(d) none of these

29. $\int \sin^{-1}(\cos x)\,dx = ?$

(a) $\operatorname{cosec} x + C$
(b) $\frac{\pi x}{2} + \frac{x^2}{2} + C$
(c) $\frac{\pi x}{2} - \frac{x^2}{2} + C$
(d) $\frac{x^2}{2} - \frac{\pi x}{2} + C$

30. $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\} dx = ?$

(a) $\frac{1}{(1+x^2)}+C$ (b) $\frac{1}{\sqrt{1+x^2}}+C$ (c) $\frac{1}{\sqrt{1-x^2}}+C$ (d) $\frac{x^2}{2}+C$

31. $\int \cot^{-1}\left(\frac{\sin 2x}{1-\cos 2x}\right) dx = ?$

(a) $\frac{-1}{(1+x^2)}+C$ (b) $\frac{-1}{(1-x^2)}+C$ (c) $\frac{x^2}{2}+C$ (d) $2x^2+C$

32. $\int \sin^{-1}\left(\frac{2\tan x}{1+\tan^2 x}\right) dx = ?$

(a) $-x^2+C$ (b) x^2+C (c) $\frac{x^2}{2}+C$ (d) $2x^2+C$

33. $\int \cos^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right) dx = ?$

(a) x^2+C (b) $-x^2+C$ (c) $\frac{1}{\sqrt{1+x^2}}+C$ (d) $\frac{1}{\sqrt{1-x^2}}+C$

34. $\int \tan^{-1}(\operatorname{cosec} x - \cot x) dx = ?$

(a) $\frac{x^2}{4}+C$ (b) $\frac{-x^2}{4}+C$ (c) $\frac{x^2}{2}+C$ (d) $\frac{-x^2}{2}+C$

35. $\int \left(\frac{(x^4+1)}{(x^2+1)}\right) dx = ?$

(a) $\frac{x^3}{3}+x-\tan^{-1}x+C$ (b) $\frac{x^3}{3}-x-2\tan^{-1}x+C$

(c) $\frac{x^3}{3}+x-2\tan^{-1}x+C$ (d) none of these

36. $\int \frac{(ax+b)}{(cx+d)} dx = ?$

(a) $\frac{ax}{c}+\log|cx+d|+C$ (b) $\frac{a}{c}+\log|cx+d|+C$

(c) $\frac{ax}{c}+\frac{(bc-ad)}{c^2}\log|cx+d|+C$ (d) none of these

37. $\int \frac{(\sin^3 x+\cos^3 x)}{\sin^2 x\cos^2 x} dx = ?$

(a) $\sin x-\cos x+C$ (b) $\tan x-\cos x+C$

(c) $\sec x-\operatorname{cosec} x+C$ (d) none of these

38. $\int \frac{\sin x}{\sin(x-\alpha)}\,dx = ?$

(a) $x\cos\alpha + (\sin\alpha)\log|\sin(x-\alpha)| + C$

(b) $x\sin\alpha + (\sin\alpha)\log|\sin(x-\alpha)| + C$

(c) $x\cos\alpha - (\sin\alpha)\log|\sin(x-\alpha)| + C$

(d) $x\sin\alpha - (\sin\alpha)\log|\sin(x-\alpha)| + C$

39. $\int \sin 3x \sin 2x\,dx = ?$

(a) $-\frac{1}{5}\cos 5x + C$ (b) $\frac{1}{2}\sin x + \frac{1}{10}\sin 5x - C$

(c) $\frac{1}{2}\sin x - \frac{1}{10}\sin 5x - C$ (d) $-\frac{1}{3}\cos 3x - \frac{1}{2}\sin 2x + C$

40. $\int \cos 3x \sin 2x\,dx = ?$

(a) $\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$ (b) $-\frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C$

(c) $-\frac{1}{2}\cos x + \frac{1}{10}\cos 5x + C$ (d) none of these

41. $\int \cos 4x \cos x\,dx = ?$

(a) $\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x + C$ (b) $\frac{1}{5}\cos 5x - \frac{1}{3}\cos 3x + C$

(c) $\frac{1}{10}\sin 5x + \frac{1}{6}\sin 3x + C$ (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (b)	**2.** (c)	**3.** (b)	**4.** (c)	**5.** (a)	**6.** (b)	**7.** (c)	**8.** (b)	**9.** (d)	**10.** (b)
11. (b)	**12.** (c)	**13.** (a)	**14.** (a)	**15.** (b)	**16.** (c)	**17.** (d)	**18.** (c)	**19.** (a)	**20.** (b)
21. (b)	**22.** (c)	**23.** (b)	**24.** (b)	**25.** (a)	**26.** (b)	**27.** (b)	**28.** (a)	**29.** (c)	**30.** (d)
31. (c)	**32.** (b)	**33.** (a)	**34.** (a)	**35.** (b)	**36.** (c)	**37.** (c)	**38.** (a)	**39.** (c)	**40.** (a)
41. (c)									

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

8. $2^{\log x} = x^{\log 2}$.

10. $I = \int \left\{ \frac{\sec x}{(\sec x + \tan x)} \times \frac{(\sec x - \tan x)}{(\sec x - \tan x)} \right\} dx.$

12. Write $1 = (\sin^2 x + \cos^2 x)$.

13. $\cos 2x = (\cos^2 x - \sin^2 x)$.

14. $\cos 2x = (2\cos^2 x - 1)$ and $\cos 2\alpha = (2\cos^2\alpha - 1)$.

15. $\sqrt{1+\cos 2x} = \sqrt{2\cos^2 x} = \sqrt{2}\cos x$.

16. $(1+\sin 2x) = \sin^2 x + \cos^2 x + 2\sin x\cos x = (\sin x + \cos x)^2$

$\therefore \quad \sqrt{1+\sin 2x} = (\sin x + \cos x)$.

17. $I = \int \frac{(\cos^2 x - \sin^2 x)}{(\sin^2 x \cos^2 x)} dx = \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx$

$= \int (\text{cosec}^2 x - \sec^2 x)\, dx.$

18. $\frac{1}{(1 - \cos 2x)} = \frac{1}{2 \sin^2 x} = \frac{1}{2} \text{cosec}^2 x.$ **21.** $\cot^2 x = (\text{cosec}^2 x - 1).$

23. $\frac{\sec^2 x}{\text{cosec}^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x = (\sec^2 x - 1).$

25. $I = \int \left\{ \frac{\cot x}{(\text{cosec}\, x - \cot x)} \times \frac{(\text{cosec}\, x + \cot x)}{(\text{cosec}\, x + \cot x)} \right\} dx$

$= \int (\text{cosec}\, x \cot x + \cot^2 x) dx = \int (\text{cosec}\, x \cot x + \text{cosec}^2 x - 1) dx.$

26. $I = \int \left\{ \frac{\sin x}{(1 + \sin x)} \times \frac{(1 - \sin x)}{(1 - \sin x)} \right\} dx = \int \frac{\sin x (1 - \sin x)}{(1 - \sin^2 x)} dx$

$= \int \frac{(\sin x - \sin^2 x)}{\cos^2 x} dx = \int \left(\frac{\sin x}{\cos^2 x} - \tan^2 x \right) dx$

$= \int (\sec x \tan x - \sec^2 x + 1)\, dx.$

27. $I = \int \left\{ \frac{(1 + \sin x)}{(1 - \sin x)} \times \frac{(1 - \sin x)}{(1 + \sin x)} \right\} dx = \int \frac{(1 + \sin x)^2}{(1 - \sin^2 x)} dx$

$= \int \frac{(1 + \sin^2 x + 2 \sin x)}{\cos^2 x} dx = \int (\sec^2 x + \tan^2 x + 2 \sec x \tan x)\, dx$

$= \int (2 \sec^2 x - 1 + 2 \sec x \tan x)\, dx.$

28. $I = \int \left\{ \frac{1}{(1 + \cos x)} \times \frac{(1 - \cos x)}{(1 - \cos x)} \right\} dx = \int \frac{(1 - \cos x)}{\sin^2 x} dx$

$= \left\{ \frac{-1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right\} dx = \int (\text{cosec}^2 x - \text{cosec}\, x \cot x)\, dx.$

29. $\sin^{-1}(\cos x) = \sin^{-1}\left\{ \sin\left(\frac{\pi}{2} - x \right) \right\} = \left(\frac{\pi}{2} - x \right) \cdot$

32. $\sin^{-1}\left(\frac{2 \tan x}{1 + \tan^2 x} \right) = \sin^{-1}(\sin 2x) = 2x.$

33. $\cos^{-1}\left(\frac{1 - \tan^2 x}{1 + \tan^2 x} \right) = \cos^{-1}\left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} \right)$

$= \cos^{-1}(\cos^2 x - \sin^2 x) = \cos^{-1}(\cos 2x) = 2x.$

34. $\tan^{-1}(\text{cosec}\, x - \cot x) = \tan^{-1}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) = \tan^{-1}\left(\frac{1 - \cos x}{\sin x} \right)$

$= \tan^{-1}\left\{ \frac{2 \sin^2\left(\frac{x}{2} \right)}{2 \sin\left(\frac{x}{2} \right) \cos\left(\frac{x}{2} \right)} \right\} = \tan^{-1} \tan\left(\frac{x}{2} \right) = \frac{x}{2} \cdot$

35. On dividing $(x^4 + 1)$ by $(x^2 + 1)$, we get

$$\frac{(x^4 + 1)}{(x^2 + 1)} = (x^2 - 1) + \frac{2}{(1 + x^2)}.$$

36. On dividing $(ax + b)$ by $(cx + d)$, we get

$$\int \frac{(ax + b)}{(cx + d)}\,dx = \int \left\{ \frac{a}{c} + \frac{(bc - ad)}{c(cx + d)} \right\} dx$$

$$= \int \frac{a}{c}\,dx + \frac{(bc - ad)}{c^2} \cdot \int \frac{c}{(cx + d)}\,dx$$

$$= \frac{ax}{c} + \frac{(bc - ad)}{c^2} \log|cx + d| + C.$$

37. $I = \int \frac{\sin^3 x}{\sin^2 x \cos^2 x}\,dx + \int \frac{\cos^3 x}{\sin^2 x \cos^2 x}\,dx$

$= \int \sec x \tan x\,dx + \int \operatorname{cosec} x \cot x\,dx$

$= \sec x - \operatorname{cosec} x + C.$

38. $I = \frac{\sin(x - \alpha + \alpha)}{\sin(x - \alpha)}\,dx = \int \frac{\sin(x - \alpha)\cos\alpha + \cos(x - \alpha)\sin\alpha}{\sin(x - \alpha)}\,dx$

$= (\cos\alpha)\int dx + (\sin\alpha)\int \cot(x - \alpha)\,dx$

$= x\cos\alpha + (\sin\alpha)\log|\sin(x - \alpha)| + C.$

39. $I = \frac{1}{2}\int 2\sin 3x \sin 2x\,dx$

$= \frac{1}{2}\int(\cos x - \cos 5x)\,dx = \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C.$

40. $I = \frac{1}{2}\,2\cos 3x \sin 2x\,dx$

$= \frac{1}{2}\int(\sin 5x - \sin x)\,dx = \frac{-1}{10}\cos 5x + \frac{1}{2}\cos x + C.$

41. $I = \frac{1}{2}2\cos 4x \cos x\,dx = \frac{1}{2}\int(\cos 5x + \cos 3x)\,dx$

$= \frac{1}{10}\sin 5x + \frac{1}{6}\sin 3x + C.$

13. METHODS OF INTEGRATION

Integration by Substitution

If we have to evaluate an integral of the type $\int f\{\phi(x)\} \cdot \phi'(x)\, dx$ then we put $\phi(x) = t$ and $\phi'(x) dx = dt$. With this substitution, the integrand becomes easily integrable.

Case I When the integrand is of the form $f(ax + b)$, we put $(ax + b) = t$ and $dx = \frac{1}{a} dt$.

Case II When the integrand is of the form $x^{n-1} \cdot f(x^n)$, we put $x^n = t$ and $nx^{n-1} dx = dt$.

Case III When the integrand is of the form $\{f(x)\}^n \cdot f'(x)$, we put $f(x) = t$ and $f'(x)\, dx = dt$.

Case IV When the integrand is of the form $\frac{f'(x)}{f(x)}$, we put $f(x) = t$ and $f'(x)\, dx = dt$.

THEOREM 1 $\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + C, \quad \text{where } n \neq -1.$

PROOF Putting $ax + b = t$, we get $a\, dx = dt$ or $dx = \frac{1}{a} dt$.

$$\therefore \quad \int (ax + b)^n dx = \frac{1}{a} \int t^n\, dt = \frac{1}{a} \cdot \frac{t^{n+1}}{(n+1)} + C = \frac{(ax + b)^{n+1}}{a(n+1)} + C.$$

THEOREM 2 (i) $\int \cos(ax + b)\, dx = \frac{1}{a} \sin(ax + b) + C$

(ii) $\int \operatorname{cosec}^2(ax + b)\, dx = -\frac{1}{a} \cot(ax + b) + C$

PROOF (i) Put $(ax + b) = t$ so that $dx = \frac{1}{a}\, dt$.

$$\therefore \quad \int \cos(ax + b)\, dx = \frac{1}{a} \int \cos t\, dt$$

$$= \frac{1}{a} \sin t + C$$

$$= \frac{1}{a} \sin(ax + b) + C.$$

(ii) Put $(ax+b)=t$ so that $dx=\frac{1}{a}\,dt$.

$$\therefore\ \int \operatorname{cosec}^2(ax+b)\,dx=\frac{1}{a}\int \operatorname{cosec}^2 t\,dt$$
$$=-\frac{1}{a}\cot t+C=-\frac{1}{a}\cot(ax+b)+C.$$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:* (i) $\int (3x+5)^7\,dx$ (ii) $\int (4-9x)^5\,dx$

(iii) $\int \frac{1}{(2-3x)^4}\,dx$ (iv) $\int \sqrt{ax+b}\,dx$

SOLUTION (i) Put $(3x+5)=t$ so that $3dx=dt$ or $dx=\frac{1}{3}dt$.

$$\therefore\ \int (3x+5)^7\,dx=\frac{1}{3}\int t^7\,dt=\frac{1}{3}\cdot\frac{t^8}{8}+C=\frac{(3x+5)^8}{24}+C.$$

(ii) Put $(4-9x)=t$ so that $-9\,dx=dt$ or $dx=-\frac{1}{9}\,dt$.

$$\therefore\ \int (4-9x)^5\,dx=-\frac{1}{9}\int t^5\,dt=-\frac{1}{9}\cdot\frac{t^6}{6}+C=\frac{-(4-9x)^6}{54}+C.$$

(iii) Put $(2-3x)=t$ so that $-3\,dx=dt$ or $dx=-\frac{1}{3}\,dt$.

$$\therefore\ \int \frac{1}{(2-3x)^4}\,dx=-\frac{1}{3}\int \frac{1}{t^4}\,dt=-\frac{1}{3}\cdot\frac{1}{(-3t^3)}+C=\frac{1}{9(2-3x)^3}+C.$$

(iv) Put $(ax+b)=t$ so that $a\,dx=dt$.

$$\therefore\ \int \sqrt{ax+b}\,dt=\frac{1}{a}\int \sqrt{t}\,dt=\frac{2}{3a}t^{3/2}+C=\frac{2(ax+b)^{3/2}}{3a}+C.$$

EXAMPLE 2 *Evaluate:* (i) $\int \cos 2x\,dx$ (ii) $\int e^{(5x+3)}\,dx$

(iii) $\int \sec^2(3x+5)\,dx$ (iv) $\int \sin^3 x\,dx$

SOLUTION (i) Put $2x=t$ so that $2dx=dt$ or $dx=\frac{1}{2}dt$.

$$\therefore\ \int \cos 2x\,dx=\frac{1}{2}\int \cos t\,dt=\frac{1}{2}\sin t+C=\frac{1}{2}\sin 2x+C.$$

(ii) Put $(5x+3)=t$ so that $5dx=dt$ or $dx=\frac{1}{5}dt$.

$$\therefore\ \int e^{(5x+3)}\,dx=\frac{1}{5}\int e^t\,dt=\frac{1}{5}\cdot e^t+C=\frac{1}{5}e^{(5x+3)}+C.$$

(iii) Put $(3x+5) = t$ so that $3dx = dt$ or $dx = \frac{1}{3}\,dt$.

$$\therefore \int \sec^2(3x+5)\,dx = \frac{1}{3}\int \sec^2 t\,dt = \frac{1}{3}\tan t + C$$
$$= \frac{1}{3}\tan(3x+5) + C.$$

(iv) We know that $\sin 3x = 3\sin x - 4\sin^3 x$.

$$\therefore \quad \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x).$$

So, $$\int \sin^3 x\,dx = \int\left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right)dx$$
$$= \frac{3}{4}\int \sin x\,dx - \frac{1}{4}\int \sin 3x\,dx$$
$$= \frac{3}{4}(-\cos x) - \frac{1}{4}\cdot\frac{(-\cos 3x)}{3} + C$$
$$= -\frac{3}{4}\cos x + \frac{\cos 3x}{12} + C.$$

<u>EXAMPLE 3</u> *Evaluate:* *(i)* $\int \frac{\log x}{x}\,dx$ *(ii)* $\int \frac{\sec^2(\log x)}{x}\,dx$

(iii) $\int \frac{e^{\tan^{-1} x}}{(1+x^2)}\,dx$ *(iv)* $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

SOLUTION (i) Put $\log x = t$ so that $\frac{1}{x}\,dx = dt$.

$$\therefore \int \frac{\log x}{x}\,dx = \int t\,dt = \frac{1}{2}t^2 + C = \frac{1}{2}(\log x)^2 + C.$$

(ii) Put $\log x = t$ so that $\frac{1}{x}\,dx = dt$.

$$\therefore \int \frac{\sec^2(\log x)}{x}\,dx = \int \sec^2 t\,dt = \tan t + C = \tan(\log x) + C.$$

(iii) Put $\tan^{-1} x = t$ so that $\frac{1}{(1+x^2)}\,dx = dt$.

$$\therefore \int \frac{e^{\tan^{-1} x}}{(1+x^2)}\,dx = \int e^t\,dt = e^t + C = e^{\tan^{-1} x} + C.$$

(iv) Put $\sqrt{x} = t$ so that $\frac{1}{2}x^{-1/2}dx = dt$

or $\frac{1}{\sqrt{x}}\,dx = 2\,dt$.

$$\therefore \int \frac{\sin \sqrt{x}}{\sqrt{x}}\, dx = 2\int \sin t\, dt$$

$$= 2(-\cos t) + C = -2\cos t + C = -2\cos \sqrt{x} + C.$$

EXAMPLE 4 *Evaluate:* (i) $\int \cos^3 x \sin x\, dx$ (ii) $\int (\sqrt{\sin x}) \cos x\, dx$

(iii) $\int \frac{\operatorname{cosec}^2 x}{(1 + \cot x)}\, dx$ (iv) $\int \frac{\sin x}{(3 + 4\cos x)^2}\, dx$

SOLUTION (i) Put $\cos x = t$ so that $\sin x\, dx = -dt$.

$$\therefore \int \cos^3 x \sin x\, dx = -\int t^3 dt = -\frac{t^4}{4} + C = -\frac{1}{4}\cos^4 x + C.$$

(ii) Put $\sin x = t$ so that $\cos x\, dx = dt$.

$$\therefore \int (\sqrt{\sin x}) \cos x\, dx = \int \sqrt{t}\, dt = \frac{2}{3} t^{3/2} + C = \frac{2}{3}(\sin x)^{3/2} + C.$$

(iii) Put $(1 + \cot x) = t$ so that $-\operatorname{cosec}^2 x\, dx = dt$.

$$\therefore \int \frac{\operatorname{cosec}^2 x}{(1 + \cot x)}\, dx = -\int \frac{1}{t}\, dt$$

$$= -\log t + C = -\log |(1 + \cot x)| + C.$$

(iv) Put $(3 + 4\cos x) = t$ so that $-4\sin x\, dx = dt$.

$$\therefore \int \frac{\sin x}{(3 + 4\cos x)^2}\, dx = -\frac{1}{4}\int \frac{1}{t^2}\, dt = \frac{1}{4t} + C = \frac{1}{4(3 + 4\cos x)} + C.$$

EXAMPLE 5 *Evaluate:* (i) $\int \frac{2x}{(2x+1)^2}\, dx$ (ii) $\int \frac{(2 + 3x)}{(3 - 2x)}\, dx$

SOLUTION (i) Put $(2x + 1) = t$ so that $2x = (t - 1)$ and $dx = \frac{1}{2}\, dt$.

$$\therefore \int \frac{2x}{(2x+1)^2}\, dx = \frac{1}{2}\int \frac{(t-1)}{t^2}\, dt$$

$$= \frac{1}{2}\int \frac{1}{t}\, dt - \frac{1}{2}\int \frac{1}{t^2}\, dt = \frac{1}{2}\log |t| + \frac{1}{2t} + C$$

$$= \frac{1}{2}\log |(2x+1)| + \frac{1}{2(2x+1)} + C.$$

(ii) Put $(3 - 2x) = t$ so that $x = \left(\frac{3-t}{2}\right)$ and $dx = -\frac{1}{2}\, dt$.

$$\therefore \int \frac{(2 + 3x)}{(3 - 2x)}\, dt = -\frac{1}{2}\int \frac{\left[2 + \left(\frac{9 - 3t}{2}\right)\right]}{t}\, dt = -\frac{1}{4}\int \frac{(13 - 3t)}{t}\, dt$$

$$= \frac{-13}{4}\int \frac{1}{t}\, dt + \frac{3}{4}\int dt = -\frac{13}{4}\log |t| + \frac{3}{4}t + C$$

$$= -\frac{13}{4}\log |(3 - 2x)| + \frac{3}{4}(3 - 2x) + C.$$

EXAMPLE 6 *Evaluate:*

(i) $\int \frac{3x^2}{(1+x^6)}\,dx$ (ii) $\int \frac{x^3}{(x^2+1)^3}\,dx$ (iii) $\frac{x^8}{(1-x^3)^{1/3}}\,dx$

SOLUTION (i) Put $x^3 = t$ so that $3x^2\,dx = dt$.

$$\therefore \int \frac{3x^2}{(1+x^6)}\,dx = \int \frac{dt}{(1+t^2)} = \tan^{-1}t + C = \tan^{-1}x^3 + C.$$

(ii) Put $(x^2+1) = t$ so that $x^2 = (t-1)$ and $x\,dx = \frac{1}{2}\,dt$.

$$\therefore \int \frac{x^3}{(x^2+1)^3}\,dx = \int \frac{x^2 \cdot x}{(x^2+1)^3}\,dx$$

$$= \frac{1}{2}\int \frac{(t-1)}{t^3}\,dt = \frac{1}{2}\int \frac{1}{t^2}\,dt - \frac{1}{2}\int \frac{1}{t^3}\,dt$$

$$= \frac{-1}{2t} + \frac{1}{4t^2} + C = \frac{-1}{2(x^2+1)} + \frac{1}{4(x^2+1)^2} + C$$

$$= \frac{-(1+2x^2)}{4(x^2+1)^2} + C.$$

(iii) Put $(1-x^3) = t$ so that $x^3 = (1-t)$ and $x^2dx = -\frac{1}{3}\,dt$.

$$\therefore \int \frac{x^8}{(1-x^3)^{1/3}}\,dx = \int \frac{x^6 \cdot x^2}{(1-x^3)^{1/3}}\,dx$$

$$= -\frac{1}{3}\int \frac{(1-t)^2}{t^{1/3}}\,dt = -\frac{1}{3}\int \frac{(1+t^2-2t)}{t^{1/3}}\,dt$$

$$= -\frac{1}{3}\int t^{-1/3}\,dt - \frac{1}{3}\int t^{5/3}dt + \frac{2}{3}\int t^{2/3}dt$$

$$= -\frac{1}{2}t^{2/3} - \frac{1}{8}t^{8/3} + \frac{2}{5}t^{5/3} + C$$

$$= -\frac{1}{2}(1-x^3)^{2/3} - \frac{1}{8}(1-x^3)^{8/3} + \frac{2}{5}(1-x^3)^{5/3} + C.$$

EXAMPLE 7 *Evaluate* $\int \frac{dx}{x \cdot \sqrt{x^6-1}}$.

SOLUTION Put $x^3 = t$ so that $3x^2dx = dt$ or $x^2dx = \frac{1}{3}\,dt$.

$$\therefore \int \frac{dx}{x \cdot \sqrt{x^6-1}} = \int \frac{x^2}{x^3 \cdot \sqrt{x^6-1}}\,dx$$

[multiplying numerator and denominator by x^2]

$$= \frac{1}{3}\int \frac{1}{t\sqrt{t^2-1}}\,dt = \frac{1}{3}\sec^{-1}t + C = \frac{1}{3}\sec^{-1}x^3 + C.$$

EXAMPLE 8 *Evaluate* $\int \frac{1}{(\sqrt{x}+x)}\,dx.$ **[CBSE 2003]**

SOLUTION $\int \frac{1}{(\sqrt{x}+x)}\,dx = \int \frac{1}{\sqrt{x}\,(1+\sqrt{x})}\,dx$

Now, put $(1+\sqrt{x}) = t$ so that $\frac{1}{\sqrt{x}}\,dx = 2dt.$

$$\therefore \int \frac{1}{(\sqrt{x}+x)}\,dx = \int \frac{1}{\sqrt{x}\,(1+\sqrt{x})}\,dx$$

$$= 2\int \frac{1}{t}\,dt = 2\log|t| + C = 2\log|(1+\sqrt{x})| + C.$$

EXAMPLE 9 *Evaluate:* *(i)* $\int \frac{(x-1)}{\sqrt{x+4}}\,dx$ *(ii)* $\int x\sqrt{x+2}\,dx$

(iii) $\int (4x+2)\sqrt{x^2+x+1}\,dx$ *(iv)* $\int \frac{(4x+3)}{\sqrt{2x^2+3x+1}}\,dx$

SOLUTION (i) Put $(x+4) = t^2$ so that $x = (t^2-4)$ and $dx = 2t\,dt$.

$$\therefore \int \frac{(x-1)}{\sqrt{x+4}}\,dx = 2\int \frac{(t^2-5)t}{t}\,dt$$

$$= 2\int t^2 dt - 10\int dt = \frac{2t^3}{3} - 10t + C$$

$$= \frac{2}{3}(x+4)^{3/2} - 10(x+4)^{1/2} + C.$$

(ii) Put $(x+2) = t^2$ so that $x = (t^2-2)$ and $dx = 2t\,dt$.

$$\therefore \int x\sqrt{x+2}\,dx = \int (t^2-2)\,2t^2 dt = 2\int t^4 dt - 4\int t^2 dt$$

$$= \frac{2t^5}{5} - \frac{4t^3}{3} + C = \frac{2(x+2)^{5/2}}{5} - \frac{4(x+2)^{3/2}}{3} + C.$$

(iii) Put $(x^2+x+1) = t$ so that $(2x+1)\,dx = dt$.

$$\therefore \int (4x+2)(\sqrt{x^2+x+1})\,dx = 2\int \sqrt{t}\,dt$$

$$= \frac{4}{3}t^{3/2} + C = \frac{4}{3}(x^2+x+1)^{3/2} + C.$$

(iv) Put $(2x^2+3x+1) = t$ so that $(4x+3)\,dx = dt$.

$$\therefore \int \frac{(4x+3)}{\sqrt{2x^2+3x+1}}\,dx = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C = 2\sqrt{2x^2+3x+1} + C.$$

EXAMPLE 10 *Evaluate:*

(i) $\int \frac{(2x+5)}{(x^2+5x+9)}\,dx$ (ii) $\int \frac{(6x-7)}{(3x^2-7x+5)^2}\,dx$

(iii) $\int \frac{(\cos x - \sin x)}{(\cos x + \sin x)}\,dx$ (iv) $\int \frac{\sec x}{\log(\sec x + \tan x)}\,dx$

SOLUTION (i) Put $(x^2+5x+9) = t$ so that $(2x+5)\,dx = dt$.

$$\therefore \int \frac{(2x+5)}{(x^2+5x+9)}\,dx = \int \frac{1}{t}\,dt = \log|t| + C$$
$$= \log|(x^2+5x+9)| + C.$$

(ii) Put $(3x^2-7x+5) = t$ so that $(6x-7)\,dx = dt$.

$$\therefore \int \frac{(6x-7)}{(3x^2-7x+5)^2}\,dx = \int \frac{1}{t^2}\,dt = -\frac{1}{t} + C = \frac{-1}{(3x^2-7x+5)} + C.$$

(iii) Put $(\cos x + \sin x) = t$ so that $(\cos x - \sin x)\,dx = dt$.

$$\therefore \int \frac{(\cos x - \sin x)}{(\cos x + \sin x)}\,dx = \int \frac{1}{t}\,dt$$
$$= \log|t| + C = \log|(\cos x + \sin x)| + C.$$

(iv) Put $\log(\sec x + \tan x) = t$.

Then, on differentiation, we get

$$\frac{1}{(\sec x + \tan x)} \cdot (\sec x \tan x + \sec^2 x)\,dx = dt$$

or $\sec x\,dx = dt$.

$$\therefore \int \frac{\sec x}{\log(\sec x + \tan x)}\,dx = \int \frac{1}{t}\,dt$$
$$= \log|t| + C = \log|\log(\sec x + \tan x)| + C.$$

EXAMPLE 11 *Evaluate* $\int \frac{\sin 2x}{(a^2\sin^2 x + b^2\cos^2 x)}\,dx.$ **[CBSE 2005]**

SOLUTION Put $(a^2\sin^2 x + b^2\cos^2 x) = t$ so that

$$2(a^2-b^2)\sin x\cos x\,dx = dt \Leftrightarrow \sin 2x\,dx = \frac{dt}{(a^2-b^2)}.$$

$$\therefore \quad I = \int \frac{\sin 2x}{(a^2\sin^2 x + b^2\cos^2 x)}\,dx = \frac{dt}{(a^2-b^2)t}$$
$$= \frac{1}{(a^2-b^2)}\log|t| + C$$
$$= \frac{1}{(a^2-b^2)}\log(a^2\sin^2 x + b^2\cos^2 x) + C.$$

EXAMPLE 12 *Evaluate* $\int \frac{x^2\tan^{-1}x^3}{(1+x^6)}\,dx.$

SOLUTION Put $\tan^{-1}x^3 = t$ so that $\frac{3x^2}{(1+x^6)}\,dx = dt$ or $\frac{x^2}{(1+x^6)}\,dx = \frac{1}{3}\,dt$

$$\therefore \int \frac{x^2 \tan^{-1}x^3}{(1+x^6)}\,dx = \frac{1}{3}\int t\,dt = \frac{1}{6}t^2 + C = \frac{1}{6}(\tan^{-1}x^3)^2 + C.$$

EXAMPLE 13 *Evaluate* $\int \frac{\sqrt{\tan x}}{\sin x \cos x}\,dx.$

SOLUTION $\int \frac{\sqrt{\tan x}}{\sin x \cos x}\,dx = \int \frac{\tan x}{(\sqrt{\tan x})\cdot \sin x \cos x}\,dx = \int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx$

$$= \int \frac{1}{\sqrt{t}}\,dt, \text{ where } \tan x = t \text{ and } \sec^2 x\,dx = dt$$

$$= 2\sqrt{t} + C = 2\sqrt{\tan x} + C.$$

EXAMPLE 14 *Evaluate* $\int \frac{dx}{(\sqrt{2x+3} + \sqrt{2x-3})}.$

SOLUTION $\int \frac{dx}{(\sqrt{2x+3} + \sqrt{2x-3})}$

$$= \int \frac{1}{(\sqrt{2x+3} + \sqrt{2x-3})} \times \frac{(\sqrt{2x+3}) - \sqrt{2x-3})}{(\sqrt{2x+3}) - \sqrt{2x-3})}\,dx$$

$$= \int \frac{(\sqrt{2x+3} - \sqrt{2x-3})}{[(2x+3) - (2x-3)]}\,dx = \frac{1}{6}\int (2x+3)^{1/2}dx - \frac{1}{6}\int (2x-3)^{1/2}dx$$

$$= \frac{1}{18}(2x+3)^{3/2} - \frac{1}{18}(2x-3)^{3/2} + C.$$

EXAMPLE 15 *Evaluate:*

(i) $\int \frac{1}{(1+\tan x)}\,dx$ *(ii)* $\int \frac{1}{(1+\cot x)}\,dx$

(iii) $\int \left(\frac{1-\tan x}{1+\tan x}\right)dx$ **[CBSE 2000C]** *(iv)* $\int \frac{\tan x}{(\sec x + \cos x)}\,dx$

SOLUTION (i) $\int \frac{1}{(1+\tan x)}\,dx = \int \frac{1}{\left(1 + \frac{\sin x}{\cos x}\right)}\,dx$

$$= \int \frac{\cos x}{(\cos x + \sin x)}\,dx = \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2(\cos x + \sin x)}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{(\cos x - \sin x)}{(\cos x + \sin x)}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{t}\,dt, \text{ where } (\cos x + \sin x) = t \text{ and } (\cos x - \sin x)\,dx = dt$$

$$= \frac{1}{2}x + \frac{1}{2}\log|t| + C = \frac{1}{2}x + \frac{1}{2}\log|\cos x + \sin x| + C.$$

(ii) $\int \frac{1}{(1+\cot x)}\,dx = \int \frac{1}{\left(1+\frac{\cos x}{\sin x}\right)}\,dx = \int \frac{\sin x}{(\sin x+\cos x)}\,dx$

$= \int \frac{(\sin x+\cos x)-(\cos x-\sin x)}{2(\sin x+\cos x)}\,dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \frac{(\cos x-\sin x)}{(\sin x+\cos x)}\,dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \frac{1}{t}\,dt,$

where $\sin x+\cos x = t$ and $(\cos x-\sin x)\,dx = dt$

$= \frac{1}{2}x - \frac{1}{2}\log|t| + C = \frac{1}{2}x - \frac{1}{2}\log|\sin x+\cos x| + C.$

(iii) $\int \left(\frac{1-\tan x}{1+\tan x}\right)dx = \int \frac{\left(1-\frac{\sin x}{\cos x}\right)}{\left(1+\frac{\sin x}{\cos x}\right)}\,dx = \int \frac{(\cos x-\sin x)}{(\cos x+\sin x}\,dx$

$= \int \frac{1}{t}\,dt,$ where $(\cos x+\sin x) = t$ and $(\cos x-\sin x)\,dx = dt$

$= \log|t| + C = \log|(\cos x+\sin x)| + C.$

(iv) $\int \frac{\tan x}{(\sec x+\cos x)}\,dx = \int \frac{\sin x}{1+\cos^2 x}\,dx$

$= -\int \frac{1}{(1+t^2)}\,dt,$ where $\cos x = t$ and $\sin x\,dx = -dt$

$= -\tan^{-1}t + C = -\tan^{-1}(\cos x) + C.$

EXAMPLE 16 *Evaluate:* *(i)* $\int \tan x\,dx$ *(ii)* $\int \cot x\,dx$ *(iii)* $\int \sec x\,dx$ *(iv)* $\int \operatorname{cosec} x\,dx$

SOLUTION (i) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$= -\int \frac{1}{t}\,dt,$ where $\cos x = t$ and $\sin x\,dx = -dt$

$= -\log|t| + C = -\log|\cos x| + C.$

$\therefore \int \tan x\,dx = -\log|\cos x| + C.$

(ii) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$

$= \int \frac{1}{t}\,dt,$ where $\sin x = t$ and $\cos x\,dx = dt$

$= \log|t| + C = \log|\sin x| + C.$

$\therefore \int \cot x\,dx = \log|\sin x| + C.$

(iii) $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\, dx$

[multiplying numerator and denominator by $(\sec x + \tan x)$]

$$= \int \frac{1}{t}\, dt, \quad \text{where } (\sec x + \tan x) = t$$

and $\sec x(\sec x + \tan x)\, dx = dt$

$$= \log|t| + C = \log|(\sec x + \tan x)| + C.$$

$\therefore \int \sec x\, dx = \log|(\sec x + \tan x)| + C.$

Alternative form

$$\sec x + \tan x = \left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) = \frac{(1 + \sin x)}{\cos x}$$

Putting $\sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$ and $\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$.

$$\therefore \sec x + \tan x = \frac{1 + \tan(x/2)}{1 - \tan(x/2)} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

$$\therefore \int \sec x\, dx = \log|\sec x + \tan x| + C$$

$$= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C.$$

(iv) $\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)}\, dx$

[multiplying numerator and denominator by $(\operatorname{cosec} x - \cot x)$]

$$= \int \frac{1}{t}\, dt, \quad \text{where } (\operatorname{cosec} x - \cot x) = t$$

and $\operatorname{cosec} x\,(\operatorname{cosec} x - \cot x)\, dx = dt$

$$= \log|t| + C = \log|\operatorname{cosec} x - \cot x| + C.$$

$\therefore \int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C.$

Alternative form

$$\operatorname{cosec} x - \cot x = \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \frac{1 - \cos x}{\sin x}$$

$$= \frac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)} = \tan\frac{x}{2}.$$

$$\therefore \int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$= \log\left|\tan\frac{x}{2}\right| + C.$$

As a consequence of the above results, the integral of trigonometric functions may be listed as given below:

(i) $\int \sin x\,dx = -\cos x + C$ (ii) $\int \cos x\,dx = \sin x + C$

(iii) $\int \tan x\,dx = -\log|\cos x| + C$

(iv) $\int \cot x\,dx = \log|\sin x| + C$

(v) $\int \sec x\,dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(vi) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

EXAMPLE 17 *Evaluate:*

(i) $\int \frac{(1+\cos x)}{(1-\cos x)}\,dx$ **[CBSE 2000C]** *(ii)* $\int \frac{(1+\sin x)}{(1+\cos x)}\,dx$

SOLUTION (i) $\int \frac{(1+\cos x)}{(1-\cos x)}\,dx = \int \frac{2\cos^2(x/2)}{2\sin^2(x/2)}\,dx$

$$= \int \cot^2\left(\frac{x}{2}\right)dx = \int \left(\operatorname{cosec}^2\frac{x}{2} - 1\right)dx$$

$$= \int \operatorname{cosec}^2\frac{x}{2}\,dx - \int dx$$

$$= 2\int \operatorname{cosec}^2 t\,dt - \int dx, \text{ where } \frac{x}{2} = t \text{ and } dx = 2dt$$

$$= -2\cot t - x + C = -2\cot\left(\frac{x}{2}\right) - x + C.$$

(ii) $\int \left(\frac{1+\sin x}{1+\cos x}\right)dx = \int \frac{1}{(1+\cos x)}\,dx + \int \frac{\sin x}{(1+\cos x)}\,dx$

$$= \int \frac{1}{2\cos^2(x/2)}\,dx + \int \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\,dx$$

$$= \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right)dx + \int \tan\frac{x}{2}\,dx$$

$$= \int \sec^2 t\,dt + 2\int \tan t\,dt, \text{ where } \frac{x}{2} = t$$

$$= \tan t - 2\log|\cos t| + C$$

$$= \tan\left(\frac{x}{2}\right) - 2\log\left|\cos\left(\frac{x}{2}\right)\right| + C.$$

EXAMPLE 18 *Evaluate:*

(i) $\int \frac{dx}{1+\sqrt{x}}$ (ii) $\int \frac{x+\sqrt{x+1}}{x+2}\,dx$

SOLUTION (i) Put $\sqrt{x}=t$ so that $x=t^2$ and $dx=2t\ dt$.

$$\therefore \int \frac{dx}{1+\sqrt{x}} = \int \frac{2t}{(1+t)}\,dt = \int \frac{2(1+t)-2}{(1+t)}\,dt$$

$$= 2\int dt - 2\int \frac{dt}{1+t} = 2t - 2\log|1+t| + C$$

$$= 2\sqrt{x} - 2\log|1+\sqrt{x}| + C.$$

(ii) Put $\sqrt{x+1}=t$ so that $x+1=t^2$ and $dx=2t\,dt$.

$$\therefore \int \frac{x+\sqrt{x+1}}{(x+2)}\,dx = 2\int \frac{(t^2-1+t)t}{(t^2+1)}\,dt$$

$$= 2\int \left(\frac{t^3+t^2-t}{t^2+1}\right)dt$$

$$= 2\int \left(t+1-\frac{2t+1}{t^2+1}\right)dt \qquad \text{[by division]}$$

$$= 2\int \left(t+1-\frac{2t}{t^2+1}-\frac{1}{t^2+1}\right)dt$$

$$= 2\int t\,dt + 2\int dt - 2\int \frac{2t}{t^2+1}\,dt - 2\int \frac{1}{t^2+1}\,dt$$

$$= t^2 + 2t - 2\log|t^2+1| - 2\tan^{-1}t + C$$

$$= (x+1) + 2\sqrt{x+1} - 2\log|x+2| - 2\tan^{-1}\sqrt{x+1} + C.$$

EXAMPLE 19 *Evaluate* $\int \sqrt{\frac{1+x}{1-x}}\,dx.$ **[CBSE 2006]**

SOLUTION $\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx$

$$= \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt, \text{ where } (1-x^2)=t$$

$$= \sin^{-1}x - \sqrt{t} + C$$

$$= \sin^{-1}x - \sqrt{1-x^2} + C.$$

EXAMPLE 20 *Evaluate* $\int \frac{(3\sin x-2)\cos x}{(5-\cos^2 x-4\sin x)}\,dx$ **[CBSE 2013C]**

SOLUTION We have

$$I = \int \frac{(3\sin x - 2)\cos x}{\{5-(1-\sin^2 x) - 4\sin x\}}\,dx$$

$$= \int \frac{(3\sin x - 2)\cos x}{\{4+\sin^2 x - 4\sin x\}}\,dx = \int \frac{(3\sin x - 2)\cos x}{(2-\sin x)^2}\,dx \qquad \dots \text{(i)}$$

Putting $2-\sin x = t$, we get $\sin x = 2-t$ and $\cos x\,dx = -dt$.

$$\therefore \quad I = -\int \frac{\{3(2-t)-2\}}{t^2}\,dt = -\int \frac{(4-3t)}{t^2}\,dt = \int \frac{(3t-4)}{t^2}\,dt$$

$$= \int \left(\frac{3}{t} - \frac{4}{t^2}\right) dt = 3\log|t| + \frac{4}{t} + C$$

$$= 3\log|(2-\sin x)| + \frac{4}{(2-\sin x)} + C.$$

EXERCISE 13A

Evaluate the following integrals:

Very-Short-Answer Questions

1. $\int (2x+9)^5\,dx$
2. $\int (7-3x)^4\,dx$
3. $\int \sqrt{3x-5}\,dx$
4. $\int \frac{1}{\sqrt{4x+3}}\,dx$
5. $\int \frac{1}{\sqrt{3-4x}}\,dx$
6. $\int \frac{1}{(2x-3)^{3/2}}\,dx$
7. $\int e^{(2x-1)}\,dx$
8. $\int e^{(1-3x)}\,dx$
9. $\int 3^{(2-3x)}\,dx$
10. $\int \sin 3x\,dx$

Short-Answer Questions

11. $\int \cos(5+6x)\,dx$
12. $\int \sin x\sqrt{1+\cos 2x}\,dx$ **[CBSE 2006C]**
13. $\int \operatorname{cosec}^2(2x+5)\,dx$
14. $\int \sin x\cos x\,dx$
15. $\int \sin^3 x\cos x\,dx$
16. $\int (\sqrt{\cos x})\sin x\,dx$
17. $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$
18. $\int \frac{\sin(2\tan^{-1}x)}{(1+x^2)}\,dx$ **[CBSE 2002]**
19. $\int \frac{\cos(\log x)}{x}\,dx$
20. $\int \frac{\operatorname{cosec}^2(\log x)}{x}\,dx$ **[CBSE 2001]**
21. $\int \frac{1}{x\log x}\,dx$
22. $\int \frac{(x+1)(x+\log x)^2}{x}\,dx$ **[CBSE 2002]**
23. $\int \frac{(\log x)^2}{x}\,dx$
24. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

25. $\int e^{\tan x} \sec^2 x \, dx$

26. $\int e^{\cos^2 x} \sin 2x \, dx$

27. $\int \sin(ax+b)\cos(ax+b)\, dx$

28. $\int \cos^3 x \, dx$

29. $\int \frac{1}{x^2} e^{-1/x} \, dx$

30. $\int \frac{1}{x^2} \cos\left(\frac{1}{x}\right) dx$

31. $\int \frac{dx}{(e^x + e^{-x})}$

32. $\int \frac{e^{2x}}{(e^{2x} - 2)} dx$

33. $\int \cot x \log(\sin x)\, dx$

34. $\int \frac{\cot x}{\log(\sin x)} dx$

35. $\int 2x \sin(x^2+1)\, dx$

36. $\int \sec x \log(\sec x + \tan x)\, dx$

37. $\int \frac{\tan\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}} dx$

38. $\int \frac{x \tan^{-1} x^2}{(1+x^4)} dx$

39. $\int \frac{x \sin^{-1} x^2}{\sqrt{1-x^4}} dx$

40. $\int \frac{1}{(\sqrt{1-x^2}) \sin^{-1} x} dx$

41. $\int \frac{\sqrt{(2+\log x)}}{x} dx$

42. $\int \frac{\sec^2 x}{(1+\tan x)} dx$

43. $\int \frac{\sin x}{(1+\cos x)} dx$

44. $\int \left(\frac{1+\tan x}{1-\tan x}\right) dx$ **[CBSE 2000C]**

45. (i) $\int \frac{(1+\tan x)}{(x+\log \sec x)} dx$ (ii) $\int \frac{(1-\sin 2x)}{(x+\cos^2 x)} dx$ **[CBSE 2000]**

46. $\int \frac{\sin 2x}{(a^2 + b^2 \sin^2 x)} dx$

47. $\int \frac{\sin 2x}{(a^2\cos^2 x + b^2 \sin^2 x)} dx$ **[CBSE 2005]**

48. $\int \left(\frac{2\cos x - 3\sin x}{3\cos x + 2\sin x}\right) dx$

49. $\int \frac{4x}{(2x^2+3)} dx$

50. $\int \frac{(x+1)}{(x^2+2x-3)} dx$

51. $\int \frac{(4x-5)}{(2x^2-5x+1)} dx$

52. $\int \frac{(9x^2-4x+5)}{(3x^3-2x^2+5x+1)} dx$

53. $\int \frac{\sec x \operatorname{cosec} x}{\log(\tan x)} dx$

54. $\int \frac{(1+\cos x)}{(x+\sin x)^3} dx$

55. $\int \frac{\sin x}{(1+\cos x)^2} dx$

56. $\int \frac{(2x+3)}{\sqrt{x^2+3x-2}} dx$

57. $\int \frac{(2x-1)}{\sqrt{x^2-x-1}} dx$

58. $\int \frac{dx}{(\sqrt{x+a}+\sqrt{x+b})}$

59. $\int \frac{dx}{(\sqrt{1-3x}-\sqrt{5-3x})}$

60. $\int \frac{x^2}{(1+x^6)} dx$

61. $\int \frac{x^3}{(1+x^8)} dx$

62. $\int \frac{x}{(1+x^4)}\,dx$

63. $\int \frac{x^5}{\sqrt{1+x^3}}\,dx$

64. $\int \frac{x}{\sqrt{1+x}}\,dx$

65. $\int \frac{1}{x\sqrt{x^4-1}}\,dx$

66. $\int x\sqrt{x-1}\,dx$

67. $\int (1-x)\sqrt{1+x}\,dx$

68. $\int x\sqrt{x^2-1}\,dx$

69. $\int x\sqrt{3x-2}\,dx$

70. $\int \frac{dx}{x\cos^2(1+\log x)}$

71. $\int x^2 \sin x^3\,dx$

72. $\int (2x+4)\sqrt{x^2+4x+3}\,dx$

73. $\int \frac{\sin x}{(\sin x - \cos x)}\,dx$

74. $\int \frac{dx}{(1-\tan x)}$

75. $\int \frac{dx}{(1-\cot x)}$

76. $\int \frac{\cos 2x}{(\sin x + \cos x)^2}\,dx$

77. $\int \frac{(\cos x - \sin x)}{(1+\sin 2x)}\,dx$

78. $\int \frac{(x+1)(x+\log x)^2}{x}\,dx$

79. $\int x \sin^3 x^2 \cos x^2\,dx$

80. $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$

81. $\int e^{-x}\operatorname{cosec}^2(2e^{-x}+5)\,dx$

82. $\int 2x\sec^3(x^2+3)\tan(x^2+3)\,dx$

83. $\int \frac{\sin 2x}{(a+b\cos x)^2}\,dx$ **[CBSE 2005]**

84. $\int \frac{dx}{(3-5x)}$

85. $\int \sqrt{1+x}\,dx$

86. $\int x^2 e^{x^3}\cos(e^{x^3})\,dx$

87. $\int \frac{e^{m\tan^{-1}x}}{(1+x^2)}\,dx$

88. $\int \frac{(x+1)e^x}{\cos^2(xe^x)}\,dx$

89. $\int \frac{e^{\sqrt{x}}\cos(e^{\sqrt{x}})}{\sqrt{x}}\,dx$

90. $\int \sqrt{e^x-1}\,dx$

91. $\int \frac{dx}{(x-\sqrt{x})}$

92. $\int \frac{\sec^2(2\tan^{-1}x)}{(1+x^2)}\,dx$

93. $\left(\frac{1+\sin 2x}{x+\sin^2 x}\right)dx$ **[CBSE 2000]**

94. $\int \left(\frac{1-\tan x}{x+\log\cos x}\right)dx$ **[CBSE 2000]**

95. $\int \frac{(1+\cot x)}{(x+\log\sin x)}\,dx$ **[CBSE 2000]**

96. $\int \frac{\tan x\sec^2 x}{(1-\tan^2 x)}\,dx$

97. $\int \frac{\sin(2\tan^{-1}x)}{(1+x^2)}\,dx$ **[CBSE 2002]**

98. $\int \frac{dx}{(x^{1/2}+x^{1/3})}$

99. $\int (\sin^{-1}x)^2\,dx$ **[CBSE 2004]**

100. $\int \frac{2x\tan^{-1}x^2}{(1+x^4)}\,dx$ **[CBSE 2006]**

101. $\int \frac{(x^2+1)}{(x^4+1)}\,dx$ **[CBSE 2006C, '07]**

102. $\int \frac{(\sin x + \cos x)}{\sqrt{\sin 2x}} dx$ **[CBSE 2009C, '10C]**

ANSWERS (EXERCISE 13A)

1. $\frac{(2x+9)^6}{12} + C$ **2.** $\frac{-(7-3x)^5}{15} + C$ **3.** $\frac{2}{9}(3x-5)^{3/2} + C$

4. $\frac{1}{2}\sqrt{4x+3} + C$ **5.** $-\frac{1}{2}\sqrt{3-4x} + C$ **6.** $\frac{-1}{\sqrt{2x-3}} + C$

7. $\frac{1}{2}e^{(2x-1)} + C$ **8.** $-\frac{1}{3}e^{(1-3x)} + C$ **9.** $\frac{-3^{(2-3x)}}{3 \log 3} + C$

10. $\frac{-\cos 3x}{3} + C$ **11.** $\frac{1}{6}\sin(5+6x) + C$ **12.** $\frac{1}{\sqrt{2}}\sin^2 x + C$

13. $-\frac{1}{2}\cot(2x+5) + C$ **14.** $\frac{1}{2}\sin^2 x + C$ **15.** $\frac{1}{4}\sin^4 x + C$

16. $-\frac{2}{3}(\cos x)^{3/2} + C$ **17.** $\frac{1}{2}(\sin^{-1} x)^2 + C$ **18.** $-\frac{1}{2}\cos(2\tan^{-1} x) + C$

19. $\sin(\log x) + C$ **20.** $-\cot(\log x) + C$ **21.** $\log|\log x| + C$

22. $\frac{1}{3}(x + \log x)^3 + C$ **23.** $\frac{1}{3}(\log x)^3 + C$ **24.** $2\sin\sqrt{x} + C$

25. $e^{\tan x} + C$ **26.** $-e^{\cos^2 x} + C$ **27.** $\frac{\sin^2(ax+b)}{2a} + C$

28. $\frac{\sin 3x}{12} + \frac{3}{4}\sin x + C$ **29.** $e^{-1/x} + C$ **30.** $-\sin\left(\frac{1}{x}\right) + C$

31. $\tan^{-1}(e^x) + C$ **32.** $\frac{1}{2}\log|e^{2x} - 2| + C$ **33.** $\frac{1}{2}[\log(\sin x)]^2 + C$

34. $\log|\log(\sin x)| + C$ **35.** $-\cos(x^2+1) + C$

36. $\frac{1}{2}[\log(\sec x + \tan x)]^2 + C$ **37.** $\tan^2\sqrt{x} + C$ **38.** $\frac{1}{4}(\tan^{-1} x^2)^2 + C$

39. $\frac{1}{4}(\sin^{-1} x^2)^2 + C$ **40.** $\log|\sin^{-1} x| + C$ **41.** $\frac{2}{3}(2 + \log x)^{3/2} + C$

42. $\log|1 + \tan x| + C$ **43.** $-\log|1 + \cos x| + C$

44. $-\log|\cos x - \sin x| + C$

45. (i) $\log|x + \log(\sec x)| + C$ (ii) $\log|x + \cos^2 x| + C$

46. $\frac{1}{b^2}\log|a^2 + b^2\sin^2 x| + C$ **47.** $\frac{1}{(b^2 - a^2)}\log|a^2\cos^2 x + b^2\sin^2 x| + C$

48. $\log|3\cos x + 2\sin x| + C$ **49.** $\log|2x^2 + 3| + C$

50. $\frac{1}{2}\log|x^2 + 2x - 3| + C$ **51.** $\log|2x^2 - 5x + 1| + C$

52. $\log|3x^3-2x^2+5x+1|+C$ **53.** $\log|\log\tan x|+C$

54. $\dfrac{-1}{2(x+\sin x)^2}+C$ **55.** $\dfrac{1}{(1+\cos x)}+C$ **56.** $2\sqrt{x^2+3x-2}+C$

57. $2\sqrt{x^2-x-1}+C$ **58.** $\dfrac{2}{3(a-b)}[(x+a)^{3/2}-(x+b)^{3/2}]+C$

59. $\dfrac{1}{18}[(1-3x)^{3/2}+(5-3x)^{3/2}]+C$ **60.** $\dfrac{1}{3}\tan^{-1}x^3+C$

61. $\dfrac{1}{4}\tan^{-1}x^4+C$ **62.** $\dfrac{1}{2}\tan^{-1}x^2+C$

63. $\dfrac{2}{9}(1+x^3)^{3/2}-\dfrac{2}{3}(1+x^3)^{1/2}+C$ **64.** $\dfrac{2}{3}(1+x)^{3/2}-2\sqrt{1+x}+C$

65. $\dfrac{1}{2}\sec^{-1}x^2+C$ **66.** $\dfrac{2}{5}(x-1)^{5/2}+\dfrac{2}{3}(x-1)^{3/2}+C$

67. $\dfrac{4}{3}(1+x)^{3/2}-\dfrac{2}{5}(1+x)^{5/2}+C$ **68.** $\dfrac{1}{3}(x^2-1)^{3/2}+C$

69. $\dfrac{2}{45}(3x-2)^{5/2}+\dfrac{4}{27}(3x-2)^{3/2}+C$ **70.** $\tan(1+\log x)+C$

71. $-\dfrac{1}{3}\cos x^3+C$ **72.** $\dfrac{2}{3}(x^2+4x+3)^{3/2}+C$

73. $\dfrac{1}{2}x+\dfrac{1}{2}\log|\sin x-\cos x|+C$ **74.** $\dfrac{1}{2}x-\dfrac{1}{2}\log|\sin x-\cos x|+C$

75. $\dfrac{1}{2}x+\dfrac{1}{2}\log|\sin x-\cos x|+C$ **76.** $\log|\sin x+\cos x|+C$

77. $\dfrac{-1}{(\sin x+\cos x)}+C$ **78.** $\dfrac{1}{3}(x+\log x)^3+C$ **79.** $\dfrac{1}{8}\sin^4 x^2+C$

80. $\sin^{-1}(\tan x)+C$ **81.** $\dfrac{1}{2}\cot(2e^{-x}+5)+C$ **82.** $\dfrac{\sec^3(x^2+3)}{3}+C$

83. $\dfrac{-2}{b^2}\left[\log|a+b\cos x|+\dfrac{a}{(a+b\cos x)}\right]+C$ **84.** $\dfrac{-\log|3-5x|}{5}+C$

85. $\dfrac{2}{3}(1+x)^{3/2}+C$ **86.** $\dfrac{1}{3}\sin(e^{x^3})+C$ **87.** $\dfrac{e^{m\tan^{-1}x}}{m}+C$

88. $\tan(xe^x)+C$ **89.** $2\sin(e^{\sqrt{x}})+C$ **90.** $2\sqrt{e^x-1}-2\tan^{-1}\sqrt{e^x-1}+C$

91. $2\log|\sqrt{x}-1|+C$ **92.** $\dfrac{1}{2}\tan(2\tan^{-1}x)+C$ **93.** $\log|x+\sin^2 x|+C$

94. $\log|x+\log\cos x|+C$ **95.** $\log|x+\log\sin x|+C$

96. $-\dfrac{1}{2}\log(1-\tan^2 x)+C$ **97.** $-\dfrac{1}{2}\cos(2\tan^{-1}x)+C$

98. $2\sqrt{x}-3x^{1/3}+6x^{1/6}-6\log|1+x^{1/6}|+C$

99. $x(\sin^{-1}x)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+C$ **100.** $\dfrac{1}{2}(\tan^{-1}x^2)^2+C$

101. $\frac{1}{\sqrt{2}}\tan^{-1}\left\{\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right)\right\}+C$ **102.** $\tan^{-1}\left\{\frac{\tan x-1}{\sqrt{2\tan x}}\right\}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 13A)

12. $I=\int \sin x\sqrt{2\cos^2 x}\,dx=\sqrt{2}\int \sin x\cos x\,dx.$ Put $\sin x=t$ and $\cos x\,dx=dt$.

22. $I=\int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx.$ Put $(x+\log x)=t$ and $\left(1+\frac{1}{x}\right)dx=dt$

24. Put $\sqrt{x}=t$. 25. Put $\tan x=t$. 26. Put $\cos^2 x=t$. 27. Put $\sin(ax+b)=t$.

28. $\cos^3 x=\frac{3}{4}\cos x+\frac{1}{4}\cos 3x.$ 29. Put $\frac{-1}{x}=t$. 32. Put $(e^{2x}-2)=t$.

33. Put $\log(\sin x)=t$. 35. Put $(x^2+1)=t$. 36. Put $\log(\sec x+\tan x)=t$.

37. Put $\tan\sqrt{x}=t$. 38. Put $\tan^{-1}x^2=t$.

39. Put $\sin^{-1}x^2=t$. 40. Put $\sin^{-1}x=t$.

41. Put $(2+\log x)=t$. 44. $I=\int\frac{(\cos x+\sin x)}{(\cos x-\sin x)}\,dx.$ Put $(\cos x-\sin x)=t$.

45. to 53. In each of these questions, put the denominator equal to t.

60. Put $x^3=t$. 63. Put $(1+x^3)=t^2$. Then, $I=\frac{2}{3}\int\frac{t(t^2-1)}{\sqrt{t}}\,dt=\frac{2}{3}\int(t^{5/2}-t^{1/2})\,dt.$

64. Put $(1+x)=t^2$. 65. Multiply num. and denom. by x and put $x^2=t$.

73. $I=\int\frac{(\sin x-\cos x)+(\sin x+\cos x)}{2(\sin x-\cos x)}\,dx.$

74. $I=\int\frac{-\cos x}{(\sin x-\cos x)}\,dx=\int\frac{(\sin x-\cos x)-(\sin x+\cos x)}{2(\sin x-\cos x)}\,dx.$

76. $I=\int\frac{(\cos^2 x-\sin^2 x)}{(\sin x+\cos x)^2}\,dx=\int\frac{(\cos x-\sin x)}{(\sin x+\cos x)}\,dx.$

77. $I=\int\frac{(\cos x-\sin x)}{(\sin x+\cos x)^2}\,dx.$ Put $(\sin x+\cos x)=t$.

78. $I=\int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx.$ Put $(x+\log x)=t$. 79. Put $\sin x^2=t$.

80. Put $\tan x=t$. 81. Put $(2e^{-x}+5)=t$. 82. Put $\sec(x^2+3)=t$.

83. Put $(a+b\cos x)=t$. Then, $\cos x=\frac{(t-a)}{b}$ and $-\sin x\,dx=\frac{1}{b}\,dt.$

$\therefore\ I=-\frac{2}{b}\cdot\int\frac{1}{t^2}\left(\frac{t-a}{b}\right)dt=-\frac{2}{b^2}\cdot\int\left\{\frac{1}{t}-\frac{a}{t^2}\right\}dt.$

86. Put $e^{x^3}=t$. 88. Put $(xe^x)=t$. 89. Put $e^{\sqrt{x}}=t$.

90. Put $(e^x - 1) = t^2$ and $dx = \frac{2t}{(t^2+1)} dt$. Then,

$$I = \int \frac{2t^2}{(1+t^2)} dt = 2\int \left(1 - \frac{1}{1+t^2}\right) dt.$$

91. $I = \int \frac{dx}{\sqrt{x}(\sqrt{x}-1)}$. Put $\sqrt{x} - 1 = t$ and $\frac{1}{\sqrt{x}} dx = 2dt$.

92. Put $2\tan^{-1} x = t$.

96. Put $(1 - \tan^2 x) = t$

97. Put $\tan^{-1} x = t$ and $\frac{1}{(1+x^2)} dx = dt$.

98. $I = \int \frac{dx}{x^{1/3}(1+x^{1/6})}$. Now put $x = t^6$ and $dx = 6t^5 dt$.

Then, $I = 6\int \frac{t^3}{1+t} dt = 6 \cdot \int \left(t^2 - t + 1 - \frac{1}{1+t}\right) dt.$

101. Divide num. and denom. by x^2. Put $\left(x - \frac{1}{x}\right) = t$.

$\therefore \quad I = \int \frac{dt}{\{t^2 + (\sqrt{2})^2\}}$.

OBJECTIVE QUESTIONS I

Mark (✓) against the correct answer in each of the following:

1. $\int (2x+3)^5 dx = ?$

(a) $\frac{(2x+3)^6}{6} + C$ (b) $\frac{(2x+3)^4}{8} + C$ (c) $\frac{(2x+3)^6}{12} + C$ (d) none of these

2. $\int (3-5x)^7 dx = ?$

(a) $-5(3-5x)^6 + C$ (b) $\frac{(3-5x)^8}{-40} + C$

(c) $\frac{-5(3-5x)^8}{8} + C$ (d) none of these

3. $\int \frac{1}{(2-3x)^4} dx = ?$

(a) $\frac{1}{15(2-3x)^5} + C$ (b) $\frac{1}{-12(2-3x)^3} + C$

(c) $\frac{1}{9(2-3x)^3} + C$ (d) none of these

4. $\int \sqrt{ax+b}\, dx = ?$

(a) $\frac{2(ax+b)^{3/2}}{3a} + C$ (b) $\frac{3(ax+b)^{3/2}}{2a} + C$

(c) $\frac{1}{2\sqrt{ax+b}} + C$ (d) none of these

5. $\int \sec^2(7-4x)\,dx = ?$

(a) $\frac{1}{4}\tan(7-4x)+C$ (b) $\frac{-1}{4}\tan(7-4x)+C$

(c) $4\tan(7-4x)+C$ (d) $-4\tan(7-4x)+C$

6. $\int \cos 3x\,dx = ?$

(a) $-\frac{1}{3}\sin 3x+C$ (b) $\frac{1}{3}\sin 3x+C$ (c) $3\sin 3x+C$ (d) $-3\sin 3x+C$

7. $\int e^{(5-3x)}dx = ?$

(a) $-3e^{(5-3x)}+C$ (b) $\frac{1}{3}e^{(5-3x)}+C$ (c) $\frac{e^{(5-3x)}}{-3}+C$ (d) none of these

8. $\int 2^{(3x+4)}dx = ?$

(a) $\frac{3}{(\log 2)}\cdot 2^{(3x+4)}+C$ (b) $\frac{2^{(3x+4)}}{3(\log 2)+C}$

(c) $\frac{2^{(3x+4)}}{2(\log 3)}+C$ (d) none of these

9. $\int \tan^2\frac{x}{2}\,dx = ?$

(a) $\tan\frac{x}{2}-x+C$ (b) $\tan\frac{x}{2}+x+C$

(c) $2\tan\frac{x}{2}+x+C$ (d) $2\tan\frac{x}{2}-x+C$

10. $\int \sqrt{1-\cos x}\,dx = ?$

(a) $-\sqrt{2}\cos\frac{x}{2}+C$ (b) $-2\sqrt{2}\cos\frac{x}{2}+C$ (c) $\frac{-1}{2}\cos\frac{x}{2}+C$ (d) $\frac{-1}{\sqrt{2}}\cos\frac{x}{2}+C$

11. $\int \sqrt{1+\sin x}\,dx = ?$

(a) $-\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)+C$ (b) $\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)+C$

(c) $-2\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)+C$ (d) none of these

12. $\int \sin^3 x\,dx = ?$

(a) $-\frac{3}{4}\cos x+\frac{\cos 3x}{12}+C$ (b) $\frac{3}{4}\cos x+\frac{\cos 3x}{12}+C$

(c) $-\frac{3}{4}\cos x-\frac{\cos 3x}{12}+C$ (d) none of these

13. $\int \frac{\log x}{x} dx = ?$

(a) $\frac{1}{2}(\log x)^2 + C$ (b) $-\frac{1}{2}(\log x)^2 + C$

(c) $\frac{2}{x^2} + C$ (d) $\frac{-2}{x^2} + C$

14. $\int \frac{\sec^2(\log x)}{x} dx = ?$

(a) $\log(\tan x) + C$ (b) $-\log(\tan x) + C$

(c) $\tan(\tan x) + C$ (d) $-\tan(\log x) + C$

15. $\int \frac{1}{x(\log x)} dx = ?$

(a) $\log |x| + C$ (b) $\frac{-2}{x^2} + C$ (c) $(\log x)^2 + C$ (d) $\log |\log x| + C$

16. $\int e^{x^3} x^2 dx = ?$

(a) $e^{x^3} + C$ (b) $\frac{1}{3}e^{x^3} + C$ (c) $\frac{1}{6}e^{x^3} + C$ (d) none of these

17. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = ?$

(a) $e^{\sqrt{x}} + C$ (b) $\frac{1}{2}e^{\sqrt{x}} + C$ (c) $2e^{\sqrt{x}} + C$ (d) none of these

18. $\int \frac{e^{\tan^{-1} x}}{(1+x^2)} dx = ?$

(a) $\frac{e^{\tan^{-1} x}}{x} + C$ (b) $e^{\tan^{-1} x} + C$ (c) $e^x \tan^{-1} x + C$ (d) none of these

19. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx = ?$

(a) $2\cos \sqrt{x} + C$ (b) $-2\cos \sqrt{x} + C$ (c) $-\frac{\cos \sqrt{x}}{2} + C$ (d) $\frac{\cos \sqrt{x}}{2} + C$

20. $\int (\sqrt{\sin x}) \cos x \, dx = ?$

(a) $\frac{2}{3}(\cos x)^{3/2} + C$ (b) $\frac{3}{2}(\cos x)^{3/2} + C$

(c) $\frac{2}{3}(\sin x)^{3/2} + C$ (d) $\frac{3}{2}(\sin x)^{3/2} + C$

21. $\int \frac{1}{(1+x^2)\sqrt{\tan^{-1} x}} dx = ?$

(a) $\frac{1}{2} \log |\tan^{-1} x| + C$ (b) $2\sqrt{\tan^{-1} x} + C$

(c) $\frac{1}{2\sqrt{\tan^{-1} x}} + C$ (d) none of these

22. $\int \frac{\cot x}{\log(\sin x)} dx = ?$

(a) $\log|\cot x| + C$

(b) $\log|\cot x \operatorname{cosec} x| + C$

(c) $\log|\log \sin x| + C$

(d) none of these

23. $\int \frac{1}{x\cos^2(1+\log x)} dx = ?$

(a) $\tan(1+\log x) + C$

(b) $\cot(1+\log x) + C$

(c) $\sec(1+\log x) + C$

(d) none of these

24. $\int \frac{x^2 \tan^{-1} x^3}{(1+x^6)} dx = ?$

(a) $\frac{1}{3}(\tan^{-1} x^3)^2 + C$

(b) $\log|\tan^{-1} x^3| + C$

(c) $\frac{1}{6}(\tan^{-1} x^3)^2 + C$

(d) none of these

25. $\int \sec^5 x \tan x \, dx = ?$

(a) $5\tan^5 x + C$

(b) $\frac{1}{5}\tan^5 x + C$

(c) $5\log|\cos x| + C$

(d) none of these

26. $\int \operatorname{cosec}^3(2x+1)\cot(2x+1)\, dx = ?$

(a) $\frac{1}{4}\operatorname{cosec}^4(2x+1) + C$

(b) $-\frac{1}{3}\operatorname{cosec}^3(2x+1) + C$

(c) $-\frac{1}{6}\operatorname{cosec}^3(2x+1) + C$

(d) $\frac{1}{2}\operatorname{cosec}(2x+1)\cot(2x+1) + C$

27. $\int \frac{\tan(\sin^{-1} x)}{\sqrt{1-x^2}} dx = ?$

(a) $\log|\sec(\sin^{-1} x)| + C$

(b) $\log|\cos(\sin^{-1} x)| + C$

(c) $\tan(\sin^1 x) + C$

(d) none of these

28. $\int \frac{\tan(\log x)}{x} dx = ?$

(a) $x\tan(\log x) + C$

(b) $\log|\tan x| + C$

(c) $\log|\cos(\log x)| + C$

(d) $-\log|\cos(\log x)| + C$

29. $\int e^x \cot(e^x)\, dx = ?$

(a) $\cot(e^x) + C$

(b) $\log|\sin e^x| + C$

(c) $\log|\operatorname{cosec} e^x| + C$

(d) none of these

30. $\int \frac{e^x}{\sqrt{1+e^x}}\,dx = ?$

(a) $2\sqrt{1+e^x}+C$ (b) $\frac{1}{2}\sqrt{1+e^x}+C$

(c) $\frac{1}{\sqrt{1+e^x}}+C$ (d) none of these

31. $\int \frac{x}{\sqrt{1-x^2}}\,dx = ?$

(a) $\sin^{-1}x+C$ (b) $\sin^{-1}\sqrt{x}+C$ (c) $\sqrt{1-x^2}+C$ (d) $-\sqrt{1-x^2}+C$

32. $\int \frac{e^x(1+x)}{\cos^2(xe^x)}\,dx = ?$

(a) $\tan(xe^x)+C$ (b) $\cot(xe^x)+C$ (c) $ex^x \tan x + C$ (d) none of these

33. $\int \frac{dx}{(e^x+e^{-x})} = ?$

(a) $\cot^{-1}(e^x)+C$ (b) $\tan^{-1}(e^x)+C$

(c) $\log|e^x+1|+C$ (d) none of these

34. $\int \frac{2^x}{1-4^x}\,dx = ?$

(a) $\sin^{-1}(2^x)+C$ (b) $(\log e^2)\sin^{-1}(2^x)+C$

(c) $(\log e^2)\cos^{-1}(2^x)+C$ (d) $(\log_2 e)\sin^{-1}(2^x)+C$

35. $\int \frac{dx}{(e^x-1)} = ?$

(a) $\log|e^x-1|+C$ (b) $\log|1-e^{-x}|+C$

(c) $\log|e^x-1|+C$ (d) none of these

36. $\int \frac{1}{(\sqrt{x}+x)}\,dx = ?$

(a) $\log|1+\sqrt{x}|+C$ (b) $2\log|1+\sqrt{x}|+C$

(c) $\frac{1}{\sqrt{x}}\tan^{-1}\sqrt{x}+C$ (d) none of these

37. $\int \frac{dx}{(1+\sin x)} = ?$

(a) $\tan x+\sec x+C$ (b) $\tan x-\sec x+C$

(c) $\frac{1}{2}\tan\frac{x}{2}+C$ (d) none of these

38. $\int \frac{\sin x}{(1+\sin x)}\,dx = ?$

(a) $x+\tan x-\sec x+C$ (b) $x-\tan x-\sec x+C$

(c) $x-\tan x+\sec x+C$ (d) none of these

39. $\int \frac{\sin x}{(1-\sin x)}\, dx = ?$

(a) $-x+\sec x-\tan x+C$
(b) $x+\cos x-\sin x+C$
(c) $-\log|1-\sin x|+C$
(d) none of these

40. $\int \frac{dx}{(1+\cos x)} = ?$

(a) $\frac{1}{2}\tan\frac{x}{2}+C$
(b) $-\cot\frac{x}{2}+C$
(c) $\tan\frac{x}{2}+C$
(d) none of these

41. $\int \frac{dx}{(1-\cos x)} = ?$

(a) $\frac{1}{(x-\sin x)}+C$
(b) $\log|x-\sin x|+C$
(c) $\log\left|\tan\frac{x}{2}\right|+C$
(d) $-\cot\frac{x}{2}+C$

42. $\int \left\{\frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right\} dx = ?$

(a) $2\log\left|\sec\frac{x}{2}\right|+C$
(b) $2\log\left|\cos\frac{x}{2}\right|+C$
(c) $2\log\left|\sec\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|+C$
(d) $2\log\left|\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|+C$

43. $\int \sqrt{e^x}\, dx = ?$

(a) $\sqrt{e^x}+C$
(b) $2\sqrt{e^x}+C$
(c) $\frac{1}{2}\sqrt{e^x}+C$
(d) none of these

44. $\int \frac{\cos x}{(1+\cos x)}\, dx = ?$

(a) $x+\tan\frac{x}{2}+C$
(b) $-x+\tan\frac{x}{2}+C$
(c) $x-\tan\frac{x}{2}+C$
(d) none of these

45. $\int \sec^2 x\,\mathrm{cosec}^2 x\, dx = ?$

(a) $\tan x-\cot x+C$
(b) $\tan x+\cot x+C$
(c) $-\tan x+\cot x+C$
(d) none of these

46. $\int \frac{(1-\cos 2x)}{(1+\cos 2x)}\,dx = ?$

(a) $\tan x + x + C$ (b) $\tan x - x + C$ (c) $-\tan x + x + C$ (d) none of these

47. $\int \frac{(1+\cos x)}{(1-\cos x)}\,dx = ?$

(a) $-2\cot \frac{x}{2} - x + C$ (b) $-2\cot \frac{x}{2} + x + C$

(c) $2\cot \frac{x}{2} + x + C$ (d) none of these

48. $\int \frac{1}{\sin^2 x \cos^2 x}\,dx = ?$

(a) $\tan x + \cot x + C$ (b) $\tan x - \cot x + C$

(c) $-\tan x + \cot x + C$ (d) none of these

49. $\int \frac{\cos 2x}{\cos^2 x \sin^2 x}\,dx = ?$

(a) $\cot x + \tan x + C$ (b) $-\cot x + \tan x + C$

(c) $\cot x - \tan x + C$ (d) $-\cot x - \tan x + C$

50. $\int \frac{(\cos 2x - \cos 2\alpha)}{(\cos x - \cos \alpha)}\,dx = ?$

(a) $\sin x + x\cos\alpha + C$ (b) $2\sin x + x\cos\alpha + C$

(c) $2\sin x + 2x\cos\alpha + C$ (d) none of these

51. $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\} dx = ?$

(a) $2x^2 + C$ (b) $\frac{x^2}{2} + C$ (c) $\frac{2}{(1+x^2)} + C$ (d) none of these

52. $\int \tan^{-1}(\sec x + \tan x)\,dx = ?$

(a) $\frac{\pi x}{4} + \frac{x^2}{4} + C$ (b) $\frac{\pi x}{4} - \frac{x^2}{4} + C$ (c) $\frac{1}{(1+x^2)} + C$ (d) none of these

53. $\int \frac{(1+\sin x)}{(1-\sin x)}\,dx = ?$

(a) $2\tan x + x - 2\sec x + C$ (b) $2\tan x - x + 2\sec x + C$

(c) $2\tan x - x - 2\sec x + C$ (d) none of these

54. $\int \frac{x^4}{(1+x^2)}\,dx = ?$

(a) $\frac{x^3}{3} + x + \tan^{-1}x + C$ (b) $\frac{-x^3}{3} + x - \tan^{-1}x + C$

(c) $\frac{x^3}{3} - x + \tan^{-1}x + C$ (d) none of these

55. $\int \frac{\sin(x-\alpha)}{\sin(x+\alpha)}\,dx = ?$

(a) $x\cos 2\alpha - \sin 2\alpha \cdot \log|\sin(x+\alpha)| + C$
(b) $x\cos 2\alpha + \sin 2\alpha \cdot \log|\sin(x+\alpha)| + C$
(c) $x\cos 2\alpha + \sin \alpha \cdot \log|\sin(x+\alpha)| + C$
(d) none of these

56. $\int \frac{1}{(\sqrt{x+3}-\sqrt{x+2})}\,dx = ?$

(a) $\frac{2}{3}(x+3)^{3/2} - \frac{2}{3}(x+2)^{3/2} + C$
(b) $\frac{2}{3}(x+3)^{3/2} + \frac{2}{3}(x+2)^{3/2} + C$
(c) $\frac{3}{2}(x+3)^{3/2} - \frac{3}{2}(x+2)^{3/2} + C$
(d) none of these

57. $\int \frac{(1+\tan x)}{(1-\tan x)}\,dx = ?$

(a) $-\log|\cos x - \sin x| + C$
(b) $\log|\cos x - \sin x| + C$
(c) $\log|\cos x + \sin x| + C$
(d) none of these

58. $\int \frac{3x^2}{(1+x^6)}\,dx = ?$

(a) $\sin^{-1} x^3 + C$
(b) $\cos^{-1} x^3 + C$
(c) $\tan^{-1} x^3 + C$
(d) $\cot^{-1} x^3 + C$

59. $\int \frac{dx}{x\sqrt{x^6-1}} = ?$

(a) $\frac{1}{3}\sec^{-1} x^3 + C$
(b) $\frac{1}{3}\operatorname{cosec}^{-1} x^3 + C$
(c) $\frac{1}{3}\cot^{-1} x^3 + C$
(d) none of these

60. $\int \{(2x+1)\sqrt{x^2+x+1}\}\,dx = ?$

(a) $\frac{3}{2}(x^2+x+1)^{3/2} + C$
(b) $\frac{2}{3}(x^2+x+1)^{3/2} + C$
(c) $\frac{3}{2}(2x+1)^{3/2} + C$
(d) none of these

61. $\int \frac{dx}{\{\sqrt{2x+3}+\sqrt{2x-3}\}} = ?$

(a) $\frac{1}{18}(2x+3)^{3/2} + \frac{1}{18}(2x-3)^{3/2} + C$
(b) $\frac{1}{18}(2x+3)^{3/2} - \frac{1}{18}(2x-3)^{3/2} + C$
(c) $\frac{1}{12}(2x+3)^{3/2} - \frac{1}{12}(2x-3)^{3/2} + C$
(d) none of these

62. $\int \tan x\,dx = ?$

(a) $\log|\cos x| + C$
(b) $-\log|\cos x| + C$
(c) $\log|\sin x| + C$
(d) $-\log|\sin x| + C$

63. $\int \sec x\, dx = ?$

(a) $\log|\sec x - \tan x| + C$ (b) $-\log|\sec x + \tan x| + C$

(c) $\log|\sec x + \tan x| + C$ (d) none of these

64. $\int \operatorname{cosec} x\, dx = ?$

(a) $\log|\operatorname{cosec} x - \cot x| + C$ (b) $-\log|\operatorname{cosec} x - \cot x| + C$

(c) $\log|\operatorname{cosec} x + \cot x| + C$ (d) none of these

65. $\int \frac{(1+\sin x)}{(1+\cos x)}\, dx = ?$

(a) $\tan\frac{x}{2} + 2\log\left|\cos\frac{x}{2}\right| + C$ (b) $-\tan\frac{x}{2} + 2\log\left|\cos\frac{x}{2}\right| + C$

(c) $\tan\frac{x}{2} - 2\log\left|\cos\frac{x}{2}\right| + C$ (d) none of these

66. $\int \frac{\tan x}{(\sec x + \cos x)}\, dx = ?$

(a) $\tan^{-1}(\cos x) + C$ (b) $-\tan^{-1}(\cos x) + C$

(c) $\cot^{-1}(\cos x) + C$ (d) none of these

67. $\int \sqrt{\frac{1+x}{1-x}}\, dx = ?$

(a) $\sin^{-1}x + \sqrt{1-x^2} + C$ (b) $\sin^{-1}x + (1+x^2) + C$

(c) $\sin^{-1}x - \sqrt{1-x^2} + C$ (d) none of these

68. $\int \frac{1}{x^2} e^{-1/x}\, dx = ?$

(a) $e^{-1/x} + C$ (b) $-e^{-1/x} + C$ (c) $\frac{e^{-1/x}}{x} + C$ (d) none of these

69. $\int \frac{x^3}{(1+x^8)}\, dx = ?$

(a) $\tan^{-1}x^4 + C$ (b) $4\tan^{-1}x^4 + C$

(c) $\frac{1}{4}\tan^{-1}x^4 + C$ (d) none of these

70. $\int \frac{(x+1)(x+\log x)^2}{x}\, dx = ?$

(a) $\frac{1}{3}(x+\log x)^3 + C$ (b) $\frac{x^2}{2} + x + C$

(c) $\frac{x^3}{3} + \frac{x^2}{2} + x + C$ (d) none of these

71. $\int \frac{2x \tan^{-1} x^2}{(1+x^4)} dx = ?$

(a) $(\tan^{-1} x^2)^2 + C$ (b) $2\tan^{-1} x^2 + C$

(c) $\frac{1}{2} (\tan^{-1} x^2)^2 + C$ (d) none of these

72. $\int \frac{dx}{(2-3x)} = ?$

(a) $-3 \log |2-3x| + C$ (b) $-\frac{1}{3} \log |2-3x| + C$

(c) $-\log |2-3x| + C$ (d) none of these

73. $\int x\sqrt{x^2-1}\, dx = ?$

(a) $\frac{2}{3}(x^2-1)^{3/2} + C$ (b) $\frac{1}{3}(x^2-1)^{3/2} + C$

(c) $\frac{1}{\sqrt{x^2-1}} + C$ (d) none of these

74. $\int 3^{(5-3x)} dx = ?$

(a) $\frac{-3^{(5-3x)}}{3(\log 3)} + C$ (b) $\frac{3^{(4-3x)}}{(\log 3)} + C$

(c) $-3^{(5-3x)} \log 3 + C$ (d) none of these

75. $\int e^{\tan x} \sec^2 x\, dx = ?$

(a) $e^{\tan x} + \tan x + C$ (b) $e^{\tan x} \cdot \tan x + C$

(c) $e^{\tan x} + C$ (d) none of these

76. $\int e^{\cos^2 x} \sin 2x\, dx = ?$

(a) $e^{\cos^2 x} + C$ (b) $-e^{\cos^2 x} + C$ (c) $e^{\sin^2 x} + C$ (d) none of these

77. $\int x \sin^3 x^2 \cos x^2 dx = ?$

(a) $\frac{1}{4}\sin^4 x^2 + C$ (b) $\frac{1}{8}\sin^4 x^2 + C$ (c) $\frac{1}{2}\sin^4 x^2 + C$ (d) none of these

78. $\int \frac{e^{\sqrt{x}} \cos (e^{\sqrt{x}})}{\sqrt{x}} dx = ?$

(a) $\sin (e^{\sqrt{x}}) + C$ (b) $\frac{1}{2} \sin (e^{\sqrt{x}}) + C$

(c) $2\sin (e^{\sqrt{x}}) + C$ (d) none of these

79. $\int x^2 \sin x^3 dx = ?$

(a) $\cos x^3 + C$ (b) $-\cos x^3 + C$ (c) $-\frac{1}{3}\cos x^3 + C$ (d) none of these

80. $\int \frac{(x+1)e^x}{\cos^2(xe^x)}dx = ?$

(a) $\tan(xe^x) + C$ (b) $-\tan(xe^x) + C$

(c) $\cot(xe^x) + C$ (d) none of these

81. $\int \frac{1}{x\sqrt{x^4-1}}dx = ?$

(a) $\sec^{-1}x^2 + C$ (b) $\frac{1}{2}\sec^{-1}x^2 + C$ (c) $\text{cosec}^{-1}x^2 + C$ (d) none of these

82. $\int x\sqrt{x-1}\,dx = ?$

(a) $\frac{2}{3}(x-1)^{3/2} + C$ (b) $\frac{2}{5}(x-1)^{5/2} + C$

(c) $\frac{2}{5}(x-1)^{5/2} + \frac{3}{2}(x-1)^{3/2} + C$ (d) none of these

83. $\int x\sqrt{x^2-x}\,dx = ?$

(a) $\frac{1}{3}(x^2-1)^{3/2} + C$ (b) $\frac{2}{3}(x^2-1)^{3/2} + C$

(c) $\frac{1}{\sqrt{x^2-1}} + C$ (d) none of these

84. $\int \frac{dx}{(1+\sqrt{x})} = ?$

(a) $\sqrt{x} - \log|1+\sqrt{x}| + C$ (b) $\sqrt{x} + \log|1+\sqrt{x}| + C$

(c) $2\sqrt{x} - 2\log|1+\sqrt{x}| + C$ (d) none of these

85. $\int \sqrt{e^x-1}\,dx$

(a) $\frac{3}{2}(e^x-1)^{3/2} + C$ (b) $\frac{1}{2}(e^x-1)^{1/2} + C$

(c) $\frac{2}{3}(e^x-1)^{3/2} + C$ (d) none of these

86. $\int \frac{\sin x}{(\sin x - \cos x)}dx = ?$

(a) $\frac{1}{2}x - \frac{1}{2}\log|\sin x - \cos x| + C$ (b) $\frac{1}{2}x + \frac{1}{2}\log|\sin x - \cos x| + C$

(c) $\log|\sin x - \cos x| + C$ (d) none of these

87. $\int \frac{dx}{(1-\tan x)} = ?$

(a) $\frac{1}{2}\log|\sin x - \cos x| + C$ (b) $\frac{1}{2}x + \frac{1}{2}\log|\sin x - \cos x| + C$

(c) $\frac{1}{2}x - \frac{1}{2}\log|\sin x - \cos x| + C$ (d) none of these

88. $\int \frac{dx}{(1-\cot x)} = ?$

(a) $\log|\sin x - \cos x| + C$

(b) $\frac{1}{2}\log|\sin x - \cos x| + C$

(c) $\frac{1}{2}x - \frac{1}{2}\log|\sin x - \cos x| + C$

(d) $\frac{1}{2}x + \frac{1}{2}\log|\sin x - \cos x| + C$

89. $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx = ?$

(a) $\sin^{-1}(\tan x) + C$

(b) $\cos^{-1}(\sin x) + C$

(c) $\tan^{-1}(\cos x) + C$

(d) $\tan^{-1}(\sin x) + C$

90. $\int \frac{(x^2+1)}{(x^4+1)}\,dx = ?$

(a) $\frac{1}{\sqrt{2}}\tan^{-1}\left(x - \frac{1}{x}\right) + C$

(b) $\frac{1}{\sqrt{2}}\cot^{-1}\left\{\left(x - \frac{1}{x}\right)\right\} + C$

(c) $\frac{1}{\sqrt{2}}\tan^{-1}\left\{\frac{1}{\sqrt{2}}\left(x - \frac{1}{x}\right)\right\} + C$

(d) none of these

91. $\int \frac{\sin^6 x}{\cos^8 x}\,dx = ?$

(a) $\frac{1}{7}\tan^7 x + C$

(b) $\frac{1}{7}\sec^7 x + C$

(c) $\log|\cos^6 x| + C$

(d) none of these

92. $\int \sec^5 x \tan x\,dx = ?$

(a) $\frac{1}{5}\tan^5 x + C$

(b) $\frac{1}{5}\sec^5 x + C$

(c) $5\log|\cos x| + C$

(d) none of these

93. $\int \tan^5 x\,dx = ?$

(a) $\frac{1}{6}\tan^6 x + C$

(b) $\frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + \log|\sec x| + C$

(c) $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \log|\sec x| + C$

(d) none of these

94. $\int \sin^3 x \cos^3 x\,dx = ?$

(a) $-\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x + C$

(b) $\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$

(c) $\frac{1}{4}\sin^4 x + \frac{1}{6}\cos^6 x + C$

(d) none of these

95. $\int \sec^4 x \tan x \, dx = ?$

(a) $\frac{1}{2}\sec^2 x + \frac{1}{4}\sec^4 x + C$ (b) $\frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$

(c) $\frac{1}{2}\sec x + \log|\sec x + \tan x| + C$ (d) none of these

96. $\int \frac{\log \tan x}{\sin x \cos x} dx = ?$

(a) $\log\{\log(\tan x)\} + C$ (b) $\frac{1}{2}(\log \tan x)^2 + C$

(c) $\log(\sin x \cos x) + C$ (d) none of these

97. $\int \sin^3(2x+1)\, dx = ?$

(a) $\frac{1}{8}\sin^4(2x+1) + C$

(b) $\frac{1}{2}\cos(2x+1) + \frac{1}{3}\cos^3(2x+1) + C$

(c) $-\frac{1}{2}\cos(2x+1) + \frac{1}{6}\cos^3(2x+1) + C$

(d) none of these

98. $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx = ?$

(a) $2\sqrt{\tan x} + C$ (b) $2\sqrt{\cot x} + C$ (c) $2\sqrt{\sec x} + C$ (d) none of these

99. $\int \frac{(\cos + \sin x)}{(1 - \sin 2x)} dx = ?$

(a) $\log|\sin x - \cos x| + C$ (b) $\frac{1}{(\cos x - \sin x)} + C$

(c) $\log|\cos x + \sin x| + C$ (d) none of these

100. $\int \sqrt{e^x - 1}\, dx = ?$

(a) $\frac{2}{3}(e^x - 1)^{3/2} + C$ (b) $\frac{1}{2} \cdot \frac{e^x}{\sqrt{e^x - 1}} + C$

(c) $2\sqrt{e^x - 1} - 2\tan^{-1}\sqrt{e^x - 1} + C$ (d) none of these

101. $\int \frac{dx}{\sqrt{\sin^3 x \cos x}} = ?$

(a) $2\sqrt{\tan x} + C$ (b) $2\sqrt{\cot x} + C$ (c) $-2\sqrt{\tan x} + C$ (d) $\frac{-2}{\sqrt{\tan x}} + C$

ANSWERS (OBJECTIVE QUESTIONS I)

1. (c) **2.** (b) **3.** (c) **4.** (a) **5.** (b) **6.** (b) **7.** (c) **8** (b) **9.** (d) **10.** (b)
11. (c) **12.** (a) **13.** (a) **14.** (c) **15.** (d) **16.** (b) **17.** (c) **18.** (b) **19.** (b) **20.** (c)

21. (b)	**22.** (c)	**23.** (a)	**24.** (c)	**25.** (b)	**26.** (c)	**27.** (a)	**28.** (d)	**29.** (b)	**30.** (a)
31. (d)	**32.** (a)	**33.** (b)	**34.** (d)	**35.** (b)	**36.** (b)	**37.** (b)	**38.** (c)	**39.** (a)	**40.** (c)
41. (d)	**42.** (d)	**43.** (b)	**44.** (c)	**45.** (a)	**46.** (b)	**47.** (a)	**48.** (b)	**49.** (d)	**50.** (c)
51. (b)	**52.** (a)	**53.** (b)	**54.** (c)	**55.** (a)	**56.** (b)	**57.** (a)	**58.** (c)	**59.** (a)	**60.** (b)
61. (b)	**62.** (b)	**63.** (c)	**64.** (a)	**65.** (c)	**66.** (b)	**67.** (c)	**68.** (a)	**69.** (c)	**70.** (a)
71. (c)	**72.** (b)	**73.** (b)	**74.** (a)	**75.** (c)	**76.** (b)	**77.** (b)	**78.** (c)	**79.** (c)	**80.** (a)
81. (b)	**82.** (c)	**83.** (a)	**84.** (c)	**85.** (d)	**86.** (b)	**87.** (c)	**88.** (d)	**89.** (a)	**90.** (c)
91. (a)	**92.** (b)	**93.** (c)	**94.** (b)	**95.** (b)	**96.** (b)	**97.** (c)	**98.** (a)	**99.** (b)	**100.** (c)
101. (d)									

HINTS TO THE GIVEN OBJECTIVE QUESTIONS I

1. Put $(2x+3)=t$ and $2dx=dt$.
2. Put $(3-5x)=t$ and $-5dx=dt$.
3. Put $(2-3x)=t$ and $-3dx=dt$.
4. Put $(ax+b)=t$ and $a\,dx=dt$.

$\therefore \quad I=\frac{1}{a}\int\sqrt{t}\,dt=\frac{1}{a}\cdot\frac{t^{3/2}}{(3/2)}+C.$

5. Put $(7-4x)=t$ and $-4dx=dt$.

$\therefore \quad I=-\frac{1}{4}\int\sec^2 t\,dt=-\frac{1}{4}\tan t+C.$

6. Put $3x=t$ and $3dx=dt$.
7. Put $(5-3x)=t$ and $-3dx=dt$.
8. Put $(3x+4)=t$ and $3dx=dt$.

$\therefore \quad I=\frac{1}{3}\int 2^t\,dt=\frac{1}{3}\cdot\frac{2^t}{\log 2}+C.$

9. $I=\int\left(\sec^2\frac{x}{2}-1\right)dx=\int\sec^2\frac{x}{2}\,dx-\int dx.$

$=2\int\sec^2 t\,dt-x+C,\quad \text{where } \frac{x}{2}=t$

$=2\tan t-x+C=2\tan\frac{x}{2}+C.$

10. $I=\int\sqrt{2\sin^2(x/2)}\,dx=\sqrt{2}\int\sin(x/2)\,dx.$

Put $\frac{x}{2}=t$ and $dx=2dt$.

$\therefore \quad I=2\sqrt{2}\int\sin t\,dt=-2\sqrt{2}\cos t+C.$

11. $(1+\sin x)=1+\cos\left(\frac{\pi}{2}-x\right)=2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right).$

$\therefore \quad I=\sqrt{2}\int\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)dx.$ Put $\left(\frac{\pi}{4}-\frac{x}{2}\right)=t$ and $\frac{-1}{2}dx=dt.$

$I=-2\sqrt{2}\int\cos t\,dt=-2\sqrt{2}\sin t+C.$

12. $\sin 3x = 3\sin x - 4\sin^3 x \Rightarrow \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$.

$\therefore \quad \int \sin^3 x\, dx = \frac{3}{4}\int \sin x\, dx - \frac{1}{4}\int \sin 3x\, dx$

$$= \frac{3}{4}(-\cos x) - \frac{1}{4}\frac{(-\cos 3x)}{3} + C = \frac{\cos 3x}{12} - \frac{3}{4}\cos x + C.$$

13. Put $\log x = t$ and $\frac{1}{x}dx = dt$.

14. Put $\log x = t$ and $\frac{1}{x}dx = dt$.

15. Put $\log x = t$ and $\frac{1}{x}dx = dt$.

16. Put $x^3 = t$ and $3x^2 dx = dt$.

17. Put $\sqrt{x} = t$ and $\frac{1}{2}x^{-1/2}dx = dt \Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$.

18. Put $\tan^{-1} x = t$ and $\frac{1}{(1+x^2)}dx = dt$.

19. Put $\sqrt{x} = t$ and $\frac{1}{\sqrt{x}}dx = 2dt$.

$\therefore \quad I = 2\int \sin dt = -2\cos t + C = -2\cos\sqrt{x} + C.$

20. Put $\sin x = t$ and $\cos x\, dx = dt$.

$\therefore \quad I = \int \sqrt{t}\, dt = \frac{t^{3/2}}{(3/2)} + C = \frac{2}{3}(\sin x)^{3/2} + C.$

21. Put $\tan^{-1} x = t$ and $\frac{1}{(1+x^2)}dx = dt$.

$\therefore \quad I = \int \frac{1}{\sqrt{t}}dt = 2\sqrt{t} + C.$

22. Put $\log(\sin x) = t$ and $\cot x\, dx = dt$.

23. Put $(1+\log x) = t$ and $\frac{1}{x}dx = dt$.

$\therefore \quad I = \int \frac{1}{\cos^2 t}dt = \int \sec^2 t\, dt = \tan t + C.$

24. Put $\tan^{-1} x^3 = t$ and $\frac{3x^2}{(1+x^6)}dx = dt$.

$\therefore \quad I = \frac{1}{3}\int t\, dt = \frac{1}{6}t^2 + C = \frac{1}{6}(\tan^{-1} x^3)^2 + C.$

25. Put $\sec x = t$ and $\sec x \tan x\, dx = dt$.

$\therefore \quad I = \int \sec^4 x(\sec x \tan x)\, dx = \int t^4 dt = \frac{t^5}{5} + C.$

26. Put $(2x+1) = t$ and $dx = \frac{1}{2}dt$.

$\therefore \quad I = \frac{1}{2}\int \operatorname{cosec}^2 t(\operatorname{cosec} t \cot t)\, dt$

$$= -\frac{1}{2}\int u^2 du, \text{ where } \operatorname{cosec} t = u$$

$$= -\frac{1}{6}u^3 + C = -\frac{1}{6}\operatorname{cosec}^3 t + C = \frac{-1}{6}\operatorname{cosec}^3(2x+1) + C.$$

27. Put $\sin^{-1} x = t$ and $\frac{1}{\sqrt{1-x^2}}\,dx = dt$.

$\therefore \quad I = \int \tan t\, dt = \log|\sec t| + C.$

28. Put $\log x = t$ and $\frac{1}{x}\,dx = dt$.

$\therefore \quad I = \int \tan t\, dt = -\log|\cos t| + C$

$= -\log|\cos(\log x)| + C.$

29. Put $e^x = t$ and $e^x\, dx = dt$.

$\therefore \quad I = \int \cot t\, dt = \log|\sin t| + C.$

30. Put $(1 + e^x) = t$ and $e^x\, dx = dt$.

$\therefore \quad I = \int \frac{1}{\sqrt{t}}\,dt = 2\sqrt{t} + C.$

31. Put $(1 - x^2) = t$ and $-2x\, dx = dt$.

$\therefore \quad I = -\frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt = -\frac{1}{2}\int t^{-1/2}\,dt = -\frac{1}{2}\cdot\frac{\sqrt{t}}{(1/2)} + C = -\sqrt{t} + C.$

32. Put $xe^x = t$ and $(xe^x + e^x)\,dx = dt$.

$\therefore \quad I = \int \frac{1}{\cos^2 t}\,dt = \int \sec^2 t\, dt = \tan t + C.$

33. $I = \int \frac{dx}{\left(e^x + \frac{1}{e^x}\right)} = \int \frac{e^x}{(1 + e^{2x})}\,dx.$

$= \int \frac{1}{(1+t^2)}\,dt$, where $e^x = t$ and $e^x\, dx = dt$.

$= \tan^{-1}(t) + C = \tan^{-1}(e^x) + C.$

34. Put $2^x = t$ and $2^x(\log 2)\,dx + dt$.

$\therefore \quad I = \int \frac{1}{(\log 2)}\cdot\frac{1}{\sqrt{1-t^2}}\,dt = \frac{1}{(\log_e 2)}\sin^{-1}(t) + C$

$= (\log_2 e)\sin^{-1}(2^x) + C.$

35. $I = \int \frac{dx}{e^x(e^x(1-e^{-x})} = \int \frac{e^{-x}}{(1-e^{-x})}\,dx.$

Put $(1 - e^{-x}) = t$ and $e^{-x}dx = dt$

$\therefore \quad I = \int \frac{1}{t}\,dt = \log|t| + C + \log|1 - e^{-x}| + C.$

36. $I = \int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$ $\cdot$ Put $(1 + \sqrt{x}) = t$ and $\frac{1}{2\sqrt{x}}\,dx = dt$.

$\therefore \quad I = 2\cdot\int \frac{1}{t}\,dt = 2\log|t| + C.$

37. $I = \int \left\{ \frac{1}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)} \right\} dx = \int \frac{(1-\sin x)}{(1-\sin^2 x)} dx.$

$= \int \frac{(1-\sin x)}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx$

$= \int (\sec^2 x - \sec x \tan x)\, dx = \tan x - \sec x + C.$

38. $I = \int \left\{ \frac{\sin x}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)} \right\} dx = \int \frac{\sin x(1-\sin x)}{\cos^2 x} dx.$

$= \int \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right) dx = \int (\sec x \tan x - \tan^2 x)\, dx$

$= \int \sec x \tan x\, dx - \int (\sec^2 x - 1)\, dx = \sec x - \tan x + x + C.$

39. $I = \int \left\{ \frac{\sin x}{(1-\sin x)} \times \frac{(1+\sin x)}{(1+\sin x)} \right\} dx = \int \left(\frac{\sin x + \sin^2 x}{\cos^2 x} \right) dx.$

$= \int \left\{ \frac{\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} \right\} dx = \int (\sec x \tan x + \tan^2 x)\, dx$

$= \int (\sec x \tan x + \sec^2 x - 1)\, dx = \sec x + \tan x - x + C.$

40. $I = \int \frac{dx}{2\cos^2(x/2)} = \frac{1}{2} \int \sec^2 \left(\frac{x}{2} \right) dx.$

$= \int \sec^2 t\, dt$, where $\frac{x}{2} = t$ and $dx = 2dt$

$= \tan t + C = \tan \frac{x}{2} + C.$

41. $I = \int \frac{dx}{2\sin^2(x/2)} = \frac{1}{2} \int \text{cosec}^2 \left(\frac{x}{2} \right) dx.$

$= \int \text{cosec}^2 t\, dt$, where $\frac{x}{2} = t$ and $dx = 2dt$

$= -\cot t + C = -\cot(x/2) + C.$

42. $I = \int \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) dx.$ Put $\frac{\pi}{4} - \frac{x}{2} = t$ and $-\frac{1}{2} dx = dt.$

$\therefore \quad I = -2 \int \tan t\, dt = 2 \log |\cos t| + C = 2 \log \left| \cos \left(\frac{\pi}{4} - \frac{x}{2} \right) \right| + C.$

43. Put $e^x = t$ and $e^x dx = dt \Rightarrow dx = \frac{1}{t} dt.$

$\therefore \quad I = \int \frac{\sqrt{t}}{t} dt = \int \frac{1}{\sqrt{t}} dt = \int t^{-1/2} dt = \frac{t^{1/2}}{(1/2)} + C = 2\sqrt{t} + C = 2\sqrt{e^x} + C.$

44. $I = \int \left\{ \frac{(1+\cos x) - 1}{(1+\cos x)} \right\} = \int \left\{ 1 - \frac{1}{(1+\cos x)} \right\} dx.$

$= \int dx - \int \frac{dx}{(1+\cos x)} = x - \int \frac{dx}{2\cos^2(x/2)} = x - \frac{1}{2} \int \sec^2 \frac{x}{2}\, dx$

$$= x - \frac{1}{2} \times 2 \times \int \sec^2 t \, dt, \quad \text{where } \frac{x}{2} = t$$

$$= x - \tan t + C = x - \tan \frac{x}{2} + C.$$

45. $I = \int \sec^2 x (1 + \cot^2 x) \, dx = \int \sec^2 x \, dx + \int \sec^2 x \cot^2 x \, dx.$

$$= \tan x + \int \frac{1}{\cos^2 x} \cdot \frac{\cos^2 x}{\sin^2 x} \, dx = \tan x + \int \operatorname{cosec}^2 x \, dx$$

$$= \tan x - \cot x + C.$$

46. $I = \int \frac{2\sin^2 x}{2\cos^2 x} \, dx = \int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C.$

47. $I = \frac{2\cos^2 (x/2)}{2\sin^2 (x/2)} \, dx = \int \cot^2 \frac{x}{2} \, dx = \int \left(\operatorname{cosec}^2 \frac{x}{2} - 1 \right) dx.$

$$= \int \operatorname{cosec}^2 \frac{x}{2} \, dx - \int dx = 2\int \operatorname{cosec}^2 t \, dt - x + C$$

$$= -2\cot t - x + C = -2\cot \frac{x}{2} - x + C.$$

48. $I = \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} \, dx = \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} \, dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} \, dx.$

$$= \int \frac{1}{\cos^2 x} \, dx + \int \frac{1}{\sin^2 x} \, dx = \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx = \tan x - \cot x + C.$$

49. $I = \int \frac{(\cos^2 x - \sin^2 x)}{\cos^2 x \sin^2 x} \, dx = \int \frac{\cos^2 x}{\cos^2 x \sin^2 x} \, dx - \int \frac{\sin^2 x}{\cos^2 x \sin^2 x} \, dx.$

$$= \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2 x} = \int \operatorname{cosec}^2 x \, dx - \int \sec^2 x \, dx$$

$$= -\cot x - \tan x + C.$$

50. $I = \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{(\cos x - \cos \alpha)} \, dx = 2\int \frac{(\cos^2 x - \cos^2 \alpha)}{(\cos x - \cos \alpha)} \, dx.$

$$= 2\int (\cos x + \cos \alpha) \, dx = \int 2\cos x \, dx + 2\cos \alpha \cdot \int dx$$

$$= 2\sin x + 2x \cos \alpha + C.$$

51. $I = \int \tan^{-1} \sqrt{\frac{2\sin^2 x}{2\cos^2 x}} \, dx = \int \tan^{-1}(\tan x) \, dx.$

$$= \int x \, dx = \frac{x^2}{2} + C.$$

52. $\tan^{-1}(\sec x + \tan x) = \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x} \right) = \tan^{-1}\left\{ \frac{1 + \sin x}{\cos x} \right\}.$

$$= \tan^{-1}\left\{ \frac{1 - \cos\left(\frac{\pi}{2} + x\right)}{\sin\left(\frac{\pi}{2} + x\right)} \right\} = \tan^{-1}\left\{ \frac{2\sin^2\left(\frac{\pi}{4} + \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)} \right\}$$

$$= \tan^{-1}\left\{ \tan\left(\frac{\pi}{4} + \frac{x}{2} \right) \right\} = \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

$$\therefore \quad I = \int \left(\frac{\pi}{4} + \frac{x}{2} \right) dx = \frac{\pi x}{4} + \frac{x^2}{4} + C.$$

53. $I = \int \frac{(1+\sin x)}{(1-\sin x)} \times \frac{(1+\sin x)}{(1+\sin x)}\, dx.$

$$= \int \frac{(1+\sin x)^2}{(1-\sin^2 x)}\, dx = \int \frac{1+\sin^2 x + 2\sin x}{\cos^2 x}\, dx$$

$$= \int \left\{ \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} \right\} dx$$

$$= \int (\sec^2 x + \tan^2 x + 2\sec x \tan x)\, dx$$

$$= \int (2\sec^2 x - 1 + 2\sec x \tan x)\, dx = 2\tan x - x + 2\sec x + C.$$

54. On dividing x^4 by (x^2+1), we get:

$$I = \int \left\{ (x^2 - 1) + \frac{1}{(x^2+1)} \right\} dx = \frac{x^3}{3} - x + \tan^{-1} x + C.$$

55. $\frac{\sin(x-\alpha)}{\sin(x+\alpha)} = \frac{\sin(x+\alpha-2\alpha)}{\sin(x+\alpha)} = \frac{\sin(x+\alpha)\cos 2\alpha - \cos(x+\alpha)\sin 2\alpha}{\sin(x+\alpha)}.$

$$= \cos 2\alpha - \sin 2\alpha \cot(x+\alpha)$$

$$\therefore \quad I = \int \{\cos 2\alpha - \sin 2\alpha \cdot \cot(x+\alpha)\}\, dx$$

$$= (\cos 2\alpha) \cdot \int dx - \sin 2\alpha \cdot \int \cot(x+\alpha)\, dx$$

$$= x\cos 2\alpha - \sin 2\alpha \cdot \log|\sin(x+\alpha)| + C.$$

56. $I = \int \frac{1}{\{\sqrt{x+3} - \sqrt{x+2}\}} \times \frac{\{\sqrt{x+3} + \sqrt{x+2}\}}{\{\sqrt{x+3} + \sqrt{x+2}\}}\, dx.$

$$= \int \left\{ \frac{\sqrt{x+3} + \sqrt{x+2}}{(x+3) - (x+2)} \right\} dx = \int \sqrt{x+3}\, dx + \int \sqrt{x+2}\, dx$$

$$= \frac{(x+3)^{3/2}}{(3/2)} + \frac{(x+2)^{3/2}}{(3/2)} + C = \frac{2}{3}(x+3)^{3/2} + \frac{2}{3}(x+2)^{3/2} + C.$$

57. $I = \int \frac{\left(1 + \frac{\sin x}{\cos x}\right)}{\left(1 - \frac{\sin x}{\cos x}\right)}\, dx = \int \frac{(\cos x + \sin x)}{(\cos x - \sin x)}\, dx.$

$$= -\int \frac{dt}{t}, \text{ where } (\cos x - \sin x) = t$$

$$= -\log|t| + C = -\log|\cos x - \sin x| + C.$$

58. Putting $x^3 = t$ and $3x^2 dx = dt$, we get:

$$I = \int \frac{dt}{(1+t^2)} = \tan^{-1} t + C = \tan^{-1} x^3 + C.$$

59. $I = \int \frac{x^2}{x^3\sqrt{x^6 - 1}}\, dx.$

$= \frac{1}{3}\int \frac{dt}{t\sqrt{t^2 - 1}}$, where $x^3 = t$

$= \frac{1}{3}\sec^{-1} t + C = \frac{1}{3}\sec^{-1} x^3 + C.$

60. Put $x^2 + x + 1 = t$ and $(2x + 1)\, dx = dt$.

$I = \int \sqrt{t}\, dt = \frac{t^{3/2}}{(3/2)} + C = \frac{2}{3}(x^2 + x + 1)^{3/2} + C.$

61. $I = \int \frac{1}{\{\sqrt{2x+3} + \sqrt{2x-3}\}} \times \frac{\{\sqrt{2x+3} - \sqrt{2x-3}\}}{\{\sqrt{2x+3} - \sqrt{2x-3}\}}\, dx.$

$= \int \frac{\{\sqrt{2x+3} - \sqrt{2x-3}\}}{\{(2x+3) - (2x-3)\}}\, dx = \frac{1}{6}\int \{\sqrt{2x+3} - \sqrt{2x-3}\}\, dx$

$= \frac{1}{18}(2x+3)^{3/2} - \frac{1}{18}(2x+3)^{3/2} + C.$

62. $I = \int \frac{\sin x}{\cos x}\, dx = \int \frac{-dt}{t}$, where $\cos x = t$.

$= -\log|t| + C = -\log|\cos x| + C.$

63. $I = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\, dx = \int \frac{1}{t}\, dt$, where $(\sec x + \tan x) = t$.

$= \log|t| + C = \log|\sec x + \tan x| + C.$

64. $I = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)}\, dx = \int \frac{1}{t}\, dt$, where $(\operatorname{cosec} x - \cot x) = t$.

$= \log|t| + C = \log|\operatorname{cosec} x - \cot x| + C.$

65. $I = \int \frac{1}{(1 + \cos x)}\, dx + \int \frac{\sin x}{(1 + \cos x)}\, dx.$

$= \int \frac{dx}{2\cos^2(x/2)} + \int \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\, dx$

$= \frac{1}{2}\int \sec^2 \frac{x}{2}\, dx + \int \tan\frac{x}{2}\, dx$

$= \int \sec^2 t\, dt + 2\int \tan t\, dt$, where $\frac{x}{2} = t$

$= \tan t - 2\log|\cos t| + C = \tan\frac{x}{2} - 2\log\left|\cos\frac{x}{2}\right| + C.$

66. $I = \int \frac{\sin x}{(1 + \cos^2 x)}\, dx = -\int \frac{1}{(1 + t^2)}\, dt$, where $\cos x = t$.

$= -\tan^{-1} t + C = -\tan^{-1}(\cos x) + C.$

67. $I = \int \left\{\frac{\sqrt{1+x}}{\sqrt{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}}\right\} dx = \int \frac{(1+x)}{\sqrt{1-x^2}}\, dx$

$= \int \frac{1}{\sqrt{1-x^2}}\, dx - \frac{1}{2}\int \frac{2x}{\sqrt{1-x^2}}\, dx = \sin^{-1} x - \frac{1}{2}\int \frac{1}{\sqrt{t}}\, dt$, where $(1 - x^2) = t$

$= \sin^{-1} x - \sqrt{t} + C = \sin^{-1} x - \sqrt{1 - x^2} + C.$

68. Put $-\frac{1}{x} = t$ and $\frac{1}{x^2}\,dx = dt$.

$\therefore \quad I = \int e^t dt = e^t + C = e^{-1/x} + C.$

69. Put $x^4 = t$ and $4x^3\,dx = dt$.

$\therefore \quad I = \frac{1}{4}\int \frac{dt}{(1+t^2)} = \frac{1}{4}\tan^{-1}t + C = \frac{1}{4}\tan^{-1}x^4 + C.$

70. Put $(x + \log x) = t$ and $\left(1 + \frac{1}{x}\right)dx = dt$.

$\therefore \quad I = \int \left(\frac{x+1}{x}\right)\cdot(x + \log x)^2\,dx = \int \left(1 + \frac{1}{x}\right)(x + \log x)^2\,dx$

$= \int t^2 dt = \frac{t^3}{3} + C = \frac{1}{3}(x + \log x)^3 + C.$

71. Put $\tan^{-1}x^2 = t$ and $\frac{2x}{(1+x^4)}\,dx = dt$.

$\therefore \quad I = \int t\,dt = \frac{t^2}{2} + C = \frac{1}{2}(\tan^{-1}x^2)^2 + C.$

72. Put $(2 - 3x) = t$ and $-3dx = dt$.

$\therefore \quad I = -\frac{1}{3}\int \frac{1}{t}\,dt = -\frac{1}{3}\log|t| + C = -\frac{1}{3}\log|2 - 3x| + C.$

73. Put $(x^2 - 1) = t$ and $2x\,dx = dt$.

$\therefore \quad I = \frac{1}{2}\int \sqrt{t}\,dt = \frac{1}{2}\cdot\frac{t^{3/2}}{(3/2)} + C = \frac{1}{3}t^{3/2} + C = \frac{1}{3}(x^2 - 1)^{3/2} + C.$

74. Put $5 - 3x = t$ and $-3dx = dt$.

$\therefore \quad I = -\frac{1}{3}\int 3^t dt = -\frac{1}{3}\times\frac{3^t}{\log 3} + C = -\frac{1}{3}\times\frac{3^{(5-3x)}}{\log 3} + C.$

75. Put $\tan x = t$ and $\sec^2 x\,dx = dt$.

$\therefore \quad I = \int e^t dt = e^t + C = e^{\tan x} + C.$

76. Put $\cos^2 x = t$ and $-2\cos x \sin x\,dx = dt$.

$\therefore \quad I = -\int e^t dt = -e^t + C = -e^{\cos^2 x} + C.$

77. Put $\sin x^2 = t$ and $2x\cos x^2 dx = dt$.

$\therefore \quad I = \frac{1}{2}\int t^3 dt = \frac{1}{2}\times\frac{t^4}{4} + C = \frac{t^4}{8} + C = \frac{1}{8}\sin^4 x^2 + C.$

78. Put $e^{\sqrt{x}} = t$ and $e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\,dx = dt$.

$\therefore \quad I = 2\int \cos t\,dt + C = 2\sin t + C = 2\sin(e^{\sqrt{x}}) + C.$

79. Put $x^3 = t$ and $3x^2 dx = dt$.

$I = \frac{1}{3}\int \sin t\,dt = \frac{1}{3}(-\cos t) + C = -\frac{1}{3}\cos x^3 + C.$

80. Put $xe^x = t$ and $(xe^x + e^x \cdot 1)\, dx = dt$, i.e., $(x+1)e^x\, dx = dt$.

$$\therefore \quad I = \int \frac{1}{\cos^2 t}\, dt = \int \sec^2 t\, dt = \tan t + C = \tan(xe^x) + C.$$

81. $$I = \int \frac{x}{x^2\sqrt{x^4 - 1}}\, dx = \frac{1}{2}\int \frac{dt}{t\sqrt{t^2 - 1}}, \text{ where } x^2 = t.$$

$$= \frac{1}{2}\sec^{-1} t + C = \frac{1}{2}\sec^{-1} x^2 + C.$$

82. $$I = \int \{(x - 1 + 1)\sqrt{x - 1}\}\, dx = \int \{(x-1)^{3/2} + (x-1)^{1/2}\}\, dx.$$

$$= \frac{2}{5}(x-1)^{5/2} + \frac{3}{2}(x-1)^{3/2} + C.$$

83. Put $x^2 - 1 = t$ and $2x\, dx = dt$.

$$I = \frac{1}{2}\int \sqrt{t}\, dt = \frac{1}{2}\cdot\frac{t^{3/2}}{3/2} + C = \frac{1}{3}t^{3/2} + C = \frac{1}{3}(x^2 - 1)^{3/2} + C.$$

84. Put $\sqrt{x} = t$, so that $x = t^2$ and $dx = 2t\, dt$.

$$\therefore \quad I = \int \frac{dx}{(1 + \sqrt{x})} = \int \frac{2t}{(1+t)}\, dt = \int \frac{\{2(1+t) - 2\}}{(1+t)}\, dt$$

$$= \int \left\{2 - \frac{2}{(1+t)}\right\} dt = 2t - 2\log|1+t| + C$$

$$= 2\sqrt{x} - 2\log|1 + \sqrt{x}| + C.$$

85. Put $(e^x - 1) = t^2$. Then $e^x\, dx = 2t\, dt$. So, $dx = \dfrac{2t}{(t^2+1)}\cdot$

$$\therefore \quad I = \int \frac{2t^2}{(t^2+1)}\, dt = 2\int \frac{\{(t^2+1) - 1\}}{(t^2+1)}\, dt = 2\int \left(1 - \frac{1}{t^2+1}\right) dt$$

$$= 2[t - \tan^{-1} t] + C = 2[\sqrt{e^x - 1} - \tan^{-1}\{\sqrt{e^x - 1}\} + C.$$

86. $$I = \int \frac{(\sin x - \cos x) + (\cos x + \sin x)}{2(\sin x - \cos x)}\, dx.$$

$$= \int \frac{1}{2}\, dx + \frac{1}{2}\int \frac{(\cos x + \sin x)}{(\sin x - \cos x)}\, dx = \frac{1}{2}x + \frac{1}{2}\int \frac{1}{t}\, dt, \text{ where } t = (\sin x - \cos x)$$

$$= \frac{1}{2}x + \frac{1}{2}\log|t| + C = \frac{1}{2}x + \frac{1}{2}\log|\sin x - \cos x| + C.$$

87. $$I = \int \frac{\cos x}{(\cos x - \sin x)}\, dx = \int \frac{-\cos x}{(\sin x - \cos x)}\, dx.$$

$$= \int \frac{(\sin x - \cos x) - (\sin x + \cos x)}{2(\sin x - \cos x)}\, dx = \int \left\{\frac{1}{2} - \frac{1}{2}\cdot\frac{(\sin x + \cos x)}{(\sin x - \cos x)}\right\} dx$$

$$= \frac{1}{2}x - \frac{1}{2}\log|\sin x - \cos x| + C.$$

88. $I = \displaystyle\int \frac{\sin x}{(\sin x - \cos x)}\, dx$, which is the same as Q. 86.

89. Put $\tan x = t$ and $\sec^2 x\, dx = dt$.

$\therefore \quad I = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(t) + C = \sin^{-1}(\tan x) + C.$

90. On dividing the numerator and denominator by x^2, we get

$$I = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)} dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left\{\left(x-\frac{1}{x}\right)^2+2\right\}} dx$$

$$= \int \frac{dt}{(t^2+2)}, \text{ where } \left(x-\frac{1}{x}\right) = t \text{ and } \left(1+\frac{1}{x^2}\right) dx = dt$$

$$= \int \frac{dt}{[t^2+(\sqrt{2})^2]} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}} \tan^{-1}\left\{\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right)\right\} + C.$$

91. $I = \int \frac{\sin^6 x}{\cos^6 x} \cdot \frac{1}{\cos^2 x} dx = \int \tan^6 x \sec^2 x\, dx = \int t^6 dt$, where $\tan x = t$.

$$= \frac{t^7}{7} + C = \frac{1}{7} \tan^7 x + C.$$

92. $I = \sec^4 x \cdot \sec x \tan x\, dx = \int t^4 dt$, where $\sec x = t$.

$= \frac{1}{5} t^5 + C = \frac{1}{5} \sec^5 x + C.$

93. $I = \int \tan^3 x \tan^2 x\, dx = \int \tan^3 x (\sec^2 x - 1)\, dx.$

$= \int \tan^3 x \sec^2 x\, dx - \int \tan^3 x\, dx = \int t^3 dt - \int \tan x (\sec^2 x - 1)\, dx$, where $\tan x = t$

$= \frac{t^4}{4} - \int \tan x \sec^2 x\, dx + \int \tan x\, dx$

$= \frac{1}{4} \tan^4 x - \int u\, du + \log|\sec x| + C$, where $\tan x = u$

$= \frac{1}{4} \tan^4 x - \frac{1}{2} u^2 + \log|\sec x| + C = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \log|\sec x| + C.$

94. $I = \int \sin^3 \cos^2 x \cos x\, dx = \int \sin^3 x (1 - \sin^2 x) \cos x\, dx.$

$= \int t^3 (1 - t^2) dt$, where $\sin x = t$

$= \int (t^3 - t^5)\, dt = \frac{t^4}{4} - \frac{t^6}{6} + C = \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C.$

95. $I = \int \sec^2 x \cdot \sec^2 x \cdot \tan x\, dx.$

$= \int \sec^2 x (1 + \tan^2 x) \tan x\, dx = \int (1 + t^2) t\, dt$, where $\tan x = t$

$= \int (t + t^3)\, dt = \frac{t^2}{2} + \frac{t^4}{4} + C = \frac{1}{2} \tan^2 x + \frac{1}{4} \tan^4 x + C.$

96. Put $\log \tan x = t$. Then, $\frac{1}{\tan x} \cdot \sec^2 x\, dx = dt$, i.e., $\frac{1}{\sin x \cos x} dx = dt$.

$\therefore \quad I = \int t\, dt = \frac{1}{2} t^2 + C = \frac{1}{2} (\log \tan x)^2 + C.$

97. Put $(2x+1)=t$ and $dx=\frac{1}{2}\,dt$.

$$\therefore \quad I=\frac{1}{2}\int \sin^3 t\,dt=\frac{1}{2}\int(1-\cos^2 t)\sin t\,dt$$

$$=\frac{1}{2}\int(1-u^2)(-du)=\frac{1}{2}\int(u^2-1)\,du=\frac{1}{2}\left[\frac{u^3}{3}-u\right]+C,\text{ where } \cos t=u$$

$$=\frac{1}{6}u^3-\frac{1}{2}u+C=\frac{1}{6}\cos^3 t-\frac{1}{2}\cos t+C=\frac{1}{6}\cos^3(2x+1)-\frac{1}{2}\cos(2x+1)+C.$$

98. Dividing Nr and Dr by $\cos^2 x$, we get

$$I=\int\frac{\sqrt{\tan x}}{\tan x}\cdot\sec^2 x\,dx=\int\frac{\sec^2 x}{\sqrt{\tan x}}\,dx$$

$$=\int\frac{dt}{\sqrt{t}},\text{ where } \tan x=t$$

$$=2\sqrt{t}+C=2\sqrt{\tan x}+C.$$

99. $$I=\int\frac{(\cos x+\sin x)}{(\cos^2 x+\sin^2 x-2\sin x\cos x)}\,dx$$

$$=\int\frac{(\cos x+\sin x)}{(\sin x-\cos x)^2}\,dx=\int\frac{1}{t^2}\,dt,\text{ where } (\sin x-\cos x)=t$$

$$=\frac{-1}{t}+C=\frac{-1}{(\sin x-\cos x)}+C=\frac{1}{(\cos x-\sin x)}+C.$$

100. Putting $(e^x-1)=t^2$ and $e^x dx=2t\,dt$.

$$\therefore \quad dx=\frac{2t}{(t^2+1)}\,dt.$$

$$\therefore \quad I=\int\frac{2t^2}{(t^2+1)}\,dt=\int\left\{2-\frac{2}{(t^2+1)}\right\}dt=2t-2\tan^{-1}t+C$$

$$=2\sqrt{e^x-1}-2\tan^{-1}\sqrt{e^x-1}+C.$$

101. On dividing Nr and Dr by $\cos^2 x$, we get

$$I=\int\frac{\sec^2 x}{\sqrt{\tan^3 x}}\,dx=\int\frac{dt}{\sqrt{t^3}},\text{ where } \tan x=t.$$

Integration Using Trigonometric Identities

When the integrand consists of trigonometric functions, we use known identities to convert it into a form which can easily be integrated. Some of the identities useful for this purpose are given below:

(i) $2\sin^2\left(\frac{x}{2}\right)=(1-\cos x)$ (ii) $2\cos^2\left(\frac{x}{2}\right)=(1+\cos x)$

(iii) $2\sin a\cos b=\sin(a+b)+\sin(a-b)$

(iv) $2\cos a\sin b=\sin(a+b)-\sin(a-b)$

(v) $2\cos a \cos b = \cos(a+b) + \cos(a-b)$

(vi) $2\sin a \sin b = \cos(a-b) - \cos(a+b)$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:* (i) $\int \sin^2 \frac{x}{2}\,dx$ (ii) $\int \tan^2 \frac{x}{2}\,dx$

(iii) $\int \cos^2 nx\,dx$ (iv) $\int \cos^5 x\,dx$

(v) $\int \sin^7 x\,dx$ (vi) $\int \sin^3(2x+1)\,dx$

SOLUTION (i)
$$\int \sin^2 \frac{x}{2}\,dx = \frac{1}{2}\int 2\sin^2 \frac{x}{2}\,dx$$
$$= \frac{1}{2}\int (1-\cos x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos x\,dx$$
$$= \frac{1}{2}x - \frac{1}{2}\sin x + C.$$

(ii)
$$\int \tan^2 \frac{x}{2}\,dx = \int \left(\sec^2 \frac{x}{2} - 1\right) dx = \int \sec^2 \frac{x}{2}\,dx - \int dx$$
$$= 2\int \sec^2 t\,dt - \int dx, \text{ where } \frac{x}{2} = t$$
$$= 2\tan t - x + C = 2\tan \frac{x}{2} - x + C.$$

(iii)
$$\int \cos^2 nx\,dx = \frac{1}{2}\int 2\cos^2 nx\,dx$$
$$= \frac{1}{2}\int (1+\cos 2nx)\,dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2nx\,dx$$
$$= \frac{x}{2} + \frac{1}{4n}\sin 2nx + C.$$

(iv)
$$\int \cos^5 x\,dx = \int \cos^4 x \cdot \cos x\,dx$$
$$= \int (1-\sin^2 x)^2 \cdot \cos x\,dx = \int (1-t^2)^2\,dt, \text{ where } \sin x = t$$
$$= \int (1+t^4-2t^2)\,dt = \int dt + \int t^4 dt - 2\int t^2 dt$$
$$= t + \frac{t^5}{5} - \frac{2t^3}{3} + C = \sin x + \frac{1}{5}\sin^5 x - \frac{2}{3}\sin^3 x + C.$$

(v)
$$\int \sin^7 x\,dx = \int \sin^6 x \cdot \sin x\,dx$$
$$= \int (1-\cos^2 x)^3 \sin x\,dx$$
$$= -\int (1-t^2)^3\,dt, \text{ where } \cos x = t$$
$$= \int (t^6 - 3t^4 + 3t^2 - 1)\,dt = \frac{t^7}{7} - \frac{3t^5}{5} + t^3 - t + C$$
$$= \frac{1}{7}\cos^7 x - \frac{3}{5}\cos^5 x + \cos^3 x - \cos x + C.$$

(vi) $\int \sin^3(2x+1)\,dx = \int \{1-\cos^2(2x+1)\}\cdot \sin(2x+1)\,dx$

$$= -\frac{1}{2}\int (1-t^2)\,dt, \quad \text{where } \cos(2x+1) = t$$

$$= -\frac{1}{2}\int dt + \frac{1}{2}\int t^2 dt = -\frac{1}{2}t + \frac{1}{6}t^3 + C$$

$$= -\frac{1}{2}\cos(2x+1) + \frac{1}{6}\cos^3(2x+1) + C.$$

EXAMPLE 2 *Evaluate* $\int \cos mx \cos nx\,dx$, *when (i)* $m \neq n$ *(ii)* $m = n$.

SOLUTION (i) When $m \neq n$, we have

$$\int \cos mx \cos nx\,dx = \frac{1}{2}\int [\cos(m+n)x + \cos(m-n)x]\,dx$$

$$= \frac{1}{2}\int \cos(m+n)x\,dx + \frac{1}{2}\int \cos(m-n)x\,dx$$

$$= \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C.$$

(ii) When $m = n$, we have

$$\int \cos mx \cos nx\,dx = \int \cos^2 nx\,dx$$

$$= \frac{1}{2}\int 2\cos^2 nx\,dx = \frac{1}{2}\int (1+\cos 2nx)\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2nx\,dx = \frac{x}{2} + \frac{\sin 2nx}{4n} + C.$$

EXAMPLE 3 *Evaluate:*

(i) $\int \sin 3x \sin 2x\,dx$ *(ii)* $\int \cos 3x \sin 2x\,dx$

(iii) $\int \cos 4x \cos x\,dx$ **[CBSE 2001]** *(iv)* $\int \sin^3 x \cos^3 x\,dx$

SOLUTION (i) Using $2\sin a \sin b = \cos(a-b) - \cos(a+b)$, we have

$$\int \sin 3x \sin 2x\,dx = \frac{1}{2}\int 2\sin 3x \sin 2x\,dx$$

$$= \frac{1}{2}\int (\cos x - \cos 5x)\,dx$$

$$= \frac{1}{2}\int \cos x\,dx - \frac{1}{2}\int \cos 5x\,dx$$

$$= \frac{1}{2}\sin x - \frac{\sin 5x}{10} + C.$$

(ii) Using $2\cos a \sin b = \sin(a+b) - \sin(a-b)$, we get

$$\int \cos 3x \sin 2x\,dx = \frac{1}{2}\int 2\cos 3x \sin 2x\,dx$$

$$= \frac{1}{2}\int (\sin 5x - \sin x)\,dx$$

$$= \frac{1}{2}\int \sin 5x\,dx - \frac{1}{2}\int \sin\ x\,dx$$

$$= \frac{-\cos 5x}{10} + \frac{\cos x}{2} + C.$$

(iii) Using $2\cos a \cos b = \cos(a+b) + \cos(a-b)$, we get

$$\int \cos 4x \cos x\,dx = \frac{1}{2}\int 2\cos 4x \cos x\,dx$$

$$= \frac{1}{2}\int (\cos 5x + \cos\ 3x)\,dx$$

$$= \frac{1}{2}\int \cos 5x\,dx + \frac{1}{2}\int \cos\ 3x\,dx$$

$$= \frac{\sin 5x}{10} + \frac{\sin\ 3x}{6} + C.$$

(iv) $\int \sin^3 x \cos^3 x\,dx = \int \sin^3 x \cos^2 x\ \cos x\,dx$

$$= \int \sin^3 x(1-\sin^2 x) \cos x\,dx$$

$$= \int t^3(1-t^2)\,dt, \quad \text{where } \sin\ x = t$$

$$= \int t^3 dt - \int t^5 dt = \frac{t^4}{4} - \frac{t^6}{6} + C$$

$$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C.$$

EXAMPLE 4 *Evaluate* $\int \cos x \cos 2x \cos 3x\,dx.$

SOLUTION $\int \cos x \cos 2x \cos 3x\,dx$

$$= \frac{1}{2}\int (2\cos x \cos 2x) \cos 3x\,dx$$

$$= \frac{1}{2}\int (\cos 3x + \cos x) \cos 3x\,dx = \frac{1}{2}\int (\cos^2 3x + \cos x \cos 3x)\,dx$$

$$= \frac{1}{4}\int (2\cos^2 3x)\,dx + \frac{1}{4}\int (2\cos x \cos 3x)\,dx$$

$$= \frac{1}{4}\int (1 + \cos 6x)\,dx + \frac{1}{4}\int (\cos 4x + \cos 2x)\,dx$$

$$= \frac{1}{4}\int dx + \frac{1}{4}\int \cos 6x\,dx + \frac{1}{4}\int \cos 4x\,dx + \frac{1}{4}\int \cos 2x\,dx$$

$$= \frac{1}{4}x + \frac{1}{4}\cdot\frac{\sin 6x}{6} + \frac{1}{4}\cdot\frac{\sin 4x}{4} + \frac{1}{4}\cdot\frac{\sin 2x}{2} + C$$

$$= \frac{x}{4} + \frac{\sin 6x}{24} + \frac{\sin 4x}{16} + \frac{\sin 2x}{8} + C.$$

EXAMPLE 5 *Evaluate* $\int \sec^4 x \tan x \, dx$.

SOLUTION $\int \sec^4 x \tan x \, dx = \int \sec^2 x \cdot \sec^2 x \tan x \, dx$

$$= \int (1 + \tan^2 x) \sec^2 x \tan x \, dx$$

$$= \int (1 + t^2) t \, dt, \text{ where } \tan x = t$$

$$= \int t \, dt + \int t^3 dt = \frac{t^2}{2} + \frac{t^4}{4} + C$$

$$= \frac{1}{2} \tan^2 x + \frac{1}{4} \tan^4 x + C.$$

EXAMPLE 6 *Evaluate* $\int \sin^4 x \, dx$. **[CBSE 2004]**

SOLUTION $\int \sin^4 x \, dx = \frac{1}{4} \int (2 \sin^2 x)^2 dx$

$$= \frac{1}{4} \int (1 - \cos 2x)^2 dx = \frac{1}{4} \int (1 + \cos^2 2x - 2 \cos 2x) \, dx$$

$$= \frac{1}{8} \int (2 + 2 \cos^2 2x - 4 \cos 2x) \, dx$$

$$= \frac{1}{8} \int [2 + (1 + \cos 4x) - 4 \cos 2x] dx$$

$$= \frac{3}{8} \int dx + \frac{1}{8} \int \cos 4x \, dx - \frac{1}{2} \int \cos 2x \, dx$$

$$= \frac{3}{8} x + \frac{\sin 4x}{32} - \frac{\sin 2x}{4} + C.$$

EXAMPLE 7 *Evaluate* $\int \frac{\sin x}{\sin (x - \alpha)} dx$. **[CBSE 2003C, '04]**

SOLUTION Put $(x - \alpha) = t$ so that $x = (t + \alpha)$ and $dx = dt$.

$$\therefore \quad \int \frac{\sin x}{\sin (x - \alpha)} dx = \int \frac{\sin (t + \alpha)}{\sin t} dt$$

$$= \int \frac{\sin t \cos \alpha + \cos t \sin \alpha}{\sin t} dt$$

$$= \cos \alpha \cdot \int dt + \sin \alpha \cdot \int \cot t \, dt$$

$$= \cos \alpha \cdot t + \sin \alpha \cdot \log | \sin t | + C$$

$$= (\cos \alpha)(x - \alpha) + \sin \alpha \cdot \log | \sin (x - \alpha) | + C.$$

EXAMPLE 8 *Evaluate:* (i) $\int \frac{\sin 4x}{\cos 2x} dx$ (ii) $\int \frac{\sin 4x}{\sin x} dx$

SOLUTION (i) $\int \frac{\sin 4x}{\cos 2x} dx = \int \frac{2 \sin 2x \cos 2x}{\cos 2x} dx$

$$= 2 \int \sin 2x \, dx = -\cos 2x + C.$$

(ii) $\int \frac{\sin 4x}{\sin x}\,dx = \int \frac{2\sin 2x \cos 2x}{\sin x}\,dx$

$$= \int \frac{4\sin x \cos x \cos 2x}{\sin x}\,dx = 2\int 2\cos x \cos 2x\,dx$$

$$= 2\int (\cos 3x + \cos x)\,dx = 2\int \cos 3x\,dx + 2\int \cos x\,dx$$

$$= \frac{2\sin 3x}{3} + 2\sin x + C.$$

EXAMPLE 9 *Evaluate* $\int \sqrt{1+\sin x}\,dx.$

SOLUTION $\int \sqrt{1+\sin x}\,dx = \int \sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$

$$= \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)dx = \int \sin\frac{x}{2}\,dx + \int \cos\frac{x}{2}\,dx$$

$$= -2\cos\frac{x}{2} + 2\sin\frac{x}{2} + C.$$

EXAMPLE 10 *Evaluate* $\int \frac{\sin^2 x}{(1+\cos x)^2}\,dx.$

SOLUTION $\int \frac{\sin^2 x}{(1+\cos x)^2}\,dx = \int \left(\frac{\sin x}{1+\cos x}\right)^2 dx = \int \left[\frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right]^2 dx$

$$= \int \tan^2(x/2)\,dx = \int \left(\sec^2\frac{x}{2} - 1\right)dx$$

$$= \int \sec^2(x/2)\,dx - \int dx = 2\tan(x/2) - x + C.$$

EXAMPLE 11 *Evaluate:* *(i)* $\int \sin x\sqrt{1-\cos 2x}\,dx$ *(ii)* $\int \frac{\cos 2x}{\sqrt{1+\cos 4x}}\,dx$

SOLUTION (i) $\int \sin x\sqrt{1-\cos 2x}\,dx$

$$= \int \sin x \cdot \sqrt{2\sin^2 x}\,dx = \sqrt{2}\int \sin^2 x\,dx = \frac{1}{\sqrt{2}}\int 2\sin^2 x\,dx$$

$$= \frac{1}{\sqrt{2}}\int (1-\cos 2x)\,dx = \frac{1}{\sqrt{2}}\int dx - \frac{1}{\sqrt{2}}\int \cos 2x\,dx$$

$$= \frac{1}{\sqrt{2}}x - \frac{\sin 2x}{2\sqrt{2}} + C.$$

(ii) $\int \frac{\cos 2x}{\sqrt{1+\cos 4x}}\,dx = \int \frac{\cos 2x}{\sqrt{2\cos^2 2x}}\,dx = \frac{1}{\sqrt{2}}\int dx = \frac{x}{\sqrt{2}} + C.$

EXAMPLE 12 *Evaluate:* *(i)* $\int \frac{dx}{a\sin x + b\cos x}$ *(ii)* $\int \frac{dx}{\sin x + \cos x}$

SOLUTION (i) Put $a = r\cos\theta$ and $b = r\sin\theta$ so that

$$r^2 = (a^2 + b^2) \quad\text{and}\quad \theta = \tan^{-1}(b/a).$$

$$\therefore \int \frac{dx}{a\sin x + b\cos x} = \int \frac{dx}{r\cos\theta\sin x + r\sin\theta\cos x}$$

$$= \frac{1}{r}\int \frac{dx}{\sin(x+\theta)} = \frac{1}{r}\cdot\int \operatorname{cosec}(x+\theta)\,dx$$

$$= \frac{1}{r}\log\left\{\tan\left(\frac{\theta+x}{2}\right)\right\} + C$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log\left[\tan\left\{\frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right) + \frac{x}{2}\right\}\right] + C.$$

(ii) We can write,

$$\int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}}\int \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\left(\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x\right)}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sin\left(\frac{\pi}{4}+x\right)} = \frac{1}{\sqrt{2}}\int \operatorname{cosec}\left(\frac{\pi}{4}+x\right)dx$$

$$= \frac{1}{\sqrt{2}}\log\tan\left(\frac{\pi}{8}+\frac{x}{2}\right) + C.$$

EXAMPLE 13 *Evaluate:* (i) $\int \frac{dx}{4\cos x + 3\sin x}$ (ii) $\int \frac{dx}{(2\sin x + 3\cos x)^2}$

SOLUTION (i) Put $4 = r\sin\theta$ and $3 = r\cos\theta$ so that

$$r^2 = 25 \text{ and } \theta = \tan^{-1}\left(\frac{4}{3}\right).$$

$$\therefore \int \frac{dx}{4\cos x + 3\sin x} = \int \frac{dx}{r\sin\theta\cos x + r\cos\theta\sin x}$$

$$= \frac{1}{r}\int \frac{dx}{\sin(\theta+x)} = \frac{1}{r}\int \operatorname{cosec}(\theta+x)\,dx$$

$$= \frac{1}{r}\log\left\{\tan\left(\frac{\theta+x}{2}\right)\right\} + C$$

$$= \frac{1}{5}\log\left[\tan\left\{\frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right) + \frac{x}{2}\right\}\right] + C.$$

(ii) Put $2 = r\cos\theta$ and $3 = r\sin\theta$ so that $r^2 = 13$ and $\theta = \tan^{-1}\left(\frac{3}{2}\right)$.

$$\because \int \frac{dx}{(2\sin x + 3\cos x)^2} = \int \frac{dx}{(r\cos\theta\sin x + r\sin\theta\cos x)^2}$$

$$= \frac{1}{r^2}\int \frac{dx}{\sin^2(\theta + x)} = \frac{1}{13}\cdot\int \operatorname{cosec}^2(\theta + x)\,dx$$

$$= -\frac{1}{13}\cot(\theta + x) + C$$

$$= -\frac{1}{13}\cot\left\{\tan^{-1}\left(\frac{3}{2}\right) + x\right\} + C.$$

EXAMPLE 14 *Evaluate* $\displaystyle\int \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^3}dx.$

SOLUTION $\displaystyle\int \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^3}dx = \int \frac{\cos^2(x/2) - \sin^2(x/2)}{\{\cos(x/2) + \sin(x/2)\}^3}dx$

$$= \int \frac{\cos(x/2) - \sin(x/2)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}dx = 2\int \frac{1}{t^2}dt, \text{ where } t = \cos\frac{x}{2} + \sin\frac{x}{2}$$

$$= \frac{-2}{t} + C = \frac{-2}{\cos(x/2) + \sin(x/2)} + C.$$

EXAMPLE 15 *Evaluate* $\displaystyle\int \frac{dx}{\sqrt{1 - \sin x}}.$

SOLUTION $\displaystyle\int \frac{dx}{\sqrt{1 - \sin x}} = \int \frac{dx}{\left[\sin^2(x/2) + \cos^2(x/2) - 2\sin\frac{x}{2}\cos\frac{x}{2}\right]^{1/2}}$

$$= \int \frac{dx}{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)} = \frac{1}{\sqrt{2}}\int \frac{dx}{\left(\frac{1}{\sqrt{2}}\cdot\sin\frac{x}{2} - \cos\frac{x}{2}\cdot\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{1}{\sqrt{2}}\cdot\int \frac{dx}{\left(\sin\frac{x}{2}\cos\frac{\pi}{4} - \cos\frac{x}{2}\sin\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\int \operatorname{cosec}\left(\frac{x}{2} - \frac{\pi}{4}\right)dx = \frac{1}{\sqrt{2}}2\cdot\log\left[\tan\left(\frac{x}{4} - \frac{\pi}{8}\right)\right] + C$$

$$= \sqrt{2}\log\tan\left(\frac{x}{4} - \frac{\pi}{8}\right) + C.$$

EXAMPLE 16 *Evaluate* $\displaystyle\int \frac{\sin x}{\sqrt{1 + \sin x}}dx.$

SOLUTION $\displaystyle\int \frac{\sin x}{\sqrt{1 + \sin x}}dx = \int \frac{(1 + \sin x) - 1}{\sqrt{1 + \sin x}}dx$

$$= \int \sqrt{1 + \sin x}\, dx - \int \frac{dx}{\sqrt{1 + \sin x}}$$

$$= \int \sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}}\, dx$$

$$- \int \frac{dx}{\sqrt{\cos^2 (x/2) + \sin^2 (x/2) + 2\sin (x/2) \cos (x/2)}}$$

$$= \int [\cos (x/2) + \sin (x/2)] dx - \int \frac{dx}{[\cos (x/2) + \sin (x/2)]}$$

$$= \left(2\sin \frac{x}{2} - 2\cos \frac{x}{2}\right) - \frac{1}{\sqrt{2}} \cdot \int \frac{dx}{\frac{1}{\sqrt{2}} \cos \frac{x}{2} + \frac{1}{\sqrt{2}} \sin \frac{x}{2}}$$

$$= \left(2\sin \frac{x}{2} - 2\cos \frac{x}{2}\right) - \frac{1}{\sqrt{2}} \cdot \int \frac{dx}{\sin \left(\frac{x}{2} + \frac{\pi}{4}\right)}$$

$$= \left(2\sin \frac{x}{2} - 2\cos \frac{x}{2}\right) - \frac{1}{\sqrt{2}} \int \operatorname{cosec} \left(\frac{x}{2} + \frac{\pi}{4}\right) dx$$

$$= 2\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) - \frac{1}{\sqrt{2}} \times 2\log \left|\tan \left(\frac{x}{4} + \frac{\pi}{8}\right)\right| + C$$

$$= 2\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) - \sqrt{2}\log \left|\tan \left(\frac{x}{4} + \frac{\pi}{8}\right)\right| + C.$$

EXERCISE 13B

Evaluate the following integrals:

1. (i) $\int \sin^2 x\, dx$ (ii) $\int \cos^2 x\, dx$
2. (i) $\int \cos^2 (x/2)\, dx$ (ii) $\int \cot^2 (x/2)\, dx$
3. (i) $\int \sin^2 nx\, dx$ (ii) $\int \sin^5 x\, dx$
4. $\int \cos^3 (3x + 5)\, dx$

5. $\int \sin^7 (3 - 2x)\, dx$

6. (i) $\int \left(\frac{1 - \cos 2x}{1 + \cos 2x}\right) dx$ (ii) $\int \left(\frac{1 + \cos 2x}{1 - \cos 2x}\right) dx$
7. (i) $\int \left(\frac{1 - \cos x}{1 + \cos x}\right) dx$ (ii) $\int \left(\frac{1 + \cos x}{1 - \cos x}\right) dx$
8. $\int \sin 3x \cos 4x\, dx$
9. $\int \cos 4x \cos 3x\, dx$ **[CBSE 2007]**
10. $\int \sin 4x \sin 8x\, dx$ **[CBSE 2007]**
11. $\int \sin 6x \cos x\, dx$

12. $\int \sin x\sqrt{1+\cos 2x}\, dx$

13. $\int \cos^4 x\, dx$ **[CBSE 2000, '03]**

14. $\int \cos 2x \cos 4x \cos 6x\, dx$

15. $\int \sin^3 x \cos x\, dx$

16. $\int \sec^4 x\, dx$

17. $\int \cos^3 x \sin^4 x\, dx$

18. $\int \cos^4 x \sin^3 x\, dx$

19. $\int \sin^{2/3} x \cos^3 x\, dx$

20. $\int \cos^{3/5} x \sin^3 x\, dx$

21. $\int \operatorname{cosec}^4 2x\, dx$

22. $\int \frac{\cos 2x}{\cos x} dx$

23. $\int \frac{\cos x}{\cos (x+\alpha)} dx$

24. $\int \cos^3 x \sin 2x\, dx$

25. $\int \frac{\cos^9 x}{\sin x} dx$

26. $\int \cos^4 2x\, dx$

27. $\int \frac{\sin^2 x}{(1+\cos x)^2} dx$

28. $\int \frac{dx}{(3\cos x + 4\sin x)}$

29. $\int \frac{dx}{(a\cos x + b\sin x)^2}$, $a > 0$ and $b > 0$

30. $\int \frac{dx}{(\cos x - \sin x)}$

31. $\int (2\tan x - 3\cot x)^2 dx$

32. $\int \sin x \sin 2x \sin 3x\, dx$ **[CBSE 2012]**

33. $\int \left(\frac{1-\cot x}{1+\cot x}\right) dx$

34. $\int \frac{dx}{(2\sin x + \cos x + 3)}$ **[CBSE 2000C, '04]**

ANSWERS (EXERCISE 13B)

1. (i) $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$ (ii) $\frac{1}{2}x + \frac{1}{4}\sin 2x + C$

2. (i) $\frac{1}{2}x + \frac{1}{2}\sin x + C$ (ii) $-2\cot (x/2) - x + C$

3. (i) $\frac{1}{2}x - \frac{\sin 2nx}{4n} + C$ (ii) $-\cos x - \frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} + C$

4. $\frac{1}{3}\sin (3x+5) - \frac{1}{9}\sin^3 (3x+5) + C$

5. $\frac{1}{2}\cos (3-2x) - \frac{1}{14}\cos^7 (3-2x) - \frac{1}{2}\cos^3 (3-2x) + \frac{3}{10}\cos^5 (3-2x) + C$

6. (i) $\tan x - x + C$ (ii) $-\cot x - x + C$

7. (i) $2\tan \frac{x}{2} - x + C$ (ii) $-2\cot \frac{x}{2} - x + C$

8. $\frac{-\cos 7x}{14} + \frac{\cos x}{2} + C$

9. $\frac{\sin 7x}{14} + \frac{1}{2}\sin x + C$

10. $\dfrac{\sin 4x}{8} - \dfrac{\sin 12x}{24} + C$

11. $\dfrac{-\cos 7x}{14} - \dfrac{\cos 5x}{10} + C$

12. $\dfrac{\sin^2 x}{\sqrt{2}} + C$

13. $\dfrac{3x}{8} + \dfrac{\sin 4x}{32} + \dfrac{\sin 2x}{4} + C$

14. $\dfrac{x}{4} + \dfrac{\sin 4x}{16} + \dfrac{\sin 8x}{32} + \dfrac{\sin 12x}{48} + C$

15. $\dfrac{\sin^4 x}{4} + C$

16. $\tan x + \dfrac{1}{3}\tan^3 x + C$

17. $\dfrac{1}{5}\sin^5 x - \dfrac{1}{7}\sin^7 x + C$

18. $\dfrac{1}{7}\cos^7 x - \dfrac{1}{5}\cos^5 x + C$

19. $\dfrac{3}{5}\sin^{5/3} x - \dfrac{3}{11}\sin^{11/3} x + C$

20. $-\dfrac{5}{8}\cos^{8/5} x + \dfrac{5}{18}\cos^{18/5} x + C$

21. $-\dfrac{1}{2}\cot 2x - \dfrac{1}{6}\cot^3 2x + C$

22. $2\sin x - \log|\sec x + \tan x| + C$

23. $x\cos\alpha - \sin\alpha \cdot \log|\cos(x+\alpha)| + C$

24. $-\dfrac{2}{5}\cos^5 x + C$

25. $\log|\sin x| - 2\sin^2 x + \dfrac{3}{2}\sin^4 x - \dfrac{2}{3}\sin^6 x + \dfrac{1}{8}\sin^8 x + C$

26. $\dfrac{3x}{8} + \dfrac{\sin 4x}{8} + \dfrac{\sin 8x}{64} + C$

27. $2\tan\dfrac{x}{2} - x + C$

28. $\dfrac{1}{5}\log\tan\left[\dfrac{1}{2}\tan^{-1}\dfrac{3}{4} + \dfrac{x}{2}\right] + C$

29. $\dfrac{1}{(a^2+b^2)}\tan\left\{x - \tan^{-1}\dfrac{b}{a}\right\} + C$

30. $\dfrac{-1}{\sqrt{2}}\log\left|\tan\left(\dfrac{\pi}{8} - \dfrac{x}{2}\right)\right| + C$

31. $4\tan x - 9\cot x - 25x + C$

32. $\dfrac{1}{24}\cos 6x - \dfrac{1}{16}\cos 4x - \dfrac{1}{8}\cos 2x + C$

33. $-\log|\sin x + \cos x| + C$

34. $\tan^{-1}\left(1 + \tan\dfrac{x}{2}\right) + C$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 13B)

12. $I = \int \sin x \cdot \sqrt{2\cos^2 x}\, dx = \sqrt{2}\int \sin x \cos x\, dx.$ Put $\sin x = t$.

13. $I = \dfrac{1}{4}\int (2\cos^2 x)^2\, dx = \dfrac{1}{4}\int (1+\cos 2x)^2\, dx$

$$= \frac{1}{4}\int\left[(1+2\cos 2x) + \left(\frac{1+\cos 4x)}{2}\right)\right]dx = \int\left[\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right]dx.$$

16. $I = \int (1+\tan^2 x)\sec^2 x\, dx.$ Put $\tan x = t$.

17. $I = \int \cos x(1-\sin^2 x)\sin^4 x\, dx.$ Put $\sin x = t$.

19. $I = \int \sin^{2/3} x(1-\sin^2 x)\cos x\, dx.$ Put $\sin x = t$.

21. $I = \int (1+\cot^2 2x)\operatorname{cosec}^2 2x\, dx.$ Put $\cot 2x = t$.

22. $I = \int \dfrac{(2\cos^2 x - 1)}{\cos x}\, dx = 2\int \cos x\, dx - \int \sec x\, dx.$

24. $I = 2\int \cos^4 x \sin x \, dx$. Put $\cos x = t$.

25. $I = \int \dfrac{\cos^8 x \cos x}{\sin x} dx = \int \dfrac{(1-\sin^2 x)^4 \cos x}{\sin x} dx$. Put $\sin x = t$.

27. $I = \int \dfrac{(1-\cos^2 x)}{(1+\cos x)^2} dx = \int \dfrac{(1-\cos x)}{(1+\cos x)} dx = \int \tan^2 \dfrac{x}{2} dx = \int \left(\sec^2 \dfrac{x}{2} - 1 \right) dx.$

28. Put $3 = r \sin \theta$ and $4 = r \cos \theta$.

29. Put $a = r \sin \theta$ and $b = r \cos \theta$.

30. $I = \dfrac{1}{\sqrt{2}} \cdot \int \dfrac{dx}{\left(\dfrac{1}{\sqrt{2}} \cos x - \dfrac{1}{\sqrt{2}} \sin x \right)} = \dfrac{1}{\sqrt{2}} \int \dfrac{dx}{\sin \left(\dfrac{\pi}{4} - x \right)} = \dfrac{1}{\sqrt{2}} \cdot \int \operatorname{cosec} \left(\dfrac{\pi}{4} - x \right) dx.$

32. $I = \dfrac{1}{4} \int (\sin 2x + \sin 4x - \sin 6x) \, dx.$

33. $I = -\int \dfrac{(\cos x - \sin x)}{(\sin x + \cos x)} dx$. Put $(\sin x + \cos x) = t$.

34. $\tan \dfrac{x}{2} = t,\ dx = \dfrac{2dt}{(1+t^2)};\ \sin x = \dfrac{2t}{(1+t^2)}$ and $\cos x = \dfrac{1-t^2}{1+t^2}\cdot$

Integration by Parts

THEOREM *If u and v are two functions of x then*

$$\int (uv)\, dx = [u \cdot \int v\, dx] - \int \left\{ \frac{du}{dx} \cdot \int v\, dx \right\} dx.$$

PROOF For any two functions $f_1(x)$ and $f_2(x)$, we have

$$\frac{d}{dx}[f_1(x) \cdot f_2(x)] = f_1(x) \cdot f_2'(x) + f_2(x) \cdot f_1'(x).$$

$\therefore$ $\int \{f_1(x) \cdot f_2'(x) + f_2(x) \cdot f_1'(x)\}\, dx = f_1(x) \cdot f_2(x)$

or $\int f_1(x) \cdot f_2'(x)\, dx + \int f_2(x) \cdot f_1'(x)\, dx = f_1(x) \cdot f_2(x)$

or $\int f_1(x) \cdot f_2'(x)\, dx = f_1(x) \cdot f_2(x) - \int f_2(x) \cdot f_1'(x)\, dx.$

Let $f_1(x) = u$ and $f_2'(x) = v$ so that $f_2(x) = \int v\, dx.$

$\therefore$ $\int (uv)\, dx = u \cdot \int v\, dx - \int \left\{ \dfrac{du}{dx} \cdot \int v\, dx \right\} dx.$

We can express this result as given below:

Integral of product of two functions

= (1st function) × (integral of 2nd)

$-\int$ {(derivative of 1st) × (integral of 2nd)} dx.

REMARKS (i) If the integrand is of the form $f(x)\, x^n$, we consider x^n as the first function and $f(x)$ as the second function.

(ii) If the integrand contains a *logarithmic* or an *inverse trigonometric* function, we take it as the first function. In all such cases, if the second function is not given, we take it as 1.

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\int x \sec^2 x\, dx$ *(ii)* $\int x \sin 2x\, dx$

SOLUTION (i) Integrating by parts, taking x as the first function, we get

$$\int x \sec^2 x\, dx = x \cdot \int \sec^2 x\, dx - \int \left\{ \frac{d}{dx}(x) \cdot \int \sec^2 x\, dx \right\} dx$$

$$= x \tan x - \int 1 \cdot \tan x\, dx = x \tan x + \log |\cos x| + C.$$

(ii) Integrating by parts, taking x as the first function, we get

$$\int x \sin 2x\, dx = x \cdot \int \sin 2x\, dx - \int \left\{ \frac{d}{dx}(x) \cdot \int \sin 2x\, dx \right\} dx$$

$$= x \cdot \left(\frac{-\cos 2x}{2} \right) - \int 1 \cdot \left(\frac{-\cos 2x}{2} \right) dx$$

$$= \frac{-x \cos 2x}{2} + \frac{1}{2} \int \cos 2x\, dx$$

$$= \frac{-x \cos 2x}{2} + \frac{1}{2} \cdot \frac{\sin 2x}{2} + C$$

$$= \frac{-x \cos 2x}{2} + \frac{1}{4} \sin 2x + C.$$

EXAMPLE 2 *Evaluate* $\int x^n \log x\, dx$.

SOLUTION Integrating by parts, taking $\log x$ as the first function, we get

$$\int x^n \log x = (\log x) \cdot \int x^n dx - \int \left\{ \frac{d}{dx}(\log x) \cdot \int x^n dx \right\} dx$$

$$= (\log x) \cdot \frac{x^{n+1}}{(n+1)} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{(n+1)} dx$$

$$= \frac{x^{n+1} \log x}{(n+1)} - \frac{1}{(n+1)} \int x^n dx$$

$$= \frac{x^{n+1} \log x}{(n+1)} - \frac{x^{n+1}}{(n+1)^2} + C.$$

EXAMPLE 3 *Evaluate* $\int x^2 \sin x \, dx.$

SOLUTION Integrating by parts, taking x^2 as the first function, we get

$$\int x^2 \sin x \, dx = x^2 \int \sin x \, dx - \int \left[\frac{d}{dx}(x^2) \cdot \int \sin x \, dx \right] dx$$

$$= x^2(-\cos x) - \int 2x(-\cos x) \, dx$$

$$= -x^2 \cos x + 2\int x \cos x \, dx$$

$$= -x^2 \cos x + 2\left[x(\sin x) - \int \left\{ \frac{d}{dx}(x) \cdot \int \cos x \, dx \right\} dx \right]$$

[integrating x cos x by parts]

$$= -x^2 \cos x + 2[x \sin x - \int \sin x \, dx]$$

$$= -x^2 \cos x + 2[x \sin x + \cos x] + C.$$

EXAMPLE 4 *Evaluate* $\int x \cos^2 x \, dx.$

SOLUTION

$$\int x \cos^2 x \, dx = \int x \left(\frac{1 + \cos 2x}{2} \right) dx = \frac{1}{2} \int x \, dx + \frac{1}{2} \int x \cos 2x \, dx$$

$$= \frac{x^2}{4} + \frac{1}{2} \cdot \left[x \cdot \int \cos 2x \, dx - \int \left\{ \frac{d}{dx}(x) \cdot \int \cos 2x \, dx \right\} dx \right]$$

[integrating $x \cos 2x$ by parts]

$$= \frac{x^2}{4} + \frac{1}{2} \left[\frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} dx \right]$$

$$= \frac{x^2}{4} + \frac{x \sin 2x}{4} - \frac{1}{4} \cdot \frac{(-\cos 2x)}{2} + C$$

$$= \frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C.$$

EXAMPLE 5 *Evaluate* $\int \log x \, dx.$

SOLUTION Integrating by parts, taking log x as the first function and 1 as the second function, we get

$$\int \log x \, dx = \int (\log x \cdot 1) \, dx$$

$$= (\log x) \cdot \int 1 \, dx - \int \left\{ \frac{d}{dx}(\log x) \cdot \int 1 \, dx \right\} dx$$

$$= (\log x) \cdot x - \int \left(\frac{1}{x} \cdot x \right) dx = x \log x - \int dx$$

$$= x \log x - x + C = x(\log x - 1) + C.$$

EXAMPLE 6 *Evaluate* $\int \log (1 + x^2) \, dx.$

SOLUTION Integrating by parts, taking $\log (1 + x^2)$ as the first function and 1 as the second function, we get

$$\int \log(1+x^2)\,dx = \int \{\log(1+x^2)\cdot 1\}\,dx$$

$$= \log(1+x^2)\cdot\int 1\,dx - \int\left[\frac{d}{dx}\{\log(1+x^2)\}\cdot\int 1\,dx\right]dx$$

$$= \log(1+x^2)\cdot x - \int \frac{2x}{(1+x^2)}\cdot x\,dx$$

$$= x\log(1+x^2) - 2\int \frac{x^2}{(1+x^2)}\,dx$$

$$= x\log(1+x^2) - 2\int\left(1-\frac{1}{1+x^2}\right)dx$$

$$= x\log(1+x^2) - 2\int dx + 2\int \frac{dx}{(1+x^2)}$$

$$= x\log(1+x^2) - 2x + 2\tan^{-1}x + C.$$

EXAMPLE 7 *Evaluate* $\int (\log x)^2\,dx.$

SOLUTION Integrating by parts, taking $(\log x)^2$ as the first function and 1 as the second function, we get

$$\int (\log x)^2\,dx = \int \{(\log x)^2\cdot 1\}\,dx$$

$$= (\log x)^2\cdot\int 1\,dx - \int\left\{\frac{d}{dx}(\log x)^2\cdot\int 1\,dx\right\}dx$$

$$= x(\log x)^2 - \int\left(\frac{2\log x}{x}\cdot x\right)dx$$

$$= x(\log x)^2 - 2\int(\log x\cdot 1)\,dx$$

$$= x(\log x)^2 - 2\left[(\log x)\int dx - \int\left\{\frac{d}{dx}(\log x)\cdot\int dx\right\}dx\right]$$

$$= x(\log x)^2 - 2\left[x\log x - \int\frac{1}{x}\cdot x\,dx\right]$$

$$= x(\log x)^2 - 2x\log x + 2x + C.$$

EXAMPLE 8 *Evaluate* $\int \frac{\log x}{x^2}\,dx.$

SOLUTION Integrating by parts, taking $\log x$ as the first function and $\frac{1}{x^2}$ as the second function, we get

$$\int \frac{\log x}{x^2}\,dx = \int (\log x)\cdot\frac{1}{x^2}\,dx$$

$$= (\log x)\cdot\int\frac{1}{x^2}\,dx - \int\left\{\frac{d}{dx}(\log x)\cdot\int\frac{1}{x^2}\,dx\right\}dx$$

$$= (\log x)\left(-\frac{1}{x}\right) - \int\frac{1}{x}\cdot\left(-\frac{1}{x}\right)dx = -\frac{\log x}{x} + \int\frac{1}{x^2}\,dx$$

$$= -\frac{\log x}{x} - \frac{1}{x} + C = \frac{-(\log x + 1)}{x} + C.$$

EXAMPLE 9 *Evaluate* $\int e^{2x} \sin x \, dx.$

SOLUTION Integrating by parts, we get

$$\int e^{2x} \sin x \, dx = (e^{2x} \cdot \int \sin x \, dx) - \int \left\{ \frac{d}{dx}(e^{2x}) \cdot \int \sin x \, dx \right\} dx$$

$$= e^{2x} \cdot (-\cos x) - 2\int e^{2x}(-\cos x) \, dx$$

$$= -e^{2x} \cos x + 2\int e^{2x} \cos x \, dx$$

$$= -e^{2x} \cos x + 2\left[(e^{2x} \cdot \int \cos x \, dx) - \int \left\{ \frac{d}{dx}(e^{2x}) \cdot \int \cos x \, dx \right\} dx \right]$$

[integrating $e^{2x} \cos x$ by parts]

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4\int e^{2x} \sin x \, dx + C.$$

$\therefore \quad 5\int e^{2x} \sin x \, dx = e^{2x}(2\sin x - \cos x) + C$

or $\quad \int e^{2x} \sin x \, dx = \frac{1}{5} e^{2x}(2\sin x - \cos x) + C.$

EXAMPLE 10 *Evaluate* $\int \left(\frac{x - \sin x}{1 - \cos x} \right) dx.$

SOLUTION

$$\int \left(\frac{x - \sin x}{1 - \cos x} \right) dx = \int \frac{x}{(1 - \cos x)} dx - \int \frac{\sin x}{(1 - \cos x)} dx$$

$$= \int \frac{x}{2\sin^2(x/2)} dx - \int \frac{2\sin(x/2)\cos(x/2)}{2\sin^2(x/2)} dx$$

$$= \frac{1}{2}\int x \operatorname{cosec}^2(x/2)\, dx - \int \cot(x/2)\, dx$$

$$= \frac{1}{2}\left[x \cdot \int \operatorname{cosec}^2(x/2)\, dx - \int \left\{ \frac{d}{dx}(x) \cdot \int \operatorname{cosec}^2(x/2)\, dx \right\} dx \right]$$

$$- \int \cot(x/2)\, dx \quad \text{[integrating by parts]}$$

$$= \frac{1}{2}\left[x \cdot \left(-2\cot\frac{x}{2} \right) - \int \left[1 \cdot \left(-2\cot\frac{x}{2} \right) \right] dx - \int \cot(x/2)\, dx + C \right.$$

$$= -x\cot\frac{x}{2} + \int \cot\frac{x}{2} dx - \int \cot\frac{x}{2} dx + C = -x\cot\frac{x}{2} + C.$$

EXAMPLE 11 *Evaluate* $\int x \tan^{-1} x \, dx.$ **[CBSE 2000C, '04C]**

SOLUTION Integrating by parts, taking $\tan^{-1} x$ as the first function, we get

$$\int x \tan^{-1} x \, dx = (\tan^{-1} x) \cdot \int x \, dx - \int \left\{ \frac{d}{dx}(\tan^{-1} x) \cdot \int x \, dx \right\} dx$$

$$= (\tan^{-1} x) \cdot \frac{x^2}{2} - \int \frac{1}{(1 + x^2)} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int \left(1 - \frac{1}{1 + x^2} \right) dx \quad \text{[on dividing } x^2 \text{ by } 1 + x^2\text{]}$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{(1+x^2)}\,dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C$$

$$= \frac{1}{2}(1+x^2)\tan^{-1} x - \frac{1}{2}x + C.$$

EXAMPLE 12 *Evaluate* $\int x^2 \sin^{-1} x\, dx$.

SOLUTION Integrating by parts, taking $\sin^{-1} x$ as the first function we get

$$\int x^2 \sin^{-1} x\, dx = (\sin^{-1} x)\cdot \frac{x^3}{3} - \int \frac{1}{\sqrt{1-x^2}}\cdot \frac{x^3}{3}\,dx$$

$$= \frac{x^3 \sin^{-1} x}{3} - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^3 \sin^{-1} x}{3} - \frac{1}{3}\int \frac{x\cdot x^2}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^3 \sin^{-1} x}{3} + \frac{1}{3}\int \frac{t(1-t^2)}{t}\,dt, \text{ where } (1-x^2) = t^2$$

$$= \frac{x^3 \sin^{-1} x}{3} + \frac{1}{3}\int dt - \frac{1}{3}\int t^2 dt$$

$$= \frac{x^3 \sin^{-1} x}{3} + \frac{1}{3}t - \frac{1}{9}t^3 + C$$

$$= \frac{x^3 \sin^{-1} x}{3} + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{3/2} + C.$$

EXAMPLE 13 *Evaluate:*

(i) $\int \cos^{-1} x\, dx$ (ii) $\int \tan^{-1} x\, dx$ (iii) $\int \sec^{-1} x\, dx$

SOLUTION (i) Put $\cos^{-1} x = t$ so that $x = \cos t$ and $dx = -\sin t\; dt$.

$$\therefore \int \cos^{-1} x\, dx = -\int t \sin t\, dt$$

$$= -[t\cdot(-\cos t) - \int 1\cdot(-\cos t)\,dt]$$

[integrating by parts]

$$= t\cos t - \int \cos t\, dt = t\cos t - \sin t + C$$

$$= x\cos^{-1} x - \sqrt{1-x^2} + C$$

$[\because \cos t = x \Rightarrow \sin t = \sqrt{1-x^2}]$.

(ii) Put $\tan^{-1} x = t$ so that $x = \tan t$ and $dx = \sec^2 t\, dt$.

$$\therefore \int \tan^{-1} x\, dx = \int t \sec^2 t\, dt$$

$$= t\cdot \tan t - \int 1\cdot \tan t\, dt \quad \text{[integrating by parts]}$$

$$= t\cdot \tan t + \log|\cos t| + C$$

$$= (\tan^{-1}x)\cdot x + \log\left|\frac{1}{\sqrt{1+x^2}}\right| + C$$

$$\left[\because\ \tan t = x \ \Rightarrow\ \cos t = \frac{1}{\sqrt{1+x^2}}\right]$$

$$= x(\tan^{-1}x) - \frac{1}{2}\log|1+x^2| + C.$$

(iii) Put $\sec^{-1}x = t$ so that $x = \sec t$ and $dx = \sec t \tan t\, dt$.

$$\therefore\ \int \sec^{-1}x\, dx = \int t(\sec t \tan t)\, dt$$

$$= t(\sec t) - \int 1\cdot \sec t\, dt \qquad \text{[integrating by parts]}$$

$$= t(\sec t) - \log|\sec t + \tan t| + C$$

$$= t(\sec t) - \log|\sec t + \sqrt{\sec^2 t - 1}| + C$$

$$= x(\sec^{-1}x) - \log|x + \sqrt{x^2-1}| + C.$$

EXAMPLE 14 *Evaluate* $\int (\sin^{-1}x)^2 dx.$ **[CBSE 2004]**

SOLUTION Putting $x = \sin t$ and $dx = \cos t\, dt$, we get

$$\int (\sin^{-1}x)^2 dx = \int t^2 \cos t\, dt$$

$$= t^2\cdot(\sin t) - \int 2t(\sin t)\, dt \qquad \text{[integrating by parts]}$$

$$= t^2 \sin t - 2[t(-\cos t) - \int 1\cdot(-\cos t)\, dt]$$

[integrating $t(\sin t)$ by parts]

$$= t^2 \sin t + 2t\cos t - 2\sin t + C$$

$$= x(\sin^{-1}x)^2 + 2(\sin^{-1}x)\sqrt{1-x^2} - 2x + C.$$

EXAMPLE 15 *Evaluate* $\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}} dx.$

SOLUTION Put $x = \sin t$ so that $dx = \cos t\, dt$ and $t = \sin^{-1}x$.

$$\therefore\ \int \frac{\sin^{-1}x}{(1-x^2)^{3/2}} dx = \int \frac{t\cos t}{(1-\sin^2 t)^{3/2}} dt = \int \frac{t\cos t}{\cos^3 t} dt$$

$$= \int t \sec^2 t\, dt$$

$$= t\cdot(\tan t) - \int 1\cdot \tan t\, dt \qquad \text{[integrating by parts]}$$

$$= t\cdot(\tan t) + \log|\cos t| + C$$

$$= (\sin^{-1}x)\cdot\frac{x}{\sqrt{1-x^2}} + \log|\sqrt{1-x^2}| + C.$$

$$\left[\because\ \cos t = \sqrt{1-x^2} \text{ and } \tan t = \frac{x}{\sqrt{1-x^2}}\right]$$

$$= \frac{x(\sin^{-1}x)}{\sqrt{1-x^2}} + \frac{1}{2}\log|(1-x^2)| + C.$$

EXAMPLE 16 *Evaluate* $\int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx.$

SOLUTION Put $x = \tan t$ so that $dx = \sec^2 t\, dt$.

$$\therefore \int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx = \int \frac{(\tan t)\, t}{(1+\tan^2 t)^{3/2}} \cdot \sec^2 t\, dt$$

$$= \int \frac{(\tan t) t}{\sec t} dt = \int t \sin t\, dt$$

$$= t(-\cos t) - \int 1 \cdot (-\cos t)\, dt \quad \text{[integrating by parts]}$$

$$= -t \cos t + \sin t + C = \frac{-\tan^{-1} x}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + C$$

$$\left[\because \quad \sin t = \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad \cos t = \frac{1}{\sqrt{1+x^2}}\right].$$

EXAMPLE 17 *Evaluate:*

(i) $\int \sin^{-1}(3x - 4x^3)\, dx$ *(ii)* $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$ **[CBSE 2002]**

(iii) $\int \tan^{-1} \sqrt{\frac{1-x}{1+x}}\, dx$ *(iv)* $\int \sin^{-1} \sqrt{\frac{x}{a+x}}\, dx$

SOLUTION (i) Put $x = \sin t$ so that $dx = \cos t\ dt$.

$$\therefore \int \sin^{-1}(3x - 4x^3)\, dx = \int \sin^{-1}(3 \sin t - 4 \sin^3 t) \cos t\, dt$$

$$= \int \sin^{-1}(\sin\ 3t) \cos t\, dt$$

$$= 3 \int t \cos t\, dt$$

$$= 3[t(\sin t) - \int 1 \cdot \sin t\, dt]$$

[integrating by parts]

$$= 3t \sin t + 3 \cos t + C$$

$$= 3x(\sin^{-1} x) + 3\sqrt{1-x^2} + C.$$

(ii) Put $x = \tan t$ so that $dx = \sec^2 t\, dt$.

$$\therefore \int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = \int \sin^{-1}\left(\frac{2 \tan t}{1+\tan^2 t}\right) \sec^2 t\, dt$$

$$= \int \sin^{-1}(\sin 2t) \sec^2 t\, dt = 2\int t \cdot \sec^2 t\, dt$$

$$= 2[t \cdot \tan t - \int 1 \cdot \tan t\, dt]$$

$$= 2t \cdot \tan t + 2 \log|\cos t| + C$$

$$= 2x(\tan^{-1} x) + 2 \log \left|\frac{1}{\sqrt{1+x^2}}\right| + C$$

$$= 2x(\tan^{-1} x) + 2 \cdot \left(-\frac{1}{2}\right) \log|1+x^2| + C$$

$$= 2x(\tan^{-1} x) - \log|1+x^2| + C.$$

(iii) Put $x = \cos t$ so that $dx = -\sin t\, dt$.

$$\therefore \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx = \int \tan^{-1}\sqrt{\frac{1-\cos t}{1+\cos t}}(-\sin t)\,dt$$

$$= \int \tan^{-1}\sqrt{\frac{2\sin^2(t/2)}{2\cos^2(t/2)}}(-\sin t)\,dt$$

$$= \int\left[\tan^{-1}\left(\tan\frac{t}{2}\right)\right](-\sin t)\,dt = -\frac{1}{2}\int t(\sin t)\,dt$$

$$= -\frac{1}{2}[t(-\cos t) - \int 1\cdot(-\cos t)\,dt] \quad \text{[integrating by parts]}$$

$$= \frac{1}{2}t\cdot\cos t - \frac{1}{2}\sin t + C = \frac{1}{2}x(\cos^{-1}x) - \frac{1}{2}\sqrt{1-x^2} + C.$$

(iv) Put $x = a\tan^2 t$ so that $dx = (2a\sec^2 t\tan t)\,dt$.

$$\therefore \int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx = \int \sin^{-1}\left\{\sqrt{\frac{a\tan^2 t}{a(1+\tan^2 t)}}\right\}2a\sec^2 t\tan t\,dt$$

$$= 2a\int t(\sec^2 t\cdot\tan t)\,dt$$

$$= 2a\left[t\cdot\frac{1}{2}\tan^2 t - \int 1\cdot\frac{1}{2}\tan^2 t\,dt\right]$$

$$\text{[integrating by parts and using } \int \sec^2 t\tan t\,dt = \frac{1}{2}\tan^2 t\text{]}$$

$$= at(\tan^2 t) - a\int(\sec^2 t - 1)\,dt$$

$$= at(\tan^2 t) - a\int \sec^2 t\,dt + a\int dt$$

$$= at(\tan^2 t) - a\tan t + at + C$$

$$= a\left(\tan^{-1}\sqrt{\frac{x}{a}}\right)\cdot\left(\frac{x}{a}\right) - a\cdot\sqrt{\frac{x}{a}} + a\tan^{-1}\sqrt{\frac{x}{a}} + C$$

$$= x\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + a\tan^{-1}\sqrt{\frac{x}{a}} + C.$$

EXAMPLE 18 *Evaluate* $\int x\cos^3 x\sin x\,dx$.

SOLUTION Take x as the first function and $(\cos^3 x\sin x)$ as the second.

Putting $\cos x = t$, we can evaluate $\int \cos^3 x\sin x\,dx$ as $-\frac{1}{4}\cos^4 x$.

So, integrating by parts, we get

$$\int x\cos^3 x\sin x\,dx = x\cdot\left(\frac{-1}{4}\cos^4 x\right) - \int 1\cdot\left(-\frac{1}{4}\right)\cos^4 x\,dx$$

$$= -\frac{x}{4}\cos^4 x + \frac{1}{4}\int\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= -\frac{x\cos^4 x}{4} + \frac{1}{4}\int\left(\frac{1}{4} + \frac{\cos^2 2x}{4} + \cos 2x\right)dx$$

$$= -\frac{x\cos^4 x}{4} + \frac{1}{16}\int dx + \frac{1}{4}\int \cos 2x\, dx + \frac{1}{32}\int 2\cos^2 2x\, dx$$

$$= -\frac{x\cos^4 x}{4} + \frac{x}{16} + \frac{\sin 2x}{8} + \frac{1}{32}\int(1 + \cos 4x)\, dx + C$$

$$= -\frac{x\cos^4 x}{4} + \frac{x}{16} + \frac{\sin 2x}{8} + \frac{1}{32}\int dx + \frac{1}{32}\int \cos 4x\, dx$$

$$= -\frac{x\cos^4 x}{4} + \frac{3x}{32} + \frac{\sin 2x}{8} + \frac{\sin 4x}{128} + C.$$

EXAMPLE 19 *Evaluate* $\int \sin(\log x)\, dx$.

SOLUTION Put $\log x = t$ so that $x = e^t$ and $\frac{1}{x}dx = dt$ or $dx = e^t dt$.

$\therefore \int \sin(\log x)\, dx = \int e^t \sin t\, dt$ ... (i)

Now, $\int e^t \sin t\, dt = e^t(-\cos t) - \int e^t \cdot(-\cos t)\, dt$ [integrating by parts]

$$= -e^t \cos t + \int e^t \cos t\, dt$$

$$= -e^t \cos t + [e^t \sin t - \int e^t \sin t\, dt]$$

[integrating $e^t \cos t$ by parts]

$$= -e^t \cos t + e^t \sin t - \int e^t \sin t\, dt.$$

$\therefore \quad 2\int e^t \sin t\, dt = -e^t \cos t + e^t \sin t$

or $\quad \int e^t \sin t\, dt = \frac{1}{2}(-e^t \cos t + e^t \sin t) + C.$

Putting this value in (i), we get

$$\int \sin(\log x)\, dx = \int e^t \sin t\, dt$$

$$= \frac{1}{2}(-e^t \cos t + e^t \sin t) + C$$

$$= \frac{1}{2}[-x\cos(\log x) + x\sin(\log x)] + C$$

$$= -\frac{1}{2}x\cos(\log x) + \frac{1}{2}x\sin(\log x) + C.$$

EXAMPLE 20 *Evaluate* $\int \sin\sqrt{x}\, dx$.

SOLUTION Put $\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}}dx = dt$ or $dx = 2t\, dt$.

$\therefore \quad \int \sin\sqrt{x}\, dx = 2\int t\sin t\, dt = 2[t(-\cos t) - \int 1\cdot(-\cos t)\, dt]$

[integrating $t \sin t$ by parts]

$$= -2t\cos t + 2\sin t + C$$
$$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C.$$

EXAMPLE 21 *Evaluate* $\int \sec^3 x\,dx.$ **[CBSE 2003]**

SOLUTION $\int \sec^3 x\,dx = \int \sec x \cdot \sec^2 x\,dx$

$$= \sec x \cdot (\tan x) - \int \sec x \tan x (\tan x)\,dx$$
[integrating by parts]
$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\therefore\quad 2\int \sec^3 x\,dx = \sec x \tan x + \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$$

or $\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C'.$

EXAMPLE 22 *Evaluate* $\int \tan^{-1}\sqrt{x}\,dx.$

SOLUTION Put $\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}}dx = dt$ or $dx = 2t\,dt.$

$$\therefore\quad \int \tan^{-1}\sqrt{x}\,dx = 2\int t(\tan^{-1}t)\,dt$$
$$= 2\left[(\tan^{-1}t)\cdot\frac{t^2}{2} - \int\left\{\frac{1}{(1+t^2)}\cdot\frac{t^2}{2}\right\}dt\right] + C$$
$$= t^2(\tan^{-1}t) - \int \frac{t^2}{(1+t^2)}dt + C$$
$$= t^2(\tan^{-1}t) - \int \frac{[(1+t^2)-1]}{(1+t^2)}dt + C$$
$$= t^2(\tan^{-1}t) - \int dt + \int \frac{1}{(1+t^2)}dt + C$$
$$= t^2(\tan^{-1}t) - t + \tan^{-1}t + C = (t^2+1)\tan^{-1}t - t + C$$
$$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C.$$

EXAMPLE 23 *Evaluate* $\int \frac{\tan^{-1}x}{(1+x)^2}dx.$ **[CBSE 2003]**

SOLUTION Integrating by parts, taking $\tan^{-1}x$ as the first function and $\frac{1}{(1+x)^2}$ as the second function, we get

$$I = \tan^{-1}x \cdot \frac{(-1)}{(1+x)} - \int \frac{1}{(1+x^2)}\cdot\frac{(-1)}{(1+x)}dx$$
$$= \frac{-\tan^{-1}x}{(1+x)} + \int \frac{dx}{(1+x)(1+x^2)} = \frac{-\tan^{-1}x}{(1+x)} + \frac{1}{2}\cdot\int\left\{\frac{1}{(1+x)} + \frac{(1-x)}{(1+x^2)}\right\}$$
[by partial fractions]

$$= \frac{-\tan^{-1}x}{(1+x)} + \frac{1}{2}\log|1+x| + \frac{1}{2}\tan^{-1}x - \frac{1}{4}\log(1+x^2) + C.$$

EXAMPLE 24 *Evaluate* $\int \left\{ \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} \right\} dx.$ **[CBSE 2014C]**

SOLUTION We have

$$I = \int \frac{\left\{ \sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x} \right) \right\}}{\left(\frac{\pi}{2} \right)} dx \quad [\because \sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}]$$

$$= \frac{2}{\pi} \int \left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2} \right) dx$$

$$= \frac{4}{\pi} \int \sin^{-1}\sqrt{x}\, dx - \int dx = \frac{4}{\pi} \int \sin^{-1}\sqrt{x}\, dx - x + C. \quad \ldots \text{(i)}$$

Putting $x = \sin^2\theta$ and $dx = 2\sin\theta\cos\theta\, d\theta = \sin 2\theta\, d\theta$, we get

$$\int \sin^{-1}\sqrt{x}\, dx = \int \underset{\text{I}}{\theta} \underset{\text{II}}{\sin 2\theta}\, d\theta$$

$$= \theta\left(\frac{-\cos 2\theta}{2} \right) - \int 1 \cdot \frac{(-\cos 2\theta)}{2} d\theta$$

[integrating by parts]

$$= -\frac{\theta}{2}\cos 2\theta + \int \frac{1}{2}\cos 2\theta\, d\theta$$

$$= -\frac{1}{2}\theta\cos 2\theta + \frac{1}{4}\sin 2\theta$$

$$= -\frac{1}{2}\theta(1 - 2\sin^2\theta) + \frac{1}{2}\sin\theta\sqrt{1 - \sin^2\theta}$$

$$= -\frac{1}{2}\sin^{-1}\sqrt{x}(1 - 2x) + \frac{1}{2}\sqrt{x} \cdot \sqrt{1-x}. \quad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$I = \frac{4}{\pi} \cdot \left\{ -\frac{1}{2}\sin^{-1}\sqrt{x}(1-2x) + \frac{1}{2}\sqrt{x}\sqrt{1-x} \right\} - x + C$$

$$\therefore \quad I = \frac{-2}{\pi}\sin^{-1}\sqrt{x}(1-2x) + \frac{2}{\pi}\sqrt{x}\sqrt{1-x} - x + C.$$

EXAMPLE 25 *Evaluate* $\int \frac{\sqrt{x^2+1}\{\log(x^2+1) - 2\log x\}}{x^4} dx.$ **[CBSE 2012, '14C]**

SOLUTION We have

$$I = \int \frac{\sqrt{x^2+1}}{x^4} \cdot \log\left(\frac{x^2+1}{x^2} \right) dx$$

$$= \int \sqrt{1 + \frac{1}{x^2}} \cdot \frac{1}{x^3} \log\left(1 + \frac{1}{x^2} \right) dx \quad \ldots \text{(i)}$$

Putting $\left(1+\frac{1}{x^2}\right)=t$ and $\frac{-2}{x^3}dx=dt$, i.e., $\frac{1}{x^3}dx=-\frac{1}{2}dt$ in (i), we get

$$I=-\frac{1}{2}\int \underset{\text{I}}{(\log t)}\,\underset{\text{II}}{\sqrt{t}}\,dt$$

$$=-\frac{1}{2}\left\{\frac{2}{3}(\log t)t^{3/2}-\frac{2}{3}\int\left(\frac{1}{t}\times t^{3/2}\right)dt\right\}$$

$$=-\frac{1}{3}t^{3/2}(\log t)+\frac{1}{3}\int t^{1/2}dt$$

$$=-\frac{1}{3}t^{3/2}(\log t)+\frac{2}{9}t^{3/2}+C$$

$$=-\frac{1}{3}t^{3/2}\left\{\log t-\frac{2}{3}\right\}+C$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left\{\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right\}+C.$$

EXAMPLE 26 *Evaluate* $\int \frac{\sqrt{1-\sin x}}{(1+\cos x)}e^{-x/2}dx.$ **[CBSE 2013C]**

SOLUTION Putting $\frac{-x}{2}=t$, we get $x=-2t$ and $dx=-2dt$.

$$\therefore \quad I=\int \frac{\sqrt{1-\sin x}}{(1+\cos x)}e^{-x/2}dx$$

$$=\int \frac{\sqrt{1-\sin(-2t)}}{\{1+\cos(-2t)\}}e^t(-2dt)=-2\int \frac{\sqrt{1+\sin 2t}}{(1+\cos 2t)}e^t dt$$

$$=-2\int \frac{\sqrt{\cos^2 t+\sin^2 t+2\sin t\cos t}}{2\cos^2 t}e^t dt$$

$$=-2\int \frac{(\cos t+\sin t)}{2\cos^2 t}e^t dt=-\int(\sec t+\sec t\tan t)\,e^t dt$$

$$=-\int e^t\{f(t)+f'(t)\}\,dt,\text{ where } f(t)=\sec t$$

$$=-e^t f(t)+C=-e^{-x/2}\sec\left(\frac{-x}{2}\right)+C=-e^{-x/2}\sec\frac{x}{2}+C.$$

Integrals of the form $\int e^x[f(x)+f'(x)]dx$

THEOREM 1 $\int e^x\{f(x)+f'(x)\}\,dx=e^x\cdot f(x)+C.$

PROOF $\int e^x\{f(x)+f'(x)\}\,dx=\int e^x\cdot f(x)\,dx+\int e^x\cdot f'(x)\,dx$

$$=f(x)\cdot\int e^x dx-\int\{f'(x)\cdot\int e^x dx\}\,dx+\int e^x f'(x)\,dx+C$$

[evaluating the first integral by parts]

$$= e^x f(x) - \int e^x f'(x)\,dx + \int e^x f'(x)\,dx + C$$

$$= e^x f(x) + C.$$

$$\therefore \quad \int e^x \{f(x) + f'(x)\}\,dx = e^x f(x) + C.$$

EXAMPLE 1 *Evaluate:*

(i) $\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$ (ii) $\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx$

(iii) $\int e^x \left\{\sin^{-1} x + \frac{1}{\sqrt{1-x^2}}\right\} dx$ (iv) $\int e^x (\tan x + \log \sec x)\,dx$

SOLUTION We have

(i) $I = \int e^x \left\{\frac{1}{x} + \left(-\frac{1}{x^2}\right)\right\} dx$

$= \int e^x \{f(x) + f'(x)\} dx$, where $f(x) = \frac{1}{x}$ and $f'(x) = \frac{-1}{x^2}$

$= e^x \cdot f(x) + C = e^x \cdot \frac{1}{x} + C = \frac{e^x}{x} + C.$

(ii) $I = \int e^x \left\{\frac{1}{x^2} + \left(\frac{-2}{x^3}\right)\right\} dx$

$= \int e^x \{f(x) + f'(x)\} dx$, where $f(x) = \frac{1}{x^2}$ and $f'(x) = \frac{-2}{x^3}$

$= e^x \cdot f(x) + C = e^x \cdot \frac{1}{x^2} + C = \frac{e^x}{x^2} + C.$

(iii) $I = \int e^x \left\{\sin^{-1} x + \frac{1}{\sqrt{1-x^2}}\right\} dx$

$= \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \sin^{-1} x$ and $f'(x) = \frac{1}{\sqrt{1-x^2}}$

$= e^x \cdot f(x) + C = e^x \cdot \sin^{-1} x + C = e^x \sin^{-1} x + C.$

(iv) $I = \int e^x (\tan x + \log \sec x)\,dx$

$= \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \log(\sec x)$

and $f'(x) = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$

$= e^x f(x) + C = e^x \log(\sec x) + C.$

EXAMPLE 2 *Evaluate* $\int \frac{x e^x}{(1+x)^2}\,dx.$

SOLUTION We have

$$I = \int e^x \cdot \left\{\frac{x}{(1+x)^2}\right\} dx = \int e^x \cdot \left\{\frac{(1+x) - 1}{(1+x)^2}\right\} dx$$

$$= \int e^x \cdot \left\{ \frac{(1+x)}{(1+x)^2} - \frac{1}{(1+x)^2} \right\} dx = \int e^x \cdot \left\{ \frac{1}{(1+x)} - \frac{1}{(1+x)^2} \right\} dx$$

$$= \int e^x \cdot \{f(x) + f'(x)\}\, dx, \text{ where } f(x) = \frac{1}{1+x} \text{ and } f'(x) = \frac{-1}{(1+x)^2}$$

$$= e^x \cdot f(x) + C = e^x \cdot \frac{1}{(1+x)} + C = \frac{e^x}{(1+x)} + C.$$

EXAMPLE 3 *Evaluate* $\int e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx.$

SOLUTION We have $I = \int e^x \cdot \left(\frac{1 - \sin x}{1 - \cos x} \right) dx = \int e^x \cdot \left\{ \frac{1}{(1 - \cos x)} - \frac{\sin x}{(1 - \cos x)} \right\} dx$

$$= \int e^x \cdot \left\{ \frac{1}{2\sin^2(x/2)} - \frac{2\sin(x/2)\cos(x/2)}{2\sin^2(x/2)} \right\} dx$$

$$= \int e^x \cdot \left\{ \frac{1}{2}\operatorname{cosec}^2 \frac{x}{2} - \cot \frac{x}{2} \right\} dx$$

$$= \int e^x \cdot \left\{ -\cot \frac{x}{2} + \frac{1}{2}\operatorname{cosec}^2 \frac{x}{2} \right\} dx$$

$$= \int e^x \cdot \{f(x) + f'(x)\}\, dx,$$

where $f(x) = -\cot \frac{x}{2}$ and $f'(x) = \frac{1}{2}\operatorname{cosec}^2 \frac{x}{2}$

$$= e^x \cdot f(x) + C = e^x \left(-\cot \frac{x}{2} \right) + C = -e^x \cot \frac{x}{2} + C.$$

EXAMPLE 4 *Evaluate* $\int e^x \cdot \left\{ \frac{2 + \sin 2x}{1 + \cos 2x} \right\} dx.$

SOLUTION We have

$$I = \int e^x \cdot \left\{ \frac{2 + \sin 2x}{1 + \cos 2x} \right\} dx$$

$$= \int e^x \cdot \left\{ \frac{2}{(1 + \cos 2x)} + \frac{\sin 2x}{(1 + \cos 2x)} \right\} dx$$

$$= \int e^x \cdot \left\{ \frac{2}{2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x} \right\} dx = \int e^x \cdot \{\sec^2 x + \tan x\}\, dx$$

$$= \int e^x \cdot \{\tan x + \sec^2 x\}\, dx$$

$$= \int e^x \cdot \{f(x) + f'(x)\}\, dx, \text{ where } f(x) = \tan x \text{ and } f'(x) = \sec^2 x$$

$$= e^x \cdot f(x) + C = e^x \tan x + C.$$

EXAMPLE 5 *Evaluate* $\int \frac{(x^2+1)e^x}{(x+1)^2}\, dx.$ **[CBSE 2005C, '06]**

SOLUTION We have

$$I = \int e^x \cdot \frac{(x^2+1)}{(x+1)^2}\,dx = \int e^x \cdot \left\{\frac{(x+1)^2 - 2x}{(x+1)^2}\right\} dx$$

$$= \int e^x \cdot \left\{1 - \frac{2x}{(x+1)^2}\right\} dx = \int e^x\,dx - 2\int e^x \cdot \frac{x}{(x+1)^2}\,dx$$

$$= e^x - 2\cdot\int e^x \cdot \frac{\{(x+1)-1\}}{(x+1)^2}\,dx = e^x - 2\cdot\int e^x \cdot \left\{\frac{1}{(x+1)} - \frac{1}{(x+1)^2}\right\} dx$$

$$= e^x - 2\cdot\int e^x \cdot \{f(x) + f'(x)\}\,dx,$$

where $f(x) = \frac{1}{(x+1)}$ and $f'(x) = \frac{-1}{(x+1)^2}$

$$= e^x - 2e^x \cdot f(x) + C = e^x - 2e^x \cdot \frac{1}{(x+1)} + C$$

$$= e^x \cdot \left\{1 - \frac{2}{(x+1)}\right\} + C = e^x\left(\frac{x-1}{x+1}\right) + C.$$

EXAMPLE 6 *Evaluate* $\int e^{2x}\left(\frac{\sin 4x - 2}{1 - \cos 4x}\right) dx.$ **[CBSE 2007]**

SOLUTION Putting $2x = t$ and $dx = \frac{1}{2}\,dt$, we get

$$I = \frac{1}{2}\int e^t\left(\frac{\sin 2t - 2}{1 - \cos 2t}\right) dt = \frac{1}{2}\int e^t\left(\frac{2\sin t\cos t - 2}{2\sin^2 t}\right) dt$$

$$= \frac{1}{2}\int e^t\left\{\frac{\sin t\cos t - 1}{\sin^2 t}\right\} dt = \frac{1}{2}\int e^t(\cot t - \operatorname{cosec}^2 t)\,dt$$

$$= \frac{1}{2}\int e^t\{f(t) + f'(t)\}\,dt, \quad \text{where } f(t) = \cot t$$

$$= \frac{1}{2}e^t \cdot f(t) + C = \frac{1}{2}e^t \cot t + C = \frac{1}{2}e^{2x}\cot 2x + C.$$

Integrals of the form $\int e^{kx} \cdot \{k \cdot f(x) + f'(x)\}\,dx$

THEOREM 2 $\int e^{kx}\{k \cdot f(x) + f'(x)\}\,dx = e^{kx} \cdot f(x) + C.$

PROOF $\int e^{kx} \cdot \{k \cdot f(x) + f'(x)\}\,dx$

$$= k \cdot \int e^{kx} f(x)\,dx + \int e^{kx} f'(x)\,dx$$

$$= k \cdot \left[f(x) \cdot \frac{e^{kx}}{k} - \int f'(x) \cdot \frac{e^{kx}}{k}\,dx\right] + \int e^{kx} f'(x)\,dx + C$$

[evaluating the first integral by parts]

$$= e^{kx} \cdot f(x) - \int e^{kx} \cdot f'(x)\,dx + \int e^{kx} f'(x)\,dx + C = e^{kx} \cdot f(x) + C.$$

$$\therefore \int e^{kx} \cdot \{k \cdot f(x) + f'(x)\}\,dx = e^{kx} \cdot f(x) + C.$$

EXAMPLE 7 *Evaluate* $\int e^{2x} \cdot (-\sin x + 2\cos x)\, dx.$

SOLUTION We have

$$I = \int e^{2x} \cdot \{2\cos x - \sin x\}\, dx = 2\int e^{2x} \cos x\, dx - \int e^{2x} \sin x\, dx$$

$$= 2 \cdot \left[\cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \cdot \frac{e^{2x}}{2} dx \right] - \int e^{2x} \sin x\, dx$$

[integrating $e^{2x} \cos x$ by parts]

$$= e^{2x} \cos x + \int e^{2x} \sin x\, dx - \int e^{2x} \sin x\, dx + C$$

$$= e^{2x} \cos x + C.$$

Integrals of the form $e^{ax} \cos (bx + c)$ and $e^{ax} \sin (bx + c)$

EXAMPLE 8 *Evaluate* $\int e^{ax} \cos (bx + c)\, dx.$ **[CBSE 2002]**

SOLUTION Integrating by parts, taking e^{ax} as the second function, we get

$$\int e^{ax} \cos (bx + c) dx = \cos (bx + c) \cdot \frac{e^{ax}}{a} - \int \left\{ -b \sin (bx + c) \cdot \frac{e^{ax}}{a} \right\} dx$$

$$= \frac{e^{ax}}{a} \cos (bx + c) + \frac{b}{a} \int e^{ax} \sin (bx + c)\, dx$$

$$= \frac{e^{ax}}{a} \cdot \cos (bx + c) + \frac{b}{a} \left[\sin (bx + c) \cdot \frac{e^{ax}}{a} - \int \left\{ b \cos (bx + c) \cdot \frac{e^{ax}}{a} \right\} \right] dx + C$$

[integrating $e^{ax} \sin (bx + c)$ by parts]

$$= \frac{e^{ax}}{a} \cdot \cos (bx + c) + \frac{b}{a^2} e^{ax} \sin (bx + c) - \frac{b^2}{a^2} \int e^{ax} \cos (bx + c)\, dx + C$$

$$\therefore \left(1 + \frac{b^2}{a^2} \right) \int e^{ax} \cos (bx + c) dx = \frac{e^{ax}}{a} \cos (bx + c) + \frac{b}{a^2} e^{ax} \sin (bx + c) + C$$

$$\text{or } \int e^{ax} \cos (bx + c) dx = e^{ax} \left[\frac{a \cos (bx + c) + b \sin (bx + c)}{(a^2 + b^2)} \right] + C'.$$

REMARK Put $a = r \cos \theta$ and $b = r \sin \theta$ so that

$$r = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \tan^{-1} \left(\frac{b}{a} \right).$$

$$\therefore \quad \int e^{ax} \cos (bx + c)\, dx = \frac{r e^{ax} \cos (bx + c - \theta)}{(a^2 + b^2)}$$

$$= e^{ax} \cdot \frac{\cos [bx + c - \tan^{-1}(b/a)]}{\sqrt{a^2 + b^2}}.$$

Similarly, $\int e^{ax} \sin (bx + c)\, dx = e^{ax} \cdot \dfrac{\sin [bx + c - \tan^{-1}(b/a)]}{\sqrt{a^2 + b^2}}.$

EXERCISE 13C

Evaluate the following integrals:

1. $\int x e^x \, dx$
2. $\int x \cos x \, dx$
3. $\int x e^{2x} \, dx$
4. $\int x \sin 3x \, dx$
5. $\int x \cos 2x \, dx$
6. $\int x \log 2x \, dx$ **[CBSE 2007]**
7. $\int x \operatorname{cosec}^2 x \, dx$
8. $\int x^2 \cos x \, dx$
9. $\int x \sin^2 x \, dx$
10. $\int x \tan^2 x \, dx$
11. $\int x^2 e^x \, dx$
12. $\int x^2 \cos^3 x \, dx$
13. $\int x^2 e^{3x} \, dx$
14. $\int x^2 \sin^2 x \, dx$
15. $\int x^3 \log 2x \, dx$
16. $\int x \cdot \log (x+1) \, dx$ **[CBSE 2008C]**
17. $\int \frac{\log x}{x^n} \, dx$
18. $\int 2x^3 e^{x^2} \, dx$
19. $\int x \sin^3 x \, dx$
20. $\int x \cos^3 x \, dx$
21. $\int x^3 \cos x^2 dx$
22. $\int \sin x \log (\cos x) \, dx$
23. $\int x \sin x \cos x \, dx$
24. $\int \cos \sqrt{x} \, dx$
25. $\int \operatorname{cosec}^3 x \, dx$
26. $\int x \sin^3 x \cos x \, dx$
27. $\int \sin x \log (\cos x) \, dx$
28. $\int \frac{\log (\log x)}{x} \, dx$
29. $\int \log (2 + x^2) \, dx$ **[CBSE 2001]**
30. $\int \frac{x}{(1+\sin x)} \, dx$
31. $\int \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$ **[CBSE 2005]**
32. $\int e^{-x} \cos 2x \cos 4x \, dx$
33. $\int e^{\sqrt{x}} \, dx$
34. $\int e^{\sin x} \sin 2x \, dx$
35. $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx$ **[CBSE 2012]**
36. $\int \frac{x^2 \tan^{-1} x}{(1+x^2)} \, dx$
37. $\int \frac{\log (x+2)}{(x+2)^2} \, dx$
38. $\int x \sin^{-1} x \, dx$ **[CBSE 2000C, '09]**
39. $\int x \cos^{-1} x \, dx$
40. $\int \cot^{-1} x \, dx$
41. $\int x \cot^{-1} x \, dx$
42. $\int x^2 \cot^{-1} x \, dx$ **[CBSE 2006C]**
43. $\int \sin^{-1} \sqrt{x} \, dx$
44. $\int \cos^{-1} \sqrt{x} \, dx$
45. $\int \cos^{-1} (4x^3 - 3x) \, dx$
46. $\int \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) dx$
47. $\int \tan^{-1} \left(\frac{2x}{1-x^2} \right) dx$

48. $\int \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) dx$

49. $\int \frac{\sin^{-1}x}{x^2} dx$ **[CBSE 2002C, '04]**

50. $\int \frac{\tan x \sec^2 x}{(1 - \tan^2 x)} dx$

51. $\int e^{3x} \sin 4x \, dx$

52. $\int e^{2x} \sin x \, dx$

53. $\int e^{2x} \sin x \cos x \, dx$

54. $\int e^{2x} \cos(3x + 4) \, dx$

55. $\int e^{-x} \cos x \, dx$

56. $\int e^x (\sin x + \cos x) \, dx$

57. $\int e^x (\cot x - \operatorname{cosec}^2 x) \, dx$

58. $\int e^x \sec x (1 + \tan x) \, dx$ **[CBSE 2012]**

59. $\int e^x \left(\tan^{-1} x + \frac{1}{1 + x^2}\right) dx$

60. $\int e^x (\cot x + \log \sin x) \, dx$

61. $\int e^x (\tan x - \log \cos x) \, dx$ **[CBSE 2000]**

62. $\int e^x [\sec x + \log(\sec x + \tan x)] \, dx$

63. $\int e^x \left(\frac{1 + \sin x \cos x}{\cos^2 x}\right) dx$

64. $\int e^x \left(\frac{\sin x \cos x - 1}{\sin^2 x}\right) dx$

65. $\int e^x \left(\frac{\cos x + \sin x}{\cos^2 x}\right) dx$

66. $\int e^x \left(\frac{2 - \sin 2x}{1 - \cos 2x}\right) dx$

67. $\int e^x \left(\frac{1 + \sin x}{1 + \cos x}\right) dx$

68. $\int e^x \left(\frac{\sin 4x - 4}{1 - \cos 4x}\right) dx$ **[CBSE 2010]**

69. $\int \frac{e^x [\sqrt{1 - x^2} \sin^{-1} x + 1]}{\sqrt{1 - x^2}} dx$

70. $\int e^x \left(\frac{1 + x \log x}{x}\right) dx$

71. $\int e^x \cdot \frac{x}{(1 + x)^2} dx$

72. $\int e^x \frac{(x - 1)}{(x + 1)^3} dx$ **[CBSE 2006]**

73. $\int e^x \frac{(2 - x)}{(1 - x)^2} dx$

74. $\int e^x \cdot \frac{(x - 3)}{(x - 1)^3} dx$

75. $\int e^{3x} \left(\frac{3x - 1}{9x^2}\right) dx$

76. $\int \frac{(x + 1)}{(x + 2)^2} e^x \, dx$

77. $\int \frac{x e^{2x}}{(1 + 2x)^2} dx$

78. $\int e^{2x} \left(\frac{2x - 1}{4x^2}\right) dx$

79. $\int e^x \left(\log x + \frac{1}{x^2}\right) dx$

80. $\int \frac{\log x}{(1 + \log x)^2} dx$

81. $\int \{\sin(\log x) + \cos(\log x)\} \, dx$

82. $\int \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\} dx$ **[CBSE 2005]**

83. $\int \left\{\log(\log x) + \frac{1}{(\log x)^2}\right\} dx$

84. $\int \left(\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\right) dx$

85. $\int 5^{5^{5^x}} \cdot 5^{5^x} \cdot 5^x \, dx$

86. $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right)dx$ **[CBSE 2010]** 87. $\int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$ **[CBSE 2013C]**

ANSWERS (EXERCISE 13C)

1. $e^x(x-1)+C$

2. $x\sin x+\cos x+C$

3. $\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$

4. $\frac{-x\cos 3x}{3}+\frac{\sin 3x}{9}+C$

5. $\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+C$

6. $\frac{1}{2}x^2\log 2x-\frac{1}{4}x^2+C$

7. $-x\cot x+\log|\sin x|+C$

8. $x^2\sin x+2x\cos x-2\sin x+C$

9. $\frac{x^2}{4}-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+C$

10. $x\tan x+\log|\cos x|-\frac{x^2}{2}+C$

11. $e^x(x^2-2x+2)+C$

12. $\frac{2x}{9}\cos 3x+\frac{\sin 3x}{36}(4+3x^2)$

13. $e^{3x}\left(\frac{x^2}{3}-\frac{2x}{9}+\frac{2}{27}\right)+C$

14. $\frac{1}{6}x^3-\frac{1}{4}x^2\sin 2x-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+C$

15. $\frac{x^4\log 2x}{4}-\frac{x^4}{16}+C$

16. $\frac{1}{2}(x^2-1)\log(x+1)-\frac{1}{4}x^2+\frac{1}{2}x+C$

17. $\frac{x^{1-n}}{(1-n)}\cdot\log|x|-\frac{x^{1-n}}{(1-n)^2}+C$

18. $e^{x^2}(x^2-1)+C$

19. $\frac{-3x\cos x}{4}+\frac{3\sin x}{4}+\frac{x\cos 3x}{12}-\frac{\sin 3x}{36}+C$

20. $\frac{1}{12}x\sin 3x+\frac{1}{36}\cos 3x+\frac{3}{4}x\sin x+\frac{3}{4}\cos x+C$

21. $\frac{1}{2}x^2\sin x^2+\frac{1}{2}\cos x^2+C$

22. $-\cos x\log(\cos x)+\cos x+C$

23. $-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+C$

24. $2[\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}]+C$

25. $-\frac{1}{2}\operatorname{cosec} x\cot x+\frac{1}{2}\log\left|\tan\frac{x}{2}\right|+C$

26. $\frac{1}{4}x\sin^4 x-\frac{3}{32}x+\frac{1}{16}\sin 2x-\frac{1}{128}\sin 4x+C$

27. $\cos x(1-\log\cos x)+C$

28. $(\log x)[\log(\log x)-1]+C$

29. $x\log(x^2+2)-2x+2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$

30. $-x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+2\log\left|\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|+C$

31. $\frac{x}{(\log x)}+C$

32. $e^{-x}\left[\frac{1}{74}(6\sin 6x-\cos 6x)+\frac{1}{10}(2\sin 2x-\cos 2x)\right]+C$

33. $2e^{\sqrt{x}}(\sqrt{x}-1)+C$ **34.** $2e^{\sin x}(\sin x-1)+C$ **35.** $-\sqrt{1-x^2}\sin^{-1}x+x+C$

36. $x\tan^{-1}x-\frac{1}{2}\log(1+x^2)-\frac{1}{2}(\tan^{-1}x)^2+C$ **37.** $\frac{-1}{(x+2)}-\frac{\log(x+2)}{(x+2)}+C$

38. $\frac{1}{2}x^2\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}-\frac{1}{4}\sin^{-1}x+C$

39. $\frac{1}{2}x^2\cos^{-1}x-\frac{1}{4}x\sqrt{1-x^2}+\frac{1}{4}\sin^{-1}x+C$ **40.** $x\cot^{-1}x+\frac{1}{2}\log(1+x^2)+C$

41. $\frac{1}{2}x^2(\cot^{-1}x)+\frac{1}{2}(x-\tan^{-1}x)+C$

42. $\frac{1}{3}x^3\cot^{-1}x+\frac{1}{6}x^2-\frac{1}{6}\log|x^2+1|+C$

43. $\frac{1}{2}(2x-1)\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x(1-x)}+C$

44. $\frac{1}{2}(2x-1)\cos^{-1}\sqrt{x}-\frac{1}{2}\sqrt{x(1-x)}+C$ **45.** $3x\cos^{-1}x-3\sqrt{1-x^2}+C$

46. $2x\tan^{-1}x-\log(1+x^2)+C$ **47.** $2x\tan^{-1}x-\log(1+x^2)+C$

48. $3x\tan^{-1}x-\frac{3}{2}\log(1+x^2)+C$ **49.** $\frac{-\sin^{-1}x}{x}+\log\left|\frac{1}{x}-\frac{\sqrt{1-x^2}}{x}\right|+C$

50. $-\frac{1}{2}\log|1-\tan^2x|+C$ **51.** $\frac{e^{3x}}{25}\cdot(3\sin 4x-4\cos 4x)+C$

52. $\frac{1}{5}\cdot e^{2x}(2\sin x-\cos x)+C$ **53.** $\frac{1}{8}\cdot e^{2x}(\sin 2x-\cos 2x)+C$

54. $\frac{e^{2x}}{13}\cdot\{2\cos(3x+4)+3\sin(3x+4)\}+C$

55. $\frac{1}{2}e^{-x}(\sin x-\cos x)+C$ **56.** $e^x\sin x+C$

57. $e^x\cot x+C$ **58.** $e^x\sec x+C$ **59.** $e^x\tan^{-1}x+C$

60. $e^x\log(\sin x)+C$ **61.** $e^x\log(\sec x)+C$

62. $e^x\log(\sec x+\tan x)+C$ **63.** $e^x\tan x+C$ **64.** $e^x\cot x+C$

65. $e^x\sec x+C$ **66.** $-e^x\cot x+C$ **67.** $e^x\tan\frac{x}{2}+C$ **68.** $e^x\cot 2x+C$

69. $e^x\sin^{-1}x+C$ **70.** $e^x\log x+C$ **71.** $\frac{e^x}{(1+x)}+C$ **72.** $\frac{e^x}{(x+1)^2}+C$

73. $\frac{e^x}{(1-x)}+C$ **74.** $\frac{e^x}{(x-1)^2}+C$ **75.** $\frac{e^{3x}}{9x}+C$ **76.** $\frac{e^x}{(x+2)}+C$

77. $\frac{e^{2x}}{4(1+2x)}+C$ **78.** $\frac{e^{2x}}{4x}+C$ **79.** $e^x\left(\log x-\frac{1}{x}\right)+C$

80. $\dfrac{x}{(1+\log x)}+C$

81. $e^{\log x}\sin(\log x)+C$

82. $\dfrac{x}{(\log x)}+C$

83. $x\cdot\left\{\log(\log x)-\dfrac{1}{\log x}\right\}+C$

84. $\dfrac{2}{\pi}\{\sqrt{x-x^2}-(1-2x)\sin^{-1}\sqrt{x}\}-x+C$

85. $\dfrac{5^{5^{5^x}}}{(\log 5)^3}+C$

86. $\dfrac{1}{2}e^{2x}\tan x+C$

87. $\dfrac{1}{2}e^{2x}\tan x+C$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 13C)

9. Write $\sin^2 x=\dfrac{1}{2}(1-\cos 2x)$.

10. $\tan^2 x=(\sec^2 x-1)$.

15. Integrate by parts, taking $(\log 2x)$ as the 2nd function.

16. Integrate by parts, taking x as the second function.

18. Put $x^2=t$.

19. $\sin^3 x=\dfrac{3}{4}\sin x-\dfrac{1}{4}\sin 3x$.

20. $\cos^3 x=\dfrac{1}{4}\cos 3x+\dfrac{3}{4}\cos x$.

21. Put $x^2=t$.

22. Put $\cos x=t$.

23. $I=\dfrac{1}{2}\int x\sin 2x\,dx$.

24. Put $\sqrt{x}=t$ and $dx=2t\,dt$.

25. $I=\int \operatorname{cosec} x(\operatorname{cosec}^2 x)\,dx$. Now, integrate by parts.

26. Take $(\sin^3 x\cos x)$ as 2nd function. Its integral is $\dfrac{1}{4}\sin^4 x$.

Further, use $\sin^4 x=\left(\dfrac{1-\cos 2x}{2}\right)^2$.

27. Put $\cos x=t$.

29. Take $\log(2+x^2)$ as 1st function.

30. $I=\displaystyle\int\frac{x}{1+\cos\left(\frac{\pi}{2}-x\right)}dx=\frac{1}{2}\int x\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$

31. Integrate $\dfrac{1}{(\log x)}$ by parts, taking 1 as 2nd function.

32. $\cos 2x\cos 4x=\dfrac{1}{2}(\cos 6x+\cos 2x)$.

33. Put $\sqrt{x}=t$ and $dx=2t\,dt$.

34. $I=2\int e^{\sin x}\sin x\cos x\,dx=2\int t\,e^t dt$, where $\sin x=t$.

35. Put $x=\sin t$ and $dx=\cos t\,dt$.

36. Put $x=\tan t$, $dx=\sec^2 t\,dt$ and then write $\tan^2 t=(\sec^2 t-1)$.

37. Integrate by parts, taking $\log(x+2)$ as the 1st function and $\dfrac{1}{(x+2)^2}$ as the 2nd function.

38. Integrate by parts, taking $\sin^{-1} x$ as the 1st function and x as 2nd.

$$I = (\sin^{-1}x)\cdot\frac{x^2}{2} - \int \frac{1}{1-x^2}\cdot\frac{x^2}{2}\,dx = \frac{1}{2}x^2(\sin^{-1}x) + \frac{1}{2}\cdot\int\frac{(1-x^2)-1}{\sqrt{1-x^2}}dx$$

$$= \frac{1}{2}x^2(\sin^{-1}x) + \frac{1}{2}\cdot\int\sqrt{1-x^2}\,dx - \frac{1}{2}\int\frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{2}x^2(\sin^{-1}x) + \frac{1}{2}\cdot\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right] - \frac{1}{2}\sin^{-1}x + C.$$

43. Put $\sqrt{x} = \sin t$. Then, $x = \sin^2 t$ and $dx = \sin 2t\,dt$.

$$I = \int t\sin 2t\,dt = -\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t + C$$

$$= \frac{1}{2}t(2\sin^2 t - 1) + \frac{1}{2}\sin t\cos t = \frac{1}{2}\sin^{-1}\sqrt{x}\,(2x-1) + \frac{1}{2}\sqrt{x(1-x)} + C.$$

45. Put $x = \cos t$ and $dx = -\sin t\,dt$. Use $(4\cos^3 t - 3\cos t) = \cos 3t$.

46. Put $x = \tan t$ and $dx = \sec^2 t\,dt$. Use $\left(\frac{1-\tan^2 t}{1+\tan^2 t}\right) = \cos 2t$

47. Put $x = \tan t$ and $dx = \sec^2 t\,dt$.

48. Put $x = \tan t$ and $dx = \sec^2 t\,dt$. Use $\left(\frac{3\tan t - \tan^3 t}{1 - 3\tan^2 t}\right) = \tan 3t$.

49. Put $x = \sin t$ and $dx = \cos t\,dt$. Then,

$$I = \int t(\operatorname{cosec} t\cot t)\,dt = -t\operatorname{cosec} t + \log|\operatorname{cosec} t - \cot t| + C$$

$$= \frac{-\sin^{-1}x}{x} + \log\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + C.$$

50. Put $\tan x = t$ and $\sec^2 x\,dx = dt$.

51. Integrating by parts, taking e^{3x} as 2nd function, we get

$$I = (\sin 4x)\cdot\frac{e^{3x}}{3} - \int 4\cos 4x\cdot\frac{e^{3x}}{3}dx = \frac{1}{3}e^{3x}(\sin 4x) - \frac{4}{3}\cdot\int e^{3x}\cos 4x\,dx$$

$$= \frac{1}{3}e^{3x}(\sin 4x) - \frac{4}{3}\cdot\left[(\cos 4x)\cdot\frac{e^{3x}}{3} - \int(-4\sin 4x) - \frac{e^{3x}}{3}dx\right]$$

$$= \frac{1}{3}e^{3x}(\sin 4x) - \frac{4}{9}e^{3x}(\cos 4x) - \frac{16}{9}I$$

$$\Leftrightarrow \left(1 + \frac{16}{9}\right)I = \frac{1}{3}e^{3x}(\sin 4x) - \frac{4}{9}e^{3x}(\cos 4x).$$

53. $I = \frac{1}{2}\cdot\int e^{2x}\sin 2x\,dx.$

61. $I = \int e^x(\log\sec x + \tan x)\,dx = \int e^x\{f(x) + f'(x)\}\,dx$, where $f(x) = \log\sec x$.

62. $I = \int e^x[\log(\sec x + \tan x) + \sec x]\,dx = \int e^x\{f(x) + f'(x)\}\,dx$,
where $f(x) = \log(\sec x + \tan x)$.

63. $I = \int e^x(\sec^2 x + \tan x)\,dx = \int e^x(\tan x + \sec^2 x)\,dx = e^x\tan x + C$.

64. $I = \int e^x(\cot x - \operatorname{cosec}^2 x)\,dx = e^x\cot x + C$.

65. $I = \int e^x(\sec x + \sec x \tan x)\,dx = e^x \sec x + C.$

66. $I = \int e^x\left\{\dfrac{2}{2\sin^2 x} - \dfrac{2\sin x\cos x}{2\sin^2 x}\right\}dx = \int e^x(-\cot x + \operatorname{cosec}^2 x)\,dx = -e^x \cot x + C.$

67. $I = \int e^x\left\{\dfrac{1}{2\cos^2(x/2)} + \dfrac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right\}dx$

$= \int e^x \cdot \left\{\dfrac{1}{2}\sec^2\dfrac{x}{2} + \tan\dfrac{x}{2}\right\}dx = \int e^x\left\{\tan\dfrac{x}{2} + \dfrac{1}{2}\sec^2\dfrac{x}{2}\right\}dx = e^x \tan\dfrac{x}{2} + C.$

68. $I = \int e^x\left\{\dfrac{2\sin 2x\cos 2x}{2\sin^2 2x} - \dfrac{4}{2\sin^2 2x}\right\}dx = \int e^x\{\cot 2x - 2\operatorname{cosec}^2 2x\}\,dx = e^x \cot 2x + C.$

69 $I = \int e^x\left\{\sin^{-1}x + \dfrac{1}{\sqrt{1-x^2}}\right\}dx = e^x \sin^{-1}x + C.$

70. $I = \int e^x\left(\dfrac{1}{x} + \log x\right) = \int e^x\left(\log x + \dfrac{1}{x}\right)dx = e^x \log x + C.$

71. $I = \int e^x \dfrac{(1+x-1)}{(1+x)^2}\,dx = \int e^x\left[\dfrac{1}{(1+x)} - \dfrac{1}{(1+x)^2}\right]dx$

$= \int e^x\{f(x) + f'(x)\}\,dx$, where $f(x) = \dfrac{1}{(1+x)}\cdot$

72. $I = \int \dfrac{e^x(x+1-2)}{(x+1)^3}\,dx = \int e^x\left\{\dfrac{(x+1)}{(x+1)^3} - \dfrac{2}{(x+1)^3}\right\}dx$

$= \int e^x\left\{\dfrac{1}{(x+1)^2} - \dfrac{2}{(x+1)^3}\right\}dx = \int e^x \cdot \{f(x) + f'(x)\}\,dx$, where $f(x) = \dfrac{1}{(x+1)^2}\cdot$

73. $I = \int e^x \dfrac{(1-x+1)}{(1-x)^2}\,dx = \int e^x\left\{\dfrac{(1-x)}{(1-x)^2} + \dfrac{1}{(1-x)^2}\right\}dx$

$= \int e^x\left\{\dfrac{1}{(1-x)} + \dfrac{1}{(1-x)^2}\right\}dx = \int e^x \cdot \{f(x) + f'(x)\}\,dx$, where $f(x) = \dfrac{1}{(1-x)}\cdot$

74. $I = \int \dfrac{e^x(x-1-2)}{(x-1)^3}\,dx = \int e^x\left\{\dfrac{(x-1)}{(x-1)^3} - \dfrac{2}{(x-1)^3}\right\}dx$

$= \int e^x\left\{\dfrac{1}{(x-1)^2} - \dfrac{2}{(x-1)^3}\right\}dx = \int e^x \cdot \{f(x) + f'(x)\}\,dx$, where $f(x) = \dfrac{1}{(x-1)^2}\cdot$

75. Putting $3x = t$ and $dx = \dfrac{1}{3}\,dt$, we get

$I = \dfrac{1}{3}\cdot \int e^t\left(\dfrac{t-1}{t^2}\right)dt = \dfrac{1}{3}\cdot \int e^t\left(\dfrac{1}{t} - \dfrac{1}{t^2}\right)dt = \dfrac{1}{3}e^t \cdot \dfrac{1}{t} + C = \dfrac{e^{3x}}{9x} + C.$

76. $I = \int e^x \cdot \left\{\dfrac{(x+2)-1}{(x+2)^2}\right\}dx = \int e^x \cdot \left\{\dfrac{1}{(x+2)} - \dfrac{1}{(x+2)^2}\right\}dx = e^x \cdot \dfrac{1}{(x+2)} + C.$

77. Putting $2x = t$ and $dx = \frac{1}{2}dt$, we get

$$I = \frac{1}{2}\cdot\int \frac{2x\cdot e^{2x}}{(1+2x)^2}dx = \frac{1}{4}\cdot\int \frac{te^t}{(1+t)^2}dt = \frac{1}{4}\cdot\int \frac{(1+t-1)}{(1+t)^2}e^t dt$$

$$= \frac{1}{4}\cdot e^t\cdot\left\{\frac{1}{1+t} - \frac{1}{(1+t)^2}\right\}dt = \frac{e^t}{4(1+t)} + C = \frac{e^{2x}}{4(1+2x)} + C.$$

78. Putting $2x = t$ and $dx = \frac{1}{2}dt$, we get

$$I = \frac{1}{2}\cdot\int e^t\cdot\left(\frac{t-1}{t^2}\right)dt = \frac{1}{2}\cdot\int e^t\cdot\left\{\frac{1}{t} - \frac{1}{t^2}\right\}dt = \frac{1}{2}e^t\cdot\frac{1}{t} + C = \frac{1}{4}\cdot\frac{e^{2x}}{x} + C.$$

79. $I = \int e^x\left\{(\log x) + \frac{1}{x} - \frac{1}{x} + \frac{1}{x^2}\right\}dx = \int e^x\left\{\log x + \frac{1}{x}\right\}dx - \int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$

$$= e^x(\log x) - e^x\cdot\frac{1}{x} + C = e^x\left(\log x - \frac{1}{x}\right) + C.$$

80. Putting $\log x = t, x = e^t$ and $dx = e^t dt$, we get

$$I = \int \frac{t}{(1+t)^2}\cdot e^t dt = \int e^t\cdot\frac{(1+t-1)}{(1+t)^2}dt = \int e^t\cdot\left\{\frac{1}{(1+t)} - \frac{1}{(1+t)^2}\right\}dt$$

$$= e^t\frac{1}{(1+t)} + C = \frac{x}{(1+\log x)} + C.$$

81. Putting $\log x = t, x = e^t$ and $dx = e^t dt$, we get

$$I = \int e^t(\sin t + \cos t)\,dt = e^t\sin t + C = e^{\log x}\cdot\sin(\log x) + C.$$

82. Putting $\log x = t, x = e^t$ and $dx = e^t dt$, we get

$$I = \int e^t\left(\frac{1}{t} - \frac{1}{t^2}\right)dt = e^t\cdot\frac{1}{t} + C = \frac{x}{\log x} + C.$$

83. Putting $\log x = t, x = e^t$ and $dx = e^t dt$, we get

$$I = \int\left\{\log t + \frac{1}{t^2}\right\}e^t dt = \int e^t\left\{\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2}\right\}dt$$

$$= \int e^t\left(\log t + \frac{1}{t}\right)dt - \int e^t\left(\frac{1}{t} - \frac{1}{t^2}\right)dt = e^t\log t - e^t\cdot\frac{1}{t} + C$$

$$= x\left\{\log(\log x) - \frac{1}{\log x}\right\} + C.$$

84. $I = \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}x\right)}{(\pi/2)}dx = \frac{2}{\pi}\cdot\int\left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right)dx$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - \int dx = \frac{4}{\pi}\cdot\int \sin^{-1}\sqrt{x}\,dx - x + C$$

$$= \frac{4}{\pi}\cdot\left[\int t\sin 2t\,dt\right] - x + C, \text{ where } \sin^{-1}\sqrt{x} = t.$$

85. Putting $5^x = t \Leftrightarrow 5^x\log 5\,dx = dt \Leftrightarrow 5^x dx = \frac{1}{\log 5}dt.$

$$\therefore\ I = \int 5^{5^t} 5^t \frac{1}{\log 5} dt = \int 5^u \cdot \frac{1}{(\log 5)^2} du, \text{ where } 5^t = u \Leftrightarrow 5^t \log 5\, dt = du$$

$$= \int \frac{1}{(\log 5)^3} dv, \text{ where } 5^u = v \Leftrightarrow 5^u \log 5\, du = dv$$

$$= \frac{v}{(\log 5)^3} + C = \frac{5^u}{(\log 5)^3} + C = \frac{5^{5^t}}{(\log 5)^3} + C = \frac{5^{5^{5^x}}}{(\log 5)^3} + C.$$

87. $I = \int e^{2x} \left\{ \frac{1 + 2\sin x \cos x}{2\cos^2 x} \right\} dx$

$$= \int e^{2x} \left\{ \frac{1}{2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x} \right\} dx$$

$$= \frac{1}{2} \int e^{2x} (\sec^2 x)\, dx + \int e^{2x} \tan x\, dx$$

$$= \frac{1}{2} \{ e^{2x} \tan x - \int 2e^{2x} \tan x\, dx \} + \int e^{2x} \tan x\, dx + C$$

$$= \frac{1}{2} e^{2x} \tan x + C.$$

OBJECTIVE QUESTIONS II

Mark (✓) against the correct answer in each of the following:

1. $\int x e^x\, dx = ?$

   (a) $e^x(1 - x) + C$ (b) $e^x(x + 1) + C$ (c) $e^x(x - 1) + C$ (d) none of these

2. $\int x e^{2x} dx = ?$

   (a) $\frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + C$ (b) $\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$

   (c) $2x e^{2x} + 4e^{2x} + C$ (d) none of these

3. $\int x \cos 2x\, dx = ?$

   (a) $\frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C$ (b) $\frac{1}{2} x \sin 2x - \frac{1}{4} \cos 2x + C$

   (c) $2x \sin 2x + 4\cos 2x + C$ (d) none of these

4. $\int x \sec^2 x\, dx = ?$

   (a) $x \tan x - \log |\cos x| + C$ (b) $x \tan x + \log |\cos x| + C$

   (c) $x \tan x + \log |\sec x| + C$ (d) none of these

5. $\int x \sin 2x\, dx = ?$

   (a) $\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$ (b) $-\frac{1}{2} x \cos 2x - \frac{1}{4} \sin 2x + C$

   (c) $-\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$ (d) none of these

6. $\int x \log x\, dx = ?$

(a) $x \log x + \frac{1}{2}x^2 + C$

(b) $\frac{1}{2}x^2 \log x + \frac{1}{4}x^2 + C$

(c) $\frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + C$

(d) none of these

7. $\int x \operatorname{cosec}^2 x\, dx = ?$

(a) $x \cot x - \log |\sin x| + C$

(b) $-x \cot x + \log |\sin x| + C$

(c) $x \tan x - \log |\sec x| + C$

(d) none of these

8. $\int x \sin x \cos x\, dx = ?$

(a) $-\frac{1}{4}x \sin 2x + \frac{1}{8}\cos 2x + C$

(b) $\frac{1}{4}x \cos 2x - \frac{1}{8}\sin 2x + C$

(c) $\frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x + C$

(d) $-\frac{1}{4}x \cos 2x + \frac{1}{8}\sin 2x + C$

9. $\int x \cos^2 x\, dx = ?$

(a) $\frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C$

(b) $\frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C$

(c) $\frac{x^2}{4} + \frac{x \sin 2x}{4} - \frac{\cos 2x}{8} + C$

(d) none of these

10. $\int \frac{\log x}{x^2}\, dx = ?$

(a) $-\frac{1}{x}(\log x + 1) + C$

(b) $\frac{1}{x}(\log x - 1) + C$

(c) $\frac{1}{x}(\log x + 1) + C$

(d) none of these

11. $\int \log x\, dx = ?$

(a) $\frac{1}{x} + C$

(b) $\frac{1}{2}(\log x)^2 + C$

(c) $x(\log x + 1) + C$

(d) $x(\log x - 1) + C$

12. $\int \log_{10} x\, dx = ?$

(a) $\frac{1}{x}\log_e 10 + C$

(b) $\frac{1}{x}\log_{10} e + C$

(c) $x(\log x - 1)\log_e 10 + C$

(d) $x(\log x - 1)\log_{10} e + C$

13. $\int (\log x)^2 dx = ?$

(a) $\frac{2 \log x}{x} + C$

(b) $\frac{1}{3}(\log x)^3 + C$

(c) $x(\log x)^2 - 2x \log x + 2x + C$

(d) $x(\log x)^2 + 2x \log x - 2x + C$

14. $\int e^{\sqrt{x}}\,dx = ?$

(a) $e^{\sqrt{x}} + \sqrt{x} + C$

(b) $\frac{1}{2}e^{\sqrt{x}}(\sqrt{x}+1) + C$

(c) $2e^{\sqrt{x}}(\sqrt{x}-1) + C$

(d) none of these

15. $\int \cos\sqrt{x}\,dx = ?$

(a) $\sin\sqrt{x} + \cos\sqrt{x} + C$

(b) $\frac{1}{2}(\sqrt{x}\sin\sqrt{x} - \cos\sqrt{x}) + C$

(c) $2[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}] + C$

(d) none of these

16. $\int \cos(\log x)\,dx = ?$

(a) $\frac{x}{2}[\cos(\log x) - \sin(\log x)] + C$

(b) $\frac{x}{2}[\cos(\log x) + \sin(\log x)] + C$

(c) $2x[\cos(\log x) + \sin(\log x)] + C$

(d) $2x[\cos(\log x) - \sin(\log x)] + C$

17. $\int \sec^3 x\,dx = ?$

(a) $\frac{1}{2}\{\sec x\tan x - \log|\sec x + \tan x|\} + C$

(b) $\frac{1}{2}\{\sec x\tan x + \log|\sec x + \tan x|\} + C$

(c) $2\{\sec x\tan x + \log|\sec x + \tan x|\} + C$

(d) none of these

18. $\int \left\{\frac{1}{(\log x)} - \frac{1}{(\log x)^2}\right\} dx = ?$

(a) $x\log x + C$

(b) $\frac{x}{\log x} + C$

(c) $x + \frac{1}{\log x} + C$

(d) none of these

19. $\int 2x^3 e^{x^2}\,dx = ?$

(a) $e^{x^2}(x^2-1) + C$ (b) $e^{x^2}(x^2+1) + C$ (c) $e^{x^2}(x+1) + C$ (d) none of these

20. $\int (x2^x)\,dx = ?$

(a) $\frac{2^x}{(\log 2)}(x + \log 2) + C$

(b) $\frac{2^x}{(\log 2)^2}(x\log 2 - 1) + C$

(c) $\frac{x \cdot 2^x}{(\log 2)} + \frac{2^x}{(\log 2)^2} + C$

(d) none of these

21. $\int x\cot^2 x\,dx = ?$

(a) $-x\cot x + \frac{x^2}{2} + \log|\sin x| + C$ (b) $-x\cot x - \frac{x^2}{2} + \log|\sin x| + C$

(c) $-x\cot x + \frac{x^2}{2} - \log|\sin x| + C$ (d) none of these

22. $\int \sin\sqrt{x}\,dx = ?$

(a) $-\sqrt{x}\cos\sqrt{x} + C$ (b) $-\sqrt{x}\cos\sqrt{x} - 2\sin\sqrt{x} - C$

(c) $-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$ (d) none of these

23. $\int e^{\sin x}\sin 2x\,dx = ?$

(a) $(2\sin x)e^{\sin x} + C$ (b) $(2\cos x)e^{\sin x} + C$

(c) $2e^{\sin x}(\sin x + 1) + C$ (d) $2e^{\sin x}(\sin x - 1) + C$

24. $\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx = ?$

(a) $\frac{\sin^{-1}x}{\sqrt{1-x^2}} - \frac{1}{2}\log|1-x^2| + C$ (b) $x\sin^{-1}x + \frac{1}{2}\log|1-x^2| + C$

(c) $\frac{x\sin^{-1}x}{\sqrt{1-x^2}} + \frac{1}{2}\log|1-x^2| + C$ (d) none of these

25. $\int \frac{x\tan^{-1}x}{(1-x^2)^{3/2}}\,dx = ?$

(a) $\frac{\tan^{-1}x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}} + C$ (b) $\frac{-\tan^{-1}x}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + C$

(c) $\frac{x\tan^{-1}x}{\sqrt{1+x^2}} + \frac{1}{2}\log\left|\frac{x}{\sqrt{1+x^2}}\right| + C$ (d) none of these

26. $\int x\tan^{-1}x\,dx = ?$

(a) $\frac{1}{2}\tan^{-1}x + \log(1+x^2) - \frac{1}{2}x + C$ (b) $\frac{1}{2}x^2\tan^{-1}x + \frac{1}{2}x + C$

(c) $\frac{1}{2}(1+x^2)\tan^{-1}x - \frac{1}{2}x + C$ (d) none of these

27. $\int \tan^{-1}\sqrt{x}\,dx = ?$

(a) $(x-1)\tan^{-1}\sqrt{x} + \sqrt{x} + C$ (b) $(x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

(c) $\frac{1}{2}\sqrt{x}\tan^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x} + C$ (d) none of these

28. $\int \cos^{-1}x\,dx = ?$

(a) $x\cos^{-1}x - \sqrt{1-x^2} + C$ (b) $x\cos^{-1}x + \sqrt{1-x^2} + C$

(c) $x\sin^{-1}x - \sqrt{1-x^2} + C$ (d) none of these

29. $\int \tan^{-1} x\, dx = ?$

(a) $x \tan^{-1} x + \frac{1}{2} \log |1 + x^2| + C$

(b) $x \tan^{-1} x - \frac{1}{2} \log |1 + x^2| + C$

(c) $-x \tan^{-1} x + \frac{1}{2} \log |1 + x^2| + C$

(d) none of these

30. $\int \sec^{-1} x\, dx = ?$

(a) $x \sec^{-1} x + \log |x + \sqrt{x^2 - 1}| + C$

(b) $x \sec^{-1} x - \log |x + \sqrt{x^2 - 1}| + C$

(c) $x \sec^{-1} x + \log |x - \sqrt{x^2 - 1}| + C$

(d) none of these

31. $\int \sin^{-1}(3x - 4x^3)\, dx = ?$

(a) $3[x \sin^{-1} x + \sqrt{1 - x^2}] + C$

(b) $3[x \sin^{-1} x - \sqrt{1 - x^2}] + C$

(c) $\frac{3x^2}{2} + C$

(d) none of these

32. $\int \sin^{-1}\left(\frac{2x}{1 + x^2}\right) dx = ?$

(a) $2x \tan^{-1} x + \log |1 + x^2| + C$

(b) $2x \tan^{-1} x - \log |1 + x^2| + C$

(c) $2x \sin^{-1} x + \log |1 + x^2| + C$

(d) none of these

33. $\int \tan^{-1} \sqrt{\frac{1 - x}{1 + x}} dx = ?$

(a) $\frac{1}{2} x (\cos^{-1} x) + \frac{1}{2} \sqrt{1 - x^2} + C$

(b) $\frac{1}{2} x (\sin^{-1} x) + \frac{1}{2} \sqrt{1 - x^2} + C$

(c) $\frac{1}{2} x (\cos^{-1} x) - \frac{1}{2} \sqrt{1 - x^2} + C$

(d) none of these

34. $\int \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) dx = ?$

(a) $3x \tan^{-1} x + \frac{3}{2} \log (1 + x^2) + C$

(b) $3x \tan^{-1} x - \frac{3}{2} \log (1 + x^2) + C$

(c) $3x \cos^{-1} x - \frac{3}{2} \sqrt{1 - x^2} + C$

(d) $3x \sin^{-1} x + \frac{3}{2} \sqrt{1 - x^2} + C$

35. $\int x^2 \cos x\, dx = ?$

(a) $x^2 \sin x + 2x \cos x - 2 \sin x + C$

(b) $2x \cos x - x \sin x + 2 \sin x + C$

(c) $x^2 \sin x - 2x \sin x + 2 \sin x + C$

(d) none of these

36. $\int \sin x \log (\cos x)\, dx = ?$

(a) $\cos x \log (\cos x) - \cos x + C$

(b) $-\cos x \log (\cos x) + \cos x + C$

(c) $\cos x \log (\cos x) + \cos x + C$

(d) none of these

37. $\int x \sin x \cos x \, dx = ?$

(a) $-\frac{1}{4}x \cos 2x + \frac{1}{8}\sin 2x + C$
(b) $\frac{1}{4}x \cos 2x + \frac{1}{8}\sin 2x + C$
(c) $\frac{1}{4}x \cos 2x - \frac{1}{8}\sin 2x + C$
(d) none of these

38. $\int x^3 \cos x^2 dx = ?$

(a) $x^2 \sin x^2 + \cos x^2 + C$
(b) $\frac{1}{2}x^2 \sin x^2 + \frac{1}{2}\cos x^2 + C$
(c) $-\frac{1}{2}x^2 \sin x^2 + \frac{1}{2}\cos x^2 + C$
(d) none of these

39. $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx = ?$

(a) $2x \tan^{-1}x + \log(1+x^2) + C$
(b) $-2x \tan^{-1}x - 2\log(1+x^2) + C$
(c) $2x \tan^{-1}x - \log(1+x^2) + C$
(d) none of these

40. $\int x \tan^{-1}x \, dx = ?$

(a) $\frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x + C$
(b) $\frac{1}{2}(x^2-1)\tan^{-1}x - \frac{1}{2}x + C$
(c) $\frac{1}{2}(x^2+1)\tan^{-1}x + \frac{1}{2}x + C$
(d) none of these

41. $\int \sin(\log x) \, dx = ?$

(a) $\frac{1}{2}x \sin \log x + \frac{1}{2}x \cos(\log x) + C$
(b) $\frac{1}{2}x \sin \log x - \frac{1}{2}x \cos(\log x) + C$
(c) $-\frac{1}{2}x \sin(\log x) + \frac{1}{2}x \cos(\log x) + C$
(d) none of these

42. $\int (\sin^{-1}x)^2 dx = ?$

(a) $\frac{2\sin^{-1}x}{\sqrt{1-x^2}} + C$
(b) $\frac{1}{3}(\sin^{-1}x)^3 + \frac{1}{\sqrt{1-x^2}} + C$
(c) $x(\sin^{-1}x)^2 + (\sin^{-1}x)\sqrt{1-x^2} + 2x + C$
(d) $x(\sin^{-1}x)^2 + 2(\sin^{-1}x)\sqrt{1-x^2} - 2x + C$

43. $\int e^x \left\{\frac{1}{x} - \frac{1}{x^2}\right\} dx = ?$

(a) $e^x \left\{\log x + \frac{1}{x}\right\} + C$
(b) $xe^x - e^x + C$
(c) $e^x \cdot \frac{1}{x} + C$
(d) none of these

44. $\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx = ?$

(a) $\frac{-e^x}{x^2} + C$
(b) $\frac{e^x}{x^2} + C$
(c) $e^x \left(\frac{-1}{x} + \frac{1}{x^2}\right) + C$
(d) none of these

45. $\int e^x \left\{\sin^{-1} x + \frac{1}{\sqrt{1-x^2}}\right\} dx = ?$

(a) $e^x \cdot \frac{1}{\sqrt{1-x^2}} + C$
(b) $e^x \sin^{-1} x + C$
(c) $\frac{-e^x}{\sin^{-1} x} + C$
(d) none of these

46. $\int e^x (\tan x + \log \sec x)\, dx = ?$

(a) $e^x \log \sec x + C$
(b) $e^x \tan x + C$
(c) $e^x (\log \cos x) + C$
(d) none of these

47. $\int e^x (\cot x + \log \sin x)\, dx = ?$

(a) $e^x \cot x + C$
(b) $e^x \log \sin x + C$
(c) $e^x \sin x + C$
(d) none of these

48. $\int e^x [\sec x + \log (\sec x + \tan x)]\, dx = ?$

(a) $e^x \log (\sec x + \tan x) + C$
(b) $e^x \sec x + C$
(c) $e^x \log \tan x + C$
(d) none of these

49. $\int e^x \left\{\tan^{-1} x + \frac{1}{(1+x^2)}\right\} dx = ?$

(a) $e^x \cdot \frac{1}{(1+x^2)} + C$
(b) $e^x \tan^{-1} x + C$
(c) $-e^x \cot^{-1} x + C$
(d) none of these

50. $\int e^x (\tan x - \log \cos x)\, dx = ?$

(a) $e^x \tan x + C$
(b) $e^x \log \cos x + C$
(c) $e^x \log \sec x + C$
(d) none of these

51. $\int e^x(\cot x - \text{cosec}^2 x)\,dx = ?$

(a) $-e^x \text{cosec}^2 x + C$ (b) $e^x \cot x + C$ (c) $-e^x \cot x + C$ (d) none of these

52. $\int e^x(\sin x + \cos x)\,dx = ?$

(a) $e^x \sin x + C$ (b) $e^x \cos x + C$ (c) $e^x \tan x + C$ (d) none of these

53. $\int e^x \sec x(1 + \tan x)\,dx = ?$

(a) $e^x(1 + \tan x) + C$ (b) $e^x \sec x + C$

(c) $e^x \tan x + C$ (d) none of these

54. $\int e^x \left(\frac{1 + x\log x}{x}\right) dx = ?$

(a) $e^x \cdot \frac{1}{x} + C$ (b) $e^x \log x + C$ (c) $x e^x \log x + C$ (d) none of these

55. $\int e^x \cdot \frac{x}{(1+x)^2}\,dx = ?$

(a) $e^x \cdot \frac{1}{(1+x)} + C$ (b) $e^x \cdot \frac{1}{x} + C$ (c) $e^x \cdot \frac{x}{(1+x)} + C$ (d) none of these

56. $\int e^x \left(\frac{1 + \sin x}{1 + \cos x}\right) dx = ?$

(a) $e^x \sin \frac{x}{2} + C$ (b) $e^x \cos \frac{x}{2} + C$ (c) $e^x \tan \frac{x}{2} + C$ (d) none of these

ANSWERS (OBJECTIVE QUESTIONS II)

1. (c) **2.** (b) **3.** (a) **4.** (b) **5.** (c) **6.** (c) **7.** (b) **8** (d) **9.** (b) **10.** (a)
11. (d) **12.** (d) **13.** (c) **14.** (c) **15.** (c) **16.** (b) **17.** (b) **18.** (b) **19.** (a) **20.** (b)
21. (b) **22.** (c) **23.** (d) **24.** (c) **25.** (b) **26.** (c) **27.** (b) **28.** (a) **29.** (b) **30.** (b)
31. (a) **32.** (b) **33.** (c) **34.** (b) **35.** (a) **36.** (b) **37.** (a) **38.** (b) **39.** (c) **40.** (a)
41. (b) **42.** (d) **43.** (c) **44.** (b) **45.** (b) **46.** (a) **47.** (b) **48.** (a) **49.** (b) **50.** (c)
51. (b) **52.** (a) **53.** (b) **54.** (b) **55.** (a) **56.** (c)

HINTS TO THE GIVEN OBJECTIVE QUESTIONS II

1. $\int \underset{I}{x}\, \underset{II}{e^x}\,dx$ 2. $\int \underset{I}{x}\, \underset{II}{e^{2x}}\,dx$ 3. $\int \underset{I}{x}\, \underset{II}{\cos 2x}\,dx$ 4. $\int \underset{I}{x}\, \underset{II}{\sec^2 x}\,dx$

5. $\int \underset{I}{x}\, \underset{II}{\sin 2x}\,dx$ 6. $\int \underset{II}{x}\, \underset{I}{\log x}\,dx$ 7. $\int \underset{I}{x}\, \underset{II}{\text{cosec}^2 x}\,dx$ 8. $I = \frac{1}{2}\int \underset{I}{x}\, \underset{II}{\sin 2x}\,dx$

9. $I = \int x \cdot \frac{1}{2}(1 + \cos 2x)\,dx = \frac{1}{2}\int x\,dx + \frac{1}{2}\int \underset{I}{x}\, \underset{II}{\cos 2x}\,dx$

10. $\int \frac{\log x}{x^2}\,dx = \int \underset{\text{I}}{(\log x)} \cdot \underset{\text{II}}{\frac{1}{x^2}}\,dx$

11. $I = \int \underset{\text{I}}{(\log x)} \cdot \underset{\text{II}}{1}\,dx$

12. $I = \int \frac{\log x}{\log 10}\,dx = \frac{1}{\log_e 10} \cdot \int \underset{\text{I}}{(\log x)} \cdot \underset{\text{II}}{1}\,dx = (\log_{10} e) \cdot \int \left\{ \underset{\text{I}}{(\log x)} \cdot \underset{\text{II}}{1} \right\} dx$

13. $I = \left\{ \underset{\text{I}}{(\log x)^2} \cdot \underset{\text{II}}{1} \right\} dx = (\log x)^2 \cdot x - \int \frac{2\log x}{x} \cdot x\,dx$

$= x(\log x)^2 - 2\int \log x\,dx = x(\log x)^2 - 2[x(\log x - 1)] + C.$

14. Putting $\sqrt{x} = t$ and $\frac{1}{2\sqrt{x}}\,dx = dt$, we get $I = 2\int \underset{\text{I}}{t}\, \underset{\text{II}}{e^t}\,dt$

15. Putting $\sqrt{x} = t$ and $\frac{1}{\sqrt{x}}\,dx = 2dt$, we get $dx = 2t\,dt$

$\therefore \quad I = 2\int \underset{\text{I}}{t}\, \underset{\text{II}}{\cos t}\,dt.$

16. Putting $\log x = t$, $x = e^t$ and $dx = e^t dt$, we get

$I = \int \underset{\text{I}}{e^t}\, \underset{\text{II}}{\cos t}\,dt = e^t \cos t + e^t \sin t - \int e^t \cos t\,dt$

$\therefore \quad 2I = (e^t \cos t + e^t \sin t) \Rightarrow I = \frac{1}{2}(e^t \cos t + e^t \sin t) + C.$

17. $I = \int \underset{\text{II}}{\sec^2 x} \cdot \underset{\text{I}}{\sec x}\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$

$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx = \sec x \tan x - I + \int \sec x\,dx$

$\therefore \quad 2I = \sec x \tan x + \log|\sec x + \tan x| + C.$

18. $I = \int \left\{ \underset{\text{I}}{\frac{1}{(\log x)}} \cdot \underset{\text{II}}{1} \right\} dx - \int \frac{1}{(\log x)^2}\,dx$

$= \frac{1}{(\log x)} \cdot x - \int (-1) \cdot \frac{1}{(\log x)^2} \cdot \frac{1}{x} \cdot x\,dx - \int \frac{1}{(\log x)^2}\,dx + C$

$= \frac{x}{(\log x)} + C.$

19. $I = \int (2x)x^2 e^{x^2}\,dx = \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^t}\,dt$, where $x^2 = t$.

20. $I = \int \underset{\text{I}}{x} \cdot \underset{\text{II}}{2^x}\,dx = x \cdot \frac{2^x}{(\log 2)} - \int \frac{2^x}{(\log 2)}\,dx$

$= \frac{x \cdot 2^x}{(\log 2)} - \frac{2^x}{(\log 2)^2} + C$

21. $I = x(\operatorname{cosec}^2 x - 1)\,dx = \int \underset{\text{I}}{x}\, \underset{\text{II}}{\operatorname{cosec}^2 x}\,dx - \int x\,dx$

$= x(-\cot x) - \int (-\cot x)\,dx - \frac{x^2}{2} + C = -x\cos x + \log|\sin x| - \frac{x^2}{2} + C.$

22. Put $\sqrt{x} = t$ and $\frac{1}{2\sqrt{x}}\,dx = dt$ or $dx = 2t\,dt$

$\therefore \quad I = 2\int \underset{\text{I}}{t}\, \underset{\text{II}}{\sin t}\,dt.$

23. $I = 2\int e^{\sin x} \sin x \cos x\, dx = 2\int \underset{\text{I}}{t}\, \underset{\text{II}}{e^t}\, dt$, where $\sin x = t$.

24. Put $x = \sin t$ so that $dx = \cos t\ dt$ and $t = \sin^{-1} x$.

$$\therefore \quad I = \int \frac{t \cos t}{(1-\sin^2 t)^{3/2}} dt = \int \frac{t \cos t}{\cos^3 t} dt = \int \underset{\text{I}}{t}\, \underset{\text{II}}{\sec^2 t}\, dt.$$

25. Put $x = \tan t$, so that $dx = \sec^2 t\ dt$ and $t = \tan^{-1} x$.

$$\therefore \quad I = \int \frac{t(\tan t)}{(1+\tan^2 t)^{3/2}} \sec^2 t\, dt = \int \underset{\text{I}}{t}\, \underset{\text{II}}{\sin t}\, dt.$$

26. $I = \int \underset{\text{II}}{x} (\underset{\text{I}}{\tan^{-1} x})\, dx.$

27. Put $\sqrt{x} = t$ and $\dfrac{1}{2\sqrt{x}} dx = dt$ or $dx = 2t\ dt$.

$$\therefore \quad I = 2\int \underset{\text{II}}{t}\, (\underset{\text{I}}{\tan^{-1} t})\, dt.$$

28. Put $\cos^{-1} x = t$, so that $x = \cos t$ and $dx = -\sin t\, dt$.

$$\therefore \quad I = -\int \underset{\text{I}}{t}\, \underset{\text{II}}{\sin t}\, dt.$$

29. Put $\tan^{-1} x = t$, so that $x = \tan t$ and $dx = \sec^2 t\, dt$.

$$\therefore \quad I = \int \underset{\text{I}}{t}\, \underset{\text{II}}{\sec^2 t}\, dt.$$

30. Put $\sec^{-1} x = t$, so that $x = \sec t$ and $dx = \sec t \tan t\, dt$.

$$\therefore \quad I = \int \underset{\text{I}}{t} (\underset{\text{II}}{\sec t \tan t})\, dt.$$

31. Put $x = \sin t$ and $dx = \cos t\, dt$.

$$\therefore \quad I = \int \sin^{-1}(\sin 3t) \cos t\, dt = 3\int \underset{\text{I}}{t}\, \underset{\text{II}}{\cos t}\, dt.$$

32. Put $x = \tan t$ and $dx = \sec^2 t\, dt$.

$$\therefore \quad I = \int \sin^{-1}\left(\frac{2\tan t}{1+\tan^2 t}\right) \sec^2 t\, dt = \int \sin^{-1}(\sin 2t) \sec^2 t\, dt$$

$$= 2\int \underset{\text{I}}{t}\, \underset{\text{II}}{\sec^2 t}\, dt.$$

33. Put $x = \cos t$ and $dx = -\sin t\, dt$. Then,

$$I = \int \tan^{-1} \sqrt{\frac{1-\cos t}{1+\cos t}} \cdot (-\sin t)\, dt = \int \tan^{-1} \sqrt{\frac{2\sin^2(t/2)}{2\cos^2(t/2)}} (-\sin t)\, dt$$

$$= -\int \tan^{-1}\left(\tan \frac{t}{2}\right)(-\sin t)\, dt = \frac{1}{2}\int \underset{\text{I}}{t}\, (\underset{\text{II}}{\sin t})\, dt.$$

34. Put $x = \tan t$ and $dx = \sec^2 t\ dt$.

$$I = \int \tan^{-1}(\tan 3t) \sec^2 t\, dt = 3\int \underset{\text{I}}{t}\, \underset{\text{II}}{\sec^2 t}\, dt.$$

35. $I = \int \underset{\text{I}}{x^2} \underset{\text{II}}{\cos x}\, dx = x^2 \sin x - \int \underset{\text{I}}{2x} \underset{\text{II}}{\sin x}\, dx$

$= (x^2 \sin x) - 2[x(-\cos x) - \int (-\cos x)\, dx] + C$

$= (x^2 \sin x) + 2x \cos x - 2 \sin x + C.$

36. Put $\cos x = t$ and $-\sin x\, dx = dt$.

$\therefore \quad I = -\int \log t\, dt = -\int \left(\underset{\text{I}}{\log t} \cdot \underset{\text{II}}{1} \right) dt = -\left[(\log t)t - \int \frac{1}{t} \cdot t\, dt \right]$

$= -t(\log t) + t + C = -\cos t[\log(\cos t)] + \cos x + C.$

37. $I = \frac{1}{2} \int \underset{\text{I}}{x} \underset{\text{II}}{\sin 2x}\, dx.$

38. Put $x^2 = t$, so that $x\, dx = \frac{1}{2} dt$.

$\therefore \quad I = \int \frac{1}{2} t \cos t\, dt = \frac{1}{2} \int \underset{\text{I}}{t} \underset{\text{II}}{\cos t}.$

39. Put $x = \tan t$ and $dx = \sec^2 t\, dt$.

Then, $I = \int \cos^{-1} \left(\frac{1 - \tan^2 t}{1 + \tan^2 t} \right) \sec^2 t\, dt = \int \cos^{-1}(\cos 2t) \sec^2 t\, dt$

$= 2 \int \underset{\text{I}}{t} \underset{\text{II}}{\sec^2 t}\, dt.$

40. $I = \int \underset{\text{II}}{x} \underset{\text{I}}{\tan^{-1} x}\, dx = (\tan^{-1} x) \cdot \frac{x^2}{2} - \int \frac{1}{(1 + x^2)} \cdot \frac{x^2}{2}\, dx$

$= \frac{1}{2} x^2 (\tan^{-1} x) - \frac{1}{2} \int \left(1 - \frac{1}{1 + x^2} \right) dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C$

$= \frac{1}{2} (x^2 + 1) \tan^{-1} x - \frac{1}{2} x + C.$

41. $I = \int \left\{ \underset{\text{I}}{\sin(\log x)} \cdot \underset{\text{II}}{1} \right\} dx = (\sin \log x)x - \int \frac{\cos(\log x)}{x} \cdot x\, dx$

$= x \sin(\log x) - \int \left\{ \underset{\text{I}}{\cos(\log x)} \cdot \underset{\text{II}}{1} \right\} dx$

$= x \sin(\log x) - \cos(\log x) \cdot x + \int \frac{-\sin(\log x)}{x} \cdot x\, dx$

$= x \sin(\log x) - x \cos(\log x) - I$

$\therefore \quad 2I = x \sin(\log x) - x \cos(\log x) + C.$

42. Putting $x = \sin t$ and $dx = \cos t\, dt$, we get

$I = \int \{\sin^{-1}(\sin t)\}^2 \cos t\, dt = \int \underset{\text{I}}{t^2} \underset{\text{II}}{\cos t}\, dt$

$= t^2 \sin t - 2 \int \underset{\text{I}}{t} \underset{\text{II}}{\sin t}\, dt = t^2 \sin t - 2[t(-\cos t) - \int (-\cos t)\, dt]$

$= t^2 \sin t + 2t \cos t - 2 \sin t + C = x(\sin^{-1} x)^2 + 2(\sin^{-1} x)\sqrt{1 - x^2} - 2x + C.$

43. $I = \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \frac{1}{x}$

$= e^x f(x) + C = e^x \cdot \frac{1}{x} + C.$

44. $I = \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \dfrac{1}{x^2}$

$= e^x f(x) + C = e^x \cdot \dfrac{1}{x^2} + C.$

45. $I = \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \sin^{-1} x.$

46. $I = \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \log \sec x$

$= e^x f(x) + C = e^x \log \sec x + C.$

47. $I = \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \log \sin x$

$= e^x f(x) + C = e^x \log \sin x + C.$

48. $I = \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \log(\sec x + \tan x)$

$= e^x f(x) + C = e^x \log(\sec x + \tan x) + C.$

49. $I = \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \tan^{-1} x$

$= e^x f(x) + C = e^x \tan^{-1} x + C.$

50. $I = \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \log \sec x$

$= e^x f(x) + C = e^x \log \sec x + C.$

51. $I = \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \cot x$

$= e^x f(x) + C = e^x \cot x + C.$

52. $I = \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \sin x$

$= e^x f(x) + C = e^x \sin x + C.$

53. $I = \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \sec x$

$= e^x \sec x + C.$

54. $I = \int e^x \left(\dfrac{1}{x} + \log x \right) dx$

$= \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \log x$

$= e^x f(x) + C = e^x \log x + C.$

55. $I = \int e^x \left\{ \dfrac{(1+x) - 1}{(1+x)^2} \right\} dx = \int e^x \left\{ \dfrac{1}{(1+x)} - \dfrac{1}{(1+x)^2} \right\} dx$

$= \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \dfrac{1}{(1+x)}$

$= e^x f(x) + C = e^x \cdot \dfrac{1}{(1+x)} + C.$

56. $I = \int e^x \left(\dfrac{1 + 2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)} \right) dx = \int e^x \left(\tan \dfrac{x}{2} + \dfrac{1}{2} \sec^2 \dfrac{x}{2} \right) dx$

$= \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \tan \dfrac{x}{2}$

$= e^x f(x) + C = e^x \tan \dfrac{x}{2} + C.$

14. SOME SPECIAL INTEGRALS

Three Special Integrals

THEOREM (i) $\int \frac{dx}{(a^2-x^2)} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C.$

(ii) $\int \frac{dx}{(x^2-a^2)} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C.$

(iii) $\int \frac{dx}{(x^2+a^2)} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$

PROOF (i) $\int \frac{dx}{(a^2-x^2)} = \int \frac{dx}{(a+x)(a-x)}$

$$= \int \frac{1}{2a}\cdot\left\{\frac{(a-x)+(a+x)}{(a+x)(a-x)}\right\}dx = \frac{1}{2a}\cdot\left[\int\frac{dx}{(a+x)} + \int\frac{dx}{(a-x)}\right]$$

$$= \frac{1}{2a}\cdot[\log|a+x| - \log|a-x|] + C$$

$$= \frac{1}{2a}\cdot\log\left|\frac{a+x}{a-x}\right| + C.$$

$\therefore \quad \int \frac{dx}{(a^2-x^2)} = \frac{1}{2a}\cdot\log\left|\frac{a+x}{a-x}\right| + C.$

(ii) $\int \frac{dx}{(x^2-a^2)} = \int \frac{dx}{(x-a)(x+a)}$

$$= \int \frac{1}{2a}\cdot\left\{\frac{(x+a)-(x-a)}{(x-a)(x+a)}\right\}dx = \frac{1}{2a}\cdot\left[\int\frac{dx}{(x-a)} - \int\frac{dx}{(x+a)}\right]$$

$$= \frac{1}{2a}\cdot[\log|x-a| - \log|x+a|] + C = \frac{1}{2a}\cdot\log\left|\frac{x-a}{x+a}\right| + C.$$

$\therefore \quad \int \frac{dx}{(x^2-a^2)} = \frac{1}{2a}\cdot\log\left|\frac{x-a}{x+a}\right| + C.$

(iii) $\int \frac{dx}{(x^2+a^2)} = \frac{1}{a^2}\cdot\int\frac{dx}{\left(1+\frac{x^2}{a^2}\right)}$

$$= \frac{1}{a^2}\cdot\int\frac{a\,dt}{(1+t^2)} \quad \left[\text{putting } \frac{x}{a} = t \text{ and } dx = a\,dt\right]$$

$$= \frac{1}{a}\tan^{-1}t + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

$$\therefore \quad \int \frac{dx}{(x^2+a^2)} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

REMARK If we have an integral of the form $\int \frac{dx}{(ax^2+bx+c)}$ then we put the denominator in the form $[(x+\alpha)^2 \pm \beta^2]$ and then integrate.

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{dx}{(1-4x^2)}$ (ii) $\int \frac{dx}{(32-2x^2)}$ (iii) $\int \frac{x^2}{(1-x^6)}dx$

SOLUTION We have

(i) $$\int \frac{dx}{(1-4x^2)} = \frac{1}{4}\cdot\int \frac{dx}{\left(\frac{1}{4}-x^2\right)}$$

$$= \frac{1}{4}\cdot\int \frac{dx}{\left\{\left(\frac{1}{2}\right)^2 - x^2\right\}}$$

$$= \frac{1}{4}\cdot\frac{1}{\left(2\times\frac{1}{2}\right)}\cdot\log\left|\frac{\frac{1}{2}+x}{\frac{1}{2}-x}\right| + C$$

$$\left[\because \int \frac{dx}{(a^2-x^2)} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C\right]$$

$$= \frac{1}{4}\log\left|\frac{1+2x}{1-2x}\right| + C.$$

(ii) $$\int \frac{dx}{(32-2x^2)} = \frac{1}{2}\cdot\int \frac{dx}{(16-x^2)}$$

$$= \frac{1}{2}\cdot\int \frac{dx}{\{(4)^2-x^2\}}$$

$$= \frac{1}{2}\cdot\frac{1}{(2\times 4)}\log\left|\frac{4+x}{4-x}\right| + C$$

$$\left[\because \int \frac{dx}{(a^2-x^2)} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C\right]$$

$$= \frac{1}{16}\log\left|\frac{4+x}{4-x}\right| + C.$$

(iii) Putting $x^3 = t$ and $3x^2dx = dt$, we get

$$\int \frac{x^2}{(1-x^6)}dx = \frac{1}{3}\cdot\int \frac{1}{(1-t^2)}dt$$

$$= \frac{1}{3} \cdot \frac{1}{(2 \times 1)} \cdot \log \left| \frac{1+t}{1-t} \right| + C$$

$$\left[\because \int \frac{dx}{(a^2 - x^2)} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C \right]$$

$$= \frac{1}{6} \log \left| \frac{1+x^3}{1-x^3} \right| + C.$$

EXAMPLE 2 *Evaluate:* (i) $\int \frac{\sin x}{(1 - 4\cos^2 x)} dx$ (ii) $\int \frac{\text{cosec}^2 x}{(1 - \cot^2 x)} dx$

SOLUTION (i) Putting $\cos x = t$ and $-\sin x \, dx = dt$, we get

$$\int \frac{\sin x}{(1 - 4\cos^2 x)} dx = -\int \frac{dt}{(1 - 4t^2)}$$

$$= -\frac{1}{4} \cdot \int \frac{dt}{\left(\frac{1}{4} - t^2 \right)} = -\frac{1}{4} \cdot \int \frac{dt}{\left\{ \left(\frac{1}{2} \right)^2 - t^2 \right\}}$$

$$= -\frac{1}{4} \times \frac{1}{\left(2 \times \frac{1}{2} \right)} \log \left| \frac{\frac{1}{2} + t}{\frac{1}{2} - t} \right| + C$$

$$= -\frac{1}{4} \log \left| \frac{1+2t}{1-2t} \right| + C = -\frac{1}{4} \log \left| \frac{1 + 2\cos x}{1 - 2\cos x} \right| + C.$$

(ii) Putting $\cot x = t$ and $-\text{cosec}^2 x \, dx = dt$, we get

$$\int \frac{\text{cosec}^2 x}{(1 - \cot^2 x)} dx = -\int \frac{dt}{(1 - t^2)} = -\frac{1}{(2 \times 1)} \log \left| \frac{1+t}{1-t} \right| + C$$

$$= -\frac{1}{2} \log \left| \frac{1 + \cot x}{1 - \cot x} \right| + C.$$

EXAMPLE 3 *Evaluate:*

(i) $\int \frac{dx}{(9x^2 - 1)}$ (ii) $\int \frac{x}{(x^4 - 9)} dx$ (iii) $\int \frac{x^2}{(x^2 - 9)} dx$

SOLUTION We have

(i) $$\int \frac{dx}{(9x^2 - 1)} = \frac{1}{9} \cdot \int \frac{dx}{\left(x^2 - \frac{1}{9} \right)}$$

$$= \frac{1}{9} \cdot \int \frac{dx}{\left\{ x^2 - \left(\frac{1}{3} \right)^2 \right\}}$$

$$= \frac{1}{9} \cdot \frac{1}{\left(2 \times \frac{1}{3}\right)} \log \left| \frac{x - \frac{1}{3}}{x + \frac{1}{3}} \right| + C$$

$$\left[\because \int \frac{dx}{(x^2 - a^2)} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \right]$$

$$= \frac{1}{6} \log \left| \frac{3x-1}{3x+1} \right| + C.$$

(ii) Putting $x^2 = t$ and $2x\ dx = dt$, we get

$$\int \frac{x}{(x^4 - 9)} dx = \frac{1}{2} \int \frac{dt}{(t^2 - 9)} = \frac{1}{2} \cdot \int \frac{dt}{\{t^2 - (3)^2\}}$$

$$= \frac{1}{2} \cdot \frac{1}{(2 \times 3)} \log \left| \frac{t-3}{t+3} \right| + C$$

$$= \frac{1}{12} \log \left| \frac{x^2 - 3}{x^2 + 3} \right| + C.$$

(iii) $$\int \frac{x^2}{(x^2 - 9)} dx = \int \left\{ 1 + \frac{9}{x^2 - 9} \right\} dx$$

$$= \int dx + 9 \int \frac{dx}{\{x^2 - (3)^2\}}$$

$$= x + 9 \cdot \left[\frac{1}{(2 \times 3)} \log \left| \frac{x-3}{x+3} \right| \right] + C$$

$$= x + \frac{3}{2} \log \left| \frac{x-3}{x+3} \right| + C.$$

EXAMPLE 4 *Evaluate* $\displaystyle\int \frac{dx}{(4 + 25x^2)}$.

SOLUTION We have

$$\int \frac{dx}{(4 + 25x^2)} = \frac{1}{25} \cdot \int \frac{dx}{\left(\frac{4}{25} + x^2\right)}$$

$$= \frac{1}{25} \cdot \int \frac{dx}{\{(2/5)^2 + x^2\}}$$

$$= \frac{1}{25} \cdot \frac{1}{(2/5)} \tan^{-1} \frac{x}{(2/5)} + C \quad \left[\because \int \frac{dx}{(a^2 + x^2)} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \right]$$

$$= \frac{1}{10} \tan^{-1} \left(\frac{5x}{2} \right) + C.$$

EXAMPLE 5 *Evaluate* $\displaystyle\int \frac{3x}{(1 + 2x^4)} dx.$

SOLUTION Putting $x^2 = t$ and $2x\,dx = dt$, we get

$$\int \frac{3x}{(1+2x^4)}dx = \frac{3}{2}\cdot\int\frac{dt}{(1+2t^2)}$$

$$= \frac{3}{4}\cdot\int\frac{dt}{\left(\frac{1}{2}+t^2\right)} = \frac{3}{4}\cdot\int\frac{dt}{\left\{\left(\frac{1}{\sqrt{2}}\right)^2+t^2\right\}}$$

$$= \frac{3}{4}\cdot\frac{1}{(1/\sqrt{2})}\tan^{-1}\frac{t}{(1/\sqrt{2})}+C$$

$$= \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}t)+C = \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}x^2)+C.$$

EXAMPLE 6 *Evaluate* $\displaystyle\int\frac{dx}{(1+5\sin^2 x)}\cdot$

SOLUTION On dividing num. and denom. by $\cos^2 x$, we get

$$\int\frac{dx}{(1+5\sin^2 x)} = \int\frac{(1/\cos^2 x)}{\left(\frac{1}{\cos^2 x}+5\cdot\frac{\sin^2 x}{\cos^2 x}\right)}dx$$

$$= \int\frac{\sec^2 x}{(\sec^2 x+5\tan^2 x)}dx = \int\frac{\sec^2 x}{\{(1+\tan^2 x)+5\tan^2 x\}}dx$$

$$= \int\frac{\sec^2 x}{(1+6\tan^2 x)}dx = \int\frac{dt}{(1+6t^2)},\ \text{where } \tan x = t$$

$$= \frac{1}{6}\int\frac{dt}{\left(\frac{1}{6}+t^2\right)} = \frac{1}{6}\cdot\int\frac{dt}{\left\{\left(\frac{1}{\sqrt{6}}\right)^2+t^2\right\}}$$

$$= \frac{1}{6}\cdot\frac{1}{(1/\sqrt{6})}\tan^{-1}\frac{t}{(1/\sqrt{6})}+C = \frac{1}{\sqrt{6}}\tan^{-1}(\sqrt{6}t)+C$$

$$= \frac{1}{\sqrt{6}}\tan^{-1}(\sqrt{6}\tan x)+C.$$

EXAMPLE 7 *Evaluate* $\displaystyle\int\frac{dx}{(2+\sin^2 x)}\cdot$

SOLUTION On dividing num. and denom. by $\cos^2 x$, we get

$$\int\frac{dx}{(2+\sin^2 x)} = \int\frac{\sec^2 x}{(2\sec^2 x+\tan^2 x)}dx = \int\frac{\sec^2 x}{\{2(1+\tan^2 x)+\tan^2 x\}}dx$$

$$= \int\frac{\sec^2 x}{2+3\tan^2 x}dx = \int\frac{dt}{(2+3t^2)},\ \text{where } \tan x = t$$

$$= \frac{1}{3}\cdot\int\frac{dt}{\left(t^2+\frac{2}{3}\right)} = \frac{1}{3}\cdot\int\frac{dt}{\left(\sqrt{\frac{2}{3}}\right)^2+t^2}$$

$$= \frac{1}{3} \cdot \frac{1}{\sqrt{\frac{2}{3}}} \cdot \tan^{-1} \frac{t}{\left(\sqrt{\frac{2}{3}}\right)} + C = \frac{1}{\sqrt{6}} \tan^{-1} \frac{\sqrt{3}t}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{6}} \tan^{-1}\left(\frac{\sqrt{3} \tan x}{\sqrt{2}} \right) + C.$$

EXAMPLE 8 *Evaluate* $\int \frac{dx}{(x^2 + 6x + 13)}$.

SOLUTION We have

$$\int \frac{dx}{(x^2 + 6x + 13)} = \int \frac{dx}{\{(x^2 + 6x + 9) + 4\}}$$

$$= \int \frac{dx}{\{(x+3)^2 + 2^2\}} = \int \frac{dt}{(t^2 + 2^2)}, \text{ where } (x+3) = t$$

$$= \frac{1}{2} \tan^{-1} \frac{t}{2} + C$$

$$= \frac{1}{2} \tan^{-1} \frac{1}{2}(x+3) + C.$$

EXAMPLE 9 *Evaluate* $\int \frac{dx}{(x^2 + 8x + 20)}$. **[CBSE 2002C]**

SOLUTION We have

$$\int \frac{dx}{(x^2 + 8x + 20)} = \int \frac{dx}{\{(x+4)^2 + 2^2\}} = \int \frac{dt}{(t^2 + 2^2)}, \text{ where } (x+4) = t$$

$$= \frac{1}{2} \tan^{-1} \frac{t}{2} + C$$

$$= \frac{1}{2} \tan^{-1} \frac{1}{2}(x+4) + C.$$

EXAMPLE 10 *Evaluate* $\int \frac{dx}{(9x^2 - 12x + 8)}$.

SOLUTION We have

$$(9x^2 - 12x + 8) = 9\left(x^2 - \frac{4}{3}x + \frac{8}{9} \right)$$

$$= 9\left\{ \left(x^2 - \frac{4}{3}x + \frac{4}{9} \right) - \frac{4}{9} + \frac{8}{9} \right\} = 9\left\{ \left(x - \frac{2}{3} \right)^2 + \left(\frac{2}{3} \right)^2 \right\}$$

$$\therefore \int \frac{dx}{(9x^2 - 12x + 8)} = \frac{1}{9} \cdot \int \frac{dx}{\left\{ \left(x - \frac{2}{3} \right)^2 + \left(\frac{2}{3} \right)^2 \right\}}$$

$$= \frac{1}{9} \cdot \frac{1}{\left(\frac{2}{3} \right)} \tan^{-1} \left\{ \frac{\left(x - \frac{2}{3} \right)}{\left(\frac{2}{3} \right)} \right\} + C = \frac{1}{6} \tan^{-1}\left(\frac{3x - 2}{2} \right) + C.$$

EXAMPLE 11 *Evaluate* $\int \frac{x}{(x^4-x^2+1)}dx.$ **[CBSE 2003, '07C]**

SOLUTION Putting $x^2=t$ and $2x\,dx=dt$, we get

$$\int \frac{x}{(x^4-x^2+1)}dx = \frac{1}{2}\cdot\int \frac{dt}{(t^2-t+1)} = \frac{1}{2}\cdot\int \frac{dt}{\left\{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right\}}$$

$$= \frac{1}{2}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\frac{\left(t-\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}+C$$

$$= \frac{1}{\sqrt{3}}\cdot\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)+C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2-1}{\sqrt{3}}\right)+C.$$

EXAMPLE 12 *Evaluate* $\int \frac{dx}{(2x^2+x-1)}\cdot$

SOLUTION We have

$$\int \frac{dx}{(2x^2+x-1)} = \frac{1}{2}\cdot\int \frac{dx}{\left(x^2+\frac{1}{2}x-\frac{1}{2}\right)}$$

$$= \frac{1}{2}\int \frac{dx}{\left[\left\{x^2+\frac{1}{2}x+\left(\frac{1}{4}\right)^2\right\}-\frac{1}{16}-\frac{1}{2}\right]} = \frac{1}{2}\int \frac{dx}{\left[\left(x+\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]}$$

$$= \frac{1}{2}\cdot\frac{1}{2\cdot\frac{3}{4}}\log\left|\frac{\left(x+\frac{1}{4}\right)-\frac{3}{4}}{\left(x+\frac{1}{4}\right)+\frac{3}{4}}\right|+C = \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right|+C.$$

EXAMPLE 13 *Evaluate* $\int \frac{dx}{(3x^2+13x-10)}\cdot$

SOLUTION We have

$$(3x^2+13x-10) = 3\left(x^2+\frac{13}{3}x-\frac{10}{3}\right)$$

$$= 3\left\{\left(x+\frac{13}{6}\right)^2-\frac{169}{36}-\frac{10}{3}\right\} = 3\left\{\left(x+\frac{13}{6}\right)^2-\frac{289}{36}\right\}$$

$$= 3\left\{\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2\right\}.$$

$$\therefore \int \frac{dx}{(3x^2+13x-10)} = \int \frac{dx}{3\left\{\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2\right\}}$$

$$= \frac{1}{3}\int \frac{dt}{\left\{t^2 - \left(\frac{17}{6}\right)^2\right\}}, \text{ where } \left(x + \frac{13}{6}\right) = t$$

$$= \frac{1}{3} \cdot \frac{1}{\left(2 \times \frac{17}{6}\right)} \log \left|\frac{t - \frac{17}{6}}{t + \frac{17}{6}}\right| + C$$

$$\left[\because \int \frac{dx}{(x^2 - a^2)} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right|\right]$$

$$= \frac{1}{17} \log \left|\frac{6t - 17}{6t + 17}\right| + C$$

$$= \frac{1}{17} \log \left|\frac{6\left(x + \frac{13}{6}\right) - 17}{6\left(x + \frac{13}{6}\right) + 17}\right| = \frac{1}{17} \log \left|\frac{6x - 4}{6x + 30}\right| + C$$

$$= \frac{1}{17} \log \left|\frac{3x - 2}{3x + 15}\right| + C = \frac{1}{17} \log \left|\frac{(3x - 2)}{3(x + 5)}\right| + C$$

$$= \frac{1}{17}\left\{\log \frac{1}{3} + \log \left|\frac{3x - 2}{x + 5}\right|\right\} + C$$

$$= \frac{1}{17} \log \left|\frac{3x - 2}{x + 5}\right| + k,$$

$$\text{where } \frac{1}{17} \log \frac{1}{3} + C = k = \text{constant.}$$

EXAMPLE 14 *Evaluate* $\int \frac{dx}{(1 + x - x^2)}$.

SOLUTION We have

$$\int \frac{dx}{(1 + x - x^2)} = -\int \frac{dx}{(x^2 - x - 1)}$$

$$= -\int \frac{dx}{\left\{\left(x^2 - x + \frac{1}{4}\right) - \frac{5}{4}\right\}} = -\int \frac{dx}{\left\{\left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2\right\}}$$

$$= \int \frac{dx}{\left\{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2\right\}} = \int \frac{dx}{\left\{\left(\frac{\sqrt{5}}{2}\right)^2 - u^2\right\}}, \text{ where } \left(x - \frac{1}{2}\right) = u$$

$$= \frac{1}{\left(2 \times \frac{\sqrt{5}}{2}\right)} \cdot \log \left|\frac{\frac{\sqrt{5}}{2} + u}{\frac{\sqrt{5}}{2} - u}\right| + C$$

$$= \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} + 2u}{\sqrt{5} - 2u} \right| + C = \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} + 2\left(x - \frac{1}{2}\right)}{\sqrt{5} - 2\left(x - \frac{1}{2}\right)} \right| + C$$

$$= \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} + 2x - 1}{\sqrt{5} - 2x + 1} \right| + C = \frac{1}{\sqrt{5}} \log \left| \frac{(\sqrt{5} - 1) + 2x}{(\sqrt{5} + 1) - 2x} \right| + C.$$

EXAMPLE 15 *Evaluate* $\int \frac{dx}{(5 - 8x - x^2)}$.

SOLUTION We have

$$\int \frac{dx}{(5 - 8x - x^2)} = -\int \frac{dx}{(x^2 + 8x - 5)}$$

$$= -\int \frac{dx}{\{(x^2 + 8x + 16) - 21\}} = -\int \frac{dx}{\{(x+4)^2 - (\sqrt{21})^2\}}$$

$$= \int \frac{dx}{\{(\sqrt{21})^2 - (x+4)^2\}} = \int \frac{dt}{\{(\sqrt{21})^2 - t^2\}},$$

where $(x + 4) = t$

$$= \frac{1}{2\sqrt{21}} \cdot \log \left| \frac{\sqrt{21} + t}{\sqrt{21} - t} \right| + C$$

$$= \frac{1}{2\sqrt{21}} \cdot \log \left| \frac{\sqrt{21} + 4 + x}{\sqrt{21} - 4 - x} \right| + C.$$

EXAMPLE 16 *Evaluate* $\int \frac{dx}{(1 - 6x - 9x^2)}$.

SOLUTION We have

$$\int \frac{dx}{(1 - 6x - 9x^2)} = -\int \frac{dx}{(9x^2 + 6x - 1)} = -\frac{1}{9} \int \frac{dx}{\left(x^2 + \frac{2}{3}x - \frac{1}{9}\right)}$$

$$= -\frac{1}{9} \cdot \int \frac{dx}{\left\{\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) - \frac{2}{9}\right\}}$$

$$= -\frac{1}{9} \cdot \int \frac{dx}{\left\{\left(x + \frac{1}{3}\right)^2 - \left(\frac{\sqrt{2}}{3}\right)^2\right\}} = \frac{1}{9} \cdot \int \frac{dx}{\left\{\left(\frac{\sqrt{2}}{3}\right)^2 - \left(x + \frac{1}{3}\right)^2\right\}}$$

$$= \frac{1}{9} \cdot \int \frac{dx}{\left\{\left(\frac{\sqrt{2}}{3}\right)^2 - t^2\right\}}, \text{ where } \left(x + \frac{1}{3}\right) = t$$

$$= \frac{1}{9} \cdot \frac{1}{2 \cdot \frac{\sqrt{2}}{3}} \log \left| \frac{\frac{\sqrt{2}}{3} + t}{\frac{\sqrt{2}}{3} - t} \right| + C = \frac{1}{6\sqrt{2}} \log \left| \frac{\sqrt{2} + 3t}{\sqrt{2} - 3t} \right| + C$$

$$= \frac{1}{6\sqrt{2}} \log \left| \frac{\sqrt{2} + 3\left(x + \frac{1}{3}\right)}{\sqrt{2} - 3\left(x + \frac{1}{3}\right)} \right| + C$$

$$= \frac{1}{6\sqrt{2}} \log \left| \frac{\sqrt{2} + 1 + 3x}{\sqrt{2} - 1 - 3x} \right| + C.$$

EXAMPLE 17 *Evaluate* $\int \frac{\cos x}{(\sin^2 x + 4\sin x + 5)} dx.$

SOLUTION Putting $\sin x = t$ and $\cos x\, dx = dt$, we get

$$\int \frac{\cos x}{(\sin^2 x + 4\sin x + 5)} dx = \int \frac{dt}{(t^2 + 4t + 5)} = \int \frac{dt}{\{(t^2 + 4t + 4) + 1\}}$$

$$= \int \frac{dt}{\{(t+2)^2 + 1^2\}} = \int \frac{du}{(u^2 + 1)},$$

where $u = (t + 2)$

$$= \tan^{-1} u + C = \tan^{-1}(t + 2) + C$$

$$= \tan^{-1}(\sin x + 2) + C.$$

EXAMPLE 18 *Evaluate* $\int \frac{e^x}{(e^{2x} + 6e^x + 5)} dx.$

SOLUTION Putting $e^x = t$ and $e^x dx = dt$, we get

$$\int \frac{e^x}{e^{2x} + 6e^x + 5} dx = \int \frac{dt}{(t^2 + 6t + 5)} = \int \frac{dt}{\{(t^2 + 6t + 9) - 4\}}$$

$$= \int \frac{dt}{\{(t+3)^2 - 2^2\}} = \int \frac{du}{(u^2 - 2^2)}, \text{ where } (t+3) = u$$

$$= \frac{1}{(2 \times 2)} \log \left| \frac{u-2}{u+2} \right| + C = \frac{1}{4} \log \left| \frac{t+3-2}{t+3+2} \right| + C$$

$$= \frac{1}{4} \log \left| \frac{t+1}{t+5} \right| + C = \frac{1}{4} \log \left| \frac{e^x + 1}{e^x + 5} \right| + C.$$

Integrals of the form $\int \frac{(px+q)}{(ax^2+bx+c)} dx.$

METHOD Let $(px + q) = A \cdot \frac{d}{dx}(ax^2 + bx + c) + B$

Find A and B.

Now, the integrand so obtained can be integrated easily.

EXAMPLE 19 *Evaluate* $\int \frac{(5x-2)}{(1 + 2x + 3x^2)} dx.$ **[CBSE 2013]**

SOLUTION Let $(5x - 2) = A \cdot \frac{d}{dx}(1 + 2x + 3x^2) + B.$

Then, $(5x-2) = A(6x+2) + B$. … (i)

Comparing the coefficients of like powers of x on both sides, we get

$6A = 5$ and $2A + B = -2$.

This gives $A = \frac{5}{6}$ and $B = \frac{-11}{3}$.

$$\therefore \quad I = \int \frac{\left\{\frac{5}{6}(6x+2) - \frac{11}{3}\right\}}{(1+2x+3x^2)} dx$$

$$= \frac{5}{6} \cdot \int \frac{6x+2}{(1+2x+3x^2)} dx - \frac{11}{3} \int \frac{dx}{(3x^2+2x+1)}$$

$$= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{3} \cdot \frac{1}{3} \int \frac{dx}{\left(x^2 + \frac{2}{3}x + \frac{1}{3}\right)}$$

$$= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{9} \cdot \int \frac{dx}{\left\{\left(x+\frac{1}{3}\right)^2 + \left(\frac{1}{3} - \frac{1}{9}\right)\right\}}$$

$$= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{9} \cdot \int \frac{dx}{\left\{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right\}} + C$$

$$= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{9} \cdot \frac{1}{\left(\frac{\sqrt{2}}{3}\right)} \tan^{-1}\left\{\frac{x+\frac{1}{3}}{\left(\frac{\sqrt{2}}{3}\right)}\right\} + C$$

$$= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{3\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C.$$

EXAMPLE 20 *Evaluate* $\int \frac{(3x+1)}{(2x^2-2x+3)} dx.$ **[CBSE 2006]**

SOLUTION Let $(3x+1) = A \cdot \frac{d}{dx}(2x^2 - 2x + 3) + B$. Then,

$(3x+1) = A(4x-2) + B$ … (i)

Comparing the coefficients of like powers of x, we get

$$(4A = 3 \text{ and } B - 2A = 1) \Rightarrow \left(A = \frac{3}{4} \text{ and } B = \frac{5}{2}\right).$$

$$\therefore \quad \int \frac{(3x+1)}{(2x^2-2x+3)} dx = \int \frac{A \cdot (4x-2) + B}{(2x^2-2x+3)}$$

$$= \int \frac{\frac{3}{4} \cdot (4x-2) + \frac{5}{2}}{(2x^2-2x+3)} dx = \frac{3}{4} \cdot \int \frac{(4x-2)}{(2x^2-2x+3)} dx + \frac{5}{2} \int \frac{dx}{2\left(x^2 - x + \frac{3}{2}\right)}$$

$$= \frac{3}{4}\log|2x^2 - 2x + 3| + \frac{5}{4}\cdot\int \frac{dx}{\left\{\left(x^2 - x + \frac{1}{4}\right) + \frac{5}{4}\right\}}$$

$$= \frac{3}{4}\log|2x^2 - 2x + 3| + \frac{5}{4}\cdot\int \frac{dx}{\left\{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{5}}{2}\right)^2\right\}}$$

$$= \frac{3}{4}\log|2x^2 - 2x + 3| + \frac{5}{4}\cdot\frac{1}{\left(\frac{\sqrt{5}}{2}\right)}\tan^{-1}\left\{\frac{\left(x - \frac{1}{2}\right)}{\left(\frac{\sqrt{5}}{2}\right)}\right\} + C$$

$$= \frac{3}{4}\log|2x^2 - 2x + 3| + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C.$$

EXAMPLE 21 *Evaluate* $\int \frac{(2x+1)}{(4-3x-x^2)}\,dx.$ **[CBSE 2004C]**

SOLUTION Let $(2x+1) = A\cdot\frac{d}{dx}(4-3x-x^2) + B.$

Then, $(2x+1) = A(-3-2x) + B$ … (i)

Comparing the coefficients of like terms, we get

$$(-2A = 2 \ \text{ and } \ -3A + B = 1) \Rightarrow (A = -1,\ B = -2).$$

$$\therefore \int \frac{(2x+1)}{(4-3x-x^2)}\,dx = \int\left\{\frac{(-1)\cdot(-3-2x) - 2}{(4-3x-x^2)}\right\}dx$$

$$= -\int \frac{(-3-2x)}{(4-3x-x^2)}\,dx - 2\int \frac{dx}{(4-3x-x^2)}$$

$$= -\log|4-3x-x^2| + 2\int \frac{dx}{(x^2+3x-4)}$$

$$= -\log|4-3x-x^2| + 2\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - \left(4+\frac{9}{4}\right)}$$

$$= -\log|4-3x-x^2| + 2\int \frac{dx}{\left\{\left(x+\frac{3}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right\}}$$

$$= -\log|4-3x-x^2| + \frac{2}{\left(2\times\frac{5}{2}\right)}\log\left|\frac{\left(x+\frac{3}{2}\right) - \frac{5}{2}}{\left(x+\frac{3}{2}\right) + \frac{5}{2}}\right| + C$$

$$= -\log|4-3x-x^2| + \frac{2}{5}\log\left|\frac{x-1}{x+4}\right| + C.$$

EXAMPLE 22 *Evaluate* $\int \left(\frac{x^2+5x+3}{x^2+3x+2} \right) dx.$

SOLUTION We have

$$\frac{(x^2+5x+3)}{(x^2+3x+2)} = \left\{ 1 + \frac{(2x+1)}{x^2+3x+2} \right\}$$

$$\Rightarrow \int \frac{(x^2+5x+3)}{(x^2+3x+2)} dx = \int dx + \int \frac{(2x+1)}{(x^2+3x+2)} dx. \qquad \ldots \text{(i)}$$

Let $(2x+1) = A \cdot \frac{d}{dx}(x^2+3x+2) + B$. Then,

$$(2x+1) = A(2x+3) + B. \qquad \ldots \text{(ii)}$$

Comparing the coefficients of like powers of x, we get

$$(2A = 2 \text{ and } 3A + B = 1) \Rightarrow (A = 1 \text{ and } B = -2).$$

$\therefore \quad (2x+1) = (2x+3) - 2.$

$$\therefore \quad I = x + \int \frac{(2x+1)}{(x^2+3x+2)} dx = x + \int \frac{\{(2x+3)-2\}}{(x^2+3x+2)} dx$$

$$= x + \int \frac{(2x+3)}{(x^2+3x+2)} dx - 2\int \frac{dx}{(x^2+3x+2)}$$

$$= x + \log|x^2+3x+2| - 2\int \frac{dx}{\left\{ \left(x^2+3x+\frac{9}{4} \right) + \left(2 - \frac{9}{4} \right) \right\}}$$

$$= x + \log|x^2+3x+2| - 2\int \frac{dx}{\left\{ \left(x+\frac{3}{2} \right)^2 - \left(\frac{1}{2} \right)^2 \right\}}$$

$$= x + \log|x^2+3x+2| - 2 \cdot \frac{1}{\left(2 \times \frac{1}{2} \right)} \log \left| \frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}} \right| + C$$

$$= x + \log|x^2+3x+2| - 2\log\left| \frac{x+1}{x+2} \right| + C.$$

Integrals of the form $\int \frac{dx}{a+b\cos^2 x}$, $\int \frac{dx}{a+b\sin^2 x}$

and $\int \frac{dx}{a\cos^2 x + b\sin x\cos x + c\sin^2 x}$.

METHOD In each such an integral, we divide the numerator and denominator by $\cos^2 x$ and put $\tan x = t$, $\sec^2 x\, dx = dt$ and then integrate.

EXAMPLE 23 *Evaluate* $\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$. **[CBSE 2003C]**

SOLUTION Dividing the numerator and the denominator of the given integrand by $\cos^2 x$, we get

$$\int \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)} = \int \frac{\sec^2 x}{a^2 \tan^2 x + b^2} dx$$

$$= \int \frac{dt}{(a^2 t^2 + b^2)} \qquad [\text{putting } \tan x = t]$$

$$= \frac{1}{a^2} \int \frac{dt}{\left[t^2 + \left(\frac{b}{a}\right)^2\right]} = \frac{1}{a^2} \cdot \frac{1}{\left(\frac{b}{a}\right)} \tan^{-1} \frac{t}{\left(\frac{b}{a}\right)} + C$$

$$= \frac{1}{ab} \tan^{-1}\left(\frac{at}{b}\right) + C = \frac{1}{ab} \tan^{-1}\left(\frac{a \tan x}{b}\right) + C.$$

EXAMPLE 24 *Evaluate:*

(i) $\int \frac{dx}{(1 + 3\sin^2 x)}$ (ii) $\int \frac{dx}{(3 + 2\cos^2 x)}$

SOLUTION (i) Dividing the numerator and denominator by $\cos^2 x$, we get

$$\int \frac{dx}{(1 + 3\sin^2 x)} = \int \frac{\sec^2 x}{\sec^2 x + 3\tan^2 x} dx = \int \frac{\sec^2 x}{(1 + 4\tan^2 x)} dx$$

$$= \int \frac{dt}{(1 + 4t^2)} \quad [\text{putting } \tan x = t]$$

$$= \frac{1}{4} \int \frac{dt}{\left[t^2 + \left(\frac{1}{2}\right)^2\right]} = \frac{1}{4} \cdot \frac{1}{(1/2)} \tan^{-1} \frac{t}{(1/2)} + C$$

$$= \frac{1}{2} \tan^{-1}(2t) + C = \frac{1}{2} \tan^{-1}(2 \tan x) + C.$$

(ii) Dividing the numerator and denominator by $\cos^2 x$, we get

$$\int \frac{dx}{(3 + 2\cos^2 x)} = \int \frac{\sec^2 x}{(3\sec^2 x + 2)} dx = \int \frac{\sec^2 x}{5 + 3\tan^2 x} dx$$

$$= \int \frac{dt}{(5 + 3t^2)} \qquad [\text{putting } \tan x = t]$$

$$= \frac{1}{3} \int \frac{dt}{\left[t^2 + \left(\frac{5}{3}\right)\right]} = \frac{1}{3} \cdot \int \frac{dt}{\left[t^2 + \left(\frac{\sqrt{5}}{\sqrt{3}}\right)^2\right]}$$

$$= \frac{1}{3} \cdot \frac{1}{\left(\frac{\sqrt{5}}{\sqrt{3}}\right)} \tan^{-1} \frac{t}{\left(\frac{\sqrt{5}}{\sqrt{3}}\right)} + C = \frac{1}{\sqrt{15}} \tan^{-1}\left(\frac{\sqrt{3}t}{\sqrt{5}}\right) + C$$

$$= \frac{1}{\sqrt{15}} \tan^{-1}\left(\frac{\sqrt{3} \tan x}{\sqrt{5}}\right) + C.$$

EXAMPLE 25 *Evaluate* $\int \frac{dx}{(4\sin^2 x + 5\cos^2 x)}$.

SOLUTION On dividing the numerator and denominator by $\cos^2 x$, we get

$$\int \frac{dx}{(4\sin^2 x + 5\cos^2 x)} = \int \frac{\sec^2 x}{(4\tan^2 x + 5)} dx$$

$$= \int \frac{dt}{4t^2 + 5} \quad [\text{putting } \tan x = t]$$

$$= \frac{1}{4} \int \frac{dt}{\left(t^2 + \frac{5}{4}\right)} = \frac{1}{4} \cdot \int \frac{dt}{\left[t^2 + \left(\frac{\sqrt{5}}{2}\right)^2\right]}$$

$$= \frac{1}{4} \cdot \frac{1}{\left(\frac{\sqrt{5}}{2}\right)} \tan^{-1} \frac{t}{\left(\frac{\sqrt{5}}{2}\right)} + C$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{2t}{\sqrt{5}}\right) + C = \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + C.$$

EXAMPLE 26 *Evaluate* $\int \frac{dx}{(1 + 3\sin^2 x + 8\cos^2 x)}$.

SOLUTION On dividing the numerator and denominator by $\cos^2 x$, we get

$$\int \frac{dx}{(1 + 3\sin^2 x + 8\cos^2 x)} = \int \frac{\sec^2 x\, dx}{\sec^2 x + 3\tan^2 x + 8}$$

$$= \int \frac{\sec^2 x}{9 + 4\tan^2 x} dx = \int \frac{dt}{9 + 4t^2}$$

$$[\text{putting } \tan x = t]$$

$$= \frac{1}{4} \int \frac{dt}{\left(t^2 + \frac{9}{4}\right)} = \frac{1}{4} \cdot \int \frac{dt}{\left[t^2 + \left(\frac{3}{2}\right)^2\right]}$$

$$= \frac{1}{4} \cdot \frac{1}{(3/2)} \cdot \tan^{-1}\left\{\frac{t}{(3/2)}\right\} + C$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{2t}{3}\right) + C = \frac{1}{6} \tan^{-1}\left(\frac{2\tan x}{3}\right) + C.$$

EXAMPLE 27 *Evaluate* $\int \frac{\sin x}{\sin 3x} dx$.

SOLUTION We have

$$\int \frac{\sin x}{\sin 3x} dx = \int \frac{\sin x}{(3\sin x - 4\sin^3 x)} dx$$

$$= \int \frac{1}{(3 - 4\sin^2 x)} dx \quad [\text{dividing num. and denom. by } \sin x]$$

$$= \int \frac{\sec^2 x}{3\sec^2 x - 4\tan^2 x}\,dx \quad \text{[dividing num. and denom. by } \cos^2 x]$$

$$= \int \frac{\sec^2 x}{3(1+\tan^2 x) - 4\tan^2 x}\,dx = \int \frac{\sec^2 x}{(3-\tan^2 x)}\,dx$$

$$= \int \frac{dt}{(3-t^2)}, \text{ where } \tan x = t \text{ and } \sec^2 x\,dx = dt$$

$$= \int \frac{dt}{[(\sqrt{3})^2 - t^2]} = \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}+t}{\sqrt{3}-t} \right| + C$$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x} \right| + C.$$

EXAMPLE 28 *Evaluate* $\int \frac{\cos x}{\cos 3x}\,dx.$

SOLUTION We have

$$\int \frac{\cos x}{\cos 3x}\,dx = \int \frac{\cos x}{(4\cos^3 x - 3\cos x)}\,dx$$

$$= \int \frac{dx}{4\cos^2 x - 3} = \int \frac{dx}{4\cos^2 x - 3(\sin^2 x + \cos^2 x)}$$

$$= \int \frac{dx}{\cos^2 x - 3\sin^2 x} = \int \frac{\sec^2 x}{1 - 3\tan^2 x}\,dx$$

[on dividing the num. and denom. by $\cos^2 x$]

$$= \int \frac{dt}{1-3t^2}, \text{ where } \tan x = t \text{ and } \sec^2 x\,dx = dt$$

$$= \frac{1}{3}\cdot\int \frac{dt}{\left(\frac{1}{3} - t^2\right)} = \frac{1}{3}\cdot\int \frac{dt}{\left[\left(\frac{1}{\sqrt{3}}\right)^2 - t^2\right]} = \frac{1}{3}\cdot\frac{1}{2\cdot\frac{1}{\sqrt{3}}} \log \left| \frac{\frac{1}{\sqrt{3}}+t}{\frac{1}{\sqrt{3}}-t} \right| + C$$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{1+\sqrt{3}\,t}{1-\sqrt{3}\,t} \right| + C = \frac{1}{2\sqrt{3}} \log \left| \frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x} \right| + C.$$

EXAMPLE 29 *Evaluate* $\int \frac{dx}{(2+\cos x)}.$

SOLUTION We have

$$\int \frac{dx}{(2+\cos x)} = \int \frac{dx}{1+(1+\cos x)} = \int \frac{dx}{1+2\cos^2(x/2)} = \int \frac{\sec^2(x/2)\,dx}{\sec^2(x/2)+2}$$

[dividing the num. and denom. by $\cos^2(x/2)$]

$$= \int \frac{\sec^2(x/2)}{3 + \tan^2(x/2)}\,dx = 2\int \frac{dt}{3+t^2}, \text{ where } \tan(x/2) = t$$

$$= 2\cdot \int \frac{dt}{(\sqrt{3})^2 + t^2} = 2\cdot \frac{1}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left[\frac{\tan(x/2)}{\sqrt{3}}\right] + C.$$

Some More Special Integrals

EXAMPLE 30 *Evaluate* $\int \frac{(x^2+1)}{(x^4+1)}\,dx.$ **[CBSE 2006C, '07, '11C]**

SOLUTION We have

$$\int \left(\frac{x^2+1}{x^4+1}\right) dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)} dx \quad \text{[dividing num. and denom. by } x^2\text{]}$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2 + 2} dx$$

$$= \int \frac{dt}{[t^2 + (\sqrt{2})^2]}, \text{ where } \left(x - \frac{1}{x}\right) = t \text{ and } \left(1+\frac{1}{x^2}\right)dx = dt$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + C.$$

EXAMPLE 31 *Evaluate* $\int \frac{(x^2+4)}{(x^4+16)}\,dx.$ **[CBSE 2007C, '11C]**

SOLUTION We have

$$\int \left(\frac{x^2+4}{x^4+16}\right) dx = \int \frac{\left(1+\frac{4}{x^2}\right)}{\left(x^2+\frac{16}{x^2}\right)} dx \quad \text{[dividing num. and denom. by } x^2\text{]}$$

$$= \int \frac{\left(1+\frac{4}{x^2}\right)}{\left(x-\frac{4}{x}\right)^2 + 8} dx$$

$$= \int \frac{dt}{(t^2+8)} \quad \left[\text{putting } \left(x - \frac{4}{x}\right) = t \text{ and } \left(1 + \frac{4}{x^2}\right) dx = dt\right]$$

$$= \int \frac{dt}{t^2 + (\sqrt{8})^2} = \frac{1}{\sqrt{8}} \tan^{-1}\left(\frac{t}{\sqrt{8}}\right) + C$$

$$= \frac{1}{\sqrt{8}} \tan^{-1} \frac{\left(x - \frac{4}{x}\right)}{\sqrt{8}} + C = \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2-4}{2\sqrt{2}x}\right) + C.$$

EXAMPLE 32 *Evaluate* $\int \frac{(x^2-1)}{(x^4+x^2+1)} dx.$

SOLUTION We have

$$\int \frac{(x^2-1)}{(x^4+x^2+1)} dx = \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x^2 + \frac{1}{x^2} + 1\right)} dx$$

[dividing num. and denom. by x^2]

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left[\left(x + \frac{1}{x}\right)^2 - 1\right]} dx = \int \frac{dt}{(t^2-1)}$$

$\left[\text{putting } \left(x + \frac{1}{x}\right) = t \text{ and } \left(1 - \frac{1}{x^2}\right) dx = dt\right]$

$$= \frac{1}{2} \log \left|\frac{t-1}{t+1}\right| + C = \frac{1}{2} \log \left|\frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1}\right| + C$$

$$= \frac{1}{2} \log \left|\frac{x^2 - x + 1}{x^2 + x + 1}\right| + C.$$

EXAMPLE 33 *Evaluate* $\int \frac{dx}{(x^4+1)}\cdot$

SOLUTION We have

$$\int \frac{dx}{(x^4+1)} = \int \frac{(x^2+1) - (x^2-1)}{2(x^4+1)} dx$$

$$= \frac{1}{2} \int \frac{(x^2+1)}{(x^4+1)} dx - \frac{1}{2} \int \frac{(x^2-1)}{(x^4+1)} dx$$

$$= \frac{1}{2}\left[\int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}dx - \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}dx\right]$$

[dividing num. and denom. of each integral by x^2]

$$= \frac{1}{2}\left[\int \frac{\left(1+\frac{1}{x^2}\right)}{\left[\left(x-\frac{1}{x}\right)^2+2\right]}dx - \int \frac{\left(1-\frac{1}{x^2}\right)}{\left[\left(x+\frac{1}{x}\right)^2-2\right]}dx\right]$$

$$= \frac{1}{2}\left[\int \frac{dt}{[t^2+(\sqrt{2})^2]} - \int \frac{du}{[u^2-(\sqrt{2})^2]}\right]$$

[putting $\left(x-\frac{1}{x}\right)=t$ in the 1st integral, and $\left(x+\frac{1}{x}\right)=u$ in the 2nd]

$$= \frac{1}{2}\left\{\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|\right\}+C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right|+C.$$

EXAMPLE 34 *Evaluate* $\displaystyle\int \frac{x^2}{(x^4+x^2+1)}dx.$ **[CBSE 2008C]**

SOLUTION We have

$$I = \frac{1}{2}\int \frac{2x^2}{(x^4+x^2+1)}dx$$

$$= \frac{1}{2}\cdot\int \frac{(x^2-1)+(x^2+1)}{(x^4+x^2+1)}dx$$

$$= \frac{1}{2}\int \frac{(x^2-1)}{(x^4+x^2+1)}dx + \frac{1}{2}\int \frac{(x^2+1)}{(x^4+x^2+1)}dx$$

$$= \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+1+\frac{1}{x^2}\right)}dx + \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+1+\frac{1}{x^2}\right)}dx$$

[on dividing num. and denom. of each by x^2]

$$= \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)}{\left\{\left(x+\frac{1}{x}\right)^2-1\right\}}dx + \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)}{\left\{\left(x-\frac{1}{x}\right)^2+(\sqrt{3})^2\right\}}dx$$

$$= \frac{1}{2}\int \frac{du}{(u^2-1)} + \frac{1}{2}\int \frac{dv}{\{v^2+(\sqrt{3})^2\}}$$

$$\left\{\begin{array}{l}\text{putting } \left(x+\frac{1}{x}\right)=u \text{ and } \left(1-\frac{1}{x^2}\right)=du \text{ in } I_1, \\ \quad\text{and } \left(x-\frac{1}{x}\right)=v \text{ and } \left(1+\frac{1}{x^2}\right)dx = dv \text{ in } I_2\end{array}\right\}$$

$$= \frac{1}{2}\log\left|\frac{u-1}{u+1}\right| + \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{v}{\sqrt{3}} + C$$

$$= \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + C$$

$$= \frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2+1}{\sqrt{3}x}\right) + C.$$

EXAMPLE 35 *Evaluate* $\int \sqrt{\cot x}\,dx$.

SOLUTION Put $\cot x = t^2$ so that $-\operatorname{cosec}^2 x\,dx = 2t\,dt$ or $dx = \frac{-2t}{(1+t^4)}dt$.

$$\therefore \int \sqrt{\cot x}\,dx = -\int \frac{2t^2}{(t^4+1)}dt$$

$$= -\int \frac{[(t^2+1)+(t^2-1)]}{(t^4+1)}dt = -\int \frac{(t^2+1)}{(t^4+1)}dt - \int \frac{(t^2-1)}{(t^4+1)}dt$$

$$= -\int \frac{\left(1+\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)}dt - \int \frac{\left(1-\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)}dt$$

$$= -\int \frac{\left(1+\frac{1}{t^2}\right)}{\left[\left(t-\frac{1}{t}\right)^2+2\right]}dt - \int \frac{\left(1-\frac{1}{t^2}\right)}{\left[\left(t+\frac{1}{t}\right)^2-2\right]}dt$$

$$= -\int \frac{du}{[u^2+(\sqrt{2})^2]} - \int \frac{dv}{[v^2-(\sqrt{2})^2]}$$

[putting $\left(t-\frac{1}{t}\right)=u$ in the 1st integral, and $\left(t+\frac{1}{t}\right)=v$ in the 2nd]

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| + C$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right| + C$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot x-1}{\sqrt{2\cot x}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{\cot x-\sqrt{2\cot x}+1}{\cot x+\sqrt{2\cot x}+1}\right| + C.$$

EXAMPLE 36 *Evaluate* $\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx.$ **[CBSE 2010C, '13C]**

SOLUTION We have

$$\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx$$

$$= \int\left(\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}\right)dx = \int\frac{(\tan x+1)}{\sqrt{\tan x}}\,dx$$

$$= \int\frac{(t^2+1)}{t}\cdot\frac{2t}{(1+t^4)}\,dt, \text{ where } \tan x = t^2 \Rightarrow x = \tan^{-1}t^2$$

$$\Rightarrow dx = \frac{2t}{(1+t^4)}\,dt$$

$$= 2\int\frac{(t^2+1)}{(t^4+1)}\,dt = 2\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)}\,dt \text{ [on dividing num. and denom. by } t^2]$$

$$= 2\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}\,dt$$

$$= 2\int\frac{du}{(u^2+2)}, \text{ where } \left(t-\frac{1}{t}\right) = u \text{ and } \left(1+\frac{1}{t^2}\right)dt = du$$

$$= 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + C = \sqrt{2}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}} + C \qquad \left[\because\ u = \left(t-\frac{1}{t}\right)\right]$$

$$= \sqrt{2}\tan^{-1}\frac{(t^2-1)}{(\sqrt{2}t)} + C = \sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right) + C \qquad [\because\ t^2 = \tan x].$$

EXAMPLE 37 *Evaluate* $\int\sqrt{\tan\theta}\,d\theta.$ **[CBSE 2006]**

SOLUTION Putting $\tan\theta = t^2$, we get $\theta = \tan^{-1}t^2 \Rightarrow d\theta = \frac{2t}{(1+t^4)}\,dt.$

$$\therefore \quad I = \int t \cdot \frac{2t}{(1+t^4)}\,dt = \int \frac{2t^2}{(t^4+1)}\,dt$$

$$= \int \frac{(t^2+1)+(t^2-1)}{(t^4+1)}\,dt = \int \frac{(t^2+1)}{(t^4+1)}\,dt + \int \frac{(t^2-1)}{(t^4+1)}\,dt$$

$$= \int \frac{\left(1+\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)}\,dt + \int \frac{\left(1-\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)}\,dt$$

$$= \int \frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}\,dt + \int \frac{\left(1-\frac{1}{t^2}\right)}{\left\{\left(t+\frac{1}{t}\right)^2-2\right\}}\,dt$$

$$= \int \frac{du}{(u^2+2)} + \int \frac{dv}{(v^2-2)},$$

where $\left(t-\frac{1}{t}\right) = u$ and $\left(t+\frac{1}{t}\right) = v$ in I_1 and I_2

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\cdot\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\cdot\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right) + \frac{1}{2\sqrt{2}}\cdot\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right| + C, \text{ where } t = \sqrt{\tan\theta}.$$

EXERCISE 14A

Evaluate:

1. $\int \frac{dx}{(1-9x^2)}$
2. $\int \frac{dx}{(25-4x^2)}$
3. $\int \frac{dx}{(x^2+16)}$ **[CBSE 2011]**
4. $\int \frac{dx}{(4+9x^2)}$
5. $\int \frac{dx}{(50+2x^2)}$
6. $\int \frac{dx}{(16x^2-25)}$
7. $\int \frac{(x^2-1)}{(x^2+4)}\,dx$
8. $\int \frac{x^2}{(9+4x^2)}\,dx$
9. $\int \frac{e^x}{(e^{2x}+1)}\,dx$
10. $\int \frac{\sin x}{(1+\cos^2 x)}\,dx$
11. $\int \frac{\cos x}{(1+\sin^2 x)}\,dx$
12. $\int \frac{3x^5}{(1+x^{12})}\,dx$
13. $\int \frac{2x^3}{(4+x^8)}\,dx$
14. $\int \frac{dx}{(e^x+e^{-x})}$
15. $\int \frac{x}{(1-x^4)}\,dx$

16. $\int \frac{x^2}{(a^6 - x^6)} dx$ 17. $\int \frac{dx}{(x^2 + 4x + 8)}$ 18. $\int \frac{dx}{(4x^2 - 4x + 3)}$

19. $\int \frac{dx}{(2x^2 + x + 3)}$ 20. $\int \frac{dx}{(2x^2 - x - 1)}$ 21. $\int \frac{dx}{(3 - 2x - x^2)}$

22. $\int \frac{x}{(x^2 + 3x + 2)} dx$ 23. $\int \frac{(x - 3)}{(x^2 + 2x - 4)} dx$ 24. $\int \frac{(2x - 3)}{(x^2 + 3x - 18)} dx$

25. $\int \frac{x^2}{(x^2 + 6x - 3)} dx$ 26. $\int \frac{(2x - 1)}{(2x^2 + 2x + 1)} dx$ 27. $\int \frac{(1 - 3x)}{(3x^2 + 4x + 2)} dx$

28. $\int \frac{2x}{(2 + x - x^2)} dx$ 29. $\int \frac{dx}{(1 + \cos^2 x)}$ 30. $\int \frac{dx}{(2 + \sin^2 x)}$

31. $\int \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)}$ 32. $\int \frac{dx}{(\cos^2 x - 3\sin^2 x)}$ 33. $\int \frac{dx}{(\sin^2 x - 4\cos^2 x)}$

34. $\int \frac{dx}{(\sin x \cos x + 2\cos^2 x)}$ 35. $\int \frac{\sin 2x}{(\sin^4 x + \cos^4 x)} dx$

36. $\int \frac{(2\sin 2\phi - \cos \phi)}{(6 - \cos^2 \phi - 4\sin \phi)} d\phi$ 37. $\int \frac{dx}{(\sin x - 2\cos x)(2\sin x + \cos x)}$

38. $\int \frac{(1 - x^2)}{(1 + x^4)} dx$ **[CBSE 2007]** 39. $\int \frac{(x^2 + 1)}{(x^4 + x^2 + 1)} dx$ 40. $\int \frac{dx}{(\sin^4 x + \cos^4 x)}$

ANSWERS (EXERCISE 14A)

1. $\frac{1}{6} \log \left| \frac{1 + 3x}{1 - 3x} \right| + C$ **2.** $\frac{1}{20} \log \left| \frac{5 + 2x}{5 - 2x} \right| + C$ **3.** $\frac{1}{4} \tan^{-1} \frac{x}{4} + C$

4. $\frac{1}{6} \tan^{-1} \left(\frac{3x}{2} \right) + C$ **5.** $\frac{1}{10} \tan^{-1} \frac{x}{5} + C$ **6.** $\frac{1}{40} \log \left| \frac{4x - 5}{4x + 5} \right| + C$

7. $x - \frac{5}{2} \tan^{-1} \frac{x}{2} + C$ **8.** $\frac{x}{4} - \frac{3}{8} \tan^{-1} \left(\frac{2x}{3} \right) + C$ **9.** $\tan^{-1}(e^x) + C$

10. $-\tan^{-1}(\cos x) + C$ **11.** $\tan^{-1}(\sin x) + C$ **12.** $\frac{1}{2} \tan^{-1}(x^6) + C$

13. $\frac{1}{4} \tan^{-1} \left(\frac{x^4}{2} \right) + C$ **14.** $\tan^{-1}(e^x) + C$ **15.** $\frac{1}{4} \log \left| \frac{1 + x^2}{1 - x^2} \right| + C$

16. $\frac{1}{6a^3} \log \left| \frac{a^3 + x^3}{a^3 - x^3} \right| + C$ **17.** $\frac{1}{2} \tan^{-1} \left(\frac{x + 2}{2} \right) + C$ **18.** $\frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{2x - 1}{\sqrt{2}} \right) + C$

19. $\frac{2}{\sqrt{23}} \tan^{-1} \left(\frac{4x + 1}{\sqrt{23}} \right) + C$ **20.** $\frac{1}{3} \log \left| \frac{2(x - 1)}{2x + 1} \right| + C$ **21.** $\frac{1}{4} \log \left| \frac{3 + x}{1 - x} \right| + C$

22. $\frac{1}{2} \log |x^2 + 3x + 2| - \frac{3}{2} \log \left| \frac{x + 1}{x + 2} \right| + C$

23. $\frac{1}{2}\log|x^2+2x-4|-\frac{2}{\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right|+C$

24. $\log|x^2+3x-18|-\frac{2}{3}\log\left|\frac{x-3}{x+6}\right|+C$

25. $x-3\log|x^2+6x-3|+\frac{7\sqrt{3}}{4}\log\left|\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}}\right|+C$

26. $\frac{1}{2}\log|2x^2+2x+1|-2\tan^{-1}(2x+1)+C$

27. $-\frac{1}{2}\log|3x^2+4x+2|+\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+2}{\sqrt{2}}\right)+C$

28. $-\log|2+x-x^2|+\frac{1}{3}\log\left|\frac{1+x}{2-x}\right|+C$

29. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)+C$

30. $\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right)+C$

31. $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right)+C$

32. $\frac{1}{2\sqrt{3}}\log\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right|+C$

33. $\frac{1}{4}\log\left|\frac{\tan x-2}{\tan x+2}\right|+C$

34. $\log|\tan x+2|+C$

35. $\tan^{-1}(\tan^2 x)+C$

36. $2\log|\sin^2\phi-4\sin\phi+5|+7\tan^{-1}(\sin\phi-2)+C$

37. $\frac{1}{5}\log\left|\frac{\tan x-2}{2\tan x+1}\right|+C$

38. $\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}x+x^2+1}{\sqrt{2}x-x^2-1}\right|+C$

39. $\frac{1}{\sqrt{3}}\tan^{-1}\frac{(x^2-1)}{\sqrt{3}x}+C$

40. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x-1}{\sqrt{2}\tan x}\right)+C$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 14A)

1. On dividing, we get $\frac{(x^2-1)}{(x^2+4)}=\left\{1-\frac{5}{(x^2+4)}\right\}$.

8. On dividing x^2 by $(4x^2+9)$, we get $\frac{x^2}{(9+4x^2)}=\frac{1}{4}-\frac{(9/4)}{4x^2+9}$.

   $\therefore\quad I=\frac{1}{4}\int dx-\frac{9}{19}\cdot\int\frac{dx}{\left\{x^2+\left(\frac{3}{2}\right)^2\right\}}$.

9. Put $e^x=t$ and $e^x dx=dt$.

10. Put $\cos x=t$ and $-\sin x\,dx=dt$.

12. Put $x^6=t$ and $6x^5 dx=dt$.

14. Multiply numerator and denominator by e^x and put $e^x=t$.

15. Put $x^2 = t$ and $2x\,dx = dt$.

16. Put $x^3 = t$ and $3x^2 dx = dt$.

22. Let $x = A \cdot \frac{d}{dx}(x^2 + 3x + 2) + B$. Then, $x = A \cdot (2x + 3) + B$.

$\therefore \; (2A = 1 \text{ and } 3A + B = 0) \Rightarrow \left(A = \frac{1}{2} \text{ and } B = \frac{-3}{2}\right)$.

23. Let $(x - 3) = A \cdot \frac{d}{dx}(x^2 + 2x - 4) + B \Rightarrow (x - 3) = A(2x + 2) + B$.

$\therefore \; (2A = 1 \text{ and } 2A + B = -3) \Rightarrow \left(A = \frac{1}{2} \text{ and } B = -4\right)$.

24. Let $(2x - 3) = A \cdot \frac{d}{dx}(x^2 + 3x - 18) + B \Rightarrow (2x - 3) = A\,(2x + 3) + B$.

25. On dividing x^2 by $(x^2 + 6x - 3)$, we get

$$I = \int \left\{1 - \frac{6x - 3}{x^2 + 6x - 3}\right\} dx = x - 3\int \frac{(2x - 1)}{(x^2 + 6x - 3)}\,dx.$$

Let $(2x - 1) = A \cdot \frac{d}{dx}(x^2 + 6x - 3) + B \Rightarrow (2x - 1) = A\,(2x + 6) + B$.

29. On dividing num. and denom. by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x}{(2 + \tan^2 x)}\,dx = \int \frac{dt}{(2 + t^2)}, \text{ where } \tan x = t.$$

30. Divide num. and denom. by $\cos^2 x$.

31. Divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

32. Divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

33. Divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

34. Divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

35. On dividing num. and denom. by $\cos^4 x$ and putting $\tan x = t$, $\sec^2 x\,dx = dt$, we get

$$I = \int \frac{2t}{(t^4 + 1)}\,dt. \text{ Now put } t^2 = u.$$

36. $I = \int \frac{(4\sin\phi - 1)\cos\phi}{(\sin^2\phi - 4\sin\phi + 5)}\,d\phi = \int \frac{(4t - 1)}{(t^2 - 4t + 5)}\,dt$, where $\sin\phi = t$.

Let $(4t - 1) = A \cdot \frac{d}{dt}(t^2 - 4t + 5) + B$.

37. $I = \int \frac{dx}{(2\sin^2 x - 3\sin x\cos x - 2\cos^2 x)}$.

Now, divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

38. $I = -\int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x^2 + \frac{1}{x^2}\right)}\,dx = -\int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 2}$

$$= \int \frac{-dt}{(t^2 - 2)}, \text{ where } \left(x + \frac{1}{x}\right) = t \text{ and } \left(1 - \frac{1}{x^2}\right) dx = dt$$

$$= \int \frac{dt}{(\sqrt{2})^2 - t^2} = \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2} + t}{\sqrt{2} - t} \right| + C$$

$$= \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2} + x + \frac{1}{x}}{\sqrt{2} - x - \frac{1}{x}} \right| + C = \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2}x + x^2 + 1}{\sqrt{2}x - x^2 - 1} \right| + C.$$

39. $\dfrac{1}{(\sin^4 x + \cos^4 x)} = \dfrac{\sec^4 x}{(\tan^4 x + 1)} = \dfrac{(1 + \tan^2 x)\sec^2 x}{(\tan^4 x + 1)}$.

Now, put $\tan x = t$.

Three More Special Integrals

THEOREM (i) $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C.$

(ii) $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C.$

(iii) $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C.$

PROOF (i) Put $x = a \sin\theta$ so that $dx = a\cos\theta\, d\theta$.

$$\therefore \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta}{a\cos\theta} d\theta = \int d\theta = \theta + C = \sin^{-1}\frac{x}{a} + C.$$

Hence, $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C.$

(ii) Put $x = a\sec\theta$ so that $dx = a\sec\theta\tan\theta\, d\theta$.

$$\therefore \quad \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta}{a\tan\theta} d\theta$$

$$= \int \sec\theta\, d\theta = \log|\sec\theta + \tan\theta| + c$$

$$= \log \left| \sec\theta + \sqrt{\sec^2\theta - 1} \right| + c$$

$$= \log \left| \frac{x}{a} + \sqrt{\left(\frac{x^2}{a^2} - 1\right)} \right| + c = \log \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + c$$

$$= \log \left| x + \sqrt{x^2 - a^2} \right| - \log a + c$$

$$= \log \left| x + \sqrt{x^2 - a^2} \right| + C \quad [\text{taking } -\log a + c = C].$$

Hence, $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C.$

(iii) Put $x = a\tan\theta$ so that $dx = a\sec^2\theta\, d\theta$.

$$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta}{a\sec\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta = \log|\sec\theta + \tan\theta| + c$$

$$= \log\left|\sqrt{1+\tan^2\theta} + \tan\theta\right| + c$$

$$= \log\left|\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right| + c$$

$$= \log\left|x + \sqrt{x^2+a^2}\right| - \log a + c$$

$$= \log\left|x + \sqrt{x^2+a^2}\right| + C \quad [\text{taking } -\log a + c = C].$$

Hence, $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C.$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\displaystyle\int \frac{dx}{\sqrt{9-25x^2}}$ (ii) $\displaystyle\int \frac{dx}{\sqrt{4x^2-9}}$ (iii) $\displaystyle\int \frac{dx}{\sqrt{16x^2+25}}$

SOLUTION We have

(i) $$\int \frac{dx}{\sqrt{9-25x^2}} = \frac{1}{5}\cdot\int \frac{dx}{\sqrt{\frac{9}{25}-x^2}} = \frac{1}{5}\cdot\int \frac{dx}{\sqrt{\left(\frac{3}{5}\right)^2 - x^2}}$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{x}{3/5}\right) + C = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C.$$

(ii) $$\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2-\frac{9}{4}}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2-\left(\frac{3}{2}\right)^2}}$$

$$= \frac{1}{2}\log\left|x+\sqrt{x^2-\frac{9}{4}}\right| + C$$

$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2-9}\right| + C.$$

(iii) $$\int \frac{dx}{\sqrt{16x^2+25}} = \frac{1}{4}\cdot\int \frac{dx}{\sqrt{x^2+\frac{25}{16}}} = \frac{1}{4}\cdot\int \frac{dx}{\sqrt{x^2+\left(\frac{5}{4}\right)^2}}$$

$$= \frac{1}{4}\cdot\log\left|x+\sqrt{x^2+\frac{25}{16}}\right| + C$$

$$= \frac{1}{4}\log\left|4x+\sqrt{16x^2+25}\right| + C.$$

EXAMPLE 2 *Evaluate* $\int \frac{dx}{\sqrt{15-8x^2}}\cdot$ **[CBSE 2002C]**

SOLUTION We have

$$\int \frac{dx}{\sqrt{15-8x^2}} = \frac{1}{\sqrt{8}}\cdot\int \frac{dx}{\sqrt{\frac{15}{8}-x^2}}$$

$$= \frac{1}{2\sqrt{2}}\sin^{-1}\left\{\frac{x}{\sqrt{\frac{15}{8}}}\right\} + C = \frac{1}{2\sqrt{2}}\sin^{-1}\left(\sqrt{\frac{8}{15}}x\right) + C.$$

EXAMPLE 3 *Evaluate* $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx.$

SOLUTION Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx = \int \frac{dt}{\sqrt{4-t^2}} = \int \frac{dt}{\sqrt{2^2-t^2}}$$

$$= \sin^{-1}\frac{t}{2} + C = \sin^{-1}\left(\frac{\sin x}{2}\right) + C.$$

EXAMPLE 4 *Evaluate* $\int \frac{dx}{\sqrt{1-e^{2x}}}\cdot$

SOLUTION Multiplying the numerator and denominator by e^{-x}, we get

$$\int \frac{dx}{\sqrt{1-e^{2x}}} = \int \frac{e^{-x}\,dx}{\sqrt{e^{-2x}(1-e^{2x})}} = \int \frac{e^{-x}}{\sqrt{e^{-2x}-1}}\,dx$$

$$= -\int \frac{dt}{\sqrt{t^2-1}} \quad [\text{putting } e^{-x} = t]$$

$$= -\log\left|t + \sqrt{t^2-1}\right| + C$$

$$= -\log\left|e^{-x} + \sqrt{e^{-2x}-1}\right| + C.$$

EXAMPLE 5 *Evaluate* $\int \frac{2^x}{\sqrt{1-4^x}}\,dx.$

SOLUTION Putting $2^x = t$ and $(2^x \log 2)\,dx = dt$, we get

$$\int \frac{2^x}{\sqrt{1-4^x}}\,dx = \frac{1}{(\log 2)}\cdot\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{(\log 2)}\cdot\sin^{-1}t + C = \frac{1}{(\log 2)}\cdot\sin^{-1}(2^x) + C.$$

EXAMPLE 6 *Evaluate* $\int \frac{x^2}{\sqrt{x^6-1}}\,dx.$

SOLUTION Putting $x^3 = t$ and $x^2 dx = \frac{1}{3}\,dt$, we get

$$\int \frac{x^2}{\sqrt{x^6-1}}\,dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2-1}} = \frac{1}{3}\log\left|t+\sqrt{t^2-1}\right| + C$$
$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6-1}\right| + C.$$

EXAMPLE 7 *Evaluate:*

(i) $\int \frac{\sin x}{\sqrt{4\cos^2 x - 1}}\,dx$ *(ii)* $\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 4}}\,dx$

SOLUTION (i) Putting $\cos x = t$ and $-\sin x\,dx = dt$, we get

$$\int \frac{\sin x}{\sqrt{4\cos^2 x-1}}\,dx = \int \frac{-\,dt}{\sqrt{4t^2-1}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t^2-(1/2)^2}}$$
$$= -\frac{1}{2}\cdot\log\left|t+\sqrt{t^2-\frac{1}{4}}\right| + C$$
$$= -\frac{1}{2}\log\left|2t+\sqrt{4t^2-1}\right| + C$$
$$= -\frac{1}{2}\log\left|2\cos x+\sqrt{4\cos^2 x-1}\right| + C.$$

(ii) Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x-4}}\,dx = \int \frac{dt}{\sqrt{t^2-4}} = \log\left|t+\sqrt{t^2-4}\right| + C$$
$$= \log\left|\tan x+\sqrt{\tan^2 x-4}\right| + C.$$

EXAMPLE 8 *Evaluate* $\int \frac{x^2}{\sqrt{x^6+a^6}}\,dx.$

SOLUTION Putting $x^3 = t$ and $x^2 dx = \frac{1}{3}\,dt$, we get

$$\int \frac{x^2}{\sqrt{x^6+a^6}}\,dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+(a^3)^2}} = \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right| + C$$
$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C.$$

EXAMPLE 9 *Evaluate* $\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}}\,dx.$

SOLUTION Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}}\,dx = \int \frac{dt}{\sqrt{16+t^2}} = \log\left|t+\sqrt{t^2+16}\right| + C$$
$$= \log\left|\tan x+\sqrt{\tan^2 x+16}\right| + C.$$

EXAMPLE 10 *Evaluate* $\int \sqrt{\frac{1-x}{1+x}}\,dx.$ **[CBSE 2006]**

SOLUTION We have

$$\int \sqrt{\frac{1-x}{1+x}}\,dx = \int \left\{ \frac{\sqrt{1-x}}{\sqrt{1+x}} \times \frac{\sqrt{1-x}}{\sqrt{1-x}} \right\} dx$$

$$= \int \frac{(1-x)}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1}x + \frac{1}{2} \cdot \int \frac{(-2x)}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1}x + \frac{1}{2}\int \frac{dt}{\sqrt{t}}, \text{ where } (1-x^2) = t \text{ and } (-2x)\,dx = dt$$

$$= \sin^{-1}x + \frac{1}{2}\int t^{-1/2}\,dt = \sin^{-1}x + \frac{1}{2} \cdot \frac{t^{1/2}}{(1/2)} + C$$

$$= \sin^{-1}x + \sqrt{1-x^2} + C.$$

Integrals of the form $\int \frac{dx}{\sqrt{(ax^2+bx+c)}}$.

METHOD Put (ax^2+bx+c) in the form $a\{(x+\alpha)^2 \pm \beta^2\}$ and then integrate.

EXAMPLE 11 *Evaluate* $\int \frac{dx}{\sqrt{x^2-3x+2}}$. **[CBSE 2006]**

SOLUTION We have

$$\int \frac{dx}{\sqrt{x^2-3x+2}} = \int \frac{dx}{\sqrt{\left(x^2-3x+\frac{9}{4}\right)-\frac{1}{4}}} = \int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$= \int \frac{dz}{\sqrt{z^2-\left(\frac{1}{2}\right)^2}}, \text{ where } \left(x-\frac{3}{2}\right) = z$$

$$= \log\left|z + \sqrt{z^2-\frac{1}{4}}\right| + C$$

$$\left[\because \int \frac{dz}{\sqrt{z^2-a^2}} = \log\left|z+\sqrt{z^2-a^2}\right| + C\right]$$

$$= \log\left|\left(x-\frac{3}{2}\right) + \sqrt{x^2-3x+2}\right| + C.$$

EXAMPLE 12 *Evaluate* $\int \frac{dx}{\sqrt{5x^2-2x}}$.

SOLUTION We have

$$\int \frac{dx}{\sqrt{5x^2-2x}} = \frac{1}{\sqrt{5}} \cdot \int \frac{dx}{\sqrt{x^2-\frac{2}{5}x}} = \frac{1}{\sqrt{5}} \cdot \int \frac{dx}{\sqrt{x^2-\frac{2}{5}x+\left(\frac{1}{5}\right)^2-\left(\frac{1}{5}\right)^2}}$$

$$= \frac{1}{\sqrt{5}} \cdot \int \frac{dx}{\sqrt{\left(x - \frac{1}{5}\right)^2 - \left(\frac{1}{5}\right)^2}} = \frac{1}{\sqrt{5}} \cdot \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{5}\right)^2}},$$

where $\left(x - \frac{1}{5}\right) = t$

$$= \frac{1}{\sqrt{5}} \cdot \log \left| t + \sqrt{t^2 - \left(\frac{1}{5}\right)^2} \right| + C$$

$$= \frac{1}{\sqrt{5}} \log \left| \left(x - \frac{1}{5}\right) + \sqrt{x^2 - \frac{2x}{5}} \right| + C.$$

EXAMPLE 13 *Evaluate* $\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}} dx.$ **[CBSE 2006C]**

SOLUTION Putting $\sin x = t$ and $\cos x\, dx = dt$, we get

$$\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}} dx = \int \frac{dt}{\sqrt{t^2 - 2t - 3}} = \int \frac{dt}{\sqrt{(t-1)^2 - 2^2}}$$

$$= \log \left| (t-1) + \sqrt{(t-1)^2 - 2^2} \right| + C$$

$$= \log \left| (\sin x - 1) + \sqrt{\sin^2 x - 2\sin x - 3} \right| + C.$$

EXAMPLE 14 *Evaluate* $\int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}} dx.$ **[CBSE 2005C, '09]**

SOLUTION Putting $e^x = t$ and $e^x dx = dt$, we get

$$\int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}} dx = \int \frac{dt}{\sqrt{5 - 4t - t^2}} = \int \frac{dt}{\sqrt{5 - (t^2 + 4t + 4) + 4}}$$

$$= \int \frac{dt}{\sqrt{9 - (t+2)^2}} = \int \frac{dt}{\sqrt{3^2 - (t+2)^2}}$$

$$= \int \frac{dz}{\sqrt{3^2 - z^2}}, \text{ where } (t+2) = z$$

$$= \sin^{-1} \frac{z}{3} + C = \sin^{-1} \frac{(t+2)}{3} + C$$

$$= \sin^{-1} \frac{(e^x + 2)}{3} + C.$$

EXAMPLE 15 *Evaluate* $\int \frac{dx}{\sqrt{2 - 4x + x^2}}.$

SOLUTION We have

$$\int \frac{dx}{\sqrt{2 - 4x + x^2}} = \int \frac{dx}{\sqrt{x^2 - 4x + 4 - 2}} = \int \frac{dx}{\sqrt{(x-2)^2 - (\sqrt{2})^2}}$$

$$= \log \left| (x-2) + \sqrt{(x-2)^2 - 2} \right| + C$$

$$= \log \left| x - 2 + \sqrt{x^2 - 4x + 2} \right| + C.$$

EXAMPLE 16 *Evaluate* $\int \frac{dx}{\sqrt{3x^2+6x+12}}$. **[CBSE 2004C]**

SOLUTION We have

$$\int \frac{dx}{\sqrt{3x^2+6x+12}} = \frac{1}{\sqrt{3}} \cdot \int \frac{dx}{\sqrt{x^2+2x+4}}$$
$$= \frac{1}{\sqrt{3}} \cdot \int \frac{dx}{\sqrt{(x+1)^2+(\sqrt{3})^2}}$$
$$= \frac{1}{\sqrt{3}} \cdot \int \frac{dt}{\sqrt{t^2+(\sqrt{3})^2}}, \text{ where } (x+1)=t$$
$$= \frac{1}{\sqrt{3}} \log \left| t+\sqrt{t^2+3} \right| + C$$
$$= \frac{1}{\sqrt{3}} \log \left| (x+1)+\sqrt{x^2+2x+4} \right| + C.$$

EXAMPLE 17 *Evaluate* $\int \frac{dx}{\sqrt{8+3x-x^2}}$.

SOLUTION We have

$$\int \frac{dx}{\sqrt{8+3x-x^2}} = \int \frac{dx}{\sqrt{8-(x^2-3x)}} = \int \frac{dx}{\sqrt{8-\left(x^2-3x+\frac{9}{4}\right)+\frac{9}{4}}}$$
$$= \int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}} = \sin^{-1}\left\{\frac{\left(x-\frac{3}{2}\right)}{\left(\frac{\sqrt{41}}{2}\right)}\right\}+C$$
$$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C.$$

EXAMPLE 18 *Evaluate* $\int \frac{dx}{\sqrt{2x-x^2}}$.

SOLUTION We have

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x^2-2x+1)}}$$
$$= \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}(x-1)+C.$$

EXAMPLE 19 *Evaluate* $\int \frac{dx}{\sqrt{x(1-2x)}}$.

SOLUTION We have

$$\int \frac{dx}{\sqrt{x(1-2x)}} = \int \frac{dx}{\sqrt{x-2x^2}}$$
$$= \frac{1}{\sqrt{2}} \cdot \int \frac{dx}{\sqrt{\frac{x}{2}-x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left(x^2-\frac{x}{2}+\frac{1}{16}\right)+\frac{1}{16}}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{1}{16} - \left\{x^2 - \frac{x}{2} + \frac{1}{16}\right\}}} = \frac{1}{\sqrt{2}} \cdot \int \frac{dx}{\sqrt{\left(\frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\left(\frac{1}{4}\right)^2 - t^2}}, \text{ where } \left(x - \frac{1}{4}\right) = t$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \frac{t}{(1/4)} + C = \frac{1}{\sqrt{2}} \sin^{-1}(4t) + C = \frac{1}{\sqrt{2}} \sin^{-1} 4\left(x - \frac{1}{4}\right) + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}(4x - 1) + C.$$

Integrals of the form $\int \frac{(px + q)}{\sqrt{(ax^2 + bx + c)}}\, dx.$

METHOD Let $(px + q) = A \cdot \frac{d}{dx}(ax^2 + bx + c) + B.$

Now, the value of the integral can be obtained easily.

EXAMPLE 20 *Evaluate* $\int \frac{(5x + 3)}{\sqrt{x^2 + 4x + 10}}\, dx.$ **[CBSE 2011, 12]**

SOLUTION Let $(5x + 3) = A \cdot \frac{d}{dx}(x^2 + 4x + 10) + B.$

Then, $(5x + 3) = A(2x + 4) + B.$

Comparing the coefficients of like powers of x, we get

$$(2A = 5 \text{ and } 4A + B = 3) \Rightarrow \left(A = \frac{5}{2} \text{ and } B = -7\right)$$

$$\therefore \int \frac{(5x + 3)}{\sqrt{x^2 + 4x + 10}}\, dx = \int \left\{ \frac{\frac{5}{2} \cdot (2x + 4) - 7}{\sqrt{x^2 + 4x + 10}} \right\} dx$$

$$= \frac{5}{2} \cdot \int \frac{(2x + 4)}{\sqrt{x^2 + 4x + 10}}\, dx - 7 \int \frac{dx}{\sqrt{x^2 + 4x + 10}}$$

$$= \frac{5}{2} \cdot \int \frac{1}{\sqrt{t}}\, dt - 7 \int \frac{dx}{\sqrt{(x + 2)^2 + (\sqrt{6})^2}}, \text{ where } t = x^2 + 4x + 10$$

$$= \left(\frac{5}{2} \times 2\sqrt{t}\right) - 7 \log \left|(x + 2) + \sqrt{x^2 + 4x + 10}\right| + C$$

$$= 5\sqrt{x^2 + 4x + 10} - 7 \log \left|(x + 2) + \sqrt{x^2 + 4x + 10}\right| + C.$$

EXAMPLE 21 *Evaluate* $\int \frac{(x + 1)}{\sqrt{4 + 5x - x^2}}\, dx.$

SOLUTION Let $(x + 1) = A \cdot \frac{d}{dx}(4 + 5x - x^2) + B.$ Then,

$$(x + 1) = A(5 - 2x) + B.$$

Comparing the coefficients of like powers of x, we get

$$(-2A = 1 \text{ and } 5A + B = 1) \;\Rightarrow\; \left(A = -\frac{1}{2} \text{ and } B = \frac{7}{2}\right)\cdot$$

$$\therefore\; (x+1) = -\frac{1}{2}(5-2x) + \frac{7}{2}\cdot$$

$$\therefore\; \int \frac{(x+1)}{\sqrt{4+5x-x^2}}\,dx = \int \frac{-\frac{1}{2}(5-2x)+\frac{7}{2}}{\sqrt{4+5x-x^2}}\,dx$$

$$= -\frac{1}{2}\int \frac{(5-2x)}{\sqrt{4+5x-x^2}}\,dx + \frac{7}{2}\int \frac{1}{\sqrt{4+5x-x^2}}\,dx$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt + \frac{7}{2}\cdot\int \frac{1}{\sqrt{4-(x^2-5x)}}\,dx, \text{ where } t = 4+5x-x^2$$

$$= -\frac{1}{2}\cdot 2\sqrt{t} + \frac{7}{2}\cdot\int \frac{dx}{\sqrt{4-\left(x^2-5x+\frac{25}{4}\right)+\frac{25}{4}}}$$

$$= -\sqrt{t} + \frac{7}{2}\cdot\int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{5}{2}\right)^2}}$$

$$= -\sqrt{4+5x-x^2} + \frac{7}{2}\cdot\sin^{-1}\frac{\left(x-\frac{5}{2}\right)}{\left(\frac{\sqrt{41}}{2}\right)} + C$$

$$= -\sqrt{4+5x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right) + C.$$

EXAMPLE 22 *Evaluate* $\displaystyle\int \frac{(x+3)}{\sqrt{5-4x-x^2}}\,dx.$

SOLUTION Let $(x+3) = A\cdot\dfrac{d}{dx}(5-4x-x^2) + B$. Then,

$$(x+3) = A(-4-2x) + B.$$

Comparing the coefficients of like powers of x, we get

$$(-2A = 1 \text{ and } -4A + B = 3) \;\Rightarrow\; \left(A = \frac{-1}{2},\, B = 1\right)\cdot$$

$$\therefore\; \int \frac{(x+3)}{\sqrt{5-4x-x^2}}\,dx = -\frac{1}{2}\cdot\int \frac{(-4-2x)}{\sqrt{5-4x-x^2}}\,dx + \int \frac{dx}{\sqrt{5-4x-x^2}}$$

$$= -\frac{1}{2}\int \frac{dt}{\sqrt{t}} + \int \frac{dx}{\sqrt{5-(x^2+4x+4)+4}},$$

where $(5-4x-x^2) = t$

$$= -\frac{1}{2} \cdot \frac{t^{1/2}}{(1/2)} + \int \frac{dx}{\sqrt{3^2 - (x+2)^2}}$$

$$= -\sqrt{t} + \sin^{-1} \frac{(x+2)}{3} + C$$

$$= -\sqrt{5 - 4x - x^2} + \sin^{-1} \frac{(x+2)}{3} + C.$$

EXERCISE 14B

Evaluate:

1. $\int \frac{dx}{\sqrt{16 - x^2}}$
2. $\int \frac{dx}{\sqrt{1 - 9x^2}}$
3. $\int \frac{dx}{\sqrt{15 - 8x^2}}$
4. $\int \frac{dx}{\sqrt{x^2 - 4}}$
5. $\int \frac{dx}{\sqrt{4x^2 - 1}}$
6. $\frac{dx}{\sqrt{9x^2 - 7}}$
7. $\int \frac{dx}{\sqrt{x^2 + 9}}$
8. $\int \frac{dx}{\sqrt{1 + 4x^2}}$
9. $\int \frac{dx}{\sqrt{9 + 4x^2}}$
10. $\int \frac{x}{\sqrt{9 - x^4}}\, dx$
11. $\int \frac{3x^2}{\sqrt{9 - 16x^6}}\, dx$
12. $\int \frac{\sec^2 x}{\sqrt{16 + \tan^2 x}}\, dx$
13. $\int \frac{\sin x}{\sqrt{4 + \cos^2 x}}\, dx$
14. $\int \frac{\cos x}{\sqrt{9 \sin^2 x - 1}}\, dx$
15. $\int \frac{e^x}{\sqrt{4 + e^{2x}}}\, dx$
16. $\int \frac{2e^x}{\sqrt{4 - e^{2x}}}\, dx$
17. $\int \frac{dx}{\sqrt{1 - e^x}}$
18. $\int \sqrt{\frac{a - x}{a + x}}\, dx$
19. $\int \frac{dx}{\sqrt{x^2 + 6x + 5}}$ **[CBSE 2001C]**
20. $\int \frac{dx}{\sqrt{(2 - x)^2 + 1}}$ **[CBSE 2001C]**
21. $\int \frac{dx}{\sqrt{(x - 3)^2 - 1}}$
22. $\int \frac{dx}{\sqrt{x^2 - 6x + 10}}$
23. $\int \frac{dx}{\sqrt{2 + 2x - x^2}}$
24. $\int \frac{dx}{\sqrt{8 - 4x - 2x^2}}$ **[CBSE 2003C]**
25. $\int \frac{dx}{\sqrt{16 - 6x - x^2}}$
26. $\int \frac{dx}{\sqrt{7 - 6x - x^2}}$ **[CBSE 2002]**
27. $\int \frac{dx}{\sqrt{x - x^2}}$
28. $\int \frac{dx}{\sqrt{8 + 2x - x^2}}$ **[CBSE 2002]**
29. $\int \frac{dx}{\sqrt{x^2 - 3x + 2}}$
30. $\int \frac{dx}{\sqrt{2x^2 + 3x - 2}}$
31. $\int \frac{dx}{\sqrt{2x^2 + 4x + 6}}$ **[CBSE 2004C]**
32. $\int \frac{dx}{\sqrt{1 + 2x - 3x^2}}$

33. $\int \frac{dx}{\sqrt{x}\sqrt{5-x}}$

34. $\int \frac{dx}{\sqrt{3+4x-2x^2}}$

35. $\int \frac{x^2}{\sqrt{x^6+2x^3+3}}\,dx$

36. $\int \frac{(2x+3)}{\sqrt{x^2+x+1}}\,dx$

37. $\int \frac{(5x+3)}{\sqrt{x^2+4x+10}}\,dx$ **[CBSE 2012C]**

38. $\int \frac{(4x+3)}{\sqrt{2x^2+2x-3}}$ **[CBSE 2001]**

39. $\int \frac{(3-2x)}{\sqrt{2+x-x^2}}\,dx$

40. $\int \frac{(x+2)}{\sqrt{x^2+2x-1}}\,dx$

41. $\int \frac{(3x+1)}{\sqrt{5-2x-x^2}}\,dx$

42. $\int \frac{(6x+5)}{\sqrt{6+x-2x^2}}\,dx$

43. $\int \sqrt{\frac{1+x}{x}}\,dx$

44. $\int \frac{(x+2)}{\sqrt{x^2+5x+6}}\,dx$ **[CBSE 2014]**

ANSWERS (EXERCISE 14B)

1. $\sin^{-1}\frac{x}{4}+C$

2. $\frac{1}{3}\sin^{-1}3x+C$

3. $\frac{1}{2\sqrt{2}}\sin^{-1}\sqrt{\frac{8}{15}}\,x+C$

4. $\log\left|x+\sqrt{x^2-4}\right|+C$

5. $\frac{1}{2}\log\left|2x+\sqrt{4x^2-1}\right|+C$

6. $\frac{1}{3}\log\left|3x+\sqrt{9x^2-7}\right|+C$

7. $\log\left|x+\sqrt{x^2+9}\right|+C$

8. $\frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right|+C$

9. $\frac{1}{2}\log\left|2x+\sqrt{9+4x^2}\right|+C$

10. $\frac{1}{2}\sin^{-1}\left(\frac{x^2}{3}\right)+C$

11. $\frac{1}{4}\sin^{-1}\left(\frac{4x^3}{3}\right)+C$

12. $\log\left|\tan x+\sqrt{16+\tan^2 x}\right|+C$

13. $-\log\left|\cos x+\sqrt{4+\cos^2 x}\right|+C$

14. $\frac{1}{3}\log\left|3\sin x+\sqrt{9\sin^2 x-1}\right|+C$

15. $\log\left|e^x+\sqrt{4+e^{2x}}\right|+C$

16. $2\sin^{-1}\left(\frac{e^x}{2}\right)+C$

17. $-2\log\left|e^{-x/2}+\sqrt{e^{-x}-1}\right|+C$

18. $a\sin^{-1}\frac{x}{a}+\sqrt{a^2-x^2}+C$

19. $\log\left|(x+3)+\sqrt{x^2+6x+5}\right|+C$

20. $-\log\left|(2-x)+\sqrt{x^2-4x+5}\right|+C$

21. $\log\left|(x-3)+\sqrt{x^2-6x+8}\right|+C$

22. $\log\left|(x-3)+\sqrt{x^2-6x+10}\right|+C$

23. $\sin^{-1}\left(\frac{x-1}{\sqrt{3}}\right)+C$

24. $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right)+C$

25. $\sin^{-1}\left(\frac{x+3}{5}\right)+C$

26. $\sin^{-1}\left(\frac{x+3}{4}\right)+C$

27. $\sin^{-1}(2x-1)+C$

28. $\sin^{-1}\left(\frac{x-1}{3}\right)+C$

29. $\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$

30. $\frac{1}{\sqrt{2}}\log\left|x+\frac{3}{4}+\sqrt{x^2+\frac{3}{2}x-1}\right|+C$

31. $\frac{1}{\sqrt{2}}\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$

32. $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{3x-1}{2}\right)+C$

33. $\sin^{-1}\left(\frac{2x-5}{5}\right)+C$

34. $\frac{1}{\sqrt{2}}\sin^{-1}\left\{\frac{\sqrt{2}(x-1)}{\sqrt{5}}\right\}+C$

35. $\frac{1}{3}\log\left|(x^3+1)+\sqrt{x^6+2x^3+3}\right|+C$

36. $2\sqrt{x^2+x+1}+2\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+C$

37. $5\sqrt{x^2+4x+10}-7\log\left|(x+2)+\sqrt{x^2+4x+10}\right|+C$

38. $2\sqrt{2x^2+2x-3}+\frac{1}{\sqrt{2}}\log\left|x+\frac{1}{2}+\sqrt{x^2+x-\frac{3}{2}}\right|+C$

39. $2\sqrt{2+x-x^2}+2\sin^{-1}\left(\frac{2x-1}{3}\right)+C$

40. $\sqrt{x^2+2x-1}+\log\left|(x+1)+\sqrt{x^2+2x-1}\right|+C$

41. $-3\sqrt{5-2x-x^2}-2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+C$

42. $-3\sqrt{6+x-2x^2}+\frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+C$

43. $\sqrt{x+x^2}+\frac{1}{2}\log\left|x+\frac{1}{2}+\sqrt{x+x^2}\right|+C$

44. $\sqrt{x^2+5x+6}-\frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right|+C$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 14B)

10. Put $x^2=t$ and $x\,dx=\frac{1}{2}\,dt$.
11. Put $x^3=t$ and $3x^2dx=dt$.
12. Put $\tan x=t$ and $\sec^2 x\,dx=dt$.
13. Put $\cos x=t$ and $\sin x\,dx=-\,dt$.
14. Put $\sin x=t$ and $\cos x\,dx=dt$.
15. Put $e^x=t$ and $e^x dx=dt$.
16. Put $e^x=t$ and $e^x dx=dt$.

17. Multiplying the numerator and denominator by $e^{-x/2}$, we get

$$I=\int\frac{e^{-x/2}}{\sqrt{e^{-x}-1}}\,dx.\text{ Now, put } e^{-x/2}=t \text{ and } e^{-x/2}\cdot\left(\frac{-1}{2}\right)dx=dt.$$

18. $$\int\sqrt{\frac{a-x}{a+x}}\,dx=\int\left\{\frac{\sqrt{a-x}}{\sqrt{a+x}}\times\frac{\sqrt{a-x}}{\sqrt{a-x}}\right\}dx=\int\frac{(a-x)}{\sqrt{a^2-x^2}}\,dx$$

$$=a\int\frac{dx}{\sqrt{a^2-x^2}}+\frac{1}{2}\cdot\int\frac{(-2x)}{\sqrt{a^2-x^2}}\,dx=a\sin^{-1}\frac{x}{a}+\frac{1}{2}\int\frac{1}{\sqrt{t}}\,dt\text{, where } (a^2-x^2)=t$$

$$=a\sin^{-1}\frac{x}{a}+\frac{1}{2}\times2\sqrt{t}+C=a\sin^{-1}\frac{x}{a}+\sqrt{a^2-x^2}+C.$$

20. Put $(2-x)=t$ and $dx=-\,dt$.

25. $(16-6x-x^2)=\{16-(x^2+6x+9)+9\}=[25-(x+3)^2]$.

27. $$\sqrt{(x-x^2)}=\sqrt{-\left(x^2-x+\frac{1}{4}\right)+\frac{1}{4}}=\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}.$$

32. $$I=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{-x^2+\frac{2}{3}x+\frac{1}{3}}}=\frac{1}{\sqrt{3}}\cdot\int\frac{dx}{\sqrt{\frac{1}{3}-\left(x^2-\frac{2}{3}x+\frac{1}{9}\right)+\frac{1}{9}}}$$

$$=\frac{1}{\sqrt{3}}\cdot\int\frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2-\left(x-\frac{1}{3}\right)^2}}.$$

33. $$I=\int\frac{dx}{\sqrt{5x-x^2}}=\int\frac{dx}{\sqrt{-\left(x^2-5x+\frac{25}{4}\right)+\frac{25}{4}}}=\int\frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2-\left(x-\frac{5}{2}\right)^2}}.$$

34. $$I=\frac{1}{\sqrt{2}}\cdot\int\frac{dx}{\sqrt{-x^2+2x+3}}=\frac{1}{\sqrt{2}}\cdot\int\frac{dx}{\{3-(x^2-2x+1)+1\}}=\frac{1}{\sqrt{2}}\cdot\int\frac{dx}{\sqrt{2^2-(x+1)^2}}.$$

35. Put $x^3=t$ and $3x^2dx=dt$.

43. $$I=\int\left(\frac{\sqrt{1+x}}{\sqrt{x}}\times\frac{\sqrt{1+x}}{\sqrt{1+x}}\right)dx=\int\frac{(1+x)}{\sqrt{x(1+x)}}\,dx=\int\frac{(x+1)}{\sqrt{x^2+x}}\,dx.$$

44. Let $(x+2)=A\cdot\frac{d}{dx}(x^2+5x+6)+B=A(2x+5)+B$

$\therefore\ (2A=1 \text{ and } 5A+B=2)\Rightarrow A=\frac{1}{2}$ and $B=\frac{-1}{2}$.

$$\therefore\ I=\int\frac{\left\{\frac{1}{2}(2x+5)-\frac{1}{2}\right\}}{\sqrt{x^2+5x+6}}\,dx=\frac{1}{2}\int\frac{(2x+5)}{\sqrt{x^2+5x+6}}\,dx-\frac{1}{2}\int\frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$=\sqrt{x^2+5x+6}-\frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right|+C.$$

Three More Special Integrals

THEOREM

(i) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

(ii) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C.$

(iii) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C.$

PROOF (i) Integrating by parts, taking 1 as the second function, we get

$$I = \int \sqrt{a^2 - x^2}\, dx = \int (\sqrt{a^2 - x^2} \cdot 1)\, dx$$

$$= (\sqrt{a^2 - x^2}) \cdot x - \int \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x) \cdot x\, dx$$

$$= x(\sqrt{a^2 - x^2}) + \int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx = x(\sqrt{a^2 - x^2}) + \int \frac{a^2 - (a^2 - x^2)}{\sqrt{a^2 - x^2}}\, dx$$

$$= x(\sqrt{a^2 - x^2}) + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} - \int \sqrt{a^2 - x^2}\, dx$$

$$= x(\sqrt{a^2 - x^2}) + a^2 \sin^{-1}\frac{x}{a} - I + c.$$

$$\therefore \quad 2I = x(\sqrt{a^2 - x^2}) + a^2 \sin^{-1}\frac{x}{a} + c$$

$$\Rightarrow \; I = \frac{x}{2}(\sqrt{a^2 - x^2}) + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C \quad \left[\text{taking } \frac{c}{2} = C\right].$$

Hence, $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}(\sqrt{a^2 - x^2}) + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

(ii) Integrating by parts, taking 1 as the second function, we get

$$I = \int \sqrt{x^2 - a^2}\, dx = \int (\sqrt{x^2 - a^2} \cdot 1)\, dx$$

$$= (\sqrt{x^2 - a^2}) \cdot x - \int \frac{1}{2}(x^2 - a^2)^{-1/2}(2x) \cdot x\, dx$$

$$= x(\sqrt{x^2 - a^2}) - \int \frac{x^2}{\sqrt{x^2 - a^2}}\, dx = x(\sqrt{x^2 - a^2}) - \int \frac{(x^2 - a^2) + a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x(\sqrt{x^2 - a^2}) - \int \sqrt{x^2 - a^2}\, dx - \int \frac{a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x(\sqrt{x^2 - a^2}) - I - a^2 \log\left|x + \sqrt{x^2 - a^2}\right| + c.$$

$$\therefore \quad 2I = x(\sqrt{x^2 - a^2}) - a^2 \log\left|x + \sqrt{x^2 - a^2}\right| + c$$

$$\Rightarrow \; I = \frac{x}{2}(\sqrt{x^2 - a^2}) - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C \quad \left[\text{taking } \frac{c}{2} = C\right].$$

Hence, $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}(\sqrt{x^2 - a^2}) - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C.$

(iii) Integrating by parts, taking 1 as the second function, we get

$$I = \int \sqrt{x^2 + a^2}\,dx = \int (\sqrt{x^2 + a^2} \cdot 1)\,dx$$

$$= (\sqrt{x^2 + a^2})x - \int \frac{1}{2}(x^2 + a^2)^{-1/2}(2x) \cdot x\,dx$$

$$= x(\sqrt{x^2 + a^2}) - \int \frac{x^2}{\sqrt{x^2 + a^2}}\,dx = x(\sqrt{x^2 + a^2}) - \int \frac{(x^2 + a^2) - a^2}{\sqrt{x^2 + a^2}}\,dx$$

$$= x(\sqrt{x^2 + a^2}) - \int \sqrt{x^2 + a^2}\,dx + a^2 \int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$= x \cdot (\sqrt{x^2 + a^2}) - I + a^2 \log\left|x + \sqrt{x^2 + a^2}\right| + c.$$

$$\therefore \quad 2I = x(\sqrt{x^2 + a^2}) + a^2 \log\left|x + \sqrt{x^2 + a^2}\right| + c$$

or $\quad I = \frac{x}{2}(\sqrt{x^2 + a^2}) + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C \qquad$ [taking $\frac{c}{2} = C$].

Hence, $\quad \int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}(\sqrt{x^2 + a^2}) + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C.$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\int \sqrt{9 - x^2}\,dx \qquad$ *(ii)* $\int \sqrt{1 - 4x^2}\,dx \qquad$ *(iii)* $\int \sqrt{16 - 9x^2}\,dx$

SOLUTION We know that $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

$\therefore$ (i) $\int \sqrt{9 - x^2}\,dx = \int \sqrt{3^2 - x^2}\,dx$

$$= \frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + C.$$

(ii) $\sqrt{1 - 4x^2}\,dx = 2\int \sqrt{\left(\frac{1}{4} - x^2\right)}\,dx = 2\int \left\{\sqrt{\left(\frac{1}{2}\right)^2 - x^2}\right\}dx$

$$= 2\left[\frac{x}{2}\sqrt{\frac{1}{4} - x^2} + \frac{1}{8}\sin^{-1}\left(\frac{x}{(1/2)}\right)\right] + C$$

$$= \frac{x}{2}\sqrt{1 - 4x^2} + \frac{1}{4}\sin^{-1}(2x) + C.$$

(iii) $\int \sqrt{16 - 9x^2}\,dx = 3\int \left\{\sqrt{\left(\frac{16}{9} - x^2\right)}\right\}dx = 3\int \left\{\sqrt{\left(\frac{4}{3}\right)^2 - x^2}\right\}dx$

$$= 3\left[\frac{x}{2}\sqrt{\frac{16}{9} - x^2} + \frac{8}{9}\sin^{-1}\frac{x}{(4/3)}\right] + C$$

$$= \frac{x}{2}\sqrt{16 - 9x^2} + \frac{8}{3}\sin^{-1}\left(\frac{3x}{4}\right) + C.$$

EXAMPLE 2 *Evaluate:*

(i) $\int \sqrt{x^2-16}\,dx$ (ii) $\int \sqrt{4x^2-5}\,dx$ (iii) $\int \sqrt{17x^2-11}\,dx$

SOLUTION We know that $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C.$

$\therefore$ (i) $\int \sqrt{x^2-16}\,dx = \int \sqrt{x^2-4^2}\,dx$

$$= \frac{x}{2}\cdot\sqrt{x^2-4^2} - \frac{16}{2}\log\left|x+\sqrt{x^2-16}\right| + C$$

$$= \frac{x}{2}\cdot\sqrt{x^2-16} - 8\log\left|x+\sqrt{x^2-16}\right| + C.$$

(ii) $\int \sqrt{4x^2-5}\,dx = 2\int \sqrt{x^2-\frac{5}{4}}\,dx = 2\cdot\int \sqrt{x^2-\left(\frac{\sqrt{5}}{2}\right)^2}\,dx$

$$= 2\cdot\left[\frac{x}{2}\sqrt{x^2-\frac{5}{4}} - \frac{5}{8}\log\left|x+\sqrt{x^2-\frac{5}{4}}\right|\right] + C$$

$$= x\sqrt{x^2-\frac{5}{4}} - \frac{5}{4}\log\left|x+\sqrt{x^2-\frac{5}{4}}\right| + C.$$

(iii) $\int \sqrt{17x^2-11}\,dx = \sqrt{17}\cdot\int \sqrt{x^2-\frac{11}{17}}\,dx$

$$= \sqrt{17}\cdot\left\{\frac{x}{2}\sqrt{x^2-\frac{11}{17}} - \frac{11}{34}\log\left|x+\sqrt{x^2-\frac{11}{17}}\right|\right\} + C$$

$$= \frac{x}{2}\sqrt{17x^2-11} - \frac{11\sqrt{17}}{34}\log\left|x+\sqrt{x^2-\frac{11}{17}}\right| + C.$$

EXAMPLE 3 *Evaluate* $\int \sqrt{16x^2+25}\,dx.$

SOLUTION We know that $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C.$

$\therefore$ $\int \sqrt{16x^2+25}\,dx = 4\int\left\{\sqrt{x^2+\frac{25}{16}}\right\}dx = 4\int\left\{\sqrt{x^2+\left(\frac{5}{4}\right)^2}\right\}dx$

$$= 4\cdot\left\{\frac{x}{2}\sqrt{x^2+\frac{25}{16}} + \frac{25}{32}\log\left|x+\sqrt{x^2+\frac{25}{16}}\right|\right\} + C$$

$$= \frac{x}{2}\cdot\sqrt{16x^2+25} + \frac{25}{8}\log\left|x+\sqrt{x^2+\frac{25}{16}}\right| + C.$$

EXAMPLE 4 *Evaluate* $\int \sqrt{\frac{16+(\log x)^2}{x}}\,dx.$ **[CBSE 2005]**

SOLUTION Putting $\log x = t$ and $\frac{1}{x}\,dx = dt$, we get

$I = \int \sqrt{16+t^2}\,dt = \int \sqrt{4^2+t^2}\,dt$

$$= \frac{t}{2}\sqrt{16+t^2} + \frac{16}{2}\log\left|t+\sqrt{16+t^2}\right| + C$$

$$= \frac{1}{2}\log x \cdot \sqrt{16+(\log x)^2} + 8\log\left|\log x + \sqrt{16+(\log x)^2}\right| + C.$$

EXAMPLE 5 *Evaluate* $\int x\sqrt{x^4+1}\,dx.$ **[CBSE 2003]**

SOLUTION Putting $x^2 = t$ and $x\,dx = \frac{1}{2}\,dt$, we get

$$I = \frac{1}{2}\int\sqrt{t^2+1}\,dt$$

$$= \frac{1}{2}\cdot\left[\frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\log\left|t+\sqrt{t^2+1}\right|\right] + C$$

$$= \frac{x^2}{4}\sqrt{x^4+1} + \frac{1}{4}\log\left|x^2+\sqrt{x^4+1}\right| + C.$$

EXAMPLE 6 *Evaluate* $\int e^x\sqrt{e^{2x}+4}\,dx.$

SOLUTION Putting $e^x = t$ and $e^x dx = dt$, we get

$$I = \int\sqrt{t^2+4}\,dx$$

$$= \frac{t}{2}\cdot\sqrt{t^2+4} + \frac{4}{2}\log\left|t+\sqrt{t^2+4}\right| + C$$

$$= \frac{1}{2}e^x\sqrt{e^{2x}+4} + 2\log\left|e^x+\sqrt{e^{2x}+4}\right| + C.$$

EXAMPLE 7 *Evaluate* $\int \cos x\sqrt{4-\sin^2 x}\,dx.$

SOLUTION Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$I = \int\sqrt{4-t^2}\,dt$$

$$= \frac{t}{2}\sqrt{4-t^2} + \frac{4}{2}\sin^{-1}\frac{t}{2} + C$$

$$= \frac{1}{2}\sin x\sqrt{4-\sin^2 x} + 2\sin^{-1}\left(\frac{1}{2}\sin x\right) + C.$$

Integrals of the form $\int\sqrt{(ax^2+bx+c)}\,dx$

METHOD Express (ax^2+bx+c) as $a[(x+\alpha)^2 \pm \beta^2]$ and obtain an integral which can be evaluated easily.

EXAMPLE 8 *Evaluate* $\int\sqrt{x^2+3x}\,dx.$

SOLUTION We have

$$(x^2+3x) = \left\{x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\right\} = \left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2.$$

$$\therefore \quad I = \int \sqrt{x^2 + 3x}\ dx$$

$$= \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\ dx = \int \sqrt{t^2 - \left(\frac{3}{2}\right)^2}\ dt, \text{ where } \left(x + \frac{3}{2}\right) = t$$

$$= \frac{1}{2} t \sqrt{t^2 - \frac{9}{4}} - \frac{9}{8} \log \left| t + \sqrt{t^2 - \frac{9}{4}} \right| + C$$

$$\left\{ \text{using } \int \sqrt{x^2 - a^2}\ dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C \right\}$$

$$= \frac{1}{2} \left(x + \frac{3}{2}\right) \sqrt{x^2 + 3x} - \frac{9}{8} \log \left| \left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x} \right| + C.$$

EXAMPLE 9 *Evaluate* $\int \sqrt{2x^2 + 3x + 4}\ dx.$

SOLUTION We have

$$(2x^2 + 3x + 4) = 2\left(x^2 + \frac{3}{2}x + 2\right)$$

$$= 2 \cdot \left\{ \left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) + \left(2 - \frac{9}{16}\right) \right\} = 2 \cdot \left\{ \left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2 \right\}.$$

$$\therefore \quad \sqrt{2x^2 + 3x + 4} = \sqrt{2} \cdot \sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2}$$

$$\Rightarrow \int \sqrt{2x^2 + 3x + 4}\, dx = \sqrt{2} \cdot \int \sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2}\ dx$$

$$= \sqrt{2} \int \sqrt{t^2 + \left(\frac{\sqrt{23}}{4}\right)^2}\ dt, \text{ where } \left(x + \frac{3}{4}\right) = t$$

$$= \sqrt{2} \cdot \left\{ \frac{t}{2} \cdot \sqrt{t^2 + \frac{23}{16}} + \frac{23}{32} \log \left| t + \sqrt{t^2 + \frac{23}{16}} \right| \right\} + C$$

$$\left[\because \quad \int \sqrt{t^2 + a^2}\, dt = \frac{t}{2} \sqrt{t^2 + a^2} + \frac{a^2}{2} \log \left| t + \sqrt{t^2 + a^2} \right| + C \right]$$

$$= \frac{\sqrt{2}}{2} \left(x + \frac{3}{4}\right) \sqrt{\left(x + \frac{3}{4}\right)^2 + \frac{23}{16}}$$

$$+ \frac{23\sqrt{2}}{32} \log \left| \left(x + \frac{3}{4}\right) + \sqrt{x^2 + \frac{3}{2}x + 2} \right| + C$$

$$= \frac{(4x + 3)}{4\sqrt{2}} \cdot \sqrt{x^2 + \frac{3}{2}x + 2} + \frac{23\sqrt{2}}{32} \log \left| \frac{(4x + 3)}{4} + \frac{\sqrt{2x^2 + 3x + 4}}{\sqrt{2}} \right| + C$$

$$= \frac{(4x + 3)\sqrt{2x^2 + 3x + 4}}{8} + \frac{23\sqrt{2}}{32} \log \left| \frac{(4x + 3)}{4} + \frac{\sqrt{2x^2 + 3x + 4}}{\sqrt{2}} \right| + C$$

EXAMPLE 10 *Evaluate* $\int \sqrt{3-2x-2x^2}\,dx.$

SOLUTION We have

$$(3-2x-2x^2) = 2\cdot\left\{\frac{3}{2}-x-x^2\right\}$$

$$= 2\cdot\left[\frac{3}{2}-\left(x^2+x+\frac{1}{4}\right)+\frac{1}{4}\right]$$

$$= 2\cdot\left[\frac{7}{4}-\left(x+\frac{1}{2}\right)^2\right] = 2\cdot\left\{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2\right\}.$$

$$\therefore \quad \sqrt{3-2x-2x^2} = \sqrt{2}\cdot\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}$$

$$\Rightarrow \int\sqrt{3-2x-2x^2}\,dx = \sqrt{2}\cdot\int\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}\,dx$$

$$= \sqrt{2}\cdot\int\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2-t^2}\,dt, \text{ where } \left(x+\frac{1}{2}\right)=t \text{ and } dx = dt$$

$$= \sqrt{2}\cdot\left\{\frac{t}{2}\cdot\sqrt{\frac{7}{4}-t^2}+\frac{7}{8}\sin^{-1}\frac{t}{(\sqrt{7}/2)}\right\}+C$$

$$\left[\because \int\sqrt{a^2-t^2}\,dt = \frac{t}{2}\sqrt{a^2-t^2}+\frac{a^2}{2}\cdot\sin^{-1}\frac{t}{a}+C\right]$$

$$= \sqrt{2}\cdot\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\frac{7}{4}-\left(x+\frac{1}{2}\right)^2}+\frac{7}{8}\sin^{-1}\frac{\left(x+\frac{1}{2}\right)}{(\sqrt{7}/2)}\right\}+C$$

$$= \sqrt{2}\left\{\frac{1}{4}(2x+1)\cdot\sqrt{\frac{3}{2}-x-x^2}+\frac{7}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)\right\}+C$$

$$= \frac{1}{4}(2x+1)\sqrt{3-2x-2x^2}+\frac{7}{4\sqrt{2}}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)+C.$$

Integrals of the form $\int (px+q)\sqrt{(ax^2+bx+c)}\,dx$

METHOD Let $(px+q) = A\cdot\frac{d}{dx}(ax^2+bx+c)+B.$

Find A and B.

Then, we get the integrand which is easily integrable.

EXAMPLE 11 *Evaluate* $\int(x+3)\sqrt{3-4x-x^2}\,dx.$ **[CBSE 2005C]**

SOLUTION Let $(x+3) = A\cdot\frac{d}{dx}(3-4x-x^2)+B \Leftrightarrow (x+3) = A(-4-2x)+B.$

Comparing the coefficients of like powers of x, we get

$$-2A = 1 \text{ and } -4A + B = 3 \Leftrightarrow A = -\frac{1}{2} \text{ and } B = 1.$$

$$\therefore \quad (x+3) = -\frac{1}{2}(-4-2x) + 1$$

$$\Rightarrow \quad I = \int \left\{ -\frac{1}{2}(-4-2x) + 1 \right\} \sqrt{3-4x-x^2}\, dx$$

$$= -\frac{1}{2}\int (-4-2x)\sqrt{3-4x-x^2}\, dx + \int \sqrt{3-4x-x^2}\, dx$$

$$= -\frac{1}{2}\int \sqrt{t}\, dt + \int \sqrt{7-(4x+x^2+4)}\, dx, \text{ where } 3-4x-x^2 = t$$

$$= \left(-\frac{1}{2} \times t^{3/2} \times \frac{2}{3} \right) + \int \sqrt{(\sqrt{7})^2 - (x+2)^2}\, dx$$

$$= -\frac{1}{3}(3-4x-x^2)^{3/2} + \frac{1}{2}(x+2)\sqrt{3-4x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}} \right) + C$$

$$\left[\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \cdot \sin^{-1}\frac{x}{a} + C \right] \cdot$$

EXAMPLE 12 *Evaluate* $\int (3x-2)\sqrt{x^2+x+1}\, dx.$

SOLUTION Let $(3x-2) = A \cdot \frac{d}{dx}(x^2+x+1) + B.$

Then, $(3x-2) = A(2x+1) + B.$

Comparing the coefficients of like powers of x, we get

$$2A = 3 \text{ and } A + B = -2. \text{ So, } A = \frac{3}{2} \text{ and } B = \frac{-7}{2} \cdot$$

$$\therefore \quad (3x-2) = \frac{3}{2}(2x+1) - \frac{7}{2} \cdot$$

So, $\int (3x-2)\sqrt{x^2+x+1}\, dx$

$$= \int \left\{ \frac{3}{2}(2x+1) - \frac{7}{2} \right\} \sqrt{x^2+x+1}\, dx$$

$$= \frac{3}{2}\int (2x+1)\sqrt{x^2+x+1}\, dx - \frac{7}{2}\int \sqrt{x^2+x+1}\, dx$$

$$= \frac{3}{2}\int \sqrt{t}\, dt - \frac{7}{2}\int \sqrt{\left(x^2 + x + \frac{1}{4} \right) + \frac{3}{4}}\, dx,$$

where $x^2 + x + 1 = t$ in the 1st integral

$$= t^{3/2} - \frac{7}{2}\int \sqrt{\left\{ \left(x + \frac{1}{2} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2 \right\}}\, dx$$

$$= (x^2+x+1)^{3/2} - \frac{7}{2} \cdot \int \sqrt{u^2 + \left(\frac{\sqrt{3}}{2} \right)^2}\, du, \text{ where } u = \left(x + \frac{1}{2} \right)$$

$$= (x^2 + x + 1)^{3/2} - \frac{7}{2}\left\{\frac{u}{2}\cdot\sqrt{u^2 + \frac{3}{4}} + \frac{3}{8}\log\left|u + \sqrt{u^2 + \frac{3}{4}}\right|\right\} + C$$

$$\left\{\because \int\sqrt{u^2 + a^2}\,du = \frac{u}{2}\sqrt{u^2 + a^2} + \frac{a^2}{2}\log\left|u + \sqrt{u^2 + a^2}\right| + C\right\}$$

$$= (x^2 + x + 1)^{3/2} - \frac{7u}{8}\sqrt{4u^2 + 3} - \frac{21}{16}\log\left|u + \sqrt{u^2 + \frac{3}{4}}\right| + C$$

$$= (x^2 + x + 1)^{3/2} - \frac{7}{8}\left(x + \frac{1}{2}\right)\sqrt{4\left(x + \frac{1}{2}\right)^2 + 3}$$

$$- \frac{21}{16}\log\left|\left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}\right| + C$$

$$= (x^2 + x + 1)^{3/2} - \frac{7(2x + 1)}{8}\sqrt{x^2 + x + 1}$$

$$- \frac{21}{16}\log\left|\frac{(2x + 1)}{2} + \sqrt{x^2 + x + 1}\right| + C.$$

EXAMPLE 13 *Evaluate* $\int x\sqrt{x + x^2}\,dx.$ **[CBSE 2005C]**

SOLUTION Let $x = A\cdot\frac{d}{dx}(x + x^2) + B.$ Then,

$$x = A(1 + 2x) + B \qquad \dots \text{(i)}$$

Comparing the coefficients of various powers of x, we get

$$(2A = 1 \;\text{ and }\; A + B = 0) \Rightarrow \left(A = \frac{1}{2} \;\text{ and }\; B = \frac{-1}{2}\right).$$

$$\therefore \quad x = \frac{1}{2}(1 + 2x) - \frac{1}{2}$$

$$\Rightarrow \int x\sqrt{x + x^2}\,dx$$

$$= \int\left\{\frac{1}{2}(1 + 2x) - \frac{1}{2}\right\}\sqrt{x + x^2}\,dx$$

$$= \frac{1}{2}\int(1 + 2x)\sqrt{x + x^2}\,dx - \frac{1}{2}\int\sqrt{x + x^2}\,dx$$

$$= \frac{1}{2}\int\sqrt{t}\,dt - \frac{1}{2}\int\sqrt{\left\{\left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4}\right\}}\,dx,$$

where $(x + x^2) = t$ in the first integral

$$= \frac{1}{2}\cdot\frac{t^{3/2}}{(3/2)} - \frac{1}{2}\int\sqrt{\left\{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\}}\,dx$$

$$= \frac{1}{3}(x + x^2)^{3/2} - \frac{1}{2}\cdot\int\sqrt{u^2 - \left(\frac{1}{2}\right)^2}\,du, \text{ where } \left(x + \frac{1}{2}\right) = u$$

$$= \frac{1}{3}(x+x^2)^{3/2} - \frac{1}{2}\left\{\frac{u}{2}\cdot\sqrt{u^2 \frac{-1}{4}} - \frac{1}{8}\log\left|u+\sqrt{u^2-\frac{1}{4}}\right|\right\}+C$$

$$\left\{\because \int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C\right\}$$

$$= \frac{1}{3}(x+x^2)^{3/2} - \frac{1}{4}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}}$$

$$+\frac{1}{16}\log\left|\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}}\right|+C$$

$$= \frac{1}{3}(x+x^2)^{3/2} - \frac{1}{8}(2x+1)\sqrt{x+x^2} + \frac{1}{16}\log\left|\frac{(2x+1)}{2}\cdot\sqrt{x+x^2}\right| + C.$$

EXERCISE 14C

Evaluate the following integrals:

1. $\int\sqrt{4-x^2}\,dx$ **[CBSE 2003C, '06]**
2. $\int\sqrt{4-9x^2}\,dx$
3. $\int\sqrt{x^2-2}\,dx$
4. $\int\sqrt{2x^2-3}\,dx$
5. $\int\sqrt{x^2+5}\,dx$
6. $\int\sqrt{4x^2+9}\,dx$
7. $\int\sqrt{3x^2+4}\,dx$
8. $\int\cos x\sqrt{9-\sin^2 x}\,dx$
9. $\int\sqrt{x^2-4x+2}\,dx$
10. $\sqrt{x^2+6x-4}\,dx$
11. $\int\sqrt{2x-x^2}\,dx$
12. $\int\sqrt{1-4x-x^2}\,dx$
13. $\int\sqrt{2ax-x^2}\,dx$
14. $\int\sqrt{2x^2+3x+4}\,dx$
15. $\int\sqrt{x^2+x}\,dx$
16. $\int\sqrt{x^2+x+1}\,dx$
17. $\int(2x-5)\sqrt{x^2-4x+3}\,dx$ **[CBSE 2005C]**
18. $\int(x+2)\sqrt{x^2+x+1}\,dx$
19. $\int(x-5)\sqrt{x^2+x}\,dx$
20. $\int(4x+1)\sqrt{x^2-x-2}\,dx$
21. $\int(x+1)\sqrt{2x^2+3}\,dx$
22. $\int x\sqrt{1+x-x^2}\,dx$
23. $\int(2x-5)\sqrt{2+3x-x^2}\,dx$
24. $\int(6x+5)\sqrt{6+x-2x^2}\,dx$
25. $\int(x+1)\sqrt{1-x-x^2}\,dx$
26. $\int(x-3)\sqrt{x^2+3x-18}\,dx$ **[CBSE 2014]**

ANSWERS (EXERCISE 14C)

1. $\frac{x}{2}\cdot\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+C$

2. $\frac{x}{2}\cdot\sqrt{4-9x^2}+2\sin^{-1}\left(\frac{3x}{2}\right)+C$

3. $\frac{1}{2}x\sqrt{x^2-2}-\log\left|x+\sqrt{x^2-2}\right|+C$

4. $\frac{x}{2}\cdot\sqrt{2x^2-3}-\frac{3}{2\sqrt{2}}\log\left|\sqrt{2}x+\sqrt{2x^2-3}\right|+C$

5. $\frac{x}{2}\cdot\sqrt{x^2+5}+\frac{5}{2}\log\left|x+\sqrt{x^2+5}\right|+C$

6. $\frac{x}{2}\cdot\sqrt{4x^2+9}+\frac{9}{4}\log\left|2x+\sqrt{4x^2+9}\right|+C$

7. $\frac{x}{2}\cdot\sqrt{3x^2+4}+\frac{2}{\sqrt{3}}\log\left|\sqrt{3}x+\sqrt{3x^2+4}\right|+C$

8. $\frac{\sin x}{2}\sqrt{9-\sin^2 x}+\frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right)+C$

9. $\frac{(x-2)}{2}\cdot\sqrt{x^2-4x+2}-\log\left|(x-2)+\sqrt{x^2-4x+2}\right|+C$

10. $\frac{1}{2}(x+3)\sqrt{x^2+6x-4}-\frac{13}{2}\log\left|(x+3)+\sqrt{x^2+6x-4}\right|+C$

11. $\frac{1}{2}(x-1)\sqrt{2x-x^2}+\frac{1}{2}\sin^{-1}(x-1)+C$

12. $\frac{1}{2}(x+2)\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$

13. $\frac{1}{2}(x-a)\sqrt{2ax-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)+C$

14. $\frac{1}{8}(4x+3)\sqrt{2x^2+3x+4}+\frac{23\sqrt{2}}{32}\log\left|\left(x+\frac{3}{4}\right)+\frac{1}{\sqrt{2}}\cdot\sqrt{2x^2+3x+4}\right|+C$

15. $\frac{1}{4}(2x+1)\sqrt{x^2+x}-\frac{1}{8}\log\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C$

16. $\frac{1}{4}(2x+1)\sqrt{x^2+x+1}+\frac{3}{8}\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+C$

17. $\frac{2}{3}(x^2-4x+3)^{3/2}-\frac{1}{2}(x-2)\sqrt{x^2-4x+3}+\frac{1}{2}\log\left|x-2+\sqrt{x^2-4x+3}\right|+C$

18. $\frac{1}{3}(x^2+x+1)^{3/2}+\frac{3}{8}(2x+1)\sqrt{x^2+x+1}+\frac{9}{16}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+C$

19. $\frac{1}{3}(x^2+x)^{3/2}-\frac{11}{8}(2x+1)\sqrt{x^2+x}+\frac{11}{16}\log\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C$

20. $\frac{4}{3}(x^2-x-2)^{3/2}+\frac{3}{4}(2x-1)\sqrt{x^2-x-2}-\frac{27}{8}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2-x-2}\right|+C$

21. $\frac{1}{6}(2x^2+3)^{3/2}+\frac{x}{2}\sqrt{2x^2+3}+\frac{3\sqrt{2}}{4}\log\left|\sqrt{2}x+\sqrt{2x^2+3}\right|+C$

22. $-\frac{1}{3}(1+x-x^2)^{3/2}+\frac{1}{8}(2x-1)\sqrt{1+x-x^2}+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+C$

23. $-\frac{2}{3}(2+3x-x^2)^{3/2}-\frac{1}{2}(2x-3)\sqrt{2+3x-x^2}-\frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+C$

24. $-(6+x-2x^2)^{3/2}+\frac{13}{16}(4x-1)\sqrt{6+x-2x^2}+\frac{637}{32\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+C$

25. $-\frac{1}{3}(1-x-x^2)^{3/2}+\frac{1}{8}(2x+1)\sqrt{1-x-x^2}+\frac{5}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)+C$

26. $\frac{1}{3}(x^2+3x-18)^{3/2}-\frac{9}{8}\{(2x+3)\sqrt{x^2+3x-18}\}$

$$-\frac{81}{2}\log\left|x+\frac{3}{2}+\sqrt{x^2+3x-18}\right|+C$$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 14C)

2. $I=3\cdot\int\sqrt{\frac{4}{9}-x^2}\,dx.$

4. $I=\sqrt{2}\cdot\int\sqrt{x^2-\frac{3}{2}}\,dx.$

6. $I=2\cdot\int\sqrt{x^2+\frac{9}{4}}\,dx.$

7. $I=\sqrt{3}\cdot\int\sqrt{x^2+\frac{4}{3}}\,dx.$

8. Put $\sin x=t$ and $\cos x\,dx=dt$.

9. $I=\int\sqrt{(x-2)^2-(\sqrt{2})^2}\,dx.$

11. $I=\int\sqrt{-(x^2-2x+1)+1}\,dx=\int\sqrt{1-(x-1)^2}\,dx.$

12. $I=\int\sqrt{1-(x^2+4x+4)+4}\,dx=\int\sqrt{5-(x+2)^2}\,dx.$

13. $I=\int\sqrt{-(x^2-2ax+a^2)+a^2}\,dx=\int\sqrt{a^2-(x-a)^2}\,dx.$

14. $I=\sqrt{2}\cdot\int\sqrt{\left(x^2+\frac{3}{2}x+2\right)}\,dx=\sqrt{2}\int\sqrt{\left\{\left(x+\frac{3}{4}\right)^2+\frac{23}{16}\right\}}\,dx.$

17. Let $(2x-5)=A\cdot\frac{d}{dx}(x^2-4x+3)+B=A(2x-4)+B.$

26. Let $(x-3)=A\cdot\frac{d}{dx}(x^2+3x-18)+B$. Then,

$(x-3)=A(2x+3)+B$. This gives $A=\frac{1}{2}$ and $B=\frac{-9}{2}\cdot$

$$\therefore\quad I=\frac{1}{2}\int(2x+3)\sqrt{x^2+3x-18}\,dx-\frac{9}{2}\int\sqrt{x^2+3x-18}\,dx$$

$$=\frac{1}{2}\int\sqrt{t}\,dt-\frac{9}{2}\int\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}\,dx.$$

15. INTEGRATION USING PARTIAL FRACTIONS

Partial Fractions

RATIONAL FUNCTIONS *If* $f(x)$ and $g(x)$ *are polynomial functions such that* $g(x) \neq 0$ *then* $\frac{f(x)}{g(x)}$ *is called a rational function.*

If degree $f(x) <$ degree $g(x)$ then $\frac{f(x)}{g(x)}$ is called a *proper rational function.*

If degree $f(x) \geq$ degree $g(x)$ then $\frac{f(x)}{g(x)}$ is called an *improper rational function.*

If $\frac{f(x)}{g(x)}$ is an improper rational function then by dividing $f(x)$ by $g(x)$, we can express $\frac{f(x)}{g(x)}$ as the sum of a polynomial and a proper rational function.

PARTIAL FRACTIONS *Any proper rational function* $\frac{p(x)}{q(x)}$ *can be expressed as the sum of rational functions, each having a simplest factor* $q(x)$. Each such fraction is known as a partial fraction and the process of obtaining them is called the decomposition or resolving of the given function into partial fractions.

METHOD We first resolve the denominator of the given fraction into simplest factors. On the basis of these factors, we obtain the corresponding partial fraction as per rules given below:

Factor in the denominator	Corresponding partial fraction
(i) $(x-a)$	$\frac{A}{(x-a)}$
(ii) $(x-b)^2$	$\frac{A}{(x-b)}+\frac{B}{(x-b)^2}$
(iii) $(x-c)^3$	$\frac{A}{(x-c)}+\frac{B}{(x-c)^2}+\frac{C}{(x-c)^3}$
(iv) (ax^2+bx+c)	$\frac{Ax+B}{(ax^2+bx+c)}$

The values of A, B, C, etc., can be obtained as shown below.

SOLVED EXAMPLES ON PARTIAL FRACTIONS

EXAMPLE 1 *Resolve* $\dfrac{2x+3}{(x-3)(x+1)}$ *into partial fractions.*

SOLUTION Let $\dfrac{(2x+3)}{(x-3)(x+1)} = \dfrac{A}{(x-3)} + \dfrac{B}{(x+1)}\cdot$ Then,

$$\frac{2x+3}{(x-3)(x+1)} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

or $(2x+3) \equiv A(x+1) + B(x-3)$. … (i)

Putting $(x-3)=0$ or $x=3$ in (i), we get $A=(9/4)$.

Putting $(x+1)=0$ or $x=-1$ in (i), we get $B=(-1/4)$.

$$\therefore \quad \frac{(2x+3)}{(x-3)(x+1)} = \frac{9}{4(x-3)} - \frac{1}{4(x+1)}.$$

EXAMPLE 2 *Resolve* $\dfrac{x^3-2x^2-13x-12}{x^2-3x-10}$ *into partial fractions.*

SOLUTION On dividing, we get

$$\frac{x^3-2x^2-13x-12}{x^2-3x-10} = (x+1) - \frac{2}{(x^2-3x-10)}. \quad \text{… (i)}$$

Let $\dfrac{2}{(x^2-3x-10)} = \dfrac{2}{(x-5)(x+2)} = \dfrac{A}{x-5} + \dfrac{B}{x+2}$

Then, $\dfrac{2}{(x-5)(x+2)} = \dfrac{A(x+2)+B(x-5)}{(x-5)(x+2)}$

or $2 \equiv A(x+2) + B(x-5)$. … (ii)

Putting $(x-5)=0$ or $x=5$ in (ii), we get $A=(2/7)$.

Putting $(x+2)=0$ or $x=-2$ in (ii), we get $B=(-2/7)$.

$$\therefore \quad \frac{2}{(x^2-3x-10)} = \frac{2}{7(x-5)} - \frac{2}{7(x+2)}.$$

Hence, $\dfrac{x^3-2x^2-13x-12}{x^2-3x-10} = (x+1) - \dfrac{2}{7(x-5)} + \dfrac{2}{7(x+2)}.$

EXAMPLE 3 *Resolve* $\dfrac{16}{(x-2)(x+2)^2}$ *into partial fractions.*

SOLUTION Let $\dfrac{16}{(x-2)(x+2)^2} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$

or $\dfrac{16}{(x-2)(x+2)^2} = \dfrac{A(x+2)^2 + B(x-2)(x+2) + C(x-2)}{(x-2)(x+2)^2}$

$$\therefore \qquad 16 \equiv A(x+2)^2 + B(x-2)(x+2) + C(x-2) \qquad \ldots \text{(i)}$$

$$\text{or} \qquad 16 \equiv (A+B)x^2 + (4A+C)x + (4A-4B-2C). \qquad \ldots \text{(ii)}$$

Putting $(x-2)=0$ or $x=2$ in (i), we get $A=1$.

Putting $(x+2)=0$ or $x=-2$ in (i), we get $C=-4$.

Comparing the coefficients of x^2 on both sides of (ii), we get

$$A+B=0 \text{ or } B=-A=-1.$$

Thus $A=1$, $B=-1$ and $C=-4$.

$$\therefore \qquad \frac{16}{(x-2)(x+2)^2} \equiv \left[\frac{1}{(x-2)} - \frac{1}{(x+2)} - \frac{4}{(x+2)^2}\right].$$

EXAMPLE 4 *Resolve* $\dfrac{2x+1}{(x-1)(x^2+1)}$ *into partial fractions.*

SOLUTION Let $\dfrac{2x+1}{(x-1)(x^2+1)} = \dfrac{A}{(x-1)} + \dfrac{Bx+C}{(x^2+1)}$

$$\text{or} \quad \frac{(2x+1)}{(x-1)(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}.$$

$$\therefore \qquad 2x+1 \equiv A(x^2+1)+(Bx+C)(x-1)$$

$$\text{or} \qquad 2x+1 \equiv (A+B)x^2 + (C-B)x + (A-C). \qquad \ldots \text{(i)}$$

Equating the like powers of x on both sides of (i), we get

$$A+B=0,\ C-B=2 \text{ and } A-C=1.$$

On solving these equations, we get

$$A=\frac{3}{2},\ B=\frac{-3}{2} \text{ and } C=\frac{1}{2}.$$

$$\therefore \quad \frac{2x+1}{(x-1)(x^2+1)} = \frac{3}{2(x-1)} + \frac{\left(\frac{-3}{2}x+\frac{1}{2}\right)}{x^2+1} = \left[\frac{3}{2(x-1)} + \frac{(1-3x)}{2(x^2+1)}\right].$$

Integration Using Partial Fractions

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate* $\displaystyle\int \frac{(x-1)}{(x+1)(x-2)}\,dx$. **[CBSE 2001]**

SOLUTION Let $\dfrac{(x-1)}{(x+1)(x-2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-2)}$.

Then, $(x-1) \equiv A(x-2) + B(x+1)$. $\ldots$ (i)

Putting $x=-1$ in (i), we get $A=\dfrac{2}{3}$.

Putting $x=2$ in (i), we get $B=\dfrac{1}{3}$.

$$\therefore\ \frac{(x-1)}{(x+1)(x-2)} = \frac{2}{3(x+1)} + \frac{1}{3(x-2)}$$

$$\Rightarrow \int \frac{(x-1)}{(x+1)(x-2)}dx = \frac{2}{3}\int \frac{dx}{(x+1)} + \frac{1}{3}\int \frac{dx}{(x-2)}$$

$$= \frac{2}{3}\log|x+1| + \frac{1}{3}\log|x-2| + C.$$

EXAMPLE 2 *Evaluate* $\displaystyle\int \frac{(x^2+1)}{(x^2-5x+6)}dx.$

SOLUTION Here the integrand is not a proper rational function; on dividing (x^2+1) by (x^2-5x+6), we get

$$\frac{(x^2+1)}{(x^2-5x+6)} = 1 + \frac{(5x-5)}{(x^2-5x+6)} = 1 + \frac{(5x-5)}{(x-2)(x-3)}.$$

Now, let $\displaystyle\frac{(5x-5)}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$

$$\Rightarrow \frac{(5x-5)}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$\Rightarrow (5x-5) \equiv A(x-3) + B(x-2).$ … (i)

Putting $x=2$ on both sides of (i), we get $A=-5$.

Putting $x=3$ on both sides of (i), we get $B=10$.

$$\therefore\ \frac{(x^2+1)}{(x^2-5x+6)} = 1 - \frac{5}{(x-2)} + \frac{10}{(x-3)}$$

$$\Rightarrow \int \frac{(x^2+1)}{(x^2-5x+6)}dx = \int dx - 5\int \frac{dx}{(x-2)} + 10\int \frac{dx}{(x-3)}$$

$$= x - 5\log|x-2| + 10\log|x-3| + C.$$

EXAMPLE 3 *Evaluate* $\displaystyle\int \frac{(3x-2)}{(x+1)^2(x+3)}dx.$ **[CBSE 2006C]**

SOLUTION Let $\displaystyle\frac{(3x-2)}{(x+1)^2(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+3)}$

$\Rightarrow (3x-2) \equiv A(x+1)(x+3) + B(x+3) + C(x+1)^2.$ … (i)

Putting $x=-3$ on both sides of (i), we get $C = \dfrac{-11}{4}\cdot$

Putting $x=-1$ on both sides of (i), we get $B = \dfrac{-5}{2}\cdot$

Comparing the coefficients of x^2 on both sides of (i), we get

$$A + C = 0 \Rightarrow A = -C = \frac{11}{4}\cdot$$

$$\therefore\ \frac{(3x-2)}{(x+1)^2(x+3)} = \frac{11}{4(x+1)} - \frac{5}{2(x+1)^2} - \frac{11}{4(x+3)}$$

$$\Rightarrow \int \frac{(3x-2)}{(x+1)^2(x+3)}dx = \frac{11}{4}\cdot\int\frac{dx}{(x+1)} - \frac{5}{2}\cdot\int\frac{1}{(x+1)^2}dx - \frac{11}{4}\cdot\int\frac{dx}{(x+3)}$$

$$= \frac{11}{4}\cdot\log|x+1| + \frac{5}{2(x+1)} - \frac{11}{4}\cdot\log|x+3| + C$$

$$= \frac{11}{4}\cdot\log\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)} + C.$$

EXAMPLE 4 *Evaluate* $\int \frac{dx}{(x^3+x^2+x+1)}\cdot$ **[CBSE 2006]**

SOLUTION We have $\frac{1}{(x^3+x^2+x+1)} = \frac{1}{x^2(x+1)+(x+1)} = \frac{1}{(x+1)(x^2+1)}\cdot$

Let $\frac{1}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+1)}$

$\Rightarrow 1 \equiv A(x^2+1) + (Bx+C)(x+1).$ … (i)

Putting $x = -1$ on both sides of (i), we get $A = \frac{1}{2}\cdot$

Comparing the coefficients of x^2 on both sides of (i), we get

$$A + B = 0 \Rightarrow B = -A = \frac{-1}{2}\cdot$$

Comparing the coefficients of x on both sides of (i), we get

$$B + C = 0 \Rightarrow C = -B = \frac{1}{2}\cdot$$

$$\therefore \frac{1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} + \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1}$$

$$\therefore \int\frac{dx}{(x^3+x^2+x+1)} = \int\frac{dx}{(x+1)(x^2+1)}$$

$$= \frac{1}{2}\cdot\int\frac{dx}{(x+1)} - \frac{1}{2}\int\frac{x}{(x^2+1)}dx + \frac{1}{2}\int\frac{dx}{(x^2+1)}$$

$$= \frac{1}{2}\cdot\int\frac{dx}{(x+1)} - \frac{1}{4}\cdot\int\frac{2x}{(x^2+1)}dx + \frac{1}{2}\int\frac{dx}{(x^2+1)}$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C.$$

EXAMPLE 5 *Evaluate* $\int \frac{x^4}{(x-1)(x^2+1)}dx.$

SOLUTION $\frac{x^4}{(x-1)(x^2+1)} = \frac{x^4}{(x^3-x^2+x-1)} = (x+1) + \frac{1}{(x^3-x^2+x-1)}$

$$\Rightarrow \frac{x^4}{(x-1)(x^2+1)} = (x+1) + \frac{1}{(x-1)(x^2+1)}. \quad \dots \text{(i)}$$

Let $\frac{1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$. Then,

$$1 \equiv A(x^2+1) + (Bx+C)(x-1). \quad \dots \text{(ii)}$$

Putting $x = 1$ in (ii), we get $A = \frac{1}{2}$.

Comparing the coefficients of x^2 on both sides of (ii), we get

$$A + B = 0 \Rightarrow B = -A = -\frac{1}{2}.$$

Comparing the constant terms on both sides of (ii), we get

$$A - C = 1 \Rightarrow C = (A-1) = \left(\frac{1}{2} - 1\right) = \frac{-1}{2}.$$

$$\therefore \frac{1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{-\frac{1}{2}x - \frac{1}{2}}{(x^2+1)}$$

$$\therefore \frac{x^4}{(x-1)(x^2+1)} = (x+1) + \frac{1}{2(x-1)} - \frac{1}{2} \cdot \frac{(x+1)}{(x^2+1)}$$

$$\Rightarrow \int \frac{x^4}{(x-1)(x^2+1)} dx = \int (x+1)\, dx + \frac{1}{2} \int \frac{dx}{(x-1)}$$

$$- \frac{1}{4} \cdot \int \frac{2x}{(x^2+1)} dx - \frac{1}{2} \int \frac{dx}{(x^2+1)}$$

$$= \frac{x^2}{2} + x + \frac{1}{2} \log |x-1| - \frac{1}{4} \log (x^2+1) - \frac{1}{2} \tan^{-1} x + C.$$

EXAMPLE 6 *Evaluate* $\int \frac{(3x+5)}{(x^3 - x^2 - x + 1)} dx.$ **[CBSE 2005C, '13C]**

SOLUTION $(x^3 - x^2 - x + 1) = x^2(x-1) - (x-1) = (x-1)(x^2-1) = (x-1)^2(x+1).$

Let $\frac{3x+5}{(x^3 - x^2 - x + 1)} = \frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$

$$\Rightarrow (3x+5) \equiv A(x-1)(x+1) + B(x+1) + C(x-1)^2. \quad \dots \text{(i)}$$

Putting $x = 1$ on both sides of (i), we get $B = 4$.

Putting $x = -1$ on both sides of (i), we get $C = \frac{1}{2}$.

Comparing the coefficient of x^2 on both sides of (i), we get

$$A + C = 0 \Rightarrow A = -C = \frac{-1}{2}.$$

$$\therefore \frac{(3x+5)}{(x^3 - x^2 - x + 1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$$

$$\Rightarrow \int \frac{(3x+5)}{(x^3-x^2-x+1)}dx = -\frac{1}{2}\int\frac{dx}{(x-1)} + 4\int\frac{dx}{(x-1)^2} + \frac{1}{2}\int\frac{dx}{(x+1)}$$

$$= -\frac{1}{2}\log|x-1| - \frac{4}{(x-1)} + \frac{1}{2}\log|x+1| + C.$$

EXAMPLE 7 *Evaluate* $\int \frac{(x^3-1)}{(x^3+x)}dx.$

SOLUTION We have

$$\frac{(x^3-1)}{(x^3+x)} = 1 - \frac{(x+1)}{(x^3+x)} \qquad \text{[on dividing]}$$

$$= 1 - \frac{(x+1)}{x(x^2+1)}. \qquad \ldots \text{(i)}$$

Let $\frac{(x+1)}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)}$.

Then, $(x+1) \equiv A(x^2+1) + (Bx+C)x.$ ... (ii)

Putting $x=0$ in (ii), we get $A=1$.

Comparing the coefficients of x in (ii), we get $C=1$.

Comparing the coefficients of x^2 in (ii), we get

$A+B=0 \Rightarrow B=-A=-1.$

$\therefore \quad A=1,\ B=-1$ and $C=1$.

Thus, $\frac{(x+1)}{x(x^2+1)} = \frac{1}{x} + \frac{(1-x)}{(x^2+1)}$

$$\Rightarrow \int \frac{(x^3-1)}{(x^3+x)}dx = \int dx - \int \frac{(x+1)}{x(x^2+1)}dx$$

$$= x - \left\{\int\frac{dx}{x} + \int\frac{(1-x)}{(x^2+1)}dx\right\}$$

$$= x - \int\frac{dx}{x} - \int\frac{dx}{(x^2+1)} + \frac{1}{2}\int\frac{2x}{(x^2+1)}dx$$

$$= x - \log|x| - \tan^{-1}x + \frac{1}{2}\log(x^2+1) + C.$$

EXAMPLE 8 *Evaluate* $\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx.$ **[CBSE 2006C]**

SOLUTION Putting $\sin x = t$ and $\cos x\, dx = dt$, we get

$$I = \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int\frac{dt}{(1-t)(2-t)}.$$

Let $\frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)} + \frac{B}{(2-t)}$

$\Rightarrow \ 1 \equiv A(2-t) + B(1-t).$ ... (i)

Putting $t=1$ in (i), we get $A=1$.

Putting $t = 2$ in (i), we get $B = -1$.

$$\therefore \quad \frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)} - \frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \frac{dt}{(1-t)(2-t)}$$

$$= \int \left\{ \frac{1}{(1-t)} - \frac{1}{(2-t)} \right\} dt$$

$$= \int \frac{dt}{(1-t)} - \int \frac{dt}{(2-t)}$$

$$= -\log|1-t| + \log|2-t| + C$$

$$= \log\left|\frac{2-t}{1-t}\right| + C = \log\left|\frac{2-\sin x}{1-\sin x}\right| + C.$$

EXAMPLE 9 *Evaluate* $\int \frac{dx}{x\{6(\log x)^2 + 7\log x + 2\}}$.

SOLUTION Putting $\log x = t$ and $\frac{1}{x}dx = dt$, we get

$$I = \int \frac{dx}{x\{6(\log x)^2 + 7\log x + 2\}} = \int \frac{dt}{(6t^2 + 7t + 2)} = \int \frac{dt}{(2t+1)(3t+2)}.$$

Let $\frac{1}{(2t+1)(3t+2)} = \frac{A}{(2t+1)} + \frac{B}{(3t+2)}$.

Then, $1 \equiv A(3t+2) + B(2t+1)$. ... (i)

Putting $t = -\frac{1}{2}$ in (i), we get $A = 2$.

Putting $t = \frac{-2}{3}$ in (i), we get $B = -3$.

$$\therefore \quad \frac{1}{(2t+1)(3t+2)} = \frac{2}{(2t+1)} - \frac{3}{(3t+2)}$$

$$\Rightarrow I = \int \frac{dt}{(2t+1)(3t+2)}$$

$$= \int \frac{2dt}{(2t+1)} - \int \frac{3dt}{(3t+2)}$$

$$= \log|2t+1| - \log|3t+2| + C$$

$$= \log\left|\frac{2t+1}{3t+2}\right| + C$$

$$= \log\left|\frac{2\log x + 1}{3\log x + 2}\right| + C.$$

EXAMPLE 10 *Evaluate* $\int \frac{x^2}{(1+x^3)(2+x^3)}\,dx.$ **[CBSE 2003C]**

SOLUTION Putting $x^3 = t$ and $x^2 dx = \frac{1}{3}dt$, we get

$$I = \int \frac{x^2}{(1+x^3)(2+x^3)}\,dx = \frac{1}{3} \cdot \int \frac{dt}{(1+t)(2+t)}\cdot$$

Let $\frac{1}{(1+t)(2+t)} = \frac{A}{(1+t)} + \frac{B}{(2+t)}\cdot$ Then,

$$1 \equiv A(2+t) + B(1+t). \qquad \dots \text{(i)}$$

Putting $t = -1$ in (i), we get $A = 1$.
Putting $t = -2$ in (i), we get $B = -1$.

$$\therefore \quad \frac{1}{(1+t)(2+t)} = \frac{1}{(1+t)} - \frac{1}{(2+t)}$$

$$\Rightarrow \quad I = \int \frac{dt}{(1+t)(2+t)} = \int \frac{dt}{(1+t)} - \int \frac{dt}{(2+t)}$$

$$= \log|1+t| - \log|2+t| + C$$

$$= \log\left|\frac{1+t}{2+t}\right| + C$$

$$= \log\left|\frac{1+x^3}{2+x^3}\right| + C.$$

EXAMPLE 11 *Evaluate* $\int \frac{dx}{(e^x - 1)}\cdot$ **[CBSE 2003]**

SOLUTION Putting $e^x = t$ and $e^x dx = dt$, i.e., $dx = \frac{1}{t}dt$, we get

$$I = \int \frac{dx}{(e^x - 1)} = \int \frac{dt}{t(t-1)}\cdot$$

Let $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}\cdot$

Then, $1 \equiv A(t-1) + Bt.$ $\qquad \dots$ (i)

Putting $t = 0$ in (i), we get $A = -1$.
Putting $t = 1$ in (i), we get $B = 1$.

$$\therefore \quad \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{(t-1)}\cdot$$

Hence, $I = \int \frac{dx}{(e^x - 1)}$

$$= \int \frac{dt}{t(t-1)} = \int \frac{-1}{t}\,dt + \int \frac{1}{(t-1)}\,dt$$

$$= -\log|t| + \log|t-1| + C$$

$$= \log\left|\frac{t-1}{t}\right| + C$$

$$= \log\left|\frac{e^x-1}{e^x}\right| + C.$$

EXAMPLE 12 *Evaluate* $\int \frac{dx}{x(x^n+1)}\cdot$

SOLUTION Putting $x^n = t$, we get $nx^{n-1}dx = dt$.

$\therefore \quad \frac{nx^n}{x}dx = dt \Rightarrow \frac{1}{x}dx = \frac{1}{nt}dt$ (note).

$$\therefore \quad \int \frac{dx}{x(x^n+1)} = \int \frac{dt}{nt(t+1)} = \frac{1}{n}\cdot\int \frac{dt}{t(t+1)}\cdot \qquad \ldots \text{(i)}$$

Let $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{(t+1)}\cdot$

Then, $1 \equiv A(t+1) + Bt$. … (ii)

Putting $t = 0$ in (i), we get $A = 1$.

Putting $t = -1$ in (i), we get $B = -1$.

$$\therefore \quad \frac{1}{t(t+1)} = \left\{\frac{1}{t} - \frac{1}{(t+1)}\right\}$$

$$\Rightarrow \int \frac{dx}{x(x^n+1)} = \frac{1}{n}\int \frac{dt}{t(t+1)}$$

$$= \frac{1}{n}\left[\int \frac{1}{t}dt - \int \frac{1}{(t+1)}dt\right]$$

$$= \frac{1}{n}\cdot\{\log|t| - \log|t+1|\} + C$$

$$= \frac{1}{n}\cdot\log\left|\frac{t}{t+1}\right| + C$$

$$= \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C.$$

EXAMPLE 13 *Evaluate* $\int \frac{dx}{x(x^5+3)}\cdot$ **[CBSE 2013]**

SOLUTION Let $I = \int \frac{dx}{x(x^5+3)} = \int \frac{x^4}{x^5(x^5+3)}dx.$ … (i)

Putting $x^5 = t$ and $5x^4dx = dt$ in (i), we get

$$I = \frac{1}{5}\int \frac{dt}{t(t+3)}\cdot \qquad \ldots \text{(ii)}$$

Let $\frac{1}{t(t+3)} = \frac{A}{t} + \frac{B}{t+3}\cdot$

Then, $1 \equiv A(t+3) + Bt$. ... (iii)

Putting $t = 0$ on both sides of (iii), we get $A = \frac{1}{3}\cdot$

Putting $t = -3$ on both sides of (iii), we get $B = \frac{-1}{3}\cdot$

$$\therefore \quad \frac{1}{t(t+3)} = \frac{1}{3t} - \frac{1}{3(t+3)}$$

$$\Rightarrow \quad I = \frac{1}{5}\int\left\{\frac{1}{3t} - \frac{1}{3(t+3)}\right\}dt$$

$$= \frac{1}{15}\int\frac{1}{t}\,dt - \frac{1}{15}\int\frac{1}{(t+3)}\,dt$$

$$= \frac{1}{15}\log|t| - \frac{1}{15}\log|t+3| + C$$

$$= \frac{1}{15}\log|x^5| - \frac{1}{15}\log|x^5+3| + C.$$

EXAMPLE 14 *Evaluate* $\int\frac{x^2}{(x^2+4)(x^2+9)}\,dx.$ **[CBSE 2013]**

SOLUTION Let $\frac{x^2}{(x^2+4)(x^2+9)} = \frac{y}{(y+4)(y+9)}$, where $x^2 = y$.

Let $\frac{y}{(y+4)(y+9)} = \frac{A}{(y+4)} + \frac{B}{(y+9)}\cdot$

Then, $y \equiv A(y+9) + B(y+4)$. ... (i)

Putting $y = -4$ on both sides of (i), we get $A = \frac{-4}{5}\cdot$

Putting $y = -9$ on both sides of (i), we get $B = \frac{9}{5}\cdot$

$$\therefore \quad \frac{y}{(y+4)(y+9)} = \frac{-4}{5(y+4)} + \frac{9}{5(y+9)}$$

$$\Rightarrow \quad \frac{x^2}{(x^2+4)(x^2+9)} = \frac{-4}{5(x^2+4)} + \frac{9}{5(x^2+9)}$$

$$\Rightarrow \quad \int\frac{x^2}{(x^2+4)(x^2+9)}\,dx = \frac{-4}{5}\int\frac{dx}{(x^2+4)} + \frac{9}{5}\int\frac{dx}{(x^2+9)}$$

$$= \left(\frac{-4}{5}\times\frac{1}{2}\right)\tan^{-1}\frac{x}{2} + \left(\frac{9}{5}\times\frac{1}{3}\right)\tan^{-1}\frac{x}{3} + C$$

$$= \frac{-2}{5}\tan^{-1}\frac{x}{2} + \frac{3}{5}\tan^{-1}\frac{x}{3} + C.$$

EXAMPLE 15 *Evaluate* $\int\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}\,dx.$

SOLUTION Let $\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + \frac{A}{(x-4)} + \frac{B}{(x-5)} + \frac{C}{(x-6)}\cdot$

Then, $(x-1)(x-2)(x-3)$

$$\equiv (x-4)(x-5)(x-6) + A(x-5)(x-6) + B(x-4)(x-6) + C(x-4)(x-5). \quad \ldots \text{(i)}$$

Putting $x = 4$ on both sides of (i), we get $A = 3$.

Putting $x = 5$ on both sides of (i), we get $B = -24$.

Putting $x = 6$ on both sides of (i), we get $C = 30$.

$$\therefore \quad I = \int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}\, dx$$

$$= \int \left\{1 + \frac{3}{(x-4)} - \frac{24}{(x-5)} + \frac{30}{(x-6)}\right\} dx$$

$$= \int dx + 3\int \frac{dx}{(x-4)} - 24\int \frac{dx}{(x-5)} + 30\int \frac{dx}{(x-6)}$$

$$= x + 3\log|x-4| - 24\log|x-5| + 30\log|x-6| + C.$$

EXAMPLE 16 *Evaluate* $\displaystyle\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\, dx.$

SOLUTION We have

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(t+1)(t+2)}{(t+3)(t+4)}, \text{ where } x^2 = t$$

$$= \frac{(t^2+3t+2)}{(t^2+7t+12)} = 1 - \frac{(4t+10)}{(t+3)(t+4)}.$$

Let $\displaystyle\frac{(4t+10)}{(t+3)(t+4)} = \frac{A}{(t+3)} + \frac{B}{(t+4)}$

$$\Rightarrow \quad (4t+10) \equiv A(t+4) + B(t+3). \quad \ldots \text{(i)}$$

Putting $t = -3$ in (i), we get $A = -2$.

Putting $t = -4$ in (i), we get $B = 6$.

$$\therefore \quad \frac{(4t+10)}{(t+3)(t+4)} = \frac{-2}{(t+3)} + \frac{6}{(t+4)}. \quad \ldots \text{(ii)}$$

Thus, $\displaystyle\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(t+1)(t+2)}{(t+3)(t+4)}$, where $x^2 = t$

$$= \frac{(t^2+3t+2)}{(t^2+7t+12)} = 1 - \frac{(4t+10)}{(t+3)(t+4)}$$

$$= 1 - \left\{\frac{-2}{(t+3)} + \frac{6}{(t+4)}\right\} \quad \text{[from (ii)]}$$

$$= \left\{1 + \frac{2}{(t+3)} - \frac{6}{(t+4)}\right\}$$

$$= \left\{1 + \frac{2}{(x^2+3)} - \frac{6}{(x^2+4)}\right\}.$$

$$\therefore \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx = \int\left\{1+\frac{2}{(x^2+3)}-\frac{6}{(x^2+4)}\right\}dx$$

$$= \int dx + 2\int\frac{dx}{(x^2+3)} - 6\int\frac{dx}{(x^2+4)}$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{6}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 3\tan^{-1}\left(\frac{x}{2}\right) + C.$$

EXAMPLE 17 *Evaluate* $\int \frac{(3\sin\theta-2)\cos\theta}{(5-\cos^2\theta-4\sin\theta)}d\theta.$

SOLUTION We have

$$I = \int \frac{(3\sin\theta-2)\cos\theta}{\{5-(1-\sin^2\theta)-4\sin\theta\}}d\theta$$

$$= \int \frac{(3\sin\theta-2)\cos\theta}{(4+\sin^2\theta-4\sin\theta)}d\theta$$

$$= \int \frac{(3\sin\theta-2)\cos\theta}{(\sin\theta-2)^2}d\theta = \int\frac{(3t-2)}{(t-2)^2}dt, \text{ where } \sin\theta = t.$$

Let $\frac{(3t-2)}{(t-2)^2} = \frac{A}{(t-2)} + \frac{B}{(t-2)^2}\cdot$ Then,

$$(3t-2) \equiv A(t-2)+B. \qquad \ldots \text{(i)}$$

Putting $t=2$ in (i), we get $B=4$.

Comparing the coefficients of t on both sides of (i), we get $A=3$.

Thus, $A=3$ and $B=4$.

$$\therefore \frac{(3t-2)}{(t-2)^2} = \frac{3}{(t-2)} + \frac{4}{(t-2)^2}$$

$$\Rightarrow I = \int\frac{(3t-2)}{(t-2)^2}dt = \int\frac{3}{(t-2)}dt + \int\frac{4}{(t-2)^2}dt$$

$$= 3\log|t-2| - \frac{4}{(t-2)} + C$$

$$= 3\log|\sin\theta-2| - \frac{4}{(\sin\theta-2)} + C$$

$$= 3\log(2-\sin\theta) + \frac{4}{(2-\sin\theta)} + C \quad [\because (2-\sin\theta)>0].$$

EXAMPLE 18 *Evaluate* $\int \frac{(\tan\theta+\tan^3\theta)}{(1+\tan^3\theta)}d\theta.$ **[CBSE 2009C]**

SOLUTION We have

$$\frac{(\tan\theta+\tan^3\theta)}{(1+\tan^3\theta)} = \frac{\tan\theta(1+\tan^2\theta)}{(1+\tan^3\theta)} = \frac{\tan\theta\sec^2\theta}{(1+\tan^3\theta)}\cdot$$

$$\therefore\quad I = \int \frac{(\tan\theta + \tan^3\theta)}{(1+\tan^3\theta)}\,d\theta$$

$$= \int \frac{\tan\theta\sec^2\theta}{(1+\tan^3\theta)}\,d\theta$$

$$= \int \frac{t}{(1+t^3)}\,dt = \int \frac{t}{(1+t)(1-t+t^2)}\,dt, \text{ where } \tan\theta = t.$$

Let $\dfrac{t}{(1+t)(1-t+t^2)} = \dfrac{A}{(1+t)} + \dfrac{(Bt+C)}{(1-t+t^2)}\cdot$ Then,

$$t \equiv A(1-t+t^2) + (Bt+C)(1+t). \qquad \ldots \text{(i)}$$

Putting $t = -1$ on both sides of (i), we get $A = \dfrac{-1}{3}\cdot$

Comparing the coefficients of t^2 on both sides of (i), we get

$$A + B = 0 \Rightarrow B = -A = \frac{1}{3}\cdot$$

Comparing the constant terms on both sides of (i), we get

$$A + C = 0 \Rightarrow C = -A = \frac{1}{3}\cdot$$

$$\therefore\quad \frac{t}{(1+t)(1-t+t^2)} = \frac{-1}{3(1+t)} + \frac{\left(\frac{1}{3}t + \frac{1}{3}\right)}{(1-t+t^2)}\cdot$$

Now, $I = \displaystyle\int \frac{t}{(1+t)(1-t+t^2)}\,dt$

$$= -\frac{1}{3}\int \frac{dt}{(1+t)} + \frac{1}{6}\int \frac{2t}{(t^2-t+1)}\,dt + \frac{1}{3}\int \frac{dt}{(t^2-t+1)}$$

$$= -\frac{1}{3}\int \frac{dt}{(1+t)} + \frac{1}{6}\int \frac{(2t-1)+1}{(t^2-t+1)}\,dt + \frac{1}{3}\int \frac{dt}{(t^2-t+1)}$$

$$= -\frac{1}{3}\int \frac{dt}{(1+t)} + \frac{1}{6}\int \frac{(2t-1)}{(t^2-t+1)}\,dt + \frac{1}{2}\int \frac{dt}{(t^2-t+1)}$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\int \frac{dt}{\left(t^2 - t + \frac{1}{4}\right) + \frac{3}{4}}$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\int \frac{dt}{(t-1/2)^2 + (\sqrt{3}/2)^2}$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{\left(t-\frac{1}{2}\right)}{(\sqrt{3}/2)} + C$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2 - t + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

$$= -\frac{1}{3}\log|1+\tan\theta| + \frac{1}{6}\log|\tan^2\theta - \tan\theta + 1|$$

$$+ \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C.$$

EXAMPLE 19 *Evaluate* $\int \frac{dx}{(\sin x - \sin 2x)}$. **[CBSE 2010]**

SOLUTION $\int \frac{dx}{(\sin x - \sin 2x)} = \int \frac{dx}{(\sin x - 2\sin x \cos x)}$

$$= \int \frac{dx}{\sin x(1 - 2\cos x)} = \int \frac{\sin x}{\sin^2 x(1 - 2\cos x)}dx$$

$$= \int \frac{\sin x}{(1-\cos^2 x)(1 - 2\cos x)}dx$$

$$= -\int \frac{dt}{(1-t^2)(1-2t)}, \text{ where } \cos x = t$$

$$= \int \frac{dt}{(t-1)(t+1)(1-2t)}. \qquad \ldots \text{(i)}$$

Let $\frac{1}{(t-1)(t+1)(1-2t)} = \frac{A}{(t-1)} + \frac{B}{(t+1)} + \frac{C}{(1-2t)}$.

Then, $1 \equiv A(t+1)(1-2t) + B(t-1)(1-2t) + C(t-1)(t+1)$. ... (ii)

Putting $t = 1$ in (ii), we get $A = \frac{-1}{2}$.

Putting $t = -1$ in (ii), we get $B = \frac{-1}{6}$.

Putting $t = \frac{1}{2}$ in (ii), we get $C = \frac{-4}{3}$.

$$\therefore \quad I = -\frac{1}{2}\int \frac{dt}{(t-1)} - \frac{1}{6}\cdot\int \frac{dt}{(t+1)} - \frac{4}{3}\cdot\int \frac{dt}{(1-2t)}$$

$$= -\frac{1}{2}\log|t-1| - \frac{1}{6}\log|t+1| + \frac{2}{3}\cdot\int \frac{-2dt}{(1-2t)}$$

$$= -\frac{1}{2}\log|t-1| - \frac{1}{6}\log|t+1| + \frac{2}{3}\log|1-2t| + C$$

$$= -\frac{1}{2}\log|\cos x - 1| - \frac{1}{6}\log|\cos x + 1| + \frac{2}{3}\log|1 - 2\cos x| + C.$$

EXAMPLE 20 *Evaluate* $\int \frac{(1-\cos x)}{\cos x(1+\cos x)}dx$.

SOLUTION Let $\frac{(1-\cos x)}{\cos x(1+\cos x)} = \frac{(1-t)}{t(1+t)} = \frac{A}{t} + \frac{B}{(1+t)}$, where $t = \cos x$.

Then, $(1-t) \equiv A(1+t) + Bt.$

Putting $t = 0$ in this identity, we get $A = 1.$

Putting $t = -1$ in the identity, we get $B = -2.$

$$\therefore \quad \frac{(1-t)}{t(1+t)} = \frac{1}{t} - \frac{2}{1+t}$$

$$\text{or} \quad \frac{(1-\cos x)}{\cos x(1+\cos x)} = \frac{1}{\cos x} - \frac{2}{(1+\cos x)}$$

$$\therefore \quad \int \frac{(1-\cos x)}{\cos x(1+\cos x)}\,dx = \int \frac{dx}{\cos x} - 2\int \frac{dx}{(1+\cos x)}$$

$$= \int \sec x\,dx - \int \sec^2 \frac{x}{2}\,dx$$

$$= \log|\sec x + \tan x| - 2\tan\frac{x}{2} + C.$$

EXAMPLE 21 *Evaluate* $\int \frac{(x^2+1)}{(x+1)^2}\,dx.$ **[CBSE 2006]**

SOLUTION On dividing (x^2+1) by (x^2+2x+1), we get

$$\frac{(x^2+1)}{(x+1)^2} = \left\{1 - \frac{2x}{(x+1)^2}\right\}.$$

$$\text{Let} \quad \frac{2x}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2}$$

$\Rightarrow\ 2x \equiv A(x+1) + B.$ … (i)

On equating the coefficients of x, we get $A = 2.$

On equating constant terms, we get $A + B = 0 \ \Rightarrow\ B = -A = -2.$

$$\therefore \quad \frac{2x}{(x+1)^2} = \frac{2}{(x+1)} - \frac{2}{(x+1)^2}$$

$$\therefore \quad I = \int \left\{1 - \frac{2x}{(x+1)^2}\right\}$$

$$= \int \left\{1 - \frac{2}{(x+1)} + \frac{2}{(x+1)^2}\right\} dx$$

$$= x - 2\log|x+1| - \frac{2}{(x+1)} + C.$$

EXERCISE 15A

Evaluate:

1. $\int \frac{dx}{x(x+2)}$

2. $\int \frac{(2x+1)}{(x+2)(x-3)}\,dx$ **[CBSE 2007]**

3. $\int \frac{x}{(x+2)(3-2x)}dx$ **[CBSE 2003]**

4. $\int \frac{dx}{x(x-2)(x-4)}$

5. $\int \frac{(2x-1)}{(x-1)(x+2)(x-3)}dx$ **[CBSE 2005]**

6. $\int \frac{(2x-3)}{(x^2-1)(2x+3)}dx$ **[CBSE 2002]**

7. $\int \frac{(2x+5)}{(x^2-x-2)}dx$

8. $\int \frac{(x^2+5x+3)}{(x^2+3x+2)}dx$

9. $\int \frac{(x^2+1)}{(x^2-1)}dx$

10. $\int \frac{x^3}{(x^2-4)}dx$

11. $\int \frac{(3+4x-x^2)}{(x+2)(x-1)}dx$

12. $\int \frac{x^3}{(x-1)(x-2)}dx$

13. $\int \frac{(x^3-x-2)}{(1-x^2)}dx$

14. $\frac{(2x+1)}{(4-3x-x^2)}dx$ **[CBSE 2004C]**

15. $\int \frac{2x}{(x^2+1)(x^2+3)}dx$ **[CBSE 2011]**

16. $\int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$ **[CBSE 2003C, '07]**

17. $\int \frac{\sec^2 x}{(2+\tan x)(3+\tan x)}dx$

18. $\int \frac{\sin x \cos x}{(\cos^2 x-\cos x-2)}dx$

19. $\int \frac{e^x}{(e^{2x}+5e^x+6)}dx$

20. $\int \frac{e^x}{(e^{3x}-3e^{2x}-e^x+3)}dx$

21. $\int \frac{2\log x}{x[2(\log x)^2-\log x-3]}dx$

22. $\int \frac{\operatorname{cosec}^2 x}{(1-\cot^2 x)}dx$

23. $\int \frac{\sec^2 x}{(\tan^3 x+4\tan x)}dx$

24. $\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)}dx$ **[CBSE 2004]**

25. $\frac{e^x}{e^x(e^x-1)}dx$

26. $\int \frac{dx}{x(x^4-1)}$

27. $\int \frac{(1-x^2)}{x(1-2x)}dx$ **[CBSE 2010]**

28. $\int \frac{(x^2+x+1)}{(x+2)(x+1)^2}dx$ **[CBSE 2006C]**

29. $\int \frac{(2x+9)}{(x+2)(x-3)^2}dx$

30. $\int \frac{(x^2+1)}{(x-1)^2(x+3)}dx$ **[CBSE 2012]**

31. $\int \frac{(x^2+1)}{(x+3)(x-1)^2}dx$

32. $\int \frac{(x^2+x+1)}{(x+2)(x^2+1)}dx$ **[CBSE 2009C]**

33. $\int \frac{2x}{(2x+1)^2}dx$

34. $\int \frac{3x+1}{(x+2)(x-2)^2}dx$ **[CBSE 2007C]**

35. $\int \frac{(5x+8)}{x^2(3x+8)}dx$

36. $\int \frac{(5x^2-18x+17)}{(x-1)^2(2x-3)}dx$

37. $\int \frac{8}{(x+2)(x^2+4)} dx$

38. $\int \frac{(3x+5)}{(x^3-x^2+x-1)} dx$

39. $\int \frac{2x}{(x^2+1)(x^2+3)} dx$

40. $\int \frac{x^2}{(x^4-1)} dx$

41. $\int \frac{dx}{(x^3-1)}$

42. $\int \frac{dx}{(x^3+1)}$

43. $\int \frac{dx}{(x+1)^2(x^2+1)}$ **[CBSE 2002]**

44. $\int \frac{17}{(2x+1)(x^2+4)} dx$

45. $\int \frac{dx}{(x^2+2)(x^2+4)}$

46. $\int \frac{x^2}{(x^2+4)(x^2+25)} dx$ **[CBSE 2013]**

47. $\int \frac{dx}{(e^x-1)^2}$

48. $\int \frac{dx}{x(x^5+1)}$

49. $\int \frac{dx}{x(x^6+1)}$

50. $\int \frac{dx}{\sin x(3+2\cos x)}$

51. $\int \frac{dx}{\cos x(5-4\sin x)}$

52. $\int \frac{dx}{\sin x \cos^2 x}$

53. $\int \frac{\tan x}{(1-\sin x)} dx$

54. $\int \frac{dx}{(\sin x + \sin 2x)}$

55. $\int \frac{x^2}{(x^4-x^2-12)} dx$

56. $\int \frac{x^4}{(x^2+1)(x^2+9)(x^2+16)} dx$

57. $\int \frac{\sin 2x}{(1-\cos 2x)(2-\cos 2x)} dx$ **[CBSE 2007]**

58. $\int \frac{2}{(1-x)(1+x^2)} dx$ **[CBSE 2012]**

59. $\int \frac{2x^2+1}{x^2(x^2+4)} dx$ **[CBSE 2013]**

ANSWERS (EXERCISE 15A)

1. $\frac{1}{2}\log\left|\frac{x}{x+2}\right| + C$

2. $\frac{3}{5}\log|x+2| + \frac{7}{5}\log|x-3| + C$

3. $\frac{-2}{7}\log|x+2| - \frac{3}{14}\log|3-2x| + C$

4. $\frac{1}{8}\log|x| - \frac{1}{4}\log|x-2| + \frac{1}{8}\log|x-4| + C$

5. $-\frac{1}{6}\log|x-1| - \frac{1}{3}\log|x+2| + \frac{1}{2}\log|x-3| + C$

6. $-\frac{1}{10}\log|x-1| + \frac{5}{2}\log|x+1| - \frac{12}{5}\log|2x+3| + C$

7. $3\log|x-2| - \log|x+1| + C$

8. $x + 3\log|x+2| - \log|x+1| + C$

9. $x + \log|x-1| - \log|x+1| + C$

10. $\frac{x^2}{2} + 2\log|x^2-4| + C$

11. $-x + 3\log|x+2| + 2\log|x-1| + C$

12. $\frac{x^2}{2} + 3x - \log|x-1| + 8\log|x-2| + C$

13. $-\frac{x^2}{2} + \log\left|\frac{1-x}{1+x}\right| + C$

14. $-\frac{3}{5}\log|1-x|-\frac{7}{5}\log|4+x|+C$

15. $\frac{1}{2}\log\left(\frac{x^2+1}{x^2+3}\right)+C$

16. $\log\left|\frac{1+\sin x}{2+\sin x}\right|+C$

17. $\log\left|\frac{2+\tan x}{3+\tan x}\right|+C$

18. $-\frac{1}{3}\log|\cos x+1|-\frac{2}{3}\log|\cos x-2|+C$

19. $\log\left|\frac{2+e^x}{3+e^x}\right|+C$

20. $\frac{1}{8}\log|e^x-3|-\frac{1}{4}\log|e^x-1|+\frac{1}{8}\log|e^x+1|+C$

21. $\frac{2}{5}\log|1+\log x|+\frac{3}{5}\log|2\log x-3|+C$

22. $-\frac{1}{2}\log\left|\frac{1+\cot x}{1-\cot x}\right|+C$

23. $\frac{1}{4}\log|\tan x|-\frac{1}{8}\log|\tan^2 x+4|+C$

24. $-2\log|1+\sin x|+4\log|2+\sin x|+C$

25. $\log\left|\frac{e^x-1}{e^x}\right|+C$

26. $\log|x|-\frac{1}{4}\log|x^4+1|+C$

27. $\frac{1}{2}x+\log|x|-\frac{3}{4}\log|1-2x|+C$

28. $3\log|x+2|-2\log|x+1|-\frac{1}{(x+1)}+C$

29. $\frac{1}{5}\log|x+2|-\frac{1}{5}\log|x-3|-\frac{3}{(x-3)}+C$

30. $\frac{3}{8}\log|x-1|-\frac{1}{2(x-1)}+\frac{5}{8}\log|x+3|+C$

31. $\frac{5}{2}\log|x-3|-\frac{3}{2}\log|x-1|+\frac{1}{(x-1)}+C$

32. $\frac{3}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C$

33. $\frac{1}{2}\log|2x+1|+\frac{1}{2(2x+1)}+C$

34. $\frac{-5}{16}\log|x+2|+\frac{5}{16}\log|x-2|-\frac{7}{4(x-2)}+C$

35. $\frac{1}{4}\log|x|-\frac{1}{x}-\frac{1}{4}\log|3x+8|+C$

36. $\frac{4}{(x-1)}+\frac{5}{2}\log|2x-3|+C$

37. $\log|x+2|-\frac{1}{2}\log|x^2+4|+\tan^{-1}\frac{x}{2}+C$

38. $4\log|x-1|-2\log|x^2+1|-\tan^{-1}x+C$

39. $\frac{1}{2}\log|x^2+1|-\frac{1}{2}\log|x^2+3|+C$

40. $\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|+\frac{1}{2}\tan^{-1}x+C$

41. $\frac{1}{3}\log|x-1|-\frac{1}{6}\log|x^2+x+1|-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$

42. $\frac{1}{3}\log|x+1|-\frac{1}{6}\log|x^2-x+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$

43. $\frac{1}{2}\log|x+1|-\frac{1}{2(x+1)}-\frac{1}{4}\log|x^2+1|+C$

44. $2\log|2x+1|-\log|x^2+4|+2\tan^{-1}\frac{x}{2}+C$

45. $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)-\frac{1}{4}\tan^{-1}\frac{x}{2}+C$

46. $\frac{-1}{14}\tan^{-1}\left(\frac{x}{2}\right)+\frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right)+C$

47. $x-\log|e^x-1|-\frac{1}{(e^x-1)}+C$

48. $\log|x|-\frac{1}{5}\log|x^5+1|+C$

49. $\log|x|-\frac{1}{6}\log|x^6+1|+C$

50. $-\frac{1}{2}\log|\cos x+1|+\frac{1}{10}\log|\cos x-1|+\frac{2}{5}\log|2\cos x+3|+C$

51. $-\frac{1}{2}\log|1-\sin x|+\frac{1}{18}\log|1+\sin x|+\frac{4}{9}\log|5-4\sin x|+C$

52. $\sec x-\frac{1}{2}\log\left|\frac{\cos x+1}{\cos x-1}\right|+C$

53. $\frac{1}{4}\log\left|\frac{1-\sin x}{1+\sin x}\right|+\frac{1}{2(1-\sin x)}+C$

54. $\frac{1}{2}\log|\cos x+1|+\frac{1}{6}\log|\cos x-1|-\frac{2}{3}\log|2\cos x+1|+C$

55. $\frac{1}{7}\log\left|\frac{x-2}{x+2}\right|+\frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}}+C$

56. $\frac{1}{120}\tan^{-1}x-\frac{27}{56}\tan^{-1}\frac{x}{3}+\frac{64}{105}\tan^{-1}\frac{x}{4}+C$

57. $\frac{1}{2}\log|1-\cos 2x|-\frac{1}{2}\log|2-\cos 2x|+C$

58. $-\log(1-x)+\frac{1}{2}\log(1+x^2)+\tan^{-1}x+C$

59. $\frac{-1}{4x}+\frac{7}{8}\tan^{-1}\frac{x}{2}+C$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 15A)

7. $I=\int\frac{(2x+5)}{(x-2)(x+1)}dx.$

8. $I=\int\left\{1+\frac{(2x+1)}{(x+2)(x+1)}\right\}dx.$

9. $I=\int\left\{1+\frac{2}{(x^2-1)}\right\}dx=\int\left\{1+\frac{2}{(x-1)(x+1)}\right\}dx.$

10. $I=\int\left\{x+\frac{4x}{(x^2-4)}\right\}dx=\int\left\{x+\frac{4x}{(x-2)(x+2)}\right\}dx.$

11. $\frac{(3+4x-x^2)}{(x+2)(x-1)}=\frac{(-x^2+4x+3)}{(x^2+x-2)}=\left\{-1+\frac{(5x+1)}{(x+2)(x-1)}\right\}.$

12. $\dfrac{x^3}{(x-1)(x-2)} = \dfrac{x^3}{(x^2-3x+2)} = \left[x + 3 + \dfrac{7x-6}{x^2-3x+2}\right] = \left\{x + 3 + \dfrac{7x-6}{(x-2)(x-1)}\right\}.$

13. $\dfrac{(x^3-x-2)}{(1-x^2)} = \dfrac{(x^3-x-2)}{(-x^2+1)} = \left\{-x - \dfrac{2}{(1-x^2)}\right\} = \left\{-x - \dfrac{2}{(1-x)(1+x)}\right\}.$

15. Putting $x^2 = t$ and $2x\,dx = dt$, we get

$$I = \int \frac{dt}{(2+t)(3+t)}.$$

16. Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$I = \int \frac{dt}{(1+t)(2+t)}.$$

17. Put $\tan x = t$ and $\sec^2 x\,dx = dt$.

18. Put $\cos x = t$ and $-\sin x\,dx = dt$.

19. Put $e^x = t$ and $e^x dx = dt$.

20. Put $e^x = t$ and $e^x dx = dt$.

21. Put $\log x = t$ and $\dfrac{1}{x}\,dx = dt$.

22. Put $\cot x = t$ and $-\mathrm{cosec}^2 x\,dx = dt$.

23. Put $\tan x = t$ and $\sec^2 x\,dx = dt$.

24. Write $\sin 2x = 2\sin x \cos x$. Put $\sin x = t$ and $\cos x\,dx = dt$.

25. Put $e^x = t$ and $e^x dx = dt$.

26. $I = \dfrac{x^3}{x^4(x^4+1)}\,dx$. Put $x^4 = t$ and $4x^3 dx = dt$.

27. $\dfrac{(1-x^2)}{x(1-2x)} = \dfrac{(-x^2+1)}{(-2x^2+x)} = \left\{\dfrac{1}{2} + \dfrac{\left(1-\dfrac{1}{2x}\right)}{x(1-2x)}\right\}.$

Let $\dfrac{\left(1-\dfrac{1}{2x}\right)}{x(1-2x)} = \dfrac{A}{x} + \dfrac{B}{(1-2x)}$. Then, $\left(1-\dfrac{1}{2x}\right) \equiv A(1-2x) + Bx.$

$(x = 0 \Rightarrow A = 1)$ and $\left(x = \dfrac{1}{2} \Rightarrow B = \dfrac{3}{2}\right)$

$\therefore \quad I = \displaystyle\int \left\{\frac{1}{2} + \frac{1}{x} + \frac{3}{2(1-2x)}\right\} dx.$

32. Let $\dfrac{(x^2+x+1)}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{(x+2)}.$

33. $\dfrac{2x}{(2x+1)^2} = \dfrac{A}{(2x+1)} + \dfrac{B}{(2x+1)^2}.$

34. Let $\dfrac{(3x+5)}{(x-2)^2} = \dfrac{A}{(x-2)} + \dfrac{B}{(x-2)^2}.$

35. Let $\frac{(5x+8)}{x^2(3x+8)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(3x+8)}\cdot$

39. Put $x^2 = t$ and $2x\,dx = dt$. Then,

$$I = \int \frac{dt}{(t+1)(t+3)}\cdot \quad \text{... (i)}$$

Let $\frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)} + \frac{B}{(t+3)}\cdot$ Then,

$$1 \equiv A(t+3) + B(t+1). \quad \text{... (ii)}$$

Putting $t = -1$ in (ii), we get $A = \frac{1}{2}\cdot$

Putting $t = -3$ in (ii), we get $B = \frac{-1}{2}\cdot$

43. Let $\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)}\cdot$

$$\therefore \quad 1 \equiv A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2. \quad \text{... (i)}$$

Putting $x = -1$ in (i), we get $B = \frac{1}{2}\cdot$

Comparing coefficients of various powers of x, we get

$$A + B + D = 0,\ A + C + 2D = 0,\ A + B + 2C + D = 0,\ A + C = 0$$

$$(A + C = 0,\ A + C + 2D = 0) \Rightarrow D = 0.$$

$$\left[A + B + D = 0 \Rightarrow A + B = 0 \Rightarrow A = \frac{-1}{2}\right] \text{ and } \left(C = -A = \frac{1}{2}\right)\cdot$$

45. Put $x^2 = t$. Then,

$$\frac{1}{(x^2+2)(x^2+4)} = \frac{1}{(t+2)(t+4)} = \frac{A}{(t+2)} + \frac{B}{(t+4)}\cdot$$

$$\therefore \quad 1 \equiv A(t+4) + B(t+2). \quad \text{... (i)}$$

Putting $t = -2$ in (i), we get $A = \frac{1}{2}\cdot$

Putting $t = -4$ in (i), we get $B = \frac{-1}{2}\cdot$

$$\therefore \quad I = \frac{1}{2}\int \frac{dx}{(x^2+2)} - \frac{1}{2}\int \frac{dx}{(x^2+4)}\cdot$$

46. Let $\frac{x^2+1}{(x^2+4)(x^2+25)} = \frac{y+1}{(y+4)(y+25)}$, where $x^2 = y$.

Let $\frac{y+1}{(y+4)(y+25)} = \frac{A}{(y+4)} + \frac{B}{(y+25)}\cdot$

Then, $y + 1 \equiv A(y+25) + B(y+4)$. ... (i)

Putting $y = -4$ in (i), we get $A = \frac{-3}{21} = \frac{-1}{7}\cdot$

Putting $y = -25$ in (i), we get $B = \frac{-24}{-21} = \frac{8}{7}\cdot$

$$\therefore \quad I = \frac{-1}{7}\int \frac{dx}{(x^2+4)} + \frac{8}{7}\int \frac{dx}{(x^2+25)}\cdot$$

$$= \left(\frac{-1}{7} \times \frac{1}{2}\right) \tan^{-1} \frac{x}{2} + \left(\frac{8}{7} \times \frac{1}{5}\right) \tan^{-1} \frac{x}{5} + C$$

$$= \frac{-1}{14} \tan^{-1} \frac{x}{2} + \frac{8}{35} \tan^{-1} \frac{x}{5} + C.$$

47. Put $e^x - 1 = t$ and $e^x dx = dt$, i.e., $dx = \dfrac{dt}{(t+1)}\cdot$

$\therefore \quad I = \displaystyle\int \frac{dt}{t^2(t+1)} \cdot$ Let $\dfrac{1}{t^2(t+1)} = \dfrac{A}{t} + \dfrac{B}{t^2} + \dfrac{C}{t+1}\cdot$

Then, $1 \equiv At(t+1) + B(t+1) + Ct^2$. … (i)

48. $I = \displaystyle\int \frac{x^4}{x^5(x^5+1)} dx$. Put $x^5 = t$ and $5x^4\, dx = dt$.

49. $I = \displaystyle\int \frac{x^5}{x^6(x^6+1)} dx$. Now, put $x^6 = t$ and $6x^5\, dx = dt$.

50. $I = \displaystyle\int \frac{\sin x}{\sin^2 x(3+2\cos x)} dx = \int \frac{\sin x}{(1-\cos^2 x)(3+2\cos x)} dx.$

Now, put $\cos x = t$ and $-\sin x\, dx = dt$.

51. $I = \displaystyle\int \frac{\cos x}{\cos^2 x(5-4\sin x)} dx = \int \frac{\cos x}{(1-\sin^2 x)(5-4\sin x)} dx.$

Now, put $\sin x = t$ and $\cos x\, dx = dt$.

52. $I = \displaystyle\int \frac{\sin x}{\sin^2 x \cos^2 x} dx = \int \frac{\sin x}{(1-\cos^2 x)\cos^2 x} dx.$

Put $\cos x = t$ and $-\sin x\, dx = dt$.

53. $I = \displaystyle\int \frac{\sin x}{\cos x(1-\sin x)} dx = \int \frac{\sin x \cos x}{\cos^2 x(1-\sin x)} dx$

$= \displaystyle\int \frac{\sin x \cos x}{(1-\sin^2 x)(1-\sin x)} dx = \frac{t}{(1-t^2)(1-t)} dt$, where $\sin x = t$

$= \displaystyle\int \frac{t}{(1+t)(1-t)^2} dt.$

54. $I = \displaystyle\int \frac{dx}{(\sin x + 2\sin x \cos x)} = \int \frac{dx}{\sin x(1+2\cos x)}$

$= \displaystyle\int \frac{\sin x}{(1-\cos^2 x)(1+2\cos x)} dx$. Now put $\cos x = t$.

55. Let $x^2 = t$. Then, $\dfrac{x^2}{(x^4 - x^2 - 12)} = \dfrac{t}{(t^2 - t - 12)} = \dfrac{t}{(t-4)(t+3)}\cdot$

Let $\dfrac{t}{(t-4)(t+3)} = \dfrac{A}{(t-4)} + \dfrac{B}{(t+3)}\cdot$

Then, $t \equiv A(t+3) + B(t-4)$.

Putting $t = -3$, we get $B = \dfrac{3}{7}\cdot$

Putting $t = 4$, we get $A = \dfrac{4}{7}\cdot$

$\therefore \quad \dfrac{x^2}{(x^4 - x^2 - 12)} = \dfrac{4}{7} \cdot \dfrac{1}{(x^2 - 4)} + \dfrac{3}{7} \cdot \dfrac{1}{(x^2 + 3)}$.

Now, integrate.

56. Let $x^2 = t$. Then, $\dfrac{x^4}{(x^2 + 1)(x^2 + 9)(x^2 + 16)} = \dfrac{t^2}{(t + 1)(t + 9)(t + 16)}$.

EXERCISE 15B

Very-Short-Answer Questions

Evaluate:

1. $\int x^{-6} dx$

2. $\int \left(\sqrt{x} + \dfrac{1}{\sqrt{x}} \right) dx$

3. $\int \sin 3x \, dx$

4. $\int \dfrac{x^2}{(1 + x^3)} dx$ **[CBSE 2008]**

5. $\int \dfrac{2 \cos x}{3 \sin^2 x} dx$ **[CBSE 2008]**

6. $\int \dfrac{(3 \sin \phi - 2) \cos \phi}{(5 - \cos^2 \phi - 4 \sin \phi)} d\phi$ **[CBSE 2007]**

7. $\int \sin^2 x \, dx$

8. $\int \dfrac{(\log x)^2}{x} dx$

9. $\int \dfrac{(x + 1)(x + \log x)^2}{x} dx$

10. $\int \dfrac{\sin x}{(1 + \cos x)} dx$

11. $\int \dfrac{(1 + \tan x)}{(1 - \tan x)} dx$

12. $\int \dfrac{(1 - \cot x)}{(1 + \cot x)} dx$

13. $\int \dfrac{(1 + \cot x)}{(x + \log \sin x)} dx$

14. $\int \dfrac{(1 - \sin 2x)}{(x + \cos^2 x)} dx$

15. $\int \dfrac{\sec^2(\log x)}{x} dx$

16. $\int \dfrac{\sin (2 \tan^{-1} x)}{(1 + x^2)} dx$

17. $\int \dfrac{\tan x \sec^2 x}{(1 - \tan^2 x)} dx$

18. $\int \dfrac{(x^4 + 1)}{(x^2 + 1)} dx$

19. $\int \tan^{-1} \sqrt{\dfrac{1 - \sin x}{1 + \sin x}} \, dx$

20. $\int \log (1 + x^2) \, dx$

21. $\int \cos x \cos 3x \, dx$

22. $\int \sin 3x \sin x \, dx$

23. $\int \dfrac{xe^x}{(x + 1)^2} dx$

24. $\int e^x \{\tan x - \log \cos x\} \, dx$

25. $\int \dfrac{dx}{(1 - \sin x)}$

26. $\int x \cos x^2 dx$

27. $\int \frac{\cot x}{\sqrt{\sin x}}\,dx$

28. $\int \frac{\sec^2 x}{\text{cosec}^2 x}\,dx$

29. $\int \sin^{-1}(\cos x)\,dx$

30. $\int \frac{dx}{(\sqrt{x+2}+\sqrt{x+1})}$

31. $\int 2^x\,dx$

32. $\int \frac{(1+\tan x)}{(x+\log \sec x)}\,dx$

33. $\int \frac{\sec^2(\log x)}{x}\,dx$

34. $\int (2x+1)(\sqrt{x^2+x+1})\,dx$

35. $\int \frac{dx}{\sqrt{9x^2+16}}$

36. $\int \frac{dx}{\sqrt{4-9x^2}}$

37. $\int \frac{dx}{\sqrt{4x^2-25}}$

38. $\int \sqrt{4-x^2}\,dx$

39. $\int \sqrt{9+x^2}\,dx$

40. $\int \sqrt{x^2-16}\,dx$

ANSWERS (EXERCISE 15B)

1. $\frac{-1}{5x^5}+C$

2. $\frac{2}{3}x^{3/2}+2\sqrt{x}+C$

3. $\frac{-1}{3}\cos 3x+C$

4. $\frac{1}{3}\log|1+x^3|+C$

5. $\frac{-2}{3}\text{cosec } x+C$

6. $3\log|\sin\phi-2|-\frac{4}{(\sin\phi-2)}+C$

7. $\frac{1}{2}x-\frac{1}{4}\sin 2x+C$

8. $\frac{1}{3}(\log x)^3+C$

9. $\frac{1}{3}(x+\log x)^3+C$

10. $-\log|1+\cos x|+C$

11. $-\log|\cos x+\sin x|+C$

12. $-\log|\sin x+\cos x|+C$

13. $\log|x+\log \sin x|+C$

14. $\log|x+\cos^2 x|+C$

15. $\tan(\log x)+C$

16. $-\frac{1}{2}\cos(2\tan^{-1}x)+C$

17. $-\frac{1}{2}\log|1-\tan^2 x|+C$

18. $\frac{x^3}{3}-x+2\tan^{-1}x+C$

19. $\frac{\pi x}{4}-\frac{x^2}{4}+C$

20. $x\log(1+x^2)-2x+2\tan^{-1}x+C$

21. $\frac{\sin 4x}{8}+\frac{\sin 2x}{4}+C$

22. $\frac{\sin 2x}{4}-\frac{\sin 4x}{8}+C$

23. $\frac{e^x}{(x+1)}+C$

24. $-e^x\log\cos x+C$

25. $\tan x+\sec x+C$

26. $\frac{1}{2}\sin x^2+C$

27. $\frac{-2}{\sqrt{\sin x}}+C$

28. $\tan x - x + C$

29. $\frac{\pi x}{2}-\frac{x^2}{2}+C$

30. $\frac{2}{3}(x+2)^{3/2}-\frac{2}{3}(x+1)^{3/2}+C$

31. $\frac{2^x}{\log 2}+C$

32. $\log|x+\log \sec x|+C$

33. $\tan(\log x)+C$

34. $\frac{2}{3}(x^2+x+1)^{3/2}+C$

35. $\frac{1}{3}\log\left|3x+\sqrt{9x^2+16}\right|+C$

36. $\frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)+C$

37. $\frac{1}{2}\log\left|2x+\sqrt{4x^2-25}\right|+C$

38. $\frac{x}{2}\cdot\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+C$

39. $\frac{x}{2}\sqrt{9+x^2}+\frac{9}{2}\log\left|x+\sqrt{9+x^2}\right|+C$

40. $\frac{1}{2}x\sqrt{x^2-16}-8\log\left|x+\sqrt{x^2-16}\right|+C$

OBJECTIVE QUESTIONS I

Mark (✓) against the correct answer in each of the following:

1. $\int\frac{dx}{(9+x^2)}=?$

   (a) $\tan^{-1}\frac{x}{3}+C$ (b) $\frac{1}{3}\tan^{-1}\frac{x}{3}+C$ (c) $3\tan^{-1}\frac{x}{3}+C$ (d) none of these

2. $\int\frac{dx}{(4+16x^2)}=?$

   (a) $\frac{1}{32}\tan^{-1}4x+C$ (b) $\frac{1}{16}\tan^{-1}\frac{x}{2}+C$

   (c) $\frac{1}{8}\tan^{-1}2x+C$ (d) $\frac{1}{4}\tan^{-1}\frac{x}{2}+C$

3. $\int\frac{dx}{(9+4x^2)}dx=?$

   (a) $\frac{1}{2}\tan^{-1}\frac{2x}{3}+C$ (b) $\frac{1}{6}\tan^{-1}\frac{2x}{3}+C$

   (c) $\frac{1}{6}\tan^{-1}\frac{3x}{2}+C$ (d) none of these

4. $\int\frac{\sin x}{(1+\cos^2 x)}dx=?$

   (a) $-\tan^{-1}(\cos x)+C$ (b) $\cot^{-1}(\cos x)+C$

   (c) $-\cot^{-1}(\cos x)+C$ (d) $\tan^{-1}(\cos x)+C$

5. $\int \frac{\cos x}{(1+\sin^2 x)}\,dx = ?$

(a) $-\tan^{-1}(\sin x) + C$ (b) $\tan^{-1}(\cos x) + C$

(c) $\tan^{-1}(\sin x) + C$ (d) $-\tan^{-1}(\cos x) + C$

6. $\int \frac{e^x}{(e^{2x}+1)}\,dx = ?$

(a) $\cot^{-1}(e^x) + C$ (b) $\tan^{-1}(e^x) + C$ (c) $2\tan^{-1}(e^x) + C$ (d) none of these

7. $\int \frac{3x^5}{(1+x^{12})}\,dx = ?$

(a) $\tan^{-1}x^6 + C$ (b) $\frac{1}{4}\tan^{-1}x^6 + C$ (c) $\frac{1}{2}\tan^{-1}x^6 + C$ (d) none of these

8. $\int \frac{2x^3}{(4+x^8)}\,dx = ?$

(a) $\frac{1}{2}\tan^{-1}\frac{x^4}{2} + C$ (b) $\frac{1}{4}\tan^{-1}\frac{x^4}{2} + C$

(c) $\frac{1}{2}\tan^{-1}x^4 + C$ (d) none of these

9. $\int \frac{dx}{(x^2+4x+8)} = ?$

(a) $\frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C$ (b) $\frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C$

(c) $\frac{1}{2}\tan^{-1}(x+2) + C$ (d) $\tan^{-1}\left(\frac{x+2}{2}\right) + C$

10. $\int \frac{dx}{(2x^2+x+3)} = ?$

(a) $\frac{1}{\sqrt{23}}\tan^{-1}\left(\frac{4x+1}{\sqrt{23}}\right) + C$ (b) $\frac{1}{\sqrt{23}}\tan^{-1}\left(\frac{x+1}{\sqrt{23}}\right) + C$

(c) $\frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{4x+1}{\sqrt{23}}\right) + C$ (d) none of these

11. $\int \frac{dx}{(e^x+e^{-x})} = ?$

(a) $\tan^{-1}(e^x) + C$ (b) $\tan^{-1}(e^{-x}) + C$

(c) $-\tan^{-1}(e^{-x}) + C$ (d) none of these

12. $\int \frac{x^2}{(9+4x^2)} = ?$

(a) $\frac{x}{4} - \frac{1}{8}\tan^{-1}\frac{x}{3} + C$ (b) $\frac{x}{4} - \frac{3}{8}\tan^{-1}\frac{x}{3} + C$

(c) $\frac{x}{4} - \frac{3}{8}\tan^{-1}\frac{2x}{3} + C$ (d) none of these

13. $\int \frac{(x^2-1)}{(x^2+4)}dx = ?$

(a) $x - 5\tan^{-1}\frac{x}{2} + C$

(b) $x - \frac{5}{2}\tan^{-1}\frac{x}{2} + C$

(c) $x - \frac{5}{2}\tan^{-1}\frac{5x}{2} + C$

(d) none of these

14. $\int \frac{dx}{(4+9x^2)} = ?$

(a) $\frac{2}{3}\tan^{-1}\frac{3x}{2} + C$

(b) $\frac{1}{6}\tan^{-1}3x + C$

(c) $\frac{1}{6}\tan^{-1}\frac{3x}{2} + C$

(d) none of these

15. $\int \frac{dx}{(4x^2-4x+3)} = ?$

(a) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$

(b) $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$

(c) $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$

(d) none of these

16. $\int \frac{dx}{(\sin^4 x + \cos^4 x)} = ?$

(a) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2}\tan x}\right) + C$

(b) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\tan x}\right) + C$

(c) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}\tan x}\right) + C$

(d) none of these

17. $\int \frac{(x^2+1)}{(x^4+x^2+1)}dx = ?$

(a) $\tan^{-1}\frac{(x^2-1)}{\sqrt{3}} + C$

(b) $\frac{1}{\sqrt{3}}\tan^{-1}\frac{(x^2-1)}{\sqrt{3}} + C$

(c) $\frac{1}{\sqrt{3}}\tan^{-1}\frac{(x^2-1)}{\sqrt{3}x} + C$

(d) none of these

18. $\int \frac{\sin 2x}{(\sin^4 x + \cos^4 x)}dx = ?$

(a) $\tan^{-1}(\tan^2 x) + C$

(b) $x^2 + C$

(c) $-\tan^{-1}(\tan^2 x) + C$

(d) none of these

19. $\int \frac{dx}{(1-9x^2)} = ?$

(a) $\frac{1}{3}\log\left|\frac{1+3x}{1-3x}\right| + C$

(b) $\frac{1}{3}\log\left|\frac{1-3x}{1+3x}\right| + C$

(c) $\frac{1}{6}\log\left|\frac{1+3x}{1-3x}\right| + C$

(d) $\frac{1}{6}\log\left|\frac{1-3x}{1+3x}\right| + C$

20. $\int \frac{dx}{(16-4x^2)} = ?$

(a) $\frac{1}{8}\log\left|\frac{2-x}{2+x}\right| + C$

(b) $\frac{1}{16}\log\left|\frac{2-x}{2+x}\right| + C$

(c) $\frac{1}{8}\log\left|\frac{2+x}{2-x}\right| + C$

(d) $\frac{1}{16}\log\left|\frac{2+x}{2-x}\right| + C$

21. $\int \frac{x^2}{(1-x^6)}dx = ?$

(a) $\frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$

(b) $\frac{1}{6}\log\left|\frac{1-x^3}{1+x^3}\right| + C$

(c) $\frac{1}{3}\log\left|\frac{1-x^3}{1+x^3}\right| + C$

(d) none of these

22. $\int \frac{x}{(1-x^4)}dx = ?$

(a) $\frac{1}{4}\log\left|\frac{1+x^2}{1-x^2}\right| + C$

(b) $\frac{1}{4}\log\left|\frac{1-x^2}{1+x^2}\right| + C$

(c) $\frac{1}{2}\log\left|\frac{1+x^2}{1-x^2}\right| + C$

(d) none of these

23. $\int \frac{x^2}{(a^6-x^6)}dx = ?$

(a) $\frac{1}{3a^3}\log\left|\frac{a^3+x^3}{a^3-x^3}\right| + C$

(b) $\frac{1}{6a^3}\log\left|\frac{a^3+x^3}{a^3-x^3}\right| + C$

(c) $\frac{1}{6a^3}\log\left|\frac{a^3-x^3}{a^3+x^3}\right| + C$

(d) none of these

24. $\int \frac{dx}{(3-2x-x^2)} = ?$

(a) $\frac{1}{4}\log\left|\frac{3+x}{3-x}\right| + C$

(b) $\frac{1}{4}\log\left|\frac{1+x}{1-x}\right| + C$

(c) $\frac{1}{4}\log\left|\frac{3+x}{1-x}\right| + C$

(d) none of these

25. $\int \frac{dx}{(\cos^2 x - 3\sin^2 x)} = ?$

(a) $\frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C$

(b) $\frac{1}{\sqrt{3}}\log\left|\frac{1-\sqrt{3}\tan x}{1+\sqrt{3}\tan x}\right| + C$

(c) $\frac{1}{2\sqrt{3}}\log\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right| + C$

(d) none of these

26. $\int \frac{\operatorname{cosec}^2 x}{(1-\cot^2 x)}\,dx = ?$

(a) $\frac{1}{2}\log\left|\frac{1+\cot x}{1-\cot x}\right| + C$

(b) $-\frac{1}{2}\log\left|\frac{1+\cot x}{1-\cot x}\right| + C$

(c) $\frac{1}{2}\log\left|\frac{1-\cot x}{1+\cot x}\right| + C$

(d) none of these

27. $\int \frac{dx}{(4x^2-1)} = ?$

(a) $\frac{1}{2}\log\left|\frac{2x-1}{2x+1}\right| + C$

(b) $\frac{1}{2}\log\left|\frac{2x+1}{2x-1}\right| + C$

(c) $\frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right| + C$

(d) none of these

28. $\int \frac{x}{(x^4-16)}\,dx = ?$

(a) $\frac{1}{4}\log\left|\frac{x^2+4}{x^2-4}\right| + C$

(b) $\frac{1}{16}\log\left|\frac{x^2+4}{x^2-4}\right| + C$

(c) $\frac{1}{16}\log\left|\frac{x^2-4}{x^2+4}\right| + C$

(d) none of these

29. $\int \frac{dx}{(\sin^2 x - 4\cos^2 x)} = ?$

(a) $\frac{1}{4}\log\left|\frac{\tan x-2}{\tan x+2}\right| + C$

(b) $\frac{1}{4}\log\left|\frac{\tan x+2}{\tan x-2}\right| + C$

(c) $\frac{1}{4}\log\left|\frac{1-\tan x}{1+\tan x}\right| + C$

(d) none of these

30. $\int \frac{dx}{(4\sin^2 x + 5\cos^2 x)} = ?$

(a) $\frac{1}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{5}}\right) + C$

(b) $\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{\tan x}{\sqrt{5}}\right) + C$

(c) $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + C$

(d) none of these

31. $\int \frac{\sin x}{\sin 3x}\,dx = ?$

(a) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x}{\sqrt{3}-\sin x}\right| + C$

(b) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\cos x}{\sqrt{3}-\cos x}\right| + C$

(c) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C$

(d) none of these

32. $\int \frac{(x^2+1)}{(x^4+1)}\,dx = ?$

(a) $\frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{\sqrt{2}x}\right)+C$

(b) $\frac{1}{2}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)+C$

(c) $\frac{1}{\sqrt{2}}\log\left(\frac{x^2+1}{x^2-1}\right)+C$

(d) none of these

ANSWERS (OBJECTIVE QUESTIONS I)

1. (b) **2.** (c) **3.** (b) **4.** (a) **5.** (c) **6.** (b) **7.** (c) **8** (b) **9.** (a) **10.** (c)
11. (a) **12.** (c) **13.** (b) **14.** (c) **15.** (b) **16.** (a) **17.** (c) **18.** (a) **19.** (c) **20.** (d)
21. (a) **22.** (a) **23.** (b) **24.** (c) **25.** (c) **26.** (b) **27.** (c) **28.** (c) **29.** (a) **30.** (c)
31. (c) **32.** (b)

HINTS TO THE GIVEN OBJECTIVE QUESTIONS I

1. $I = \int \frac{dx}{(3^2+x^2)} = \frac{1}{3}\tan^{-1}\frac{x}{3} + C.$

2. $I = \frac{1}{16}\cdot\int \frac{dx}{\left(\frac{1}{4}+x^2\right)} = \frac{1}{16}\cdot\int \frac{dx}{\left\{\left(\frac{1}{2}\right)^2+x^2\right\}} = \frac{1}{16}\cdot\frac{1}{(1/2)}\tan^{-1}\frac{x}{(1/2)} + C = \frac{1}{8}\tan^{-1}2x + C.$

3. $I = \frac{1}{4}\int \frac{dx}{\left(\frac{9}{4}+x^2\right)} = \frac{1}{4}\cdot\int \frac{dx}{\left\{\left(\frac{3}{2}\right)^2+x^2\right\}}$

$= \frac{1}{4}\cdot\frac{1}{(3/2)}\tan^{-1}\frac{x}{(3/2)} + C = \frac{1}{6}\tan^{-1}\frac{2x}{3} + C.$

4. Putting $\cos x = t$ and $-\sin x\,dx = dt$, we get

$$I = -\int \frac{dt}{(1+t^2)} = -\tan^{-1}t + C = -\tan^{-1}(\cos x) + C.$$

5. Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$I = \int \frac{dt}{(1+t^2)} = \tan^{-1}t + C = \tan^{-1}(\sin x) + C.$$

6. Putting $e^x = t$ and $e^x dx = dt$, we get

$$I = \int \frac{dt}{(t^2+1)} = \tan^{-1}t + C = \tan^{-1}(e^x) + C.$$

7. Putting $x^6 = t$ and $6x^5 dx = dt$, i.e., $3x^5 dx = \frac{1}{2}dt$, we get

$$I = \frac{1}{2}\int \frac{dt}{(1+t^2)} = \frac{1}{2}\tan^{-1}t + C = \frac{1}{2}\tan^{-1}x^6 + C.$$

8. Putting $x^4 = t$ and $4x^3 dx = dt$, i.e., $2x^3 dx = \frac{1}{2}dt$, we get

$$I = \frac{1}{2}\int \frac{dt}{(2^2 + t^2)} = \frac{1}{2} \times \frac{1}{2}\tan^{-1}\frac{t}{2} + C = \frac{1}{4}\tan^{-1}\frac{x^4}{2} + C.$$

9. $I = \int \frac{dx}{\{(x+2)^2 + 2^2\}} = \int \frac{dt}{(t^2 + 2^2)}$, where $x + 2 = t$

$$= \frac{1}{2}\tan^{-1}\frac{t}{2} + C = \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C.$$

10. $I = \frac{1}{2}\int \frac{dx}{\left(x^2 + \frac{x}{2} + \frac{3}{2}\right)} = \frac{1}{2}\int \frac{dx}{\left\{x^2 + \frac{x}{2} + \left(\frac{1}{4}\right)^2\right\} + \left(\frac{3}{2} - \frac{1}{16}\right)}$

$$= \frac{1}{2}\int \frac{dx}{\left\{\left(x + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2\right\}} = \frac{1}{2}\int \frac{dx}{\left\{t^2 + \left(\frac{\sqrt{23}}{4}\right)\right\}},\ \text{where } \left(x + \frac{1}{4}\right) = t$$

$$= \frac{1}{2} \times \frac{4}{\sqrt{23}}\tan^{-1}\frac{t}{\left(\frac{\sqrt{23}}{4}\right)} + C = \frac{2}{\sqrt{23}}\tan^{-1}\frac{4t}{\sqrt{23}} + C = \frac{2}{\sqrt{23}}\tan^{-1}\frac{4\left(x + \frac{1}{4}\right)}{\sqrt{23}} + C$$

$$= \frac{2}{\sqrt{23}}\tan^{-1}\frac{(4x+1)}{\sqrt{23}} + C.$$

11. $I = \int \frac{dx}{\left(e^x + \frac{1}{e^x}\right)} = \int \frac{e^x dx}{(e^{2x} + 1)} = \int \frac{dt}{(t^2 + 1)}$, where $e^x = t$

$$= \tan^{-1}t + C = \tan^{-1}(e^x) + C.$$

12. On dividing x^2 by $(4x^2 + 9)$, we get $\frac{1}{4} - \frac{(9/4)}{(4x^2 + 9)}$.

$$\therefore \quad I = \frac{1}{4}\int dx - \frac{9}{4}\cdot\frac{1}{4}\int \frac{dx}{\left(x^2 + \frac{9}{4}\right)} = \frac{x}{4} - \frac{9}{16}\cdot\frac{1}{(3/2)}\tan^{-1}\frac{x}{(3/2)} + C$$

$$= \frac{x}{4} - \frac{3}{8}\tan^{-1}\frac{2x}{3} + C.$$

13. On dividing, we get $\frac{(x^2 - 1)}{(x^2 + 4)} = \left\{1 - \frac{5}{(x^2 + 4)}\right\}$.

$$\therefore \quad I = \int dx - 5\int \frac{dx}{(x^2 + 4)} = x - \frac{5}{2}\tan^{-1}\frac{x}{2} + C.$$

14. $$I = \frac{1}{9}\int \frac{dx}{\left(\frac{4}{9} + x^2\right)} = \frac{1}{9}\cdot\int \frac{dx}{\left\{\left(\frac{1}{3}\right)^2 + x^2\right\}}$$

$$= \frac{1}{9}\cdot\frac{1}{(2/3)}\tan^{-1}\frac{x}{(2/3)} + C = \frac{1}{6}\tan^{-1}\frac{3x}{2} + C.$$

15. $$I = \frac{1}{4}\int \frac{dx}{\left(x^2 - x + \frac{3}{4}\right)} = \frac{1}{4}\int \frac{dx}{\left(x^2 - x + \frac{1}{4}\right) + \frac{1}{2}} = \frac{1}{4}\int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{4}\int \frac{dt}{\left\{t^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right\}} = \frac{1}{4}\cdot\frac{1}{\left(\frac{1}{\sqrt{2}}\right)}\tan^{-1}\frac{t}{\left(\frac{1}{\sqrt{2}}\right)} + C = \frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}t) + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left\{\frac{\sqrt{2}(2x-1)}{2}\right\} + C = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C.$$

16. $$\frac{1}{(\sin^4 x + \cos^4 x)} = \frac{\sec^4 x}{(\tan^4 x + 1)} = \frac{(1 + \tan^2 x)\sec^2 x}{(\tan^4 x + 1)}$$

Put $\tan x = t$ and $\sec^2 x\, dx = dt$.

$$\therefore \quad I = \int \frac{(1+t^2)}{(1+t^4)}dt = \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + (\sqrt{2})^2}dt = \int \frac{du}{u^2 + (\sqrt{2})^2}, \text{ where } \left(t - \frac{1}{t}\right) = u$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{(t^2 - 1)}{\sqrt{t}} + C = \frac{1}{\sqrt{2}}\tan^{-1}\frac{(\tan^2 x - 1)}{\sqrt{2}\tan x} + C.$$

17. $$\frac{(x^2+1)}{(x^4 + x^2 + 1)} = \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x^2 + 1 + \frac{1}{x^2}\right)} = \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{3})^2}.$$

Put $\left(x - \frac{1}{x}\right) = t$ and $\left(1 + \frac{1}{x^2}\right)dx = dt$.

$$\therefore \quad I = \int \frac{dt}{t^2 + (\sqrt{3})^2} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + C = \frac{1}{\sqrt{3}}\tan^{-1}\frac{\left(x - \frac{1}{x}\right)}{\sqrt{3}} + C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{(x^2 - 1)}{\sqrt{3}x} + C.$$

18. On dividing Nr and Dr by $\cos^4 x$, we get

$$I = \int \frac{\frac{2\sin x}{\cos^3 x}}{(\tan^4 x + 1)}dx = 2\int \frac{\tan x \sec^2 x}{(\tan^4 x + 1)}dx$$

$$= 2\int \frac{t}{(t^4 + 1)} dt, \text{ where } \tan x = t$$

$$= \frac{du}{(u^2 + 1)} = \tan^{-1} u + C, \text{ where } t^2 = u$$

$$= \tan^{-1} t^2 + C = \tan^{-1}(\tan^2 x) + C.$$

19. $I = \frac{1}{9} \cdot \int \frac{dx}{\left(\frac{1}{9} - x^2\right)} = \frac{1}{9} \cdot \int \frac{dx}{\left\{\left(\frac{1}{3}\right)^2 - x^2\right\}}$

$$= \frac{1}{9} \cdot \frac{1}{\left(2 \times \frac{1}{3}\right)} \log \left| \frac{\frac{1}{3} + x}{\frac{1}{3} - x} \right| + C = \frac{1}{6} \log \left| \frac{1 + 3x}{1 - 3x} \right| + C.$$

20. $I = \frac{1}{4} \int \frac{dx}{(4 - x^2)} = \frac{1}{4} \cdot \frac{dx}{(2^2 - x^2)} = \frac{1}{4} \cdot \frac{1}{(2 \times 2)} \log \left| \frac{2 + x}{2 - x} \right| + C = \frac{1}{16} \log \left| \frac{2 + x}{2 - x} \right| + C.$

21. Put $x^3 = t$ and $3x^2 dx = dt$, we get

$$I = \frac{1}{3} \int \frac{dt}{(1 - t^2)} = \frac{1}{3} \cdot \frac{1}{2} \log \left| \frac{1 + t}{1 - t} \right| + C = \frac{1}{6} \log \left| \frac{1 + x^3}{1 - x^3} \right| + C.$$

22. Put $x^2 = t$ and $2x\, dx = dt$.

$$\therefore \quad I = \frac{1}{2} \int \frac{dt}{(1 - t^2)} = \frac{1}{2} \cdot \frac{1}{2} \log \left| \frac{1 + t}{1 - t} \right| + C = \frac{1}{4} \log \left| \frac{1 + x^2}{1 - x^2} \right| + C.$$

23. Put $x^3 = t$ and $3x^2 dx = dt$.

$$\therefore \quad I = \frac{1}{3} \int \frac{dt}{[(a^3)^2 - t^2]} = \frac{1}{3} \times \frac{1}{3a^3} \log \left| \frac{a^3 + t}{a^3 - t} \right| + C = \frac{1}{6a^3} \log \left| \frac{a^3 + x^3}{a^3 - x^3} \right| + C.$$

24. $(3 - 2x - x^2) = 4 - (1 + 2x + x^2) = [2^2 - (1 + x)^2].$

$$\therefore \quad I = \int \frac{dx}{\{2^2 - (1 + x)^2\}} = \int \frac{dt}{(4 - t^2)}, \text{ where } (1 + x) = t$$

$$= \int \frac{dt}{(2^2 - t^2)} = \frac{1}{2 \times 2} \log \left| \frac{2 + t}{2 - t} \right| + C = \frac{1}{4} \log \left| \frac{2 + (1 + x)}{2 - (1 + x)} \right| + C$$

$$= \frac{1}{4} \log \left| \frac{3 + x}{1 - x} \right| + C.$$

25. On dividing Nr and Dr by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x}{(1 - 3\tan^2 x)} dx = \int \frac{dt}{(1 - 3t^2)}, \text{ where } \tan x = t$$

$$= \frac{1}{3} \int \frac{dt}{\left(\frac{1}{3} - t^2\right)} = \frac{1}{3} \cdot \int \frac{dt}{\left\{\left(\frac{1}{\sqrt{3}}\right)^2 - t^2\right\}} = \frac{1}{3} \cdot \frac{1}{2 \times \frac{1}{\sqrt{3}}} \log \left| \frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t} \right| + C$$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{1 + \sqrt{3} \tan x}{1 - \sqrt{3} \tan x} \right| + C.$$

26. Put $\cot x = t$ and $-\operatorname{cosec}^2 x\,dx = dt$.

$$\therefore \quad I = -\int \frac{dt}{(1-t^2)} = -\frac{1}{(2\times 1)} \log\left|\frac{1+t}{1-t}\right| + C = \frac{-1}{2}\log\left|\frac{1+\cot x}{1-\cot x}\right| + C.$$

27. $$I = \frac{1}{4}\int \frac{dx}{\left\{x^2 - \left(\frac{1}{2}\right)^2\right\}} = \frac{1}{4}\cdot\frac{1}{\left(2\times\frac{1}{2}\right)}\log\left|\frac{x-\frac{1}{2}}{x+\frac{1}{2}}\right| + C = \frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right| + C.$$

28. Put $x^2 = t$ and $2x\,dx = dt$.

$$\therefore \quad I = \frac{1}{2}\int \frac{dt}{(t^2 - 4^2)} = \frac{1}{2}\cdot\frac{1}{2\times 4}\log\left|\frac{t-4}{t+4}\right| = \frac{1}{16}\log\left|\frac{x^2-4}{x^2+4}\right| + C.$$

29. On dividing Nr and Dr by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x}{(\tan^2 x - 4)}\,dx = \int \frac{dt}{(t^2-4)}, \text{ where } \tan x = t \text{ and } \sec^2 x\,dx = dt$$

$$= \int \frac{dt}{\{t^2 - 2^2\}} = \frac{1}{(2\times 2)}\log\left|\frac{t-2}{t+2}\right| + C = \frac{1}{4}\log\left|\frac{\tan x - 2}{\tan x + 2}\right| + C.$$

30. On dividing Nr and Dr by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x}{(4\tan^2 x + 5)}\,dx = \int \frac{dt}{(4t^2+5)}, \text{ where } \tan x = t$$

$$= \frac{1}{4}\int \frac{dt}{\left(t^2 + \frac{5}{4}\right)} = \frac{1}{4}\int \frac{dt}{\left\{t^2 + \left(\frac{\sqrt{5}}{2}\right)^2\right\}} = \frac{1}{4}\cdot\frac{1}{\left(\frac{\sqrt{5}}{2}\right)}\tan^{-1}\frac{t}{\left(\frac{\sqrt{5}}{2}\right)} + C$$

$$= \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + C.$$

31. $$I = \int \frac{\sin x}{(3\sin x - 4\sin^3 x)}\,dx = \frac{1}{(3 - 4\sin^2 x)}\,dx$$

$$= \int \frac{\sec^2 x}{3\sec^2 x - 4\tan^2 x}\,dx \quad [\text{on dividing Nr and Dr by } \cos^2 x]$$

$$= \int \frac{\sec^2 x}{(3 - \tan^2 x)}\,dx = \int \frac{dt}{(3-t^2)}, \text{ where } \tan x = t$$

$$= \int \frac{dt}{\{(\sqrt{3})^2 - t^2\}} = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + C = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C.$$

32. On dividing Nr and Dr by x^2, we get

$$I = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}\,dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left\{\left(x-\frac{1}{x}\right)^2 + 2\right\}}\,dx = \frac{dt}{t^2+2}, \text{ where } x - \frac{1}{x} = t$$

$$= \int \frac{dt}{t^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\tan^{-1}\frac{(x^2-1)}{\sqrt{2}x} + C.$$

OBJECTIVE QUESTIONS II

Mark (✓) against the correct answer in each of the following:

1. $\int \frac{dx}{\sqrt{4-9x^2}} = ?$

(a) $\frac{1}{3}\sin^{-1}\frac{x}{3}+C$ (b) $\frac{2}{3}\sin^{-1}\left(\frac{2x}{3}\right)+C$

(c) $\frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)+C$ (d) none of these

2. $\int \frac{dx}{\sqrt{16-4x^2}} = ?$

(a) $\frac{1}{2}\sin^{-1}\frac{x}{2}+C$ (b) $\frac{1}{4}\sin^{-1}\frac{x}{2}+C$ (c) $\frac{1}{2}\sin^{-1}\frac{x}{4}+C$ (d) none of these

3. $\int \frac{\cos x}{\sqrt{4-\sin^2 x}} = ?$

(a) $\sin^{-1}\frac{x}{2}+C$ (b) $\sin^{-1}\left(\frac{1}{2}\cos x\right)+C$

(c) $\sin^{-1}(2\sin x)+C$ (d) $\sin^{-1}\left(\frac{1}{2}\sin x\right)+C$

4. $\int \frac{2^x}{\sqrt{1-4^x}}\,dx = ?$

(a) $\sin^{-1}(2^x)\log 2 + C$ (b) $\frac{\sin^{-1}(2^x)}{\log 2} + C$

(c) $\sin^{-1}(2^x) + C$ (d) none of these

5. $\int \frac{dx}{\sqrt{2x-x^2}} = ?$

(a) $\sin^{-1}(x+1)+C$ (b) $\sin^{-1}(x-2)+C$

(c) $\sin^{-1}(x-1)+C$ (d) none of these

6. $\int \frac{dx}{\sqrt{x(1-2x)}} = ?$

(a) $\frac{1}{\sqrt{2}}\sin^{-1}(2x-1)+C$ (b) $\frac{1}{\sqrt{2}}\sin^{-1}(2x+1)+C$

(c) $\frac{1}{\sqrt{2}}\sin^{-1}(4x+1)+C$ (d) $\frac{1}{\sqrt{2}}\sin^{-1}(4x-1)+C$

7. $\int \frac{3x^2}{\sqrt{9-16x^6}}\,dx = ?$

(a) $\frac{1}{4}\sin^{-1}\left(\frac{x^3}{3}\right)+C$

(b) $\frac{1}{4}\sin^{-1}\left(\frac{4x^3}{3}\right)+C$

(c) $4\sin^{-1}\left(\frac{x^3}{4}\right)+C$

(d) none of these

8. $\int \frac{dx}{\sqrt{2+2x-x^2}} = ?$

(a) $\sin^{-1}\left(\frac{x-1}{\sqrt{3}}\right)+C$

(b) $\sin^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C$

(c) $\sin^{-1}\sqrt{3}(x-1)+C$

(d) none of these

9. $\int \frac{dx}{\sqrt{16-6x-x^2}} = ?$

(a) $\sin^{-1}\left(\frac{x-3}{5}\right)+C$

(b) $\sin^{-1}\left(\frac{x+3}{5}\right)+C$

(c) $\frac{1}{5}\sin^{-1}(x+3)+C$

(d) none of these

10. $\int \frac{dx}{\sqrt{x-x^2}} = ?$

(a) $\sin^{-1}(x-1)+C$

(b) $\sin^{-1}(x+1)+C$

(c) $\sin^{-1}(2x-1)+C$

(d) none of these

11. $\int \frac{dx}{\sqrt{1+2x-3x^2}} = ?$

(a) $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{3x-1}{2}\right)+C$

(b) $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{2x-1}{3}\right)+C$

(c) $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{2x-1}{3}\right)+C$

(d) none of these

12. $\int \frac{dx}{\sqrt{x^2-16}} = ?$

(a) $\sin^{-1}\left(\frac{x}{4}\right)+C$

(b) $\log\left|x+\sqrt{x^2-16}\right|+C$

(c) $\log\left|x-\sqrt{x^2-16}\right|+C$

(d) none of these

13. $\int \frac{dx}{\sqrt{4x^2-9}} = ?$

(a) $\frac{1}{2}\log\left|2x+\sqrt{4x^2-9}\right|+C$

(b) $\frac{1}{4}\log\left|x+\sqrt{4x^2-9}\right|+C$

(c) $\log\left|2x+\sqrt{4x^2-9}\right|+C$

(d) none of these

14. $\int \frac{x^2}{\sqrt{x^6-1}}\,dx = ?$

(a) $\frac{1}{2}\log\left|x^3+\sqrt{x^6-1}\right|+C$

(b) $\frac{1}{3}\log\left|x^3+\sqrt{x^6-1}\right|+C$

(c) $\frac{1}{3}\log\left|x^3-\sqrt{x^6-1}\right|+C$

(d) none of these

15. $\int \frac{\sin x}{\sqrt{4\cos^2 x-1}} = ?$

(a) $-\frac{1}{2}\log\left|2\cos x+\sqrt{4\cos^2 x-1}\right|+C$

(b) $-\frac{1}{3}\log\left|2\cos x+\sqrt{4\cos^2 x-1}\right|+C$

(c) $-\frac{1}{6}\log\left|\cos x+\sqrt{2\cos^2 x-1}\right|+C$

(d) none of these

16. $\int \frac{\sec^2 x}{\sqrt{\tan^2 x-4}}\,dx = ?$

(a) $\log\left|\tan x-\sqrt{\tan^2 x-4}\right|+C$

(b) $\log\left|\tan x+\sqrt{\tan^2 x-4}\right|+C$

(c) $\frac{1}{2}\log\left|\tan x+\sqrt{\tan^2 x-4}\right|+C$

(d) none of these

17. $\int \frac{dx}{(1-e^{2x})} = ?$

(a) $\log\left|e^x+\sqrt{e^{2x}-1}\right|+C$

(b) $\log\left|e^{-x}+\sqrt{e^{-2x}-1}\right|+C$

(c) $-\log\left|e^{-x}+\sqrt{e^{-2x}-1}\right|+C$

(d) none of these

18. $\int \frac{dx}{\sqrt{x^2-3x+2}} = ?$

(a) $\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$

(b) $\log\left|x+\sqrt{x^2-3x+2}\right|+C$

(c) $\log\left|x-\sqrt{x^2-3x+2}\right|+C$

(d) none of these

19. $\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}} dx = ?$

(a) $\log \left| \sin x + \sqrt{\sin^2 x - 2\sin x - 3} \right| + C$

(b) $\log \left| (\sin x - 1) + \sqrt{\sin^2 x - 2\sin x - 3} \right| + C$

(c) $\log \left| (\sin x - 1) - \sqrt{\sin^2 x - 2\sin x - 3} \right| + C$

(d) none of these

20. $\int \frac{dx}{\sqrt{2 - 4x + x^2}} = ?$

(a) $\log \left| (x-2) + \sqrt{x^2 - 4x + 2} \right| + C$ (b) $\log \left| x + \sqrt{x^2 - 4x + 2} \right| + C$

(c) $\log \left| x - \sqrt{x^2 - 4x + 2} \right| + C$ (d) none of these

21. $\int \frac{dx}{\sqrt{x^2 + 6x + 5}} = ?$

(a) $\log \left| x + \sqrt{x^2 + 6x + 5} \right| + C$ (b) $\log \left| x - \sqrt{x^2 + 6x + 5} \right| + C$

(c) $\log \left| (x+3) + \sqrt{x^2 + 6x + 5} \right| + C$ (d) none of these

22. $\int \frac{dx}{\sqrt{(x-3)^2 - 1}} = ?$

(a) $\log \left| (x-3) + \sqrt{x^2 - 6x + 8} \right| + C$ (b) $\log \left| x + \sqrt{x^2 - 6x + 8} \right| + C$

(c) $\log \left| (x-3) - \sqrt{x^2 - 6x + 8} \right| + C$ (d) none of these

23. $\int \frac{dx}{\sqrt{x^2 - 6x + 10}} = ?$

(a) $\log \left| x + \sqrt{x^2 - 6x + 10} \right| + C$ (b) $\log \left| (x-3) + \sqrt{x^2 - 6x + 10} \right| + C$

(c) $\log \left| x - \sqrt{x^2 - 6x + 10} \right| + C$ (d) none of these

24. $\int \frac{x^2 dx}{\sqrt{x^6 + a^6}} dx = ?$

(a) $\frac{1}{3} \log |x^6 + a^6| + C$ (b) $\frac{1}{3} \tan^{-1}\left(\frac{x^3}{a^3}\right) + C$

(c) $\frac{1}{3} \log \left| x^3 + \sqrt{x^6 + a^6} \right| + C$ (d) none of these

25. $\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}}\,dx = ?$

(a) $\log\left|\tan x+\sqrt{\tan^2 x+16}\right|+C$

(b) $\log\left|x+\sqrt{\tan^2 x+16}\right|+C$

(c) $\log\left|\tan x-\sqrt{\tan^2 x+16}\right|+C$

(d) none of these

26. $\int \frac{dx}{\sqrt{3x^2+6x+12}} = ?$

(a) $\log\left|(x+1)+\sqrt{x^2+2x+4}\right|+C$

(b) $\frac{1}{3}\log\left|(x+1)+\sqrt{x^2+2x+4}\right|+C$

(c) $\frac{1}{\sqrt{3}}\log\left|(x+1)+\sqrt{x^2+2x+4}\right|+C$

(d) none of these

27. $\int \frac{dx}{\sqrt{2x^2+4x+6}} = ?$

(a) $\frac{1}{2}\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$

(b) $\frac{1}{\sqrt{2}}\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$

(c) $\frac{1}{\sqrt{2}}\log\left|x+\sqrt{x^2+2x+3}\right|+C$

(d) none of these

28. $\int \frac{x^2}{\sqrt{x^6+2x^3+3}}\,dx = ?$

(a) $\frac{1}{3}\log\left|(x^3+1)+\sqrt{x^6+2x^3+3}\right|+C$

(b) $\log\left|x^3+\sqrt{x^6+2x^3+3}\right|+C$

(c) $\frac{1}{3}\log\left|(x^3+1)-\sqrt{x^6+2x^3+3}\right|+C$

(d) none of these

29. $\int \sqrt{4-x^2}\,dx = ?$

(a) $\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+C$

(b) $x\sqrt{4-x^2}+\sin^{-1}\frac{x}{2}+C$

(c) $\frac{1}{2}x\sqrt{4-x^2}-2\sin^{-1}\frac{x}{2}+C$

(d) none of these

30. $\int \sqrt{1-9x^2}\,dx = ?$

(a) $\frac{x}{2}\sqrt{1-9x^2}+\frac{1}{18}\sin^{-1}3x+C$

(b) $\frac{3x}{2}\sqrt{1-9x^2}+\frac{1}{6}\sin^{-1}3x+C$

(c) $\frac{x}{2}\sqrt{1-9x^2}+\frac{1}{6}\sin^{-1}3x+C$

(d) none of these

31. $\int \sqrt{9-4x^2}\,dx = ?$

(a) $\frac{x}{2}\sqrt{9-4x^2} + \frac{9}{4}\sin^{-1}\frac{2x}{3} + C$

(b) $x\sqrt{9-4x^2} + \frac{9}{2}\sin^{-1}\frac{2x}{3} + C$

(c) $\frac{x}{2}\sqrt{9-4x^2} - \frac{9}{4}\sin^{-1}\frac{2x}{3} + C$

(d) none of these

32. $\int \cos x\sqrt{9-\sin^2 x}\,dx = ?$

(a) $\frac{1}{2}\sin x\sqrt{9-\sin^2 x} + \frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right) + C$

(b) $\frac{\sin x}{2}\sqrt{9-\sin^2 x} + \frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right) + C$

(c) $\frac{1}{2}\cos x\sqrt{9-\sin^2 x} + \frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right) + C$

(d) none of these

33. $\int \sqrt{x^2-16}\,dx = ?$

(a) $x\sqrt{x^2-16} - 4\log\left|x+\sqrt{x^2-16}\right| + C$

(b) $\frac{x}{2}\sqrt{x^2-16} - 8\log\left|x+\sqrt{x^2-16}\right| + C$

(c) $\frac{x}{2}\sqrt{x^2-16} + 8\log\left|x+\sqrt{x^2-16}\right| + C$

(d) none of these

34. $\int \sqrt{x^2-4x+2}\,dx = ?$

(a) $\frac{1}{2}(x-2)\sqrt{x^2-4x+2} + \log\left|(x-2)+\sqrt{x^2-4x+2}\right| + C$

(b) $(x-2)\sqrt{x^2-4x+2} + \frac{1}{2}\log\left|(x-2)+\sqrt{x^2-4x+2}\right| + C$

(c) $\frac{1}{2}(x-2)\sqrt{x^2-4x+2} - \log\left|(x-2)+\sqrt{x^2-4x+2}\right| + C$

(d) none of these

35. $\int \sqrt{9x^2+16}\,dx = ?$

(a) $\frac{x}{2}\sqrt{9x^2+16} + \frac{8}{3}\log\left|3x+\sqrt{9x^2+16}\right| + C$

(b) $\frac{x}{2}\sqrt{9x^2+16} - \frac{8}{3}\log\left|3x+\sqrt{9x^2+16}\right| + C$

(c) $x\sqrt{9x^2+16} + 24\log\left|3x+\sqrt{9x^2+16}\right| + C$

(d) none of these

36. $\int e^x \sqrt{e^{2x}+4}\, dx = ?$

(a) $\frac{1}{2}e^x\sqrt{e^{2x}+4} - 2\log\left|e^x + \sqrt{e^{2x}+4}\right| + C$

(b) $\frac{1}{2}e^x\sqrt{e^{2x}+4} + 2\log\left|e^x + \sqrt{e^{2x}+4}\right| + C$

(c) $e^x\sqrt{e^{2x}+4} + \frac{1}{2}\log\left|e^x + \sqrt{e^{2x}+4}\right| + C$

(d) none of these

37. $\int \frac{\sqrt{16+(\log x)^2}}{x}\, dx = ?$

(a) $\frac{1}{2}\log x \cdot \sqrt{16+(\log x)^2} + 8\log\left|\log x + \sqrt{16+(\log x)^2}\right| + C$

(b) $\frac{1}{2}\log x \cdot \sqrt{16+(\log x)^2} + 4\log\left|\log x + \sqrt{16+(\log x)^2}\right| + C$

(c) $\log x \cdot \sqrt{16+(\log x)^2} + 16\log\left|\log x + \sqrt{16+(\log x)^2}\right| + C$

(d) none of these

ANSWERS (OBJECTIVE QUESTIONS II)

1. (c) **2.** (a) **3.** (d) **4.** (b) **5.** (c) **6.** (d) **7.** (b) **8** (a) **9.** (b) **10.** (c)
11. (a) **12.** (b) **13.** (a) **14.** (b) **15.** (a) **16.** (b) **17.** (c) **18.** (a) **19.** (b) **20.** (a)
21. (c) **22.** (a) **23.** (b) **24.** (c) **25.** (a) **26.** (c) **27.** (b) **28.** (a) **29.** (a) **30.** (c)
31. (a) **32.** (a) **33.** (b) **34.** (c) **35.** (a) **36.** (b) **37.** (a)

HINTS TO THE GIVEN OBJECTIVE QUESTIONS II

1. $I = \int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\cdot\int \frac{dx}{\sqrt{\frac{4}{9}-x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2 - x^2}}$

$= \frac{1}{3}\sin^{-1}\left(\frac{x}{2/3}\right) + C.$

2. $I = \int \frac{dx}{\sqrt{16-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{4-x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{2^2-x^2}} = \frac{1}{2}\sin^{-1}\frac{x}{2} + C.$

3. Putting $\sin x = t$ and $\cos x\, dx = dt$, we get

$$I = \int \frac{dt}{\sqrt{4-t^2}} = \int \frac{dt}{\sqrt{2^2-t^2}} = \sin^{-1}\frac{t}{2} + C = \sin^{-1}\left(\frac{1}{2}\sin x\right) + C.$$

4. Put $2^x = t$ and $2^x \log 2\, dx = dt$.

$$\therefore \quad I = \frac{1}{\log 2}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\log 2}\sin^{-1}t + C = \frac{1}{(\log 2)}\sin^{-1}(2^x) + C.$$

5. $\sqrt{2x - x^2} = \sqrt{1 - (1 - 2x + x^2)} = \sqrt{1 - (x-1)^2}$

$\therefore \quad I = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \int \frac{dt}{\sqrt{1-t^2}}$, where $(x-1) = t$

$= \sin^{-1} t + C = \sin^{-1}(x-1) + C.$

6. $x(1-2x) = (x - 2x^2) = 2\left(\frac{x}{2} - x^2\right) = 2\left[\frac{1}{16} - \left(x^2 - \frac{x}{2} + \frac{1}{16}\right)\right] = 2\left[\left(\frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2\right]$

$\therefore \quad I = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}} = \frac{1}{\sqrt{2}} \cdot \int \frac{dx}{\sqrt{\left(\frac{1}{4}\right)^2 - t^2}}$, where $\left(x - \frac{1}{4}\right) = t$

$= \frac{1}{\sqrt{2}} \sin^{-1} \frac{t}{(1/4)} + C = \frac{1}{\sqrt{2}} \sin^{-1} 4t + C = \frac{1}{\sqrt{2}} \sin^{-1}\left\{4\left(x - \frac{1}{4}\right)\right\} + C$

$= \frac{1}{\sqrt{2}} \sin^{-1}(4x-1) + C.$

7. Put $x^3 = t$ and $3x^2 dx = dt$.

$\therefore \quad I = \int \frac{dt}{\sqrt{9 - 16t^2}} = \frac{1}{4} \int \frac{dt}{\sqrt{\frac{9}{16} - t^2}} = \frac{1}{4} \cdot \int \frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2 - t^2}}$

$= \frac{1}{4} \sin^{-1} \frac{t}{(3/4)} + C = \frac{1}{4} \sin^{-1}\left(\frac{4x^3}{3}\right) + C.$

8. $(2 + 2x - x^2) = 3 - (1 + x^2 - 2x) = (\sqrt{3})^2 - (x-1)^2$

$\therefore \quad I = \int \frac{dx}{\sqrt{(\sqrt{3})^2 - (x-1)^2}} = \int \frac{dt}{\sqrt{(\sqrt{3})^2 - t^2}}$, where $(x-1) = t$ and $dx = dt$

$= \sin^{-1} \frac{t}{\sqrt{3}} + C = \sin^{-1} \frac{(x-1)}{\sqrt{3}} + C.$

9. $(16 - 6x - x^2) = [-(x^2 + 6x + 9) + 25] = \{25 - (x+3)^2\}$

$\therefore \quad I = \int \frac{dx}{\sqrt{5^2 - (x+3)^2}} = \int \frac{dt}{\sqrt{5^2 - t^2}} = \sin^{-1} \frac{t}{5} + C = \sin^{-1}\left(\frac{x+3}{5}\right) + C.$

10. $(x - x^2) = \frac{1}{4} - \left(x^2 - x + \frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2$

$\therefore \quad I = \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}} = \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - t^2}} = \sin^{-1} \frac{t}{(1/2)} + C$

$= \sin^{-1} 2t + C = \sin^{-1} 2\left(x - \frac{1}{2}\right) + C = \sin^{-1}(2x-1) + C.$

11. $(1 + 2x - 3x^2) = -3\left(x^2 - \frac{2}{3}x - \frac{1}{3}\right) = 3\left\{\frac{4}{9} - \left(x^2 - \frac{2}{3}x + \frac{1}{9}\right)\right\}$

$= 3\left\{\left(\frac{2}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2\right\}.$

$$\therefore \quad I = \frac{1}{\sqrt{3}} \cdot \int \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2}} = \frac{1}{\sqrt{3}} \cdot \int \frac{dt}{\sqrt{\left(\frac{2}{3}\right)^2 - t^2}} = \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{t}{2/3}\right) + C$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3t}{2}\right) + C = \frac{1}{\sqrt{3}} \sin^{-1} \frac{(3x-1)}{2} + C.$$

12. $I = \int \frac{dx}{\sqrt{x^2 - 4^2}} = \log\left|x + \sqrt{x^2 - 16}\right| + C.$

13. $I = \frac{1}{2}\int \frac{dx}{\sqrt{x^2 - \left(\frac{3}{2}\right)^2}} = \frac{1}{2}\log\left|x + \sqrt{x^2 - \frac{9}{4}}\right| + C = \frac{1}{2}\log\left|2x + \sqrt{4x^2 - 9}\right| + C.$

14. Put $x^3 = t$ and $3x^2 dx = dt$. Then,

$$I = \frac{1}{3}\int \frac{dt}{\sqrt{t^2 - 1}} = \frac{1}{3}\log\left|t + \sqrt{t^2 - 1}\right| + C = \frac{1}{3}\log\left|x^3 + \sqrt{x^6 - 1}\right| + C.$$

15. Put $\cos x = t$ and $\sin x\, dx = -\,dt$. Then,

$$I = -\int \frac{dt}{\sqrt{4t^2 - 1}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t^2 - \frac{1}{4}}} = -\frac{1}{2}\log\left|t + \sqrt{t^2 - \frac{1}{4}}\right| + C$$

$$= -\frac{1}{2}\log\left|2t + \sqrt{4t^2 - 1}\right| + C = -\frac{1}{2}\log\left|2\cos x + \sqrt{4\cos^2 x - 1}\right| + C.$$

16. Put $\tan x = t$ and $\sec^2 x\, dx = dt$. Then,

$$I = \int \frac{dt}{\sqrt{t^2 - 2^2}} = \log\left|t + \sqrt{t^2 - 4}\right| + C = \log\left|\tan x + \sqrt{\tan^2 x - 4}\right| + C.$$

17. $I = \int \frac{e^{-x}dx}{\sqrt{e^{-2x}(1 - e^{2x})}} = \int \frac{e^{-x}}{\sqrt{e^{-2x} - 1}}\,dx = -\int \frac{dt}{\sqrt{(t^2 - 1)}}$, where $e^{-x} = t$

$$= -\log\left|t + \sqrt{t^2 - 1}\right| + C = -\log\left|e^{-x} + \sqrt{e^{-2x} - 1}\right| + C.$$

18. $I = \int \frac{dx}{\sqrt{\left(x^2 - 3x + \frac{9}{4}\right) - \frac{1}{4}}} = \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$

$$= \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{2}\right)^2}} = \log\left|t + \sqrt{t^2 - \frac{1}{4}}\right| + C, \text{ where } \left(x - \frac{3}{2}\right) = t$$

$$= \log\left|\left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2}\right| + C.$$

19. Putting $\sin x = t$ and $\cos x\, dx = dt$, we get

$$I = \int \frac{dt}{\sqrt{t^2 - 2t - 3}} = \int \frac{dt}{\sqrt{(t^2 - 2t + 1) - 4}} = \int \frac{dt}{\sqrt{(t-1)^2 - 2^2}}$$

$$= \log\left|(t-1) + \sqrt{(t-1)^2 - 2^2}\right| + C = \log\left|(t-1) + \sqrt{t^2 - 2t - 3}\right| + C$$

$$= \log\left|(\sin x - 1) + \sqrt{\sin^2 x - 2\sin x - 3}\right| + C.$$

20. $I = \int \frac{dx}{\sqrt{(x^2 - 4x + 4) - 2}} = \int \frac{dx}{\sqrt{(x-2)^2 - (\sqrt{2})^2}}$

$= \log\left|(x-2) + \sqrt{x^2 - 4x + 2}\right| + C.$

21. $I = \int \frac{dx}{\sqrt{(x^2 + 6x + 9) - 4}} = \int \frac{dx}{\sqrt{(x+3)^2 - 2^2}} = \log\left|(x+3) + \sqrt{x^2 + 6x + 5}\right| + C.$

22. $I = \log\left|(x-3) + \sqrt{x^2 - 6x + 8}\right| + C.$

23. $I = \int \frac{dx}{\sqrt{(x-3)^2 + 1^2}} = \log\left|(x-3) + \sqrt{x^2 - 6x + 10}\right| + C.$

24. Put $x^3 = t$ and $3x^2 dx = dt$. Then,

$$I = \frac{1}{3}\int \frac{dt}{\sqrt{t^2 + (a^3)^2}} = \frac{1}{3}\log\left|t + \sqrt{t^2 + a^6}\right| + C = \frac{1}{3}\log\left|x^3 + \sqrt{x^6 + a^6}\right| + C.$$

25. Put $\tan x = t$ and $\sec^2 x\, dx = dt$. Then,

$$I = \int \frac{dt}{\sqrt{t^2 + 4^2}} = \log\left|t + \sqrt{t^2 + 16}\right| = \log\left|\tan x + \sqrt{\tan^2 x + 16}\right| + C.$$

26. $I = \int \frac{dx}{\sqrt{3(x^2 + 2x + 4)}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{x^2 + 2x + 4}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{(x+1)^2 + (\sqrt{3})^2}}$

$= \frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{t^2 + (\sqrt{3})^2}} = \frac{1}{\sqrt{3}}\log\left|t + \sqrt{t^2 + 3}\right| + C$

$= \frac{1}{\sqrt{3}} \cdot \log\left|(x+1) + \sqrt{x^2 + 2x + 4}\right| + C.$

27. $I = \int \frac{dx}{\sqrt{2(x^2 + 2x + 3)}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2 + 2x + 3}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{(x+1)^2 + (\sqrt{2})^2}}$

$= \frac{1}{\sqrt{2}}\log\left|(x+1) + \sqrt{x^2 + 2x + 3}\right| + C.$

28. Put $x^3 = t$ and $3x^2 dx = dt$. Then,

$$I = \frac{1}{3}\int \frac{dt}{\sqrt{t^2 + 2t + 3}} = \frac{1}{3}\int \frac{dt}{\sqrt{(t+1)^2 + (\sqrt{2})^2}}$$

$$= \frac{1}{3}\log\left|(t+1) + \sqrt{t^2 + 2t + 3}\right| + C$$

$$= \frac{1}{3}\log\left|(x^3 + 1) + \sqrt{x^6 + 2x^3 + 3}\right| + C.$$

29. Using $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$,

$$\int \sqrt{4 - x^2}\,dx = \int \sqrt{2^2 - x^2}\,dx = \frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2} + C.$$

30. $\int \sqrt{1 - 9x^2}\,dx = 3\int \sqrt{\frac{1}{9} - x^2}\,dx = 3\left[\frac{x}{2}\sqrt{\frac{1}{9} - x^2} + \frac{1}{18}\sin^{-1}\frac{x}{(1/3)}\right] + C$

$$= \frac{3x}{2}\sqrt{\frac{1}{9} - x^2} = \frac{1}{6}\sin^{-1} 3x + C = \frac{x}{2}\sqrt{1 - 9x^2} + \frac{1}{6}\sin^{-1} 3x + C.$$

31. $\int \sqrt{9-4x^2}\,dx = 2\int \sqrt{\frac{9}{4}-x^2}\,dx = 2\left[\frac{x}{2}\sqrt{\frac{9}{4}-x^2}+\frac{9}{8}\sin^{-1}\frac{x}{(3/2)}\right]+C$

$$= \frac{x}{2}\sqrt{9-4x^2}+\frac{9}{4}\sin^{-1}\frac{2x}{3}+C.$$

32. Put $\sin x = t$ and $\cos x\,dx = dt$. Then,

$$I = \int \sqrt{9-t^2}\,dt = \frac{t}{2}\sqrt{9-t^2}+\frac{9}{2}\sin^{-1}\frac{t}{3}+C$$

$$= \frac{\sin x}{2}\sqrt{9-\sin^2 x}+\frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right)+C.$$

33. $\int \sqrt{x^2-16}\,dx = \int \sqrt{x^2-4^2}\,dx = \frac{x}{2}\sqrt{x^2-4^2}-\frac{16}{2}\log\left|x+\sqrt{x^2-16}\right|+C$

$$= \frac{x}{2}\sqrt{x^2-16}-8\cdot\log\left|x+\sqrt{x^2-16}\right|+C.$$

34. $I = \int \sqrt{(x-2)^2-(\sqrt{2})^2}\,dx$

$$= \frac{1}{2}(x-2)\sqrt{x^2-4x+2}-\frac{2}{2}\log\left|(x-2)+\sqrt{x^2-4x+2}\right|+C$$

$$= \frac{1}{2}(x-2)\sqrt{x^2-4x+2}-\log\left|(x-2)+\sqrt{x^2-4x+2}\right|+C.$$

35. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$$\therefore \quad I = 3\int \sqrt{x^2+\frac{16}{9}}\,dx = 3\int \sqrt{x^2+\left(\frac{4}{3}\right)^2}\,dx$$

$$= 3\left\{\frac{x}{2}\sqrt{x^2+\frac{16}{9}}+\frac{8}{9}\log\left|x+\sqrt{x^2+\frac{16}{9}}\right|\right\}+C$$

$$= \frac{x}{2}\sqrt{9x^2+16}+\frac{8}{3}\log\left|3x+\sqrt{9x^2+16}\right|+C.$$

36. Put $e^x = t$ and $e^x dx = dt$. Then,

$$I = \int \sqrt{t^2+4}\,dt = \int \sqrt{t^2+2^2}\,dt = \frac{t}{2}\sqrt{t^2+4}+\frac{4}{2}\log\left|t+\sqrt{t^2+4}\right|+C$$

$$= \frac{1}{2}e^x\sqrt{e^{2x}+4}+2\log\left|e^x+\sqrt{e^{2x}+4}\right|+C.$$

37. Put $\log x = t$ and $\frac{1}{x}\,dx = dt$. Then,

$$I = \int \sqrt{t^2+(4)^2} = \frac{t}{2}\sqrt{t^2+16}+\frac{16}{2}\log\left|t+\sqrt{t^2+16}\right|+C$$

$$= \frac{1}{2}\log x\sqrt{(\log x)^2+16}+8\log\left|\log x+\sqrt{(\log x)^2+16}\right|+C.$$

16. DEFINITE INTEGRALS

Fundamental Theorem of Integral Calculus

Let $f(x)$ be a continuous function defined on an interval $[a, b]$ and let the antiderivative of $f(x)$ be $F(x)$. Then, the definite integral of $f(x)$ over $[a, b]$, denoted by $\int_a^b f(x)\,dx$, is given by

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

NOTE Here, a and b are respectively known as the *lower limit* and the *upper limit* of the integral.

The value of a definite integral is unique, for, if $\int f(x)\,dx = F(x) + C$ then

$$\int_a^b f(x)\,dx = [F(x) + C]_a^b = \{F(b) + C\} - \{F(a) + C\} = F(b) - F(a).$$

SOLVED EXAMPLES

[Integrals based on formulae and integration by parts]

EXAMPLE 1 *Evaluate:*

(i) $\int_2^4 \frac{dx}{x}$ *(ii)* $\int_4^9 \sqrt{x}\,dx$ *(iii)* $\int_0^2 \sqrt{6x+4}\,dx$

(iv) $\int_0^1 \frac{dx}{\sqrt{5x+3}}$ *(v)* $\int_1^{\sqrt{2}} \frac{dx}{x(\sqrt{x^2-1})}$ *(vi)* $\int_0^{\pi} \sin 5x\,dx$

(vii) $\int_0^{\pi/2} \cos^2 x\,dx$ **[CBSE 2002]** *(viii)* $\int_0^{\pi/4} \tan^2 x\,dx$

(ix) $\int_0^{\pi/4} \sin 2x \sin 3x\,dx$ **[CBSE 2006]**

SOLUTION (i) $\int_2^4 \frac{dx}{x} = [\log x]_2^4 = (\log 4 - \log 2) = (2\log 2 - \log 2) = \log 2.$

(ii) $\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = \frac{2}{3}\cdot[(9)^{3/2} - (4)^{3/2}] = \frac{38}{3}\cdot$

(iii) $\int_0^2 \sqrt{6x+4}\, dx = \left[\frac{2}{3} \cdot \frac{(6x+4)^{3/2}}{6}\right]_0^2 = \frac{1}{9} \cdot [(16)^{3/2} - (4)^{3/2}] = \frac{56}{9} \cdot$

(iv) $\int_0^1 \frac{1}{\sqrt{5x+3}}\, dx = \int_0^1 (5x+3)^{-1/2} dx = \left[2 \cdot \frac{(5x+3)^{1/2}}{5}\right]_0^1$

$= \frac{2}{5}(\sqrt{8} - \sqrt{3}).$

(v) $\int_1^{\sqrt{2}} \frac{dx}{x(\sqrt{x^2-1})} = [\sec^{-1} x]_1^{\sqrt{2}} = [\sec^{-1}(\sqrt{2}) - \sec^{-1}(1)] = \left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4} \cdot$

(vi) $\int_0^{\pi} \sin 5x\, dx = \left[\frac{-\cos 5x}{5}\right]_0^{\pi} = -\frac{1}{5}[\cos 5\pi - \cos 0] = \frac{2}{5} \cdot$

(vii) $\int_0^{\pi/2} \cos^2 x\, dx = \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2x)\, dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{4} \cdot$

(viii) $\int_0^{\pi/4} \tan^2 x\, dx = \int_0^{\pi/4} (\sec^2 x - 1)\, dx = [\tan x - x]_0^{\pi/4} = \left(1 - \frac{\pi}{4}\right) \cdot$

(ix) $\int_0^{\pi/4} \sin 2x \sin 3x\, dx = \frac{1}{2} \int_0^{\pi/4} (2 \sin 2x \sin 3x)\, dx$

$= \frac{1}{2} \int_0^{\pi/4} (\cos x - \cos 5x)\, dx$

$= \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right]_0^{\pi/4}$

$= \frac{1}{2}\left[\left(\sin \frac{\pi}{4} - \frac{\sin(5\pi/4)}{5}\right)\right] = \frac{3\sqrt{2}}{10} \cdot$

EXAMPLE 2 *Evaluate:* *(i)* $\int_0^{\pi/4} \sqrt{1 + \sin 2x}\, dx$ *(ii)* $\int_0^{\pi/2} \sqrt{1 + \cos 2x}\, dx$

SOLUTION (i) $\int_0^{\pi/4} \sqrt{1 + \sin 2x}\, dx = \int_0^{\pi/4} \sqrt{\cos^2 x + \sin^2 x + 2 \sin x \cos x\, dx}$

$= \int_0^{\pi/4} (\cos x + \sin x)\, dx = [\sin x - \cos x]_0^{\pi/4} = 1.$

(ii) $\int_0^{\pi/2} \sqrt{1 + \cos 2x}\, dx = \int_0^{\pi/2} \sqrt{2 \cos^2 x}\, dx$

$= \sqrt{2} \int_0^{\pi/2} \cos x\, dx = \sqrt{2}[\sin x]_0^{\pi/2} = \sqrt{2}.$

EXAMPLE 3 *Evaluate:* (i) $\int_0^{\pi/2} \cos^3 x\,dx$ (ii) $\int_0^{\pi/2} \sin^4 x\,dx$

SOLUTION (i) $$\int_0^{\pi/2} \cos^3 x\,dx = \int_0^{\pi/2} \left(\frac{3\cos x + \cos 3x}{4}\right) dx$$

$$[\because \cos 3x = 4\cos^3 x - 3\cos x]$$

$$= \frac{3}{4} \cdot \int_0^{\pi/2} \cos x\,dx + \frac{1}{4} \cdot \int_0^{\pi/2} \cos 3x\,dx$$

$$= \frac{3}{4} \cdot [\sin x]_0^{\pi/2} + \frac{1}{4} \cdot \left[\frac{\sin 3x}{3}\right]_0^{\pi/2}$$

$$= \left(\frac{3}{4} - \frac{1}{12}\right) = \frac{8}{12} = \frac{2}{3}.$$

(ii) $$\int_0^{\pi/2} \sin^4 x\,dx = \frac{1}{4}\int_0^{\pi/2} (2\sin^2 x)^2\,dx$$

$$= \frac{1}{4} \cdot \int_0^{\pi/2} (1 - \cos 2x)^2\,dx$$

$$= \frac{1}{4} \cdot \int_0^{\pi/2} (1 - 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4} \cdot \int_0^{\pi/2} \left[1 - 2\cos 2x + \frac{(1 + \cos 4x)}{2}\right] dx$$

$$= \frac{1}{4} \cdot \int_0^{\pi/2} \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{3}{8} \cdot \int_0^{\pi/2} dx - \frac{1}{2}\int_0^{\pi/2} \cos 2x\,dx + \frac{1}{8}\int_0^{\pi/2} \cos 4x\,dx$$

$$= \frac{3}{8} \cdot [x]_0^{\pi/2} - \frac{1}{2} \cdot \left[\frac{\sin 2x}{2}\right]_0^{\pi/2} + \frac{1}{8} \cdot \left[\frac{\sin 4x}{4}\right]_0^{\pi/2}$$

$$= \left(\frac{3\pi}{16} - 0 + 0\right) = \frac{3\pi}{16}.$$

EXAMPLE 4 *Evaluate:* (i) $\int_0^4 \frac{dx}{\sqrt{x^2 + 2x + 3}}$ (ii) $\int_0^1 \frac{dx}{(1 + x + x^2)}$

SOLUTION (i) $$\int_0^4 \frac{dx}{\sqrt{x^2 + 2x + 3}} = \int_0^4 \frac{dx}{\sqrt{(x+1)^2 + (\sqrt{2})^2}}$$

$$= \left[\log \left| (x+1) + \sqrt{x^2 + 2x + 3} \right| \right]_0^4$$

$$= \{\log |5 + \sqrt{27}| - \log |1 + \sqrt{3}|\}.$$

(ii) $$\int_0^1 \frac{dx}{(1+x+x^2)} = \int_0^1 \frac{dx}{\left[\left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4}\right]} = \int_0^1 \frac{dx}{\left[\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right]}$$

$$= \left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{\left(x + \frac{1}{2}\right)}{(\sqrt{3}/2)} \right]_0^1 = \frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \right]_0^1$$

$$= \frac{2}{\sqrt{3}} \left[\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \right]$$

$$= \frac{2}{\sqrt{3}} \cdot \left[\frac{\pi}{3} - \frac{\pi}{6} \right] = \frac{\pi}{3\sqrt{3}}.$$

EXAMPLE 5 *Evaluate:*

(i) $\displaystyle\int_0^a \frac{dx}{\sqrt{ax - x^2}}$ *(ii)* $\displaystyle\int_0^{\sqrt{2}} \sqrt{2 - x^2}\, dx$

SOLUTION (i) $$\int_0^a \frac{dx}{\sqrt{ax - x^2}} = \int_0^a \frac{dx}{\sqrt{-\left(x^2 - ax + \frac{a^2}{4}\right) + \frac{a^2}{4}}}$$

$$= \int_0^a \frac{dx}{\sqrt{\left(\frac{a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2}}$$

$$= \left[\sin^{-1} \frac{\left(x - \frac{a}{2}\right)}{\left(\frac{a}{2}\right)} \right]_0^a = \left[\sin^{-1}\left(\frac{2x - a}{a}\right) \right]_0^a$$

$$= [\sin^{-1}(1) - \sin^{-1}(-1)]$$

$$= 2\sin^{-1}(1) = \left(2 \times \frac{\pi}{2}\right) = \pi.$$

(ii) $$\int_0^{\sqrt{2}} \sqrt{2 - x^2}\, dx = \int_0^{\sqrt{2}} \sqrt{(\sqrt{2})^2 - x^2}\, dx$$

$$= \left[\frac{x}{2}\sqrt{2 - x^2} + \frac{(\sqrt{2})^2}{2} \cdot \sin^{-1} \frac{x}{\sqrt{2}} \right]_0^{\sqrt{2}}$$

$$= [0 + \sin^{-1}(1)] - [0 + \sin^{-1} 0] = \frac{\pi}{2}.$$

EXAMPLE 6 *Evaluate:*

(i) $\int_0^{\pi/2} x \cos x \, dx$ (ii) $\int_0^{\pi} \cos 2x \log \sin x \, dx$

(iii) $\int_1^2 \frac{\log x}{x^2} \, dx$ (iv) $\int_0^{\pi/6} (2 + 3x^2) \cos 3x \, dx$

SOLUTION (i) Integrating by parts, we get

$$\int_0^{\pi/2} x \cos x \, dx = [x \sin x]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \sin x \, dx$$

$$= \frac{\pi}{2} + [\cos x]_0^{\pi/2} = \left(\frac{\pi}{2} - 1\right).$$

(ii) Integrating by parts, taking log (sin x) as the first function, we get

$$\int_0^{\pi} \cos 2x \log \sin x \, dx$$

$$= \left[(\log \sin x) \cdot \frac{\sin 2x}{2}\right]_0^{\pi} - \int_0^{\pi} \left(\cot x \cdot \frac{\sin 2x}{2}\right) dx$$

$$= 0 - \int_0^{\pi} \frac{\cos x}{\sin x} \cdot \frac{2 \sin x \cos x}{2} \, dx = -\int_0^{\pi} \cos^2 x \, dx$$

$$= -\frac{1}{2}\int_0^{\pi} 2\cos^2 x \, dx = -\frac{1}{2}\int_0^{\pi} (1 + \cos 2x) \, dx$$

$$= -\frac{1}{2} \cdot \left[x + \frac{\sin 2x}{2}\right]_0^{\pi} = -\frac{\pi}{2}.$$

(iii) Integrating by parts, taking (log x) as the first function, we get

$$\int_1^2 \frac{\log x}{x^2} \, dx = \int_1^2 (\log x) \cdot x^{-2} \, dx$$

$$= \left[(\log x)\left(-\frac{1}{x}\right)\right]_1^2 - \int_1^2 \frac{1}{x} \cdot \left(-\frac{1}{x}\right) dx$$

$$= \left[-\frac{\log 2}{2} + \frac{\log 1}{1}\right] + \int_1^2 \frac{dx}{x^2}$$

$$= \frac{-\log 2}{2} - \left[\frac{1}{x}\right]_1^2 = \frac{-\log 2}{2} - \left\{\frac{1}{2} - 1\right\} = \left(\frac{1 - \log 2}{2}\right).$$

(iv) $\int_0^{\pi/6} (2+3x^2)\cos 3x\, dx$

$$= 2\int_0^{\pi/6} \cos 3x\, dx + 3\int_0^{\pi/6} x^2 \cos 3x\, dx$$

$$= 2\left[\frac{\sin 3x}{3}\right]_0^{\pi/6} + 3\left\{\left[x^2\left(\frac{\sin 3x}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6} 2x\left(\frac{\sin 3x}{3}\right)dx\right\}$$

[integrating by parts]

$$= \frac{2}{3} + \frac{\pi^2}{36} - 2\int_0^{\pi/6} x \sin 3x\, dx$$

$$= \frac{2}{3} + \frac{\pi^2}{36} - 2\left\{\left[x\left(\frac{-\cos 3x}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6} 1\cdot\left(\frac{-\cos 3x}{3}\right)dx\right\}$$

[integrating by parts]

$$= \frac{2}{3} + \frac{\pi^2}{36} + \frac{2}{3}[x\cos 3x]_0^{\pi/6} - \frac{2}{3}\cdot\left[\frac{\sin 3x}{3}\right]_0^{\pi/6}$$

$$= \frac{2}{3} + \frac{\pi^2}{36} - \frac{2}{9} = \left(\frac{\pi^2}{36} + \frac{4}{9}\right) = \frac{1}{36}(\pi^2 + 16).$$

EXAMPLE 7 *Evaluate* $\int_1^2 \frac{dx}{x(1+x^2)}\cdot$

SOLUTION Let $\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{(1+x^2)}\cdot$

Then, $1 \equiv A(1+x^2) + (Bx+C)x.$ Putting $x = 0$, we get $A = 1$.

Comparing the coefficients of x^2, we get $A + B = 0$ or $B = -1$.

Comparing coefficients of x, we get $C = 0$.

$$\therefore \quad \frac{1}{x(1+x^2)} = \left[\frac{1}{x} - \frac{x}{1+x^2}\right]$$

So, $$\int_1^2 \frac{dx}{x(1+x^2)} = \int_1^2 \frac{dx}{x} - \frac{1}{2}\int_1^2 \frac{2x}{1+x^2}\, dx$$

$$= [\log x]_1^2 - \frac{1}{2}[\log(1+x^2)]_1^2$$

$$= \left[\frac{3}{2}(\log 2) - \frac{1}{2}(\log 5)\right]\cdot$$

EXAMPLE 8 *Evaluate* $\int_2^4 \frac{(x^2+x)}{\sqrt{2x+1}}\,dx.$

SOLUTION Integrating by parts, taking (x^2+x) as the first function and $\frac{1}{\sqrt{2x+1}}$ as the second function, we get

$$\int_2^4 \frac{(x^2+x)}{\sqrt{2x+1}}\,dx = [(x^2+x)\cdot\sqrt{2x+1}]_2^4 - \int_2^4 (2x+1)\cdot\sqrt{2x+1}\,dx$$

$$= (60-6\sqrt{5}) - \int_2^4 (2x+1)^{3/2}\,dx$$

$$= (60-6\sqrt{5}) - \frac{1}{5}\cdot[(2x+1)^{5/2}]_2^4$$

$$= (60-6\sqrt{5}) - \left(\frac{243}{5} - 5\sqrt{5}\right)$$

$$= \left(\frac{57}{5} - \sqrt{5}\right) = \sqrt{\frac{57-5\sqrt{5}}{5}}.$$

EXAMPLE 9 *Evaluate:* (i) $\int_0^{1/2} \frac{dx}{\sqrt{1-x}}$ (ii) $\int_0^1 \left(\frac{1-x}{1+x}\right)dx$

SOLUTION (i) $\int_0^{1/2} \frac{dx}{\sqrt{1-x}} = \int_0^{1/2} (1-x)^{-1/2}\,dx = \left[\frac{2\sqrt{1-x}}{-1}\right]_0^{1/2} = (2-\sqrt{2}).$

(ii) $\int_0^1 \left(\frac{1-x}{1+x}\right)dx = \int_0^1 \left(-1+\frac{2}{x+1}\right)dx$ [on dividing $(-x+1)$ by $(x+1)$]

$$= [-x + 2\log|x+1|]_0^1 = [(2\log 2) - 1].$$

EXERCISE 16A

Very-Short-Answer Questions

Evaluate:

1. $\int_1^3 x^4 dx$
2. $\int_1^4 \sqrt{x}\,dx$
3. $\int_1^2 x^{-5} dx$
4. $\int_0^{16} x^{3/4} dx$
5. $\int_{-4}^{-1} \frac{dx}{x}$
6. $\int_1^4 \frac{dx}{\sqrt{x}}$

7. $\int_0^1 \frac{dx}{3\sqrt{x}}$

8. $\int_1^8 \frac{dx}{x^{2/3}}$

9. $\int_2^4 3\,dx$

10. $\int_0^1 \frac{dx}{(1+x^2)}$ **[CBSE 2008]**

11. $\int_0^\infty \frac{dx}{(1+x^2)}$

12. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

13. $\int_0^{\pi/6} \sec^2 x\,dx$

14. $\int_{-\pi/4}^{\pi/4} \operatorname{cosec}^2 x\,dx$

15. $\int_{\pi/4}^{\pi/2} \cot^2 x\,dx$

16. $\int_0^{\pi/4} \tan^2 x\,dx$

17. $\int_0^{\pi/2} \sin^2 x\,dx$

18. $\int_0^{\pi/4} \cos^2 x\,dx$

19. $\int_0^{\pi/3} \tan x\,dx$

20. $\int_{\pi/6}^{\pi/4} \operatorname{cosec} x\,dx$

21. $\int_0^{\pi/3} \cos^3 x\,dx$

22. $\int_0^{\pi/2} \sin^3 x\,dx$

23. $\int_{\pi/4}^{\pi/2} \frac{(1-3\cos x)}{\sin^2 x}\,dx$

24. $\int_0^{\pi/4} \sqrt{1+\cos 2x}\,dx$

25. $\int_0^{\pi/4} \sqrt{1-\sin 2x}\,dx$ **[CBSE 2004]**

26. $\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+\sin x)}$

27. $\int_0^{\pi/4} \frac{dx}{(1+\cos 2x)}$

28. $\int_{\pi/4}^{\pi/2} \frac{dx}{1-\cos 2x}$

29. $\int_0^{\pi/4} \sin 2x \sin 3x\,dx$

30. $\int_0^{\pi/6} \cos x \cos 2x\,dx$

31. $\int_0^{\pi} \sin 2x \cos 3x\,dx$

32. $\int_0^{\pi/2} \sqrt{1+\sin x}\,dx$

33. $\int_0^{\pi/2} \sqrt{1+\cos x}\,dx$

34. $\int_0^2 \frac{(x^4+1)}{(x^2+1)}\,dx$

35. $\int_1^2 \frac{dx}{(x+1)(x+2)}$

36. $\int_1^2 \frac{(x+3)}{x(x+2)}\,dx$

37. $\int_3^4 \frac{dx}{(x^2-4)}$

38. $\int_0^4 \frac{dx}{\sqrt{x^2+2x+3}}$

39. $\int_1^2 \frac{dx}{\sqrt{x^2+4x+3}}$

40. $\int_0^1 \frac{dx}{(1+x+2x^2)}$

Short-Answer Questions

Evaluate:

41. $\int_0^{\pi/2} (a\cos^2 x + b\sin^2 x)\, dx$

42. $\int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2\, dx$

43. $\int_0^{\pi/2} \cos^4 x\, dx$

44. $\int_0^a \frac{dx}{(ax+a^2-x^2)}$

45. $\int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}}$

46. $\int_0^1 \sqrt{x(1-x)}\, dx$

47. $\int_1^3 \frac{dx}{x^2(x+1)}$

48. $\int_1^2 \frac{dx}{x(1+2x)^2}$

49. $\int_0^1 x e^x\, dx$

50. $\int_0^{\pi/2} x^2 \cos x\, dx$

51. $\int_0^{\pi/4} x^2 \sin x\, dx$

52. $\int_0^{\pi/2} x^2 \cos 2x\, dx$ **[CBSE 2006C]**

53. $\int_0^{\pi/2} x^3 \sin 3x\, dx$

54. $\int_0^{\pi/2} x^2 \cos^2 x\, dx$

55. $\int_1^2 \log x\, dx$

56. $\int_1^3 \frac{\log x}{(1+x)^2}\, dx$

57. $\int_0^{e^2} \left\{ \frac{1}{(\log x)} - \frac{1}{(\log x)^2} \right\} dx$

58. $\int_1^e e^x \left(\frac{1+x\log x}{x} \right) dx$

59. $\int_0^1 \frac{x e^x}{(1+x)^2}\, dx$

60. $\int_0^{\pi/2} 2\tan^3 x\, dx$ **[CBSE 2004]**

61. $\int_1^2 \frac{5x^2}{(x^2+4x+3)}\, dx$

ANSWERS (EXERCISE 16A)

1. $\frac{242}{5}$ **2.** $\frac{14}{3}$ **3.** $\frac{15}{64}$ **4.** $\frac{512}{7}$

5. $-\log 4$ **6.** 2 **7.** $\frac{3}{2}$ **8.** 3

9. 6 **10.** $\frac{\pi}{4}$ **11.** $\frac{\pi}{2}$ **12.** $\frac{\pi}{2}$

13. $\frac{1}{\sqrt{3}}$ **14.** -2 **15.** $\left(1-\frac{\pi}{4}\right)$ **16.** $\left(1-\frac{\pi}{4}\right)$

17. $\frac{\pi}{4}$ **18.** $\left(\frac{\pi}{8}+\frac{1}{4}\right)$ **19.** $\log 2$

20. $\log(\sqrt{2}-1)+\log(2+\sqrt{3})$ **21.** $\frac{3\sqrt{3}}{8}$ **22.** $\frac{2}{3}$

23. $(4-3\sqrt{2})$ **24.** 1 **25.** $(\sqrt{2}-1)$ **26.** 2

27. $\frac{1}{2}$ **28.** $\frac{1}{2}$ **29.** $\frac{3}{5\sqrt{2}}$ **30.** $\frac{5}{12}$

31. $\frac{-4}{5}$ **32.** 2 **33.** 2 **34.** $\left(\frac{2}{3}+2\tan^{-1}2\right)$

35. $(2\log 3-3\log 2)$ **36.** $\frac{1}{2}(\log 2+\log 3)$

37. $\frac{1}{4}(\log 5-\log 3)$ **38.** $\log\left(\frac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$

39. $\log(4+\sqrt{15})-\log(3+\sqrt{8})$ **40.** $\frac{2}{\sqrt{7}}\left\{\tan^{-1}\frac{5}{\sqrt{7}}-\tan^{-1}\frac{1}{\sqrt{7}}\right\}$

41. $\frac{\pi}{4}(a+b)$ **42.** $\frac{-2}{\sqrt{3}}$ **43.** $\frac{3\pi}{16}$ **44.** $\frac{1}{\sqrt{5}a}\log\left\{\frac{7+3\sqrt{5}}{2}\right\}$

45. $\frac{\pi}{6}$ **46.** $\frac{\pi}{8}$ **47.** $\log 2-\log 3+\frac{2}{3}$

48. $\log 6-\log 5-\frac{2}{15}$ **49.** 1 **50.** $\left(\frac{\pi^2}{4}-2\right)$

51. $\left(\sqrt{2}+\frac{\pi}{2\sqrt{2}}-\frac{\pi^2}{16\sqrt{2}}-2\right)$ **52.** $\frac{-\pi}{4}$ **53.** $\left(\frac{2}{27}-\frac{\pi^2}{12}\right)$

54. $\left(\frac{\pi^3}{48}-\frac{\pi}{8}\right)$ **55.** $(2\log 2-1)$ **56.** $\frac{3}{4}\log 3-\log 2$ **57.** $\left(\frac{e^2}{2}-e\right)$

58. e^e **59.** $\left(\frac{e}{2}-1\right)$ **60.** $(1-\log 2)$

61. $5-\frac{5}{2}\left(9\log\frac{5}{4}-\log\frac{3}{2}\right)$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 16A)

15. $\cot^2 x = (\text{cosec}^2 x - 1)$.

16. $\tan^2 x = (\sec^2 x - 1)$.

17. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

18. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

19. $\int \tan x \, dx = \log \sec x$.

20. $\int \text{cosec}\, x \, dx = \log |\text{cosec}\, x - \cot x|$.

21. $\cos 3x = 4\cos^3 x - 3\cos x \Rightarrow \cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$.

22. $\sin 3x = 3\sin x - 4\sin^3 x \Rightarrow \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$.

23. $I = \int \left(\frac{1}{\sin^2 x} - \frac{3\cos x}{\sin^2 x} \right) dx = \int (\text{cosec}^2 x - 3\text{cosec}\, x \cot x)\, dx$.

25. $\sqrt{1 - \sin 2x} = (\cos^2 x + \sin^2 x - 2\sin x \cos x)^{1/2} = (\cos x - \sin x)$

$$\left\{ \because \cos x > \sin x \text{ in } \left[0, \frac{\pi}{4}\right] \right\}.$$

26. $I = \int \frac{1}{(1 + \sin x)} \times \frac{(1 - \sin x)}{(1 - \sin x)}\, dx = \int \frac{(1 - \sin x)}{\cos^2 x}\, dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx$
$= \int (\sec^2 x - \sec x \tan x)\, dx$.

32. $\sqrt{1 + \sin x} = \left\{ \cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \right\}^{1/2} = \left(\cos\frac{x}{2} + \sin\frac{x}{2} \right)$.

34. $\frac{(x^4 + 1)}{(x^2 + 1)} = \left\{ x^2 - 1 + \frac{2}{(x^2 + 1)} \right\}$.

41. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ and $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

42. $I = \int (\tan^2 x + \cot^2 x + 2)\, dx = \int \{(\sec^2 x - 1) + (\text{cosec}^2 x - 1) + 2\}\, dx$
$= \int (\sec^2 x + \text{cosec}^2 x)\, dx$.

45. $I = \int \frac{dx}{\sqrt{\left\{ \frac{1}{4} - \left(x^2 - x + \frac{1}{4} \right) \right\}}} = \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}} = \sin^{-1}(2x - 1)$.

46. $I = \int \sqrt{\left\{ \left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2 \right\}}\, dx$.

47. Let $\frac{1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1}$. Then $Ax(x + 1) + B(x + 1) + Cx^2 \equiv 1$.
This, gives $A = -1, B = 1, C = 1$.

54. $I = \int \underset{\text{I}}{\log x} \cdot \underset{\text{II}}{(1 + x)^{-2}}\, dx$.

55. $I = \int \frac{dx}{(\log x)} - \int \frac{dx}{(\log x)^2} = I_1 - I_2$.
$I_1 = \int \{(\log x)^{-1} \cdot 1\}\, dx$. Integrate by parts.

58. $I = \int e^x \left(\log x + \frac{1}{x} \right) dx.$

59. $I = \int e^x \left\{ \frac{1}{(1+x)} - \frac{1}{(1+x)^2} \right\} dx = e^x \cdot \frac{1}{(1+x)}.$

60. $I = 2\int \tan x \tan^2 x \, dx = 2\int \tan x (\sec^2 x - 1) \, dx.$

61. Let $\frac{5x^2}{(x^2+4x+3)} = 5 + \frac{A}{(x+1)} + \frac{B}{(x+3)}.$

To Evaluate a Definite Integral by Substitution

In $\int_a^b f(x)\,dx$, when the variable x is converted into a new variable t by some relation then we put $x = a$ and $x = b$ in that relation to obtain the corresponding values of t, giving the lower limit and the upper limit respectively of the new integrand in t.

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\int_0^2 e^{x/2} dx$ (ii) $\int_2^4 \frac{x}{(x^2+1)} dx$

(iii) $\int_0^1 \cos^{-1} x \, dx$ (iv) $\int_0^1 \frac{(2x+3)}{(5x^2+1)} dx$

SOLUTION (i) Put $\frac{x}{2} = t$ so that $dx = 2\,dt$.

Also, $(x = 0 \Rightarrow t = 0)$ and $(x = 2 \Rightarrow t = 1)$.

$$\therefore \int_0^2 e^{x/2} dx = 2\int_0^1 e^t dt = 2[e^t]_0^1 = 2(e-1).$$

(ii) Put $(x^2+1) = t$ so that $x\,dx = \frac{1}{2}dt$.

Also, $(x = 2 \Rightarrow t = 5)$ and $(x = 4 \Rightarrow t = 17)$.

$$\therefore \int_2^4 \frac{x}{(x^2+1)} dx = \frac{1}{2}\int_5^{17} \frac{dt}{t} = \frac{1}{2}[\log|t|]_5^{17} = \frac{1}{2}(\log 17 - \log 5).$$

(iii) Put $x = \cos t$ so that $dx = -\sin t\,dt$.

Also, $\left(x = 0 \Rightarrow t = \frac{\pi}{2}\right)$ and $(x = 1 \Rightarrow t = 0)$.

$$\therefore \int_0^1 \cos^{-1} x\, dx = -\int_{\pi/2}^0 \cos^{-1}(\cos t) \sin t\, dt = \int_0^{\pi/2} t \sin t\, dt$$

$$= [t(-\cos t)]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot (-\cos t)\, dt$$

[integrating by parts]

$$= [\sin t]_0^{\pi/2} = 1.$$

(iv) Let $(2x + 3) \equiv A \cdot \frac{d}{dx}(5x^2 + 1) + B.$

Then, $(2x + 3) \equiv (10x)A + B.$

Comparing the coefficients of like powers of x, we get

$$10A = 2 \text{ or } A = \frac{1}{5} \text{ and } B = 3.$$

$$\therefore \quad (2x + 3) = \frac{1}{5}(10x) + 3.$$

So, $$\int_0^1 \frac{(2x+3)}{(5x^2+1)}\, dx = \int_0^1 \frac{\frac{1}{5}(10x) + 3}{(5x^2+1)}\, dx$$

$$= \frac{1}{5}\int_0^1 \frac{10x}{(5x^2+1)}\, dx + 3\int_0^1 \frac{dx}{(5x^2+1)}$$

$$= \frac{1}{5}[\log|5x^2+1|]_0^1 + \frac{3}{5}\int_0^1 \frac{dx}{x^2 + \left(\frac{1}{\sqrt{5}}\right)^2}$$

$$= \frac{1}{5}\log 6 + \frac{3}{5} \cdot \sqrt{5}\left[\tan^{-1} \frac{x}{(1/\sqrt{5})}\right]_0^1$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}(\tan^{-1}\sqrt{5}).$$

EXAMPLE 2 *Evaluate:*

(i) $\int_1^3 \frac{\cos(\log x)}{x}\, dx$ (ii) $\int_0^{\pi/2} \sqrt{\cos\theta}\, \sin^3\theta\, d\theta$

(iii) $\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)}\, dx$ **[CBSE 2004]** (iv) $\int_0^{\pi/2} \frac{dx}{(1-2\sin x)}$

(v) $\int_0^{\pi/2} \frac{dx}{(3+2\cos x)}$ (vi) $\int_0^{\pi/2} \frac{dx}{(4\sin^2 x + 5\cos^2 x)}$

SOLUTION (i) Put $\log x = t$ so that $\frac{1}{x}\, dx = dt.$

Also, $(x = 1 \Rightarrow t = \log 1 = 0)$ and $(x = 3 \Rightarrow t = \log 3).$

$$\therefore \int_1^3 \frac{\cos(\log x)}{x}\,dx = \int_0^{\log 3} \cos t\,dt = [\sin t]_0^{\log 3} = \sin(\log 3).$$

(ii) Put $\cos\theta = t$ so that $\sin\theta\,d\theta = -dt$.

Also, $(\theta = 0 \Rightarrow t = 1)$ and $\left(\theta = \frac{\pi}{2} \Rightarrow t = 0\right)$.

$$\therefore \int_0^{\pi/2} \sqrt{\cos\theta}\sin^3\theta d\theta = \int_0^{\pi/2} \sqrt{\cos\theta}\cdot(1-\cos^2\theta)\sin\theta d\theta$$

$$= -\int_1^0 \sqrt{t}(1-t^2)\,dt = \int_0^1 (t^{1/2}-t^{5/2})\,dt$$

$$= \left[\frac{2}{3}t^{3/2} - \frac{2}{7}t^{7/2}\right]_0^1 = \left(\frac{2}{3}-\frac{2}{7}\right) = \frac{8}{21}.$$

(iii) Put $\sin x = t$ so that $\cos x\,dx = dt$.

Also, $(x = 0 \Rightarrow t = 0)$ and $\left(x = \frac{\pi}{2} \Rightarrow t = 1\right)$.

$$\therefore \int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)}\,dx$$

$$= \int_0^1 \frac{dt}{(1+t)(2+t)}$$

$$= \int_0^1 \left[\frac{1}{(1+t)} - \frac{1}{(2+t)}\right] dt \qquad \text{[by partial fractions]}$$

$$= \int_0^1 \frac{dt}{(1+t)} - \int_0^1 \frac{dt}{(2+t)}$$

$$= [\log|1+t|]_0^1 - [\log|2+t|]_0^1$$

$$= [(\log 2 - \log 1) - (\log 3 - \log 2)] = (2\log 2) - (\log 3).$$

(iv) $$\int_0^{\pi/2} \frac{dx}{(1-2\sin x)} = \int_0^{\pi/2} \frac{dx}{1-2\left\{\frac{2\tan(x/2)}{1+\tan^2(x/2)}\right\}}$$

$$= \int_0^{\pi/2} \frac{\sec^2(x/2)}{[1+\tan^2(x/2) - 4\tan(x/2)]}\,dx$$

$$= 2\int_0^1 \frac{dt}{(1+t^2-4t)}, \text{ where } \tan\frac{x}{2} = t$$

$$\begin{bmatrix} x = 0 \Rightarrow t = 0 \\ x = \frac{\pi}{2} \Rightarrow t = 1 \end{bmatrix}$$

$$= 2\int_0^1 \frac{dt}{(t-2)^2 - (\sqrt{3})^2} = 2\cdot\frac{1}{2\sqrt{3}}\left[\log\left|\frac{t-2-\sqrt{3}}{t-2+\sqrt{3}}\right|\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\left[\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \log\frac{\sqrt{3}+2}{\sqrt{3}-2}\right].$$

(v) $$\int_0^{\pi/2} \frac{dx}{(3+2\cos x)} = \int_0^{\pi/2} \frac{dx}{3+2\cdot\left[\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right]}$$

$$= \int_0^{\pi/2} \frac{\sec^2(x/2)}{\tan^2(x/2)+5}\,dx$$

$$= 2\int_0^1 \frac{dt}{t^2+(\sqrt{5})^2}, \text{where } \tan\frac{x}{2} = t$$

$$\left[\begin{array}{l} x=0 \Rightarrow t=0 \\ x=\frac{\pi}{2} \Rightarrow t=1 \end{array}\right]$$

$$= 2\cdot\frac{1}{\sqrt{5}}\left[\tan^{-1}\frac{t}{\sqrt{5}}\right]_0^1 = \frac{2}{\sqrt{5}}\tan^{-1}\frac{1}{\sqrt{5}}.$$

(vi) $$\int_0^{\pi/2} \frac{dx}{(4\sin^2 x+5\cos^2 x)} = \int_0^{\pi/2} \frac{\sec^2 x}{(4\tan^2 x+5)}\,dx$$

[dividing num. and denom. by $\cos^2 x$]

$$= \int_0^{\infty} \frac{dt}{(4t^2+5)}, \text{ where } \tan x = t$$

$$\left[\begin{array}{l} x=0 \Rightarrow t=0 \\ x=\frac{\pi}{2} \Rightarrow t=\infty \end{array}\right]$$

$$= \frac{1}{4}\int_0^{\infty} \frac{dt}{t^2+\left(\frac{\sqrt{5}}{2}\right)^2} = \frac{1}{4}\cdot\frac{2}{\sqrt{5}}\left[\tan^{-1}\frac{2t}{\sqrt{5}}\right]_0^{\infty}$$

$$= \frac{1}{2\sqrt{5}}[\tan^{-1}(\infty) - \tan^{-1}(0)]$$

$$= \frac{1}{2\sqrt{5}}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4\sqrt{5}}.$$

EXAMPLE 3 *Evaluate:*

(i) $\int_0^1 \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$ *(ii)* $\int_0^{1/\sqrt{2}} \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx$ **[CBSE 1996, '97]**

SOLUTION (i) Put $x = \tan\theta$ so that $dx = \sec^2\theta\, d\theta$.

Clearly, $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = (\pi/4)$.

$$\therefore \int_0^1 \frac{x \tan^{-1} x}{(1+x^2)^{3/2}}\, dx = \int_0^{\pi/4} \frac{\theta \tan \theta}{\sec^3 \theta} \cdot \sec^2 \theta d\theta = \int_0^{\pi/4} \theta \sin \theta d\theta$$

$$= [-\theta \cos \theta]_0^{\pi/4} - \int_0^{\pi/4} (-\cos \theta)\, d\theta \qquad \text{[integrating by parts]}$$

$$= [-\theta \cos \theta]_0^{\pi/4} + [\sin \theta]_0^{\pi/4} = -\frac{\pi}{4} \cos \frac{\pi}{4} + \sin \frac{\pi}{4}$$

$$= \left(\frac{-\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} \right) = \frac{(4-\pi)}{4\sqrt{2}} = \frac{\sqrt{2}(4-\pi)}{8}.$$

(ii) Put $x = \sin \theta$ so that $dx = \cos \theta \, d\theta$.

Clearly, $(x = 0 \Rightarrow \theta = 0)$ and $\left(x = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \right)$.

$$\therefore \int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx = \int_0^{\pi/4} \frac{\theta}{\cos^3 \theta} \cdot \cos \theta \, d\theta = \int_0^{\pi/4} \theta \sec^2 \theta \, d\theta$$

$$= [\theta \tan \theta]_0^{\pi/4} - \int_0^{\pi/4} 1 \cdot \tan \theta \, d\theta \quad \text{[integrating by parts]}$$

$$= \frac{\pi}{4} + [\log (\cos \theta)]_0^{\pi/4} = \frac{\pi}{4} + \log \left(\cos \frac{\pi}{4} \right)$$

$$= \frac{\pi}{4} + \log \left(\frac{1}{\sqrt{2}} \right) = \left(\frac{\pi}{4} - \frac{1}{2} \log 2 \right).$$

EXAMPLE 4 *Evaluate:* *(i)* $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^3}\, dx$ *(ii)* $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(1 + \cos x + \sin x)}\, dx$

SOLUTION (i) $$\int_0^{\pi/2} \frac{\cos x}{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^3} dx = \int_0^{\pi/2} \frac{\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right)}{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^3} dx$$

$$= \int_0^{\pi/2} \frac{2\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)}{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2} dx = 2 \cdot \int_1^{\sqrt{2}} \frac{dt}{t^2} = \left[\frac{-2}{t} \right]_1^{\sqrt{2}} = \sqrt{2}(\sqrt{2} - 1).$$

[putting $\cos \frac{x}{2} + \sin \frac{x}{2} = t$ and $\frac{1}{2}\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right) dx = dt$;

also, $x = 0 \Rightarrow t = 1$ and $x = (\pi/2) \Rightarrow t = \sqrt{2}$]

(ii) $$\int_0^{\pi/2} \frac{\cos x}{(1 + \cos x + \sin x)} dx = \int_0^{\pi/2} \frac{\cos x}{(1 + \cos x) + \sin x} dx$$

$$= \int_0^{\pi/2} \frac{\cos^2(x/2) - \sin^2(x/2)}{[2\cos^2(x/2) + 2\sin(x/2)\cos(x/2)]}\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1 - \tan^2(x/2)}{1 + \tan(x/2)}\,dx$$

[dividing num. and denom. by $\cos^2(x/2)$]

$$= \frac{1}{2}\int_0^{\pi/2} [1 - \tan(x/2)]\,dx = \frac{1}{2}\int_0^{\pi/2} dx - \frac{1}{2}\int_0^{\pi/2} \frac{\sin(x/2)}{\cos(x/2)}\,dx$$

$$= \frac{1}{2}\cdot[x]_0^{\pi/2} + [\log\cos(x/2)]_0^{\pi/2}$$

$$= \frac{\pi}{4} + \log\cos\frac{\pi}{4} = \frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{\pi}{4} - \frac{1}{2}\log 2\right).$$

EXAMPLE 5 *Evaluate* $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\,dx.$ **[CBSE 2008, '11C]**

SOLUTION Put $x = a\cos\theta$ so that $dx = -a\sin\theta\,d\theta$.

Also, $(x = -a \Rightarrow \theta = \pi)$ and $(x = a \Rightarrow \theta = 0)$.

$$\therefore \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\,dx = \int_{\pi}^{0} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\cdot(-a\sin\theta)\,d\theta$$

$$= a\int_0^{\pi} \sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}}\cdot 2\sin(\theta/2)\cos(\theta/2)\,d\theta$$

$$= a\int_0^{\pi} 2\sin^2(\theta/2)\,d\theta = a\int_0^{\pi} (1-\cos\theta)\,d\theta$$

$$= a\int_0^{\pi} d\theta - a\int_0^{\pi} \cos\theta\,d\theta$$

$$= a\cdot[\theta]_0^{\pi} - a[\sin\theta]_0^{\pi} = a\pi.$$

EXAMPLE 6 *Evaluate* $\int_0^1 x\cdot\sqrt{\frac{1-x^2}{1+x^2}}\,dx.$ **[CBSE 2007]**

SOLUTION Put $x^2 = t$ and $x\,dx = \frac{1}{2}\,dt$. Then,

$[x = 0 \Rightarrow t = 0]$ and $[x = 1 \Rightarrow t = 1]$.

$$\therefore \quad I = \frac{1}{2}\cdot\int_0^1 \sqrt{\frac{1-t}{1+t}}\,dt$$

$$= \frac{1}{2}\cdot\int_0^1 \left\{\frac{\sqrt{1-t}}{\sqrt{1+t}} \times \frac{\sqrt{1-t}}{\sqrt{1-t}}\right\} dt = \frac{1}{2}\cdot\int_0^1 \frac{(1-t)}{\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{2}\cdot\int_0^1 \frac{dt}{\sqrt{1-t^2}} - \frac{1}{2}\cdot\int_0^1 \frac{t}{\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{2}\cdot[\sin^{-1} t]_0^1 + \frac{1}{4}\cdot\int_0^1 \frac{(-2t)}{\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{2}\cdot[\sin^{-1}1 - \sin^{-1}0] + \frac{1}{4}\cdot\int_1^0 \frac{1}{\sqrt{u}}\,du, \text{ where } (1-t^2) = u$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) - \frac{1}{4}\cdot\int_0^1 \frac{du}{\sqrt{u}} = \frac{\pi}{4} - \frac{1}{4}[2\sqrt{u}]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}[\sqrt{1} - \sqrt{0}] = \left(\frac{\pi}{4} - \frac{1}{2}\right).$$

EXAMPLE 7 *Evaluate* $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(3\cos x + \sin x)}\,dx.$

SOLUTION Let $\cos x = A(3\cos x + \sin x) + B\cdot\frac{d}{dx}(3\cos x + \sin x)$

Then, $\cos x = A(3\cos x + \sin x) + B\cdot(-3\sin x + \cos x)$

Comparing the coefficients of $\cos x$, we get $3A + B = 1$.

Comparing the coefficients of $\sin x$, we get $A - 3B = 0$.

Solving $3A + B = 1$ and $A - 3B = 0$, we get $A = \frac{3}{10}$ and $B = \frac{1}{10}$.

$$\therefore \quad \cos x = \frac{3}{10}(3\cos x + \sin x) + \frac{1}{10}(-3\sin x + \cos x).$$

So, $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(3\cos x + \sin x)}\,dx$

$$= \frac{3}{10}\int_0^{\pi/2} \frac{(3\cos x + \sin x)}{(3\cos x + \sin x)}\,dx + \frac{1}{10}\int_0^{\pi/2} \frac{(-3\sin x + \cos x)}{(3\cos x + \sin x)}\,dx$$

$$= \frac{3}{10}\int_0^{\pi/2} dx + \frac{1}{10}\int_0^{\pi/2} \frac{(-3\sin x + \cos x)}{(3\cos x + \sin x)}\,dx$$

$$= \frac{3}{10}\cdot[x]_0^{\pi/2} + \frac{1}{10}\cdot[\log|3\cos x + \sin x|]_0^{\pi/2} = \left(\frac{3\pi}{20} - \frac{\log 3}{10}\right).$$

EXAMPLE 8 *Evaluate* $\int_0^{\pi/2}\{\sqrt{\tan x}+\sqrt{\cot x}\}\,dx.$ **[CBSE 2003, '12]**

SOLUTION We have

$$I=\int_0^{\pi/2}\left\{\frac{\sqrt{\sin x}}{\sqrt{\cos x}}+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right\}dx=\int_0^{\pi/2}\frac{(\sin x+\cos x)}{\sqrt{\sin x\cos x}}\,dx$$

$$=\sqrt{2}\cdot\int_0^{\pi/2}\frac{(\sin x+\cos x)}{\sqrt{2\sin x\cos x}}\,dx=\sqrt{2}\cdot\int_0^{\pi/2}\frac{(\sin x+\cos x)}{\sqrt{1-(\sin x-\cos x)^2}}\,dx$$

Put $(\sin x-\cos x)=t$ and $(\cos x+\sin x)\,dx=dt$.

Also, $[x=0\Rightarrow t=-1]$ and $\left[x=\frac{\pi}{2}\Rightarrow t=1\right]\cdot$

$$\therefore\quad I=\sqrt{2}\cdot\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}=\sqrt{2}[\sin^{-1}t]_{-1}^{1}$$

$$=\sqrt{2}\{\sin^{-1}(1)-\sin^{-1}(-1)\}=\sqrt{2}\{2\sin^{-1}(1)\}$$

$$=\left(\sqrt{2}\times2\times\frac{\pi}{2}\right)=\sqrt{2}\pi.$$

EXERCISE 16B

Very-Short-Answer Questions

Evaluate the following integrals:

1. $\int_0^1\frac{dx}{(2x-3)}$

2. $\int_0^1\frac{2x}{(1+x^2)}\,dx$ **[CBSE 2008]**

3. $\int_1^2\frac{3x}{(9x^2-1)}\,dx$

4. $\int_0^1\frac{\tan^{-1}x}{(1+x^2)}\,dx$

5. $\int_0^1\frac{e^x}{(1+e^{2x})}\,dx$ **[CBSE 1997]**

6. $\int_0^1\frac{2x}{(1+x^4)}\,dx$

7. $\int_0^1 x\,e^{x^2}\,dx$

8. $\int_1^2\frac{e^{1/x}}{x^2}\,dx$

9. $\int_0^{\pi/6}\frac{\cos x}{(3+4\sin x)}\,dx$

10. $\int_0^{\pi/2}\frac{\sin x}{(1+\cos^2 x)}\,dx$ **[CBSE 1997]**

11. $\int_0^1\frac{dx}{(e^x+e^{-x})}$

12. $\int_{1/e}^{e}\frac{dx}{x(\log x)^{1/3}}$

Short-Answer Questions

Evaluate the following integrals:

13. $\int_0^1 \frac{\sqrt{\tan^{-1} x}}{(1+x^2)}\,dx$

14. $\int_0^{\pi/2} \frac{\sin x}{\sqrt{1+\cos x}}\,dx$

15. $\int_0^{\pi/2} \sqrt{\sin x} \cdot \cos^5 x\,dx$

16. $\int_0^{\pi/2} \frac{\sin x \cos x}{(1+\sin^4 x)}\,dx$

17. $\int_0^a \sqrt{a^2-x^2}\,dx$

18. $\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx$

19. $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}\,dx$

20. $\int_0^a \frac{x}{\sqrt{a^2+x^2}}\,dx$

21. $\int_0^2 x\sqrt{2-x}\,dx$

22. $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$ **[CBSE 2002]**

23. $\int_0^{\pi/2} \sqrt{1+\cos x}\,dx$

24. $\int_0^{\pi/2} \sqrt{1+\sin x}\,dx$

25. $\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x+b^2\sin^2 x)}$ **[CBSE 2003C]**

26. $\int_0^{\pi/2} \frac{dx}{(1+\cos^2 x)}$

27. $\int_0^{\pi/2} \frac{dx}{(4+9\cos^2 x)}$

28. $\int_0^{\pi/2} \frac{dx}{(5+4\sin x)}$

29. $\int_0^{\pi} \frac{dx}{(6-\cos x)}$

30. $\int_0^{\pi} \frac{dx}{(5+4\cos x)}$ **[CBSE 2005]**

31. $\int_0^{\pi/2} \frac{dx}{(\cos x+2\sin x)}$

32. $\int_0^{\pi} \frac{dx}{(3+2\sin x+\cos x)}$ **[CBSE 2004]**

33. $\int_0^{\pi/4} \frac{\tan^3 x}{(1+\cos 2x)}\,dx$

34. $\int_0^{\pi/2} \frac{\sin x \cos x}{(\cos^2 x+3\cos x+2)}\,dx$

35. $\int_0^{\pi/2} \frac{\sin 2x}{(\sin^4 x+\cos^4 x)}\,dx$ **[CBSE 2003]**

36. $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\,dx$

37. $\int_0^1 (\cos^{-1} x)^2\,dx$

38. $\int_0^1 x(\tan^{-1} x)^2\,dx$

39. $\int_0^1 \sin^{-1}\sqrt{x}\,dx$

40. $\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$ **[CBSE 2008]**

41. $\int_0^9 \frac{dx}{(1+\sqrt{x})}$

42. $\int_0^1 x^3\sqrt{1+3x^4}\,dx$

43. $\int_0^1 \frac{(1-x^2)}{(1+x^2)^2}\,dx$

44. $\int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$

45. $\int_0^{\pi/2} (\sqrt{\tan x}+\sqrt{\cot x})\,dx$ **[CBSE 2002, '03, '05, '12]**

46. $\int_2^3 \frac{(2-x)}{\sqrt{5x-6-x^2}}\,dx$

47. $\int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^3}\,d\theta$

48. $\int_0^{(\pi/2)^{1/3}} x^2 \sin x^3 dx$

49. $\int_1^2 \frac{dx}{x(1+\log x)^2}$ **[CBSE 2003]**

50. $\int_{\pi/6}^{\pi/2} \frac{\operatorname{cosec} x \cot x}{1+\operatorname{cosec}^2 x}\,dx$

ANSWERS (EXERCISE 16B)

1. $-\frac{1}{2}\log 3$ **2.** $\log 2$ **3.** $\frac{1}{6}(\log 35-\log 8)$

4. $\frac{\pi^2}{32}$ **5.** $\left(\tan^{-1}e-\frac{\pi}{4}\right)$ **6.** $\frac{\pi}{4}$

7. $\left(\frac{e-1}{2}\right)$ **8.** $(e-\sqrt{e})$ **9.** $\frac{1}{4}(\log 5-\log 3)$

10. $\frac{\pi}{4}$ **11.** $\left(\tan^{-1}e-\frac{\pi}{4}\right)$ **12.** 0

13. $\frac{1}{12}\pi^{3/2}$ **14.** $2(\sqrt{2}-1)$ **15.** $\frac{64}{231}$ **16.** $\frac{\pi}{8}$

17. $\frac{\pi a^2}{4}$ **18.** $\frac{\pi}{2}$ **19.** $\frac{3\pi a^4}{16}$ **20.** $a(\sqrt{2}-1)$

21. $\frac{16\sqrt{2}}{15}$ **22.** $\left(\frac{\pi}{2}-\log 2\right)$ **23.** 2 **24.** 2

25. $\frac{\pi}{2ab}$ **26.** $\frac{\pi}{2\sqrt{2}}$ **27.** $\frac{\pi}{4\sqrt{13}}$ **28.** $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$

29. $\frac{\pi}{\sqrt{35}}$ **30.** $\frac{\pi}{3}$ **31.** $\frac{1}{\sqrt{5}}\log\left|\frac{3-\sqrt{5}}{2}\right|$

32. $\frac{\pi}{4}$ **33.** $\frac{1}{8}$ **34.** $(\log 9 - \log 8)$

35. $\frac{\pi}{2}$ **36.** $\frac{3}{2}$ **37.** $(\pi - 2)$ **38.** $\frac{\pi}{4}\left(\frac{\pi}{4}-1\right)$

39. $\frac{\pi}{4}$ **40.** $a\left(\frac{\pi}{2}-1\right)$ **41.** $(6-4\log 2)$ **42.** $\frac{7}{18}$

43. $\frac{1}{2}$ **44.** $\frac{1}{\sqrt{3}}$ **45.** $\pi\sqrt{2}$ **46.** $-\frac{\pi}{2}$

47. $\frac{2}{\left(\cos\frac{\pi}{8}+\sin\frac{\pi}{8}\right)}-\sqrt{2}$ **48.** $\frac{1}{3}$ **49.** $\frac{\log 2}{(1+\log 2)}$

50. $\tan^{-1}\frac{1}{3}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 16B)

5. Put $e^x = t$.

8. Putting $\frac{1}{x} = t$, we get $I = -\int_1^{1/2} e^t dt = \int_{1/2}^1 e^t dt = [e^t]_{1/2}^1$.

9. Putting $3 + 4\sin x = t$, we get $I = \frac{1}{4}\cdot\int_3^5 \frac{dt}{t}$.

10. Put $\cos x = t$. 11. $I = \int_0^1 \frac{e^x}{(1+e^{2x})}\,dx$.

15. $I = \int_0^{\pi/2} \sqrt{\sin x}\cdot(1-\sin^2 x)^2 \cos x\,dx$. Now, put $\sin x = t$.

16. Put $\sin^2 x = t$. 18. Put $x = \sqrt{2}\sin t$. 20. Put $x = a\tan\theta$.

21. Put $(x+2) = t^2$. 22. Put $x = \tan t$ and integrate by parts.

25. Divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

32. Write $\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)}$, $\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$ and put $\tan\frac{x}{2} = t$.

33. Using $\cos 2x = (2\cos^2 x - 1)$, we get $I = \int_0^{\pi/4} \tan^3 x \sec^2 x\,dx$.

 Now, put $\tan x = t$.

34. Put $\cos x = t$.

35. Divide num. and denom. by $\cos^4 x$ and put $\tan^2 x = t$.

36. $I = \int_{\pi/3}^{\pi/2} \frac{\cos(x/2)}{\sin^5(x/2)}\,dx$. Put $\sin\frac{x}{2} = t$.

37. Put $x = \cos t$.

38. Integrate by parts with x as the second function.

39. Put $x = \sin^2 t$. 40. Put $x = a\tan^2\theta$. 41. Put $x = t^2$.

42. Put $(1 + 3x^4) = t$. 43. Put $x = \tan t$. 44. Put $(x+1) = \frac{1}{t}\cdot$

45. $I = \int_0^{\pi/2} \frac{\sqrt{2}(\sin x + \cos x)}{\sqrt{2\sin x\cos x}}\,dx = \sqrt{2}\cdot\int_0^{\pi/2} \frac{(\sin x + \cos x)}{\sqrt{1-(1-\sin 2x)}}\,dx$

$= \sqrt{2}\cdot\int_0^{\pi/2} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin x - \cos x)^2}}\,dx.$ Put $(\sin x - \cos x) = t$.

46. Let $(2 - x) = A\cdot\frac{d}{dx}(5x - 6 - x^2) + B$.

47. $I = \int_{\pi/4}^{\pi/2} \frac{\left[\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right]}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^{1/2}}\,dx.$ Now, put $\cos\frac{\theta}{2} + \sin\frac{\theta}{2} = t$.

Properties of Definite Integrals

THEOREM 1 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$

PROOF Let $\int f(x)\,dx = F(x)$. Then, $\int f(t)\,dt = F(t)$.

$\therefore \int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$

And, $\int_a^b f(t)\,dt = [F(t)]_a^b = F(b) - F(a).$

Hence, $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$

THEOREM 2 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

PROOF Let $\int f(x)\,dx = F(x)$.

Then, $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$

And, $-\int_b^a f(x)\,dx = -[F(x)]_b^a = -[F(a) - F(b)] = F(b) - F(a).$

Hence, $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

THEOREM 3 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, *where* $a < c < b$.

PROOF Let $\int f(x)\,dx = F(x)$. Then, $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = [F(x)]_a^c + [F(x)]_c^b = \{F(c) - F(a)\} + \{F(b) - F(c)\}$$

$$= F(b) - F(a) = [F(x)]_a^b = \int_a^b f(x)\,dx.$$

Hence, $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$.

REMARK If $a < c_1 < c_2 < \ldots < c_n < b$ then

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_n}^b f(x)\,dx.$$

THEOREM 4 $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **[CBSE 2002C, '05C, '06]**

PROOF In the RHS integral, put $(a - x) = t$ so that $dx = -dt$.

Now, when $x = 0$ we have $t = a$.

And, when $x = a$ we have $t = 0$.

$$\therefore \quad \int_0^a f(a-x)\,dx = -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt = \int_0^a f(x)\,dx.$$

Hence, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

THEOREM 5 $\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx$.

PROOF Putting $a + b - x = t$, we get $dx = -dt$.

Now, $x = a \Rightarrow t = b$.

And, $x = b \Rightarrow t = a$.

$$\therefore \quad \int_a^b f(a+b-x)\,dx = -\int_b^a f(t)\,dt = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx.$$

Hence, $\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx$.

THEOREM 6 $\int_a^b \{f(x) + g(x)\}\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

PROOF Let $\int f(x)\,dx = F(x)$ and $\int g(x)\,dx = G(x)$.

Then, $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx = F(x)+G(x).$

$$\therefore \quad \int_a^b \{f(x)+g(x)\}\,dx = [F(x)+G(x)]_a^b$$

$$= [F(b)+G(b)] - [F(a)+G(a)]$$

$$= [F(b)-F(a)] + [G(b)-G(a)]$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Hence, $\int_a^b \{f(x)+g(x)\}\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

THEOREM 7 $\displaystyle\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, \textit{ when } f(x) \textit{ is an odd function} \\ 2\displaystyle\int_0^a f(x)\,dx, \textit{ when } f(x) \textit{ is an even function.} \end{cases}$ **[CBSE 2005C]**

PROOF We have $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$ … (i)

In the first integral on the RHS of (i), put $x = -t$ so that $dx = -dt$.

When $x = -a$, we have $t = a$. And, when $x = 0$, we have $t = 0$.

$$\therefore \quad \int_{-a}^{0} f(x)\,dx = -\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt = \int_0^a f(-x)\,dx.$$

Thus, $\displaystyle\int_{-a}^{0} f(x)\,dx = \int_0^a f(-x)\,dx$ … (ii)

Using (ii) in (i), we have

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx = \int_0^a [f(-x)+f(x)]\,dx$$

$$= \begin{cases} 0, \text{ when } f(x) \text{ is odd} \\ 2\displaystyle\int_0^a f(x)\,dx, \text{ when } f(x) \text{ is even.} \end{cases}$$

$$\left[\begin{array}{l} \because\ f(x) \text{ is odd } \Rightarrow f(-x) = -f(x) \\ \quad f(x) \text{ is even } \Rightarrow f(-x) = f(x) \end{array}\right]$$

THEOREM 8 $\displaystyle\int_0^{2a} f(x)\,dx = \begin{cases} 0, \textit{ if } f(2a-x) = -f(x) \\ 2\displaystyle\int_0^a f(x)\,dx, \textit{ if } f(2a-x) = f(x). \end{cases}$

PROOF We have $\displaystyle\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$ … (i)

Now, in the second integral on the RHS of (i), put $x = 2a - t$ so that $dx = -dt$.

When $x = a$, we have $t = a$. When $x = 2a$, we have $t = 0$.

$$\therefore \quad \int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt = \int_0^a f(2a-t)\,dt$$

$$= \int_0^a f(2a-x)\,dx.$$

Thus, $$\int_a^{2a} f(x)\,dx = \int_0^a f(2a-x)\,dx \qquad \text{... (ii)}$$

Using (ii) in (i), we get

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$= \int_0^a \{f(x) + f(2a-x)\}\,dx$$

$$= \begin{cases} 0, \text{ when } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, \text{ when } f(2a-x) = f(x). \end{cases}$$

Hence, $$\int_0^{2a} f(x)\,dx = \begin{cases} 0, \text{ when } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, \text{ when } f(2a-x) = f(x). \end{cases}$$

SOLVED EXAMPLES

EXAMPLE 1 *Prove that* $\int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)}\,dx = \frac{\pi}{4}\cdot$ **[CBSE 2002C, '07C]**

SOLUTION Let $$I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)}\,dx \qquad \text{... (i)}$$

Using the result $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ in (i), we get

$$I = \int_0^{\pi/2} \frac{\sin[(\pi/2) - x]}{\sin[(\pi/2) - x] + \cos[(\pi/2) - x]}\,dx$$

or $$I = \int_0^{\pi/2} \frac{\cos x}{(\cos x + \sin x)}\,dx = \int_0^{\pi/2} \frac{\cos x}{(\sin x + \cos x)}\,dx \qquad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)}\,dx + \int_0^{\pi/2} \frac{\cos x}{(\sin x + \cos x)}\,dx$$

$$= \int_0^{\pi/2} \frac{(\sin x + \cos x)}{(\sin x + \cos x)}\,dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}.$$

$$\therefore\ I = \frac{\pi}{4},\ \text{i.e.,}\ \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)}\,dx = \frac{\pi}{4}.$$

EXAMPLE 2 *Prove that* $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx = \frac{\pi}{4}.$

SOLUTION Let $\displaystyle I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx$ … (i)

Using the result $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ in (i), we get

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos[(\pi/2) - x]}}{\sqrt{\sin[(\pi/2) - x]} + \sqrt{\cos[(\pi/2) - x]}}\,dx$$

$$\text{or}\quad I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \quad \text{… (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx + \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx$$

$$= \int_0^{\pi/2} \frac{(\sqrt{\sin x} + \sqrt{\cos x})}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}.$$

$$\therefore\ I = \frac{\pi}{4},\ \text{i.e.,}\ \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx = \frac{\pi}{4}.$$

EXAMPLE 3 *Evaluate:* *(i)* $\displaystyle\int_0^1 x(1-x)^n\,dx$ *(ii)* $\displaystyle\int_0^1 x(1-x)^{3/2}\,dx$

SOLUTION (i) We have $\displaystyle\int_0^1 x(1-x)^n\,dx = \int_0^1 (1-x)[1-(1-x)]^n\,dx$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)x^n\,dx = \int_0^1 x^n\,dx - \int_0^1 x^{n+1}\,dx$$

$$= \left[\frac{x^{n+1}}{(n+1)}\right]_0^1 - \left[\frac{x^{n+2}}{n+2}\right]_0^1 = \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{1}{(n+1)(n+2)}.$$

(ii) We have $\int_0^1 x(1-x)^{3/2}\,dx = \int_0^1 (1-x)[1-(1-x)]^{3/2}\,dx$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)x^{3/2}dx = \int_0^1 x^{3/2}dx - \int_0^1 x^{5/2}dx$$

$$= \left[\frac{2}{5}x^{5/2}\right]_0^1 - \left[\frac{2}{7}x^{7/2}\right]_0^1 = \left(\frac{2}{5} - \frac{2}{7}\right) = \frac{4}{35}.$$

EXAMPLE 4 *Show that* $\int_0^{\pi/2} \log(\tan x)\,dx = 0.$

SOLUTION Let $I = \int_0^{\pi/2} \log(\tan x)\,dx$ ... (i)

Then, $I = \int_0^{\pi/2} \log\left[\tan\left(\frac{\pi}{2} - x\right)\right]dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

or $I = \int_0^{\pi/2} \log(\cot x)\,dx = \int_0^{\pi/2} \log\left(\frac{1}{\tan x}\right)dx = -\int_0^{\pi/2} \log \tan x\,dx = -I.$

$\therefore$ $I = -I$ or $2I = 0$ or $I = 0.$

Hence, $\int_0^{\pi/2} \log(\tan x)\,dx = 0.$

EXAMPLE 5 *Prove that* $\int_0^{\pi/4} \log(1 + \tan x)\,dx = \frac{\pi}{8}(\log 2).$ **[CBSE 2007, '11, '13C]**

SOLUTION Let $I = \int_0^{\pi/4} \log(1 + \tan x)\,dx$ ... (i)

Then, $I = \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right]dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

or $I = \int_0^{\pi/4} \log\left[1 + \frac{1 - \tan x}{1 + \tan x}\right]dx$

or $$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right) dx$$

or $$I = \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]\, dx$$

or $$I = (\log 2)\cdot \int_0^{\pi/4} dx - \int_0^{\pi/4} \log(1+\tan x)\, dx \quad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = (\log 2) \int_0^{\pi/4} dx = (\log 2)\cdot [x]_0^{\pi/4} = \frac{\pi}{4}(\log 2).$$

$$\therefore \quad I = \frac{\pi}{8}(\log 2), \text{ i.e., } \int_0^{\pi/4} \log(1+\tan x)\, dx = \frac{\pi}{8}(\log 2).$$

EXAMPLE 6 *Prove that* $\int_0^{\pi/2} \log(\sin x)\, dx = \int_0^{\pi/2} \log(\cos x)\, dx = -\frac{\pi}{2}(\log 2).$

[CBSE 2004, '07C, '08]

SOLUTION Let $$I = \int_0^{\pi/2} \log(\sin x)\, dx \quad \text{... (i)}$$

Then, $$I = \int_0^{\pi/2} \log\left[\sin\left(\frac{\pi}{2} - x\right)\right] dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

or $$I = \int_0^{\pi/2} \log(\cos x)\, dx \quad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} [\log(\sin x) + \log(\cos x)]\, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\, dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log(\sin 2x)\, dx - \int_0^{\pi/2} (\log 2)\, dx = \frac{1}{2}\int_0^{\pi} \log \sin t\, dt - (\log 2)\cdot \int_0^{\pi/2} dx$$

[putting $2x = t$ in the 1st integral]

$$= \frac{1}{2}\int_0^{\pi} \log \sin t\; dt - (\log 2)\cdot [x]_0^{\pi/2}$$

$$= \left(\frac{1}{2}\times 2\right)\cdot \int_0^{\pi/2} \log \sin t\; dt - \frac{\pi}{2}(\log 2)$$

$$= \int_0^{\pi/2} \log(\sin x)\, dx - \frac{\pi}{2}(\log 2).$$

$$\therefore \quad 2I = I - \frac{\pi}{2}(\log 2) \quad \text{or} \quad I = -\frac{\pi}{2}(\log 2).$$

$$\therefore \quad \int_0^{\pi/2} \log(\sin x)\,dx = \int_0^{\pi/2} \log(\cos x)\,dx = -\frac{\pi}{2}(\log 2).$$

EXAMPLE 7 *Prove that:* *(a)* $\int_0^{\pi/2} \sin 2x \log(\tan x)\,dx = 0$ **[CBSE 2003C, '06C]**

(b) $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx = 0$

SOLUTION (a) Let $I = \int_0^{\pi/2} \sin 2x \log(\tan x)\,dx$ ... (i)

Then, $I = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx$

$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

or $I = \int_0^{\pi/2} \sin 2x \log(\cot x)\,dx$ ... (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} [\sin 2x \log(\tan x) + \sin 2x \log(\cot x)]\,dx$$

$$= \int_0^{\pi/2} \sin 2x\{\log(\tan x) + \log(\cot x)\}\,dx$$

$$= \int_0^{\pi/2} \sin 2x \cdot \log(\tan x \cdot \cot x)\,dx = \int_0^{\pi/2} \sin 2x \cdot \log(1)\,dx = 0$$

$[\because \quad \log 1 = 0]$

$$\therefore \quad I = 0 \Rightarrow \int_0^{\pi/2} \sin 2x \log(\tan x)\,dx = 0.$$

(b) Put $x = \cos^2 t$ so that $dx = -\sin 2t\,dt$.

When $x = 0$, we have $\cos t = 0$ or $t = \frac{\pi}{2}$.

When $x = 1$, we have $\cos t = 1$ or $t = 0$.

$$\therefore \quad \int_0^1 \log\left(\frac{1}{x} - 1\right) dx = -\int_{\pi/2}^0 \log(\tan^2 t) \cdot \sin 2t\,dt$$

$$= 2\int_0^{\pi/2} \sin 2t \cdot \log(\tan t)\,dt = 0 \quad \text{[from (a)]}.$$

EXAMPLE 8 *Prove that:* (a) $\displaystyle\int_0^{\pi/2} \frac{\sin^2 x}{(\sin x + \cos x)}\, dx = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$ **[CBSE 2012, '14C]**

(b) $\displaystyle\int_0^{\pi/2} \frac{\sin^2 x}{(1 + \sin x \cos x)}\, dx = \frac{\pi}{3\sqrt{3}}.$

SOLUTION (a) Let $\displaystyle I = \int_0^{\pi/2} \frac{\sin^2 x}{(\sin x + \cos x)}\, dx$ … (i)

Then, $$I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\left[\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)\right]}\, dx$$

or $$I = \int_0^{\pi/2} \frac{\cos^2 x}{(\cos x + \sin x)}\, dx = \int_0^{\pi/2} \frac{\cos^2 x}{(\sin x + \cos x)}\, dx \quad \dots \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)}{(\sin x + \cos x)}\, dx = \int_0^{\pi/2} \frac{dx}{(\sin x + \cos x)}$$

$$= \int_0^{\pi/2} \frac{dx}{\left[\frac{2\tan(x/2)}{1+\tan^2(x/2)} + \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right]}$$

$$= \int_0^{\pi/2} \frac{\sec^2(x/2)}{[1 - \tan^2(x/2) + 2\tan(x/2)]}\, dx$$

$$= 2\int_0^1 \frac{dt}{(1 - t^2 + 2t)}, \quad \text{where } \tan\frac{x}{2} = t$$

$$\left[x = 0 \Rightarrow t = 0 \text{ and } x = \frac{\pi}{2} \Rightarrow t = 1\right]$$

$$= 2\int_0^1 \frac{dt}{2 - (t-1)^2} = 2\int_0^1 \frac{dt}{(\sqrt{2})^2 - (t-1)^2}$$

$$= 2 \cdot \frac{1}{2\sqrt{2}} \left\{\log\left|\frac{\sqrt{2} + t - 1}{\sqrt{2} - t + 1}\right|\right\}_0^1 = \frac{1}{\sqrt{2}}\left[0 - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

$$= \frac{1}{\sqrt{2}} \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \frac{1}{\sqrt{2}} \log \frac{(\sqrt{2}+1)}{(\sqrt{2}-1)} \times \frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}$$

$$= \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)^2 = \frac{1}{\sqrt{2}} \times 2\log(\sqrt{2}+1)$$

$$= \sqrt{2}\log(\sqrt{2}+1)$$

$$\therefore \quad I = \frac{1}{2}\sqrt{2}\log(\sqrt{2}+1) = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1).$$

(b) Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{(1+\sin x \cos x)} dx$ ... (i)

Then, $I = \int_0^{\pi/2} \frac{\sin^2[(\pi/2)-x]}{1+\sin[(\pi/2)-x]\cos[(\pi/2)-x]} dx$

or $I = \int_0^{\pi/2} \frac{\cos^2 x}{(1+\sin x \cos x)} dx$ ... (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)}{(1+\sin x \cos x)} dx = \int_0^{\pi/2} \frac{dx}{(1+\sin x \cos x)}$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{(\sec^2 x + \tan x)} dx$$

[dividing num. and denom. by $\cos^2 x$]

$$= \int_0^{\pi/2} \frac{\sec^2 x}{(1+\tan^2 x + \tan x)} dx = \int_0^{\infty} \frac{dt}{(t^2+t+1)}, \text{ where } \tan x = t$$

$$\left[x = 0 \Rightarrow t = 0 \text{ and } x = \frac{\pi}{2} \Rightarrow t = \infty\right]$$

$$= \int_0^{\infty} \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \left[\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] = \frac{2}{\sqrt{3}} \cdot \left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{2\pi}{3\sqrt{3}}.$$

Hence, $I = \frac{\pi}{3\sqrt{3}}$.

EXAMPLE 9 *Prove that* $\int_0^{\pi} \frac{x \tan x}{(\sec x + \tan x)} dx = \pi\left(\frac{\pi}{2} - 1\right)$. **[CBSE 2008, '10]**

SOLUTION Let $I = \int_0^{\pi} \frac{x \tan x}{(\sec x + \tan x)} dx$ ... (i)

Then, $I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{[\sec(\pi - x) + \tan(\pi - x)]} dx$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

or $I = \int_0^{\pi} \frac{(\pi - x)\tan x}{(\sec x + \tan x)} dx$ ... (ii)

Adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \frac{\tan x}{(\sec x + \tan x)}\,dx = \pi\int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{(\sec^2 x - \tan^2 x)}\,dx$$

$$= \pi\cdot\left[\int_0^{\pi} \sec x \tan x\,dx - \int_0^{\pi} \tan^2 x\,dx\right]$$

$$= \pi\cdot\left\{[\sec x]_0^{\pi} - \int_0^{\pi} (\sec^2 x - 1)\,dx\right\}$$

$$= \pi\cdot\{-2 - [\tan x]_0^{\pi} + [x]_0^{\pi}\} = \pi(\pi - 2).$$

$$\therefore \quad I = \pi\left(\frac{\pi}{2} - 1\right) \text{ i.e., } \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\,dx = \pi\left(\frac{\pi}{2} - 1\right).$$

EXAMPLE 10 *Evaluate* $\int_0^{\pi} \frac{x}{(1 + \sin x)}\,dx.$ **[CBSE 2010, '11, '12C]**

SOLUTION Let $I = \int_0^{\pi} \frac{x}{(1 + \sin x)}\,dx$ … (i)

Then, $I = \int_0^{\pi} \frac{(\pi - x)}{1 + \sin(\pi - x)}\,dx = \int_0^{\pi} \frac{(\pi - x)}{(1 + \sin x)}\,dx$ …(ii)

Adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \frac{dx}{(1 + \sin x)} = \pi\cdot\int_0^{\pi} \frac{1}{(1 + \sin x)} \times \frac{(1 - \sin x)}{(1 - \sin x)}\,dx$$

or $$2I = \pi\int_0^{\pi} \left(\frac{1 - \sin x}{\cos^2 x}\right)dx = \pi\cdot\left[\int_0^{\pi} \sec^2 x\,dx - \int_0^{\pi} \sec x \tan x\,dx\right]$$

$$= \pi\cdot\{[\tan x]_0^{\pi} - [\sec x]_0^{\pi}\} = 2\pi$$

$$\therefore \quad I = \pi, \text{ i.e., } \int_0^{\pi} \frac{x}{(1 + \sin x)}\,dx = \pi.$$

EXAMPLE 11 *Evaluate* $\int_0^{\pi} \frac{x \sin x}{(1 + \cos^2 x)}\,dx.$ **[CBSE 2009C, '11C, '12, '14]**

SOLUTION Let $I = \int_0^{\pi} \frac{x \sin x}{(1 + \cos^2 x)}\,dx$ … (i)

Then, $I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}\,dx$

or $I = \int_0^{\pi} \frac{(\pi - x)\sin x}{(1 + \cos^2 x)}\,dx$ … (ii)

Adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \frac{\sin x}{(1+\cos^2 x)}\,dx = -\pi\int_1^{-1} \frac{dt}{(1+t^2)}, \text{ where } \cos x = t$$

$$[x = 0 \Rightarrow t = 1 \text{ and } x = \pi \Rightarrow t = -1]$$

$$= \pi\int_{-1}^{1} \frac{dt}{(1+t^2)} = \pi\,[\tan^{-1} t]_{-1}^{1}$$

$$= \pi[\tan^{-1} 1 - \tan^{-1}(-1)] = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}\cdot$$

$$\therefore \quad I = \frac{\pi^2}{4}\cdot$$

EXAMPLE 12 *Evaluate* $\displaystyle\int_0^{\pi/2} \frac{x}{(\sin x + \cos x)}\,dx.$ **[CBSE 2003C, '08]**

SOLUTION Let $\displaystyle I = \int_0^{\pi/2} \frac{x}{(\sin x + \cos x)}\,dx$ ... (i)

Then, $\displaystyle I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{\sin[(\pi/2) - x] + \cos[(\pi/2) - x]}\,dx$

or $\displaystyle I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{(\cos x + \sin x)}\,dx = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{(\sin x + \cos x)}\,dx$ ... (ii)

Adding (i) and (ii), we get

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{dx}{(\sin x + \cos x)}$$

$$\therefore \quad I = \frac{\pi}{4}\int_0^{\pi/2} \frac{dx}{(\sin x + \cos x)} = \frac{\pi}{4}\int_0^{\pi/2} \frac{dx}{\left[\frac{2\tan(x/2)}{1+\tan^2(x/2)} + \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right]}$$

$$= \frac{\pi}{4}\int_0^{\pi/2} \frac{\sec^2(x/2)}{1 - \tan^2(x/2) + 2\tan(x/2)}\,dx$$

$$= \frac{\pi}{4}\int_0^{1} \frac{2dt}{(1 - t^2 + 2t)}, \quad \text{where } t = \tan\frac{x}{2}$$

$$\left[x = 0 \Rightarrow t = 0 \text{ and } x = \frac{\pi}{2} \Rightarrow t = 1\right]$$

$$= \frac{\pi}{2}\int_0^{1} \frac{dt}{[(\sqrt{2})^2 - (t-1)^2]}\,dt$$

$$= \frac{\pi}{2}\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2} + (t-1)}{\sqrt{2} - (t-1)}\right|_0^1 = \frac{\pi}{4\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\cdot$$

EXAMPLE 13 *Prove that* $\int_0^{\pi/2} (2\log \sin x - \log \sin 2x)\, dx = -\frac{\pi}{2}(\log 2)$. **[CBSE 2009]**

SOLUTION Let $I = \int_0^{\pi/2} (2\log \sin x - \log \sin 2x)\, dx.$ ... (i)

Then, $I = \int_0^{\pi/2} \left[2 \log \sin \left(\frac{\pi}{2} - x \right) - \log \sin 2\left(\frac{\pi}{2} - x \right) \right] dx$

or $I = \int_0^{\pi/2} (2\log \cos x - \log \sin 2x)\, dx$ ... (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} [2(\log \sin x + \log \cos x) - 2\log \sin 2x]\, dx$$

or
$$I = \int_0^{\pi/2} [\log (\sin x \cos x) - \log \sin 2x]\, dx$$

$$= \int_0^{\pi/2} \left[\log \left(\frac{\sin 2x}{2} \right) - \log \sin 2x \right] dx$$

$$= \int_0^{\pi/2} (\log \sin 2x - \log 2 - \log \sin 2x)\, dx$$

$$= -\log 2 \cdot \int_0^{\pi/2} dx = (-\log 2) \cdot [x]_0^{\pi/2} = -\frac{\pi}{2}(\log 2).$$

EXAMPLE 14 *Evaluate* $\int_0^{\pi} \frac{x}{(a^2 \cos^2 x + b^2 \sin^2 x)} dx.$ **[CBSE 2003C, '09]**

SOLUTION Let $I = \int_0^{\pi} \frac{x}{(a^2 \cos^2 x + b^2 \sin^2 x)} dx$ ... (i)

Then, $I = \int_0^{\pi} \frac{(\pi - x)}{(a^2 \cos^2 (\pi - x) + b^2 \sin^2 (\pi - x)]} dx$

or $I = \int_0^{\pi} \frac{(\pi - x)}{(a^2 \cos^2 x + b^2 \sin^2 x)} dx.$ ... (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{(x + \pi - x)}{(a^2 \cos^2 x + b^2 \sin^2 x)} dx = \pi \int_0^{\pi} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)}$$

$$= 2\pi \int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)} = 2\pi \int_0^{\pi/2} \frac{\sec^2 x}{(a^2 + b^2 \tan^2 x)} dx$$

[dividing num. and denom. by $\cos^2 x$]

$$= 2\pi \int_0^{\infty} \frac{dt}{(a^2 + b^2t^2)}, \text{ where } \tan x = t$$

$$= \frac{2\pi}{b^2} \int_0^{\infty} \frac{dt}{\left(\frac{a^2}{b^2} + t^2\right)} = \left[\frac{2\pi}{b^2} \cdot \frac{b}{a} \tan^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty}$$

$$= \frac{2\pi}{ab}[\tan^{-1}(\infty) - \tan^{-1}(0)] = \frac{2\pi}{ab}\left(\frac{\pi}{2} - 0\right) = \left(\frac{2\pi}{ab} \times \frac{\pi}{2}\right) = \frac{\pi^2}{ab}.$$

$$\therefore \quad I = \frac{\pi^2}{2ab} \Rightarrow \int_0^{\pi} \frac{x}{(a^2 \cos^2 x + b^2 \sin^2 x)} dx = \frac{\pi^2}{2ab}.$$

EXAMPLE 15 *Prove that* $\int_0^{\pi} x \sin^3 x \, dx = \frac{2\pi}{3}$.

SOLUTION Let $I = \int_0^{\pi} x \sin^3 x \, dx$ … (i)

Then, $I = \int_0^{\pi} (\pi - x) \sin^3 (\pi - x) \, dx$

or $I = \int_0^{\pi} (\pi - x) \sin^3 x \, dx$ … (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \pi \sin^3 x \, dx = \pi \int_0^{\pi} \sin^2 x \cdot \sin x \, dx$$

$$= \pi \int_0^{\pi} (1 - \cos^2 x) \sin x \, dx$$

$$= -\pi \int_1^{-1} (1 - t^2) \, dt, \text{ where } \cos x = t$$

$$[x = 0 \Rightarrow t = 1 \text{ and } x = \pi \Rightarrow t = -1]$$

$$= \pi \int_{-1}^{1} (1 - t^2) dt = \pi \left[t - \frac{t^3}{3}\right]_{-1}^{1} = \frac{4\pi}{3}.$$

Hence, $I = \frac{2\pi}{3} \Rightarrow \int_0^{\pi} x \sin^3 x \, dx = \frac{2\pi}{3}$.

EXAMPLE 16 *Evaluate* $\int_0^1 \cot^{-1}\{1 - x + x^2\} \, dx$. **[CBSE 2008]**

SOLUTION We have

$$I = \int_0^1 \cot^{-1}\{1 - x + x^2\} \, dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx = \int_0^1 \tan^{-1}\left\{\frac{x+(1-x)}{1-x+x^2}\right\} dx$$

$$= \int_0^1 \{\tan^{-1}x + \tan^{-1}(1-x)\}\, dx$$

$$= \int_0^1 \tan^{-1}x\, dx + \int_0^1 \tan^{-1}(1-x)\, dx$$

$$= \int_0^1 \tan^{-1}x\, dx + \int_0^1 \tan^{-1}\{1-(1-x)\}\, dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 \tan^{-1}x\, dx + \int_0^1 \tan^{-1}x\, dx = 2\int_0^1 \tan^{-1}x\, dx$$

$$= 2\int_0^1 (\tan^{-1}x \cdot 1)\, dx$$

$$= 2\left[(\tan^{-1}x)x - \int_0^1 \frac{1}{(1+x^2)} \cdot x\, dx\right]_0^1$$

$$= 2[(\tan^{-1} x)\cdot x]_0^1 - 2\int_0^1 \frac{x}{(1+x^2)}\, dx$$

$$= 2\{(\tan^{-1}1)\cdot 1 - 0\} - [\log\,(1+x^2)]_0^1$$

$$= \left(2\times\frac{\pi}{4}\right) - (\log\,2 - \log\,1) = \left(\frac{\pi}{2} - \log\,2\right)\cdot$$

<u>EXAMPLE 17</u> *Evaluate* $\displaystyle\int_{\pi/5}^{3\pi/10} \frac{\sin\,x}{(\sin\,x+\cos\,x)}\,dx.$ **[CBSE 2009]**

SOLUTION Let $I = \displaystyle\int_{\pi/5}^{3\pi/10} \frac{\sin\,x}{(\sin\,x+\cos\,x)}\,dx$ … (i)

Here, $a = \frac{\pi}{5}$ and $b = \frac{3\pi}{10} \Rightarrow (a+b) = \left(\frac{\pi}{5}+\frac{3\pi}{10}\right) = \frac{\pi}{2}\cdot$

Using $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, we have

$$I = \int_{\pi/5}^{3\pi/10} \frac{\sin\left(\frac{\pi}{2}-x\right)dx}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$$

$$\Rightarrow \quad I = \int_{\pi/5}^{3\pi/10} \frac{\cos\,x}{(\sin\,x+\cos\,x)}\,dx \qquad \text{… (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_{\pi/5}^{3\pi/10} dx = [x]_{\pi/5}^{3\pi/10} = \left(\frac{3\pi}{10} - \frac{\pi}{5}\right) = \frac{\pi}{10}$$

$$\Rightarrow \quad I = \frac{\pi}{20}.$$

Integrals of the form $\int_{-a}^{a} f(x)\,dx$, *where* $f(x)$ *is even or odd*

We know that

(i) $f(x)$ is odd, if $f(-x) = -f(x)$;

(ii) $f(x)$ is even, if $f(-x) = f(x)$;

(iii) $\int_{-a}^{a} f(x)\,dx = 0$, when $f(x)$ is odd;

(iv) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$, when $f(x)$ is even.

EXAMPLE 18 *Show that* $\int_{-\pi/2}^{\pi/2} \sin^7 x\,dx = 0.$

SOLUTION Let $f(x) = \sin^7 x$. Then,

$f(-x) = [\sin(-x)]^7 = -\sin^7 x = -f(x).$

$\therefore$ $f(x)$ is an odd function of x.

But, $\int_{-a}^{a} f(x)\,dx = 0$, when $f(x)$ is odd.

$$\therefore \quad \int_{-\pi/2}^{\pi/2} \sin^7 x\,dx = 0.$$

EXAMPLE 19 *Evaluate* $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx.$

SOLUTION Let $f(x) = \sin^2 x$.

Then, $f(-x) = [\sin(-x)]^2 = (-\sin x)^2 = \sin^2 x = f(x)$.

$\therefore$ $f(x)$ is an even function.

$$\text{So,} \quad \int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = 2\int_{0}^{\pi/2} \sin^2 x\,dx = 2\int_{0}^{\pi/2} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \int_{0}^{\pi/2} (1-\cos 2x)\,dx = \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{2}.$$

EXAMPLE 20 *Prove that* $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx = 0.$ **[CBSE 2005C]**

SOLUTION Let $f(x) = \log\left(\frac{2-x}{2+x}\right)$.

Then, $f(-x) = \log\left(\frac{2+x}{2-x}\right) = \log\left(\frac{2-x}{2+x}\right)^{-1} = -\log\left(\frac{2-x}{2+x}\right) = -f(x)$.

$\therefore$ $f(x)$ is an odd function of x.

But, we know that $\int_{-a}^{a} f(x)\,dx = 0$, when $f(x)$ is an odd function of x.

$\therefore$ $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx = 0$.

EXAMPLE 21 *Evaluate* $\int_{1}^{4} f(x)\,dx$, *where* $f(x) = \begin{cases} 4x+3, & \text{if } 1 \le x \le 2 \\ 3x+5, & \text{if } 2 \le x \le 4. \end{cases}$

SOLUTION
$$\int_{1}^{4} f(x)\,dx = \int_{1}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx$$
$$= \int_{1}^{2} (4x+3)\,dx + \int_{2}^{4} (3x+5)\,dx$$
$$= [2x^2 + 3x]_1^2 + \left[\frac{3x^2}{2} + 5x\right]_2^4 = (9+28) = 37.$$

INTEGRALS OF MODULUS FUNCTIONS

EXAMPLE 22 *Evaluate:*

(i) $\int_{-1}^{2} |x|\,dx$ *(ii)* $\int_{0}^{1} |5x-3|\,dx$ *(iii)* $\int_{0}^{\pi} |\cos x|\,dx$

SOLUTION (i) Clearly, $|x| = \begin{cases} -x & \text{when } -1 \le x \le 0 \\ x & \text{when } 0 \le x \le 2. \end{cases}$

$$\therefore \int_{-1}^{2} |x|\,dx = \int_{-1}^{0} |x|\,dx + \int_{0}^{2} |x|\,dx$$
$$= \int_{-1}^{0} (-x)\,dx + \int_{0}^{2} x\,dx = \left[\frac{-x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2} = \left(\frac{1}{2} + 2\right) = \frac{5}{2}.$$

(ii) Clearly, $|5x-3| = \begin{cases} -(5x-3) & \text{when } 0 \le x \le \frac{3}{5} \\ (5x-3) & \text{when } \frac{3}{5} \le x \le 1. \end{cases}$

$$\therefore \int_{0}^{1} |5x-3|\,dx = \int_{0}^{3/5} |5x-3|\,dx + \int_{3/5}^{1} |5x-3|\,dx$$
$$= \int_{0}^{3/5} -(5x-3)\,dx + \int_{3/5}^{1} (5x-3)\,dx$$

$$= \left[3x - \frac{5x^2}{2}\right]_0^{3/5} + \left[\frac{5x^2}{2} - 3x\right]_{3/5}^{1}$$

$$= \left(\frac{9}{5} - \frac{9}{10}\right) + \left(\frac{-1}{2} + \frac{9}{10}\right) = \frac{13}{10}.$$

(iii) Clearly, $|\cos x| = \begin{cases} \cos x & \text{when } 0 \le x \le \frac{\pi}{2} \\ -\cos x & \text{when } \frac{\pi}{2} \le x \le \pi. \end{cases}$

$$\therefore \quad \int_0^{\pi} |\cos x|\, dx = \int_0^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{\pi} |\cos x|\, dx$$

$$= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x\, dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = (1 + 1) = 2.$$

EXAMPLE 23 *Evaluate* $\int_1^4 f(x)\, dx$, *where* $f(x) = |x-1| + |x-2| + |x-3|$. **[CBSE 2013]**

SOLUTION $\int_1^4 f(x)\, dx = \int_1^2 f(x)\, dx + \int_2^3 f(x)\, dx + \int_3^4 f(x)\, dx$

$$= \int_1^2 \{(x-1) - (x-2) - (x-3)\}\, dx + \int_2^3 \{(x-1) + (x-2) - (x-3)\}\, dx$$

$$+ \int_3^4 \{(x-1) + (x-2) + (x-3)\}\, dx$$

$$= \int_1^2 (-x+4)\, dx + \int_2^3 x\, dx + \int_3^4 (3x-6)\, dx$$

$$= \left[\frac{-x^2}{2} + 4x\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3 + \left[\frac{3x^2}{2} - 6x\right]_3^4 = \left(\frac{5}{2} + \frac{5}{2} + \frac{9}{2}\right) = \frac{19}{2}.$$

EXAMPLE 24 *Evaluate:*

(i) $\int_{-\pi/2}^{\pi/2} |\sin x|\, dx$ (ii) $\int_{-1}^{1} e^{|x|}\, dx$ (iii) $\int_{-2}^{1} |2x+1|\, dx$

SOLUTION (i) Clearly, $|\sin x|$ is an even function of x.

$$\therefore \quad \int_{-\pi/2}^{\pi/2} |\sin x|\, dx = 2\int_0^{\pi/2} |\sin x|\, dx$$

$$= 2\int_0^{\pi/2} \sin x\, dx \quad \left[\because \ \sin x \ge 0, \text{ when } 0 \le x \le \frac{\pi}{2}\right]$$

$$= [-2\cos x]_0^{\pi/2} = 2.$$

(ii) Clearly, $e^{|x|}$ is an even function of x.

$$\therefore \int_{-1}^{1} e^{|x|}\,dx = 2\int_{0}^{1} e^{|x|}\,dx$$

$$= 2\int_{0}^{1} e^{x}\,dx \qquad [\because\ |x| = x, \text{ when } 0 \le x \le 1]$$

$$= [2e^{x}]_0^1 = (2e - 2) = 2(e - 1).$$

(iii) $\left[-2 \le x < -\frac{1}{2} \Rightarrow 2x + 1 < 0\right]$ and $\left[-\frac{1}{2} \le x \le 1 \Rightarrow 2x + 1 \ge 0\right]$

$$\therefore \int_{-2}^{1} |2x+1|\,dx = \int_{-2}^{-1/2} |2x+1|\,dx + \int_{-1/2}^{1} |2x+1|\,dx$$

$$= \int_{-2}^{-1/2} -(2x+1)\,dx + \int_{-1/2}^{1} (2x+1)\,dx$$

$$= [-x^2 - x]_{-2}^{-1/2} + [x^2 + x]_{-1/2}^{1}$$

$$= \left(-\frac{1}{4} + \frac{1}{2}\right) - (-4 + 2) + \left[2 - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{4} + 2 + \frac{9}{4} = \frac{9}{2}.$$

EXAMPLE 25 *Evaluate* $\int_{0}^{2\pi} |\sin x|\,dx.$

SOLUTION We know that $\sin x$ is positive, when $0 \le x \le \pi$ and $\sin x$ is negative when $\pi \le x \le 2\pi$.

$$\therefore \quad |\sin x| = \begin{cases} \sin x, & \text{when } 0 \le x \le \pi \\ -\sin x, & \text{when } \pi \le x \le 2\pi. \end{cases}$$

$$\therefore \int_{0}^{2\pi} |\sin x|\,dx = \int_{0}^{\pi} |\sin x|\,dx + \int_{\pi}^{2\pi} |\sin x|\,dx$$

$$= \int_{0}^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} (-\sin x)\,dx$$

$$= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi} = (2 + 2) = 4.$$

EXAMPLE 26 *Evaluate* $\int_{-\pi/2}^{\pi/2} f(x)\,dx$*, where* $f(x) = \sin|x| + \cos|x|$. **[CBSE 2000]**

SOLUTION $f(x) = \sin|x| + \cos|x|$

$\Rightarrow f(-x) = \sin|-x| + \cos|-x| = \sin|x| + \cos|x| = f(x)$

$\Rightarrow f(x)$ is an even function.

$$\therefore \ I = \int_{-\pi/2}^{\pi/2} f(x)\,dx$$

$$= 2\int_0^{\pi/2} f(x)\,dx = 2\int_0^{\pi/2} \{\sin|x| + \cos|x|\}\,dx$$

$$= 2\int_0^{\pi/2} (\sin x + \cos x)\,dx \qquad [\because\ |x| = x \text{ in } 0 < x < \tfrac{\pi}{2}]$$

$$= 2\cdot[-\cos x + \sin x]_0^{\pi/2} = 2\left[\left(-\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (-\cos 0 + \sin 0)\right]$$

$$= 4.$$

EXERCISE 16C

Prove that

1. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(\sin x + \cos x)}\,dx = \frac{\pi}{4}$

2. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx = \frac{\pi}{4}$

3. (i) $\displaystyle\int_0^{\pi/2} \frac{\sin^3 x}{(\sin^3 x + \cos^3 x)}\,dx = \frac{\pi}{4}$ **[CBSE 1998, 2000C, '04]**

   (ii) $\displaystyle\int_0^{\pi/2} \frac{\cos^3 x\,dx}{(\sin^3 x + \cos^3 x)} = \frac{\pi}{4}$ **[CBSE 2004C]**

4. (i) $\displaystyle\int_0^{\pi/2} \frac{\sin^7 x}{(\sin^7 x + \cos^7 x)}\,dx = \frac{\pi}{4}$ **[CBSE 2000C]**

   (ii) $\displaystyle\int_0^{\pi/2} \frac{\sin^5 x}{(\sin^5 x + \cos^5 x)}\,dx = \frac{\pi}{4}$ **[CBSE 2000C]**

5. $\displaystyle\int_0^{\pi/2} \frac{\cos^4 x}{(\sin^4 x + \cos^4 x)}\,dx = \frac{\pi}{4}$

6. $\displaystyle\int_0^{\pi/2} \frac{\cos^{1/4} x}{(\sin^{1/4} x + \cos^{1/4} x)}\,dx = \frac{\pi}{4}$

7. $\displaystyle\int_0^{\pi/2} \frac{\sin^{3/2} x}{(\sin^{3/2} x + \cos^{3/2} x)}\,dx = \frac{\pi}{4}$

8. $\displaystyle\int_0^{\pi/2} \frac{\sin^n x}{(\sin^n x + \cos^n x)}\,dx = \frac{\pi}{4}$

9. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\tan x}}{(\sqrt{\tan x} + \sqrt{\cot x})}\,dx = \frac{\pi}{4}$

10. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\cot x}}{(\sqrt{\tan x} + \sqrt{\cot x})}\,dx = \frac{\pi}{4}$

11. $\displaystyle\int_0^{\pi/2} \frac{dx}{(1 + \tan x)} = \frac{\pi}{4}$ **[CBSE 2003C, '06C]**

12. $\displaystyle\int_0^{\pi/2} \frac{dx}{(1 + \cot x)} = \frac{\pi}{4}$

13. $\displaystyle\int_0^{\pi/2} \frac{dx}{(1 + \tan^3 x)} = \frac{\pi}{4}$

14. $\displaystyle\int_0^{\pi/2} \frac{dx}{(1 + \cot^3 x)} = \frac{\pi}{4}$

15. $\displaystyle\int_0^{\pi/2} \frac{dx}{(1 + \sqrt{\tan x})} = \frac{\pi}{4}$ **[CBSE 2006C, '07C, '11]**

16. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\cot x}}{(1 + \sqrt{\cot x})}\,dx = \frac{\pi}{4}$

17. $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{(1+\sqrt{\tan x})} dx = \frac{\pi}{4}$

18. $\int_0^{\pi/2} \frac{(\sin x - \cos x)}{(1+\sin x \cos x)} dx = 0$

19. $\int_0^1 x(1-x)^5 dx = \frac{1}{42}$

20. $\int_0^2 x\sqrt{2-x}\, dx = \frac{16\sqrt{2}}{15}$ **[CBSE 2005C]**

21. $\int_0^{\pi} x \cos^2 x\, dx = \frac{\pi^2}{4}$

22. $\int_0^{\pi} \frac{x \tan x}{(\sec x \operatorname{cosec} x)} dx = \frac{\pi^2}{4}$ **[CBSE 2008, '09C, '11C]**

23. $\int_0^{\pi/2} \frac{\cos^2 x}{(\sin x + \cos x)} dx = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$

24. $\int_0^{\pi} \frac{x \tan x}{(\sec x + \cos x)} dx = \frac{\pi^2}{4}$

25. $\int_0^{\pi} \frac{x \sin x}{(1+\sin x)} dx = \pi\left(\frac{\pi}{2} - 1\right)$

26. $\int_0^{\pi} \frac{x}{(1+\sin^2 x)} dx = \frac{\pi^2}{2\sqrt{2}}$

27. $\int_0^{\pi/2} (2\log \cos x - \log \sin 2x)\, dx = -\frac{\pi}{2}(\log 2)$

28. $\int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx = \frac{\pi}{4}$

29. $\int_0^{a} \frac{dx}{x + \sqrt{a^2 - x^2}} = \frac{\pi}{4}$

30. $\int_0^{a} \frac{\sqrt{x}}{(\sqrt{x} + \sqrt{a-x})} dx = \frac{\pi}{4}$ **[CBSE 2008C]**

31. $\int_0^{\pi} \sin^2 x \cos^3 x\, dx = 0$

32. $\int_0^{\pi} \sin^{2m} x \cos^{2m+1} x\, dx = 0$, where m is a positive integer

33. $\int_0^{\pi/2} (\sin x - \cos x) \log(\sin x + \cos x)\, dx = 0$

34. $\int_0^{\pi/2} \log(\sin 2x)\, dx = -\frac{\pi}{2}(\log 2)$

35. $\int_0^{\pi} x \log(\sin x)\, dx = -\frac{\pi^2}{2}(\log 2)$

36. $\int_0^{\pi} \log(1+\cos x)\, dx = -\pi(\log 2)$

37. $\int_0^{\pi/2} \log(\tan x + \cot x)\, dx = \pi(\log 2)$

38. $\int_{\pi/8}^{3\pi/8} \frac{\cos x}{(\cos x + \sin x)} dx = \frac{\pi}{8}$

39. $\int_{\pi/6}^{\pi/3} \frac{1}{(1+\sqrt{\tan x})} dx = \frac{\pi}{12}$ **[CBSE 2005, '06C, '07, '11, '12C]**

40. $\int_{\pi/4}^{3\pi/4} \frac{dx}{(1+\cos x)} = 2$

41. $\int_{\pi/4}^{3\pi/4} \frac{x}{(1+\sin x)}\, dx = \pi(\sqrt{2}-1)$

42. $\int_{\alpha/4}^{3\alpha/4} \frac{\sqrt{x}}{(\sqrt{a-x}+\sqrt{x})}\, dx = \frac{a}{4}$

43. $\int_{1}^{4} \frac{\sqrt{x}}{(\sqrt{5-x}+\sqrt{x})}\, dx = \frac{3}{2}$

44. $\int_{0}^{\pi/2} x \cot x\, dx = \frac{\pi}{2}(\log 2)$

45. $\int_{0}^{1} \left(\frac{\sin^{-1} x}{x}\right) dx = \frac{\pi}{2}(\log 2)$

46. $\int_{0}^{1} \frac{\log x}{\sqrt{1-x^2}}\, dx = -\frac{\pi}{2}(\log 2)$

47. $\int_{0}^{1} \frac{\log(1+x)}{(1+x^2)}\, dx = \frac{\pi}{8}(\log 2)$

[CBSE 2011C]

48. $\int_{-a}^{a} x^3\sqrt{a^2-x^2}\, dx = 0$

49. $\int_{-\pi}^{\pi} (\sin^{75} x + x^{125})\, dx = 0$

50. $\int_{-\pi}^{\pi} x^{12} \sin^9 x\, dx = 0$

51. $\int_{-1}^{1} e^{|x|}\, dx = 2(e-1)$

52. $\int_{-2}^{2} |x+1|\, dx = 6$

53. $\int_{0}^{8} |x-5|\, dx = 17$

54. $\int_{0}^{2\pi} |\cos x|\, dx = 4.$

55. $\int_{-\pi/4}^{\pi/4} |\sin x|\, dx = (2-\sqrt{2})$

56. Let $f(x) = \begin{cases} 2x+1, & \text{when } 1 \le x \le 2 \\ x^2+1, & \text{when } 2 \le x \le 3. \end{cases}$

Show that $\int_{1}^{3} f(x)\, dx = \frac{34}{3}\cdot$

57. Let $f(x) = \begin{cases} 3x^2+4, & \text{when } 0 \le x \le 2 \\ 9x-2, & \text{when } 2 \le x \le 4. \end{cases}$

Show that $\int_{0}^{4} f(x)\, dx = 66.$

58. Prove that $\int_{0}^{4} \{|x| + |x-2| + |x-4|\, dx\} = 20$ **[CBSE 2013]**

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 16C)

11. $I = \int_{0}^{\pi/2} \frac{dx}{\left(1+\frac{\sin x}{\cos x}\right)} = \int_{0}^{\pi/2} \frac{\cos x}{(\sin x + \cos x)}\, dx.$

14. $I = \int_{0}^{\pi/2} \frac{dx}{\left(1+\frac{\cos^3 x}{\sin^3 x}\right)} = \int_{0}^{\pi/2} \frac{\sin^3 x}{(\sin^3 x + \cos^3 x)}\, dx.$

17. $I = \int_0^{\pi/2} \frac{\sqrt{\frac{\sin x}{\cos x}}}{\left(1+\sqrt{\frac{\sin x}{\cos x}}\right)} dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x}+\sqrt{\cos x})} dx.$

19. $I = \int_0^1 (1-x)[1-(1-x)]^5 dx = \int_0^1 (1-x)x^5 dx = \int_0^1 (x^5 - x^6)\, dx.$

20. $I = \int_0^2 (2-x)[2-(2-x)]^{1/2} dx = \int_0^2 (2-x)\sqrt{x}\, dx = \int_0^2 (2\sqrt{x} - x^{3/2})\, dx.$

21. $I = \int_0^{\pi} x\cos^2 x\, dx$ and $I = \int_0^{\pi} (\pi - x)\cos^2(\pi - x)\, dx = \int_0^{\pi} (\pi - x)\cos^2 x\, dx.$

$\therefore\ 2I = \pi\int_0^{\pi} \cos^2 x\, dx = \frac{\pi}{2}\cdot\int_0^{\pi}(1+\cos 2x)\, dx \Rightarrow I = \frac{\pi^2}{4}\cdot$

22. $I = \int_0^{\pi} x\sin^2 x\, dx$ and $I = \int_0^{\pi} (\pi - x)\sin^2(\pi - x)\, dx = \int_0^{\pi} (\pi - x)\sin^2 x\, dx.$

$\therefore\ 2I = \pi\cdot\int_0^{\pi} \sin^2 x\, dx = \frac{\pi}{2}\cdot\int_0^{\pi}(1-\cos 2x)\, dx.$

24. $I = \int_0^{\pi} \frac{x\sin x}{(1+\cos^2 x)} dx$ and $I = \int_0^{\pi} \frac{(\pi - x)\sin x}{(1+\cos^2 x)} dx.$

$\therefore\ 2I = \pi\cdot\int_0^{\pi} \frac{\sin x}{(1+\cos^2 x)} dx.$ Now, put $\cos x = t.$

25. $2I = \pi\cdot\int_0^{\pi} \frac{\sin x}{(1+\sin x)} dx = \pi\cdot\int_0^{\pi} 20\left(1 - \frac{1}{1+\sin x}\right) dx$

$= \pi^2 - \pi\cdot\int_0^{\pi} \frac{(1-\sin x)}{(1-\sin^2 x)} dx = \pi^2 - \pi\cdot\int_0^{\pi} \frac{(1-\sin x)}{\cos^2 x} dx$

$= \pi^2 - \pi\cdot\int_0^{\pi} (\sec^2 x - \sec x\tan x)\, dx.$

26. $2I = 2\pi\cdot\int_0^{\pi/2} \frac{dx}{(1+\sin^2 x)}\cdot$

Divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

28. Putting $x = \tan\theta$, we get $I = \int_0^{\pi/2} \frac{\sin\theta}{(\cos\theta + \sin\theta)} d\theta.$

29. Put $x = a\sin\theta$ and $dx = a\cos\theta\, d\theta.$

30. Apply $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx.$

In Q. 31 to Q. 33, apply $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$ to get $I = -I$.

34. Putting $2x = t$, we get $I = \frac{1}{2} \cdot \int_0^{\pi} \log(\sin t)\, dt = \int_0^{\pi/2} \log(\sin t)\, dt.$ [by Thm. 8]

35. $2I = \pi \cdot \int_0^{\pi} \log(\sin x)\, dx = 2\pi \cdot \int_0^{\pi/2} \log(\sin x)\, dx.$ [by Thm. 8]

36. $2I = \int_0^{\pi} \log(1 - \cos^2 x)\, dx = 2\int_0^{\pi} \log(\sin x)\, dx = 4\int_0^{\pi/2} \log(\sin x)\, dx.$ [Thm. 8]

37. $I = \int_0^{\pi/2} \log\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right) dx = \int_0^{\pi/2} \log\left(\frac{1}{\sin x \cos x}\right) dx$

$$= -\left[\int_0^{\pi/2} \log(\sin x)\, dx + \int_0^{\pi/2} \log(\cos x)\, dx\right].$$

In Q. 38 to Q. 43, apply, $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx.$

44. Integrating by parts, we get

$$I = [x \log(\sin x)]_0^{\pi/2} - \int_0^{\pi/2} \log(\sin x)\, dx = -\int_0^{\pi/2} \log(\sin x)\, dx.$$

45. Put $x = \sin\theta$ and integrate by parts.

46. Put $x = \sin\theta$.

47. Put $x = \tan\theta$.

In Q.48 to Q.50, $f(-x) = -f(x) \Rightarrow I = 0.$

51. $I = \int_{-1}^{0} e^{-x} dx + \int_0^1 e^x dx.$

58. $I = \int_0^2 \{|x| + |x-2| + |x-4|\}\, dx + \int_2^4 |x| + |x-2| + |x-4|\}\, dx$

$$= \int_0^2 \{x - (x-2) - (x-4)\}\, dx + \int_2^4 \{x + (x-2) - (x-4)\}\, dx$$

$$= \int_0^2 (x - x + 2 - x + 4)\, dx + \int_2^4 (x + x - 2 - x + 4)\, dx$$

$$= \int_0^2 (-x + 2)\, dx + \int_2^4 (x + 2)\, dx.$$

Definite Integral as the Limit of a Sum

Let $f(x)$ be a continuous real-valued function, defined in the closed interval $[a, b]$. Then we define

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots + f[a + (n-1)h)],$$

where $nh = (b - a)$.

The method of evaluating $\int_a^b f(x)\,dx$ by using the above definition is called *integration from first principles*.

Some Useful Results for Direct Applications

(i) $1+2+3+\ldots+(n-1)=\dfrac{1}{2}n(n-1).$

(ii) $1^2+2^2+3^2+\ldots+(n-1)^2=\dfrac{1}{6}(n-1)n(2n-1).$

(iii) $1^3+2^3+3^3+\ldots+(n-1)^3=\left\{\dfrac{n(n-1)}{2}\right\}^2.$

(iv) $a+ar+ar^2+\ldots+ar^{n-1}=\dfrac{a(r^n-1)}{(r-1)}.$

(v) $\sin a+\sin(a+h)+\sin(a+2h)+\ldots+\sin[a+(n-1)h]$

$$=\frac{\sin\left\{a+\left(\frac{n-1}{2}\right)h\right\}\sin\left(\frac{nh}{2}\right)}{\sin(h/2)}.$$

(vi) $\cos a+\cos(a+h)+\cos(a+2h)+\ldots+\cos[a+(n-1)h]$

$$=\frac{\cos\left\{a+\frac{(n-1)}{2}h\right\}\sin\left(\frac{nh}{2}\right)}{\sin(h/2)}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate the following integrals as limit of sums:*

(i) $\int_0^5 (x+1)\,dx$ *(ii)* $\int_1^3 (2x+3)\,dx$ **[CBSE 2000C]**

SOLUTION (i) Let $f(x)=(x+1)$; $a=0$; $b=5$ and $nh=(5-0)=5$. Then,

$$\int_0^5 (x+1)\,dx=\lim_{h\to 0} h[f(0)+f(0+h)+f(0+2h)+\ldots+f\{0+(n-1)h\}]$$

$$=\lim_{h\to 0} h[1+(h+1)+(2h+1)+\ldots+\{(n-1)h+1\}]$$

$$=\lim_{h\to 0} h[n+\{h+2h+3h+\ldots+(n-1)h\}]$$

$$=\lim_{h\to 0} h[n+\{1+2+3+\ldots+(n-1)\}h]$$

$$=\lim_{h\to 0} h\left[n+\frac{n(n-1)}{2}h\right]=\lim_{h\to 0}\left[nh+\frac{nh(nh-h)}{2}\right]$$

$$= \lim_{h \to 0} \left[5 + \frac{5(5-h)}{2} \right] \quad [\because \; nh = 5]$$

$$= \frac{35}{2} \cdot$$

(ii) Let $f(x) = (2x+3)$; $a = 1$; $b = 3$ and $nh = (3-1) = 2$.

Then, $$\int_1^3 (2x+3)\,dx = \lim_{h \to 0} h[f(1) + f(1+h) + \ldots + f\{1+(n-1)h\}]$$

$$= \lim_{h \to 0} h[5 + (5+2h) + (5+4h) + \ldots + \{5 + 2(n-1)h\}]$$

$[\because \; f(1) = 5, f(1+h) = 5 + 2h, \text{etc.}]$

$$= \lim_{h \to 0} h[5n + \{2h + 4h + \ldots + 2(n-1)h\}]$$

$$= \lim_{h \to 0} h[5n + 2\{1 + 2 + 3 + \ldots + (n-1)\}h]$$

$$= \lim_{h \to 0} h\left[5n + 2 \cdot \frac{n(n-1)}{2} h \right]$$

$$= \lim_{h \to 0} [5nh + nh(nh - h)]$$

$$= \lim_{h \to 0} [10 + 2(2-h)] \quad [\because \; nh = 2]$$

$$= 14.$$

EXAMPLE 2 *Evaluate the following integrals as limit of sums:*

(i) $\int_0^2 (x^2+1)\,dx$ *(ii)* $\int_1^3 (x^2+x)\,dx$ **[CBSE 2006C]**

SOLUTION (i) Let $f(x) = (x^2+1)$; $a = 0$; $b = 2$ and $nh = (2-0) = 2$.

$$\therefore \int_0^2 (x^2+1)\,dx = \lim_{h \to 0} h[f(0) + f(0+h) + \ldots + f\{0+(n-1)h\}]$$

$$= \lim_{h \to 0} h[1 + (h^2+1) + (4h^2+1) + (9h^2+1) + \ldots + \{(n-1)^2h^2 + 1\}]$$

$[\because \; f(0) = 1, f(0+h) = (h^2+1), f(0+2h) = (4h^2+1), \text{etc.}]$

$$= \lim_{h \to 0} h[n + \{1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\}h^2]$$

$$= \lim_{h \to 0} h\left[n + \frac{(n-1)n(2n-1)}{6} \cdot h^2 \right]$$

$$= \lim_{h \to 0} \left[nh + \frac{(nh-h)nh(2nh-h)}{6} \right]$$

$$= \lim_{h \to 0} \left[2 + \frac{(2-h)2(4-h)}{6} \right] \quad [\because \; nh = 2]$$

$$= \frac{14}{3} \cdot$$

(ii) Let $f(x) = (x^2 + x)$; $a = 1$; $b = 3$ and $nh = (3 - 1) = 2$.

$$\therefore \int_1^3 (x^2 + x)\,dx = \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots + f\{1 + (n-1)h\}]$$

$$= \lim_{h\to 0} h[(1^2 + 1) + \{(1+h)^2 + (1+h)\} + \{(1+2h)^2 + (1+2h)\} + \ldots + \{1+(n-1)h\}^2 + \{1 + (n-1)h\}]$$

$$= \lim_{h\to 0} h[2n + h^2\{1^2 + 2^2 + \ldots + (n-1)^2\} + 3h\{1 + 2 + \ldots + (n-1)\}]$$

$$= \lim_{h\to 0} h\left[2n + h^2 \cdot \frac{(n-1)n\,(2n-1)}{6} + 3h \cdot \frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0} \left[2nh + \frac{(nh-h)nh\,(2nh-h)}{6} + \frac{3}{2}\cdot nh(nh-h)\right]$$

$$= \lim_{h\to 0} \left[4 + \frac{(2-h)2\,(4-h)}{6} + \frac{3}{2}\cdot 2(2-h)\right] \quad [\because\ nh = 2]$$

$$= \frac{38}{3}.$$

EXAMPLE 3 *Evaluate* $\int_0^1 (3x^2 + 2x + 1)\,dx$ *as limit of sums.* **[CBSE 2007C]**

SOLUTION Let $f(x) = (3x^2 + 2x + 1)$; $a = 0$, $b = 1$ and $nh = (1 - 0) = 1$.

Then, $\int_0^1 (3x^2 + 2x + 1)\,dx$

$$= \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots + f\{0 + (n-1)h\}]$$

$$= \lim_{h\to 0} h[1 + (3h^2 + 2h + 1) + (3\cdot 2^2h^2 + 2\cdot 2h + 1) + \ldots + \{3\cdot(n-1)^2h^2 + 2\cdot(n-1)h + 1\}]$$

$$= \lim_{h\to 0} h[n + 3h^2\{1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\} + 2h\{1 + 2 + 3 + \ldots + (n-1)\}]$$

$$= \lim_{h\to 0} h\left[n + 3h^2 \cdot \frac{(n-1)n\,(2n-1)}{6} + 2h\cdot \frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0} \left[nh + \frac{1}{2}(nh-h)(nh)(2nh-h) + (nh)(nh-h)\right]$$

$$= \lim_{h\to 0} \left[1 + \frac{1}{2}(1-h)\cdot 1\cdot(2-h) + 1\cdot(1-h)\right] \quad [\because\ nh = 1]$$

$$= 3.$$

EXAMPLE 4 *Evaluate* $\int_{-1}^1 e^x\,dx$ *as limit of sums.*

SOLUTION Let $f(x) = e^x$ and $nh = \{1 - (-1)\} = 2$. Then,

$$\int_{-1}^{1} e^x\,dx = \lim_{h\to 0} h[f(-1) + f(-1+h) + \dots + f\{-1+(n-1)h\}]$$

$$= \lim_{h\to 0} h[e^{-1} + e^{(-1+h)} + e^{(-1+2h)} + \dots + e^{\{-1+(n-1)h\}}]$$

$$[\because\ f(-1) = e^{-1},\ f(-1+h) = e^{(-1+h)}, \text{ etc.}]$$

$$= \lim_{h\to 0} e^{-1} h[1 + e^h + e^{2h} + \dots + e^{(n-1)h}]$$

$$= \frac{1}{e}\cdot \lim_{h\to 0} h\left[\frac{(e^h)^n - 1}{(e^h - 1)}\right] = \frac{1}{e}\cdot \lim_{h\to 0} \frac{h[e^{nh} - 1]}{(e^h - 1)}$$

$$= \frac{1}{e}\cdot \lim_{h\to 0} \frac{h(e^2 - 1)}{(e^h - 1)} \qquad [\because\ nh = 2]$$

$$= \frac{1}{e}\cdot \lim_{h\to 0} \frac{h(e^2 - 1)}{\left\{1 + h + \frac{h^2}{\lfloor 2} + \frac{h^3}{\lfloor 3} + \dots\right\} - 1}$$

$$= \frac{1}{e}\cdot \lim_{h\to 0} \frac{e^2 - 1}{\left(1 + \frac{h}{\lfloor 2} + \frac{h^2}{\lfloor 3} + \dots\right)} = \frac{1}{e}(e^2 - 1) = \left(e - \frac{1}{e}\right).$$

EXAMPLE 5 *Evaluate* $\int_a^b \sin x\,dx$ *from first principles.*

SOLUTION Let $f(x) = \sin x$ and let $nh = (b - a)$. Then,

$$\int_a^b \sin x\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + \dots + f\{a + (n-1)h\}]$$

$$= \lim_{h\to 0} h[\sin a + \sin(a+h) + \dots + \sin\{a + (n-1)h\}]$$

$$= \lim_{h\to 0} h\cdot \frac{\sin\left\{a + \left(\frac{n-1}{2}\right)h\right\} \sin\left(\frac{nh}{2}\right)}{\sin(h/2)}$$

$$= \lim_{h\to 0} \left[2\sin\left\{a + \frac{1}{2}nh - \frac{1}{2}h\right\} \sin\left(\frac{nh}{2}\right) \times \frac{(h/2)}{\sin(h/2)}\right]$$

$$= \lim_{h\to 0} \left[2\sin\left\{a + \frac{1}{2}(b-a) - \frac{1}{2}h\right\} \sin\left(\frac{b-a}{2}\right) \times \frac{(h/2)}{\sin(h/2)}\right]$$

$$[\because\ nh = (b-a)]$$

$$= \lim_{h\to 0} 2\sin\left\{\left(\frac{b+a}{2}\right) - \frac{1}{2}h\right\} \sin\left(\frac{b-a}{2}\right)\cdot \lim_{h\to 0} \frac{(h/2)}{\sin(h/2)}$$

$$= 2\sin\left(\frac{b+a}{2}\right) \sin\left(\frac{b-a}{2}\right)$$

$$= (\cos a - \cos b)\ [\because\ 2\sin A \sin B = \cos(A - B) - \cos(A + B)].$$

EXERCISE 16D

Evaluate each of the following integrals as the limit of sums:

1. $\int_0^2 (x+4)\,dx$
2. $\int_1^2 (3x-2)\,dx$
3. $\int_1^3 x^2 dx$
4. $\int_0^3 (x^2+1)\,dx$ **[CBSE 2001]**
5. $\int_2^5 (3x^2-5)\,dx$ **[CBSE 2009C]**
6. $\int_0^3 (x^2+2x)\,dx$
7. $\int_1^4 (3x^2+2x)\,dx$
8. $\int_1^3 (x^2+5x)\,dx$ **[CBSE 2008C, '10]**
9. $\int_1^3 (2x^2+5x)\,dx$ **[CBSE 2012C]**
10. $\int_0^2 x^3 dx$
11. $\int_2^4 (x^2-3x+2)\,dx$
12. $\int_0^2 (x^2+x)\,dx$ **[CBSE 2005]**
13. $\int_0^3 (2x^2+3x+5)\,dx$ **[CBSE 2007]**
14. $\int_0^1 |3x-1|\,dx$
15. $\int_0^2 e^x dx$
16. $\int_1^3 e^{-x} dx$
17. $\int_a^b \cos x\,dx$

ANSWERS (EXERCISE 16D)

1. 10 **2.** $\frac{5}{2}$ **3.** $\frac{26}{3}$ **4.** 12 **5.** 102 **6.** 18 **7.** 78 **8.** $\frac{86}{3}$
9. $\frac{112}{3}$ **10.** 4 **11.** $\frac{14}{3}$ **12.** $\frac{14}{3}$ **13.** $\frac{93}{2}$ **14.** 1 **15.** (e^2-1)
16. $\frac{(e^2-1)}{e^3}$ **17.** $(\sin b - \sin a)$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 16D)

14. $\int_0^1 |3x-1| = \int_0^{1/3} (1-3x)\,dx + \int_{1/3}^1 (3x-1)\,dx = I_1 + I_2.$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. $\int_{1}^{4} x\sqrt{x}\,dx = ?$

   (a) 12.8 (b) 12.4 (c) 7 (d) none of these

2. $\int_{0}^{2} \sqrt{6x+4}\,dx = ?$

   (a) $\frac{64}{9}$ (b) 7 (c) $\frac{56}{9}$ (d) $\frac{60}{9}$

3. $\int_{0}^{1} \frac{dx}{\sqrt{5x+3}} = ?$

   (a) $\frac{2}{5}(\sqrt{8}-\sqrt{3})$ (b) $\frac{2}{5}(\sqrt{8}+\sqrt{3})$ (c) $\frac{2}{5}\sqrt{8}$ (d) none of these

4. $\int_{0}^{1} \frac{1}{(1+x^2)}\,dx = ?$

   (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) none of these

5. $\int_{0}^{2} \frac{dx}{\sqrt{4-x^2}} = ?$

   (a) 1 (b) $\sin^{-1}\frac{1}{2}$ (c) $\frac{\pi}{4}$ (d) none of these

6. $\int_{\sqrt{3}}^{\sqrt{8}} x\sqrt{1+x^2}\,dx = ?$

   (a) $\frac{19}{3}$ (b) $\frac{19}{6}$ (c) $\frac{38}{3}$ (d) $\frac{9}{4}$

7. $\int_{0}^{1} \frac{x^3}{(1+x^8)}\,dx = ?$

   (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{16}$

8. $\int_{1}^{e} \frac{(\log x)^2}{x}\,dx = ?$

   (a) $\frac{1}{3}$ (b) $\frac{1}{3}e^3$ (c) $\frac{1}{3}(e^3-1)$ (d) none of these

9. $\int_{\pi/4}^{\pi/2} \cot x \, dx = ?$

(a) $\log 2$ (b) $2\log 2$ (c) $\frac{1}{2}\log 2$ (d) none of these

10. $\int_{0}^{\pi/4} \tan^2 x \, dx = ?$

(a) $\left(1-\frac{\pi}{4}\right)$ (b) $\left(1+\frac{\pi}{4}\right)$ (c) $\left(1-\frac{\pi}{2}\right)$ (d) $\left(1+\frac{\pi}{2}\right)$

11. $\int_{0}^{\pi/2} \cos^2 x \, dx = ?$

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{\pi}{4}$ (d) 1

12. $\int_{\pi/3}^{\pi/2} \operatorname{cosec} x \, dx = ?$

(a) $\frac{1}{2}\log 2$ (b) $\frac{1}{2}\log 3$ (c) $-\log 2$ (d) none of these

13. $\int_{0}^{\pi/2} \cos^3 x \, dx = ?$

(a) 1 (b) $\frac{3}{4}$ (c) $\frac{2}{3}$ (d) none of these

14. $\int_{0}^{\pi/4} \frac{e^{\tan x}}{\cos^2 x} dx = ?$

(a) $(e-1)$ (b) $(e+1)$ (c) $\left(\frac{1}{e}+1\right)$ (d) $\left(\frac{1}{e}-1\right)$

15. $\int_{0}^{\pi/2} \frac{\cos x}{(1+\sin^2 x)} dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) π (d) none of these

16. $\int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{x^2} dx = ?$

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{3}{2}$ (d) none of these

17. $\int_{0}^{\pi} \frac{dx}{(1+\sin x)} = ?$

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 0

18. $\int_0^{\pi/2} (\sqrt{\sin x}\cos x)^3 dx = ?$

(a) $\frac{2}{9}$ (b) $\frac{2}{15}$ (c) $\frac{8}{45}$ (d) $\frac{5}{2}$

19. $\int_0^1 \frac{xe^x}{(1+x)^2} dx = ?$

(a) $\left(\frac{e}{2}-1\right)$ (b) $(e-1)$ (c) $e(e-1)$ (d) none of these

20. $\int_0^{\pi/2} e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx = ?$

(a) 0 (b) $\frac{\pi}{4}$ (c) $e^{\pi/2}$ (d) $(e^{\pi/2}-1)$

21. $\int_0^{\pi/4} \sqrt{1+\sin 2x}\, dx = ?$

(a) 0 (b) 1 (c) 2 (d) $\sqrt{2}$

22. $\int_0^{\pi/2} \sqrt{1+\cos 2x}\, dx = ?$

(a) $\sqrt{2}$ (b) $\frac{3}{2}$ (c) $\sqrt{3}$ (d) 2

23. $\int_0^1 \frac{(1-x)}{(1+x)} dx = ?$

(a) $\frac{1}{2}\log 2$ (b) $(2\log 2+1)$ (c) $(2\log 2-1)$ (d) $\left(\frac{1}{2}\log 2-1\right)$

24. $\int_0^{\pi/2} \sin^2 x\, dx = ?$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

25. $\int_0^{\pi/6} \cos x \cos 2x\, dx = ?$

(a) $\frac{1}{4}$ (b) $\frac{5}{12}$ (c) $\frac{1}{3}$ (d) $\frac{7}{12}$

26. $\int_0^{\pi/2} \sin x \sin 2x\, dx = ?$

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{5}{6}$ (d) $\frac{3}{5}$

27. $\int_0^{\pi} (\sin 2x \cos 3x)\, dx = ?$

(a) $\frac{4}{5}$ (b) $-\frac{4}{5}$ (c) $\frac{5}{12}$ (d) $-\frac{12}{5}$

28. $\int_0^1 \frac{dx}{(e^x + e^{-x})} = ?$

(a) $\left(1 - \frac{\pi}{4}\right)$ (b) $\tan^{-1} e$ (c) $\tan^{-1} e + \frac{\pi}{4}$ (d) $\tan^{-1} e - \frac{\pi}{4}$

29. $\int_0^9 \frac{dx}{(1 + \sqrt{x})} = ?$

(a) $(3 - 2\log 2)$ (b) $(3 + 2\log 2)$ (c) $(6 - 2\log 4)$ (d) $(6 + 2\log 4)$

30. $\int_0^{\pi/2} x \cos x\, dx = ?$

(a) $\frac{\pi}{2}$ (b) $\left(\frac{\pi}{2} - 1\right)$ (c) $\left(\frac{\pi}{2} + 1\right)$ (d) none of these

31. $\int_0^1 \frac{dx}{(1 + x + x^2)} = ?$

(a) $\frac{\pi}{\sqrt{3}}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{3\sqrt{3}}$ (d) none of these

32. $\int_0^1 \sqrt{\frac{1 - x}{1 + x}}\, dx = ?$

(a) $\frac{\pi}{2}$ (b) $\left(\frac{\pi}{2} - 1\right)$ (c) $\left(\frac{\pi}{2} + 1\right)$ (d) none of these

33. $\int_0^1 \frac{(1 - x)}{(1 + x)}\, dx = ?$

(a) $(\log 2 + 1)$ (b) $(\log 2 - 1)$ (c) $(2\log 2 - 1)$ (d) $(2\log 2 + 1)$

34. $\int_{-a}^{a} \sqrt{\frac{a - x}{a + x}}\, dx = ?$

(a) $a\pi$ (b) $\frac{a\pi}{2}$ (c) $2a\pi$ (d) none of these

35. $\int_0^{\sqrt{2}} \sqrt{2 - x^2}\, dx = ?$

(a) π (b) 2π (c) $\frac{\pi}{2}$ (d) none of these

36. $\int_{-2}^{2} |x|\, dx = ?$

(a) 4 (b) 3.5 (c) 2 (d) 0

37. $\int_{0}^{1} |2x - 1|\, dx = ?$

(a) 2 (b) $\frac{1}{2}$ (c) 1 (d) 0

38. $\int_{-2}^{1} |2x + 1|\, dx = ?$

(a) $\frac{5}{2}$ (b) $\frac{7}{2}$ (c) $\frac{9}{2}$ (d) 4

39. $\int_{-2}^{1} \frac{|x|}{x}\, dx = ?$

(a) 3 (b) 2.5 (c) 1.5 (d) none of these

40. $\int_{-a}^{a} x|x|\, dx = ?$

(a) 0 (b) $2a$ (c) $\frac{2a^3}{3}$ (d) none of these

41. $\int_{0}^{\pi} |\cos x|\, dx = ?$

(a) 2 (b) $\frac{3}{2}$ (c) 1 (d) 0

42. $\int_{0}^{2\pi} |\sin x|\, dx = ?$

(a) 2 (b) 4 (c) 1 (d) none of these

43. $\int_{0}^{\pi/2} \frac{\sin x}{(\sin x + \cos x)}\, dx = ?$

(a) π (b) $\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{4}$

44. $\int_{0}^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\cos x} + \sqrt{\sin x})}\, dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) π (d) 0

45. $\int_{0}^{\pi/2} \frac{\sin^4 x}{(\sin^4 x + \cos^4 x)}\, dx = ?$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) 1 (d) 0

46. $\int_0^{\pi/2} \frac{\cos^{1/4} x}{(\sin^{1/4} x + \cos^{1/4} x)} dx = ?$

(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) none of these

47. $\int_0^{\pi/2} \frac{\sin^n x}{(\sin^n x + \cos^n x)} dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) 1 (d) 0

48. $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{(\sqrt{\cot x} + \sqrt{\tan x})} dx = ?$

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) none of these

49. $\int_0^{\pi/2} \frac{\sqrt[3]{\tan x}}{(\sqrt[3]{\tan x} + \sqrt[3]{\cot x})} dx = ?$

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

50. $\int_0^{\pi/2} \frac{1}{(1 + \tan x)} dx = ?$

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) π

51. $\int_0^{\pi/2} \frac{1}{(1 + \sqrt{\cot x})} dx = ?$

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

52. $\int_0^{\pi/2} \frac{1}{(1 + \tan^3 x)} dx = ?$

(a) $\frac{\pi}{4}$ (b) 0 (c) $\frac{\pi}{2}$ (d) none of these

53. $\int_0^{\pi/2} \frac{\sec^5 x}{(\sec^5 x + \operatorname{cosec}^5 x)} dx = ?$

(a) $\frac{\pi}{2}$ (b) 0 (c) $\frac{\pi}{4}$ (d) π

54. $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{(1+\sqrt{\cot x})}\, dx = ?$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) 0 (d) 1

55. $\int_0^{\pi/2} \frac{\tan x}{(1+\tan x)}\, dx = ?$

(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) π

56. $\int_{-\pi}^{\pi} x^4 \sin x\, dx = ?$

(a) 2π (b) π (c) 0 (d) none of these

57. $\int_{-\pi}^{\pi} x^3 \cos^3 x\, dx = ?$

(a) π (b) $\frac{\pi}{4}$ (c) 2π (d) 0

58. $\int_{-\pi}^{\pi} \sin^5 x\, dx = ?$

(a) $\frac{3\pi}{4}$ (b) 2π (c) $\frac{5\pi}{16}$ (d) 0

59. $\int_{-1}^{-2} x^3(1-x^2)\, dx = ?$

(a) $-\frac{40}{3}$ (b) $\frac{40}{3}$ (c) $\frac{5}{6}$ (d) 0

60. $\int_{-a}^{a} \log\left(\frac{a-x}{a+x}\right) dx = ?$

(a) $2a$ (b) a (c) 0 (d) 1

61. $\int_{-\pi}^{\pi} (\sin^{61} x + x^{123})\, dx = ?$

(a) 2π (b) 0 (c) $\frac{\pi}{2}$ (d) 125π

62. $\int_{-\pi}^{\pi} \tan x\, dx = ?$

(a) 2 (b) $\frac{1}{2}$ (c) -2 (d) 0

63. $\int_{-1}^{1} \log (x + \sqrt{x^2 + 1})\, dx = ?$

(a) $\log \frac{1}{2}$ (b) $\log 2$ (c) $\frac{1}{2} \log 2$ (d) 0

64. $\int_{-\pi/2}^{\pi/2} \cos x\, dx = ?$

(a) 0 (b) 2 (c) –1 (d) none of these

65. $\int_{0}^{a} \frac{\sqrt{x}}{(\sqrt{x} + \sqrt{a - x})}\, dx = ?$

(a) $\frac{a}{2}$ (b) $2a$ (c) $\frac{2a}{3}$ (d) $\frac{\sqrt{a}}{2}$

66. $\int_{0}^{\pi/4} \log (1 + \tan x)\, dx = ?$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{4} \log 2$ (c) $\frac{\pi}{8} \log 2$ (d) 0

67. $\int_{-a}^{a} f(x)\, dx = ?$

(a) $2\int_{0}^{a} \{f(x) + f(-x)\}\, dx$ (b) $2\int_{0}^{a} \{f(x) - f(-x)\}\, dx$

(c) $\int_{0}^{a} \{f(x) + f(-x)\}\, dx$ (d) none of these

68. Let $[x]$ denote the greatest integer less than or equal to x.

Then, $\int_{0}^{1.5} [x]\, dx = ?$

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) 2 (d) 3

69. Let $[x]$ denote the greatest integer less than or equal to x.

Then, $\int_{-1}^{1} [x]\, dx = ?$

(a) –1 (b) 0 (c) $\frac{1}{2}$ (d) 2

70. $\int_{1}^{2} |x^2 - 3x + 2|\, dx = ?$

(a) $\frac{-1}{6}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

71. $\int_{\pi}^{2\pi} |\sin x|\, dx = ?$

(a) 0 (b) 1 (c) 2 (d) none of these

72. $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx = ?$

(a) $\frac{1}{2}(\pi - \log 2)$ (b) $\left(\frac{\pi}{2} - 2\log 2\right)$

(c) $\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)$ (d) none of these

73. $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = ?$

(a) $\frac{1}{2}(\pi - \log 2)$ (b) $\left(\frac{\pi}{2} - \log 2\right)$ (c) $(\pi - 2\log 2)$ (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (b)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (d)	**6.** (a)	**7.** (d)	**8** (a)	**9.** (c)	**10.** (a)
11. (c)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (b)	**16.** (a)	**17.** (c)	**18.** (c)	**19.** (a)	**20.** (c)
21. (b)	**22.** (a)	**23.** (c)	**24.** (b)	**25.** (b)	**26.** (a)	**27.** (b)	**28.** (d)	**29.** (c)	**30.** (b)
31. (c)	**32.** (b)	**33.** (c)	**34.** (a)	**35.** (c)	**36.** (a)	**37.** (b)	**38.** (c)	**39.** (d)	**40.** (a)
41. (a)	**42.** (b)	**43.** (d)	**44.** (b)	**45.** (a)	**46.** (c)	**47.** (b)	**48.** (c)	**49.** (b)	**50.** (c)
51. (b)	**52.** (a)	**53.** (c)	**54.** (a)	**55.** (c)	**56.** (c)	**57.** (d)	**58.** (d)	**59.** (d)	**60.** (c)
61. (b)	**62.** (d)	**63.** (d)	**64.** (b)	**65.** (a)	**66.** (c)	**67.** (c)	**68.** (a)	**69.** (a)	**70.** (b)
71. (c)	**72.** (c)	**73.** (b)							

HINTS TO THE GIVEN OBJECTIVE QUESTIONS

1. $I = \int_1^4 x^{3/2} dx = \left[\frac{2}{5} x^{5/2}\right]_1^4 = \frac{62}{5} = 12.4.$

2. $I = \int_0^2 (6x+4)^{1/2} dx = \left[\frac{2}{3} \cdot \frac{(6x+4)^{3/2}}{6}\right]_0^2 = \frac{56}{9}.$

3. $I = \int_0^1 (5x+3)^{-1/2} dx = \left[2 \cdot \frac{(5x+3)^{1/2}}{5}\right]_0^1 = \frac{2}{5}(\sqrt{8} - \sqrt{3}).$

4. $I = \int_0^1 \frac{dx}{(1+x^2)} = [\tan^{-1} x]_0^1 = (\tan^{-1} 1 - \tan^{-1} 0) = \frac{\pi}{4}.$

5. $I = \int_0^2 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^2 = (\sin^{-1}1 - \sin^{-1}0) = \frac{\pi}{2}\cdot$

6. Put $(1+x^2) = t$ and $2x\,dx = dt$.

$[x = \sqrt{3} \Rightarrow t = 4]$ and $[x = \sqrt{8} \Rightarrow t = 9]$

$\therefore \quad I = \int_4^9 \sqrt{t}\,dt = \left[\frac{1}{2}\times\frac{2}{3}t^{3/2}\right]_4^9 = \frac{1}{3}(27-8) = \frac{19}{3}\cdot$

7. Put $x^4 = t$ and $4x^3 dx = dt$.

$[x = 0 \Rightarrow t = 0]$ and $[x = 1 \Rightarrow t = 1]$

$\therefore \quad I = \frac{1}{4}\int_0^1 \frac{dt}{(1+t^2)} = \frac{1}{4}[\tan^{-1}t]_0^1 = \frac{1}{4}[\tan^{-1}1 - \tan^{-1}0] = \left(\frac{1}{4}\times\frac{\pi}{4}\right) = \frac{\pi}{16}\cdot$

8. Put $\log x = t$ and $\frac{1}{x}dx = dt$.

$[x = 1 \Rightarrow t = 0]$ and $[x = e \Rightarrow t = \log e = 1]$

$\therefore \quad I = \int_0^1 t^2 dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}\cdot$

9. $I = \int_{\pi/4}^{\pi/2} \cot x\,dx = [\log \sin x]_{\pi/4}^{\pi/2} = \left[\log 1 - \log\frac{1}{\sqrt{2}}\right] = \frac{1}{2}\log 2.$

10. $I = \int_0^{\pi/2} (\sec^2 x - 1)\,dx = [\tan x - x]_0^{\pi/2} = \left(1 - \frac{\pi}{4}\right)\cdot$

11. $I = \frac{1}{2}\int_0^{\pi/2} 2\cos^2 x\,dx = \frac{1}{2}\int_0^{\pi/2}(1+\cos 2x) = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{4}\cdot$

12. $I = \int_{\pi/3}^{\pi/2} \operatorname{cosec} x\,dx = \left[\log\tan\frac{x}{2}\right]_{\pi/3}^{\pi/2} = \left(\log 1 - \log\frac{1}{\sqrt{3}}\right) = \frac{1}{2}\log 3.$

13. $I = \int_0^{\pi/2} \cos^2 x\cdot\cos x\,dx = \int_0^{\pi/2}(1-\sin^2 x)\cos x\,dx = \int_0^1 (1-t^2)\,dt$, where $\sin x = t$.

14. $I = \int_0^{\pi/4} e^{\tan x}\sec^2 x\,dx = [e^t dt]_0^1$, where $\tan x = t$.

15. Put $\sin x = t$ and $\cos x\,dx = dt$.

$[x = 0 \Rightarrow t = 0]$ and $\left[x = \frac{\pi}{2} \Rightarrow t = 1\right]\cdot$

$\therefore \quad I = \int_0^1 \frac{dt}{(1+t^2)} = [\tan^{-1}t]_0^1 = \frac{\pi}{4}\cdot$

16. Put $\frac{1}{x} = t$ and $\frac{1}{x^2}dx = -dt$.

$\left[x = \frac{1}{\pi} \Rightarrow t = \pi\right]$ and $\left[x = \frac{2}{\pi} \Rightarrow t = \frac{\pi}{2}\right]$

$$\therefore \quad I = -\int_{\pi}^{\pi/2} \sin t\, dt = \int_{\pi/2}^{\pi} \sin t\, dt = [-\cos t]_{\pi/2}^{\pi} = 1.$$

17. $$I = \int_0^{\pi} \frac{(1-\sin x)}{(1-\sin^2 x)}\, dx = \int_0^{\pi} \frac{(1-\sin x)}{\cos^2 x}\, dx$$

$$= \int_0^{\pi} [\sec^2 x - \sec x \tan x]\, dx = [\tan x - \sec x]_0^{\pi} = 2.$$

18. $$I = \int_0^{\pi/2} (\sin x)^{3/2} \cos^2 x \cos x\, dx = \int_0^1 t^{3/2}(1-t^2)\, dt, \text{ where } \sin x = t$$

$$= \int_0^1 (t^{3/2} - t^{7/2})\, dt = \left[\frac{2}{5}t^{5/2} - \frac{2}{9}t^{9/2}\right]_0^1 = \left(\frac{2}{5} - \frac{2}{9}\right) = \frac{8}{45}.$$

19. $$I = \int_0^1 e^x \left\{\frac{(1+x)-1}{(1+x)^2}\right\} dx = \int_0^1 e^x \left\{\frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right\} dx$$

$$= \int_0^1 e^x \{f(x) + f'(x)\}\, dx, \quad \text{where } f(x) = \frac{1}{(1+x)}$$

$$= [e^x f(x)]_0^1 = \left[e^x \cdot \frac{1}{(1+x)}\right]_0^1 = \left(\frac{e}{2} - 1\right).$$

20. $$I = \int_0^{\pi/2} e^x \left\{\frac{1}{(1+\cos x)} + \frac{\sin x}{(1+\cos x)}\right\} dx$$

$$= \int_0^{\pi/2} e^x \left\{\frac{1}{2\cos^2(x/2)} + \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right\} dx$$

$$= \int_0^{\pi/2} e^x \left\{\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right\} dx = \int_0^{\pi/2} e^x \{f(x) + f'(x)\}\, dx$$

$$= \left[e^x \tan\frac{x}{2}\right]_0^{\pi/2} = e^{\pi/2}.$$

21. $$I = \int_0^{\pi/4} \sqrt{\cos^2 x + \sin^2 x + 2\sin x \cos x}\, dx = \int_0^{\pi/4} (\cos x + \sin x)\, dx$$

$$= [\sin x - \cos x]_0^{\pi/4} = 1.$$

22. $$I = \int_0^{\pi/2} \sqrt{2\cos^2 x}\, dx = \sqrt{2}\int_0^{\pi/2} \cos x\, dx = \sqrt{2}\,[\sin x]_0^{\pi/2} = \sqrt{2}.$$

23. On dividing $(-x+1)$ by $(x+1)$ we get:

$$I = \int_0^1 \left\{-1 + \frac{2}{(x+1)}\right\} dx = [-x + 2\log(x+1)]_0^1 = (2\log 2 - 1).$$

24. $I = \frac{1}{2} \cdot \int_0^{\pi/2} (1 - \cos 2x)\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{4}.$

25. $I = \frac{1}{2}\int_0^{\pi/6} [\cos 3x + \cos x]\, dx = \frac{1}{2}\left[\frac{\sin 3x}{3} + \sin x\right]_0^{\pi/6} = \frac{15}{12}.$

26. $I = \frac{1}{2}\int_0^{\pi/2} [\cos(2x - x) - \cos(2x + x)]\, dx = \frac{1}{2}\int_0^{\pi/2} (\cos x - \cos 3x)\, dx$

$= \frac{1}{2}\left[\sin x - \frac{\sin 3x}{3}\right]_0^{\pi/2} = \frac{1}{2}\left(1 + \frac{1}{3}\right) = \frac{2}{3}.$

27. $2\cos A \sin B = \sin(A + B) - \sin(A - B)$

$\therefore \quad I = \frac{1}{2}\int_0^{\pi} (\sin 5x - \sin x)\, dx = \frac{1}{2} \cdot \left[\frac{-\cos 5x}{5} + \cos x\right]_0^{\pi} = \frac{-4}{5}.$

28. $I = \int_0^1 \frac{dx}{\left(e^x + \frac{1}{e^x}\right)} = \int_0^1 \frac{e^x dx}{(e^{2x} + 1)} = \int_1^e \frac{dt}{(t^2 + 1)}$, where $e^x = t$.

$= [\tan^{-1} t]_1^e = \left(\tan^{-1} e - \frac{\pi}{4}\right).$

29. Put $x = t^2$ and $dx = 2t\, dt$.

$[x = 0 \Rightarrow t = 0]$ and $[x = 9 \Rightarrow t^2 = 9 \Rightarrow t = 3]$

$\therefore \quad I = \int_0^3 \frac{2t}{(1 + t)} dt = \int_0^3 \left\{2 - \frac{2}{(1 + t)}\right\} dt = [2t - 2\log(1 + t)]_0^3 = (6 - 2\log 4).$

30. Integrating by parts, we get:

$I = \int_0^{\pi/2} x \cos x\, dx = [x \sin x]_0^{\pi/2} - \int_0^{\pi/2} (1 \cdot \sin x)\, dx = \frac{\pi}{2} + [\cos x]_0^{\pi/2} - \left(\frac{\pi}{2} - 1\right).$

31. $I = \int_0^1 \frac{dx}{\left\{\left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4}\right\}} = \int_0^1 \frac{dx}{\left\{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right\}} = \left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{\left(x + \frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}\right]_0^1 = \frac{\pi}{3\sqrt{3}}.$

32. $I = \int_0^1 \left\{\frac{\sqrt{1 - x}}{\sqrt{1 + x}} \times \frac{\sqrt{1 - x}}{\sqrt{1 - x}}\right\} dx = \int_0^1 \frac{(1 - x)}{\sqrt{1 - x^2}} dx.$

$= \int_0^1 \frac{dx}{\sqrt{1 - x^2}} + \frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{1 - x^2}} dx = [\sin^{-1} x]_0^1 + \frac{1}{2}\int_1^0 \frac{1}{\sqrt{t}} dt$, where $(1 - x^2) = t$

$= [\sin^{-1} 1 - \sin^{-1} 0] - \frac{1}{2} \cdot \int_0^1 \frac{1}{\sqrt{t}} dt = \left(\frac{\pi}{2} - 0\right) - \frac{1}{2} \cdot \left[\frac{t^{1/2}}{(1/2)}\right]_0^1 = \left(\frac{\pi}{2} - 1\right).$

33. $I = \int_0^1 \left(-1 + \frac{2}{x + 1}\right) dx = [-x + 2\log|x + 1|]_0^1 = (2\log 2 - 1).$

34. Put $x = a\cos\theta$ and $dx = -a\sin\theta\, d\theta$.

$[x = -a \Rightarrow \theta = \pi]$ and $[x = a \Rightarrow \theta = 0]$.

$$\therefore \quad I = \int_{\pi}^{0} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \cdot (-a\sin\theta)\, d\theta = a\int_{0}^{\pi} \left\{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}\right\}^{1/2} 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) d\theta$$

$$= a\int_{0}^{\pi} 2\sin^2\frac{\theta}{2}\, d\theta = a\int_{0}^{\pi}(1-\cos\theta)\, d\theta = a\pi.$$

35. Put $x = \sqrt{2}\sin t$ and $dx = \sqrt{2}\cos t\, dt$.

$[x = 0 \Rightarrow t = 0]$ and $\left[x = \sqrt{2} \Rightarrow t = \frac{\pi}{2}\right]$

$$\therefore \quad I = \int_{0}^{\pi/2} \sqrt{2-2\sin^2 t} \cdot \sqrt{2}\cos t\, dt = 2\int_{0}^{\pi/2} \cos^2 t\, dt = 2\int_{0}^{\pi/2} \frac{1}{2}(1+\cos 2t)\, dt$$

$$= \left[t + \frac{\sin 2t}{2}\right]_{0}^{\pi/2} = \frac{\pi}{2}.$$

36. $I = \int_{-2}^{0} |x|\, dx + \int_{0}^{2} |x|\, dx = \int_{-2}^{0} -x\, dx + \int_{0}^{2} x\, dx.$

$$= \left[-\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2} = 0-(-2)+(2-0) = 4.$$

37. Put $2x-1 = t$ and $dx = \frac{1}{2}dt$.

$[x = 0 \Rightarrow t = -1]$ and $[x = 1 \Rightarrow t = 1]$

$$\therefore \quad I = \frac{1}{2}\int_{-1}^{1} |t|\, dt = \frac{1}{2}\left\{\int_{-1}^{0} -t\, dt + \int_{0}^{1} t\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{-t^2}{2}\right]_{-1}^{0} + \left[\frac{t^2}{2}\right]_{0}^{1}\right\} = \frac{1}{2}\left\{\left(0+\frac{1}{2}\right)+\left(\frac{1}{2}-0\right)\right\} = \frac{1}{2}.$$

38. Put $2x+1 = t$ and $dx = \frac{1}{2}dt$

$[x = -2 \Rightarrow t = -3]$ and $[x = 1 \Rightarrow t = 3]$

$$\therefore \quad I = \frac{1}{2}\int_{-3}^{3} |t|\, dt = \frac{1}{2}\left\{\int_{-3}^{0} (-t)\, dt + \int_{0}^{3} t\, dt\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{-t^2}{2}\right]_{-3}^{0} + \left[\frac{t^2}{2}\right]_{0}^{3}\right\} = \frac{1}{2}\left[\left\{0-\left(\frac{-9}{2}\right)\right\}+\left(\frac{9}{2}-0\right)\right] = \frac{9}{2}.$$

39. $I = \int_{-2}^{0} \frac{-x}{x}\, dx + \int_{0}^{1} \frac{x}{x}\, dx = \int_{-2}^{0} -dx + \int_{0}^{1} dx = [-x]_{-2}^{0} + [x]_{0}^{1} = (-2+1) = -1.$

40. $I = \int_{-a}^{0} x(-x)\, dx + \int_{0}^{a} x \cdot x\, dx = \int_{-a}^{0} -x^2 dx + \int_{0}^{a} x^2 dx$

$$= \left[\frac{-x^3}{3}\right]_{-a}^{0} + \left[\frac{x^3}{3}\right]_{0}^{a} = +\frac{1}{3}(-a)^3 + \frac{a^3}{3} = 0.$$

41. $I = \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x\, dx.$

$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = 1 - (0 - 1) = 2.$

42. $I = \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} -\sin x\, dx.$

$= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi} = (2 + 2) = 4.$

43. $I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)} dx$ ... (i)

$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\left\{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)\right\}} dx = \int_0^{\pi/2} \frac{\cos x}{(\sin x + \cos x)} dx$ ... (ii)

$\therefore \quad 2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$

Q. 44 to 49: Similarly, each of the questions from 44 to 49 may be attempted.

50. $I = \int_0^{\pi/2} \frac{\cos x}{(\cos x + \sin x)} dx.$

51. $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x} + \sqrt{\cos x})} dx.$

52. $I = \int_0^{\pi/2} \frac{\cos^3 x}{(\cos^3 x + \sin^3 x)} dx.$

54. $I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\sin x} + \sqrt{\cos x})} dx.$

55. $I = \int_0^{\pi/2} \frac{\sin x}{(\cos x + \sin x)} dx.$

56. $f(x) = x^4 \sin x.$

$\therefore \quad f(-x) = (-x)^4 \sin(-x) = -(x^4 \sin x) = -f(x).$

$\therefore \quad f(x)$ is odd and hence $I = 0$.

57. $f(x) = x^3 \cos^3 x.$

$\therefore \quad f(-x) = (-x)^3[\cos(-x)]^3 = -x^3(\cos x)^3 = -(x^3 \cos^3 x) = -f(x)$

$\therefore \quad f(x)$ is odd and hence $I = 0$.

58. $f(x) = \sin^5 x.$

$f(-x) = [\sin(-x)]^5 = (-\sin x)^5 = -(\sin^5 x) = -f(x).$

$\therefore \quad f(x)$ is odd and hence $I = 0$.

59. $f(x) = x^3(1 - x^2) = (x^3 - x^5).$

$f(-x) = (-x)^3 - (-x)^5 = -x^3 - (-x^5) = -x^3 + x^5 = -x^3(1 - x^2) = -f(x).$

$\therefore \quad f(x)$ is odd and hence $I = 0$.

60. $f(x) = \log\left(\frac{a-x}{a+x}\right)$.

$$\Rightarrow \quad f(-x) = \log\left(\frac{a+x}{a-x}\right) = \log\left\{\left(\frac{a-x}{a+x}\right)^{-1}\right\} = -\log\left(\frac{a-x}{a+x}\right) = -f(x)$$

$\therefore$ $f(x)$ is odd and hence $I = 0$.

61. $f(x) = \sin^{61} x + x^{123}$

$$\Rightarrow \quad f(-x) = [\sin(-x)]^{61} + (-x)^{123} = (-\sin x)^{61} - x^{123} = -\sin^{61} x - x^{123}$$

$$= -[\sin^{61} x + x^{123}] = -f(x).$$

$\therefore$ $f(x)$ is odd and hence $I = 0$.

62. $f(x) = \tan x \Rightarrow f(-x) = \tan(-x) = -\tan x = -f(x)$.

$\therefore$ $f(x)$ is odd and hence $I = 0$.

63. Let $f(x) = \log(x + \sqrt{x^2+1})$. Then,

$$f(-x) = \log(-x + \sqrt{x^2+1})$$

$$= \log\left\{\frac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{(\sqrt{x^2+1} + x)}\right\} = \log\left\{\frac{1}{(\sqrt{x^2+1} + x)}\right\}$$

$$= \log(x + \sqrt{x^2+1})^{-1} = -\log(x + \sqrt{x^2+1}) = -f(x)$$

$\therefore$ $f(x)$ is odd and hence $I = 0$.

64. $f(x) = \cos x \Rightarrow f(-x) = \cos(-x) = \cos x = f(x)$.

$\therefore$ $f(x)$ is an even function.

$$\therefore \quad I = 2\int_0^{\pi/2} \cos x\, dx = 2[\sin x]_0^{\pi/2} = (2 \times 1) = 2.$$

65. $I = \int_0^a \frac{\sqrt{x}}{(\sqrt{x} + \sqrt{a-x})} dx$ and $I = \int_0^{-a} \frac{\sqrt{a-x}}{\{\sqrt{a-x} + \sqrt{a-(a-x)}\}} dx$

$$\therefore \quad 2I = \int_0^a \frac{(\sqrt{x} + \sqrt{a-x})}{(\sqrt{x} + \sqrt{a-x})} dx = \int_0^a dx = [x]_0^a = a \Rightarrow I = \frac{a}{2}.$$

66. $I = \int_0^{\pi/4} \log(1 + \tan x)\, dx$ ... (i)

$$I = \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right] dx = \int_0^{\pi/4} \log\left\{1 + \frac{1 - \tan x}{1 + \tan x}\right\} dx$$

$$= \int_0^{\pi/4} \{\log 2 - \log(1 + \tan x)\}\, dx = \int_0^{\pi/4} (\log 2)\, dx - I$$

$$\Rightarrow \quad 2I = (\log 2)[x]_0^{\pi/4} = \frac{\pi}{4}(\log 2) \Rightarrow I = \frac{\pi}{8}(\log 2).$$

67. We have: $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(x) + f(-x)\}\,dx.$

68. $\{0 \le x < 1 \Rightarrow [x] = 0\}$ and $\{1 \le x < 1.5\} \Rightarrow [x] = 1$

$$\therefore \quad I = \int_{0}^{1} [x]\,dx + \int_{1}^{1.5} [x]\,dx = \int_{0}^{1} 0\,dx + \int_{0}^{1.5} 1 \cdot dx$$

$$= 0 + [x]_1^{1.5} = (1.5 - 1) = 0.5 = \frac{1}{2}.$$

69. $\{-1 \le x < 0 \Rightarrow [x] = -1\}$ and $\{0 \le x < 1 \Rightarrow [x] = 0\}$

$$\therefore \quad I = \int_{-1}^{0} [x]\,dx + \int_{0}^{1} [x]\,dx = \int_{-1}^{0} (-1)\,dx + \int_{0}^{1} 0 \cdot dx$$

$$= [-x]_{-1}^{0} = 0 - \{-(-1)\} = -1.$$

70. $(x^2 - 3x + 2) = (x-1)(x-2)$

$1 \le x < 2 \Rightarrow (x^2 - 3x + 2) \le 0$

$$\Rightarrow |x^2 - 3x + 2| = -(x^2 - 3x + 2)$$

$$\therefore \quad I = \int_{1}^{2} -(x^2 - 3x + 1)\,dx = \int_{1}^{2} (-x^2 + 3x - 2)\,dx$$

$$= \left[\frac{-x^3}{3} + \frac{3x^2}{2} - 2x\right]_1^2 = \left(\frac{-2}{3} + \frac{5}{6}\right) = \frac{1}{6}.$$

71. $\pi \le x < 2\pi \Rightarrow \sin x \le 0 \Rightarrow |\sin x| = -\sin x$

$$\therefore \quad I = \int_{\pi}^{2\pi} -\sin x\,dx = [\cos x]_{\pi}^{2\pi} = 2.$$

72. Put $x = \sin t$ and $dx = \cos t\,dt$

$[x = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0]$ and $\left[x = \frac{1}{\sqrt{2}} \Rightarrow \sin t = \frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}\right]$

$$\therefore \quad I = \int_{0}^{\pi/4} \frac{t\cos t}{\cos^3 t}\,dt = \int_{0}^{\pi/4} t\sec^2 t = [t\tan t]_0^{\pi/4} - \int_{0}^{\pi/4} \tan t\,dt$$

$$= \frac{\pi}{4} + [\log\cos t]_0^{\pi/4} = \frac{\pi}{4} + \log\frac{1}{\sqrt{2}} = \left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)$$

73. Put $x = \tan t$ and $dx = \sec^2 t\,dt$.

$[x = 0 \Rightarrow t = 0]$ and $\left[x = 1 \Rightarrow \tan t = 1 \Rightarrow t = \frac{\pi}{4}\right]$

$$\therefore \quad I = \int_{0}^{\pi/4} \sin^{-1}(\sin 2t)\sec^2 t\,dt = 2\int_{0}^{\pi/4} t\sec^2 t\,dt$$

$$= 2\left[[t\tan t]_0^{\pi/4} - \int_{0}^{\pi/4} \tan t\,dt\right] = 2\left\{\frac{\pi}{4} + [\log\cos t]_0^{\pi/4}\right\}$$

$$= \frac{\pi}{2} + 2\log\frac{1}{\sqrt{2}} = \left(\frac{\pi}{2} - \log 2\right).$$

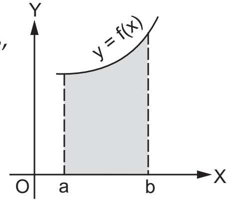
17. AREA OF BOUNDED REGIONS

1. AREA OF A CURVE BETWEEN TWO ORDINATES Let $y = f(x)$ be a continuous and finite function in $[a, b]$.

Case I *When the curve $y = f(x)$ lies above the x-axis*

The area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ is given by

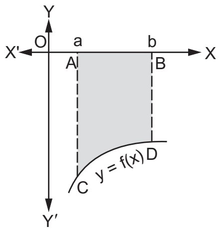
$$area = \int_a^b y\ dx.$$



Case II *When the curve $y = f(x)$ lies below the x-axis*

The area between the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ is given by

$$area = \int_a^b (-y)\ dx.$$

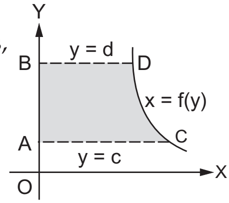


2. AREA OF A CURVE BETWEEN TWO ABSCISSAE

Case I *When the curve $x = f(y)$ lies to the right of the y-axis*

The area bounded by the curve $x = f(y)$, the y-axis, and the abscissae $y = c$ and $y = d$ is given by

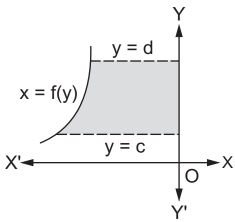
$$area = \int_c^d x\,dy.$$



Case II *When the curve $x = f(y)$ lies to the left of the y-axis*

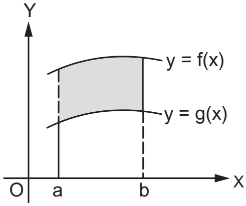
The area bounded by the curve $x = f(y)$, the y-axis, and the abscissae $y = c$ and $y = d$ is given by

$$area = \int_c^d (-x)\,dy.$$



3. AREA BETWEEN TWO CURVES The area bounded by two curves $y = f(x)$ and $y = g(x)$, which are intersected by the ordinates $x = a$ and $x = b$, is given by

$$area = \int_a^b \{f(x) - g(x)\}\, dx.$$

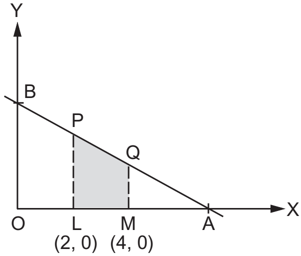


SOLVED EXAMPLES

EXAMPLE 1 *Using integration, find the area of the region bounded by the line $2y + x = 8$, the x-axis and the lines $x = 2$ and $x = 4$.* **[CBSE 2002C]**

SOLUTION The given line AB is

$$2y + x = 8 \Rightarrow y = 4 - \frac{1}{2}x. \quad \dots \text{(i)}$$



Required area = area $PLMQP$

$=$ area between the line $y = 4 - \frac{1}{2}x$,

and the x-axis between $x = 2$ and $x = 4$

$$= \int_2^4 y_{AB}\, dx = \int_2^4 \left(4 - \frac{1}{2}x\right) dx$$

$$= \left[4x - \frac{1}{4}x^2\right]_2^4 = (12 - 7) \text{ sq units}$$

$= 5$ sq units.

Hence, the required area is 5 sq units.

EXAMPLE 2 *Using integration, find the area of $\triangle ABC$ whose vertices are $A(2, 3)$, $B(4, 7)$ and $C(6, 2)$.* **[CBSE 2001]**

SOLUTION The equation of AB is

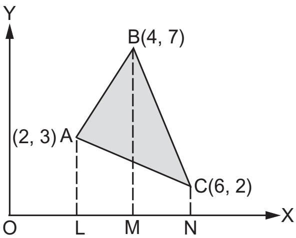
$$\frac{y-3}{x-2} = \frac{7-3}{4-2} \Rightarrow y = 2x - 1.$$

The equation of BC is

$$\frac{y-7}{x-4} = \frac{2-7}{6-4} \Rightarrow y = \frac{-5}{2}x + 17.$$

The equation of AC is

$$\frac{y-3}{x-2} = \frac{2-3}{6-2} \Rightarrow y = \frac{-1}{4}x + \frac{7}{2}\cdot$$



Draw $AL \perp OX$, $BM \perp OX$ and $CN \perp OX$.

Area of $\triangle ABC$ = (area $ALMBA$ + area $BMNCB$) − (area $ALNCA$)

$$= \int_2^4 y_{AB}\,dx + \int_4^6 y_{BC}\,dx - \int_2^6 y_{AC}\,dx$$

$$= \int_2^4 (2x-1)\,dx + \int_4^6 \left(-\frac{5}{2}x + 17\right)dx - \int_2^6 \left(-\frac{1}{4}x + \frac{7}{2}\right)dx$$

$$= [x^2 - x]_2^4 + \left[-\frac{5}{4}x^2 + 17x\right]_4^6 - \left[-\frac{1}{8}x^2 + \frac{7}{2}x\right]_2^6$$

$$= \left[(12-2) + (57-48) - \left(\frac{33}{2} - \frac{13}{2}\right)\right] \text{ sq units}$$

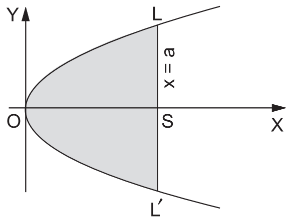
$$= 9 \text{ sq units.}$$

Hence, the required area is 9 sq units.

EXAMPLE 3 *Calculate the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.*

SOLUTION Let $S(a, 0)$ be the focus of the parabola $y^2 = 4ax$. Then, its latus rectum LSL' is the line parallel to the y-axis at a distance a from it. So, its equation is $x = a$.

Since the equation of the parabola contains only even powers of y, it is symmetrical about the x-axis.



$\therefore$ required area

$= \text{area } LOL'L$

$= (\text{area } LOSL) + (\text{area } SOL'S)$

$= 2 \times (\text{area } LOSL)$

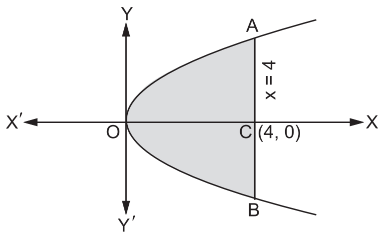
$$= 2\int_0^a y\,dx = 2 \cdot \int_0^a 2\sqrt{ax}\,dx$$

$$= 4\sqrt{a} \cdot \frac{2}{3}[x^{3/2}]_0^a = \frac{8}{3}a^2 \text{ sq units.}$$

EXAMPLE 4 *Using integration, find the area of the region bounded by the parabola $y^2 = 16x$ and the line $x = 4$.* **[CBSE 1996, '97]**

SOLUTION $y^2 = 16x$ is a right-handed parabola with its vertex at the origin.

And, $x = 4$ is the line parallel to the y-axis at a distance of 4 units from it.



Also, $y^2 = 16x$ contains only even powers of y.

So, it is symmetrical about the x-axis.

$\therefore$ required area = area $AOCA$ + area $BOCB$

$$= 2(\text{area } AOCA)$$

$$= 2\int_0^4 y\,dx = 2\int_0^4 \sqrt{16x}\,dx$$

$$= 8\int_0^4 \sqrt{x}\,dx = 8 \times \frac{2}{3} \times [x^{3/2}]_0^4$$

$$= \frac{16}{3} \times (4)^{3/2} = \left(\frac{16}{3} \times 8\right) = \frac{128}{3} \text{ sq units.}$$

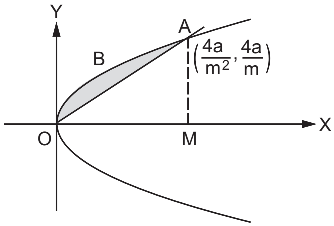
Hence, the required area is $\frac{128}{3}$ sq units.

EXAMPLE 5 *Using integration, find the area enclosed by the parabola $y^2 = 4ax$ and the chord $y = mx$.*

SOLUTION The given equations are

$$y^2 = 4ax \quad \text{... (i)}$$

and $$y = mx \quad \text{... (ii)}$$



Clearly, $y^2 = 4ax$ is a right-handed parabola, passing through the origin.

And $y = mx$ is a line passing through the origin.

In order to find the points of intersection of the given parabola and the given line, we solve (i) and (ii) simultaneously.

Putting $y = mx$ from (ii) into (i), we get

$$m^2x^2 = 4ax \Rightarrow x(m^2x - 4a) = 0$$

$$\Rightarrow x = 0 \text{ or } x = \frac{4a}{m^2}.$$

Now, $(x = 0 \Rightarrow y = 0)$ and $\left(x = \frac{4a}{m^2} \Rightarrow y = \frac{4a}{m}\right)$.

So, the points of intersection of the given parabola and the chord are

$$O(0, 0) \text{ and } A\left(\frac{4a}{m^2}, \frac{4a}{m}\right).$$

Draw $AM \perp OX$.

Required area = (area $OBAMO$) − (area $OAMO$)

$$= \int_0^{4a/m^2} (y \text{ for the parabola})\, dx - \int_0^{4a/m^2} (y \text{ for the line})\, dx$$

$$= \int_0^{4a/m^2} 2\sqrt{ax}\, dx - \int_0^{4a/m^2} mx\, dx$$

$$= 2\sqrt{a} \cdot \frac{2}{3} [x^{3/2}]_0^{4a/m^2} - \left[\frac{mx^2}{2}\right]_0^{4a/m^2}$$

$$= \left[\frac{4\sqrt{a}}{3} \cdot \frac{8}{m^3} a^{3/2} - \frac{m}{2} \cdot \frac{16a^2}{m^4}\right]$$

$$= \left(\frac{32a^2}{3m^3} - \frac{8a^2}{m^3}\right) = \left(\frac{8a^2}{3m^3}\right) \text{ sq units.}$$

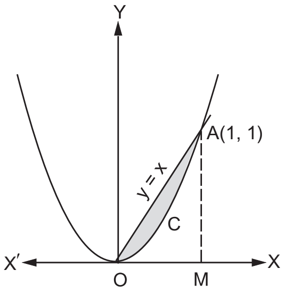
Hence, the required area is $\left(\frac{8a^2}{3m^3}\right)$ sq units.

EXAMPLE 6 *Find the area of the region* $\{(x, y) : x^2 \le y \le x\}$. **[CBSE 2007, '13C]**

SOLUTION Consider the equations

$$y = x^2 \quad \ldots \text{(i)}$$

$$\text{and } y = x \quad \ldots \text{(ii)}$$



Clearly, $y = x^2$ is an upward parabola and $y = x$ is a line passing through (0, 0).

Solving (i) and (ii) simultaneously, we get

$$x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 1.$$

From (ii), $(x = 0 \Rightarrow y = 0)$ and $(x = 1 \Rightarrow y = 1)$.

So, the points of intersection of (i) and (ii) are $O(0, 0)$ and $A(1, 1)$.

Draw $AM \perp OX$.

Required area = shaded area

$$= (\text{area } OMAO) - (\text{area } OMACO)$$

$$= \int_0^1 (y \text{ for line})\, dx - \int_0^1 (y \text{ for parabola})\, dx$$

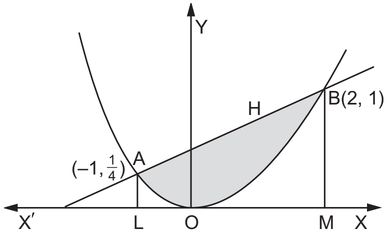
$$= \int_0^1 x\, dx - \int_0^1 x^2 dx = \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 = \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{6} \text{ sq unit.}$$

Hence, the required area is $\frac{1}{6}$ sq unit.

EXAMPLE 7 *Find the area of the region bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.* **[CBSE 2010, '13]**

SOLUTION The given curve is $x^2 = 4y$ ... (i)

The given line is $x = 4y - 2$ ... (ii)



Putting $4y = (x+2)$ from (ii) into (i), we get

$$x^2 = (x+2) \Leftrightarrow (x^2 - x - 2) = 0$$
$$\Leftrightarrow (x-2)(x+1) = 0$$
$$\Leftrightarrow x = 2 \text{ or } x = -1.$$

Putting $x = 2$ in (i), we get $y = 1$.

Putting $x = -1$ in (i), we get $y = \frac{1}{4}\cdot$

Thus, the points of intersection of the given curve (i) and the line (ii) are $A\left(-1, \frac{1}{4}\right)$ and $B(2, 1)$.

Draw AL and BM as perpendiculars on the x-axis.

$\therefore$ required area = area $AOBA$

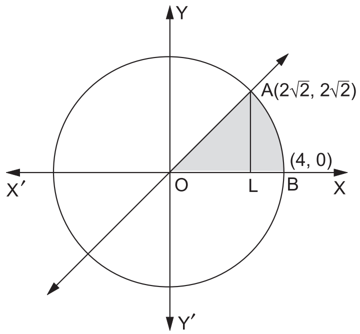
$$= (\text{area } ALMBA) - (\text{area } AOBMLA)$$
$$= \int_{-1}^{2} \left(\frac{x+2}{4}\right) dx - \int_{-1}^{2} \frac{x^2}{4} dx$$
$$= \int_{-1}^{2} \left\{\frac{(x+2)}{4} - \frac{x^2}{4}\right\} dx = \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$
$$= \frac{1}{4}\left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right] = \frac{9}{8} \text{ sq units.}$$

Hence, the required area is $\frac{9}{8}$ sq units.

EXAMPLE 8 *Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $y = x$ in the first quadrant.* **[CBSE 2004]**

SOLUTION The given circle is $x^2 + y^2 = 16$ ... (i)

The given line is $y = x$ ... (ii)



Putting $y = x$ from (ii) into (i), we get

$2x^2 = 16 \Leftrightarrow x^2 = 8 \Leftrightarrow x = 2\sqrt{2}$ [$\because$ x is + ve in the first quad.].

Thus, the point of intersection of (i) and (ii) in the first quadrant is $A(2\sqrt{2}, 2\sqrt{2})$.

Draw AL perpendicular on the x-axis.

$\therefore$ required area = (area OLA) + area($LBAL$)

$$= \int_0^{2\sqrt{2}} x\,dx + \int_{2\sqrt{2}}^{4} \sqrt{16 - x^2}\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^{2\sqrt{2}} + \left[\frac{x\sqrt{16 - x^2}}{2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_{2\sqrt{2}}^{4}$$

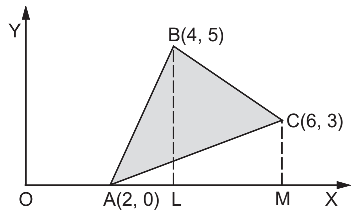
$$= \frac{1}{2}[(2\sqrt{2})^2 - 0] + \left[(0 + 8\sin^{-1}1) - (4 + 8\sin^{-1}\frac{1}{\sqrt{2}}\right]$$

$$= \left[4 + \left(8 \times \frac{\pi}{2}\right) - 4 - \left(8 \times \frac{\pi}{4}\right)\right] = (2\pi) \text{ sq units.}$$

EXAMPLE 9 *Using integration, find the area of* $\triangle ABC$, *whose vertices are* $A(2, 0)$, $B(4, 5)$ *and* $C(6, 3)$. **[CBSE 2003C, '06C]**

SOLUTION The equation of side AB is

$$\frac{y - 0}{x - 2} = \frac{(5 - 0)}{(4 - 2)} \Rightarrow y = \frac{5}{2}(x - 2) \quad \ldots \text{(i)}$$



The equation of side BC is

$$\frac{y-5}{x-4}=\frac{(3-5)}{(6-4)} \Rightarrow y=-x+9 \quad \text{... (ii)}$$

The equation of side AC is

$$\frac{y-0}{x-2}=\frac{(3-0)}{(6-2)} \Rightarrow y=\frac{3}{4}(x-2) \quad \text{... (iii)}$$

Draw perpendiculars BL and CM on the x-axis.

$\therefore$ area of $\triangle ABC$

$$= \text{ar}(\triangle ALB) + \text{ar}(\text{trap. } BLMC) - \text{ar}(\triangle AMC)$$

$$= \int_2^4 y_{AB}\,dx + \int_4^6 y_{BC}\,dx - \int_2^6 y_{AC}\,dx$$

$$= \frac{5}{2}\int_2^4 (x-2)\,dx + \int_4^6 (9-x)\,dx - \frac{3}{4}\int_2^6 (x-2)\,dx$$

$$= \frac{5}{2}\left[\frac{x^2}{2}-2x\right]_2^4 + \left[9x-\frac{x^2}{2}\right]_4^6 - \frac{3}{4}\cdot\left[\frac{x^2}{2}-2x\right]_2^6$$

$$= \frac{5}{2}[0-(-2)] + (36-28) - \frac{3}{4}[6-(-2)]$$

$$= (5+8-6) \text{ sq units} = 7 \text{ sq units.}$$

EXAMPLE 10 *Find the area cut off from the parabola* $4y=3x^2$ *by the straight line* $3x-2y+12=0$. **[CBSE 2007, '09]**

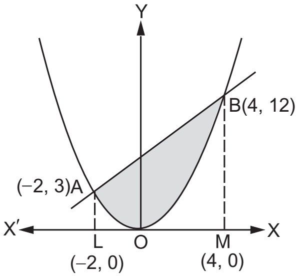
SOLUTION Clearly, $4y=3x^2$ is an upward parabola with its vertex at (0, 0).

And, $3x-2y+12=0$ is a line.

The given equations are

$$4y=3x^2 \quad \text{... (i)}$$

$$\text{and } 3x-2y+12=0 \quad \text{... (ii)}$$



The points of intersection of the given parabola and the given line will be obtained by solving (i) and (ii) simultaneously.

Putting $y=\frac{3}{4}x^2$ from (i) in (ii),

we get

$$3x-\frac{3}{2}x^2+12=0 \Rightarrow x^2-2x-8=0$$

$$\Rightarrow x^2-4x+2x-8=0$$

$$\Rightarrow x(x-4)+2(x-4)=0$$

$$\Rightarrow (x-4)(x+2)=0$$

$$\Rightarrow x=-2 \text{ or } x=4.$$

Now, ($x = -2 \Rightarrow y = 3$) and ($x = 4 \Rightarrow y = 12$).

So, the points of intersection of (i) and (ii) are $A(-2, 3)$ and $B(4, 12)$.

Draw $AL \perp OX'$ and $BM \perp OX$.

Required area = (area $ALMBA$) − (area $ALOMBOA$)

$$= \int_{-2}^{4} (y \text{ of the line})\, dx - \int_{-2}^{4} (y \text{ of the parabola})\, dx$$

$$= \int_{-2}^{4} \frac{(3x+12)}{2}\, dx - \int_{-2}^{4} \frac{3x^2}{4}\, dx = \left[\frac{3x^2}{4} + 6x\right]_{-2}^{4} - \left[\frac{x^3}{4}\right]_{-2}^{4}$$

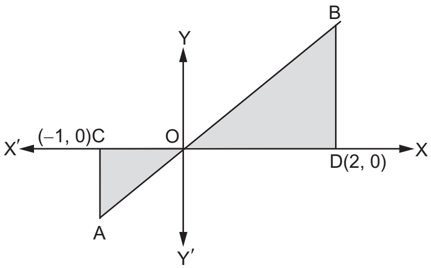
$$= (45 - 18) = 27 \text{ sq units.}$$

Hence, the required area is 27 sq units.

EXAMPLE 11 *Find the area bounded by the line $y = x$, the x-axis and the ordinates $x = -1$, $x = 2$.*

SOLUTION We know that $y = x$ is the line passing through the origin and making an angle of $45°$ with the x-axis, as shown in the given figure.

Now, we have to find the area of the shaded region.



Required area

$$= (\text{area } ODBO) + (\text{area } OACO)$$

$$= \int_{0}^{2} y\, dx + \int_{-1}^{0} (-y)\, dx \quad [\because \text{ area } OACO \text{ is below the } x\text{-axis}]$$

$$= \int_{0}^{2} x\, dx + \int_{-1}^{0} (-x)\, dx$$

$$= \left[\frac{x^2}{2}\right]_{0}^{2} + \left[\frac{-x^2}{2}\right]_{-1}^{0} = \left[2 + \frac{1}{2}\right] = \frac{5}{2} \text{ sq units.}$$

Hence, the required area is $\frac{5}{2}$ sq units.

EXAMPLE 12 *Find by integration, the area of the region bounded by the curve $y = 2x - x^2$ and the x-axis.*

SOLUTION The given curve is $y = 2x - x^2$. … (i)

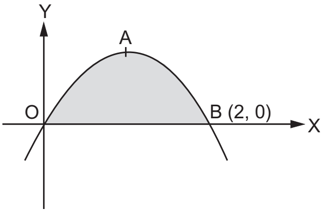
Now, $y = 2x - x^2 \Rightarrow (x^2 - 2x + 1) = (-y + 1)$

$$\Rightarrow (x-1)^2 = -1(y-1)$$

$$\Rightarrow X^2 = -Y,$$

where $(x-1) = X$ and $(y-1) = Y$.

Clearly, $X^2 = -Y$ is a downward parabola with its vertex at $(X = 0, Y = 0)$.



Now,

$$X = 0, Y = 0 \Rightarrow x - 1 = 0 \text{ and } y - 1 = 0$$

$$\Rightarrow x = 1 \text{ and } y = 1.$$

Thus, the vertex of the parabola is $A(1, 1)$.

Also, $y = 0 \Rightarrow 2x - x^2 = 0 \Rightarrow x(2-x) = 0 \Rightarrow x = 0$ or $x = 2$.

Thus, the curve cuts the x-axis at $O(0, 0)$ and $B(2, 0)$.

The rough sketch of the curve can now be drawn, as shown in the given figure.

$\therefore$ required area $= \int_0^2 y\,dx$

$$= \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2$$

$$= \left(4 - \frac{8}{3}\right) = \frac{4}{3} \text{ sq units.}$$

Hence, the required area is $\frac{4}{3}$ sq units.

EXAMPLE 13 *Using integration, find the area of the region bounded by the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

SOLUTION The given equation contains only even powers of y.

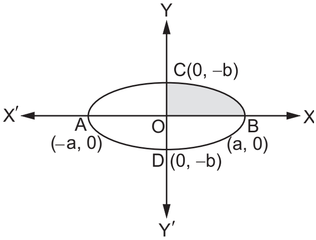
So, the curve is symmetrical about the x-axis.

Also, the given equation contains only even powers of x.

So, the curve is symmetrical about the y-axis.

A rough sketch of the ellipse can be drawn, as shown in the figure.

Now, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=\frac{b}{a}\cdot\sqrt{a^2-x^2}$.



Area of the given ellipse

$= 4\times(\text{area } OBCO)$

$= 4\times\int_0^a y\,dx = 4\times\int_0^a \frac{b}{a}\cdot\sqrt{a^2-x^2}\,dx$

$= \frac{4b}{a}\cdot\int_0^{\pi/2} a^2\cos^2\theta\,d\theta$ [putting $x=a\sin\theta$ so that $dx=a\cos\theta\,d\theta$]

$= (4ab)\cdot\int_0^{\pi/2}\frac{(1+\cos 2\theta)}{2}\,d\theta = (2ab)\cdot\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

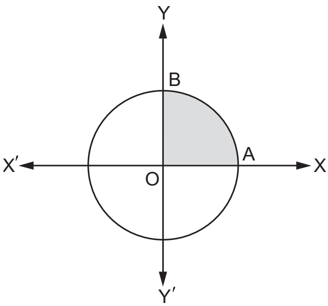
$= (\pi ab)$ sq units.

Hence, the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is (πab) sq units.

EXAMPLE 14 *By using integration, prove that the area of a circle of radius r units is πr^2 square units.*

SOLUTION The equation of a circle of radius r units with its centre at the origin is

$$x^2+y^2=r^2 \qquad \dots \text{(i)}$$



This equation contains only even powers of y.

So, the curve is symmetrical about the x-axis.

Also, the above equation contains only even powers of x.

So, the curve is symmetrical about the y-axis.

Now, $x^2+y^2=r^2 \Rightarrow y=\sqrt{r^2-x^2}$.

Area of the circle $= 4\times(\text{area } OABO)$

$= 4\times\int_0^r y\,dx = 4\times\int_0^r \sqrt{r^2-x^2}\,dx$

$= 4\times\int_0^{\pi/2} r^2\cos^2\theta\,d\theta$ [putting $x=r\sin\theta$]

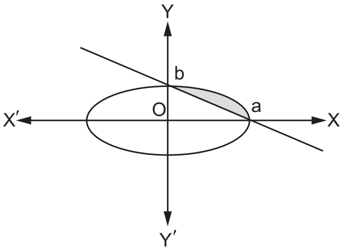
$= 4r^2\cdot\int_0^{\pi/2}\frac{(1+\cos 2\theta)}{2}\,d\theta = 2r^2\cdot\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

$= (\pi r^2)$ sq units.

Hence, the required area is (πr^2) sq units.

EXAMPLE 15 *Find the area of the smaller region bounded by the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *and the straight line* $\frac{x}{a}+\frac{y}{b}=1.$ **[CBSE 2004, '05C]**

SOLUTION The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$ can be drawn, as shown in the given figure.



Then, we have to find the area of the shaded region.

Required area

$= \{\text{area between } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and the } x\text{-axis from } x=0 \text{ to } x=a\}$

$- \{\text{area between the line } \frac{x}{a}+\frac{y}{b}=1 \text{ and the } x\text{-axis from } x=0 \text{ to } x=a\}$

$$= \int_0^a (y \text{ of the ellipse})\, dx - \int_0^a (y \text{ of the line})\, dx$$

$$= \int_0^a \frac{b}{a}\cdot\sqrt{a^2-x^2}\, dx - \int_0^a \frac{b(a-x)}{a}\, dx$$

$$= \frac{b}{a}\cdot\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a - \frac{b}{a}\cdot\left[ax-\frac{x^2}{2}\right]_0^a$$

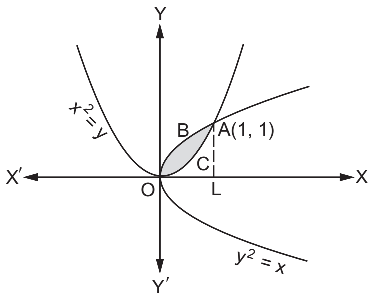
$$= \frac{ab}{2}(\sin^{-1}1-\sin^{-1}0)-\left(ab-\frac{ab}{2}\right)=\left(\frac{\pi ab}{4}-\frac{ab}{2}\right) \text{ sq units.}$$

Hence, the required area is $\left(\frac{\pi ab}{4}-\frac{ab}{2}\right)$ sq units.

EXAMPLE 16 *Find the area of the region bounded by the parabolas* $x^2=y$ *and* $y^2=x$. **[CBSE 2012C]**

SOLUTION Clearly, $x^2=y$ is an upward parabola with its vertex at (0, 0) and $y^2=x$ is a right-handed parabola with its vertex also at (0, 0).

The shaded region shows the area bounded by these parabolas.



The given equations are

$$x^2 = y \quad \ldots \text{(i)}$$

$$\text{and} \quad y^2 = x \quad \ldots \text{(ii)}$$

Using (i) in (ii), we get

$$x^4 = x \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0 \text{ or } x = 1.$$

Also, $(x = 0 \Rightarrow y = 0)$ and $(x = 1 \Rightarrow y = 1)$.

So, the given curves intersect at $O(0, 0)$ and $A(1, 1)$.

Draw $AL \perp OX$.

Required area = (area $OLABO$) − (area $OLACO$)

= {area bounded by $y^2 = x$ from $x = 0$ to $x = 1$}

− {area bounded by $x^2 = y$ from $x = 0$ to $x = 1$}

$$= \int_0^1 \sqrt{x}\, dx - \int_0^1 x^2\, dx$$

$$= \left[\frac{2}{3} x^{3/2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) = \frac{1}{3} \text{ sq unit.}$$

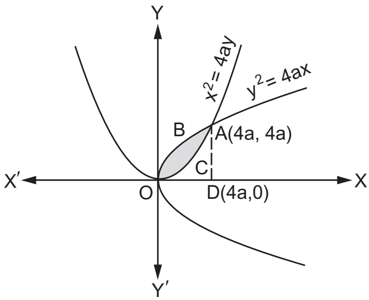
Hence, the required area is $\frac{1}{3}$ sq unit.

EXAMPLE 17 *Find the area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$.* **[CBSE 2003, '04, '08, '09, '12C]**

SOLUTION The given parabolas are

$$y^2 = 4ax \quad \ldots \text{(i)}$$

$$\text{and } x^2 = 4ay \quad \ldots \text{(ii)}$$



In order to find the points of intersection of the given curves, we solve (i) and (ii) simultaneously.

Putting $x = \frac{y^2}{4a}$ from (i) in (ii),

we get

$$\frac{y^4}{16a^2} = 4ay \Rightarrow y^4 - 64a^3 y = 0$$

$$\Rightarrow y(y^3 - 64a^3) = 0$$

$$\Rightarrow y = 0 \text{ or } y = 4a.$$

Now, $(y = 0 \Rightarrow x = 0)$ and $\left(y = 4a \Rightarrow x = \frac{16a^2}{4a} = 4a\right)$.

Thus, the points of intersection of the two parabolas are $O(0, 0)$ and $A(4a, 4a)$.

Draw $AD \perp OX$. Then, point D is $(4a, 0)$.

Required area = area $OCABO$

$$= (\text{area } OBADO) - (\text{area } OCADO)$$

$$= \int_0^{4a} y\,dx \text{ for } (y^2 = 4ax) - \int_0^{4a} y\,dx \text{ for } (x^2 = 4ay)$$

$$= \int_0^{4a} 2\sqrt{ax}\,dx - \int_0^{4a} \frac{x^2}{4a}\,dx$$

$$= \left[2\sqrt{a}\cdot\frac{2}{3}\cdot x^{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

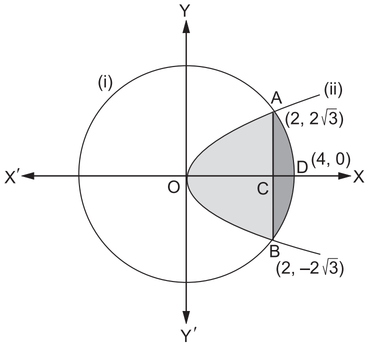
$$= \left[\frac{4\sqrt{a}}{3}\cdot(4a)^{3/2} - \frac{1}{12a}\times 64a^3\right]$$

$$= \left(\frac{32a^2}{3} - \frac{16a^2}{3}\right) = \left(\frac{16a^2}{3}\right) \text{ sq units.}$$

Hence, the required area is $\left(\frac{16a^2}{3}\right)$ sq units.

EXAMPLE 18 *Using integration, find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.* **[CBSE 2012C]**

SOLUTION $x^2 + y^2 = 16$ is a circle with its centre at (0, 0) and radius = 4 units. And, $y^2 = 6x$ is a right-handed parabola with its vertex at (0, 0).



Now, we have to find the area of the shaded region.

The given equations are

$$x^2 + y^2 = 16 \quad \ldots \text{(i)}$$

and $$y^2 = 6x \quad \ldots \text{(ii)}$$

Using (ii) in (i), we get

$$x^2 + 6x - 16 = 0 \Rightarrow (x+8)(x-2) = 0$$

$$\Rightarrow x = -8 \text{ or } x = 2.$$

Now, $x = -8 \Rightarrow y^2 = -48$

$\Rightarrow y$ is imaginary.

And, $x = 2 \Rightarrow y^2 = (6 \times 2) = 12$

$\Rightarrow y = \pm 2\sqrt{3}$.

Thus, the points of intersection of the given curves are $A(2, 2\sqrt{3})$ and $B(2, -2\sqrt{3})$.

Since each of the given equations contains only even powers of y, each one is symmetrical about the x-axis.

$\therefore$ required area = 2(area $OCDAO$)

$$= 2(\text{area } OCAO + \text{area } CDAC)$$

$$= 2\left[\int_0^2 y\,dx \text{ for curve (ii)} + \int_2^4 y\,dx \text{ for curve (i)}\right]$$

$$= 2\left[\int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16-x^2}\,dx\right]$$

$$= 2\left\{\left[\frac{2\sqrt{6}}{3}x^{3/2}\right]_0^2 + \left[x\sqrt{\frac{16-x^2}{2}} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4\right\}$$

$$= \left\{\left(\frac{2\sqrt{6}}{3}\cdot 2^{3/2} - 0\right) + 8\sin^{-1}1 - \left(2\sqrt{3} + 8\sin^{-1}\frac{1}{2}\right)\right\}$$

$$= 2\left(\frac{8\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3}\right) = 2\left(\frac{2\sqrt{3}}{3} + \frac{8\pi}{3}\right)$$

$$= \frac{4}{3}(\sqrt{3} + 4\pi) \text{ sq units.}$$

Hence, the required area is $\frac{4}{3}(\sqrt{3} + 4\pi)$ sq units.

EXAMPLE 19 *Using integration, find the area of the region enclosed between the circles* $x^2 + y^2 = 4$ *and* $(x-2)^2 + y^2 = 4$. **[CBSE 2009, '13C]**

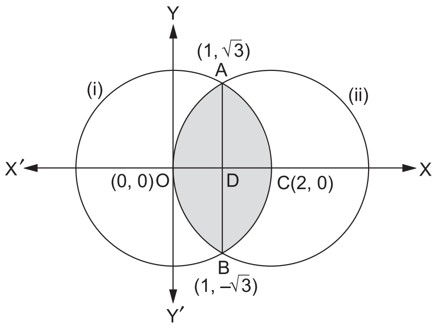
SOLUTION $x^2 + y^2 = 4$ is a circle with its centre at $O(0, 0)$ and radius = 2 units.

And, $(x-2)^2 + y^2 = 4$ is a circle with its centre at $C(2, 0)$ and radius = 2 units.

The given circles are

$$x^2 + y^2 = 4 \qquad \text{... (i)}$$

and $(x-2)^2 + y^2 = 4$ ... (ii)



Eliminating y from (i) and (ii), we get

$$4-x^2=4-(x-2)^2 \Rightarrow 4x=4 \Rightarrow x=1.$$

Putting $x=1$ in (i), we get $y^2=3 \Rightarrow y=\pm\sqrt{3}$.

Thus, the points of intersection of the two circles are $A(1, \sqrt{3})$ and $B(1, -\sqrt{3})$.

Both the circles are symmetrical about the x-axis.

Required area = 2(area $AOCA$)

$$= 2(\text{area } AODA + \text{area } CADC)$$

$$= 2\int_0^1 y\,dx \text{ for circle (ii)} + 2\int_1^2 y\,dx \text{ for circle (i)}$$

$$= 2\int_0^1 \sqrt{4-(x-2)^2}\,dx + 2\int_1^2 \sqrt{4-x^2}\,dx$$

$$= \left[\frac{(x-2)\sqrt{4-(x-2)^2}}{2} + \frac{4}{2}\cdot\sin^{-1}\frac{(x-2)}{2}\right]_0^1 + \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\cdot\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= 2\left[\left\{\frac{-\sqrt{3}}{2} + 2\sin^{-1}\left(\frac{-1}{2}\right)\right\} - \{0 + 2\sin^{-1}(-1)\} + \left\{2\sin^{-1}(1) - \frac{\sqrt{3}}{2} - 2\sin^{-1}\left(\frac{1}{2}\right)\right\}\right]$$

$$= 2\left\{\frac{-\sqrt{3}}{2} + 2\left(-\frac{\pi}{6}\right) - 2\left(-\frac{\pi}{2}\right) + \left(2\cdot\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2\times\frac{\pi}{6}\right)\right\}$$

$$= 2\left(\frac{4\pi}{3} - \sqrt{3}\right) \text{ sq units.}$$

Hence, the required area is $2\left(\frac{4\pi}{3} - \sqrt{3}\right)$ sq units.

EXAMPLE 20 *Using integration, find the area of the region*

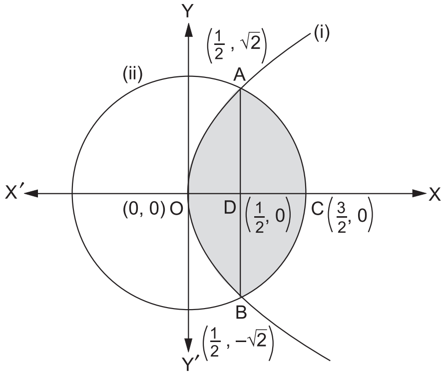
$$\{(x, y) : y^2 \le 4x,\ 4x^2 + 4y^2 \le 9\}.$$

[CBSE 2008, '13]

SOLUTION Clearly, we have to find the area enclosed between the curves $y^2 = 4x$ and $4x^2 + 4y^2 = 9$.

The curve $y^2 = 4x$ is a right-handed parabola with its vertex at (0, 0).

$4x^2 + 4y^2 = 9 \Rightarrow x^2 + y^2 = \left(\frac{3}{2}\right)^2$, which represents a circle with its centre at $O(0, 0)$ and radius equal to $(3/2)$ units.



The shaded region shows the required area.

Now, $y^2 = 4x$ … (i)

and $4x^2 + 4y^2 = 9$ … (ii)

Putting the value of y^2 from (i) into (ii), we get

$4x^2 + 16x - 9 = 0$

$\Rightarrow 4x^2 + 18x - 2x - 9 = 0$

$\Rightarrow 2x(2x+9) - (2x+9) = 0$

$\Rightarrow (2x+9)(2x-1) = 0$

$\Rightarrow x = -\dfrac{9}{2}$ or $x = \dfrac{1}{2}$.

Putting $x = -\dfrac{9}{2}$ in (i), we get

$y^2 = -18 \Rightarrow y$ is imaginary.

Putting $x = \dfrac{1}{2}$ in (i), we get $y^2 = 2 \Rightarrow y = \pm\sqrt{2}$.

So, the given curves intersect at the points $A\left(\dfrac{1}{2}, \sqrt{2}\right)$ and $B\left(\dfrac{1}{2}, -\sqrt{2}\right)$.

Since the equation of each of the given curves contains only even powers of y, each curve is symmetrical about the x-axis.

Required area = 2 (area $ODCAO$)

= 2 (area $ODAO$ + area $DCAD$)

$$= 2\left\{\int_0^{1/2} (y \text{ of the parabola})\, dx + \int_{1/2}^{3/2} (y \text{ of the circle})\, dx\right\}$$

$$= 2\left\{ \int_0^{1/2} 2\sqrt{x}\, dx + \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2}\, dx \right\}$$

$$= 2\left\{ \left[\frac{4}{3} x^{3/2} \right]_0^{1/2} + \left[\frac{x\sqrt{\frac{9}{4} - x^2}}{2} + \frac{9}{8} \sin^{-1}\left(\frac{x}{3/2} \right) \right]_{1/2}^{3/2} \right\}$$

$$= \frac{4}{3\sqrt{2}} + \left(\frac{9}{4} \sin^{-1} 1 \right) - \left(\frac{\sqrt{2}}{2} + \frac{9}{4} \sin^{-1} \frac{1}{3} \right)$$

$$= \left(\frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{2} \right) + \frac{9}{4}\left(\sin^{-1} 1 - \sin^{-1} \frac{1}{3} \right)$$

$$= \left\{ \frac{\sqrt{2}}{6} + \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1} \frac{1}{3} \right) \right\} \text{sq units}$$

$$= \left\{ \frac{\sqrt{2}}{6} + \frac{9}{4} \cos^{-1}\left(\frac{1}{3} \right) \right\} \text{sq units.}$$

Hence, the required area is $\left\{ \frac{\sqrt{2}}{6} + \frac{9}{4} \cos^{-1} \frac{1}{3} \right\}$ sq units.

EXAMPLE 21 *Find the area of the region* $\{(x, y) : x^2 + y^2 \le 1 \le x + y\}$.

SOLUTION Let $R = \{(x, y) : x^2 + y^2 \le 1 \le x + y\}$

$= \{(x, y) : x^2 + y^2 \le 1\} \cap \{(x, y) : x + y \ge 1\}$

$= R_1 \cap R_2$.

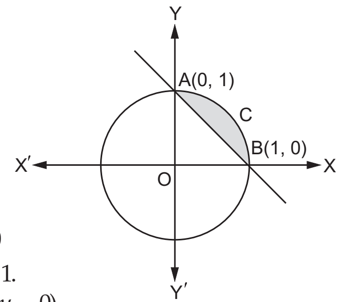
Clearly, R_1 is the interior of the circle $x^2 + y^2 = 1$ with its centre at $O(0, 0)$ and radius = 1 unit.

And, R_2 is the region lying above the line $x + y = 1$.

Consider the equations

$x^2 + y^2 = 1$ … (i)

and $x + y = 1$ … (ii)



Putting $y = (1 - x)$ from (ii) in (i), we get

$x^2 + (1 - x)^2 = 1 \Rightarrow 2x^2 - 2x = 0$

$\Rightarrow 2x(x - 1) = 0$

$\Rightarrow x = 0$ or $x = 1$.

Now, $(x = 0 \Rightarrow y = 1)$ and $(x = 1 \Rightarrow y = 0)$.

Thus, the points of intersection of (i) and (ii) are $A(0, 1)$ and $B(1, 0)$.

So, the required area is the shaded region.

Required area = area $BCAB$

$$= (\text{area } AOBCA) - (\text{area } OBAO)$$

$$= \int_0^1 \sqrt{1-x^2}\,dx - \int_0^1 (1-x)\,dx$$

$$= \left[\frac{1}{2}\sin^{-1}x + \frac{x}{2}\sqrt{1-x^2}\right]_0^1 - \left[x - \frac{x^2}{2}\right]_0^1$$

$$= \left(\frac{1}{2}\sin^{-1}1\right) - \frac{1}{2} = \left(\frac{1}{2}\times\frac{\pi}{2}\right) - \frac{1}{2} = \left(\frac{\pi}{4} - \frac{1}{2}\right) \text{ sq units.}$$

Hence, the required area is $\left(\frac{\pi}{4} - \frac{1}{2}\right)$ sq units.

EXAMPLE 22 *Find the area of the region* $\{(x, y) : x^2 + y^2 \le 2ax,\ y^2 \ge ax,\ x \ge 0,\ y \ge 0\}$.

SOLUTION Clearly, we have to find the area of the region lying in the first quadrant ($x \ge 0$, $y \ge 0$), included between the circle $x^2 + y^2 = 2ax$ and the parabola $y^2 = ax$.

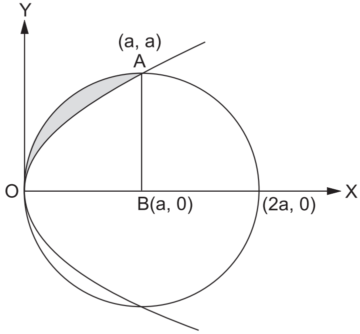
Thus, the equations of the given curves are

$$x^2 + y^2 = 2ax \quad \ldots \text{(i)}$$

and $$y^2 = ax \quad \ldots \text{(ii)}$$

Now, clearly $x^2 + y^2 = 2ax$ is a circle with its centre $B(a, 0)$ and radius $= a$ units.

And, $y^2 = ax$ is a parabola with $O(0, 0)$ as its vertex and the x-axis as its axis.



We can draw the figure, as shown.

Their points of intersection may be obtained by solving (i) and (ii) and keeping in view that $x \ge 0$ and $y \ge 0$.

Using (ii) in (i), we get

$$x^2 - ax = 0 \Rightarrow x(x-a) = 0$$
$$\Rightarrow x = 0 \text{ or } x = a.$$

Now, $(x = 0 \Rightarrow y = 0)$ and $(x = a \Rightarrow y = a)$.

Thus, the two curves intersect at $O(0, 0)$ and $A(a, a)$.

$$\therefore \quad \text{required area} = \int_0^a \sqrt{2ax - x^2}\,dx - \int_0^a \sqrt{ax}\,dx$$

$$= \int_0^a \sqrt{a^2 - (x-a)^2}\,dx - \sqrt{a}\cdot\int_0^a \sqrt{x}\,dx$$

$$= \left[\frac{(x-a)\sqrt{a^2-(x-a)^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) \right]_0^a$$

$$- \sqrt{a}\left[\frac{2}{3} x^{3/2} \right]_0^a$$

$$= \left\{ \frac{a^2}{2} \sin^{-1}(0) - \frac{a^2}{2} \sin^{-1}(-1) - \frac{2}{3} a^2 \right\}$$

$$= \left(\frac{\pi a^2}{4} - \frac{2}{3} a^2 \right) \text{ sq units.}$$

Hence, the required area is $\left(\frac{\pi a^2}{4} - \frac{2}{3} a^2 \right)$ sq units.

EXAMPLE 23 *Find the area of the region* $\{(x, y) : x^2 \le y \le |x|\}$. **[CBSE 2009, '12C, '13]**

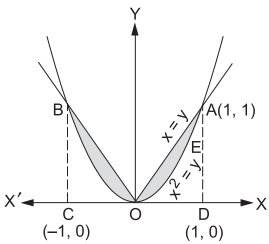
SOLUTION Consider the equations

$$x^2 = y \quad \ldots \text{(i)}$$

$$\text{and} \quad y = |x| \quad \ldots \text{(ii)}$$

Clearly, $x^2 = y$ represents an upward parabola with its vertex at $O(0, 0)$. All the points inside this parabola represent $x^2 \le y$.

Also, $y = |x| = \begin{cases} x, \text{ when } x \ge 0 \\ -x, \text{ when } x < 0. \end{cases}$



The lines OA and OB, each equally inclined to the axes, represent $y = |x|$.

All the points below the lines OA and OB, and above the x-axis represent $y \le |x|$.

Thus, the shaded portion is the required region.

In each of the given equations, the equation remains unchanged when x is replaced by $-x$.

So, each of the given curves is symmetrical about the y-axis.

$\therefore$ required area = 2(area $OEAO$).

In this region, we have

$$x^2 = y \quad \ldots \text{(iii)}$$

$$\text{and} \quad y = x \quad \ldots \text{(iv)}$$

Using (iv) in (iii), we get $x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or $x = 1$.

Now, $(x = 0 \Rightarrow y = 0)$ and $(x = 1 \Rightarrow y = 1)$.

Thus, the line $y = x$ and the curve $x^2 = y$ intersect at $O(0, 0)$ and $A(1, 1)$. Draw $AD \perp OX$ and $BC \perp OX'$.

$\therefore$ required area

$$= 2(\text{area } OEAO) = 2[(\text{area } ODAO) - (\text{area } ODAEO)]$$

$$= 2[(\text{ area between } y = x \text{ and the } x\text{-axis from } x = 0 \text{ to } x = 1)$$
$$- (\text{area between } x^2 = y \text{ and the } x\text{-axis from } x = 0 \text{ to } x = 1)]$$

$$= 2\left(\int_0^1 y\,dx \text{ for the line } OA - \int_0^1 y\,dx \text{ for the curve } OEA\right)$$

$$= 2\left[\int_0^1 x\,dx - \int_0^1 x^2\,dx\right] = 2\left\{\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right\} = 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= \left(2 \times \frac{1}{6}\right) = \frac{1}{3} \text{ sq unit.}$$

Hence, the required area is $\frac{1}{3}$ sq unit.

EXAMPLE 24 *Find the area bounded by the line $y = x$ and the curve $y = x^3$.*

SOLUTION The given equations are

$$y = x \qquad \text{... (i)}$$

and $$y = x^3 \qquad \text{... (ii)}$$

Using (i) in (ii), we get

$$x - x^3 = 0 \Rightarrow x(1 - x^2) = 0 \Rightarrow x(1 - x)(1 + x) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 1 \text{ or } x = -1.$$

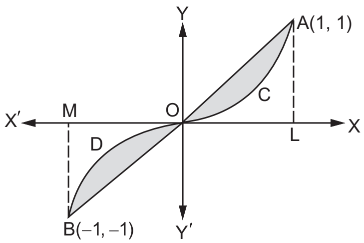
Also, $(x = 0 \Rightarrow y = 0)$, $(x = 1 \Rightarrow y = 1)$ and $(x = -1 \Rightarrow y = -1)$.

So, the given curve and the line intersect at the points $O(0, 0)$, $A(1, 1)$ and $B(-1, -1)$.

Now, $y = x$ is a line passing through the origin and making an angle of $45°$ with the x-axis. Thus, the line $y = x$ can be drawn.

For the curve $y = x^3$ some values for x and the corresponding values of y are given below:

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
y	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1



Plotting the points $(-1, -1)$, $\left(-\frac{1}{2}, -\frac{1}{8}\right)$, $(0, 0)$, $\left(\frac{1}{2}, \frac{1}{8}\right)$, and $(1, 1)$, and joining them, we get a rough sketch of $y = x^3$, as shown in the given figure.

Required area

$= (\text{area } ACOA) + (\text{area } ODBO)$

$= (\text{area } OALO) - (\text{area } OCALO) + (\text{area } OBMO) - (\text{area } ODBMO)$

$$= \int_0^1 \{y \text{ for (i)}\}\, dx - \int_0^1 \{y \text{ for (ii)}\}\, dx + \int_{-1}^0 \{(-y) \text{ for (i)}\}\, dx - \int_{-1}^0 \{(-y) \text{ for (ii)}\}\, dx$$

$$= \int_0^1 x\, dx - \int_0^1 x^3\, dx + \int_{-1}^0 -x\, dx - \int_{-1}^0 -x^3\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{-x^2}{2}\right]_{-1}^0 + \left[\frac{x^4}{4}\right]_{-1}^0$$

$$= \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2} \text{ sq unit.}$$

Hence, the required area is 0.5 sq unit.

EXAMPLE 25 *Find the area bounded by the curve* $y = \sin x$ *between* $x = 0$ *and* $x = 2\pi$.

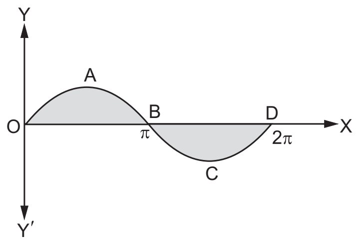
SOLUTION The given curve is $y = \sin x$.

Some values of x and the corresponding values of y are given below:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	$\frac{1}{2}$	1	0	–1	0

Taking a fixed unit (distance) for π along the x-axis, we can plot the points $(0, 0)$, $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, $\left(\frac{\pi}{2}, 1\right)$, $(\pi, 0)$, $\left(\frac{3\pi}{2}, -1\right)$ and $(2\pi, 0)$.

Join these points freehand to obtain a rough sketch of the given curve.



Required area = (area $OABO$) + (area $BCDB$)

$$= \int_0^{\pi} y\,dx + \int_{\pi}^{2\pi} (-y)\,dx \quad [\because \text{ area } BCDB \text{ is below the } x\text{-axis}]$$

$$= \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx$$

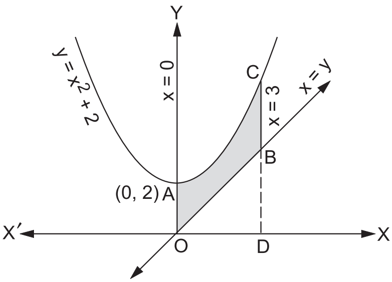
$$= [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi} = (2+2) = 4 \text{ sq units.}$$

Hence, the required area is 4 sq units.

EXAMPLE 26 *Find the area of the region bounded by the curve* $y = x^2 + 2$, *and the lines* $y = x$, $x = 0$ *and* $x = 3$.

SOLUTION $y = x^2 + 2 \Rightarrow x^2 = (y - 2)$.

Clearly, $x^2 = (y - 2)$ represents an upward parabola with its vertex at $A(0, 2)$.



Also, $y = x$ represents the straight line, making an angle of 45° with the positive direction of the x-axis.

And, $x = 0$ is the y-axis, while $x = 3$ represents a line parallel to the y-axis at a distance of 3 units from it.

Thus, the shaded region in the given figure is the required area.

$\therefore$ required area = (area $ODCAO$) − (area $ODBO$)

$$= \int_0^3 (x^2 + 2)\,dx - \int_0^3 x\,dx$$

$$= \left[\frac{x^3}{3} + 2x\right]_0^3 - \left[\frac{x^2}{2}\right]_0^3 = \left(15 - \frac{9}{2}\right) = \frac{21}{2} \text{ sq units.}$$

Hence, the required area is $\frac{21}{2}$ sq units.

EXAMPLE 27 *Find the area of the region*

$\{(x, y) : 0 \le y \le (x^2 + 1), 0 \le y \le (x + 1), 0 \le x \le 2\}$. **[CBSE 2009]**

SOLUTION Let $R = \{(x, y) : 0 \le y \le (x^2 + 1),\ 0 \le y \le (x + 1),\ 0 \le x \le 2\}$

$$= \{(x, y) : 0 \le y \le (x^2 + 1)\} \cap \{(x, y) : 0 \le y \le (x + 1)\} \cap \{(x, y) : 0 \le x \le 2\}$$

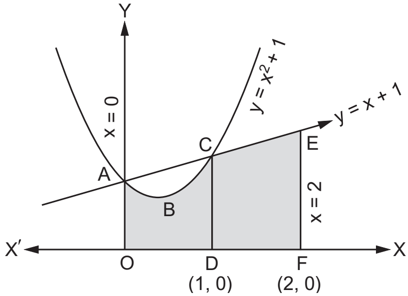
$$= R_1 \cap R_2 \cap R_3.$$

Clearly, R_1 is the region consisting of the right-hand side of the y-axis, lying below the parabola $y = x^2 + 1$.

Also, R_2 is the region consisting of the right-hand side of the y-axis, lying below the line $y = (x + 1)$.

And, R_3 is the region above the x-axis, lying between the ordinates $x = 0$ and $x = 2$.

Thus, $R_1 \cap R_2 \cap R_3$ is the shaded region.



We have, $y = x^2 + 1$ and $y = x + 1$

$$\Rightarrow x^2 + 1 = x + 1 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0 \text{ or } x = 1.$$

Now, $(x = 0 \Rightarrow y = 1)$ and $(x = 1 \Rightarrow y = 2)$.

Thus, the parabola $y = (x^2 + 1)$ and the line $y = x + 1$ intersect at the points $A(0, 1)$ and $C(1, 2)$.

$\therefore$ required area = area of the shaded region

$$= (\text{area } ODCBA) + (\text{area } CDFEC)$$

$$= \int_0^1 (y \text{ of the parabola})\, dx + \int_1^2 (y \text{ of the line})\, dx$$

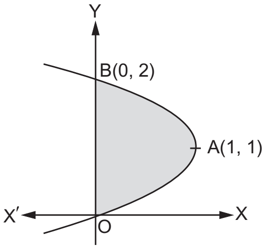
$$= \int_0^1 (x^2 + 1)\, dx + \int_1^2 (x + 1)\, dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^1 + \left[\frac{x^2}{2} + x\right]_1^2$$

$$= \left(\frac{1}{3} + 1\right) + \left(4 - \frac{3}{2}\right) = \frac{23}{6} \text{ sq units.}$$

Hence, the required area is $\frac{23}{6}$ sq units.

EXAMPLE 28 *Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis.*



SOLUTION $y^2 = 2y - x \Rightarrow y^2 - 2y = -x$

$$\Rightarrow y^2 - 2y + 1 = -x + 1$$
$$\Rightarrow (y-1)^2 = -(x-1)$$
$$\Rightarrow Y^2 = -X,$$

where $y - 1 = Y$ and $(x - 1) = X$.

This is a left-handed parabola with vertex at $(X = 0, Y = 0)$.

$X = 0, Y = 0 \Rightarrow -x + 1 = 0$ and $y - 1 = 0$
$\Rightarrow x = 1$ and $y = 1$.

Thus, the vertex of the given parabola is $A(1, 1)$.

Also, $x = 0 \Rightarrow y^2 - 2y = 0 \Rightarrow y(y-2) = 0 \Rightarrow y = 0$ or $y = 2$.

Thus, the curve meets the y-axis at $O(0, 0)$ and $B(0, 2)$.

A rough sketch of the curve can be drawn, as shown in the figure.

$$\therefore \text{ required area} = \int_0^2 x\,dy = \int_0^2 (2y - y^2)\,dy$$

$$= \left[y^2 - \frac{y^3}{3} \right]_0^2 = \left(4 - \frac{8}{3} \right) = \frac{4}{3} \text{ sq units.}$$

Hence, the required area is $\frac{4}{3}$ sq units.

EXERCISE 17

1. Find the area of the region bounded by the curve $y = x^2$, the x-axis, and the lines $x = 1$ and $x = 3$.
2. Find the area of the region bounded by the parabola $y^2 = 4x$, the x-axis, and the lines $x = 1$ and $x = 4$.
3. Find the area under the curve $y = \sqrt{6x + 4}$ (above the x-axis) from $x = 0$ to $x = 2$.
4. Determine the area enclosed by the curve $y = x^3$, and the lines $y = 0$, $x = 2$ and $x = 4$.
5. Determine the area under the curve $y = \sqrt{a^2 - x^2}$, included between the lines $x = 0$ and $x = 4$.
6. Using integration, find the area of the region bounded by the line $2y = 5x + 7$, the x-axis, and the lines $x = 2$ and $x = 8$.
7. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

8. Evaluate the area bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ above the x-axis.

9. Using integration, find the area of the region bounded by the lines $y=1+|x+1|$, $x=-2$, $x=3$ and $y=0$.

10. Find the area bounded by the curve $y=(4-x^2)$, the y-axis and the lines $y=0$, $y=3$.

11. Using integration, find the area of the region bounded by the triangle whose vertices are $A(-1, 2)$, $B(1, 5)$ and $C(3, 4)$. **[CBSE 2014]**

12. Using integration, find the area of the $\triangle ABC$, the equations of whose sides AB, BC and AC are given by

    $y=4x+5$, $x+y=5$ and $4y=x+5$ respectively.

13. Using integration, find the area of the region bounded between the line $x=2$ and the parabola $y^2=8x$.

14. Using integration, find the area of the region bounded by the line $y-1=x$, the x-axis, and the ordinates $x=-2$ and $x=3$.

15. Sketch the region lying in the first quadrant and bounded by $y=4x^2$, $x=0$, $y=2$ and $y=4$. Find the area of the region using integration.

16. Sketch the region lying in the first quadrant and bounded by $y=9x^2$, $x=0$, $y=1$ and $y=4$. Find the area of the region, using integration.

17. Find the area of the region enclosed between the circles $x^2+y^2=1$ and $(x-1)^2+y^2=1$. **[CBSE 2007, '13C]**

18. Sketch the region common to the circle $x^2+y^2=16$ and the parabola $x^2=6y$. Also, find the area of the region, using integration.

19. Sketch the region common to the circle $x^2+y^2=25$ and the parabola $y^2=8x$. Also, find the area of the region, using integration.

20. Draw a rough sketch of the region $\{(x, y) : y^2 \le 3x,\ 3x^2+3y^2 \le 16\}$ and find the area enclosed by the region, using the method of integration.

21. Draw a rough sketch and find the area of the region bounded by the parabolas $y^2=4x$ and $x^2=4y$, using the method of integration.

22. Find by integration the area bounded by the curve $y^2=4ax$ and the lines $y=2a$ and $x=0$.

23. Find the area between the curve $y=\frac{x}{\pi}+2\sin^2 x$, the x-axis and the ordinates $x=0$ and $x=\pi$.

24. Find the area bounded by the curve $y = \cos x$, the x-axis and the ordinates $x = 0$ and $x = 2\pi$.
25. Compare the areas under the curves $y = \cos^2 x$ and $y = \sin^2 x$ between $x = 0$ and $x = \pi$.
26. Using integration, find the area of the triangle, the equations of whose sides are $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.
27. Find the area of the region $\{(x, y) : x^2 \le y \le x\}$.
28. Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis.
29. Draw a rough sketch of the curves $y = \sin x$ and $y = \cos x$, as x varies from 0 to $\frac{\pi}{2}$, and find the area of the region enclosed between them and the x-axis.
30. Find the area of the region bounded by the parabola $y^2 = 2x + 1$ and the line $x - y = 1$.
31. Find the area of the region bounded by the curve $y = 2x - x^2$ and the straight line $y = -x$.
32. Find the area of the region bounded by the curve $(y-1)^2 = 4(x+1)$ and the line $y = x - 1$.
33. Find the area of the region bounded by the curve $y = \sqrt{x}$ and the line $y = x$.
34. Find the area of the region included between the parabola $y^2 = 3x$ and the circle $x^2 + y^2 - 6x = 0$, lying in the first quadrant.
35. Find the area bounded by the curve $y = \cos x$ between $x = 0$ to $x = 2\pi$.
36. Using integration, find the area of the region in the first quadrant, enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. **[CBSE 2014]**
37. Using integration, find the area of the triangle whose vertices are $A(2, 3)$, $B(4, 7)$ and $C(6, 2)$. **[CBSE 2012C]**
38. Using integration, find the area of the triangle whose vertices are $A(1, 3)$, $B(2, 5)$ and $C(3, 4)$. **[CBSE 2009C]**
39. Using integration, find the area of the triangular region bounded by the lines $y = 2x + 1$, $y = 3x + 1$ and $x = 4$. **[CBSE 2011]**

ANSWERS (EXERCISE 17)

1. $\frac{26}{3}$ sq units **2.** $\frac{28}{3}$ sq units **3.** $\frac{56}{9}$ sq units

4. 60 sq units **5.** $\frac{\pi a^2}{4}$ sq units **6.** 96 sq units

7. $8\sqrt{3}$ sq units **8.** $\frac{3\pi}{2}$ sq units **9.** $\frac{27}{2}$ sq units

10. $\frac{14}{3}$ sq units **11.** 4 sq units **12.** $\frac{15}{2}$ sq units

13. $\frac{32}{3}$ sq units **14.** 8.5 sq units **15.** $\frac{1}{3}(8 - 2\sqrt{2})$ sq units

16. $\frac{14}{9}$ sq units **17.** $\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$ sq units **18.** $\frac{16\pi+4\sqrt{3}}{3}$ sq units

19. $\left\{\frac{8\sqrt{2}}{3}\cdot a^{3/2}+\frac{25\pi}{2}-a\sqrt{25-a^2}-25\sin^{-1}\frac{a}{5}\right\}$ sq units, where $a=\sqrt{41}-4$.

20. $\left\{\frac{4}{\sqrt{3}}a^{3/2}+\frac{8\pi}{3}-a\sqrt{\frac{16}{3}-a^2}-\frac{16}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)\right\}$ sq units, where $a=\frac{-9+\sqrt{273}}{6}$

21. $\frac{16}{3}$ sq units **22.** $\frac{2a^2}{3}$ sq units **23.** $\frac{3\pi}{2}$ sq units

24. 4 sq units **25.** $\frac{\pi}{2}$ sq units each **26.** 8 sq units

27. $\frac{1}{6}$ sq unit **28.** $\frac{4}{3}$ sq units **29.** $(2-\sqrt{2})$ sq units

30. $\frac{16}{3}$ sq units **31.** $\frac{9}{2}$ sq units **32.** $\frac{64}{3}$ sq units

33. $\frac{1}{6}$ sq unit **34.** $\frac{3}{4}(3\pi-8)$ sq units **35.** 4 sq units

36. 4π sq units **37.** 9 sq units **38.** $\frac{3}{2}$ sq units

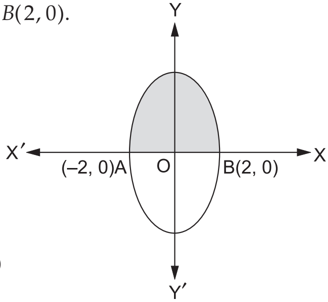
39. 8 sq units

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 17)

7. Since the equation $y^2=4x$ contains only even powers of y, the curve is symmetrical about the y-axis.

$\therefore$ required area $=2\cdot\int_0^3 2\sqrt{x}\,dx$.

8. The given ellipse meets the x-axis at $A(-2,0)$ and $B(2,0)$.



$\frac{x^2}{4}+\frac{y^2}{9}=1\Rightarrow y=\frac{3}{2}\cdot\sqrt{4-x^2}$.

$\therefore$ required area $=\int_{-2}^{2} y\,dx=\frac{3}{2}\cdot\int_{-2}^{2}\sqrt{4-x^2}\,dx$

$=2\times\frac{3}{2}\times\int_0^2\sqrt{4-x^2}\,dx$

$=6\cdot\int_0^{\pi/2}\cos^2\theta\,d\theta$, where $x=2\sin\theta$

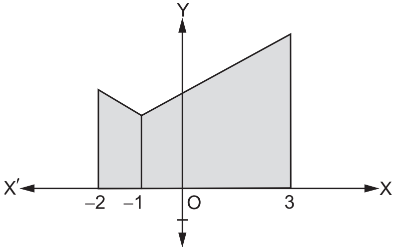
$=3\cdot\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$.

9. $y=1+|x+1|=\begin{cases}1+(x+1), \text{ when } x+1\ge 0\\ 1-(x+1), \text{ when } x+1<0\end{cases}$

$$= \begin{cases} x + 2, \text{ when } x \geq -1 \\ -x, \text{ when } x < -1 \end{cases}$$

$$\text{Required area} = \int_{-2}^{3} y\,dx = \int_{-2}^{-1} y\,dx + \int_{-1}^{3} y\,dx$$

$$= \int_{-2}^{-1} (-x)\,dx + \int_{-1}^{3} (x + 2)\,dx.$$



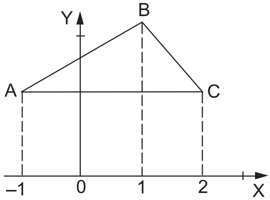
10. Required area $= \int_0^3 x\,dy = \int_0^3 \sqrt{4 - y}\,dy.$

11. Equations of AB, BC and AC are

$$y = \frac{1}{2}(3x + 7), y = \frac{1}{2}(11 - x)$$

and $y = \frac{1}{2}(x + 5)$ respectively.

$$\therefore \quad \Delta = \frac{1}{2}\int_{-1}^{1}(3x + 7)\,dx + \frac{1}{2}\int_{1}^{3}(11 - x)\,dx - \frac{1}{2}\int_{-1}^{3}(x + 5)\,dx.$$

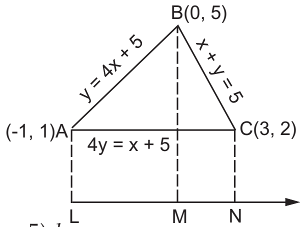


12. Solving the given equations in pairs, we get the points $A(-1, 1)$, $B(0, 5)$ and $C(3, 2)$.

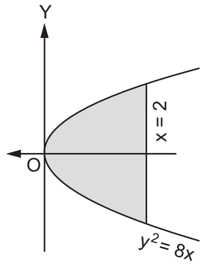
Draw AL, BM and CN perpendiculars on the x-axis.

$$\text{Required area} = \int_{-1}^{0} y_{AB}\,dx + \int_{0}^{3} y_{BC}\,dx - \int_{-1}^{3} y_{CA}\,dx$$

$$= \int_{-1}^{0} (4x + 5)\,dx + \int_{0}^{3} (-x + 5)\,dx - \frac{1}{4} \cdot \int_{-1}^{3} (x + 5)\,dx.$$



13. Required area $= 2 \cdot \int_0^2 \sqrt{8x}\,dx.$

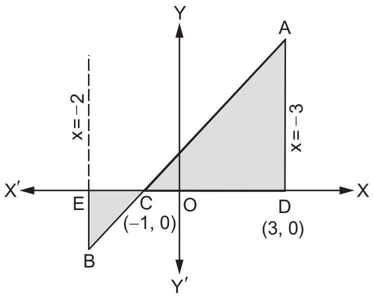


14. Let AB be the given line, intersecting the x-axis at $C(-1, 0)$.

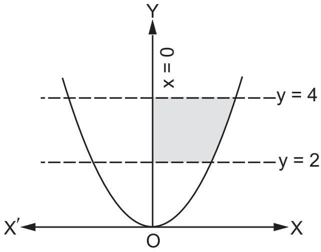
Required area = (area $CDAC$ + area $CBEC$)

$$= \int_{-1}^{3} y\,dx + \int_{-2}^{-1} (-y)\,dx$$

$$= \int_{-1}^{3} (x + 1)\,dx + \int_{-2}^{-1} -(x + 1)\,dx.$$

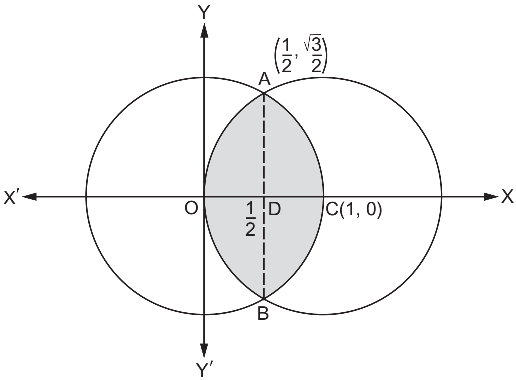


15. Clearly, we have to find the area of the region bounded by the curve $y = 4x^2$, the y-axis, and the lines $y = 2$ and $y = 4$.



Required area $= \int_2^4 x\,dy = \int_2^4 \frac{1}{2}\sqrt{y}\,dy.$

17. $(x^2 + y^2) = 1$ is a circle with its centre at (0, 0) and radius = 1 unit.



And, $(x-1)^2 + y^2 = 1$ is a circle with its centre at (1, 0) and radius = 1 unit.

The given equations are

$$(x^2 + y^2) = 1 \qquad \dots \text{(i)}$$

$$\text{and } (x-1)^2 + y^2 = 1. \qquad \dots \text{(ii)}$$

Using (i) in (ii), we get $1 - 2x = 0 \Rightarrow x = \frac{1}{2}\cdot$

Putting $x = \frac{1}{2}$ in (i), we get $y = \pm\frac{\sqrt{3}}{2}\cdot$

So, the two circles intersect at $A\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $B\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$.

Required area = area $AOBCA$

$$= 2(\text{area } AOCA) = 2(\text{area } ODAO + \text{area } DCAD)$$

$$= 2\left\{\int_0^{1/2} \sqrt{1-(x-1)^2}\,dx + \int_{1/2}^{1} \sqrt{1-x^2}\right\}$$

18. The given curves are

$$x^2 + y^2 = 16 \qquad \dots \text{(i)}$$

$$\text{and} \qquad x^2 = 6y \qquad \dots \text{(ii)}$$

On solving (i) and (ii), we get the points

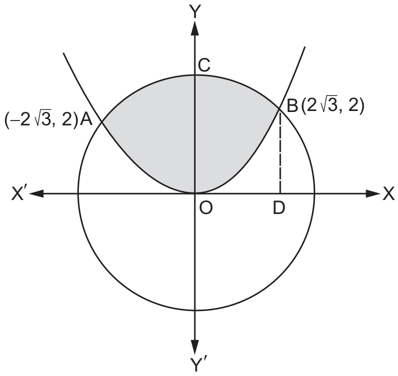
$A(-2\sqrt{3}, 2)$ and $B(2\sqrt{3}, 2)$.

Required area

$= 2(\text{area } OBCO)$

$= 2[\text{area } ODBCO - \text{area } ODBO]$

$$= 2\left\{\int_0^{2\sqrt{3}} \sqrt{16 - x^2}\, dx - \int_0^{2\sqrt{3}} \frac{x^2}{6}\, dx\right\}.$$



19. The given curves are

$$x^2 + y^2 = 25 \quad \dots \text{(i)}$$

and $$y^2 = 8x \quad \dots \text{(ii)}$$

On solving (i) and (ii), we get

$$x^2 + 8x - 25 = 0 \Rightarrow x = \frac{-8 \pm \sqrt{64 + 100}}{2}$$

$$\Rightarrow x = -4 + \sqrt{41} \quad \text{(rejecting –ve value).}$$

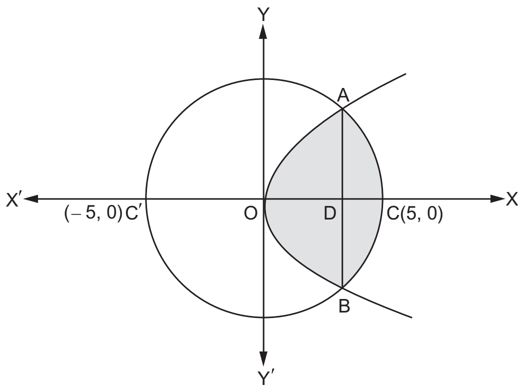
Putting $y = 0$ in (i), we get $x = \pm 5$.

Thus, the circle (i) cuts the x-axis at $C(5, 0)$ and $C'(-5, 0)$.

Required area $= 2[\text{area } ODCAO]$

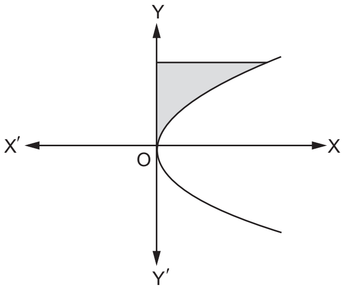
$= 2[(\text{area } ODAO) + (\text{area } ADCA)]$

$$= 2\left(\int_0^a \sqrt{8x}\, dx + \int_a^5 \sqrt{25 - x^2}\, dx\right), \text{ where } a = -4 + \sqrt{41}.$$



21. See Example 14, taking $a = 1$.

22. Required area $= \int_0^{2a} x\,dy = \int_0^{2a} \frac{y^2}{4a}\,dy = \frac{2a^2}{3}$ sq units.



28. $y^2 = 2y - x \Rightarrow y^2 - 2y + 1 = -x + 1$

$\Rightarrow (y-1)^2 = -(x-1)$.

So, the given equation represents a parabola that opens on the left, having vertex at (1, 1).

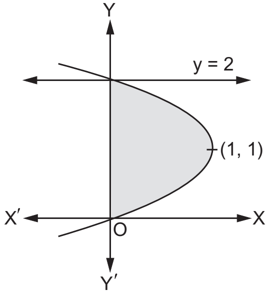
Now, $x = 0 \Rightarrow y^2 = 2y \Rightarrow y^2 - 2y = 0$

$\Rightarrow y(y-2) = 0$

$\Rightarrow y = 0$ or $y = 2$.

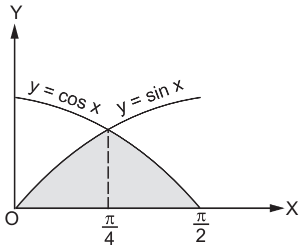
Thus, the curve meets the y-axis at (0, 0) and (0, 2)

Required area $= \int_0^2 (2y - y^2)\,dy$.



29. A rough sketch of $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$ is shown herewith.

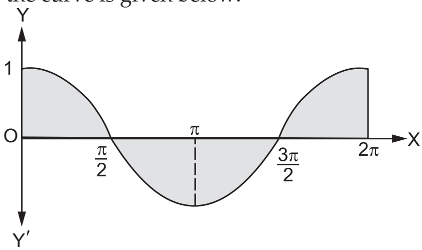
Required area $= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx$.



35. Tabular values of $y = \cos x$:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	1	0	–1	0	1

A rough sketch of the curve is given below:



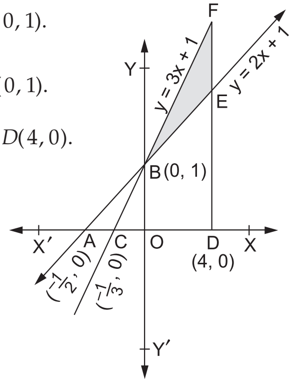
Required area $= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{3\pi/2} -\cos x\,dx + \int_{3\pi/2}^{2\pi} \cos x\,dx$

36. Analogous to Example 27.

$$\text{Required area} = \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x\sqrt{32 - x^2}}{2} + \frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$= 4\pi$ sq units.



39. $y = 2x + 1$ is the line passing through $A\left(\frac{-1}{2}, 0\right)$ and $B(0, 1)$.

$y = 3x + 1$ is the line passing through $C\left(\frac{-1}{3}, 0\right)$ and $B(0, 1)$.

$x = 4$ is the line parallel to y-axis and passing through $D(4, 0)$.

Required area = (area $ODFB$) – (area $ODEB$)

$$= \int_0^4 (3x + 1)\,dx - \int_0^4 (2x + 1)\,dx$$

$= (28 - 20)$ sq units $= 8$ sq units

18. DIFFERENTIAL EQUATIONS AND THEIR FORMATION

DIFFERENTIAL EQUATION *An equation containing an independent variable, a dependent variable and the derivatives of the dependent variable is called a differential equation.*

Examples Each of the following equations is a differential equation:

(i) $\frac{dy}{dx} + 5y = e^x$ (ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = \sin x$

(iii) $\frac{dy}{dx} = \frac{x^3 - y^3}{xy^2 - x^2y}$ (iv) $x^2dx + y^2dy = 0$

ORDER OF A DIFFERENTIAL EQUATION *The order of the highest-order derivative occurring in a differential equation is called the order of the differential equation.*

DEGREE OF A DIFFERENTIAL EQUATION *The power of the highest-order derivative occurring in a differential equation, after it is made free from radicals and fractions, is called the degree of the differential equation.*

EXAMPLE 1 *Write the order and the degree of the differential equation* $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0.$

SOLUTION In the given equation, the highest-order derivative is $\frac{d^2y}{dx^2}$ and its power is 1.

$\therefore$ its order = 2 and degree = 1.

EXAMPLE 2 *Write the order and the degree of the differential equation*

$$x\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{dy}{dx}\right)^4 + y^2 = 0.$$

SOLUTION In the given equation, the highest-order derivative is $\frac{d^3y}{dx^3}$ and its power is 2.

$\therefore$ its order = 3 and degree = 2.

EXAMPLE 3 *Write the order and the degree of the differential equation*

$$y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

SOLUTION The given equation may be written as

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \left(y - x\frac{dy}{dx}\right)$$

$$\Rightarrow \quad 1+\left(\frac{dy}{dx}\right)^2 = \left(y - x\frac{dy}{dx}\right)^2 \quad \text{[on squaring both sides]}$$

$$\Rightarrow \quad 1+\left(\frac{dy}{dx}\right)^2 = y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx}$$

$$\Rightarrow \quad (1-x^2)\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + (1-y^2) = 0.$$

Clearly, it is a differential equation of order = 1 and degree = 2.

EXAMPLE 4 *Write the order and degree of the differential equation*

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2} = k\left(\frac{d^2y}{dx^2}\right).$$

SOLUTION On squaring both sides, we get

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3} = k^2\left(\frac{d^2y}{dx^2}\right)^2.$$

In this equation, the highest-order derivative is $\frac{d^2y}{dx^2}$ whose power is 2.

$\therefore$ its order = 2 and degree = 2.

EXAMPLE 5 *Find the order and the degree of the differential equation*

$$y = px + \sqrt{a^2p^2 + b^2}, \text{ where } p = \frac{dy}{dx}.$$

SOLUTION

$$y = px + \sqrt{a^2p^2 + b^2}$$

$$\Rightarrow \quad y - px = \sqrt{a^2p^2 + b^2}$$

$$\Rightarrow \quad (y-px)^2 = a^2p^2 + b^2 \quad \text{[on squaring both sides]}$$

$$\Rightarrow \quad y^2 + x^2p^2 - 2xyp = a^2p^2 + b^2$$

$$\Rightarrow \quad (x^2 - a^2)p^2 - 2xyp + (y^2 - b^2) = 0$$

$$\Rightarrow \quad (x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\cdot\left(\frac{dy}{dx}\right) + (y^2 - b^2) = 0.$$

Clearly, it is a differential equation of order 1 and degree 2.

AN IMPORTANT NOTE *In case of differential equations involving one or more terms of the form* $e^{(dy/dx)}$, $\log\left(\frac{dy}{dx}\right)$, $\sin\left(\frac{dy}{dx}\right)$, $\cos\left(\frac{dy}{dx}\right)$, *etc., the degree is not defined.*

However, the degree of the differential equation containing terms like e^x, e^y, $\log x$, $\log y$, $\tan x$, $\tan y$, etc., is defined as usual.

EXAMPLE 6 *Find the order and degree (if any) of each of the differential equations given below:*

(i) $\frac{dy}{dx} - \tan x = 0$ (ii) $\left(\frac{dy}{dx}\right)^2 + y = e^x$

(iii) $\frac{d^2y}{dx^2} = \sin 3x + \cos 3x$ (iv) $(y'')^2 + \cos y' = 0$

(v) $y'' + 2y' + \sin y = 0$ (vi) $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$

(vii) $y''' + y^2 + e^{y'} = 0$ (viii) $3\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 = \log x$

SOLUTION (i) The given equation is $\frac{dy}{dx} - \tan x = 0$.

In this equation, the highest-order derivative is $\frac{dy}{dx}$ whose power is 1.

$\therefore$ its order = 1 and degree = 1.

(ii) The given equation is $\left(\frac{dy}{dx}\right)^2 + y = e^x$.

In this equation, the highest-order derivative is $\frac{dy}{dx}$ whose power is 2.

$\therefore$ its order = 1 and degree = 2.

(iii) The given equation is $\frac{d^2y}{dx^2} = \sin 3x + \cos 3x$.

In this equation, the highest-order derivative is $\frac{d^2y}{dx^2}$ and its power is 1.

$\therefore$ its order = 2 and degree = 1.

(iv) The given equation is $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$.

In this equation, the highest-order derivative is $\frac{d^2y}{dx^2}$, so its order is 2.

It has a term $\cos\left(\frac{dy}{dx}\right)$, so its degree is not defined.

(v) The given equation is $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \sin y = 0$.

In this equation, the highest-order derivative is $\frac{d^2y}{dx^2}$ and its power is 1.

$\therefore$ its order = 2 and degree = 1.

(vi) The given equation is $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0.$

In this equation, the highest-order derivative is $\frac{d^4y}{dx^4}$, so its order is 4.

It has a term $\sin\left(\frac{d^3y}{dx^3}\right)$, so its degree is not defined.

(vii) The given equation is $\frac{d^3y}{dx^3} + y^2 + e^{(dy/dx)} = 0.$

In this equation, the highest-order derivative is $\frac{d^3y}{dx^3}$, so its order is 3.

It has a term $e^{(dy/dx)}$, so its degree is not defined.

(viii) The given equation is $3\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 = \log x.$

In this equation, the highest-order derivative is $\frac{d^2y}{dx^2}$ and its power is 1.

$\therefore$ its order = 2 and degree = 1.

EXERCISE 18A

Write order and degree (if defined) of each of the following differential equations.

1. $\left(\frac{dy}{dx}\right)^4 + 3y\left(\frac{d^2y}{dx^2}\right) = 0$ **[CBSE 2008C, '13C]**

2. $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$ **[CBSE 2013]**

3. $\left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{ds}{dt}\right)^3 + 4 = 0$ **[CBSE 2013]**

4. $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y^5 = 0$

5. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y = 0$

6. $\frac{dy}{dx} + y = e^x$

7. $\frac{d^2y}{dx^2} + y^2 + e^{(dy/dx)} = 0$

8. $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$

9. $\frac{d^4y}{dx^4} - \cos\left(\frac{d^3y}{dx^3}\right) = 0$

10. $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$ **[CBSE 2010]**

11. $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$

12. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

13. $x\left(\frac{dy}{dx}\right) + \frac{2}{\left(\frac{dy}{dx}\right)} + 9 = y^2$

14. $\sqrt{1 - \left(\frac{dy}{dx}\right)^2} = \left(a\frac{d^2y}{dx^2}\right)^{1/3}$

15. $\sqrt{1-y^2}dx + \sqrt{1-x^2}dy = 0$

16. $(y'')^3 + (y')^2 + \sin y' + 1 = 0$

17. $(3x+5y)dy - 4x^2dx = 0$

18. $y = \frac{dy}{dx} + \frac{5}{\left(\frac{dy}{dx}\right)}$

ANSWERS (EXERCISE 18A)

1. 2, 1 **2.** 2, 2 **3.** 2, 2 **4.** 3, 2 **5.** 2, 1 **6.** 1, 1 **7.** 2, not defined
8. 1, not defined **9.** 4, not defined **10.** 2, 1 **11.** 1, 3 **12.** 3, 1 **13.** 1, 2
14. 2, 2 **15.** 1, 1 **16.** 2, not defined **17.** 1, 1 **18.** 1, 2

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 18A)

13. The given differential equation on simplification becomes

$$x\left(\frac{dy}{dx}\right)^2 + 2 + 9\left(\frac{dy}{dx}\right) = y^2\left(\frac{dy}{dx}\right).$$

So, its order = 1 and degree = 2.

14. On squaring both sides, $\left\{1 - \left(\frac{dy}{dx}\right)^2\right\} = \left(a\frac{d^2y}{dx^2}\right)^{2/3}$. ... (i)

On cubing both sides of (i), we get

$\left\{1 - \left(\frac{dy}{dx}\right)^2\right\}^3 = \left(a\frac{d^2y}{dx^2}\right)^2$ · So, its order = 2 and degree = 2.

15. $\frac{dy}{dx} = \frac{-\sqrt{1-y^2}}{\sqrt{1-x^2}}$ · So, its order =1 and degree = 1.

17. $\frac{dy}{dx} = \frac{4x^2}{(3x+5y)}$ · So, its order = 1 and degree = 1.

SOLUTION OF A DIFFERENTIAL EQUATION *A function of the form $y = f(x) + C$ which satisfies a given differential equation is called its solution.*

GENERAL SOLUTION OF A DIFFERENTIAL EQUATION *Suppose a differential equation of order n is being given. If its solution contains n arbitrary constants then it is called a general solution.*

PARTICULAR SOLUTION OF A DIFFERENTIAL EQUATION *Giving particular values to arbitrary constants in the general solution of a differential equation, we get its particular solutions.*

SOLVED EXAMPLES

EXAMPLE 1 *Verify that* $y = A\cos x - B\sin x$ *is a solution of the differential equation*

$$\frac{d^2y}{dx^2} + y = 0.$$ **[CBSE 2005C, '06]**

SOLUTION Given: $y = A\cos x - B\sin x$ ... (i)

$\Rightarrow \frac{dy}{dx} = -A\sin x - B\cos x$

$\Rightarrow \frac{d^2y}{dx^2} = -A\cos x + B\sin x$

$= -(A\cos x - B\sin x) = -y$ [from (i)]

$\Rightarrow \frac{d^2y}{dx^2} + y = 0.$

Hence, $y = A\cos x - B\sin x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0.$

EXAMPLE 2 *Verify that* $y = ae^{2x} + be^{-x}$ *is a solution of the differential equation*

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$$ **[CBSE 2003C]**

SOLUTION Given: $y = ae^{2x} + be^{-x}$ ... (i)

$\Rightarrow \frac{dy}{dx} = 2ae^{2x} - be^{-x}$ ... (ii)

$\Rightarrow \frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x}$ ... (iii)

$\therefore \left(\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y\right) = (4ae^{2x} + be^{-x}) - (2ae^{2x} - be^{-x}) - 2(ae^{2x} + be^{-x})$

$= 0$ [using (i), (ii) and (iii)].

Hence, $y = ae^{2x} + be^{-x}$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$$

EXAMPLE 3 *Verify that* $y = Ax + \frac{B}{x}$ *is a solution of the differential equation*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$

SOLUTION Given: $y = Ax + \frac{B}{x}$ ... (i)

$$\Rightarrow \quad \frac{dy}{dx} = A - \frac{B}{x^2} \qquad \ldots \text{(ii)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2B}{x^3} \qquad \ldots \text{(iii)}$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (i), (ii) and (iii) respectively, we get

$$\left(x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y \right) = x^2 \cdot \frac{2B}{x^3} + x\left(A - \frac{B}{x^2} \right) - \left(Ax + \frac{B}{x} \right)$$

$$= \left(\frac{2B}{x} + Ax - \frac{B}{x} - Ax - \frac{B}{x} \right) = 0.$$

Thus, $y = Ax + \frac{B}{x}$ satisfies $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$

Hence, $y = Ax + \frac{B}{x}$ is a solution of $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$

EXAMPLE 4 *Verify that* $y = a\cos(\log x) + b\sin(\log x)$ *is a solution of the differential equation* $x^2 + \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$ **[CBSE 2007]**

SOLUTION Given: $y = a\cos(\log x) + b\sin(\log x)$ ... (i)

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x} \quad \text{[on differentiating (i) w.r.t. } x\text{]}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x) \qquad \ldots \text{(ii)}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

[on differentiating (ii) w.r.t. x]

$$\Rightarrow \quad x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = -a\cos(\log x) - b\sin(\log x)$$

$$\Rightarrow \quad x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \quad \text{[using (i)].}$$

Hence, $y = a\cos(\log x) + b\sin(\log x)$ is a solution of

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

EXAMPLE 5 *Verify that* $y = e^{m\sin^{-1}x}$ *is a solution of the differential equation*

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0.$$

SOLUTION Given: $y = e^{m\sin^{-1}x}$ ... (i)

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{m\sin^{-1}x}}{\sqrt{1 - x^2}} \cdot m \qquad \text{[on differentiating (i)]}$$

$$\Rightarrow \quad \sqrt{1-x^2}\left(\frac{dy}{dx}\right) = my \qquad \text{... (ii)} \qquad [\because\ e^{m\sin^{-1}x} = y]$$

$$\Rightarrow \quad (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2y^2 \qquad \text{... (iii)} \quad \text{[on squaring both sides of (ii)]}$$

$$\Rightarrow \quad (1-x^2)2\frac{dy}{dx}\cdot\left(\frac{d^2y}{dx^2}\right) - 2x\left(\frac{dy}{dx}\right)^2 = 2m^2y\frac{dy}{dx}$$

[on differentiating both sides of (iii)]

$$\Rightarrow \quad 2\left(\frac{dy}{dx}\right)\cdot\left\{(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y\right\} = 0$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0.$$

Hence, $y = e^{m\sin^{-1}x}$ is a solution of the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0.$$

EXAMPLE 6 *Verify that* $v = \frac{A}{r} + B$ *is a solution of the differential equation*

$$\frac{d^2v}{dr^2} + \frac{2}{r}\cdot\frac{dv}{dr} = 0.$$

SOLUTION Since the given relation contains two arbitrary constants, we differentiate it two times w.r.t. r, and eliminate A and B.

$$v = \frac{A}{r} + B \Rightarrow \frac{dv}{dr} = \frac{-A}{r^2} \qquad \text{... (i)}$$

$$\Rightarrow \frac{d^2v}{dr^2} = \frac{2A}{r^3} \qquad \text{... (ii)}$$

On dividing (ii) by (i), we get

$$\frac{(d^2v/dr^2)}{(dv/dr)} = \left\{\frac{2A}{r^3} \times \frac{r^2}{(-A)}\right\} = \frac{-2}{r}$$

$$\Rightarrow \quad \frac{d^2v}{dr^2} = \frac{-2}{r}\cdot\frac{dv}{dr}$$

$$\Rightarrow \quad \frac{d^2v}{dr^2} + \frac{2}{r}\cdot\frac{dv}{dr} = 0.$$

Hence, $v = \frac{A}{r} + B$ is a solution of the differential equation

$$\frac{d^2v}{dr^2} + \frac{2}{r}\cdot\frac{dv}{dr} = 0.$$

EXAMPLE 7 *Prove that* $(x^2 - y^2) = c(x^2 + y^2)^2$ *is the general solution of the differential equation* $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$, *where* c *is a parameter.*

SOLUTION We have

$$(x^2 - y^2) = c(x^2 + y^2)^2. \quad \ldots \text{(i)}$$

On differentiating (i) w.r.t. x, we get

$$2x - 2y\frac{dy}{dx} = 2c(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right)$$

$$\Rightarrow \left(x - y\frac{dy}{dx}\right) = 2c(x^2 + y^2)\left(x + y\frac{dy}{dx}\right) \quad \ldots \text{(ii)}$$

$$\Rightarrow \left(x - y\frac{dy}{dx}\right) = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}(x^2 + y^2)\left(x + y\frac{dy}{dx}\right)$$

[putting the value of c from (i) in (ii)]

$$\Rightarrow (x^2 + y^2)\left(x - y\frac{dy}{dx}\right) = 2(x^2 - y^2)\left(x + y\frac{dy}{dx}\right)$$

$$\Rightarrow \{x(x^2 + y^2) - 2x(x^2 - y^2)\} = \{2y(x^2 - y^2) + y(x^2 + y^2)\}\frac{dy}{dx}$$

$$\Rightarrow (3xy^2 - x^3) = (3x^2y - y^3)\frac{dy}{dx}$$

$$\Rightarrow (x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy,$$

which is the required differential equation.

EXAMPLE 8 *Prove that* $xy = ae^x + be^{-x} + x^2$ *is the general solution of the differential equation* $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0.$

SOLUTION We have

$$xy = ae^x + be^{-x} + x^2. \quad \ldots \text{(i)}$$

On differentiating (i) w.r.t. x, we get

$$x\frac{dy}{dx} + y = ae^x - be^{-x} + 2x. \quad \ldots \text{(ii)}$$

On differentiating (ii) w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = ae^x + be^{-x} + 2$$

$$\Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - x^2 + 2 \quad [\because \text{ from (i)}, (ae^x + be^{-x}) = (xy - x^2)]$$

$$\Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0,$$

which is the required differential equation.

EXAMPLE 9 *Verify that* $y = Ae^{ax}\cos bx + Be^{ax}\sin bx$, *where A and B are arbitrary constants, is the general solution of the differential equation* $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0.$

SOLUTION We have

$$y = Ae^{ax}\cos bx + Be^{ax}\sin bx. \quad \ldots \text{(i)}$$

Differentiating (i) on both sides w.r.t. x, we get

$$\frac{dy}{dx} = A \cdot \{e^{ax}(-b\sin bx) + ae^{ax}\cos bx\} + B \cdot \{e^{ax}(b\cos bx) + ae^{ax}\sin bx\}$$

$$\Rightarrow \frac{dy}{dx} = a\{Ae^{ax}\cos bx + Be^{ax}\sin bx\} + b\{-Ae^{ax}\sin bx + Be^{ax}\cos bx\}$$

$$\Rightarrow \frac{dy}{dx} = ay + be^{ax}\{B\cos bx - A\sin bx\} \quad \text{... (ii)} \quad \text{[using (i)]}$$

Differentiating (ii) on both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = a\frac{dy}{dx} + b[e^{ax}(-bB\sin bx - bA\cos bx) + (B\cos bx - A\sin bx)ae^{ax}]$$

$$\Rightarrow \frac{d^2y}{dx^2} = a\frac{dy}{dx} - b^2\{Ae^{ax}\cos bx + Be^{ax}\sin bx\} + a\{be^{ax}(B\cos bx - A\sin bx)\}$$

$$\Rightarrow \frac{d^2y}{dx^2} = a\frac{dy}{dx} - b^2y + a\left(\frac{dy}{dx} - ay\right) \quad \text{[using (i) and (ii)]}$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0,$$

which is the required differential equation.

EXERCISE 18B

1. Verify that $x^2 = 2y^2 \log y$ is a solution of the differential equation $(x^2 + y^2)\frac{dy}{dx} - xy = 0.$
2. Verify that $y = e^x \cos bx$ is a solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$ **[CBSE 2006]**
3. Verify that $y = e^{m\cos^{-1}x}$ is a solution of the differential equation $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0.$
4. Verify that $y = (a + bx)e^{2x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$
5. Verify that $y = e^x(A\cos x + B\sin x)$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$ **[CBSE 2009]**
6. Verify that $y = A\cos 2x - B\sin 2x$ is the general solution of the differential equation $\frac{d^2y}{dx^2} + 4y = 0.$ **[CBSE 2007, '09]**

7. Verify that $y = ae^{2x} + be^{-x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$

8. Show that $y = e^x(A\cos x + B\sin x)$ is the solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$

9. Verify that $y^2 = 4a(x + a)$ is a solution of the differential equation $y\left\{1 - \left(\frac{dy}{dx}\right)^2\right\} = 2x\frac{dy}{dx}.$

10. Verify that $y = ce^{\tan^{-1}x}$ is a solution of the differential equation $(1 + x^2)\frac{d^2y}{dx^2} + (2x - 1)\frac{dy}{dx} = 0.$

11. Verify that $y = ae^{bx}$ is a solution of the differential equation $\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2.$

12. Verify that $y = \frac{a}{x} + b$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0.$

13. Verify that $y = e^{-x} + Ax + B$ is a solution of the differential equation $e^x\left(\frac{d^2y}{dx^2}\right) = 1.$

14. Verify that $Ax^2 + By^2 = 1$ is a solution of the differential equation $x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}.$

15. Verify that $y = \frac{c - x}{1 + cx}$ is a solution of the differential equation $(1 + x^2)\frac{dy}{dx} + (1 + y^2) = 0.$

16. Verify that $y = \log(x + \sqrt{x^2 + a^2})$ satisfies the differential equation $\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$

17. Verify that $y = e^{-3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0.$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 18B)

7. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Put these values in $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y$ to get 0.

14. Let $Ax^2 + By^2 = 1$ be given. Then,

$$2Ax + 2By\frac{dy}{dx} = 0 \Rightarrow Ax + By\frac{dy}{dx} = 0 \quad \ldots \text{(i)}$$

$$\Rightarrow Ax = -By\frac{dy}{dx} \Rightarrow \frac{A}{B} = \frac{-y}{x} \cdot \frac{dy}{dx}. \quad \ldots \text{(ii)}$$

On differentiating (i) w.r.t. x, we get

$$A + B\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0 \Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{-A}{B}$$

$$\Rightarrow \left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = \frac{y}{x} \cdot \frac{dy}{dx} \quad \text{[from (ii)]}.$$

15. Let $c = \tan A$ and $x = \tan B$. Then,

$$y = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan(A - B) \Rightarrow \tan^{-1} y = A - B$$

$$\Rightarrow \tan^{-1} y = \tan^{-1}(c) - \tan^{-1} x$$

$$\Rightarrow \frac{1}{(1 + y^2)}\frac{dy}{dx} = 0 - \frac{1}{(1 + x^2)}$$

$$\Rightarrow (1 + x^2)\frac{dy}{dx} + (1 + y^2) = 0.$$

16. $y = \log(x + \sqrt{x^2 + a^2})$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(x + \sqrt{x^2 + a^2})} \cdot \left\{1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right\} \Rightarrow y_1 = \frac{1}{\sqrt{x^2 + a^2}}$$

$$\Rightarrow y_1^2(x^2 + a^2) = 1 \Rightarrow y_1^2(2x) + (x^2 + a^2)2y_1y_2 = 0$$

$$\Rightarrow (x^2 + a^2)y_2 + xy_1 = 0.$$

Formation of a Differential Equation whose General Solution is Given

METHOD Suppose an equation of a family of curves contains n arbitrary constants (called parameters).

Then, we obtain its differential equation as given below.

Step 1. *Differentiate the equation of the given family of curves n times to get n more equations.*

Step 2. *Eliminate n constants, using these $(n + 1)$ equations.*

This gives us the required differential equation of order n.

SOLVED EXAMPLES

EXAMPLE 1 *Write the differential equation representing the family of curves $y = mx$, where m is an arbitrary constant.* **[CBSE 2013]**

SOLUTION The equation of the given family of curves is

$y = mx$ ... (i), where m is a constant.

Since the given equation contains one arbitrary constant, we differentiate it once only.

On differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = m. \quad \text{... (ii)}$$

Putting this value of m from (ii) in (i), we get

$$y = \left(\frac{dy}{dx}\right)x \Rightarrow x\left(\frac{dy}{dx}\right) - y = 0.$$

Hence, $x\left(\frac{dy}{dx}\right) - y = 0$ is the required differential equation.

EXAMPLE 2 *Find the differential equation of the family of curves $y = Ae^x + Be^{-x}$, where A and B are arbitrary constants.*

SOLUTION The equation of the given family of curves is

$$y = Ae^x + Be^{-x}. \quad \text{... (i)}$$

Since the given equation contains two arbitrary constants, we differentiate it two times w.r.t. x.

Now, $\frac{dy}{dx} = Ae^x - Be^{-x}$

$$\Rightarrow \frac{d^2y}{dx^2} = Ae^x + Be^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = y \Rightarrow \frac{d^2y}{dx^2} - y = 0.$$

Hence, $\frac{d^2y}{dx^2} - y = 0$ is the required differential equation.

EXAMPLE 3 *Find the differential equation of the family of curves $y = e^x(A\cos x + B\sin x)$, where A and B are arbitrary constants.*

SOLUTION The equation of the given family of curves is

$$y = e^x(A\cos x + B\sin x). \quad \text{... (i)}$$

Since the given equation contains two arbitrary constants, we differentiate it two times w.r.t. x.

Now, $\frac{dy}{dx} = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$

$$\Rightarrow \frac{dy}{dx} - y = e^x(-A\sin x + B\cos x) \quad \text{... (ii)}$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-A\cos x - B\sin x) + e^x(-A\ \sin x + B\cos x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} = -y + \left(\frac{dy}{dx} - y\right) \quad \text{[using (i) and (ii)]}$$

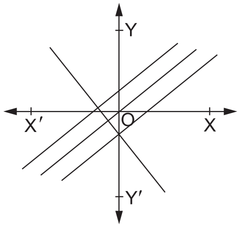
$$\Rightarrow \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0,$$ which is the required differential equation of the given family of curves.

EXAMPLE 4 *Find the differential equation of the family of all straight lines.*

SOLUTION The general equation of the family of all straight lines is given by $y = mx + c$, where m and c are parameters.

Now, $y = mx + c \Rightarrow \dfrac{dy}{dx} = m$

$$\Rightarrow \frac{d^2y}{dx^2} = 0.$$



So, the required differential equation is

$$\frac{d^2y}{dx^2} = 0.$$

EXAMPLE 5 *Form the differential equation of the family of all circles of radius r.*

[CBSE 2010]

SOLUTION The equation of the family of all circles of radius r is

$$(x-a)^2 + (y-b)^2 = r^2, \quad \text{... (i)}$$

where a and b are arbitrary constants.

Differentiating (i) w.r.t. x, we get

$$2(x-a) + 2(y-b)y_1 = 0$$

$$\Rightarrow \quad (x-a) + (y-b)y_1 = 0. \quad \text{... (ii)}$$

On differentiating (ii) again w.r.t. x, we get

$$1 + (y-b)y_2 + (y_1)^2 = 0$$

$$\Rightarrow \quad (y-b) = -\frac{\{1+(y_1)^2\}}{y_2}. \quad \text{... (iii)}$$

Putting the value of $(y-b)$ from (iii) in (ii), we get

$$(x-a) - \frac{\{1+(y_1)^2\}y_1}{y_2} = 0 \Rightarrow (x-a) = \frac{\{1+(y_1)^2\}y_1}{y_2}. \quad \text{... (iv)}$$

Putting the values of $(y-b)$ and $(x-a)$ from (iii) and (iv) in (i), we get

$$\frac{\{1+(y_1)^2\}^2 \times (y_1)^2}{(y_2)^2} + \frac{\{1+(y_1)^2\}^2}{(y_2)^2} = r^2$$

$$\Rightarrow \quad \frac{\{1+(y_1)^2\}^2 \cdot \{(y_1)^2+1\}}{(y_2)^2} = r^2$$

$$\Rightarrow \quad \{1+(y_1)^2\}^3 = r^2(y_2)^2,$$

which is the required differential equation.

EXAMPLE 6 *Form the differential equation of the family of all circles in first quadrant and touching the coordinate axes.* **[CBSE 2010]**

SOLUTION The general equation of a circle in first quadrant and touching the coordinate axes is given by

$$(x-a)^2+(y-a)^2=a^2 \qquad \text{... (i), where } a \text{ is a parameter.}$$

Since this equation contains one parameter, so we will differentiate it only once to get the differential equation. On differentiating (i) w.r.t, x, we get

$$2(x-a)+2(y-a)\frac{dy}{dx}=0$$

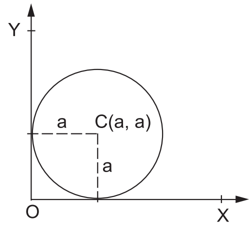
$$\Rightarrow \quad (x-a)+(y-a)\frac{dy}{dx}=0$$

$$\Rightarrow \quad (x-a)+(y-a)p=0, \text{ where } \frac{dy}{dx}=p$$

$$\Rightarrow \quad x+yp=a(1+p) \Rightarrow a=\frac{(x+yp)}{(1+p)}. \qquad \text{... (ii)}$$

Putting the value of a from (ii) in (i), we get

$$\left(x-\frac{x+yp}{1+p}\right)^2+\left(y-\frac{x+yp}{1+p}\right)^2=\left(\frac{x+yp}{1+p}\right)^2$$



$$\Rightarrow \quad (xp-yp)^2+(y-x)^2=(x+yp)^2$$

$$\Rightarrow \quad (x-y)^2p^2+(x-y)^2=(x+yp)^2$$

$$\Rightarrow \quad (x-y)^2(p^2+1)=(x+yp)^2$$

$$\Rightarrow \quad (x-y)^2\left\{\left(\frac{dy}{dx}\right)^2+1\right\}=\left(x+y\frac{dy}{dx}\right)^2,$$

which is the required differential equation.

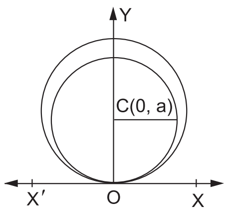
EXAMPLE 7 *Form the differential equation of the family of all circles touching the x-axis at the origin.* **[CBSE 2005, '08, '10C]**

SOLUTION The general equation of a circle touching the x-axis at the origin, is given by

$$(x-0)^2+(y-a)^2=a^2, \text{ where } a \text{ is a parameter}$$

$$\Rightarrow \quad x^2+y^2-2ay=0. \qquad \text{... (i)}$$

Since this equation contains one parameter, so we shall differentiate it only once to get the differential equation.



On differentiating (i) w.r.t. x, we get

$$2x+2y\frac{dy}{dx}-2a\frac{dy}{dx}=0$$

$$\Rightarrow \quad x+y\frac{dy}{dx}=a\frac{dy}{dx}$$

$$\Rightarrow \quad x+yp=ap \Rightarrow a=\frac{x+yp}{p} \qquad \text{... (ii), where } \frac{dy}{dx}=p.$$

Putting the value of a from (ii) in (i), we get

$$x^2 + y^2 = \frac{2(x + yp) \cdot y}{p}$$

$$\Rightarrow \quad x^2 + y^2 = \frac{2xy}{p} + 2y^2$$

$$\Rightarrow \quad x^2 - y^2 = \frac{2xy}{p}$$

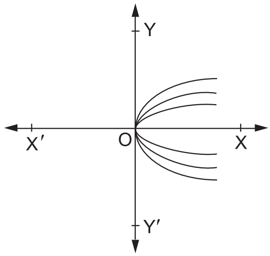
$$\Rightarrow \quad (x^2 - y^2)p = 2xy.$$

Hence, $(x^2 - y^2)\frac{dy}{dx} = 2xy$ is the required differential equation.

EXAMPLE 8 *Form the differential equation of the family of all parabolas having vertex at the origin and axis along the positive direction of the x-axis.*

SOLUTION The general equation of a parabola having vertex at the origin and axis along the positive direction of the x-axis is given by

$y^2 = 4ax$ ... (i), where a is the parameter.



Since this equation contains one parameter, so we shall differentiate it only once to get the requisite differential equation.

Differentiating (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a \Rightarrow y\frac{dy}{dx} = 2a$$

$$\Rightarrow a = \frac{1}{2}y\frac{dy}{dx}. \qquad \text{... (ii)}$$

Putting the value of a from (ii) in (i), we get

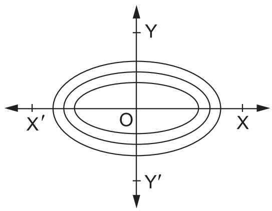
$$y^2 = 4 \times \frac{1}{2}y\frac{dy}{dx} \times x \Rightarrow y^2 - 2xy\frac{dy}{dx} = 0.$$

Hence, $y^2 - 2xy\frac{dy}{dx} = 0$ is the required differential equation.

EXAMPLE 9 *Form the differential equation of the family of all ellipses having foci on the x-axis and centre at the origin.* **[CBSE 2009C]**

SOLUTION The general equation of an ellipse having foci on the x-axis and centre at the origin, is given by

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... (i), where a and b are the parameters.



Since this equation contains two parameters, so we shall differentiate it twice to get the required differential equation.

Differentiating (i) w.r.t. x, we get:

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{y}{b^2} \cdot \frac{dy}{dx} + \frac{x}{a^2} = 0$$

$$\Rightarrow \frac{yy_1}{b^2} = \frac{-x}{a^2}, \text{ where } \frac{dy}{dx} = y_1$$

$$\Rightarrow \frac{yy_1}{x} = \frac{-b^2}{a^2}. \qquad \text{... (ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{x \cdot \frac{d}{dx}(yy_1) - yy_1 \cdot \frac{d}{dx}(x)}{x^2} = 0$$

$$\Rightarrow \quad x[yy_2 + (y_1)^2] - yy_1 = 0$$

$$\Rightarrow \quad (xy)y_2 + x(y_1)^2 - yy_1 = 0.$$

Hence, $(xy)\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) = 0$ is the required differential equation.

EXAMPLE 10 *Form the differential equation for the family of the curves $(y-b)^2 = 4(x-a)$, where a and b are parameters.*

SOLUTION The general equation of the given family of curves is

$$(y-b)^2 = 4(x-a) \qquad \text{... (i), where } a \text{ and } b \text{ are parameters.}$$

Since the given equation contains two parameters a and b, so we shall differentiate it twice to get the required differential equation.

Differentiating (i) w.r.t. x, we get

$$2(y-b)\frac{dy}{dx} = 4 \Rightarrow (y-b)y_1 = 2 \qquad \text{...(ii), where } \frac{dy}{dx} = y_1.$$

Differentiating (ii) w.r.t. x, we get

$$(y-b)y_2 + (y_1)^2 = 0 \qquad \text{... (iii), where } \frac{d^2y}{dx^2} = y_2.$$

Putting $(y-b) = \frac{2}{y_1}$ from (ii) in (iii), we get

$$\frac{2y_2}{y_1} + (y_1)^2 = 0 \Rightarrow 2y_2 + (y_1)^3 = 0.$$

Hence, $2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$ is the required differential equation.

EXAMPLE 11 *Form the differential equation for the family of the curves $ay^2 = (x-c)^3$, where c is a parameter.*

SOLUTION The general equation of the given family of curves is

$$ay^2 = (x-c)^3, \quad \text{where } c \text{ is a parameter.} \qquad \text{... (i)}$$

Since the given equation has only one parameter, so we shall differentiate it only once to get the required differential equation.

Differentiating (i) w.r.t. x, we get

$$2ayy_1 = 3(x-c)^2. \qquad \text{... (ii)}$$

On dividing (i) and (ii) on each side, we get

$$\frac{ay^2}{2ayy_1} = \frac{(x-c)^3}{3(x-c)^2} \Rightarrow \frac{y}{2y_1} = \frac{(x-c)}{3} \Rightarrow (x-c) = \frac{3y}{2y_1}. \quad \ldots \text{(iii)}$$

Putting the value of $(x-c)$ from (iii) in (i), we get

$$ay^2 = \left(\frac{3y}{2y_1}\right)^3 \Rightarrow 8ay_1^3 = 27y.$$

Hence, $8a\left(\frac{dy}{dx}\right)^3 - 27y = 0$ is the required differential equation.

EXAMPLE 12 *Form the differential equation for the family of the curves $y^2 = a(b^2 - x^2)$, where a and b are arbitrary constants.*

SOLUTION The general equation of the given family of curves is

$y^2 = a(b^2 - x^2)$ ...(i), where a and b are the parameters.

Since the given equation has two parameters, so we shall have to differentiate it two times to get the required differential equations.

Differentiating (i) w.r.t. x we get

$$2yy_1 = -2ax \Rightarrow yy_1 = -ax. \quad \ldots \text{(ii)}$$

On differentiating (ii) w.r.t. x, we get

$$yy_2 - (y_1)^2 = -a$$

$$\Rightarrow \quad yy_2 - (y_1)^2 = \frac{yy_1}{x} \quad \text{[using (ii)]}$$

$$\Rightarrow \quad (xy)y_2 - x(y_1)^2 - yy_1 = 0.$$

Hence, $(xy)\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) = 0$ is the required differential equation.

EXERCISE 18C

1. Form the differential equation of the family of straight lines $y = mx + c$, where m and c are arbitrary constants.
2. Form the differential equation of the family of concentric circles $x^2 + y^2 = a^2$, where $a > 0$ and a is a parameter.
3. Form the differential equation of the family of curves, $y = a\sin(bx + c)$, where a and c are parameters.
4. Form the differential equation of the family of curves $x = A\cos nt + B\sin nt$, where A and B are arbitrary constants. **[CBSE 2007]**
5. Form the differential equation of the family of curves $y = ae^{bx}$, where a and b are arbitrary constants.
6. Form the differential equation of the family of curves $y^2 = m(a^2 - x^2)$, where a and m are parameters.

7. Form the differential equation of the family of curves given by $(x-a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.

8. Form the differential equation of the family of curves given by $x^2 + y^2 - 2ay = a^2$, where a is an arbitrary constant. **[CBSE 2005]**

9. Form the differential equation of the family of all circles touching the y-axis at the origin. **[CBSE 2008C, '10C]**

10. Form the differential equation of the family of circles having centres on the y-axis and radius 2 units.

11. Form the differential equation of the family of circles in second quadrant and touching the coordinate axes.

12. Form the differential equation of the family of circles having centres on the x-axis and radius unity.

13. Form the differential equation of the family of circles passing through the fixed points $(a, 0)$ and $(-a, 0)$, where a is the parameter.

14. Form the differential equation of the family of parabolas having vertex at the origin and axis along positive y-axis. **[CBSE 2010, '11]**

15. Form the differential equation of the family of ellipses having foci on the y-axis and centre at the origin.

16. Form the differential equation of the family of hyperbolas having foci on the x-axis and centre at the origin.

ANSWERS (EXERCISE 18C)

1. $\frac{d^2y}{dx^2} = 0$ **2.** $x + y\frac{dy}{dx} = 0$ **3.** $\frac{d^2y}{dx^2} + b^2y = 0$ **4.** $\frac{d^2x}{dt^2} + n^2x = 0$

5. $\left(\frac{dy}{dx}\right)^2 = y \cdot \frac{d^2y}{dx^2}$ **6.** $xy\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$ **7.** $\frac{dy}{dx} = \frac{2y^2 - x^2}{4xy}$

8. $p^2(x^2 - 2y^2) - 4pxy - x^2 = 0$, where $p = \frac{dy}{dx}$ **9.** $y^2 - x^2 - 2xy\frac{dy}{dx} = 0$

10. $x^2\left\{1 + \frac{1}{(dy/dx)^2}\right\} = 4$ **11.** $(x+y)^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = \left(x + y\frac{dy}{dx}\right)^2$

12. $y^2\left(\frac{dy}{dx}\right)^2 + y^2 = 1$ **13.** $(x^2 - y^2 - a^2)\frac{dy}{dx} = 2xy$ **14.** $x\frac{dy}{dx} = 2y$

15. $xy\left(\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$ **16.** $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 18C)

6. See Example 12.

7. Given family of curves is $x^2 + 2y^2 - 2ax = 0$. ... (i)

   Differentiating (i) w.r.t. x, we get

   $$2x + 4y\frac{dy}{dx} - 2a = 0 \Rightarrow a = \left(2y\frac{dy}{dx} + x\right).$$

   Put this value of a in (i).

8. Given family of curves is $x^2 + y^2 - 2ay = a^2$. ... (i)

   Differentiating (i) w.r.t. x, we get

   $$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0 \Rightarrow x + y\frac{dy}{dx} = a\frac{dy}{dx}$$

   $$\Rightarrow a = \frac{x + yp}{p}, \text{ where } p = \frac{dy}{dx}.$$

   Putting $a = \frac{x + yp}{p}$ in (i), we get:

   $$x^2 + y^2 - \frac{2(x + yp)y}{p} = \left(\frac{x + yp}{p}\right)^2$$

   $\Rightarrow\ (x^2 + y^2)p^2 - 2yp(x + yp) = (x + yp)^2$

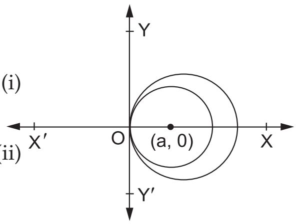
   $\Rightarrow\ x^2p^2 + y^2p^2 - 2xyp - 2y^2p^2 = x^2 + y^2p^2 + 2xyp$

   $\Rightarrow\ x^2p^2 - 2y^2p^2 - 4xyp - x^2 = 0$

   $\Rightarrow\ (x^2 - 2y^2)p^2 - 4xyp - x^2 = 0$, where $p = \frac{dy}{dx}$.

9. The equation of given family of circles is

   $$(x - a)^2 + (y - 0)^2 = a^2 \Rightarrow x^2 + y^2 - 2ax = 0. \quad \text{... (i)}$$

   Differentiating (i) w.r.t. x, we get

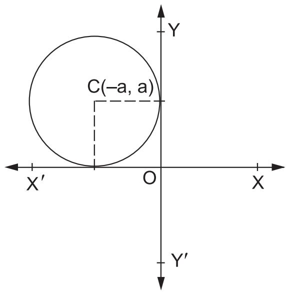
   $$2x + 2y\frac{dy}{dx} - 2a = 0 \Rightarrow x + y\frac{dy}{dx} = a. \quad \text{... (ii)}$$

   Put the value of a from (ii) in (i), we get the desired differential equation.

10. The given family is $x^2 + (y - a)^2 = 4$.

11. The equation of the given family of circles is

    $$(x + a)^2 + (y - a)^2 = a^2 \quad \text{... (i),}$$

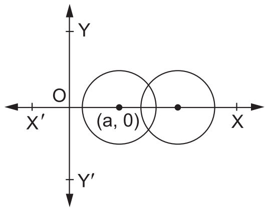
    where a is a parameter.
    Differentiating (i) w.r.t. x, we get

    $$2(x + a) + 2(y - a)\frac{dy}{dx} = 0 \Rightarrow (x + a) + (y - a)p = 0$$

    $$\Rightarrow a = \frac{(x + yp)}{(p - 1)}.$$

    Put this value of a in (i) to get the required equation.

12. The equation of the given family of circles with centre $(a, 0)$ and radius 1, is given by

    $(x - a)^2 + (y - 0)^2 = 1^2$, where a is the parameter

    $\Rightarrow\ (x - a)^2 + y^2 = 1.$ ... (i)

13. The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \text{... (i)}$$

If it passes through the points $A(-a, 0)$ and $B(a, 0)$, we have

$$a^2 - 2ga + c = 0 \quad \text{... (ii) and } a^2 + 2ga + c = 0 \quad \text{... (iii)}$$

Adding (ii) and (iii), we get $2(a^2 + c) = 0 \Rightarrow a^2 + c = 0 \Rightarrow c = -a^2$.

Putting $c = -a^2$ in (iii), we get $2ga = 0 \Rightarrow g = 0$.

Putting $g = 0$ and $c = -a^2$ in (i), we get

$$x^2 + y^2 + 2fy - a^2 = 0 \quad \text{... (iv), where } f \text{ is the parameter.}$$

Differentiating (iv) w.r.t. x, we get

$$2x + 2yy_1 + 2fy_1 = 0 \Rightarrow fy_1 = -(x + yy_1) \Rightarrow f = \frac{-(x + yy_1)}{y_1}.$$

Putting this value of f in (iv), we get

$$x^2 + y^2 - \frac{2y(x + yy_1)}{y_1} - a^2 = 0$$

$$\Rightarrow \quad x^2y_1 + y^2y_1 - 2xy - 2y^2y_1 - a^2y_1 = 0 \Rightarrow (x^2 - y^2 - a^2)y_1 = 2xy.$$

14. The equation of the given family of parabolas is given by $x^2 = 4ay$.

15. The equation of the family of ellipses having centre at the origin and foci on the y-axis, is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{... (i), where } b > a \text{ and } a, b \text{ are the parameters.}$$

Differentiating (i), w.r.t. x, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{x}{a^2} + \frac{yy_1}{b^2} = 0. \quad \text{... (ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{1}{a^2} + \frac{yy_2}{b^2} + \frac{y_1^2}{b^2} = 0 \Rightarrow \frac{x}{a^2} + \frac{xyy_2}{b^2} + \frac{xy_1^2}{b^2} = 0. \quad \text{... (iii)}$$

Subtracting (ii) from (iii), we get

$$\frac{1}{b^2}\{xyy_2 + xy_1^2 - yy_1\} = 0 \Rightarrow xy\left(\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) = 0.$$

16. The equation of the family of hyperbolas having foci on the x-axis and centre at the origin is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{... (i), where } a \text{ and } b \text{ are the parameters.}$$

Differentiating (i), w.r.t. x, we get

$$\frac{2x}{a^2} - \frac{2yy_1}{b^2} = 0 \Rightarrow \frac{x}{a^2} - \frac{yy_1}{b^2} = 0. \quad \text{... (ii)}$$

Differentiating (ii), w.r.t. x, we get

$$\frac{1}{a^2} - \frac{yy_2}{b^2} - \frac{y_1^2}{b^2} = 0. \quad \text{... (iii)}$$

Multiplying (iii) by x and subtracting from (ii), we get

$$\frac{1}{b^2}\{xyy_2 + xy_1^2 - yy_1\} = 0 \Rightarrow xy\left(\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0.$$

19. DIFFERENTIAL EQUATIONS WITH VARIABLE SEPARABLE

Method of Solving Differential Equations with Variables Separable

GENERAL SOLUTION *Let the given differential equation be expressed in the form* $g(y)\,dy = f(x)\,dx$.

Then, the ***general solution*** *of the above differential equation is given by* $\int \boldsymbol{g(y)dy} = \int \boldsymbol{f(x)dx + C}$, *where C is an arbitrary constant.*

PARTICULAR SOLUTION *Let $x = a$ be given and let the corresponding value of y be given as b. Putting $x = a$ and $y = b$ in the general solution, we get the value of C. With this value of C, we get the particular solution of the given differential equation.*

SOLVED EXAMPLES

EXAMPLE 1 *Find the general solution of the differential equation*

$$(x+2)\frac{dy}{dx} = x^2 + 5x - 3 (x \neq -2).$$

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{x^2+5x-3}{x+2}$$

$$\Rightarrow \quad dy = \left(\frac{x^2+5x-3}{x+2}\right)dx \quad \text{[separating the variables]}$$

$$\Rightarrow \quad \int dy = \int \left(\frac{x^2+5x-3}{x+2}\right)dx$$

$$\Rightarrow \quad y = \int \left\{x + 3 - \frac{9}{(x+2)}\right\}dx + C, \text{ where } C \text{ is an arbitrary constant}$$

[on dividing (x^2+5x-3) by $(x+2)$]

$$\Rightarrow \quad y = \frac{x^2}{2} + 3x - 9\log|x+2| + C.$$

Hence, $y = \frac{x^2}{2} + 3x - 9\log|x+2| + C$ is the required general solution.

EXAMPLE 2 *Find the general solution of the differential equation*

$$(1+x^2)\frac{dy}{dx} - x = 2\tan^{-1}x.$$ **[CBSE 2007]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{2\tan^{-1}x}{(1+x^2)} + \frac{x}{(1+x^2)}$$

$$\Rightarrow \quad dy = \left\{\frac{2\tan^{-1}x}{(1+x^2)} + \frac{x}{(1+x^2)}\right\}dx \quad \text{[separating the variables]}$$

$$\Rightarrow \quad \int dy = \int \frac{2\tan^{-1}x}{(1+x^2)}dx + \int \frac{x}{(1+x^2)}dx + C,$$

where C is an arbitrary constant

$$\Rightarrow \quad y = 2\int t\,dt + \frac{1}{2}\int \frac{2x}{(1+x^2)}dx + C$$

[putting $\tan^{-1}x = t$ and $\frac{1}{(1+x^2)}dx = dt$ in 1st integral]

$$\Rightarrow \quad y = t^2 + \frac{1}{2}\log|1+x^2| + C$$

$$\Rightarrow \quad y = (\tan^{-1}x)^2 + \frac{1}{2}\log|1+x^2| + C.$$

Hence, $y = (\tan^{-1}x)^2 + \frac{1}{2}\log|1+x^2| + C$ is the required solution.

EXAMPLE 3 *Find the general solution of the differential equation* $\frac{dy}{dx} = \log(x+1)$.

[CBSE 2006]

SOLUTION We have

$$\frac{dy}{dx} = \log(x+1)$$

$$\Rightarrow \quad dy = \log(x+1)\,dx$$

$$\Rightarrow \quad \int dy = \int \log(x+1)\,dx \quad \text{[integrating both sides]}$$

$$\Rightarrow \quad y = \int \{\underset{\text{I}}{\log(x+1)} \cdot \underset{\text{II}}{1}\}\,dx + C, \text{ where } C \text{ is an arbitrary constant}$$

$$= \{\log(x+1)\cdot x\} - \int \frac{1}{(x+1)}\cdot x\,dx + C \quad \text{[integrating by parts]}$$

$$= x\log(x+1) - \int \frac{(x+1)-1}{(x+1)}dx + C$$

$$= x\log(x+1) - \int\left\{1 - \frac{1}{(x+1)}\right\}dx + C$$

$$= x\log(x+1) - x + \log(x+1) + C$$

$$= (x+1)\log(x+1) - x + C.$$

Hence, $y = (x+1)\log(x+1) - x + C$ is the required solution.

EXAMPLE 4 *Find the general solution of the differential equation*

$$\frac{dy}{dx} = \sin^{-1}x.$$

SOLUTION We have

$$\frac{dy}{dx} = \sin^{-1} x$$

$$\Rightarrow \quad dy = \sin^{-1} x \, dx \qquad \text{[separating the variables]}$$

$$\Rightarrow \quad \int dy = \int \sin^{-1} x \, dx \quad \text{[integrating both sides]}$$

$$\Rightarrow \quad y = \int \{(\underset{\text{I}}{\sin^{-1} x}) \cdot \underset{\text{II}}{1}\}\, dx + C, \text{ where } C \text{ is an arbitrary constant}$$

$$= (\sin^{-1} x) \cdot x - \int \frac{1}{\sqrt{1-x^2}} \cdot x \, dx + C \quad \text{[integrating by parts]}$$

$$= (\sin^{-1} x) \cdot x + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} \, dx + C$$

$$= (\sin^{-1} x) \cdot x + \frac{1}{2} \int \frac{1}{\sqrt{t}} \, dt + C, \text{ where } (1-x^2) = t$$

$$= (\sin^{-1} x) \cdot x + \frac{1}{2} \times 2\sqrt{t} + C$$

$$= (\sin^{-1} x) x + \sqrt{1-x^2} + C.$$

Hence, $y = (\sin^{-1} x) x + \sqrt{1-x^2} + C$ is the required solution.

Hence, $y = x(\sin^{-1} x) + \sqrt{1-x^2} + C$ is the required solution of the given differential equation.

EXAMPLE 5 *Find the general solution of the differential equation*

$$\frac{dy}{dx} = \sqrt{4-y^2} \; (-2 < y < 2). \qquad \textbf{[CBSE 2002C]}$$

SOLUTION We have $\dfrac{dy}{dx} = \sqrt{4-y^2}$

$$\Rightarrow \quad \frac{dy}{\sqrt{4-y^2}} = dx \qquad \text{[on separating the variables]}$$

$$\Rightarrow \quad \int \frac{dy}{\sqrt{2^2-y^2}} = \int dx \quad \text{[integrating both sides]}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{y}{2}\right) = x + C, \text{ where } C \text{ is an arbitrary constant.}$$

Hence, $\sin^{-1}\left(\dfrac{y}{2}\right) = x + C$ is the required solution.

EXAMPLE 6 *Find the general solution of the differential equation*

$$(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x. \qquad \textbf{[CBSE 2010]}$$

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{2x^2 + x}{(x^3 + x^2 + x + 1)} = \frac{(2x^2 + x)}{x^2(x+1) + (x+1)} = \frac{(2x^2 + x)}{(x+1)(x^2+1)}$$

$$\Rightarrow \quad dy = \left\{ \frac{(2x^2 + x)}{(x+1)(x^2+1)} \right\} dx$$

$$\Rightarrow \quad \int dy = \int \frac{(2x^2 + x)}{(x+1)(x^2+1)} dx \qquad \dots \text{(i)} \quad \text{[integrating both sides]}$$

$$\Rightarrow \quad y = \int \frac{(2x^2 + x)}{(x+1)(x^2+1)} dx + C,$$ where C is an arbitrary constant.

Let $\dfrac{2x^2 + x}{(x+1)(x^2+1)} = \dfrac{A}{(x+1)} + \dfrac{Bx + C}{(x^2+1)} \cdot$

Then, $2x^2 + x \equiv A(x^2 + 1) + (Bx + C)(x + 1).$ … (ii)

Putting $x = -1$ in (ii), we get $2A = 1 \Rightarrow A = \dfrac{1}{2} \cdot$

Putting $x = 0$ in (ii), we get $A + C = 0 \Rightarrow C = -A = \dfrac{-1}{2} \cdot$

Putting $x = 1$ in (ii), we get $2A + 2B + 2C = 3$.

$\therefore \quad A + B + C = \dfrac{3}{2} \Rightarrow \dfrac{1}{2} + B - \dfrac{1}{2} = \dfrac{3}{2} \Rightarrow B = \dfrac{3}{2} \cdot$

$\therefore \quad A = \dfrac{1}{2}, B = \dfrac{3}{2}$ and $C = \dfrac{-1}{2} \cdot$

$$\therefore \quad y = \frac{1}{2} \int \frac{dx}{(x+1)} + \int \frac{\left(\frac{3}{2}x - \frac{1}{2} \right)}{(x^2+1)} dx + C$$

$$\Rightarrow \quad y = \frac{1}{2} \int \frac{dx}{(x+1)} + \frac{3}{2} \int \frac{x}{(x^2+1)} dx - \frac{1}{2} \int \frac{dx}{(x^2+1)} + C$$

$$\Rightarrow \quad y = \frac{1}{2} \log |x+1| + \frac{3}{4} \int \frac{2x}{(x^2+1)} dx - \frac{1}{2} \tan^{-1} x + C$$

$$\Rightarrow \quad y = \frac{1}{2} \log |x+1| + \frac{3}{4} \log |x^2+1| - \frac{1}{2} \tan^{-1} x + C.$$

Hence, $y = \dfrac{1}{2} \log |x+1| + \dfrac{3}{4} \log |x^2+1| - \dfrac{1}{2} \tan^{-1} x + C$ is the required solution.

EXAMPLE 7 *Find the general solution of the differential equation*

$$\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2} \cdot$$

SOLUTION $\dfrac{dy}{dx} = \dfrac{1 + y^2}{1 + x^2}$

$$\Rightarrow \quad \frac{1}{(1 + y^2)} dy = \frac{1}{(1 + x^2)} dx \quad \text{[separating the variables]}$$

$\Rightarrow \quad \int \frac{1}{(1+y^2)} dy = \int \frac{1}{(1+x^2)} dx$

$\Rightarrow \quad \tan^{-1} y = \tan^{-1} x + C_1$, where C_1 is an arbitrary constant

$\Rightarrow \quad \tan^{-1} y - \tan^{-1} x = C_1$

$\Rightarrow \quad \tan^{-1}\left(\frac{y-x}{1+yx}\right) = C_1$

$\Rightarrow \quad \frac{y-x}{1+yx} = \tan C_1 = C$, where $C = \tan C_1$.

Hence, $\frac{y-x}{1+yx} = C$ is the required solution.

EXAMPLE 8 *Find the general solution of the differential equation*

$$\log\left(\frac{dy}{dx}\right) = (ax+by).$$

SOLUTION $\log\left(\frac{dy}{dx}\right) = ax + by$

$\Rightarrow \quad \frac{dy}{dx} = e^{ax+by} = e^{ax} \cdot e^{by}$

$\Rightarrow \quad \frac{1}{e^{by}} dy = e^{ax} dx$ [on separating the variables]

$\Rightarrow \quad \int e^{-by} dy = \int e^{ax} dx$ [integrating both sides]

$\Rightarrow \quad \frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + C$

$\Rightarrow \quad ae^{-by} + be^{ax} = C'$, where $C' = -abC$.

Thus, $ae^{-by} + be^{ax} = C'$ is the required solution.

EXAMPLE 9 *Find the general solution of the differential equation*

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0.$$ **[CBSE 2010]**

SOLUTION The given differential equation may be written as

$\sqrt{(1+x^2)+y^2(1+x^2)} + xy\frac{dy}{dx} = 0$

$\Rightarrow \quad \sqrt{(1+x^2)(1+y^2)} + xy\frac{dy}{dx} = 0$

$\Rightarrow \quad (\sqrt{1+x^2})(\sqrt{1+y^2}) + xy\frac{dy}{dx} = 0$

$\Rightarrow \quad xy\frac{dy}{dx} = -(\sqrt{1+x^2})(\sqrt{1+y^2})$

$\Rightarrow \quad \frac{y}{\sqrt{1+y^2}} dy = \frac{-\sqrt{1+x^2}}{x} dx$

$$\Rightarrow \quad \int \frac{y}{\sqrt{1+y^2}}\,dy = -\int \frac{\sqrt{1+x^2}}{x^2}\cdot x\,dx. \qquad \ldots \text{(i)}$$

Putting $1+x^2=u^2$ and $1+y^2=v^2$, we get

$$2x\,dx = 2u\,du \text{ and } 2y\,dy = 2v\,dv$$

$$\Rightarrow \quad x\,dx = u\,du \text{ and } y\,dy = v\,dv.$$

Substituting these values in (i), we get

$$\int \frac{v\,dv}{v} = -\int \frac{u \times u\,du}{(u^2-1)}$$

$$\Rightarrow \quad \int dv = -\int \frac{u^2}{(u^2-1)}\,du = -\int \frac{(u^2-1)+1}{(u^2-1)}\,du$$

$$\Rightarrow \quad v = -\int \left\{1+\frac{1}{(u^2-1)}\right\}du + C, \text{ where } C \text{ is an arbitrary constant}$$

$$\Rightarrow \quad v = -u - \frac{1}{2}\log\left|\frac{u-1}{u+1}\right| + C$$

$$\Rightarrow \quad \sqrt{1+y^2} = -\sqrt{1+x^2} - \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C,$$

which is the required solution.

EXAMPLE 10 *Find the general solution of the differential equation*

$$(x\cos y)\,dy = e^x(x\log x + 1)\,dx.$$ **[CBSE 2007]**

SOLUTION The given differential equation may be written as

$$\cos y\,dy = \left(e^x \log x + \frac{e^x}{x}\right)dx$$

$$\Rightarrow \quad \int \cos y\,dy = \int (\underset{\text{I}}{\log x})\,\underset{\text{II}}{e^x}\,dx + \int \frac{e^x}{x}\,dx + C,$$

where C is an arbitrary constant

$$\Rightarrow \quad \sin y = (\log x)e^x - \int \frac{1}{x}\cdot e^x\,dx + \int \frac{e^x}{x}\,dx + C \text{ [integrating by parts]}$$

$$\Rightarrow \quad \sin y = (\log x)e^x + C, \text{ which is the required solution.}$$

EXAMPLE 11 *Find the general solution of the differential equation*

$$x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0.$$

SOLUTION We have

$$x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$$

$$\Rightarrow \quad \frac{x}{\sqrt{1-x^2}}\,dx + \frac{y}{\sqrt{1-y^2}}\,dy = 0 \quad \text{[on separating the variables]}$$

$$\Rightarrow \int \frac{x}{\sqrt{1-x^2}}\,dx + \int \frac{y}{\sqrt{1-y^2}}\,dy = C \quad \text{[integrating both sides]}$$

$$\Rightarrow -\frac{1}{2}\cdot\int \frac{(-2x)}{\sqrt{1-x^2}}\,dx - \frac{1}{2}\cdot\int \frac{(-2y)}{\sqrt{1-y^2}}\,dy = C$$

$$\Rightarrow -\frac{1}{2}\cdot\int \frac{1}{\sqrt{t}}\,dt - \frac{1}{2}\cdot\int \frac{1}{\sqrt{s}}\,ds = C, \text{ where } (1-x^2) = t \text{ and } (1-y^2) = s$$

$$\Rightarrow -\frac{1}{2}\int t^{-1/2}dt - \frac{1}{2}\int s^{-1/2}ds = C$$

$$\Rightarrow -\sqrt{t} - \sqrt{s} = C$$

$$\Rightarrow \sqrt{t} + \sqrt{s} = k \quad [\text{where } k = -C]$$

$$\Rightarrow \sqrt{1-x^2} + \sqrt{1-y^2} = k.$$

Hence, $\sqrt{1-x^2} + \sqrt{1-y^2} = k$ is the required solution.

EXAMPLE 12 *Find the general solution of the differential equation*

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right).$$ **[CBSE 2008]**

SOLUTION We have

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$$

$$\Rightarrow (y - ay^2) = (a + x)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{y(1-ay)} = \frac{dx}{(a+x)} \quad \text{[on separating the variables]}$$

$$\Rightarrow \int \frac{dy}{y(1-ay)} = \int \frac{dx}{(a+x)} \quad \text{[integrating both sides]}$$

$$\Rightarrow \int \left(\frac{1}{y} + \frac{a}{1-ay}\right)dy = \int \frac{dx}{(a+x)} \quad \text{[by partial fractions]}$$

$$\Rightarrow \log|y| - \log|1-ay| = \log|a+x| + \log|C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \log\left|\frac{y}{(1-ay)(a+x)}\right| = \log|C_1|$$

$$\Rightarrow \frac{y}{(1-ay)(a+x)} = \pm C_1 = C \text{ (say)}.$$

Hence, $y = C(1-ay)(a+x)$ is the required solution.

EXAMPLE 13 *Find the general solution of the differential equation*

$$(\sqrt{a+x})\frac{dy}{dx} + x = 0.$$

SOLUTION We have

$$\frac{dy}{dx} = \frac{-x}{\sqrt{a+x}}$$

$\Rightarrow \quad dy = \dfrac{-x}{\sqrt{a+x}}dx$ [on separating the variables]

$\Rightarrow \quad \int dy = \int \dfrac{-x}{\sqrt{a+x}}dx$ [integrating both sides]

$\Rightarrow \quad y = -\int \dfrac{[(a+x)-a]}{\sqrt{a+x}}dx$

$\Rightarrow \quad y = -\int \left(\sqrt{a+x} - \dfrac{a}{\sqrt{a+x}}\right)dx$

$\Rightarrow \quad y = -\int \sqrt{a+x}\,dx + a\int (a+x)^{-1/2}dx$

$\Rightarrow \quad y = -\dfrac{2}{3}(a+x)^{3/2} + 2a\sqrt{a+x} + C$, which is the required solution.

EXAMPLE 14 *Solve the differential equation*

$$x \cos y\, dy = (xe^x \log x + e^x)dx.$$

SOLUTION We have

$$x \cos y\, dy = (xe^x \log x + e^x)dx$$

$\Rightarrow \quad \cos y\, dy = e^x\left(\log x + \dfrac{1}{x}\right)dx$ [on separating the variables]

$\Rightarrow \quad \int \cos y\, dy = \int e^x\left(\log x + \dfrac{1}{x}\right)dx$ [integrating both sides]

$\Rightarrow \quad \sin y = e^x \log x + C \quad [\because \int e^x\{f(x) + f'(x)\}\,dx = e^x f(x)].$

Hence, $\sin y = e^x(\log x) + C$ is the required solution.

EXAMPLE 15 *Solve the differential equation*

$$\frac{dy}{dx} = \frac{e^x(\sin^2 x + \sin 2x)}{y(2\log y + 1)}.$$

SOLUTION We have

$$\frac{dy}{dx} = \frac{e^x(\sin^2 x + \sin 2x)}{y(2\log y + 1)}$$

$\Rightarrow \quad y(2\log y + 1)dy = e^x(\sin^2 x + \sin 2x)dx$

$\Rightarrow \quad 2\int \underset{\text{II}}{y} \underset{\text{I}}{\log y}\, dy + \int y\, dy = \int e^x(\sin^2 x + \sin 2x)\, dx$

$\Rightarrow \quad 2\left[(\log y)\cdot\dfrac{y^2}{2} - \int \dfrac{1}{y}\cdot\dfrac{y^2}{2}dy\right] + \dfrac{1}{2}y^2 = \int e^x(\sin^2 x + \sin 2x)dx$

[integrating by parts]

$\Rightarrow \quad y^2(\log y) - \int y\, dy + \dfrac{1}{2}y^2 = \int e^x(\sin^2 x + \sin 2x)dx$

$\Rightarrow \quad y^2(\log y) = e^x \sin^2 x + C \quad [\because \int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C].$

$\therefore \quad y^2(\log y) = e^x \sin^2 x + C$ is the required solution.

EXAMPLE 16 *Solve the differential equation*

$$(1+x)(1+y^2)\,dx + (1+y)(1+x^2)\,dy = 0.$$

SOLUTION $(1+x)(1+y^2)\,dx + (1+y)(1+x^2)\,dy = 0$

$$\Rightarrow \frac{(1+x)}{(1+x^2)}dx + \frac{(1+y)}{(1+y^2)}dy = 0 \quad \text{[on separating the variables]}$$

$$\Rightarrow \int \frac{(1+x)}{(1+x^2)}dx + \int \frac{(1+y)}{(1+y^2)}dy = C \quad \text{[integrating both sides]}$$

$$\Rightarrow \int \left\{ \frac{1}{(1+x^2)} + \frac{x}{(1+x^2)} \right\} dx + \int \left\{ \frac{1}{(1+y^2)} + \frac{y}{(1+y^2)} \right\} dy = C$$

$$\Rightarrow \int \frac{1}{(1+x^2)}dx + \frac{1}{2}\cdot\int \frac{2x}{(1+x^2)}dx + \int \frac{1}{(1+y^2)}dy + \frac{1}{2}\cdot\int \frac{2y}{(1+y^2)}dy = C$$

$$\Rightarrow \tan^{-1}x + \frac{1}{2}\log(1+x^2) + \tan^{-1}y + \frac{1}{2}\log(1+y^2) = C$$

$$\Rightarrow \tan^{-1}x + \tan^{-1}y + \frac{1}{2}\{\log(1+x^2) + \log(1+y^2)\} = C,$$

which is the required solution.

EXAMPLE 17 *Solve the differential equation*

$$\operatorname{cosec} x \log y \frac{dy}{dx} + x^2y^2 = 0. \qquad \textbf{[CBSE 2014]}$$

SOLUTION The given differential equation may be written as

$$\operatorname{cosec} x \log y \frac{dy}{dx} = -x^2y^2$$

$$\Rightarrow \frac{\log y}{y^2}dy = -x^2 \sin x\,dx \quad \text{[separating the variables]}$$

$$\Rightarrow \int \underset{\text{I}}{(\log y)} \underset{\text{II}}{\left(\frac{1}{y^2}\right)} dy = -\int \underset{\text{I}}{x^2} \underset{\text{II}}{\sin x}\,dx \quad \text{[integrating both sides].}$$

Integrating by parts on each side, we get:

$$(\log y)\left(\frac{-1}{y}\right) - \int \frac{1}{y}\cdot\left(\frac{-1}{y}\right)dy = -[x^2(-\cos x) - \int 2x(-\cos x)\,dx] + C_1,$$

where C_1 is an arbitrary constant

$$\Rightarrow \frac{-\log y}{y} + \int \frac{1}{y^2}dy = x^2\cos x - 2\int \underset{\text{I}}{x} \underset{\text{II}}{\cos x}\,dx + C_1$$

$$\Rightarrow \frac{-\log y}{y} - \frac{1}{y} = x^2\cos x - 2x\sin x + 2\int \sin x\,dx + C_1$$

[integrating by parts]

$$\Rightarrow \frac{-\log y}{y} - \frac{1}{y} = x^2\cos x - 2x\sin x - 2\cos x + C_1$$

$$\Rightarrow \frac{\log y}{y} + \frac{1}{y} = -x^2\cos x + 2x\sin x + 2\cos x + C, \text{ where } -C_1 = C.$$

This is the required solution of the given differential equation.

EXAMPLE 18 *Show that the general solution of the differential equation* $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$ *is given by* $(x+y+1)=C(1-x-y-2xy)$, *where C is an arbitrary constant.*

SOLUTION We have

$$\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-(y^2+y+1)}{(x^2+x+1)}$$

$$\Rightarrow \quad \frac{dy}{(y^2+y+1)}=-\frac{dx}{(x^2+x+1)}$$

$$\Rightarrow \quad \int\frac{dy}{(y^2+y+1)}=-\int\frac{dx}{(x^2+x+1)} \quad \text{[on integrating both sides]}$$

$$\Rightarrow \quad \int\frac{dy}{\left\{\left(y+\frac{1}{2}\right)^2 j+\left(\frac{\sqrt{3}}{2}\right)^2\right\}}=-\int\frac{dx}{\left\{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right\}}$$

$$\Rightarrow \quad \frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)=\frac{-1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C_1,$$

where C_1 is an arbitrary constant

$$\Rightarrow \quad \left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)+\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)=C_1$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)+\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)=\frac{\sqrt{3}}{2}C_1$$

$$\Rightarrow \quad \tan^{-1}a+\tan^{-1}b=\frac{\sqrt{3}}{2}C_1, \text{ where } \left(\frac{2y+1}{\sqrt{3}}\right)=a \text{ and } \left(\frac{2x+1}{\sqrt{3}}\right)=b$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{a+b}{1-ab}\right)=\frac{\sqrt{3}}{2}C_1,$$

where $(a+b)=\frac{2(x+y+1)}{\sqrt{3}}$ and $ab=\frac{(4xy+2x+2y+1)}{3}$

$$\Rightarrow \quad \left(\frac{a+b}{1-ab}\right)=\tan\left(\frac{\sqrt{3}}{2}C_1\right)$$

$$\Rightarrow \quad \frac{\left\{\frac{2(x+y+1)}{\sqrt{3}}\right\}}{\left\{1-\frac{(4xy+2x+2y+1)}{3}\right\}}=\tan\left(\frac{\sqrt{3}}{2}C_1\right)$$

$$\Rightarrow \quad \frac{(x+y+1)}{(1-x-y-2xy)} = \frac{1}{\sqrt{3}} \tan\left(\frac{\sqrt{3}}{2} C_1\right) = C \text{ (say)}$$

$$\Rightarrow \quad (x+y+1) = C(1-x-y-2xy),$$

which is the solution of the given DE.

EXAMPLE 19 *Solve the differential equation*

$$x(1+y^2)dx - y(1+x^2)dy = 0,$$

given that $y = 0$ *when* $x = 1$. **[CBSE 2006C, '14]**

SOLUTION The given differential equation is

$$x(1+y^2)dx - y(1+x^2)dy = 0$$

$$\Rightarrow \quad x(1+y^2)dx = y(1+x^2)dy$$

$$\Rightarrow \quad \frac{y}{(1+y^2)} dy = \frac{x}{(1+x^2)} dx$$

$$\Rightarrow \quad \int \frac{y}{(1+y^2)} dy = \int \frac{x}{(1+x^2)} dx$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{2y}{(1+y^2)} dy = \frac{1}{2}\int \frac{2x}{(1+x^2)} dx$$

$$\Rightarrow \quad \frac{1}{2}\log(1+y^2) = \frac{1}{2}\log(1+x^2) + \frac{1}{2}\log|C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \quad \log(1+y^2) = \log(1+x^2) + \log|C_1|$$

$$\Rightarrow \quad \frac{(1+y^2)}{(1+x^2)} = \pm C_1 = C \text{ (say)}$$

$$\Rightarrow \quad (1+y^2) = C(1+x^2). \qquad \text{... (i)}$$

Putting $x = 1$ and $y = 0$ in (i), we get $C = \frac{1}{2}\cdot$

$$\therefore \quad (1+y^2) = \frac{1}{2}(1+x^2) \Rightarrow (x^2 - 2y^2) = 1.$$

Hence, $(x^2 - 2y^2) = 1$ is the required solution.

EXAMPLE 20 *Find the particular solution of the differential equation* $xy\frac{dy}{dx} = (x+2)(y+2)$, *it being given that* $y = -1$ *when* $x = 1$.

[CBSE 2012]

SOLUTION The given differential equation is

$$xy\frac{dy}{dx} = (x+2)(y+2)$$

$$\Rightarrow \quad \frac{y}{(y+2)} dy = \frac{(x+2)}{x} dx$$

$$\Rightarrow \quad \int \frac{y}{(y+2)}\,dy = \int \frac{(x+2)}{x}\,dx + C, \text{ where } C \text{ is an arbitrary constant}$$

$$\Rightarrow \quad \int \left\{1 - \frac{2}{(y+2)}\right\} dy = \int \left(1 + \frac{2}{x}\right) dx + C$$

$$\Rightarrow \quad y - 2\log|y+2| = x + 2\log|x| + C. \qquad \ldots \text{(i)}$$

Putting $x = 1$ and $y = -1$ in (i), we get $C = -2$.

Hence, the required solution is

$$y - 2\log|y+2| = x + 2\log|x| - 2.$$

EXAMPLE 21 *Find the particular solution of the differential equation* $x(x^2-1)\dfrac{dy}{dx} = 1$, *it being given that* $y = 0$ *when* $x = 2$. **[CBSE 2012]**

SOLUTION We have

$$x(x^2-1)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad dy = \frac{1}{x(x^2-1)}\,dx$$

$$\Rightarrow \quad \int dy = \int \frac{dx}{x(x-1)(x+1)}\cdot \qquad \ldots \text{(i)}$$

Let $\dfrac{1}{x(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{(x-1)} + \dfrac{C}{(x+1)}\cdot$

Then, $A(x-1)(x+1) + Bx(x+1) + Cx(x-1) \equiv 1$. $\qquad \ldots$ (ii)

Putting $x = 0$ in (ii), we get $A = -1$.

Putting $x = 1$ in (ii), we get $B = \dfrac{1}{2}\cdot$

Putting $x = -1$ in (ii), we get $C = \dfrac{1}{2}\cdot$

$$\therefore \quad \frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}\cdot$$

Putting this value in (i), we get

$$\int dy = \int \frac{-dx}{x} + \frac{1}{2}\int \frac{dx}{(x-1)} + \frac{1}{2}\int \frac{dx}{(x+1)} + C_1,$$

where C_1 is an arbitrary constant

$$\Rightarrow \quad y = -\log|x| + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| + C_1. \qquad \ldots \text{(iii)}$$

We are given that $y = 0$ when $x = 2$.

Putting $x = 2$ and $y = 0$ in (iii), we get

$$-\log 2 + \frac{1}{2}\log 3 + C_1 = 0$$

$$\Rightarrow \quad C_1 = \log 2 - \frac{1}{2}\log 3 = \frac{1}{2}\log 4 - \frac{1}{2}\log 3 = \frac{1}{2}\log\left(\frac{4}{3}\right)\cdot$$

Putting $C_1 = \frac{1}{2}\log\left(\frac{4}{3}\right)$ in (iii) we get

$$y = -\log|x| + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| + \frac{1}{2}\log\frac{4}{3}$$

$$\Rightarrow \quad y = -\frac{1}{2}\log x^2 + \frac{1}{2}\log|(x-1)(x+1)| + \frac{1}{2}\log\frac{4}{3}$$

$$\Rightarrow \quad y = \frac{1}{2}\log\left|\frac{4(x^2-1)}{3x^2}\right|,$$ which is the required solution.

EXAMPLE 22 *Find the particular solution of the differential equation* $(x+1)\frac{dy}{dx} = 2e^{-y} - 1$, *it being given that* $y = 0$ *when* $x = 0$. **[CBSE 2012]**

SOLUTION We have

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1$$

$$\Rightarrow \quad \frac{1}{(2e^{-y}-1)}dy = \frac{1}{(x+1)}dx$$

$$\Rightarrow \quad \frac{e^y}{(2-e^y)}dy = \frac{1}{(x+1)}dx$$

$$\Rightarrow \quad \int\frac{e^y}{(e^y-2)}dy = \int\frac{-1}{(x+1)}dx$$

$$\Rightarrow \quad \log|e^y - 2| = -\log|x+1| + \log|C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \quad \log|e^y-2| + \log|x+1| = \log|C_1|$$

$$\Rightarrow \quad \log|(x+1)(e^y-2)| = \log|C_1|$$

$$\Rightarrow \quad (x+1)(e^y-2) = \pm C_1 = C. \qquad \ldots \text{(i)}$$

It is given that when $x = 0$, then $y = 0$.

Putting $x = 0$ and $y = 0$ in (i), we get $C = -1$.

Putting $C = -1$ in (i), we get

$$(x+1)(e^y-2) = -1 \Rightarrow (e^y-2) = \frac{-1}{(x+1)}$$

$$\Rightarrow \quad e^y = \left\{2 - \frac{1}{(x+1)}\right\} = \frac{(2x+1)}{(x+1)}.$$

$$\therefore \quad y = \log\left|\frac{2x+1}{x+1}\right|,$$ $x \neq -1$ is the required solution.

EXAMPLE 23 *Solve the differential equation*

$(1+y^2)(1+\log x)\,dx + x\,dy = 0$, *it being given that* $y = 1$ *when* $x = 1$.

[CBSE 2000, '03, '11]

SOLUTION We have

$$(1+y^2)(1+\log x)\,dx + x\,dy = 0$$

$$\Rightarrow \quad \frac{(1+\log x)}{x}dx + \frac{1}{(1+y^2)}dy = 0 \quad \text{[on separating the variables]}$$

$$\Rightarrow \quad \int \frac{(1+\log x)}{x}dx + \int \frac{1}{(1+y^2)}dy = C \quad \text{[integrating both sides]}$$

$$\Rightarrow \quad \int t\,dt + \tan^{-1}y = C, \text{ where } (1+\log x) = t$$

$$\Rightarrow \quad \frac{1}{2}t^2 + \tan^{-1}y = C, \text{ where } C \text{ is an arbitrary constant}$$

$$\Rightarrow \quad \frac{1}{2}(1+\log x)^2 + \tan^{-1}y = C. \qquad \ldots \text{(i)}$$

Putting $x = 1$ and $y = 1$ in (i), we get

$$C = \frac{1}{2} + \tan^{-1}1 \Rightarrow C = \left(\frac{1}{2} + \frac{\pi}{4}\right). \qquad \ldots \text{(ii)}$$

$$\therefore \quad \frac{1}{2}(1+\log x)^2 + \tan^{-1}y = \left(\frac{1}{2} + \frac{\pi}{4}\right) \quad \text{[using (ii) in (i)]}$$

$$\Rightarrow \quad \frac{1}{2}(\log x)^2 + \log x + \tan^{-1}y = \frac{\pi}{4}, \text{ which is the required solution.}$$

EXAMPLE 24 *Solve the differential equation*

$$(1+e^{2x})dy + e^x(1+y^2)dx = 0,$$

it being given that $y = 1$ *when* $x = 0$. **[CBSE 2004, '08C, '11]**

SOLUTION We have

$$(1+e^{2x})dy + e^x(1+y^2)dx = 0$$

$$\Rightarrow \quad \frac{1}{(1+y^2)}dy + \frac{e^x}{(1+e^{2x})}dx = 0 \quad \text{[separating the variables]}$$

$$\Rightarrow \quad \int \frac{1}{(1+y^2)}dy + \int \frac{e^x}{(1+e^{2x})}dx = C \quad \text{[integrating both sides]}$$

$$\Rightarrow \quad \tan^{-1}y + \int \frac{dt}{(1+t^2)} = C, \text{ where } e^x = t$$

$$\Rightarrow \quad \tan^{-1}y + \tan^{-1}t = C$$

$$\Rightarrow \quad \tan^{-1}y + \tan^{-1}e^x = C \qquad \ldots \text{(i) } [\because t = e^x]$$

Putting $x = 0$ and $y = 1$ in (i), we get

$$C = \tan^{-1}1 + \tan^{-1}e^0 = (\tan^{-1}1 + \tan^{-1}1) = \left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi}{2}.$$

$$\therefore \quad \tan^{-1}y + \tan^{-1}e^x = \frac{\pi}{2} \text{ is the required solution.}$$

EXAMPLE 25 *Find the equation of the curve that passes through the point* (1, 2) *and satisfies the differential equation* $\frac{dy}{dx} = \frac{-2xy}{(x^2+1)}$.

SOLUTION We have

$$\frac{dy}{dx} = \frac{-2xy}{(x^2+1)}$$

$\Rightarrow \quad \frac{dy}{y} = \frac{-2x}{(x^2+1)}dx$ [on separating the variables]

$\Rightarrow \quad \int \frac{dy}{y} = \int \frac{-2x}{(x^2+1)}dx$ [integrating both sides]

$\Rightarrow \quad \log y = -\log(x^2+1) + \log C,$

where $\log C$ is an arbitrary constant

$\Rightarrow \quad \log y + \log(x^2+1) = \log C$

$\Rightarrow \quad \log\{y(x^2+1)\} = \log C$

$\Rightarrow \quad y(x^2+1) = C$ ... (i)

Now, it is given that the curve passes through (1, 2).

So, putting $x = 1$ and $y = 2$ in (i), we get $C = 4$.

$\therefore \quad y(x^2+1) = 4$ is the required equation of the curve.

EXAMPLE 26 *Find the equation of a curve which passes through the point* $(-2, 3)$ *and the slope of whose tangent at any point* (x, y) *is* $\frac{2x}{y^2}\cdot$

SOLUTION We know that the slope of a curve at a point (x, y) is $\frac{dy}{dx}\cdot$

$\therefore \quad \frac{dy}{dx} = \frac{2x}{y^2}$ ... (i)

$\Rightarrow \quad y^2 dy = 2x\,dx$ [separating the variables]

$\Rightarrow \quad \int y^2 dy = \int 2x\,dx$

$\Rightarrow \quad \frac{1}{3}y^3 = x^2 + C$ ... (ii)

where C is a constant.

Thus, (ii) is the equation of the curve whose differential equation is given by (i).

Since the given curve passes through the point $(-2, 3)$, we have

$$C = \left(\frac{1}{3} \times 27\right) - (-2)^2 = (9-4) = 5.$$

Hence, the required equation of the curve is

$$\frac{1}{3}y^3 = x^2 + 5 \Rightarrow y^3 = 3x^2 + 15.$$

EXAMPLE 27 *In a bank principal increases at the rate of* 5% *per annum. In how many years will* ₹ 1000 *double itself?*

SOLUTION Let P be the principal at any time t. Then,

$$\frac{dP}{dt} = \left(\frac{5}{100}\right)P$$

$\Rightarrow \quad \frac{dP}{dt} = \frac{P}{20}$

$\Rightarrow \quad \frac{dP}{P} = \frac{1}{20}dt$

$\Rightarrow \quad \int \frac{dP}{P} = \int \frac{1}{20} dt$ [on integrating both sides]

$\Rightarrow \quad \log P = \frac{1}{20} t + \log C$, where C is an arbitrary constant

$\Rightarrow \quad \log \left(\frac{P}{C} \right) = \frac{t}{20}$

$\Rightarrow \quad \frac{P}{C} = e^{t/20}$. ... (i)

When $t = 0$, we have $P = 1000$ (given).

Putting $P = 1000$ and $t = 0$ in (i), we get $C = 1000$.

$\therefore \quad P = (1000) e^{t/20}$. ... (ii)

Let ₹ 1000 double itself in n years.

Thus, when $t = n$, then $P = 2000$.

Putting these values in (ii), we get

$$(1000) \times e^{n/20} = 2000 \Rightarrow e^{n/20} = 2$$

$$\Rightarrow \frac{n}{20} = \log_e 2 \Rightarrow n = 20(\log_e 2).$$

Hence, the required time is $\{20(\log_e 2)\}$ years.

EXERCISE 19A

Very-Short-Answer Questions

Find the general solution of each of the following differential equations:

1. $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

2. $x^4 \frac{dy}{dx} = -y^4$

3. $\frac{dy}{dx} = 1 + x + y + xy$ **[CBSE 2012]**

4. $\frac{dy}{dx} = 1 - x + y - xy$

5. $(x - 1) \frac{dy}{dx} = 2x^3 y$

6. $\frac{dy}{dx} = e^{x+y}$

7. $(e^x + e^{-x}) dy - (e^x - e^{-x}) dx = 0$

8. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ **[CBSE 2006]**

9. $e^{2x-3y} dx + e^{2y-3x} dy = 0$

10. $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$ **[CBSE 2011]**

11. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

12. $\cos x(1 + \cos y) dx - \sin y(1 + \sin x) dy = 0$

For each of the following differential equations, find a particular solution satisfying the given condition:

13. $\cos \left(\frac{dy}{dx} \right) = a$, where $a \in R$ and $y = 2$ when $x = 0$.

14. $\frac{dy}{dx} = -4xy^2$, it being given that $y = 1$ when $x = 0$.

15. $x\,dy = (2x^2+1)\,dx \ (x \neq 0)$, given that $y = 1$ when $x = 1$.

16. $\dfrac{dy}{dx} = y \tan x$, it being given that $y = 1$ when $x = 0$.

ANSWERS (EXERCISE 19A)

1. $\tan^{-1} y = x + \dfrac{x^3}{3} + C$ **2.** $\dfrac{1}{x^3} + \dfrac{1}{y^3} = C$ **3.** $\log|1+y| = x + \dfrac{x^2}{2} + C$

4. $\log|1+y| = x - \dfrac{x^2}{2} + C$ **5.** $\log|y| = \dfrac{2x^3}{3} + x^2 + 2x + 2\log|x-1| + C$

6. $e^x + e^{-y} = C$ **7.** $y = \log(e^x + e^{-x}) + C$ **8.** $e^y = e^x + \dfrac{x^3}{3} + C$

9. $e^{5x} + e^{5y} = C$ **10.** $\tan y = C(1-e^x)$ **11.** $\tan x \tan y = C$

12. $(1+\sin x)(1+\cos y) = C$ **13.** $\cos\left(\dfrac{y-2}{x}\right) = a$ **14.** $y = \dfrac{1}{(2x^2+1)}$

15. $y = x^2 + \log|x|$ **16.** $y = \sec x$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 19A)

1. $\displaystyle\int \frac{1}{(1+y^2)}\,dy = \int (1+x^2)\,dx + C.$

2. $\dfrac{-1}{y^4}\,dy = \dfrac{1}{x^4}\,dx \Rightarrow \displaystyle\int x^{-4}dx + \int y^{-4}dy = C_1 \Rightarrow \frac{1}{x^3} + \frac{1}{y^3} = -3C_1 = C.$

3. $\dfrac{dy}{dx} = (1+x) + y(1+x) = (1+x)(1+y) \Rightarrow \displaystyle\int \frac{dy}{(1+y)} = \int (1+x)\,dx + C.$

5. $\dfrac{1}{y}\,dy = \dfrac{2x^3}{(x-1)} = 2\left[x^2 + x + 1 + \dfrac{1}{(x-1)}\right]$ [on dividing x^3 by $(x-1)$]

6. $\dfrac{dy}{dx} = e^x \cdot e^y \Rightarrow \dfrac{1}{e^y}\,dy = e^x dx \Rightarrow \displaystyle\int e^{-y}dy = \int e^x dx + C$

7. $dy - \dfrac{(e^x - e^{-x})}{(e^x + e^{-x})}\,dx = 0 \Rightarrow \displaystyle\int dy - \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx + C \Rightarrow y - \log(e^x + e^{-x}) = C.$

8. $\dfrac{dy}{dx} = e^x \cdot e^{-y} + x^2 e^{-y} = (e^x + x^2)e^{-y} \Rightarrow \displaystyle\int e^y dy = \int (e^x + x^2)\,dx + C.$

9. $e^{2x} \cdot e^{-3y}dx + e^{2y} \cdot e^{-3x}dy = 0 \Rightarrow \displaystyle\int e^{5x}dx + \int e^{5y}dy = C.$

10. $-\displaystyle\int \frac{-e^x}{(1-e^x)}\,dx + \int \frac{\sec^2 y}{\tan y}\,dy = \log|C_1| \Rightarrow \log|\tan y| - \log|1-e^x| = \log|C_1|.$

$\therefore \log\left|\dfrac{\tan y}{(1-e^x)}\right| = \log|C_1| \Rightarrow \dfrac{\tan y}{(1-e^x)} = \pm C_1 = C.$

11. $\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = \log|C_1| \Rightarrow \log|\tan x| + \log|\tan y| = \log|C_1|$

$\therefore \log|\tan x \tan y| = \log|C_1| \Rightarrow |\tan x \tan y| = |C_1|.$

$\therefore \tan x \tan y = \pm C_1 = C.$

12. $\int \frac{\cos x}{(1+\sin x)} dx + \int \frac{-\sin y}{(1+\cos y)} dy = \log|C_1|$

$\Rightarrow \log|1+\sin x| + \log|1+\cos y| = \log|C_1| \Rightarrow \log|(1+\sin x)(1+\cos y)| = \log|C_1|$

$\Rightarrow |(1+\sin x)(1+\cos y)| = C_1 \Rightarrow (1+\sin x)(1+\cos y) = \pm C_1 = C.$

13. $\frac{dy}{dx} = \cos^{-1} a \Rightarrow \int dy = \int (\cos^{-1} a)\, dx \Rightarrow y = (\cos^{-1} a)x + C.$ ... (i)

Putting $x = 0$ and $y = 2$ in (i), we get $C = 2$.

$\therefore \cos^{-1} a = \left(\frac{y-2}{x}\right) \Rightarrow \cos\left(\frac{y-2}{x}\right) = a.$

14. $\int \frac{dy}{y^2} = \int -4x\, dx \Rightarrow \frac{-1}{y} = -2x^2 + C \Rightarrow y = \frac{1}{(2x^2 - C)}.$ ... (i)

Putting $x = 0$ and $y = 1$ in (i), we get $C = -1$.

Hence, $y = \frac{1}{(2x^2+1)}.$

15. $\int dy = \int \frac{(2x^2+1)}{x} dx = \int \left(2x + \frac{1}{x}\right) dx \Rightarrow y = x^2 + \log|x| + C.$ ... (i)

Putting $x = 1$ and $y = 1$ in (i), we get $C = 0$. Hence, $y = x^2 + \log|x|$.

16. $\int \frac{1}{y} dy = \int \tan x\, dx \Rightarrow \log|y| = \log|\sec x| + \log|C_1|$

$\therefore (y \cos x) = \pm C_1 = C.$

When $x = 0$ and $y = 1$, we get $C = 1$. Hence, $y = \sec x$.

EXERCISE 19B

Find the general solution of each of the following differential equations:

1. $\frac{dy}{dx} = \frac{x-1}{y+2}$

2. $\frac{dy}{dx} = \frac{x}{(x^2+1)}$

3. $\frac{dy}{dx} = (1+x)(1+y^2)$

4. $(1+x^2)\frac{dy}{dx} = xy$

5. $\frac{dy}{dx} + y = 1\ (y \neq 1)$

6. $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

7. $x\frac{dy}{dx} + y = y^2$

8. $x^2(y+1)dx + y^2(x-1)dy = 0$

9. $y(1-x^2)\frac{dy}{dx} = x(1+y^2)$

10. $y \log y\, dx - x\, dy = 0$

11. $x(x^2 - x^2y^2)dy + y(y^2 + x^2y^2)dx = 0$

12. $(1-x^2)dy + xy(1-y)dx = 0$

13. $(1-x^2)(1-y)\,dx = xy(1+y)\,dy$
14. $(y+xy)\,dx + (x-xy^2)\,dy = 0$ **[CBSE 2002C]**
15. $(x^2-yx^2)\,dy + (y^2+xy^2)\,dx = 0$ **[CBSE 2004C]**
16. $(x^2y-x^2)\,dx + (xy^2-y^2)\,dy = 0$
17. $x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0$
18. $\frac{dy}{dx} = e^{x+y} + x^2e^y$ **[CBSE 2006]**
19. $\frac{dy}{dx} = \frac{3e^{2x}+3e^{4x}}{e^x+e^{-x}}$
20. $3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$
21. $e^y(1+x^2)\,dy - \frac{x}{y}\,dx = 0$
22. $\frac{dy}{dx} = e^{x+y} + e^{x-y}$
23. $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$
24. $\frac{dy}{dx} + \frac{xy+y}{xy+x} = 0$
25. $\sqrt{1-x^4}\,dy = x\,dx$
26. $\operatorname{cosec} x \log y\frac{dy}{dx} + x^2y = 0$
27. $y\,dx + (1+x^2)\tan^{-1}x\,dy = 0$
28. $\frac{1}{x}\cdot\frac{dy}{dx} = \tan^{-1}x$
29. $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}\,dy = 0$
30. $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$
31. $(\cos x)\frac{dy}{dx} + \cos 2x = \cos 3x$
32. $\frac{dy}{dx} + \frac{(1+\cos 2y)}{(1-\cos 2x)} = 0$
33. $\frac{dy}{dx} + \frac{\cos x \sin y}{\cos y} = 0$
34. $\cos x(1+\cos y)\,dx - \sin y(1+\sin x)\,dy = 0$
35. $\sin^3 x\,dx - \sin y\,dy = 0$
36. $\frac{dy}{dx} + \sin(x+y) = \sin(x-y)$
37. $\frac{1}{x}\cos^2 y\,dy + \frac{1}{y}\cos^2 x\,dx = 0$
38. $\frac{dy}{dx} = \sin^3 x\cos^2 x + xe^x$
39. Find the particular solution of the differential equation $\frac{dy}{dx} = 1+x+y+xy$, given that $y=0$ when $x=1$. **[CBSE 2014]**
40. Find the particular solution of the differential equation $x(1+y^2)\,dx - y(1+x^2)\,dy = 0$, given that $y=1$ when $x=0$. **[CBSE 2014]**

41. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0$ when $x = 0$. **[CBSE 2014]**

42. Solve the differential equation $(x^2 - yx^2)dy + (y^2 + x^2y^2)dx = 0$, given that $y = 1$ when $x = 1$. **[CBSE 2014]**

43. Find the particular solution of the differential equation $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$, given that $y = 1$ when $x = 0$. **[CBSE 2014]**

44. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x(2\log x + 1)}{(\sin y + y\cos y)}$, given that $y = \frac{\pi}{2}$ when $x = 1$. **[CBSE 2014]**

45. Solve the differential equation $\frac{dy}{dx} = y\sin 2x$, given that $y(0) = 1$. **[CBSE 2004]**

46. Solve the differential equation $(x+1)\frac{dy}{dx} = 2xy$, given that $y(2) = 3$. **[CBSE 2004]**

47. Solve $\frac{dy}{dx} = x(2\log x + 1)$, given that $y = 0$ when $x = 2$.

48. Solve $(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$, given that $y = 1$ when $x = 0$.

49. Solve $\frac{dy}{dx} = y\tan x$, given that $y = 1$ when $x = 0$.

50. Solve $\frac{dy}{dx} = y^2\tan 2x$, given that $y = 2$ when $x = 0$.

51. Solve $\frac{dy}{dx} = y\cot 2x$, given that $y = 2$ when $x = \frac{\pi}{4}\cdot$

52. Solve $(1 + x^2)\sec^2 y\,dy + 2x\tan y\,dx = 0$, given that $y = \frac{\pi}{4}$ when $x = 1$.

53. Find the equation of the curve passing through the point $\left(0, \frac{\pi}{4}\right)$ whose differential equation is $\sin x\cos y\,dx + \cos x\sin y\,dy = 0$.

54. Find the equation of a curve which passes through the origin and whose differential equation is $\frac{dy}{dx} = e^x\sin x$.

55. A curve passes through the point (0, –2) and at any point (x, y) of the curve, the product of the slope of its tangent and y-coordinate of the point is equal to the x-coordinate of the point. Find the equation of the curve.

56. A curve passes through the point (–2, 1) and at any point (x, y) of the curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (–4, –3). Find the equation of the curve.

57. In a bank, principal increases at the rate of $r\%$ per annum. Find the value of r if ₹ 100 double itself in 10 years. (Given $\log_e 2 = 0.6931$)

58. In a bank, principal increases at the rate of 5% per annum. An amount of ₹ 1000 is deposited in the bank. How much will it worth after 10 years? (Given $e^{0.5} = 1.648$)

59. The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after t seconds.

60. In a culture the bacteria count is 100000. The number is increased by 10% in 2 hours. In how many hours will the count reach 200000, if the rate of growth of bacteria is proportional to the number present?

ANSWERS (EXERCISE 19B)

1. $y^2 + 4y - x^2 + 2x = C$ **2.** $y = \frac{1}{2}\log(x^2+1) + C$ **3.** $\tan^{-1} y = x + \frac{x^2}{2} + C$

4. $y = C_1\sqrt{1+x^2}$ **5.** $x + \log|1-y| = C$ **6.** $\sin^{-1} y + \sin^{-1} x = C$

7. $y = 1 + C_1 xy$ **8.** $\frac{x^2}{2} + \frac{y^2}{2} + x - y + \log|x-1| + \log|y+1| = C$

9. $(1+y^2)(1-x^2) = C$ **10.** $x = C\log y$ **11.** $\frac{-1}{2y^2} - \frac{1}{2x^2} + \log\left|\frac{x}{y}\right| = C$

12. $y = C(1-y)\sqrt{1-x^2}$ **13.** $\log|x(1-y^2)| = \frac{x^2}{2} - \frac{y^2}{2} - 2y + C$

14. $\log|xy| + x - \frac{y^2}{2} = C$ **15.** $\log\left|\frac{x}{y}\right| = \frac{1}{x} + \frac{1}{y} + C$

16. $\frac{1}{2}(x^2+y^2) + (x+y) + \log|(x-1)(y-1)| = C$ **17.** $\sqrt{1+y^2} + \sqrt{1+x^2} = C$

18. $e^x + e^{-y} + \frac{x^3}{3} = C$ **19.** $y = e^{3x} + C$ **20.** $\tan y = C(1-e^{-x})^3$

21. $(y-1)e^y = \frac{1}{2}\log(1+x^2) + C$ **22.** $\tan^{-1}(e^y) = e^x + C$

23. $(1+e^y)\sin x = C$ **24.** $x + y + \log|xy| = C$ **25.** $y = \frac{1}{2}\sin^{-1}(x^2) + C$

26. $\frac{1}{2}(\log y)^2 + (2-x^2)\cos x + 2x\sin x = C$ **27.** $y\tan^{-1} x = C$

28. $y = \frac{1}{2}(x^2+1)\tan^{-1} x - \frac{1}{2}x + C$ **29.** $(x-1)e^x - \sqrt{1-y^2} = C$

30. $y = 2\tan\frac{x}{2} - x + C$ **31.** $y = \sin 2x - 2\sin x - x + \log|\sec x + \tan x| + C$

32. $\tan y = \cot x + C$ **33.** $\log|\sin y| + \sin x = C$

34. $(1+\sin x)(1+\cos y)=C$ **35.** $12\cos y = 9\cos x - \cos 3x + C$

36. $\log|\text{cosec}\, y - \cot y| + 2\sin x = C$

37. $2(x^2+y^2) + 2x\sin 2x + 2y\sin 2y + \cos 2x + \cos 2y = C$

38. $y = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + xe^x - e^x + C$ **39.** $\log|1+y| = x + \frac{x^2}{2} - \frac{3}{2}$

40. $y = \sqrt{2x^2+1}$ **41.** $4e^{3x} + 3e^{-4y} = 7$ **42.** $\log|y| + \frac{1}{y} + \frac{1}{x} - x = 1$

43. $e^x(x-1) = \sqrt{1-y^2} - 1$ **44.** $y\sin y = x^2\log x + \frac{\pi}{2}$ **45.** $y = e^{\sin^2 x}$

46. $y(x+1)^2 = 27e^{(2x-4)}$ **47.** $y = x^2\log x - 4\log 2$

48. $y = \frac{1}{2}\left\{\log|x+1| + \frac{3}{2}\log(x^2+1) - \tan^{-1}x\right\} + 1$ **49.** $y\cos x = 1$

50. $y(1+\log|\cos 2x|) = 2$ **51.** $y = 2\sqrt{\sin 2x}$ **52.** $(1+x^2)\tan y = 2$

53. $y = \cos^{-1}\left(\frac{1}{\sqrt{2}}\sec x\right)$ **54.** $2y = e^x(\sin x - \cos x) + 1$ **55.** $y^2 = x^2 + 4$

56. $y + 3 = (x+4)^2$ **57.** $r = 6.931$ **58.** ₹ 1648 **59.** $r = (63t+27)^{1/3}$

60. $\dfrac{2\log 2}{\log\left(\frac{11}{10}\right)}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 19B)

4. $\int \frac{1}{y}dy = \frac{1}{2}\int \frac{2x}{(1+x^2)}dx \Rightarrow \log|y| - \frac{1}{2}\log|1+x^2| = \log|C|$

$$\Rightarrow \log\left|\frac{y}{\sqrt{1+x^2}}\right| = \log|C| \Rightarrow \frac{y}{\sqrt{1+x^2}} = \pm C = C_1.$$

5. $\frac{1}{(1-y)}dy = dx \Rightarrow -\int \frac{-dy}{(1-y)} = \int dx$

$$\Rightarrow \int dx + \int \frac{-dy}{(1-y)} = C \Rightarrow x + \log|1-y| = C.$$

7. $\int \frac{1}{y(y-1)}dy = \int \frac{1}{x}dx \Rightarrow \int\left\{\frac{1}{(y-1)} - \frac{1}{y}\right\}dy = \int \frac{1}{x}dx$ [by partial fractions]

$$\Rightarrow \log|y-1| - \log|y| = \log|x| + \log|C|$$
$$\Rightarrow (y-1) = \pm Cxy \Rightarrow (y-1) = C_1xy \Rightarrow y = 1 + C_1xy.$$

8. $\int \frac{x^2}{(x-1)}dx + \int \frac{y^2}{(y+1)}dy = C \Rightarrow \int\left(x+1+\frac{1}{x-1}\right)dx + \int\left(y-1+\frac{1}{y+1}\right)dy = C.$

9. $\int \frac{2y}{(1+y^2)}dy = -\int \frac{-2x}{(1-x^2)}dx \Rightarrow \log|1+y^2| + \log|1-x^2| = \log|C_1|$

$$\Rightarrow (1+y^2)(1-x^2) = \pm C_1 = C.$$

10. $\int \frac{1}{x} dx = \int \frac{1}{y \log y} dy \Rightarrow \int \frac{1}{x} dx = \int \frac{dt}{t}$, where $\log y = t$.

$\therefore \log|x| = \log|t| + \log|C_1| \Rightarrow \log|x| - \log|\log y| = \log|C_1|$

$$\Rightarrow \log\left|\frac{x}{\log y}\right| = \log C_1 \Rightarrow \frac{x}{\log y} = \pm C_1 = C.$$

11. $x^3(1-y^2)dy + y^3(1+x^2)dx = 0 \Rightarrow \int \frac{(1-y^2)}{y^3} dy + \int \frac{(1+x^2)}{x^3} dx = C.$

$\therefore \int \left(y^{-3} - \frac{1}{y}\right) dy + \int \left(x^{-3} + \frac{1}{x}\right) dx = C \Rightarrow \frac{y^{-2}}{-2} - \log|y| + \frac{x^{-2}}{-2} + \log|x| = C.$

12. $\int \frac{1}{y(1-y)} dy + \int \frac{x}{(1-x^2)} dx = \log|C_1|$

$\Rightarrow \int \left\{\frac{1}{y} + \frac{1}{(1-y)}\right\} dy - \frac{1}{2}\int \frac{-2x}{(1-x^2)} dx = \log|C_1|$

$\Rightarrow \log|y| - \log|1-y| - \frac{1}{2}\log|1-x^2| = \log|C_1| \Rightarrow \log\left|\frac{y}{(1-y)\sqrt{1-x^2}}\right| = \log|C_1|$

$\Rightarrow \frac{y}{(1-y)\sqrt{1-x^2}} = \pm C_1 = C.$

13. $\int \frac{(1-x^2)}{x} dx = \int \frac{y(1+y)}{(1-y)} dy \Rightarrow \int \left(\frac{1}{x} - x\right) dx = \int \frac{(y^2+y)}{(-y+1)} dy$

$\therefore \int \left(\frac{1}{x} - x\right) dx = \int \left(-y - 2 + \frac{2}{1-y}\right) dy$ [on dividing (y^2+y) by $(-y+1)$].

19. $\frac{dy}{dx} = \frac{3e^{2x}(1+e^{2x})}{\left(e^x + \frac{1}{e^x}\right)} = \frac{3e^{3x}(1+e^{2x})}{(1+e^{2x})} = 3e^{3x}.$

20. $\int \frac{3e^x}{(1-e^x)} dx + \int \frac{\sec^2 y}{\tan y} dy = \log|C_1|$

$\Rightarrow -3\int \frac{-e^x}{(1-e^x)} dx + \int \frac{\sec^2 y}{\tan y} dy = \log|C_1| \Rightarrow -3\log|1-e^{-x}| + \log|\tan y| = \log|C_1|$

$\Rightarrow \log|(1-e^{-x})^{-3}\tan y| = \log|C_1| \Rightarrow \frac{\tan y}{(1-e^{-x})^3} = \pm C_1 = C.$

22. $\int \frac{e^y}{(e^{2y}+1)} dy = \int e^x dx \Rightarrow \int \frac{dt}{(t^2+1)} = e^x + C$, where $e^y = t$

$\Rightarrow \tan^{-1}(t) = e^x + C \Rightarrow \tan^{-1}(e^y) = e^x + C.$

26. $\int \frac{\log y}{y} dy + \int \underset{\text{I}}{x^2} \underset{\text{II}}{\sin x}\, dx = C$

$\Rightarrow \int t\, dt + x^2(-\cos x) - \int 2x(-\cos x)\, dx = C$, where $\log y = t$

$\Rightarrow \frac{t^2}{2} - x^2 \cos x + 2\int \underset{\text{I}}{x} \underset{\text{II}}{\cos x}\, dx = C.$

27. $\int \frac{dx}{(1+x^2)\tan^{-1} x} + \int \frac{1}{y} dy = \log|C_1|.$

Put $\tan^{-1} x = t$ and $\frac{dx}{(1+x^2)} = dt.$

28. $\int dy = \int (\underset{\text{I}}{\tan^{-1} x})\,\underset{\text{II}}{x}\,dx \Rightarrow y = (\tan^{-1} x)\dfrac{x^2}{2} - \int \dfrac{1}{(1+x^2)}\cdot\dfrac{x^2}{2}\,dx + C$

$\Rightarrow y = \dfrac{1}{2}x^2(\tan^{-1} x) - \dfrac{1}{2}\int \left\{\dfrac{(1+x^2)-1}{(1+x^2)}\right\} dx + C$

$\Rightarrow y = \dfrac{1}{2}x^2(\tan^{-1} x) - \dfrac{1}{2}\int \left\{1 - \dfrac{1}{(1+x^2)}\right\} dx + C.$

30. $\dfrac{dy}{dx} = \dfrac{1-\cos x}{1+\cos x} = \dfrac{2\sin^2(x/2)}{2\cos^2(x/2)} = \tan^2\dfrac{x}{2} = \left(\sec^2\dfrac{x}{2} - 1\right)$

$\therefore \int dy = \int \left(\sec^2\dfrac{x}{2} - 1\right) dx.$

31. $\dfrac{dy}{dx} = \dfrac{\cos 3x - \cos 2x}{\cos x} = \dfrac{(4\cos^3 x - 3\cos x) - (2\cos^2 x - 1)}{\cos x}$

$\Rightarrow \dfrac{dy}{dx} = 4\cos^2 x - 2\cos x - 3 + \sec x$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{4(1+\cos 2x)}{2} - 2\cos x - 3 + \sec x = 2\cos 2x - 2\cos x - 1 + \sec x.$

32. $\dfrac{dy}{dx} + \dfrac{2\cos^2 y}{2\sin^2 x} = 0 \Rightarrow \dfrac{dy}{dx} + \dfrac{\operatorname{cosec}^2 x}{\sec^2 y} = 0 \Rightarrow \int (\sec^2 y)dy + \int \operatorname{cosec}^2 x\,dx = C.$

33. $\int \cot y\,dy + \int \cos x\,dx = C.$

34. $\int \dfrac{\cos x}{(1+\sin x)}dx - \int \dfrac{\sin y}{(1+\cos y)}dy = \log|C_1|$

$\Rightarrow \log|1+\sin x| + \log|1+\cos y| = \log|C_1|$

$\Rightarrow \log|(1+\sin x)(1+\cos y)| = \log|C_1| \Rightarrow (1+\sin x)(1+\cos y) = \pm C_1 = C.$

35. $\sin y\,dy = \sin^3 x\,dx = \dfrac{(3\sin x - \sin 3x)}{4}dx \quad [\because \ \sin 3x = 3\sin x - 4\sin^3 x]$

$\therefore \int \sin y\,dy = \dfrac{3}{4}\int \sin x\,dx - \dfrac{1}{4}\int \sin 3x\,dx + C.$

36. $\sin(x-y) - \sin(x+y) = 2\cos\left\{\dfrac{(x-y)+(x+y)}{2}\right\}\sin\left\{\dfrac{(x-y)-(x+y)}{2}\right\}$

$= 2\cos x\sin(-y) = -2\cos x\sin y.$

$\therefore \int \dfrac{1}{\sin y}dy = -2\int \cos x\,dx \Rightarrow \int \operatorname{cosec} y\,dy = -2\sin x + C$

$\Rightarrow \log|\operatorname{cosec} y - \cot y| = -2\sin x + C.$

37. $y\cos^2 y\,dy + x\cos^2 x\,dx = 0 \Rightarrow y(2\cos^2 y)dy + x(2\cos^2 x)dx = 0$

$\Rightarrow y(1+\cos 2y)dy + x(1+\cos 2x)dx = 0$

$\Rightarrow \int y\,dy + \int y\cos 2y\,dy + \int x\,dx + \int x\cos 2x\,dx = C$

$\Rightarrow \dfrac{x^2}{2} + \dfrac{y^2}{2} + \int \underset{\text{I}}{y}\,\underset{\text{II}}{\cos 2y}\,dy + \int \underset{\text{I}}{x}\,\underset{\text{II}}{\cos 2x}\,dx = C.$

38. $\int dy = \int \cos^2 x(1-\cos^2 x)\sin x\,dx + \int xe^x dx + C$

$\Rightarrow y = -\int t^2(1-t^2)dt + \int xe^x dx + C$, where $t = \cos x$

$\Rightarrow y = -\int t^2 dt + \int t^4 dt + \int \underset{\text{I}}{x}\,\underset{\text{II}}{e^x}\,dx + C.$

39. $\frac{dy}{dx} = (1+x)(1+y) \Rightarrow \int \frac{dy}{(1+y)} = \int (1+x)\,dx \Rightarrow \log|1+y| = x + \frac{x^2}{2} + C.$ ... (i)

Putting $x = 1$ and $y = 0$ in (i), we get $C = \frac{-3}{2}$.

$\therefore \log|1+y| = x + \frac{x^2}{2} - \frac{3}{2}$.

40. $\frac{1}{2}\int \frac{2x}{(1+x^2)}\,dx - \frac{1}{2}\int \frac{2y}{(1+y^2)}\,dy = \log|C_1|$

$\Rightarrow \frac{1}{2}\log(1+x^2) - \frac{1}{2}\log(1+y^2) = \log|C_1|$

$\Rightarrow \log(1+x^2) - \log(1+y^2) = \log|C_1^2| = \log C \Rightarrow \frac{(1+x^2)}{(1+y^2)} = C.$ ... (i)

Putting $x = 0$ and $y = 1$ in (i), we get $C = \frac{1}{2}$.

$\therefore (1+y^2) = 2(1+x^2) \Rightarrow y = \sqrt{2x^2+1}.$

41. $\frac{dy}{dx} = e^{3x+4y} = e^{3x}\cdot e^{4y} \Rightarrow \int e^{3x}dx = \int e^{-4y}dy$

$\therefore \frac{e^{3x}}{3} = \frac{e^{-4y}}{-4} + C.$ ... (i)

Putting $x = 0$ and $y = 0$ in (i), we get $C = \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{7}{12}$.

$\therefore \frac{e^{3x}}{3} = \frac{e^{-4y}}{-4} + \frac{7}{12} \Rightarrow 4e^{3x} + 3e^{-4y} = 7.$

42. $x^2(1-y)dy + y^2(1+x^2)dx = 0 \Rightarrow \int \frac{(1-y)}{y^2}dy + \int \left(\frac{1+x^2}{x^2}\right)dx = C_1.$

$\therefore -\frac{1}{y} - \log|y| - \frac{1}{x} + x = C_1 \Rightarrow \log|y| + \frac{1}{y} + \frac{1}{x} - x = C.$ ... (i)

Putting $x = 1$ and $y = 1$ in (i), we get $C = 1$.

43. $\int \underset{\text{I}}{x}\,\underset{\text{II}}{e^x}dx = \frac{1}{2}\int \frac{-2y}{\sqrt{1-y^2}}dy \Rightarrow e^x(x-1) = \sqrt{1-y^2} + C.$ ... (i)

Putting $x = 0$ and $y = 1$ in (i), we get $C = -1$.

44. $\int (\sin y + y\cos y)\,dy = \int (2x\log x + x)\,dx + C$

$\Rightarrow \int \sin y\,dy + \int y\cos y\,dy = \int 2x\log x\,dx + \int x\,dx + C$

$\Rightarrow (-\cos y) + y\sin y - \int 1\cdot\sin y\,dy = 2\left[(\log x)\cdot\frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}dx\right] + \frac{x^2}{2} + C$

$\Rightarrow y\sin y = x^2\log x + C.$ ... (i)

45. $\int \frac{1}{y}dy = \int \sin 2x\,dx + C \Rightarrow \log|y| = -\frac{1}{2}\cos 2x + C.$ ... (i)

Putting $x = 0$ and $y = 1$ in (i), we get $C = \frac{1}{2}$.

$\therefore \log|y| = \frac{1}{2}(1-\cos 2x) = \left(\frac{1}{2}\times 2\sin^2 x\right) = \sin^2 x.$

Hence, $y = e^{\sin^2 x}$.

46. $\int \frac{1}{y} dy = \int \frac{2x}{(x+1)} dx = 2\int \frac{(x+1)-1}{(x+1)} dx = 2\int \left\{1 - \frac{1}{(x+1)}\right\} dx$

$\Rightarrow \log|y| = 2x - 2\log|x+1| + \log|C_1|$

$\Rightarrow \log|y| = \log|e^{2x}| - \log|(x+1)^2| + \log|C_1|$

$\Rightarrow \log\left|\frac{y(x+1)^2}{e^{2x}}\right| = \log|C_1| \Rightarrow \frac{y(x+1)^2}{e^{2x}} = \pm C_1 = C$ (say).

Then, $y(x+1)^2 = Ce^{2x}$. ... (i)

Putting $x = 2$ and $y = 3$ in (i), we get $C = 27e^{-4}$.

$\therefore$ $y(x+1)^2 = 27e^{2x-4}$ is the required solution.

48. $dy = \frac{1}{2}\left\{\frac{1}{(x+1)} + \frac{3x-1}{(x^2+1)}\right\} dx$ [by partial fractions]

$\Rightarrow \int dy = \frac{1}{2}\int \left\{\frac{1}{(x+1)} + \frac{3}{2}\cdot\frac{2x}{(x^2+1)} - \frac{1}{(x^2+1)}\right\} dx + C$

$\Rightarrow y = \frac{1}{2}\left\{\log|x+1| + \frac{3}{2}\log|x^2+1| - \tan^{-1} x\right\} + C.$

When $x = 0$ and $y = 1$, then $C = 1$.

51. $\int \frac{dy}{y} = \int \frac{\cos 2x}{\sin 2x} dx \Rightarrow \log|y| = \frac{1}{2}\log|\sin 2x| + \log|C_1|$

$\therefore$ $\left|\frac{y}{\sqrt{\sin 2x}}\right| = C_1 \Rightarrow \frac{y}{\sqrt{\sin 2x}} = \pm C_1 = C$ (say). ... (i)

Putting $x = \frac{\pi}{4}$ and $y = 2$ in (i), we get $C = 2$.

Hence, $y = 2\sqrt{\sin 2x}$.

53. $\tan x\, dx + \tan y\, dy = 0$

$\Rightarrow \int \tan x\, dx + \int \tan y\, dy =$ constant

$\Rightarrow -\log|\cos x| - \log|\cos y| = \log|C_1|$

$\Rightarrow \log|\cos x \cos y| = \log\left|\frac{1}{C_1}\right| \Rightarrow \cos x \cos y = \pm\frac{1}{C_1} = C$ (say). ... (i)

Putting $x = 0$ and $y = \frac{\pi}{4}$ in (i), we get $C = \frac{1}{\sqrt{2}}$.

$\therefore$ $\cos x \cos y = \frac{1}{\sqrt{2}} \Rightarrow \cos y = \frac{1}{\sqrt{2}}\sec x \Rightarrow y = \cos^{-1}\left(\frac{1}{\sqrt{2}}\sec x\right)$.

54. Use the formula $\int e^{ax}\sin bx\, dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{(a^2+b^2)}$.

Put $a = 1$ and $b = 1$.

55. $y\frac{dy}{dx} = x \Rightarrow \int y\, dy = \int x\, dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C.$

Put $x = 0$ and $y = -2$ to get $C = 2$.

56. $\frac{dy}{dx} = \frac{2(y+3)}{(x+4)} \Rightarrow \int \frac{1}{(y+3)} dy = 2\int \frac{1}{(x+4)} dx$

$\therefore$ $\log|y+3| = 2\log|x+4| + \log|C_1| \Rightarrow \frac{(y+3)}{(x+4)^2} = \pm C_1 = C.$

Putting $x = -2$ and $y = 1$, we get $C = 1$.

57. $\frac{dP}{dt} = \left(\frac{r}{100}\right)P \Rightarrow \int \frac{dP}{P} = \int \left(\frac{r}{100}\right)dt \Rightarrow \log P = \frac{rt}{100} + C.$ ... (i)

At $t = 0$, we have $P = P_0$. So, $\log P_0 = C$.

$\therefore \log P - \log P_0 = \frac{rt}{100} \Rightarrow \log\left(\frac{P}{P_0}\right) = \frac{rt}{100}.$ ... (ii)

Putting $P_0 = 100$, $P = 2P_0 = 200$ and $t = 10$, we get

$\frac{r}{10} = \log 2 \Rightarrow r = (10 \times 0.6931) = 6.931.$

58. $\frac{dP}{dt} = \frac{5P}{100} \Rightarrow \int \frac{dP}{P} = \int \frac{1}{20}dt \Rightarrow \log P = \frac{t}{20} + \log C.$

At $t = 0$, we have $P = 1000$ and so $\log C = \log 1000$.

$\therefore \log P = \frac{t}{20} + \log 1000.$

Putting $t = 10$, we get $\log\left(\frac{P}{1000}\right) = 0.5 \Rightarrow \frac{P}{1000} = e^{0.5} = 1.648.$

$\therefore P = (1000 \times 1.648) = 1648.$

59. The volume of a spherical balloon of radius r is given by $V = \frac{4}{3}\pi r^3$.

Now, $\frac{dV}{dt} = -k$, where $k > 0$ [note that V is decreasing]

$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -k \Rightarrow (4\pi r^2)\frac{dr}{dt} = -k$

$\Rightarrow \int (4\pi r^2)dr = \int (-k)\,dt$

$\Rightarrow \frac{4}{3}\pi r^3 = -kt + C$ ... (i), where C is an arbitrary constant.

Putting $t = 0$ and $r = 3$ in (i), we get $C = 36\pi$.

$\therefore \frac{4}{3}\pi r^3 = -\alpha t + 36\pi.$ ... (ii)

It is being given that when $t = 3$, then $r = 6$.

Putting $t = 3$ and $r = 6$ in (ii), we get $k = -84\pi$.

Putting $k = -84\pi$ in (ii), we get

$r^3 = (63t + 27) \Rightarrow r = (63t + 27)^{1/3}.$

60. Let at any time t, the bacteria count be N. Then,

$\frac{dN}{dt} \propto N \Rightarrow \frac{dN}{dt} = kN \Rightarrow \int \frac{1}{N}dN = \int k\,dt \Rightarrow \log N = kt + \log(C).$

At $t = 0$, we have $N = 100000$.

$\therefore \log C = \log 100000$

$\Rightarrow \log N = kt + \log 100000.$ ... (i)

At $t = 2$, we have $N = 110000$.

Putting these values in (i), we get $k = \frac{1}{2}\log\frac{11}{10}.$

$\therefore \log N = \frac{1}{2}t\log\left(\frac{11}{10}\right) + \log 100000.$ ... (ii)

When $N = 200000$, let $t = T$, then

$\log 200000 = \frac{T}{2}\log\left(\frac{11}{10}\right) + \log 100000 \Rightarrow T = \frac{2\log 2}{\log\left(\frac{11}{10}\right)}.$

20. HOMOGENEOUS DIFFERENTIAL EQUATIONS

HOMOGENEOUS FUNCTION *A function $f(x, y)$ in x and y is said to be a homogeneous function of degree n, if the degree of each term is n.*

Examples (i) $f(x, y) = (x^2 + y^2 - xy)$ is a homogeneous function of degree 2.

(ii) $g(x, y) = (x^3 - 3xy^2 + 3x^2y + y^3)$ is a homogeneous function of degree 3.

In general, a homogeneous function $f(x, y)$ of degree n is expressible as

$$f(x, y) = x^n f\left(\frac{y}{x}\right).$$

HOMOGENEOUS DIFFERENTIAL EQUATION *An equation of the form* $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$, *where both $f(x, y)$ and $g(x, y)$ are homogeneous functions of degree n, is called a homogeneous differential equation of order* 1 *and degree* 1.

Such a differential equation may be expressed in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

For example, $\frac{dy}{dx} = \frac{x^2 - y^2}{xy} = \frac{f(x, y)}{g(x, y)}$, where $f(x, y)$ and $g(x, y)$ are homogeneous functions of degree 2 each. So, it is a homogeneous differential equation of order 1 and degree 1.

We may write, $\frac{dy}{dx} = \frac{x^2 - y^2}{xy} = \frac{\left\{1 - \left(\frac{y}{x}\right)^2\right\}}{\left\{\frac{y}{x}\right\}}$ [on dividing Nr and Dr by x^2]

Thus, $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

Method of Solving a Homogeneous Differential Equation

Let $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ be a homogeneous differential equation of order 1 and degree 1.

Putting $y = vx$ and $\frac{dy}{dx} = \left(v + x\frac{dv}{dx}\right)$ in the given equation, we get

$$v + x\frac{dv}{dx} = F(v)$$

$$\Rightarrow \quad \frac{dv}{[F(v)-v]}=\frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dv}{[F(v)-v)]}=\int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dv}{[F(v)-v]}=\log|x|+C.$$

Now, replace v by (y/x) to obtain the required solution.

NOTE If the differential equation is of the form $\frac{dx}{dy}=f\left(\frac{x}{y}\right)$ then we put $x=vy$ and proceed in a manner similar to as above.

SOLVED EXAMPLES

EXAMPLE 1 *Show that the differential equation* $2x^2\frac{dy}{dx}-2xy+y^2=0$ *is homogeneous and solve it.* **[CBSE 2012]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx}=\frac{2xy-y^2}{2x^2}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by $2x^2$, we get

$$\frac{dy}{dx}=\left\{\frac{y}{x}-\frac{1}{2}\left(\frac{y}{x}\right)^2\right\}=f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in (i), we get

$$v+x\frac{dv}{dx}=\frac{2vx^2-v^2x^2}{2x^2}$$

$$\Rightarrow \quad v+x\frac{dv}{dx}=\left(v-\frac{1}{2}v^2\right)$$

$$\Rightarrow \quad x\frac{dv}{dx}=-\frac{1}{2}v^2 \Rightarrow \frac{dv}{v^2}+\frac{1}{2}\frac{dx}{x}=0. \qquad \ldots \text{(ii)}$$

On integrating (ii), we get

$$\int \frac{dv}{v^2}+\frac{1}{2}\int \frac{1}{x}dx=C, \text{ where } C \text{ is an arbitrary constant}$$

$$\Rightarrow \quad \frac{-1}{v}+\frac{1}{2}\log|x|=C$$

$$\Rightarrow \quad \frac{-x}{y}+\frac{1}{2}\log|x|=C, \text{ which is the required solution.}$$

EXAMPLE 2 *Show that the differential equation* $\frac{dy}{dx}=\frac{y-x}{y+x}$ *is homogeneous and solve it.* **[CBSE 2004]**

SOLUTION The given differential equation is

$$\frac{dy}{dx}=\frac{y-x}{y+x}. \quad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x, we get

$$\frac{dy}{dx}=\left\{\frac{\frac{y}{x}-1}{\frac{y}{x}+1}\right\}=f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in (i), we get

$$v+x\frac{dv}{dx}=\frac{vx-x}{vx+x}$$

$$\Rightarrow \quad v+x\frac{dv}{dx}=\frac{v-1}{v+1}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\left(\frac{v-1}{v+1}-v\right)$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{-(1+v^2)}{(1+v)}$$

$$\Rightarrow \quad \frac{(1+v)}{(1+v^2)}dv=\frac{-1}{x}dx$$

$$\Rightarrow \quad \int\frac{(1+v)}{(1+v^2)}dv=-\int\frac{dx}{x}$$

$$\Rightarrow \quad \int\frac{1}{(1+v^2)}dv+\frac{1}{2}\int\frac{2v}{(1+v^2)}dv=-\int\frac{dx}{x}$$

$$\Rightarrow \quad \tan^{-1}v+\frac{1}{2}\log|1+v^2|=-\log|x|+C$$

$$\Rightarrow \quad \tan^{-1}v+\log\left|x\sqrt{1+v^2}\right|=C$$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x}+\log\left|\sqrt{x^2+y^2}\right|=C \quad \left[\text{putting } v=\frac{y}{x}\right]$$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x}+\frac{1}{2}\log(x^2+y^2)=C,$$ which is the required solution.

EXAMPLE 3 *Show that the differential equation* $x\frac{dy}{dx}-y=\sqrt{x^2+y^2}$ *is homogeneous and solve it.* **[CBSE 2005, '07, '11]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}. \quad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x, we get

$$\frac{dy}{dx} = \left\{ \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} \right\} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x} = v + \sqrt{1 + v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{1}{x}dx$$

$$\Rightarrow \quad \int \frac{dv}{(1 + v^2)} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \log\left|v + \sqrt{1 + v^2}\right| = \log|x| + \log|C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \quad \log\left|\frac{v + \sqrt{1 + v^2}}{x}\right| = \log|C_1|$$

$$\Rightarrow \quad \frac{v + \sqrt{1 + v^2}}{x} = \pm C_1 = C \text{ (say)}$$

$$\Rightarrow \quad v + \sqrt{1 + v^2} = Cx$$

$$\Rightarrow \quad y + \sqrt{x^2 + y^2} = Cx^2, \text{ which is the required solution } \left[\because \; v = \frac{y}{x}\right].$$

EXAMPLE 4 *Show that the differential equation* $(x\sqrt{x^2 + y^2} - y^2)dx + xydy = 0$ *is homogeneous and solve it.*

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{y^2 - x\sqrt{x^2 + y^2}}{xy}. \quad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \left\{ \frac{\left(\frac{y}{x}\right)^2 - \sqrt{1 + \left(\frac{y}{x}\right)^2}}{\left(\frac{y}{x}\right)} \right\} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x\sqrt{x^2 + v^2x^2}}{vx^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left(\frac{v^2 - \sqrt{1+v^2}}{v} - v\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-\sqrt{1+v^2}}{v}$$

$$\Rightarrow \quad \int \frac{v}{\sqrt{1+v^2}}\,dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \sqrt{1+v^2} = -\log|x| + C$$

$\Rightarrow \quad \sqrt{x^2 + y^2} + x\log|x| = Cx$, which is the required solution.

EXAMPLE 5 *Show that the differential equation* $(x^2 + xy)dy = (x^2 + y^2)dx$ *is homogeneous and solve it.* **[CBSE 2005]**

SOLUTION The given differential equation is

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \frac{\left\{1 + \left(\frac{y}{x}\right)^2\right\}}{\left\{1 + \frac{y}{x}\right\}} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{x^2 + vx^2} = \frac{1+v^2}{1+v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{(1+v^2)}{(1+v)} - v = \frac{1 + v^2 - v - v^2}{(1+v)} = \frac{(1-v)}{(1+v)}$$

$$\Rightarrow \quad \frac{(1+v)}{(1-v)}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \quad \int \frac{(1+v)}{(1-v)}\,dv = \int \frac{1}{x}\,dx \quad \text{[on integrating both sides]}$$

$$\Rightarrow \quad \int \frac{\{2 - (1-v)\}}{(1-v)}\,dv = \int \frac{1}{x}\,dx$$

$\Rightarrow \quad \int\left\{\frac{2}{(1-v)}-1\right\} dv = \int \frac{1}{x} dx$

$\Rightarrow \quad -2\log|1-v| - v = \log|x| + \log|C|,$

where C is an arbitrary constant

$\Rightarrow \quad \log|x| + \log|C| + 2\log|1-v| = -v$

$\Rightarrow \quad \log|Cx(1-v)^2| = -v$

$\Rightarrow \quad |Cx(1-v)^2| = e^{-v}$

$\Rightarrow \quad \left|Cx\left(1-\frac{y}{x}\right)^2\right| = e^{-y/x} \quad \left[\because\ v = \frac{y}{x}\right]$

$\Rightarrow \quad |C|(x-y)^2 = |x|e^{-y/x}$, which is the required solution.

EXAMPLE 6 *Show that the differential equation* $y^2dx + (x^2 - xy + y^2)dy = 0$ *is homogeneous and solve it.*

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{-y^2}{(x^2 - xy + y^2)}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \frac{-\left(\frac{y}{x}\right)^2}{\left\{1 - \frac{y}{x} + \left(\frac{y}{x}\right)^2\right\}} = f\left(\frac{y}{x}\right).$$

Thus, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$v + x\frac{dv}{dx} = \frac{-v^2}{(1-v+v^2)}$

$\Rightarrow \quad x\frac{dv}{dx} = -\left[\frac{v^2}{(1-v+v^2)} + v\right]$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{-(v+v^3)}{(1-v+v^2)}$

$\Rightarrow \quad \frac{(1-v+v^2)}{v(1+v^2)} dv = -\frac{1}{x} dx$

$\Rightarrow \quad \int \frac{(1+v^2)-v}{v(1+v^2)} dv = -\int \frac{dx}{x}$

$\Rightarrow \quad \int \frac{dv}{v} - \int \frac{dv}{(1+v^2)} + \int \frac{dx}{x} = \log C$

$\Rightarrow \quad \log|v| - \tan^{-1} v + \log|x| = \log C$

$\Rightarrow \quad \tan^{-1} v = \log \frac{|vx|}{C}$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x} = \log\left(\frac{|y|}{C}\right) \quad \left[\because \; v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \frac{|y|}{C} = e^{\tan^{-1}(y/x)}$$

$$\Rightarrow \quad |y| = Ce^{\tan^{-1}(y/x)}, \text{ which is the required solution.}$$

EXAMPLE 7 *Show that the differential equation* $x^2\left(\frac{dy}{dx}\right) = (x^2 - 2y^2 + xy)$ *is homogeneous and solve it.*

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{(x^2 - 2y^2 + xy)}{x^2}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \left\{1 - 2\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)\right\} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{x^2 - 2v^2x^2 + vx^2}{x^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = 1 - 2v^2 + v$$

$$\Rightarrow \quad x\frac{dv}{dx} = 1 - 2v^2$$

$$\Rightarrow \quad \frac{dv}{(1 - 2v^2)} = \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dv}{(1 - 2v^2)} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{dv}{\left(\frac{1}{2} - v^2\right)} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{dv}{\left\{\left(\frac{1}{\sqrt{2}}\right)^2 - v^2\right\}} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\cdot\frac{1}{\left(2 \times \frac{1}{\sqrt{2}}\right)}\log\left|\frac{\frac{1}{\sqrt{2}} + v}{\frac{1}{\sqrt{2}} - v}\right| = \log|x| + C$$

$$\Rightarrow \quad \frac{1}{2\sqrt{2}}\log\left|\frac{x + \sqrt{2}y}{x - \sqrt{2}y}\right| - \log|x| = C \quad [\because \; v = \frac{y}{x}].$$

This is the required solution.

EXAMPLE 8 *Show that the differential equation* $(x^3+y^3)dy - x^2y\,dx = 0$ *is homogeneous and solve it.* **[CBSE 2008]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{x^2y}{(x^3+y^3)}. \qquad \dots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^3, we get

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)}{\left\{1+\left(\frac{y}{x}\right)^3\right\}} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx^3}{x^3+v^3x^3}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left(\frac{v}{1+v^3} - v\right) = \frac{-v^4}{(1+v^3)}$$

$$\Rightarrow \quad \frac{(1+v^3)}{v^4}dv = \frac{-1}{x}dx$$

$$\Rightarrow \quad \int\left(\frac{1}{v^4}+\frac{1}{v}\right)dv = -\int\frac{1}{x}dx$$

$$\Rightarrow \quad \int\left(\frac{1}{v^4}+\frac{1}{v}\right)dv + \int\frac{1}{x}dx = C$$

$$\Rightarrow \quad \frac{-1}{3v^3} + \log|v| + \log|x| = C$$

$$\Rightarrow \quad \frac{-1}{3v^3} + \log|vx| = C$$

$$\Rightarrow \quad \frac{-x^3}{3y^3} + \log|y| = C, \text{ which is the required solution.}$$

EXAMPLE 9 *Show that the differential equation* $(y^2 - x^2)dy = 3xy\,dx$ *is homogeneous and solve it.* **[CBSE 2006, '08]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{3xy}{(y^2-x^2)}. \qquad \dots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \frac{3\left(\frac{y}{x}\right)}{\left\{\left(\frac{y}{x}\right)^2 - 1\right\}} = f\left(\frac{y}{x}\right).$$

Thus, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{3v}{(v^2 - 1)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left\{\frac{3v}{(v^2-1)} - v\right\} = \frac{(4v - v^3)}{(v^2-1)}$$

$$\Rightarrow \quad \frac{(v^2-1)}{(4v - v^3)}dv = \frac{1}{x}dx$$

$$\Rightarrow \quad \int \frac{(v^2-1)}{v(2-v)(2+v)}dv = \int \frac{1}{x}dx. \qquad \text{... (ii)}$$

Let $\frac{(v^2-1)}{v(2-v)(2+v)} = \frac{A}{v} + \frac{B}{(2-v)} + \frac{C}{(2+v)}.$

Then, $(v^2 - 1) \equiv A(2-v)(2+v) + Bv(2+v) + Cv(2-v).$ ... (iii)

Putting $v = 0$ on each side of (iii), we get $A = \frac{-1}{4}.$

Putting $v = 2$ on each side of (iii), we get $B = \frac{3}{8}.$

Putting $v = -2$ on each side of (iii), we get $C = \frac{-3}{8}.$

$$\therefore \quad \frac{(v^2-1)}{v(2-v)(2+v)} = \frac{-1}{4v} + \frac{3}{8(2-v)} - \frac{3}{8(2+v)}. \qquad \text{... (iv)}$$

Putting these values from (iv) in (ii), we get

$$-\frac{1}{4}\int\frac{dv}{v} + \frac{3}{8}\int\frac{dv}{(2-v)} - \frac{3}{8}\int\frac{dv}{(2+v)} = \int\frac{1}{x}dx$$

$$\Rightarrow \quad \int\frac{1}{x}dx + \frac{1}{4}\int\frac{dv}{v} + \frac{3}{8}\int\frac{-dv}{(2-v)} + \frac{3}{8}\int\frac{dv}{(2+v)} = \log|C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \quad \log|x| + \frac{1}{4}\log|v| + \frac{3}{8}\log|2-v| + \frac{3}{8}\log|2+v| = \log|C_1|$$

$$\Rightarrow \quad 8\log|x| + 2\log|v| + 3\log|2-v| + 3\log|2+v| = 8\log|C_1|$$

$$\Rightarrow \quad \log|x^8v^2(2-v)^3(2+v)^3| = \log\{(C_1)^8\}$$

$$\Rightarrow \quad |x^8v^2(2-v)^3(2+v)^3| = C_1^8 = C \text{ (say)}$$

$$\Rightarrow \quad x^8\frac{y^2}{x^2}\left(2-\frac{y}{x}\right)^3\left(2+\frac{y}{x}\right)^3 = C$$

$\Rightarrow \quad y^2(2x-y)^3(2x+y)^3 = C$, where C is an arbitrary constant.

$\Rightarrow \quad y^2(4x^2-y^2)^3 = C$, which is the required solution.

EXAMPLE 10 *Show that the differential equation* $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$ *is homogeneous and solve it.*

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y}. \quad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^3, we get

$$\frac{dy}{dx} = \frac{1 - 3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3 - 3\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right).$$

Thus, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{x^3 - 3v^2x^3}{v^3x^3 - 3vx^3}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1-3v^2}{v^3 - 3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left(\frac{1-3v^2}{v^3-3v} - v\right)$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{(1-v^4)}{(v^3-3v)}$$

$$\Rightarrow \quad \frac{(3v-v^3)}{(v^4-1)}dv = \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{(3v-v^3)}{(v^4-1)}dv = \int \frac{dx}{x}. \quad \ldots \text{(ii)}$$

Let $\frac{(3v-v^3)}{(v^4-1)} = \frac{A}{(v-1)} + \frac{B}{(v+1)} + \frac{Cv+D}{(v^2+1)}$. Then,

$(3v - v^3) \equiv A(v+1)(v^2+1) + B(v-1)(v^2+1) + (Cv+D)(v-1)(v+1)$. … (iii)

Putting $v = 1$ on each side of (iii), we get $A = \frac{1}{2}$.

Putting $v = -1$ on each side of (iii), we get $B = \frac{1}{2}$.

Comparing coefficients of v^3 on both sides of (iii), we get

$$A + B + C = -1 \Rightarrow \frac{1}{2} + \frac{1}{2} + C = -1 \Rightarrow C = -2.$$

Comparing the independent terms on both sides of (iii), we get

$$A - B - D = 0 \Rightarrow D = (A - B) = \left(\frac{1}{2} - \frac{1}{2}\right) = 0.$$

$$\therefore \quad \frac{(3v - v^3)}{(v^4 - 1)} = \frac{1}{2(v-1)} + \frac{1}{2(v+1)} - \frac{2v}{(v^2+1)}.$$

Putting this value in (ii), we get

$$\frac{1}{2}\int \frac{dv}{(v-1)} + \frac{1}{2}\int \frac{dv}{(v+1)} - \int \frac{2v}{(v^2+1)}\, dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log|v-1| + \frac{1}{2}\log|v+1| - \log|v^2+1| = \log|x| + \log|C|,$$

where C is an arbitrary constant

$$\Rightarrow \quad \log|v-1| + \log|v+1| - 2\log|v^2+1| = 2\log|x| + 2\log|C|$$

$$\Rightarrow \quad \log\left|\frac{(v-1)(v+1)}{(v^2+1)^2}\right| = \log|C^2x^2| \Rightarrow \log\left|\frac{(v^2-1)}{(v^2+1)^2}\right| = \log|C^2x^2|$$

$$\Rightarrow \quad (y^2 - x^2) = C^2(y^2 + x^2)^2 \quad \left[\because \ v = \frac{y}{x}\right].$$

Hence, $(y^2 - x^2) = C^2(y^2 + x^2)^2$ is the required solution.

EXAMPLE 11 *Show that the differential equation* $(x - y)\frac{dy}{dx} = (x + 2y)$ *is homogeneous and solve it.* **[CBSE 2013C]**

SOLUTION The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{x + 2y}{x - y}. \quad \ldots \text{(i)}$$

On dividing the Nr and Dr of (i) by x, we get

$$\frac{dy}{dx} = \frac{\left\{1 + 2\cdot\frac{y}{x}\right\}}{\left\{1 - \frac{y}{x}\right\}} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{1 + 2v}{1 - v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left\{\frac{1 + 2v}{1 - v} - v\right\} = \frac{(1 + 2v) - v + v^2}{(1 - v)} = \frac{v^2 + v + 1}{(1 - v)}$$

$$\Rightarrow \quad \frac{(v-1)}{(v^2 + v + 1)}\, dv = -\frac{1}{x}\, dx \Rightarrow \int \frac{(v-1)}{(v^2 + v + 1)}\, dv = -\int \frac{1}{x}\, dx. \quad \ldots \text{(ii)}$$

Let $(v - 1) = A \cdot \frac{d}{dv}(v^2 + v + 1) + B.$

Then, $(v-1) = A(2v+1) + B$. … (iii)

Comparing coefficients of like powers of v, we get

$$(2A = 1 \text{ and } A + B = -1) \Rightarrow \left(A = \frac{1}{2} \text{ and } B = \frac{-3}{2}\right).$$

Putting $(v-1) = \frac{1}{2}(2v+1) - \frac{3}{2}$ in (ii), we get

$$\int \frac{\frac{1}{2}(2v+1) - \frac{3}{2}}{(v^2+v+1)} dv = -\log|x| + C$$

$$\Rightarrow \frac{1}{2}\int \frac{(2v+1)}{(v^2+v+1)} dv - \frac{3}{2}\int \frac{dv}{(v^2+v+1)} = -\log|x| + C$$

$$\Rightarrow \frac{1}{2}\log|v^2+v+1| - \frac{3}{2}\int \frac{1}{\left(v+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dv = -\log|x| + C$$

$$\Rightarrow \frac{1}{2}\log|v^2+v+1| - \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = -\log|x| + C$$

$$\Rightarrow \frac{1}{2}\log|v^2+v+1| + \frac{1}{2}\log x^2 = \sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) + C$$

$$\Rightarrow \frac{1}{2}\log\left|\frac{y^2}{x^2} + \frac{y}{x} + 1\right| + \frac{1}{2}\log x^2 = \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + C$$

$$\Rightarrow \frac{1}{2}\log\left\{\frac{|(y^2+xy+x^2)|}{x^2}\cdot x^2\right\} = \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + C$$

$$\Rightarrow \log|x^2+xy+y^2| = 2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right) + C,$$

which is the required solution.

EXAMPLE 12 *Show that the differential equation* $x\frac{dy}{dx} = y - x\tan\frac{y}{x}$ *is homogeneous and solve it.* **[CBSE 2006C, '09]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{y}{x} - \tan\frac{y}{x}. \quad \text{… (i)}$$

This is of the form, $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = v - \tan v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\tan v$$

$$\Rightarrow \quad \frac{dv}{\tan v} = -\frac{dx}{x}$$

$$\Rightarrow \quad \int \cot v \, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{\cos v}{\sin v} dv + \int \frac{1}{x} dx = \log |C_1|, \text{ where } C_1 \text{ is an arbitrary constant}$$

$$\Rightarrow \quad \log |\sin v| + \log |x| = \log |C_1|$$

$$\Rightarrow \quad \log |x \sin v| = \log |C_1|$$

$$\Rightarrow \quad x \sin v = \pm C_1 = C \text{ (say)}$$

$$\Rightarrow \quad x \sin \frac{y}{x} = C, \text{ which is the required solution.}$$

EXAMPLE 13 *Show that the differential equation*

$$\left(x \cos \frac{y}{x}\right)(y\,dx + x\,dy) = \left(y \sin \frac{y}{x}\right)(x\,dy - y\,dx)$$

is homogeneous and solve it. **[CBSE 2010, '13C]**

SOLUTION The given differential equation may be written as

$$\left(xy \cos \frac{y}{x} + y^2 \sin \frac{y}{x}\right) dx = \left(xy \sin \frac{y}{x} - x^2 \cos \frac{y}{x}\right) dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left\{xy \cos \frac{y}{x} + y^2 \sin \frac{y}{x}\right\}}{\left\{xy \sin \frac{y}{x} - x^2 \cos \frac{y}{x}\right\}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(y/x) \cos (y/x) + (y/x)^2 \sin (y/x)}{(y/x) \sin (y/x) - \cos (y/x)} = f\left(\frac{y}{x}\right) \qquad \ldots \text{(i)}$$

[dividing Nr and Dr by x^2]

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{(v \cos v + v^2 \sin v)}{(v \sin v - \cos v)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left\{\frac{(v \cos v + v^2 \sin v)}{(v \sin v - \cos v)} - v\right\}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v \cos v}{(v \sin v - \cos v)}$$

$$\Rightarrow \quad \int \frac{(v \sin v - \cos v)}{v \cos v} dv = \int \frac{2}{x} dx$$

$\Rightarrow \quad \int \tan v\, dv - \int \frac{dv}{v} = \int \frac{2}{x} dx$

$\Rightarrow \quad -\log|\cos v| - \log|v| - 2\log|x| = \text{constant}$

$\Rightarrow \quad \log|\cos v| + \log|v| + 2\log|x| = \log|C_1|,$

where C_1 is an arbitrary constant

$\Rightarrow \quad \log|x^2 v \cos v| = \log|C_1|$

$\Rightarrow \quad x^2 v \cos v = \pm C_1 = C$ (say)

$\Rightarrow \quad xy\cos\frac{y}{x} = C$, which is the required solution $\left[\because\ v = \frac{y}{x}\right]$.

EXAMPLE 14 *Show that the differential equation*

$xy\log\left(\frac{y}{x}\right)dx + \left\{y^2 - x^2\log\left(\frac{y}{x}\right)\right\}dy = 0$ *is homogeneous and solve it.*

[CBSE 2010C]

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{xy\log\left(\frac{y}{x}\right)}{\left\{x^2\log\left(\frac{y}{x}\right) - y^2\right\}}. \qquad \ldots \text{(i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)\log\left(\frac{y}{x}\right)}{\left\{\log\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2\right\}} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx^2\log v}{x^2(\log v - v^2)} = \frac{v\log v}{(\log v - v^2)}$$

$\Rightarrow \quad x\frac{dv}{dx} = \left\{\frac{v\log v}{(\log v - v^2)} - v\right\} = \frac{v^3}{(\log v - v^2)}$

$\Rightarrow \quad \frac{(\log v - v^2)}{v^3} dv = \frac{1}{x} dx$

$\Rightarrow \quad \int\left\{\frac{\log v}{v^3} - \frac{1}{v}\right\} dv = \int \frac{1}{x} dx$ [integrating both sides]

$\Rightarrow \quad \int \underset{\text{I}}{(\log v)}\underset{\text{II}}{v^{-3}}\, dv - \int \frac{1}{v} dv = \int \frac{1}{x} dx + C_1,$

where C_1 is an arbitrary constant

$\Rightarrow \quad (\log v)\cdot\left(\frac{v^{-2}}{-2}\right) - \int \frac{1}{v}\cdot\left(\frac{v^{-2}}{-2}\right) dv - \log|v| = \log|x| + C_1$

[integrating by parts]

$$\Rightarrow \quad \frac{-\log v}{2v^2} + \frac{1}{2}\int v^{-3} dv - \log |v| = \log |x| + C_1$$

$$\Rightarrow \quad \frac{-\log v}{2v^2} - \frac{1}{4v^2} - \log |v| = \log |x| + C_1$$

$$\Rightarrow \quad \frac{\log v}{2v^2} + \frac{1}{4v^2} + \log |v| + \log |x| = -C_1 = C \text{ (say)}$$

$$\Rightarrow \quad \frac{\log \left|\frac{y}{x}\right|}{2\left(\frac{y^2}{x^2}\right)} + \frac{x^2}{4y^2} + \log |y| = C \quad [\because \; v = \frac{y}{x}]$$

$$\Rightarrow \quad 2x^2 \log \left|\frac{y}{x}\right| + x^2 + 4y^2 \log |y| = 4Cy^2$$

$$\Rightarrow \quad x^2\left(2\log \left|\frac{y}{x}\right| + 1\right) + 4y^2 \log |y| = 4Cy^2,$$

which is the required solution.

EXAMPLE 15 *Show that the differential equation* $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0$ *is homogeneous. Find the particular solution of this differential equation, given that* $x = 1$ *when* $y = \frac{\pi}{2}$. **[CBSE 2013]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{(y/x)\sin(y/x) - 1}{\sin(y/x)} = f\left(\frac{y}{x}\right). \qquad \text{... (i)}$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left\{\frac{v\sin v - 1}{\sin v} - v\right\} = \frac{-1}{\sin v}$$

$$\Rightarrow \quad \sin v \, dv = \frac{-1}{x}dx$$

$$\Rightarrow \quad \int \sin v \, dv = -\int \frac{1}{x}dx \quad \text{[on integrating both sides]}$$

$\Rightarrow \quad -\cos v = -\log |x| + C$, where C is an arbitrary constant

$$\Rightarrow \quad \log |x| - \cos\left(\frac{y}{x}\right) = C \qquad \text{... (ii)} \quad \left[\because \; v = \frac{y}{x}\right].$$

Putting $x = 1$ and $y = \frac{\pi}{2}$ in (ii), we get $C = 0$.

Hence, $\log|x| = \cos\left(\frac{y}{x}\right)$ is the required solution.

EXAMPLE 16 *Find the particular solution of the differential equation* $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$ *for* $x = 1$ *and* $y = 1$.

[CBSE 2004C, '07, '13C]

SOLUTION We may write the given differential equation as

$$\frac{dy}{dx} = \frac{-(3xy + y^2)}{(x^2 + xy)}. \quad \text{... (i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \frac{-\left\{3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2\right\}}{\left\{1 + \frac{y}{x}\right\}} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{-(3vx^2 + v^2x^2)}{(x^2 + vx^2)} = \frac{-(3v + v^2)}{(v + 1)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left\{\frac{-(3v + v^2)}{(v + 1)} - v\right\} = \frac{-2(v^2 + 2v)}{(v + 1)}$$

$$\Rightarrow \quad \frac{(v + 1)}{(v^2 + 2v)}\,dv = \frac{-2}{x}\,dx$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{2(v + 1)}{(v^2 + 2v)}\,dv + \int\frac{2}{x}\,dx = \log|C_1|$$

$$\Rightarrow \quad \frac{1}{2}\log|v^2 + 2v| + 2\log|x| = \log|C_1|$$

$$\Rightarrow \quad \log\left|x^2\sqrt{v^2 + 2v}\right| = \log|C_1|$$

$$\Rightarrow \quad \log\left|x\sqrt{y^2 + 2xy}\right| = \log|C_1| \quad \left[\because\ v = \frac{y}{x}\right].$$

$$\Rightarrow \quad x\sqrt{y^2 + 2xy} = \pm C_1$$

$$\Rightarrow \quad x^2(y^2 + 2xy) = C, \text{ where } C = C_1^2. \quad \text{... (ii)}$$

Putting $x = 1$ and $y = 1$ in (ii), we get $C = 3$.

$\therefore \quad x^2(y^2 + 2xy) = 3$ is the required solution.

EXAMPLE 17 *Find the particular solution of the differential equation* $x\frac{dy}{dx} - y + x\,\text{cosec}\,\frac{y}{x} = 0$, *given that* $y = 0$ *when* $x = 1$. **[CBSE 2009, '14C]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{y}{x} - \text{cosec}\,\frac{y}{x}. \quad \text{... (i)}$$

This is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$. So, it is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = v - \operatorname{cosec} v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\operatorname{cosec} v$$

$$\Rightarrow \quad -\sin v\, dv = \frac{1}{x}dx$$

$$\Rightarrow \quad \int(-\sin v)\, dv = \int\frac{1}{x}dx \quad \text{[on integrating both sides]}$$

$\Rightarrow \quad \cos v = \log|x| + C$, where C is an arbitrary constant

$$\Rightarrow \quad \cos\frac{y}{x} = \log|x| + C \qquad \ldots \text{(ii)} \quad \left[\because\ v = \frac{y}{x}\right].$$

Putting $x = 1$ and $y = 0$ in (ii), we get $C = 1$.

Hence, $\cos\frac{y}{x} = 1 + \log|x|$ is the required solution.

EXAMPLE 18 *Find the particular solution of the differential equation* $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$, *given that* $y = \frac{\pi}{4}$ *when* $x = 1$.

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right). \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$. So, it is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = v - \sin^2 v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \quad -\operatorname{cosec}^2 v\, dv = \frac{1}{x}dx$$

$$\Rightarrow \quad \int(-\operatorname{cosec}^2 v)\, dv = \int\frac{1}{x}dx \quad \text{[on integrating both sides]}$$

$\Rightarrow \quad \cot v = \log|x| + C$, where C is an arbitrary constant

$$\Rightarrow \quad \cot\frac{y}{x} = \log|x| + C \qquad \ldots \text{(ii)} \quad \left[\because\ v = \frac{y}{x}\right].$$

Putting $x = 1$ and $y = \frac{\pi}{4}$ in (ii), we get $C = 1$.

$\therefore$ $\cot \frac{y}{x} = \log |x| + 1$ is the desired solution.

EXAMPLE 19 *Find the equation of the family of curves for which the slope of tangent at any point* (x, y) *on it, is* $\frac{x^2 + y^2}{2xy}$.

SOLUTION We know that the slope of the tangent at any point (x, y) of the curve is $\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = \frac{x^2 + y^2}{2xy}. \qquad \text{... (i)}$$

On dividing the Nr and Dr of RHS of (i) by x^2, we get

$$\frac{dy}{dx} = \frac{\left\{1 + \left(\frac{y}{x}\right)^2\right\}}{2\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right).$$

So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \left\{\frac{(1 + v^2)}{2v} - v\right\} = \frac{(1 - v^2)}{2v}$$

$$\Rightarrow \quad \frac{2v}{(1 - v^2)}dv = \frac{1}{x}dx$$

$$\Rightarrow \quad \int \frac{2v}{(v^2 - 1)}dv = -\int \frac{1}{x}dx \quad \text{[on integrating both sides]}$$

$$\Rightarrow \quad \log |v^2 - 1| = -\log |x| + \log |C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \quad \log |v^2 - 1| + \log |x| = \log |C_1|$$

$$\Rightarrow \quad \log |(v^2 - 1)x| = \log |C_1|$$

$$\Rightarrow \quad (v^2 - 1)x = \pm C_1$$

$$\Rightarrow \quad \left(\frac{y^2}{x^2} - 1\right)x = \pm C_1$$

$$\Rightarrow \quad (y^2 - x^2) = \pm C_1 x$$

$$\Rightarrow \quad (x^2 - y^2) = Cx, \text{ where } \pm C_1 = C.$$

Hence, $(x^2 - y^2) = Cx$ is the equation of the required family of curves.

EXAMPLE 20 *Show that the differential equation* $2ye^{x/y}dx + (y - 2xe^{x/y})dy = 0$ *is homogeneous. Find the particular solution of this differential equation, given that* $x = 0$ *when* $y = 1$. **[CBSE 2013]**

SOLUTION The given differential equation may be written as

$$\frac{dx}{dy} = \frac{(2xe^{x/y} - y)}{2ye^{x/y}}. \qquad \text{... (i)}$$

On dividing the Nr and Dr of RHS of (i) by y, we get

$$\frac{dx}{dy} = \frac{\left\{2\frac{x}{y} \cdot e^{x/y} - 1\right\}}{2e^{x/y}} = f\left(\frac{x}{y}\right). \qquad \text{... (ii)}$$

So, the given differential equation is homogeneous.

Putting $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ in (ii), we get

$$v + y\frac{dv}{dy} = \frac{(2ve^v - 1)}{2e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \left\{\frac{(2ve^v - 1)}{2e^v} - v\right\} = \frac{-1}{2e^v}$$

$$\Rightarrow \quad 2e^v dv = \frac{-1}{y}dy$$

$$\Rightarrow \quad 2\int e^v dv = -\int \frac{1}{y}dy \quad \text{[on integrating both sides]}$$

$$\Rightarrow \quad 2e^v = -\log|y| + C$$

$$\Rightarrow \quad 2e^{x/y} + \log|y| = C \qquad \text{... (iii)} \quad \left[\because\ v = \frac{x}{y}\right].$$

Putting $y = 1$ and $x = 0$ in (iii), we get $C = 2$.

$\therefore$ $2e^{x/y} + \log|y| = 2$ is the required solution.

EXAMPLE 21 *Show that the differential equation* $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$ *is homogeneous and solve it.*

SOLUTION The given differential equation may be written as

$$\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y} - 1\right)}{(1 + e^{x/y})}. \qquad \text{... (i)}$$

This is of the form $\frac{dx}{dy} = f\left(\frac{x}{y}\right)$. So, it is homogeneous.

Putting $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ in (i), we get

$$v + y\frac{dv}{dy} = \frac{e^v(v-1)}{(1+e^v)}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \left\{\frac{e^v(v-1)}{(1+e^v)} - v\right\} = \frac{-(v+e^v)}{(1+e^v)}$$

$$\Rightarrow \quad \frac{(1+e^v)}{(v+e^v)}dv = -\frac{1}{y}dy$$

$$\Rightarrow \quad \int \frac{(1+e^v)}{(v+e^v)}dv = -\int \frac{1}{y}dy \quad \text{[on integrating both sides]}$$

$$\Rightarrow \quad \log|v+e^v| = -\log|y| + \log|C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \quad \log|v+e^v| + \log|y| = \log|C_1|$$

$$\Rightarrow \quad \log|(v+e^v)y| = \log|C_1|$$

$$\Rightarrow \quad (v+e^v)y = \pm C_1 = C \text{ (say)}$$

$$\Rightarrow \quad \left(\frac{x}{y} + e^{x/y}\right)y = C$$

$\Rightarrow \quad x + ye^{x/y} = C$, which is the required solution.

EXAMPLE 22 *Show that the differential equation* $y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$ *is homogeneous and solve it.*

SOLUTION The given differential equation may be written as

$$y\frac{dx}{dy} = 2x - x\log\left(\frac{y}{x}\right)$$

$$= 2x - x\cdot\log\left\{\frac{1}{(x/y)}\right\}$$

$$= 2x - x\cdot\left[\log 1 - \log\frac{x}{y}\right]$$

$$= 2x + x\log\frac{x}{y}.$$

$$\therefore \quad \frac{dx}{dy} = 2\left(\frac{x}{y}\right) + \left(\frac{x}{y}\right)\log\frac{x}{y} = f\left(\frac{x}{y}\right). \qquad \ldots \text{(i)}$$

So, the given differential equation is homogeneous.

Putting $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ in (i), we get

$$v + y\frac{dv}{dy} = 2v + v(\log v)$$

$$\Rightarrow \quad y\frac{dv}{dy} = v(1+\log v)$$

$$\Rightarrow \int \frac{1}{v(1+\log v)}\,dv = \int \frac{1}{y}\,dy \;\Rightarrow\; \int \frac{1}{t}\,dt = \int \frac{1}{y}\,dy, \text{ where } (1+\log v) = t$$

$$\Rightarrow \log|t| = \log|y| + \log|C_1| \;\Rightarrow\; \log|1+\log v| - \log|y| = \log|C_1|$$

$$\Rightarrow \log\left|\frac{1+\log v}{y}\right| = \log|C_1| \;\Rightarrow\; \frac{1+\log v}{y} = \pm C_1 = C$$

$\therefore$ $1+\log\dfrac{x}{y} = Cy$ is the required solution.

EXERCISE 20

In each of the following differential equations show that it is homogeneous and solve it.

1. $x\,dy = (x+y)\,dx$
2. $(x^2-y^2)\,dx + 2xy\,dy = 0$
3. $x^2dy + y(x+y)\,dx = 0$ **[CBSE 2010]**
4. $(x-y)dy - (x+y)dx = 0$
5. $(x+y)\,dy + (y-2x)\,dx = 0$
6. $(x^2+3xy+y^2)\,dx - x^2dy = 0$ **[CBSE 2006]**
7. $2xy\,dx + (x^2+2y^2)dy = 0$ **[CBSE 2008]**
8. $\dfrac{dy}{dx} + \dfrac{x-2y}{2x-y} = 0$
9. $\dfrac{dy}{dx} + \dfrac{x^2-y^2}{3xy} = 0$
10. $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2xy}$
11. $\dfrac{dy}{dx} = \dfrac{2xy}{(x^2-y^2)}$
12. $x^2\dfrac{dy}{dx} = 2xy + y^2$
13. $x^2\dfrac{dy}{dx} = x^2 + xy + y^2$
14. $y^2 + (x^2-xy)\dfrac{dy}{dx} = 0$
15. $x\dfrac{dy}{dx} - y = 2\sqrt{y^2-x^2}$
16. $y^2dx + (x^2+xy+y^2)dy = 0$
17. $(x-y)\dfrac{dy}{dx} = x+3y$
18. $(x^3+3xy^2)dx + (y^3+3x^2y)dy = 0$
19. $(x-\sqrt{xy})\,dy = y\,dx$
20. $x^2\dfrac{dy}{dx} + y^2 = xy$
21. $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$ **[CBSE 2007]**
22. $x\dfrac{dy}{dx} - y + x\sin\dfrac{y}{x} = 0$ **[CBSE 2012]**
23. $x\dfrac{dy}{dx} = y - x\cos^2\left(\dfrac{y}{x}\right)$
24. $\left(x\cos\dfrac{y}{x}\right)\dfrac{dy}{dx} = \left(y\cos\dfrac{y}{x}\right) + x$ **[CBSE 2012C]**
25. Find the particular solution of the differential equation $2xy + y^2 - 2x^2\dfrac{dy}{dx} = 0$, it being given that $y = 2$ when $x = 1$.

26. Find the particular solution of the differential equation $\left\{x \sin^2 \frac{y}{x} - y\right\} dx + x\,dy = 0$, it being given that $y = \frac{\pi}{4}$ when $x = 1$.

[CBSE 2011, '14C]

27. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{y(2y - x)}{x(2y + x)}$, given that $y = 1$ when $x = 1$.

28. Find the particular solution of the differential equation $(xe^{y/x} + y)\,dx = x\,dy$, given that $y(1) = 0$.

29. Find the particular solution of the differential equation $xe^{y/x} - y + x\frac{dy}{dx} = 0$, given that $y(e) = 0$.

30. The slope of the tangent to a curve at any point (x, y) on it is given by $\frac{y}{x} - \left(\cot \frac{y}{x}\right)\left(\cos \frac{y}{x}\right)$, where $x > 0$ and $y > 0$. If the curve passes through the point $\left(1, \frac{\pi}{4}\right)$, find the equation of the curve.

ANSWERS (EXERCISE 20)

1. $y = x \log |x| + Cx$ **2.** $x^2 + y^2 = Cx$ **3.** $x^2 y = C^2(y + 2x)$

4. $\tan^{-1} \frac{y}{x} = \frac{1}{2} \log (x^2 + y^2) + C$ **5.** $y^2 + 2xy - 2x^2 = C$

6. $\log |x| + \frac{x}{(y + x)} = C$ **7.** $3x^2 y + 2y^3 = C$ **8.** $(y - x) = C(y + x)^3$

9. $(x^2 + 2y^2)^3 = Cx^2$ **10.** $(x^2 - y^2) = Cx$ **11.** $y = C(y^2 + x^2)$

12. $y = Cx(x + y)$ **13.** $\tan^{-1} \frac{y}{x} = \log |x| + C$ **14.** $y = x\{C + \log |y|\}$

15. $y + \sqrt{y^2 - x^2} = C|x|^3$ **16.** $\log \left|\frac{y}{y + x}\right| + \log |x| + \frac{x}{(y + x)} = C$

17. $\log |x + y| + \frac{2x}{(x + y)} = C$ **18.** $y^4 + 6x^2 y^2 + x^4 = C$

19. $2\sqrt{\frac{x}{y}} + \log |y| = C$ **20.** $Cx = e^{x/y}$ **21.** $y = xe^{Cx}$

22. $x \tan\left(\frac{y}{2x}\right) = C$ **23.** $\tan \frac{y}{x} + \log |x| = C$

24. $\sin\left(\frac{y}{x}\right) = \log |x| + C$ **25.** $y = \frac{2x}{(1 - \log |x|)}$ **26.** $\cot \frac{y}{x} = \log |x| + 1$

27. $\frac{y}{x} + \frac{1}{2} \log |xy| = 1$ **28.** $\log |x| + e^{-y/x} = 1$ **29.** $y = -x \log (\log |x|)$

30. $\sec \frac{y}{x} + \log |x| = \sqrt{2}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 20)

2. $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$, which is homogeneous.

$\int \frac{2v}{(1+v^2)} dv = \int \frac{-1}{x} dx \Rightarrow \log|1+v^2| + \log|x| = \log|C|.$

3. $\int \frac{dv}{v(v+2)} = -\int \frac{1}{x} dx$

$\Rightarrow \frac{1}{2}\int \left\{\frac{1}{v} - \frac{1}{(v+2)}\right\} dv = -\int \frac{1}{x} dx$ [by partial fractions]

4. $\int \left(\frac{1-v}{1+v^2}\right) dv = \int \frac{1}{x} dx \Rightarrow \int \frac{dv}{(1+v^2)} - \frac{1}{2}\int \frac{2v}{(1+v^2)} dv = \int \frac{1}{x} dx$

$\therefore \tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log|x| + C \Rightarrow \tan^{-1}\frac{y}{x} = \frac{1}{2}\log(x^2+y^2) + C.$

5. $\int \frac{(v+1)}{(v^2+2v-2)} dv = -\int \frac{dx}{x} \Rightarrow \frac{1}{2}\int \frac{(2v+2)}{(v^2+2v-2)} dv = -\int \frac{dx}{x}$

$\therefore \frac{1}{2}\log|v^2+2v-2| + \log|x| = \log|C_1|.$

6. $x\frac{dv}{dx} = (1+2v+v^2) = (v+1)^2 \Rightarrow \int \frac{dv}{(v+1)^2} = \int \frac{1}{x} dx \Rightarrow \log|x| - \frac{1}{(v+1)} = C.$

7. $x\frac{dv}{dx} = \frac{-(3v+2v^3)}{(1+2v^2)} \Rightarrow \int \frac{(1+2v^2)}{(3v+2v^3)} dv + \int \frac{dx}{x} = \log|C_1|.$

$\therefore \frac{1}{3}\int \frac{(3+6v^2)}{(3v+2v^3)} dv + \log|x| = \log|C_1|$

$\Rightarrow \frac{1}{3}\log|3v+2v^3| + \log|x| = \log|C_1|$

$\Rightarrow \log|3v+2v^3| + 3\log|x| = 3\log|C_1|$

$\Rightarrow \log|3v+2v^3| + \log|x^3| = \log|C_1^3|.$

8. $\int \frac{(2-v)}{(v^2-1)} dv = \int \frac{1}{x} dx.$

$\frac{(2-v)}{(v^2-1)} = \left\{\frac{1}{2(v-1)} - \frac{3}{2(v+1)}\right\}$ [by partial fractions].

9. $x\frac{dv}{dx} = \frac{-(1+2v^2)}{3v} \Rightarrow \int \frac{3v}{(1+2v^2)} dv = -\int \frac{1}{x} dx$

$\Rightarrow \frac{3}{4}\int \frac{4v}{(1+2v^2)} dv = -\int \frac{1}{x} dx \Rightarrow \frac{3}{4}\log|1+2v^2| + \log|x| = \log|C_1|.$

10. $x\frac{dv}{dx}=\frac{(1-v^2)}{2v} \Rightarrow \int \frac{-2v}{(1-v^2)}dv=-\int \frac{1}{x}dx$

$\Rightarrow \log|1-v^2|+\log|x|=\log|C_1|.$

11. $x\frac{dv}{dx}=\frac{(v+v^3)}{(1-v^2)} \Rightarrow \int \frac{(1-v^2)}{v(1+v^2)}dv=\int \frac{1}{x}dx$

$\Rightarrow \int\left\{\frac{1}{v}-\frac{2v}{(1+v^2)}\right\}dv=\int \frac{1}{x}dx$ [by partial fractions].

12. $\int \frac{1}{v(1+v)}dv=\int \frac{1}{x}dx \Rightarrow \int\left\{\frac{1}{v}-\frac{1}{(1+v)}\right\}dv=\int \frac{1}{x}dx$ [by partial fractions].

13. $\int \frac{1}{(1+v^2)}dv=\int \frac{1}{x}dx \Rightarrow \tan^{-1}v=\log|x|+C.$

14. $\int \frac{(v-1)}{v(v+1)}dv=\int \frac{1}{x}dx \Rightarrow \int\left\{\frac{2}{v+1}-\frac{1}{v}\right\}dv=\int \frac{1}{x}dx.$

15. $\int \frac{1}{\sqrt{v^2-1}}dv=\int \frac{2}{x}dx \Rightarrow \log|v+\sqrt{v^2-1}|=2\log|x|+\log|C_1|.$

16. $\int \frac{1}{v(v+1)^2}dv=-\int \frac{1}{x}dx.$

Let $\frac{1}{v(v+1)^2}=\frac{A}{v}+\frac{B}{(v+1)}+\frac{C}{(v+1)^2}.$

This gives $A=1, B=-1$ and $C=-1.$

17. $x\frac{dv}{dx}=\frac{(1+v)^2}{(1-v)} \Rightarrow \int \frac{(v-1)}{(v+1)^2}dv=-\int \frac{dx}{x}$

$\Rightarrow \int\left\{\frac{(v+1)-2}{(v+1)^2}\right\}dv=-\int \frac{dx}{x} \Rightarrow \int \frac{1}{(v+1)}dv-2\int \frac{1}{(v+1)^2}dv+\int \frac{dx}{x}=C.$

18. $\int \frac{(v^3+3v)}{(v^4+6v^2+1)}dv=-\int \frac{1}{x}dx$

$\Rightarrow \frac{1}{4}\int \frac{(4v^3+12v)}{(v^4+6v^2+1)}dv=-\int \frac{1}{x}dx$

$\Rightarrow \frac{1}{4}\log|v^4+6v^2+1|+\log|x|=\log|C_1|$

$\Rightarrow (v^4+6v^2+1)x^4=C_1^4=C \text{ (say)} \Rightarrow y^4+6x^2y^2+x^4=C.$

19. $\int \frac{1-\sqrt{v}}{v^{3/2}}dv=\int \frac{1}{x}dx \Rightarrow \int\left\{\frac{1}{v^{3/2}}-\frac{\sqrt{v}}{v^{3/2}}\right\}dv=\int \frac{1}{x}dx$

$\Rightarrow \int v^{-3/2}dv-\int \frac{1}{v}dv=\log|x|-C \Rightarrow \frac{-2}{\sqrt{v}}-\log|v|=\log|x|-C$

$\Rightarrow 2\sqrt{\frac{x}{y}}+\log|y|=C.$

20. $x\dfrac{dv}{dx} = -v^2 \Rightarrow \int -\dfrac{1}{v^2}\,dv = \int \dfrac{1}{x}\,dx \Rightarrow \dfrac{1}{v} = \log x + \log C.$

$\therefore \quad \log Cx = \dfrac{x}{y} \Rightarrow Cx = e^{x/y}.$

21. $\dfrac{dy}{dx} = \dfrac{y}{x}\left\{\log\dfrac{y}{x} + 1\right\} = f\left(\dfrac{y}{x}\right).$

Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, we get

$$\int \frac{1}{v\log v}\,dv = \int \frac{1}{x}\,dx \Rightarrow \log(\log v) = \log x + \log C$$

$\therefore \quad \log v = Cx \Rightarrow v = e^{Cx} \Rightarrow y = xe^{Cx}.$

22. $\dfrac{dy}{dx} = \dfrac{y}{x} - \sin\dfrac{y}{x} = f\left(\dfrac{y}{x}\right).$

$\int \operatorname{cosec} v\,dv = \int \dfrac{-1}{x}\,dx \Rightarrow \log\tan\dfrac{v}{2} + \log x = \log C.$

25. $\dfrac{dy}{dx} = \dfrac{2xy + y^2}{2x^2} = \left\{\left(\dfrac{y}{x}\right) + \dfrac{1}{2}\left(\dfrac{y}{x}\right)^2\right\} = f\left(\dfrac{y}{x}\right).$

$2\int \dfrac{1}{v^2}\,dv = \int \dfrac{1}{x}\,dx \Rightarrow \dfrac{-2}{v} = \log|x| + C \Rightarrow \dfrac{-2x}{y} = \log|x| + C.$

Putting $x = 1$ and $y = 2$, we get $C = -1.$

30. $\dfrac{dy}{dx} = \left\{\dfrac{y}{x} - \cot\dfrac{y}{x}\cos\dfrac{y}{x}\right\} = f\left(\dfrac{y}{x}\right)$ which is homogeneous.

Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, we get

$$\int \sec v\tan v\,dv = -\int \frac{dx}{x} \Rightarrow \sec v = -\log|x| + C$$

$\therefore \quad \sec\dfrac{y}{x} + \log|x| = C.$

Putting $x = 1$ and $y = \dfrac{\pi}{4}$, we get $C = \sqrt{2}.$

21. LINEAR DIFFERENTIAL EQUATIONS

Linear Differential Equations

The most general form of linear differential equations is $\frac{dy}{dx} + Py = Q$, where P is a constant and Q is a constant or a function of x. The other common form of linear differential equations is $\frac{dx}{dy} + Px = Q$, where P is a constant and Q is a constant or a function of y.

Solution of $\frac{dy}{dx} + Py = Q$

First, we find $e^{\int P\,dx}$, which is known as the *integrating factor*, i.e., IF.

Now, $\frac{dy}{dx} + Py = Q \Rightarrow e^{\int P\,dx}\frac{dy}{dx} + Py\,e^{\int P\,dx} = Q \cdot e^{\int P\,dx}$

$$\Rightarrow e^{\int P\,dx} dy + Py \cdot e^{\int P\,dx} dx = Q \cdot e^{\int P\,dx} dx \Rightarrow d\left(y \cdot e^{\int P\,dx}\right) = Q \cdot e^{\int P\,dx} dx \quad \ldots \text{(i)}$$

$\Rightarrow y \cdot e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx} dx + C$, which is the required solution

[integrating both sides of (i)].

> ***Working Rule for Solving*** $\frac{dy}{dx} + Py = Q$
>
> (i) Find IF $= e^{\int P\,dx}$.
>
> (ii) The solution is given by
> $$y \times \text{IF} = \int \{Q \times \text{IF}\}\,dx + C.$$

SOLVED EXAMPLES

EXAMPLE 1 *Solve the differential equation*

$$x\frac{dy}{dx} - y = 2x^3,\ x > 0.$$ **[CBSE 2004, '07, '11]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \left(\frac{-1}{x}\right)y = 2x^2. \quad \ldots \text{(i)}$$

This is of the form, $\frac{dy}{dx} + Py = Q$, where $P = \frac{-1}{x}$ and $Q = 2x^2$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int -\frac{1}{x}\,dx} = e^{-\log x} = e^{\log\left(\frac{1}{x}\right)} = \frac{1}{x}.$$

So, the required solution is given by

$$y \times \text{IF} = \int \{Q \times \text{IF}\}\,dx + C, \text{ where } C \text{ is an arbitrary constant,}$$

i.e., $y \times \frac{1}{x} = \int \left(2x^2 \times \frac{1}{x}\right) dx + C = 2\int x\,dx + C = x^2 + C$

$\Rightarrow \quad y = x^3 + Cx.$

Hence, the required solution is $y = x^3 + Cx$.

EXAMPLE 2 *Solve the differential equation*

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{2}{(x^2-1)}.$$ **[CBSE 2010, '14]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \left\{\frac{2x}{(x^2-1)}\right\} y = \frac{2}{(x^2-1)^2}. \quad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$,

where $P = \frac{2x}{(x^2-1)}$ and $Q = \frac{2}{(x^2-1)^2}$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{2x}{(x^2-1)}\,dx} = e^{\log(x^2-1)} = (x^2-1).$$

So, the required solution is given by

$$y \times \text{IF} = \int \{Q \times \text{IF}\}\,dx + C$$

i.e., $y \times (x^2-1) = \int \left\{\frac{2}{(x^2-1)^2} \times (x^2-1)\right\} dx + C$

$$= 2\int \frac{dx}{(x^2-1)} + C$$

$$= 2\int \frac{1}{2}\left\{\frac{1}{(x-1)} - \frac{1}{(x+1)}\right\} dx + C \quad \text{[by partial fraction]}$$

$$= \log\left|\frac{x-1}{x+1}\right| + C.$$

Hence, $y(x^2-1) = \log\left|\frac{x-1}{x+1}\right| + C$ is the required solution.

EXAMPLE 3 *Solve* $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$. **[CBSE 2014]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \frac{1}{(1+x^2)} \cdot y = \frac{e^{\tan^{-1} x}}{(1+x^2)}.$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{(1+x^2)}$ and $Q = \frac{e^{\tan^{-1} x}}{(1+x^2)}$.

Thus, the given equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{1}{(1+x^2)} dx} = e^{\tan^{-1} x}.$$

So, the required solution is given by

$$y \times \text{IF} = \int |(Q \times (\text{IF})|\, dx + C,$$

$$\text{i.e., } y \times e^{\tan^{-1} x} = \int \left\{ \frac{e^{\tan^{-1} x}}{(1+x^2)} \times e^{\tan^{-1} x} \right\} dx + C$$

$$= \int \frac{e^{2\tan^{-1} x}}{(1+x^2)} dx + C$$

$$= \int e^{2t}\, dt + C, \text{ where } \tan^{-1} x = t$$

$$= \frac{1}{2} e^{2t} + C = \frac{1}{2} e^{2\tan^{-1} x} + C.$$

Hence, $y = \frac{1}{2} e^{\tan^{-1} x} + Ce^{-\tan^{-1} x}$ is the required solution.

EXAMPLE 4 *Solve* $(x \log x) \frac{dy}{dx} + y = \frac{2}{x} \log x.$ **[CBSE 2009, '10, '14]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \frac{1}{(x \log x)} \cdot y = \frac{2}{x^2}.$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{(x \log x)}$ and $Q = \frac{2}{x^2}$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{1}{x \log x} dx} = e^{\int \frac{1}{t} dt}, \text{ where } \log x = t$$

$$= e^{\log t} = t = \log x.$$

So, the solution of the given differential equation is

$$y \times \text{IF} = \int \{Q \times (\text{IF})\}\, dx + C,$$

$$\text{i.e., } y(\log x) = \int \left(\frac{2}{x^2} \log x \right) dx + C$$

$$= 2\int (\log x) \cdot \frac{1}{x^2} dx + C$$

$$= 2\left[(\log x)\left(-\frac{1}{x}\right) - \int \frac{1}{x} \cdot \left(-\frac{1}{x}\right) dx \right] + C$$

[integrating by parts]

$$= \frac{-2\log x}{x} - \frac{2}{x} + C.$$

Hence, $y(\log x) = \frac{-2}{x}(\log x + 1) + C$ is the required solution.

EXAMPLE 5 *Solve* $(x^2+1)\frac{dy}{dx} + 2xy = \sqrt{x^2+4}.$ **[CBSE 2008, '10]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \left(\frac{2x}{x^2+1}\right)y = \frac{\sqrt{x^2+4}}{(x^2+1)}. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$,

where $P = \frac{2x}{(x^2+1)}$ and $Q = \frac{\sqrt{x^2+4}}{(x^2+1)}$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{2x}{(x^2+1)}dx} = e^{\log(x^2+1)} = (x^2+1).$$

So, the required solution is given by

$$y \times \text{IF} = \int \{Q \times \text{IF}\}\,dx + C,$$

i.e., $y(x^2+1) = \int \frac{\sqrt{x^2+4}}{(x^2+1)} \times (x^2+1)\,dx$

$$\Rightarrow \quad y(x^2+1) = \int \sqrt{x^2+4}\,dx$$

$$= \frac{1}{2}x\sqrt{x^2+4} + \frac{1}{2} \times 2^2 \times \log|x + \sqrt{x^2+4}| + C$$

$$= \frac{1}{2}x\sqrt{x^2+4} + 2\log|x + \sqrt{x^2+4}| + C.$$

Hence, $y(x^2+1) = \frac{1}{2}x\sqrt{x^2+4} + 2\log|x + \sqrt{x^2+4}| + C$ is the required solution.

EXAMPLE 6 *Solve* $(1+x^2)\frac{dy}{dx} + y = \tan^{-1} x.$ **[CBSE 2008C, '09]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \frac{1}{(1+x^2)} \cdot y = \frac{\tan^{-1}x}{(1+x^2)}. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{(1+x^2)}$ and $Q = \frac{\tan^{-1}x}{(1+x^2)}$.

Thus, the given equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}.$$

$\therefore$ the required solution is

$$y \times \text{IF} = \int \{Q \times \text{IF}\}\, dx + C,$$

i.e., $y \times e^{\tan^{-1} x} = \int \left\{ \frac{\tan^{-1} x}{(1+x^2)} \cdot e^{\tan^{-1} x} \right\} dx + C$

$$= \int (t\, e^t)\, dt + C, \text{ where } \tan^{-1} x = t$$

$$= t\, e^t - \int 1 \cdot e^t\, dt + C \quad \text{[integrating by parts]}$$

$$= t\, e^t - e^t + C = e^t (t-1) + C$$

$$= e^{\tan^{-1} x} (\tan^{-1} x - 1) + C$$

$\Rightarrow \quad y = (\tan^{-1} x - 1) + Ce^{-\tan^{-1} x}$

Hence, $y = (\tan^{-1} x - 1) + Ce^{-\tan^{-1} x}$ is the required solution.

EXAMPLE 7 *Find the general solution of the differential equation*

$$\frac{dy}{dx} - y = \cos x.$$ **[CBSE 2012C]**

SOLUTION The given differential equation is

$$\frac{dy}{dx} - y = \cos x. \quad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = -1$ and $Q = \cos x$.

Thus, the given differential equation is linear.

$\text{IF} = e^{\int P\, dx} = e^{\int -dx} = e^{-x}$.

So, the required solution is

$$y \times \text{IF} = \int \{Q \times \text{IF}\}\, dx + C,$$

i.e., $y \times e^{-x} = \int (\cos x) e^{-x}\, dx + C. \quad \ldots \text{(ii)}$

Let $I = \int \underset{\text{I}}{(\cos x)}\, \underset{\text{II}}{e^{-x}}\, dx$

$$= (\cos x)(-e^{-x}) - \int (-\sin x)(-e^{-x})\, dx \quad \text{[integrating by parts]}$$

$$= -(\cos x) e^{-x} - \int \underset{\text{I}}{(\sin x)}\, \underset{\text{II}}{(e^{-x})}\, dx$$

$$= -(\cos x) e^{-x} - \{(\sin x)(-e^{-x})\} - \int (\cos x)(-e^{-x})\, dx$$

$$= -(\cos x) e^{-x} + (\sin x)(e^{-x}) - I.$$

$\therefore \quad 2I = (\sin x - \cos x) e^{-x} \Rightarrow I = \frac{1}{2} (\sin x - \cos x) e^{-x}.$

Putting this value in (ii), we get

$$y \times e^{-x} = \frac{1}{2} (\sin x - \cos x) e^{-x} + C \Rightarrow y = \frac{1}{2} (\sin x - \cos x) + Ce^{x}.$$

Hence, $y = \frac{1}{2} (\sin x - \cos x) + Ce^{x}$ is the required solution.

EXAMPLE 8 *Solve the differential equation*

$$x\frac{dy}{dx} + y - x + xy\cot x = 0,\ x \neq 0.$$ **[CBSE 2011, '12]**

SOLUTION The given differential equation may be written as

$$x\frac{dy}{dx} + (1 + x\cot x)y = x$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right)y = 1. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \left(\frac{1}{x} + \cot x\right)$ and $Q = 1$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int\left(\frac{1}{x}+\cot x\right)dx} = e^{\log x + \log \sin x}$$
$$= e^{\log(x \sin x)} = x \sin x.$$

So, the solution of the given differential equation is given by

$$y \times (\text{IF}) = \int (Q \times \text{IF})\,dx + C,$$

i.e., $y(x\sin x) = \int 1 \times (x \sin x)\,dx + C$

$$= \int \underset{\text{I}}{x} \underset{\text{II}}{\sin x}\,dx + C$$
$$= x(-\cos x) - \int 1 \cdot (-\cos x)\,dx + C$$
$$= -x\cos x + \int \cos x\,dx + C$$
$$= -x\cos x + \sin x + C.$$

$\therefore \quad y = \frac{1}{x} - \cot x + \frac{C}{x \sin x}$ is the required solution.

EXAMPLE 9 *Solve the differential equation*

$$\left\{\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right\}\frac{dx}{dy} = 1\ (x \neq 0).$$ **[CBSE 2012]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = -\frac{1}{\sqrt{x}} \cdot y + \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{\sqrt{x}} \cdot y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{\sqrt{x}}$ and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$.

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}.$$

So, the solution of the given differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C,$$

i.e., $y \times e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \times e^{2\sqrt{x}}\, dx + C$

$$= \int \frac{1}{\sqrt{x}}\, dx + C = 2\sqrt{x} + C.$$

Hence, $ye^{2\sqrt{x}} = 2\sqrt{x} + C$ is the required solution.

$\therefore \quad y = \frac{1}{x} - \cot x + \frac{C}{x \sin x}$ is the required solution.

EXAMPLE 10 *Solve* $\quad x\frac{dy}{dx} + 2y = x \cos x.$

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \frac{2}{x} \cdot y = \cos x. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{2}{x}$ and $Q = \cos x$.

Thus, the given equation is linear.

$\text{IF} = e^{\int P\, dx} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log (x^2)} = x^2.$

So, the required solution is

$$y \times \text{IF} = \int \{Q \times (\text{IF})\}\, dx + C,$$

i.e., $yx^2 = \int x^2 \cos x\, dx + C$

$= x^2 \sin x - \int 2x \sin x\, dx + C \quad$ [integrating by parts]

$= x^2 \sin x - 2 \cdot [x(-\cos x) - \int 1 \cdot (-\cos x)\, dx] + C$

[integrating by parts]

$= x^2 \sin x + 2x \cos x - 2 \sin x + C.$

Hence, $y = \sin x + \frac{2}{x} \cos x - \frac{2}{x^2} \sin x + \frac{C}{x^2}$ is the required solution.

EXAMPLE 11 *Solve* $\quad \frac{dy}{dx} + (\sec x) y = \tan x.$ **[CBSE 2006, '08C, '12]**

SOLUTION The given equation is of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \sec x \text{ and } Q = \tan x.$$

Thus, the given equation is linear.

$\text{IF} = e^{\int P\, dx} = e^{\int \sec x\, dx} = e^{\log (\sec x + \tan x)} = (\sec x + \tan x).$

So, the required solution is

$$y \times \text{IF} = \int \{Q \times (\text{IF})\}\, dx + C,$$

i.e., $y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\,dx + C$

$$= \int \sec x \tan x\,dx + \int \tan^2 x\,dx + C$$

$$= \sec x + \int (\sec^2 x - 1)\,dx + C$$

$$= \sec x + \tan x - x + C.$$

Hence, $y(\sec x + \tan x) = \sec x + \tan x - x + C$ is the required solution.

EXAMPLE 12 *Find the general solution of the differential equation* $\frac{dy}{dx} - 2y = \cos 3x.$

SOLUTION The given differential equation is of the form $\frac{dy}{dx} + Py = Q$, where $P = -2$ and $Q = \cos 3x$.

Thus, the given differential equation is linear.

$\text{IF} = e^{\int P\,dx} = e^{\int -2dx} = e^{-2x}.$

So, the required solution is

$$y \times \text{IF} = \int \{Q \times \text{IF}\}\,dx + C,$$

i.e., $y \times e^{-2x} = \int e^{-2x} \cos 3x\,dx + C$

$$= e^{-2x}\left[\frac{-2\cos 3x + 3\sin 3x}{\{(-2)^2 + 3^2\}}\right] + C$$

$$\left[\because \int e^{ax}\cos bx\,dx = e^{ax}\left\{\frac{a\cos bx + b\sin bx}{(a^2+b^2)}\right\}\right].$$

$\therefore\ y = \frac{(3\sin 3x - 2\cos 3x)}{13} + Ce^{2x}$, which is the required solution.

EXAMPLE 13 *Solve the differential equation*

$$(\cos^2 x)\frac{dy}{dx} + y = \tan x \left(0 \le x < \frac{\pi}{2}\right).$$ **[CBSE 2008, '11]**

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + (\sec^2 x)y = (\sec^2 x)\tan x. \quad \text{... (i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \sec^2 x$ and $Q = \sec^2 x \tan x$.

Thus, the given differential equation is linear.

$\text{IF} = e^{\int P\,dx} = e^{\int \sec^2 x\,dx} = e^{\tan x}.$

So, its solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C,$$

i.e., $y \times e^{\tan x} = \int e^{\tan x}(\sec^2 x) \tan x \, dx + C$

$= \int \underset{\text{I}}{t} \, \underset{\text{II}}{e^t} \, dt$, where $\tan x = t$ and $\sec^2 x \, dx = dt$

$= te^t - \int 1 \cdot e^t dt + C$

$= te^t - e^t + C = e^t(t-1) + C$

$= e^{\tan x}(\tan x - 1) + C.$

Hence, $y = (\tan x - 1) + Ce^{-\tan x}$ is the required solution.

EXAMPLE 14 *Solve the differential equation*

$$\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x.$$

SOLUTION The given differential equation is of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \tan x \text{ and } Q = 2x + x^2 \tan x.$$

Thus, the given differential equation is linear.

$\text{IF} = e^{\int P\,dx} = e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x.$

So, its solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) \, dx + C,$$

i.e., $y \times \sec x = \int (2x + x^2 \tan x) \sec x \, dx + C$

$= \int 2x \sec x \, dx + \int \underset{\text{I}}{x^2} \, \underset{\text{II}}{\sec x \tan x} \, dx + C$

$= \int 2x \sec x \, dx + \{x^2 \sec x - \int 2x \sec x \, dx\} + C$

$= x^2 \sec x + C.$

$\therefore$ $y = x^2 + C \cos x$ is the required solution.

EXAMPLE 15 *Solve the differential equation* $x\frac{dy}{dx} + y = x \cos x + \sin x$, *given* $y\left(\frac{\pi}{2}\right) = 1.$

[CBSE 2014C]

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + \frac{1}{x} \cdot y = \cos x + \frac{\sin x}{x}. \quad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{x}$ and

$Q = \cos x + \frac{\sin x}{x}.$

Thus, the given differential equation is linear.

$\text{IF} = e^{\int P\,dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x.$

So, its solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) \, dx + C,$$

$$\text{i.e., } y \times x = \int \left(\cos x + \frac{\sin x}{x} \right) x\,dx + C$$

$$= \int \underset{\text{I}}{x} \underset{\text{II}}{\cos x}\,dx + \int \sin x\,dx + C$$

$$= x \sin x - \int \sin x\,dx + \int \sin x\,dx + C$$

$$= x \sin x + C.$$

$$\therefore \quad y = \sin x + \frac{C}{x}. \qquad \ldots \text{(ii)}$$

It is being given that when $x = \frac{\pi}{2}$, then $y = 1$.

Putting $x = \frac{\pi}{2}$ and $y = 1$ in (ii), we get $C = 0$.

Hence, $y = \sin x$ is the required solution.

EXAMPLE 16 *Find the particular solution of the differential equation* $\cos x \frac{dy}{dx} + y = \sin x$, *given that* $y = 2$ *when* $x = 0$.

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} + (\sec x)\, y = \tan x. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \sec x$ and $Q = \tan x$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \sec x\,dx} = e^{\log(\sec x + \tan x)} = (\sec x + \tan x).$$

So, its solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C,$$

$$\text{i.e., } y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\,dx + C$$

$$= \int \sec x \tan x\,dx + \int \tan^2 x\,dx + C$$

$$= \sec x + \int (\sec^2 x - 1)\,dx + C$$

$$= \sec x + \int \sec^2 x\,dx - \int dx + C$$

$$= \sec x + \tan x - x + C.$$

Thus, $y(\sec x + \tan x) = \sec x + \tan x - x + C$. … (ii)

It is given that when $x = 0$, then $y = 2$.

$\therefore$ putting $x = 0$ and $y = 2$ in (ii), we get $C = 1$.

Hence, $y(\sec x + \tan x) = \sec x + \tan x - x + 1$ is the required solution.

EXAMPLE 17 *Find the particular solution of the differential equation* $x \frac{dy}{dx} - y = (x + 1)e^{-x}$, *given that* $y = 0$ *when* $x = 1$.

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} - \frac{1}{x} \cdot y = \frac{(x+1)e^{-x}}{x}. \quad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{-1}{x}$ and $Q = \frac{(x+1)e^{-x}}{x}$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log(1/x)} = \frac{1}{x}.$$

So, its solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C,$$

i.e., $y \times \frac{1}{x} = \int \frac{(x+1)e^{-x}}{x^2}\,dx + C$

$$\Rightarrow \quad y \times \frac{1}{x} = \int \underset{\text{I}}{\frac{1}{x}} \underset{\text{II}}{e^{-x}}\,dx + \int \frac{1}{x^2} e^{-x}\,dx + C$$

$$\Rightarrow \quad y \times \frac{1}{x} = \frac{1}{x} \cdot (-e^{-x}) + \int \frac{1}{x^2}(-e^{-x})\,dx + \int \frac{1}{x^2} e^{-x}\,dx + C$$

$$\Rightarrow \quad \frac{y}{x} = \frac{-e^{-x}}{x} + C$$

$$\Rightarrow \quad y = -e^{-x} + Cx. \quad \ldots \text{(ii)}$$

It is being given that when $x = 1$, then $y = 0$.

Putting $x = 1$ and $y = 0$ in (ii), we get $C = e^{-1}$.

$\therefore \quad y = -e^{-x} + xe^{-1}$ is the required solution.

EXAMPLE 18 *Find the particular solution of the differential equation* $\frac{dy}{dx} - 3y \cot x = \sin 2x$, *it being given that* $y = 2$ *when* $x = \frac{\pi}{2}$.

SOLUTION The given differential equation is

$$\frac{dy}{dx} - 3y \cot x = \sin 2x. \quad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = -3 \cot x$ and $Q = \sin 2x$.

Thus, the given equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int -3 \cot x\,dx} = e^{\log(\sin x)^{-3}} = (\sin x)^{-3} = \frac{1}{\sin^3 x}.$$

So, the solution of (i) is given by

$$y \times \text{IF} = \int \{Q \times (\text{IF})\}\,dx + C,$$

i.e., $y \times \frac{1}{\sin^3 x} = \int \left(\sin 2x \times \frac{1}{\sin^3 x} \right) dx + C$

$$= 2\int \frac{\cos x}{\sin^2 x}\,dx + C \quad [\because \sin 2x = 2 \sin x \cos x]$$

$$= 2\int \frac{1}{t^2}\,dt + C, \text{ where } \sin x = t$$

$$= \frac{-2}{t} + C = \frac{-2}{\sin x} + C.$$

Thus, $y = -2\sin^2 x + C\sin^3 x.$ … (ii)

It is being given that when $x = \frac{\pi}{2}$, then $y = 2$.

Putting $x = \frac{\pi}{2}$ and $y = 2$ in (ii), we get

$$-2\sin^2\frac{\pi}{2} + C\sin^3\frac{\pi}{2} = 2 \Rightarrow C - 2 = 2 \Rightarrow C = 4.$$

Hence, $y = -2\sin^2 x + 4\sin^3 x$ is the required solution.

EXAMPLE 19 *Find the particular solution of the differential equation* $(1-x^2)\frac{dy}{dx} - xy = x^2$, *given that* $y = 2$ *when* $x = 0$.

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} - \frac{x}{(1-x^2)}\cdot y = \frac{x^2}{(1-x^2)}. \qquad \text{… (i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{-x}{(1-x^2)}$ and $Q = \frac{x^2}{(1-x^2)}$.

Thus, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{-x}{(1-x^2)}dx} = e^{\frac{1}{2}\int \frac{-2x}{(1-x^2)}dx} = e^{\frac{1}{2}\log(1-x^2)}$$

$$= e^{\log\sqrt{1-x^2}} = \sqrt{1-x^2}$$

So, its solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C,$$

i.e., $y \times \sqrt{1-x^2} = \int \frac{x^2}{(1-x^2)} \times \sqrt{1-x^2}\,dx + C$

$$= \int \frac{x^2}{\sqrt{1-x^2}}\,dx + C$$

$$= \int \frac{\{-(1-x^2)+1\}}{\sqrt{1-x^2}}\,dx + C$$

$$= -\int \sqrt{1-x^2}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx + C$$

$$= -\left\{\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right\} + \sin^{-1}x + C$$

$$= \frac{-x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x + C.$$

$$\therefore \quad y = \frac{-x}{2} + \frac{\sin^{-1}x}{2\sqrt{1-x^2}} + \frac{C}{\sqrt{1-x^2}}. \qquad \text{… (ii)}$$

It is being given that when $x = 0$, then $y = 2$.

Putting $x = 0$ and $y = 2$ in (ii), we get $C = 2$.

Hence, $y = \frac{-x}{2} + \frac{\sin^{-1} x}{2\sqrt{1-x^2}} + \frac{2}{\sqrt{1-x^2}}$ is the required solution.

EXAMPLE 20 *Find the equation of a curve passing through the point (0, 1), it being given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x-coordinate and the product of the x-coordinate and y-coordinate of the point.*

SOLUTION We know that the slope of the tangent to the curve is $\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = x + xy \Rightarrow \frac{dy}{dx} - xy = x. \qquad \text{... (i)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = -x$ and $Q = x$.

So, the given differential equation is linear.

$$\text{IF} = e^{\int P\,dx} = e^{\int -x\,dx} = e^{\frac{-x^2}{2}}.$$

Hence, the solution of the given differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C,$$

$$\text{i.e., } y \times e^{\frac{-x^2}{2}} = \int x e^{\frac{-x^2}{2}}\,dx + C$$

$$= \int e^{-t} dt + C, \text{ where } \frac{x^2}{2} = t$$

$$= -e^{-t} + C = -e^{\frac{-x^2}{2}} + C.$$

$$\therefore \quad y = -1 + Ce^{\frac{x^2}{2}}. \qquad \text{...(ii)}$$

We have to find a curve satisfying (ii) and passing through (0, 1).

Putting $x = 0$ and $y = 1$ in (ii), we get $C = 2$.

Hence, $y = -1 + 2e^{\frac{x^2}{2}}$ is the equation of the required curve.

EXAMPLE 21 *An equation relating to the stability of an aeroplane is given by* $\frac{dv}{dt} = g \cos \alpha - kv$, *where v is the velocity and* g, α, k *are constants.*

Find an expression for the velocity if $v = 0$ *at* $t = 0$.

SOLUTION The given differential equation is

$$\frac{dv}{dt} + kv = g \cos \alpha. \qquad \text{... (i)}$$

This is of the form $\frac{dv}{dt} + Pv = Q$, where $P = k$ and $Q = g \cos \alpha$.

Thus, the given equation is linear.

$\text{IF} = e^{\int P\,dt} = e^{\int k\,dt} = e^{kt}$.

So, the solution of the given differential equation is

$$v \times \text{IF} = \int \{Q \times \text{IF}\}\,dt + C,$$

i.e., $ve^{kt} = \int (g\cos\alpha)\,e^{kt}\,dt + C$

$$= \frac{(g\cos\alpha)e^{kt}}{k} + C. \qquad \ldots \text{(ii)}$$

Now, it is given that $v = 0$ when $t = 0$.

Putting $t = 0$ and $v = 0$ in (ii), we get $C = \frac{-g\cos\alpha}{k}$.

$$\therefore \quad ve^{kt} = \frac{(g\cos\alpha)e^{kt}}{k} - \frac{g\cos\alpha}{k}$$

$\Rightarrow \quad v = \frac{1}{k}(g\cos\alpha)(1 - e^{-kt})$, which is the required expression.

SOLUTION OF $\frac{dx}{dy} + Px = Q$

Working Rule for Solving $\frac{dx}{dy} + Px = Q$

(i) Find $\text{IF} = e^{\int P\,dy}$.

(ii) The solution is given by $x \times \text{IF} = \int \{Q \times \text{IF}\}\,dy + C$.

EXAMPLE 22 *Find the general solution of the differential equation* $(x + 2y^3)\frac{dy}{dx} = y,\ y \neq 0$.

SOLUTION The given differential equation may be written as

$$y\frac{dx}{dy} = x + 2y^3$$

$$\Rightarrow \quad \frac{dx}{dy} - \frac{1}{y}\cdot x = 2y^2. \qquad \ldots \text{(i)}$$

This is of the form $\frac{dx}{dy} + Px = Q$, where $P = -\frac{1}{y}$ and $Q = 2y^2$.

Thus, the given equation is linear.

$$\text{IF} = e^{\int P\,dy} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = e^{\log (y^{-1})} = y^{-1} = \frac{1}{y}.$$

So, the required solution is

$$x \times \text{IF} = \int \{Q \times \text{IF}\}\,dy + C,$$

i.e., $x \times \frac{1}{y} = \int \left(2y^2 \times \frac{1}{y} \right) dy + C$

$= \int 2y\, dy + C = y^2 + C.$

Hence, $x = y^3 + Cy$ is the required solution.

EXAMPLE 23 *Find the particular solution of the differential equation* $(\tan^{-1} y - x) dy = (1 + y^2) dx$, *given that when* $x = 0$, $y = 0$.

[CBSE 2009, '13]

SOLUTION The given differential equation may be written as

$$\frac{dx}{dy} = \frac{(\tan^{-1} y - x)}{(1 + y^2)}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{(1 + y^2)} \cdot x = \frac{\tan^{-1} y}{(1 + y^2)}. \qquad \dots \text{(i)}$$

This is of the form $\frac{dx}{dy} + Px = Q$, where $P = \frac{1}{(1 + y^2)}$ and $Q = \frac{\tan^{-1} y}{(1 + y^2)}$.

Thus, the given differential equation is linear.

$\text{IF} = e^{\int P\, dy} = e^{\int \frac{1}{(1+y^2)} dy} = e^{\tan^{-1} y}.$

So, the solution of the given differential equation is given by

$x \times \text{IF} = \int \{Q \times \text{IF}\}\, dy + C$, where C is an arbitrary constant,

i.e., $x \times e^{\tan^{-1} y} = \int \left\{ \frac{\tan^{-1} y}{(1 + y^2)} \times e^{\tan^{-1} y} \right\} dy + C$

$= \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^t}\, dt + C$, where $\tan^{-1} y = t$ and $\frac{1}{(1 + y^2)} dy = dt$

$= te^t - \int 1 \cdot e^t dt + C$ [integrating by parts]

$= te^t - e^t + C$

$= (t - 1)e^t + C$

$= (\tan^{-1} y - 1)e^{\tan^{-1} y} + C.$

$$\therefore \quad xe^{\tan^{-1} y} = (\tan^{-1} y - 1)e^{\tan^{-1} y} + C. \qquad \dots \text{(ii)}$$

Putting $x = 0$ and $y = 0$ in (ii), we get $C = 1$.

$$\therefore \quad xe^{\tan^{-1} y} = (\tan^{-1} y - 1)e^{\tan^{-1} y} + 1.$$

Hence, $x = (\tan^{-1} y - 1) + e^{-\tan^{-1} y}$ is the required solution.

EXAMPLE 24 *Find the particular solution of the differential equation* $(\sqrt{1 - y^2}) dx = (\sin^{-1} y - x) dy$, *it being given that when* $y = 0$, *then* $x = 0$.

SOLUTION The given differential equation may be written as

$$\frac{dx}{dy} = \frac{\sin^{-1} y - x}{\sqrt{1 - y^2}}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{\sqrt{1-y^2}} \cdot x = \frac{\sin^{-1} y}{\sqrt{1-y^2}} \cdot \qquad \ldots \text{(i)}$$

This is of the form $\frac{dx}{dy} + Px = Q$, where $P = \frac{1}{\sqrt{1-y^2}}$ and $Q = \frac{\sin^{-1} y}{\sqrt{1-y^2}} \cdot$

Thus, the given equation is linear.

$\text{IF} = e^{\int P\,dy} = e^{\int \frac{1}{\sqrt{1-y2}}dy} = e^{\sin^{-1} y}.$

So, the solution of the given differential equation is given by

$$x \times \text{IF} = \int (Q \times \text{IF})\, dy + C,$$

i.e., $x \times e^{\sin^{-1} y} = \int \left\{ \frac{\sin^{-1} y}{\sqrt{1-y^2}} \cdot e^{\sin^{-1} y} \right\} dy + C$

$$= \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^t}\, dt + C, \text{ where } \sin^{-1} y = t \text{ and } \frac{1}{\sqrt{1-y^2}} dy = dt$$

$$= te^t - \int 1 \cdot e^t dt + C$$

$$= te^t - \int e^t dt + C = (te^t - e^t) + C$$

$$= (t-1)e^t + C$$

$$= (\sin^{-1} y - 1)e^{\sin^{-1} y} + C \qquad [\because \quad t = \sin^{-1} y].$$

$$\therefore \quad x = (\sin^{-1} y - 1) + Ce^{-\sin^{-1} y}. \qquad \ldots \text{(ii)}$$

Putting $y = 0$ and $x = 0$ in (ii), we get $C = 1$.

Hence, $x = (\sin^{-1} y - 1) + e^{-\sin^{-1} y}$ is the required solution.

EXAMPLE 25 *Find the particular solution of the differential equation* $\frac{dx}{dy} + x \cot y = 2y + y^2 \cot y \ (y \neq 0)$, *given that* $x = 0$ *when* $y = \frac{\pi}{2} \cdot$

[CBSE 2013C]

SOLUTION The given differential equation is of the form $\frac{dx}{dy} + Px = Q$, where $P = \cot y$ and $Q = (2y + y^2 \cot y)$.

So, the given differential equation is linear.

$\text{IF} = e^{\int P\,dy} = e^{\int \cot y\,dy} = e^{\log \sin y} = \sin y.$

Therefore, the solution is given by

$$x \times \text{IF} = \int (Q \times \text{IF})\, dy + C,$$

i.e., $x \times \sin y = \int (2y + y^2 \cot y) \sin y\, dy + C$

$$= \int 2y \sin y\, dy + \int \underset{\text{I}}{y^2} \underset{\text{II}}{\cos y}\, dy + C$$

$$= \int 2y \sin y \, dy + [y^2 \sin y - \int 2y \sin y \, dy] + C$$

$$= y^2 \sin y + C.$$

$\therefore \quad x = y^2 + C \operatorname{cosec} y.$ ... (i)

Putting $y = \frac{\pi}{2}$ and $x = 0$ in (i), we get $C = \frac{-\pi^2}{4}$.

Hence, the required solution is $x = y^2 - \frac{\pi^2}{4} \operatorname{cosec} y$.

EXERCISE 21

Find the general solution for each of the following differential equations.

1. $\frac{dy}{dx} + \frac{1}{x} \cdot y = x^2$
2. $x\frac{dy}{dx} + 2y = x^2$
3. $2x\frac{dy}{dx} + y = 6x^3$ **[CBSE 2003C]**
4. $x\frac{dy}{dx} + y = 3x^2 - 2,\ x > 0$
5. $x\frac{dy}{dx} - y = 2x^3$ **[CBSE 2006]**
6. $x\frac{dy}{dx} - y = x + 1$
7. $(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{(1 + x^2)}$
8. $(1 - x^2)\frac{dy}{dx} + xy = x\sqrt{1 - x^2}$
9. $(1 - x^2)\frac{dy}{dx} + xy = ax$ **[CBSE 2006C]**
10. $(x^2 + 1)\frac{dy}{dx} - 2xy = (x^2 + 1)(x^2 + 2)$
11. $\frac{dy}{dx} + 2y = 6e^x$ **[CBSE 2007]**
12. $\frac{dy}{dx} + 3y = e^{-2x}$
13. $\frac{dy}{dx} + 8y = 5e^{-3x}$ **[CBSE 2007]**
14. $x\frac{dy}{dx} - y = (x - 1)e^x,\ x > 0$
15. $\frac{dy}{dx} - y \tan x = e^x \sec x$
16. $(x \log x)\frac{dy}{dx} + y = 2 \log x$ **[CBSE 2009]**
17. $x\frac{dy}{dx} + y = x \log x$ **[CBSE 2008]**
18. $x\frac{dy}{dx} + 2y = x^2 \log x$
19. $(1 + x)\frac{dy}{dx} - y = e^{3x}(1 + x)^2$ **[CBSE 2008]**
20. $\frac{dy}{dx} + \frac{4x}{(x^2 + 1)}y + \frac{1}{(x^2 + 1)^2} = 0$
21. $(y + 3x^2)\frac{dx}{dy} = x$ **[CBSE 2011]**
22. $x\,dy - (y + 2x^2)dx = 0$ **[CBSE 2011]**
23. $x\,dy + (y - x^3)dx = 0$ **[CBSE 2011]**
24. $\frac{dy}{dx} + 2y = \sin x$
25. $\frac{dy}{dx} + y = \cos x - \sin x$ **[CBSE 2009]**
26. $\sec x\frac{dy}{dx} - y = \sin x$ **[CBSE 2009C]**
27. $(1 + x^2)dy + 2xy\,dx = \cot x$ **[CBSE 2011C]**
28. $(\sin x)\frac{dy}{dx} + (\cos x)y = \cos x \sin^2 x$
29. $\frac{dy}{dx} + 2y \cot x = 3x^2 \operatorname{cosec}^2 x$

30. $x\frac{dy}{dx} - y = 2x^2 \sec x$

31. $\frac{dy}{dx} = y\tan x - 2\sin x$

32. $\frac{dy}{dx} + y\cot x = \sin 2x$ **[CBSE 2008]**

33. $\frac{dy}{dx} + 2y\tan x = \sin x$ **[CBSE 2008]**

34. $\frac{dy}{dx} + y\cot x = x^2 \cot x + 2x$ **[CBSE 2005]**

Find a particular solution satisfying the given condition for each of the following differential equations.

35. $x\frac{dy}{dx} + y = x^3$, given that $y = 1$ when $x = 2$.

36. $\frac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x$, given that $y = 0$ when $x = \frac{\pi}{2}\cdot$ **[CBSE 2012]**

37. $\frac{dy}{dx} + 2xy = x$, given that $y = 1$ when $x = 0$.

38. $\frac{dy}{dx} + 2y = e^{-2x} \sin x$, given that $y = 0$ when $x = 0$.

39. $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$, given that $y = 0$ when $x = 0$.

40. $x\frac{dy}{dx} - y = \log x$, given that $y = 0$ when $x = 1$.

41. $\frac{dy}{dx} + y\tan x = 2x + x^2 \tan x$, given that $y = 1$ when $x = 0$.

42. A curve passes through the origin and the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point. Find the equation of the curve.

43. A curve passes through the point (0, 2) and the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5. Find the equation of the curve.

Find the general solution for each of the following differential equations.

44. $ydx - (x + 2y^2)dy = 0$

45. $ydx + (x - y^2)dy = 0$

46. $(x - y^3)\frac{dy}{dx} + y = 0$ **[CBSE 2011]**

47. $(x + 3y^2)\frac{dy}{dx} = y, (y > 0)$

48. $(x + y)\frac{dy}{dx} = 1$

49. $(x + y + 1)\frac{dy}{dx} = 1$

50. Solve $(x + 1)\frac{dy}{dx} = 2e^{-y} - 1$, given that $x = 0$ when $y = 0$.

51. Solve $(1 + y^2)dx + (x - e^{-\tan^{-1} y})\,dy = 0$, given that when $y = 0$, then $x = 0$.

ANSWERS (EXERCISE 21)

1. $y = \frac{x^3}{4} + \frac{C}{x}$ **2.** $y = \frac{x^2}{4} + \frac{C}{x^2}$ **3.** $y = \frac{6}{7}x^3 + \frac{C}{\sqrt{x}}$ **4.** $y = x^2 - 2 + \frac{C}{x}$

5. $y = x^3 + Cx$ **6.** $y = x \log x - 1 + Cx$ **7.** $y(1+x^2) = \tan^{-1} x + C$

8. $y = \frac{-1}{2}\sqrt{1-x^2} \log(1-x^2) + C\sqrt{1-x^2}$ **9.** $y = a + C\sqrt{1-x^2}$

10. $y = (x^2+1)(x + \tan^{-1} x + C)$ **11.** $y = 2e^x + Ce^{-2x}$ **12.** $y = e^{-2x} + Ce^{-3x}$

13. $y = \frac{-5}{4}e^{-3x} + Ce^{-2x}$ **14.** $y = e^x + Cx$ **15.** $y \cos x = e^x + C$

16. $y(\log x) = (\log x)^2 + C$ **17.** $4xy = 2x^2 \log x - x^2 + C$

18. $y = \frac{x^2}{16}(4\log x - 1) + \frac{C}{x^2}$ **19.** $y = \frac{1}{3}(1+x)e^{3x} + C(1+x)$

20. $y = \frac{-x}{(x^2+1)^2} + \frac{C}{(x^2+1)^2}$ **21.** $y = 3x^2 + Cx$ **22.** $y = 2x^2 + Cx$

23. $y = \frac{x^3}{4} + \frac{C}{x}$ **24.** $y = \frac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$ **25.** $y = \cos x + Ce^{-x}$

26. $y = Ce^{\sin x} - (1 + \sin x)$ **27.** $y(1+x^2) = \log|\sin x| + C$

28. $y \sin x = \frac{1}{3}\sin^3 x + C$ **29.** $y \sin^2 x = x^3 + C$

30. $y = Cx + 2x \log|\sec x + \tan x|$ **31.** $2y \cos x = \cos 2x + C$

32. $y \sin x = \frac{2}{3}\sin^3 x + C$ **33.** $y = \cos x + C\cos^2 x$

34. $y = x^2 + C \operatorname{cosec} x$ **35.** $y = \frac{x^3}{4} - \frac{2}{x}$ **36.** $y \sin x = 2x^2 - \frac{1}{2}\pi^2$

37. $2y = 1 + e^{-x^2}$ **38.** $y = \sin x$ **39.** $y = \frac{4x^3}{3(1+x^2)}$ **40.** $y = x - 1 - \log x$

41. $y = x^2 + \cos x$ **42.** $x + y + 1 = e^x$ **43.** $y = 4 - x - 2e^x$ **44.** $x = 2y^2 + Cy$

45. $x = \frac{y^2}{3} + \frac{C}{y}$ **46.** $x = \frac{y^3}{4} + \frac{C}{y}$ **47.** $x = 3y^2 + Cy$

48. $(x + y + 1)e^{-y} = C$ **49.** $x = Ce^y - (y+2)$ **50.** $y = \log\left|\frac{2x+1}{x+1}\right|$

51. $xe^{\tan^{-1} y} = \tan^{-1} y$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 21)

8. $\frac{dy}{dx} + \frac{x}{(1-x^2)} \cdot y = \frac{x}{\sqrt{1-x^2}}$.

$$\text{IF} = e^{-\frac{1}{2}\int \frac{-2x}{(1-x^2)}dx} = e^{-\frac{1}{2}\log(1-x^2)} = e^{\log \frac{1}{\sqrt{1-x^2}}} = \frac{1}{\sqrt{1-x^2}}$$

$\therefore$ its solution is

$$y \times \frac{1}{\sqrt{1-x^2}} = \int \frac{x}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-x^2}}\, dx = \frac{-1}{2} \int \frac{-2x}{(1-x^2)}\, dx + C$$

$$= -\frac{1}{2} \log(1-x^2) + C.$$

9. $\dfrac{dy}{dx} + \dfrac{x}{(1-x^2)} \cdot y = \dfrac{ax}{(1-x^2)}$.

$$\text{IF} = e^{-\frac{1}{2}\int \frac{-2x}{(1-x^2)}\, dx} = e^{-\frac{1}{2}\log(1-x^2)} = \frac{1}{\sqrt{1-x^2}}.$$

$\therefore$ its solution is given by

$$y \times \frac{1}{\sqrt{1-x^2}} = \int \frac{1}{\sqrt{1-x^2}} \times \frac{ax}{(1-x^2)}\, dx + C$$

$$= -\frac{a}{2} \int \frac{-2x}{(1-x^2)^{3/2}}\, dx + C = -\frac{a}{2} \int \frac{dt}{t^{3/2}} + C, \text{ where } (1-x^2) = t$$

$$= -\frac{a}{2} \int t^{-3/2}\, dt + C = \frac{a}{\sqrt{t}} + C = \frac{a}{\sqrt{1-x^2}} + C.$$

10. Write $(x^2 + 2) = \{(x^2 + 1) + 1\}$.

14. $\dfrac{dy}{dx} - \dfrac{1}{x} \cdot y = \dfrac{(x-1)}{x} e^x$.

$$\text{IF} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = e^{\log(1/x)} = \frac{1}{x}.$$

$\therefore$ its solution is given by

$$y \times \frac{1}{x} = \int e^x \left(\frac{x-1}{x^2} \right) dx + C = \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx + C$$

$$= e^x \cdot \frac{1}{x} + C. \quad [\because \int e^x \{f(x) + f'(x)\}\, dx = e^x f(x)].$$

16. $\dfrac{dy}{dx} + \dfrac{1}{x \log x} \cdot y = \dfrac{2}{x}$.

$$\text{IF} = e^{\int \frac{1}{x \log x} dx} = e^{\int \frac{1}{t} dt} = e^{\log t} = t = \log x.$$

21. $\dfrac{dy}{dx} = \dfrac{y + 3x^2}{x} \Rightarrow \dfrac{dy}{dx} - \dfrac{1}{x} \cdot y = 3x.$

22. $\dfrac{dy}{dx} = \dfrac{y + 2x^2}{x} \Rightarrow \dfrac{dy}{dx} - \dfrac{1}{x} \cdot y = 2x.$

23. $\dfrac{dy}{dx} = \dfrac{x^3 - y}{x} \Rightarrow \dfrac{dy}{dx} + \dfrac{1}{x} \cdot y = x^2.$

24. $\text{IF} = e^{\int 2\, dx} = e^{2x}$

$\therefore \quad y \times e^{2x} = \int e^{2x} \sin x\, dx + C$

$$= e^{2x} \left\{ \frac{2 \sin x - \cos x}{(2^2 + 1^2)} \right\} + C \quad \left[\int e^{ax} \sin bx\, dx = e^{ax} \left\{ \frac{a \sin bx + b \cos bx}{(a^2 + b^2)} \right\} \right]$$

$$\Rightarrow \quad y = \frac{(2 \sin x - \cos x)}{5} + Ce^{-2x}.$$

25. $\text{IF} = e^{\int dx} = e^x.$

$y \times e^x = \int e^x(\cos x - \sin x)\, dx + C$

$= \int \underset{\text{I}}{(\cos x)} \underset{\text{II}}{e^x}\, dx - \int (\sin x) e^x dx + C$

$= (\cos x)e^x - \int (-\sin x)e^x dx - \int (\sin x)e^x dx + C$

$= (\cos x)e^x + C.$

26. $\dfrac{dy}{dx} - (\cos x)y = \sin x \cos x.$

$\text{IF} = e^{-\int \cos x\, dx} = e^{-\sin x}.$

$\therefore\;\; y \times e^{-\sin x} = \int (\sin x \cos x)\, e^{-\sin x} dx + C$

$= \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^{-t}}\, dt + C$, where $\sin x = t$.

32. $\text{IF} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x.$

$\therefore\;\; y \times \sin x = \int \sin x \sin 2x\, dx + C$

$= 2\int \sin^2 x \cos x\, dx + C = 2\int t^2 dt + C$, where $\sin x = t$.

33. $\text{IF} = e^{\int 2 \tan x\, dx} = e^{-2 \log \cos x} = e^{\log (\cos x)^{-2}} = (\cos x)^{-2} = \dfrac{1}{\cos^2 x}\cdot$

$y \times \dfrac{1}{\cos^2 x} = \int \left\{ \sin x \times \dfrac{1}{\cos^2 x} \right\} dx + C = \int \sec x \tan x\, dx + C = \sec x + C$

$\therefore\;\; y = \cos x + C \cos^2 x.$

34. $\text{IF} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x.$

$y \times \sin x = \int (x^2 \cot x + 2x) \sin x\, dx + C$

$= \int x^2 \cos x\, dx + \int 2x \sin x\, dx + C$

$= x^2 \sin x - \int 2x \sin x\, dx + \int 2x \sin x\, dx + C = x^2 \sin x + C.$

37. $\text{IF} = e^{\int 2x\, dx} = e^{x^2}.$

$\therefore\;\; y \times e^{x^2} = \int x e^{x^2} dx = \dfrac{1}{2}\int e^t dt + C$, where $x^2 = t$

$= \dfrac{1}{2} e^{x^2} + C.$

$\therefore\;\; y = \dfrac{1}{2} + \dfrac{C}{e^{x^2}}\cdot$

Putting $y = 1$ and $x = 0$, we get $C = \dfrac{1}{2}\cdot$

40. $\text{IF} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = e^{\log (1/x)} = \dfrac{1}{x}\cdot$

$y \times \dfrac{1}{x} = \int \left(\underset{\text{I}}{\log x} \cdot \underset{\text{II}}{\dfrac{1}{x^2}} \right) dx + C.$

43. $(x + y) - \dfrac{dy}{dx} = 5.$

44. $y\, dx = (x + 2y^2) dy \;\Rightarrow\; \dfrac{dx}{dy} = \dfrac{x}{y} + 2y$

$\therefore\;\; \dfrac{dx}{dy} - \dfrac{1}{y} \cdot x = 2y.$

45. $y\,dx = (y^2 - x)dy \;\Rightarrow\; \dfrac{dx}{dy} = \dfrac{y^2 - x}{y}$

$\therefore\;\; \dfrac{dx}{dy} - \dfrac{1}{y}\cdot x = y.$

46. $(y^3 - x)\dfrac{dy}{dx} = y \;\Rightarrow\; \dfrac{dx}{dy} = \dfrac{(y^3 - x)}{y}$

$\therefore\;\; \dfrac{dx}{dy} + \dfrac{1}{y}\cdot x = y^2.$

47. $(x + 3y^2)\dfrac{dy}{dx} = y \;\Rightarrow\; \dfrac{dx}{dy} = \dfrac{x + 3y^2}{y}$

$\therefore\;\; \dfrac{dx}{dy} - \dfrac{1}{y}\cdot x = 3y.$

48. $(x + y)\dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dx}{dy} = x + y.$

$\therefore\;\; \dfrac{dx}{dy} - x = y.$

$\text{IF} = e^{-\int dy} = e^{-y}.$

$\therefore\;\; x \times e^{-y} = \int \underset{\text{I}}{y}\,\underset{\text{II}}{e^{-y}}\,dy + C.$

49. $\dfrac{dx}{dy} - x = (y + 1).$

$\text{IF} = e^{-\int dy} = e^{-y}.$

$$\begin{aligned} x \times e^{-y} &= \int (y + 1)e^{-y}dy + C \\ &= \int \underset{\text{I}}{y}\,\underset{\text{II}}{e^{-y}}\,dy + \int e^{-y}dy + C \\ &= -ye^{-y} + 2\int e^{-y}dy + C \\ &= -ye^{-y} - 2e^{-y} + C. \end{aligned}$$

$\therefore\;\; x = Ce^{y} - (y + 2).$

50. $(x + 1)\dfrac{dy}{dx} = \dfrac{(2 - e^y)}{e^y} \;\Rightarrow\; \dfrac{dx}{dy} = \dfrac{(x + 1)e^y}{(2 - e^y)}$

$\therefore\;\; \dfrac{dx}{dy} - \dfrac{e^y}{(2 - e^y)}\cdot x = \dfrac{e^y}{(2 - e^y)}\cdot$

$\text{IF} = e^{\int \frac{-e^y}{(2-e^y)}dy} = e^{\log(2-e^y)} = (2 - e^y).$

$\therefore\;\; x \times (2 - e^y) = \int \dfrac{e^y}{(2 - e^y)} \times (2 - e^y)dy + C = \int e^y dy + C = e^y + C.$

Now, $y = 0$ and $x = 0 \;\Rightarrow\; C = -1.$

$\therefore\;\; x(2 - e^y) = e^y - 1 \Rightarrow (2x + 1) = (x + 1)e^y$

$\therefore\;\; e^y = \dfrac{(2x + 1)}{(x + 1)} \Rightarrow y = \log\left(\dfrac{2x + 1}{x + 1}\right)\cdot$

51. $(1+y^2)dx = (e^{-\tan^{-1} y} - x)dy \Rightarrow \dfrac{dx}{dy} = \dfrac{e^{-\tan^{-1} y} - x}{(1+y^2)}$.

$\therefore \quad \dfrac{dx}{dy} + \dfrac{1}{(1+y^2)} \cdot x = \dfrac{e^{-\tan^{-1} y}}{(1+y^2)}$.

$\text{IF} = e^{\int \frac{1}{(1+y^2)} dy} = e^{\tan^{-1} y}$.

$\therefore \quad x \times e^{\tan^{-1} y} = \int \left\{ \dfrac{e^{-\tan^{-1} y}}{(1+y^2)} \times e^{\tan^{-1} y} \right\} dy = C = \int \dfrac{1}{(1+y^2)} dy + C$

$\Rightarrow \quad x \times e^{\tan^{-1} y} = \tan^{-1} y + C.$ ... (i)

Putting $x = 0$ and $y = 0$ in (i), we get $C = 0$.

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. The solution of the DE $\dfrac{dy}{dx} = e^{x+y}$ is

(a) $e^x + e^y = C$ (b) $e^x - e^{-y} = C$ (c) $e^x + e^{-y} = C$ (d) none of these

2. The solution of the DE $\dfrac{dy}{dx} = 2^{x+y}$ is

(a) $2^x + 2^y = C$ (b) $2^x + 2^{-y} = C$ (c) $2^x - 2^{-y} = C$ (d) none of these

3. The solution of the DE $(e^x + 1)y\,dy = (y+1)e^x\,dx$ is

(a) $e^y = C(e^x + 1)(y+1)$ (b) $e^y = e^x + y + 1$

(c) $y = (e^x + 1)(y+1)$ (d) none of these

4. The solution of the DE $x\,dy + y\,dx = 0$ is

(a) $x + y = C$ (b) $xy = C$ (c) $\log(x+y) = C$ (d) none of these

5. The solution of the DE $x\dfrac{dy}{dx} = \cot y$ is

(a) $x\cos y = C$ (b) $x\tan y = C$ (c) $x\sec y = C$ (d) none of these

6. The solution of the DE $\dfrac{dy}{dx} = \dfrac{(1+y^2)}{(1+x^2)}$ is

(a) $(y+x) = C(1-yx)$ (b) $(y-x) = C(1+yx)$

(c) $y = (1+x)C$ (d) none of these

7. The solution of the DE $\dfrac{dy}{dx} = 1 - x + y - xy$ is

(a) $\log(1+y) = x - \dfrac{x^2}{2} + C$ (b) $e^{(1+y)} = x - \dfrac{x^2}{2} + C$

(c) $e^y = x - \dfrac{x^2}{2} + C$ (d) none of these

8. The solution of the DE $\frac{dy}{dx} = e^{x+y} + x^2 \cdot e^y$ is

(a) $e^{x-y} + \frac{x^3}{3} + C$ (b) $e^x + e^{-y} + \frac{x^3}{3} = C'$

(c) $e^x - e^{-y} = \frac{x^3}{3} + C$ (d) none of these

9. The solution of the DE $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$ is

(a) $y + \sin^{-1} y = \sin^{-1} x + C$ (b) $\sin^{-1} y - \sin^{-1} x = C$

(c) $\sin^{-1} y + \sin^{-1} x = C$ (d) none of these

10. The solution of the DE $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$ is

(a) $y = 2\tan \frac{x}{2} - x + C$ (b) $y = \tan \frac{x}{2} - 2x + C$

(c) $y = \tan x - x + C$ (d) none of these

11. The solution of the DE $\frac{dy}{dx} = \frac{-2xy}{(x^2+1)}$ is

(a) $y^2(x+1) = C$ (b) $y(x^2+1) = C$ (c) $x^2(y+1) = C$ (d) none of these

12. The solution of the DE $\cos x(1 + \cos y)dx - \sin y(1 + \sin x)dy = 0$ is

(a) $1 + \sin x \cos y = C$ (b) $(1 + \sin x)(1 + \cos y) = C$

(c) $\sin x \cos y + \cos x = C$ (d) none of these

13. The solution of the DE $x \cos y\, dy = (xe^x \log x + e^x)dx$ is

(a) $\sin y = e^x + \log x + C$ (b) $\sin y - e^x + \log x = C$

(c) $\sin y = e^x (\log x) + C$ (d) none of these

14. The solution of the DE $\frac{dy}{dx} + y \log y \cot x = 0$ is

(a) $\cos x \log y = C$ (b) $\sin x \log y = C$

(c) $\log y = C \sin x$ (d) none of these

15. The general solution of the DE $(1 + x^2)dy - xy\, dx = 0$ is

(a) $y = C(1 + x^2)$ (b) $y^2 = C(1 + x^2)$ (c) $y\sqrt{1 + x^2} = C$ (d) none of these

16. The general solution of the DE $x\sqrt{1 + y^2}\, dx + y\sqrt{1 + x^2} dy = 0$ is

(a) $\sin^{-1} x + \sin^{-1} y = C$ (b) $\sqrt{1 + x^2} + \sqrt{1 + y^2} = C$

(c) $\tan^{-1} x + \tan^{-1} y = C$ (d) none of these

17. The general solution of the DE $\log\left(\frac{dy}{dx}\right) = (ax + by)$ is

(a) $\frac{-e^{-by}}{b} = \frac{e^{ax}}{a} + C$ (b) $e^{ax} - e^{-by} = C$

(c) $be^{ax} + ae^{by} = C$ (d) none of these

18. The general solution of the DE $\frac{dy}{dx} = (\sqrt{1-x^2})(\sqrt{1-y^2})$ is

(a) $\sin^{-1} y - \sin^{-1} x = x\sqrt{1-x^2} + C$ (b) $2\sin^{-1} y - \sin^{-1} x = x\sqrt{1-x^2} + C$

(c) $2\sin^{-1} y - \sin^{-1} x = C$ (d) none of these

19. The general solution of the DE $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ is

(a) $x^2 - y^2 = C_1 x$ (b) $x^2 + y^2 = C_1 y$

(c) $x^2 + y^2 = C_1 x$ (d) none of these

20. The general solution of the DE $x^2 \frac{dy}{dx} = x^2 + xy + y^2$ is

(a) $\tan^{-1} \frac{y}{x} = \log x + C$ (b) $\tan^{-1} \frac{x}{y} = \log x + C$

(c) $\tan^{-1} \frac{y}{x} = \log y + C$ (d) none of these

21. The general solution of the DE $x\frac{dy}{dx} = y + x \tan \frac{y}{x}$ is

(a) $\sin\left(\frac{y}{x}\right) = C$ (b) $\sin\left(\frac{y}{x}\right) = Cx$ (c) $\sin\left(\frac{y}{x}\right) = Cy$ (d) none of these

22. The general solution of the DE $2xy\,dy + (x^2 - y^2)dx = 0$ is

(a) $x^2 + y^2 = Cx$ (b) $x^2 + y^2 = Cy$ (c) $x^2 + y^2 = C$ (d) none of these

23. The general solution of the DE $(x - y)dy + (x + y)dx$ is

(a) $\tan^{-1} \frac{y}{x} = C\sqrt{x^2 + y^2}$ (b) $e^{\tan^{-1}(y/x)} = C\sqrt{x^2 + y^2}$

(c) $\tan^{-1}\left(\frac{y}{x}\right) = x^2 + y^2 + C$ (d) none of these

24. The general solution of the DE $\frac{dy}{dx} = \frac{y}{x} + \sin \frac{y}{x}$ is

(a) $\tan \frac{y}{2x} = Cx$ (b) $\tan \frac{y}{x} = Cx$ (c) $\tan \frac{y}{2x} = C$ (d) none of these

25. The general solution of the DE $\frac{dy}{dx} + y \tan x = \sec x$ is

(a) $y = \sin x - C\cos x$ (b) $y = \sin x + C\cos x$

(c) $y = \cos x - C\sin x$ (d) none of these

26. The general solution of the DE $\frac{dy}{dx} + y\cot x = 2\cos x$ is

(a) $(y + \sin x)\sin x = C$ (b) $(y + \cos x)\sin x = C$
(c) $(y - \sin x)\sin x = C$ (d) none of these

27. The general solution of the DE $\frac{dy}{dx} + \frac{y}{x} = x^2$ is

(a) $xy = x^4 + C$ (b) $4xy = x^4 + C$ (c) $3xy = x^3 + C$ (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (c)	**2.** (b)	**3.** (a)	**4.** (b)	**5.** (a)	**6.** (b)	**7.** (a)	**8.** (b)	**9.** (c)
10. (a)	**11.** (b)	**12.** (b)	**13.** (c)	**14.** (b)	**15.** (b)	**16.** (b)	**17.** (a)	**18.** (b)
19. (c)	**20.** (a)	**21.** (b)	**22.** (a)	**23.** (b)	**24.** (a)	**25.** (b)	**26.** (c)	**27.** (b)

HINTS TO THE GIVEN OBJECTIVE QUESTIONS

1. The given DE is $e^x dx = e^{-y} dy$.

$\therefore \int e^x dx = \int e^{-y} dy \Rightarrow e^x = -e^{-y} + C \Rightarrow e^x + e^{-y} = C.$

2. We have $2^x dx = 2^{-y} dy \Rightarrow \int 2^x dx = \int 2^{-y} dy.$

$\therefore \quad 2^x \log 2 = -2^{-y} \log 2 = \log C \Rightarrow 2^x + 2^{-y} = C$, where $\frac{\log C}{\log 2} = C.$

3. $\int \frac{e^x}{(e^x + 1)} dx = \int \left(\frac{y}{y+1}\right) dy \Rightarrow \log(e^x + 1) = \int \left(1 - \frac{1}{y+1}\right) dy$

$\Rightarrow y - \log(y+1) = \log(e^x + 1) + \log C \Rightarrow y = \log[C(e^x + 1)(y+1)]$

$\Rightarrow e^y = C(e^x + 1)(y + 1).$

4. $x\,dy + y\,dx = 0 \Rightarrow \int \frac{1}{y} dy = -\int \frac{1}{x} dx \Rightarrow \log y = -\log x + \log C.$

$\therefore \quad \log xy = \log C \Rightarrow xy = C.$

5. $\int \tan y\, dy = \int \frac{1}{x} dx \Rightarrow \log x = \log \sec y + \log C.$

$\Rightarrow x = C \sec y \Rightarrow x \cos y = C.$

6. $\int \frac{dy}{(1+y^2)} = \int \frac{dx}{(1+x^2)} \Rightarrow \tan^{-1} y = \tan^{-1} x + C_1.$

$\Rightarrow \tan^{-1} y - \tan^{-1} x = C_1 \Rightarrow \tan^{-1}\left(\frac{y-x}{1+yx}\right) = C_1 \Rightarrow \frac{y-x}{(1+yx)} = \tan C_1 = C.$

7. $\int \frac{dy}{(1+y)} = \int (1-x)\, dx \Rightarrow \log(1+y) = x - \frac{x^2}{2} + C.$

8. $\frac{dy}{dx} = e^y(e^x + x^2) \Rightarrow \int (e^x + x^2)\, dx = \int e^{-y} dy.$

$\Rightarrow -e^{-y} = e^x + \frac{x^3}{3} + C \Rightarrow e^x + e^{-y} + \frac{x^3}{3} = C'.$

9. $\int \frac{1}{\sqrt{1-y^2}}\,dy + \int \frac{1}{\sqrt{1-x^2}}\,dx = C \Rightarrow \sin^{-1} y + \sin^{-1} x = C.$

10. $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x} = \frac{2\sin^2(x/2)}{2\cos^2(x/2)} = \tan^2 \frac{x}{2} = \left(\sec^2 \frac{x}{2} - 1\right)$

$\Rightarrow \int dy = \int \left(\sec^2 \frac{x}{2} - 1\right) dx \Rightarrow y = 2\tan \frac{x}{2} - x + C.$

11. $\int \frac{1}{y}\,dy = \int \frac{-2x}{(x^2+1)}\,dx \Rightarrow \log y + \log(x^2+1) = \log C.$

$\therefore \quad y(x^2+1) = C.$

12. $\int \frac{\cos x}{(1+\sin x)}\,dx - \int \frac{\sin y}{(1+\cos y)}\,dy = \log C$

$\Rightarrow \log(1+\sin x) + \log(1+\cos y) = \log C \Rightarrow (1+\sin x)(1+\cos y) = C.$

13. $\int \cos y\,dy = \int e^x \left(\log x + \frac{1}{x}\right) dx \Rightarrow \sin y = e^x \log x + C.$

14. $\frac{1}{y\log y}\,dy + \cot x\,dx = 0 \Rightarrow \frac{dy}{y\log y} + \int \cot x\,dx = \log C$

$\Rightarrow \log(\log y) + \log \sin x = \log C \Rightarrow \log(\sin x + \log y) = \log C$

$\Rightarrow (\sin x)\log y = C.$

15. $\int \frac{1}{y}\,dy = \frac{1}{2}\int \frac{2x}{(1+x^2)}\,dx \Rightarrow \log y = \frac{1}{2}\log(1+x^2) + \log C$

$\Rightarrow \log y = \log(C\sqrt{1+x^2}) \Rightarrow y = C\sqrt{1+x^2}.$

16. $\int \frac{2x}{2\cdot\sqrt{1+x^2}}\,dx + \int \frac{2y}{2\cdot\sqrt{1+y^2}} = C \Rightarrow \frac{1}{2}\int \frac{dt}{\sqrt{t}} + \frac{1}{2}\int \frac{du}{\sqrt{u}} = C,$

where $1+x^2 = t$ and $1+y^2 = u$

$\Rightarrow \sqrt{t} + \sqrt{u} = C \Rightarrow \sqrt{1+x^2} + \sqrt{1+y^2} = C.$

17. $\frac{dy}{dx} = e^{ax+by} \Rightarrow \int e^{-by}\,dy = \int e^{ax}\,dx \Rightarrow \frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + C.$

18. $\int \frac{dy}{\sqrt{1-y^2}} = \int \sqrt{1-x^2}\,dx \Rightarrow \sin^{-1} y = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x + C$

$\Rightarrow 2\sin^{-1} y - \sin^{-1} x = x\sqrt{1-x^2} + C.$

19. The given DE is homogeneous. Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore \int \frac{2v}{(1+v^2)}\,dv = -\int \frac{1}{x}\,dx \Rightarrow \log(1+v^2) + \log x = \log C.$

$\therefore \quad x^2 + y^2 = C_1 x.$

20. $\frac{dy}{dx} = d\,\frac{x^2+xy+y^2}{x^2}$, which is homogeneous.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ to get $v + x\frac{dv}{dx} = (1+v+v^2)$.

$\therefore \int \frac{dv}{(1+v^2)} = \int \frac{1}{x}\,dx \Rightarrow \tan^{-1} v = \log x + C \Rightarrow \tan^{-1} \frac{y}{x} = \log x + C.$

21. $\frac{dy}{dx} = \frac{y}{x} + \tan \frac{y}{x}$, which is homogeneous.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ to get $\int \cot v \, dv = \int \frac{1}{x} dx$

$\Rightarrow \quad \log \sin v = \log x + \log C \;\Rightarrow\; \sin v = Cx \;\Rightarrow\; \sin \frac{y}{x} = Cx.$

22. $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$, which is homogeneous.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx} \cdot$ Then,

$$\int \frac{2v}{(1+v^2)} dv = -\int \frac{1}{x} dx \;\Rightarrow\; \log(1+v^2) = -\log x + \log C$$

$\Rightarrow \quad x(1+v^2) = C \;\Rightarrow\; x^2 + y^2 = Cx.$

23. $\frac{dy}{dx} = \frac{x+y}{x-y}$, which is clearly homogeneous.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx} \;\Rightarrow\; \int \frac{(1-v)}{(1+v^2)} dv = \int \frac{1}{x} dx$

$\Rightarrow \quad \int \frac{dv}{(1+v^2)} - \frac{1}{2}\int \frac{2v}{(1+v^2)} dv = \int \frac{1}{x} dx$

$\Rightarrow \quad \tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log x + \log C \;\Rightarrow\; \tan^{-1} v = \log(Cx\sqrt{1+v^2})$

$\Rightarrow \quad \tan^{-1}\frac{y}{x} = \log(C\sqrt{x^2+y^2}) \;\Rightarrow\; e^{\tan^{-1} y/x} = C\sqrt{x^2+y^2}.$

24. Given DE is homogeneous.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx} \cdot$

Then, $v + x\frac{dv}{dx} = v + \sin v \;\Rightarrow\; \int \operatorname{cosec} v \, dv = \int \frac{1}{x} dx$

$\Rightarrow \quad \log \tan \frac{v}{2} = \log x + \log C \;\Rightarrow\; \tan \frac{v}{2} = Cx \;\Rightarrow\; \tan \frac{y}{2x} = Cx.$

25. Given DE is linear.

$\text{IF} = e^{\int \tan x \, dx} = e^{\log(\sec x)} = \sec x.$

$\therefore$ its solution is

$y(\sec x) = \int \sec^2 x \, dx \;\Rightarrow\; y(\sec x) = \tan x + C \;\Rightarrow\; y = \sin x + C\cos x.$

26. Given DE is linear.

$\text{IF} = e^{\int \cot x \, dx} = e^{\log \sin x} = \sin x.$

$\therefore \quad y(\sin x) = \int 2\sin x \cos x \, dx \;\Rightarrow\; y\sin x = \sin^2 x + C.$

$\therefore \quad (y - \sin x)\sin x = C.$

27. Given DE is linear.

$\text{IF} = e^{\int \frac{1}{x} dx} = e^{\log x} = x.$

$\therefore \quad y \times x = \int (x \times x^2) \, dx = \int x^3 dx = \frac{x^4}{4} + C$

$\Rightarrow \quad 4xy = x^4 + C.$

22. VECTORS AND THEIR PROPERTIES

In our daily life, we generally come across two types of quantities, namely scalars and vectors.

SCALARS *A quantity that has magnitude only is known as a scalar.*

Examples Each of the quantities *mass, length, time, temperature, density, speed,* etc., is a scalar.

VECTORS *A quantity that has magnitude as well as direction is called a vector.*

Examples Each of the quantities *force, velocity, acceleration* and *momentum* is a vector.

However, we define a vector as given below.

'A directed line segment is called a vector.'

A directed line segment with ***initial point*** A and the ***terminal point*** B, is the vector denoted by $\overrightarrow{AB}$.

The magnitude of $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$.

REMARK We usually denote a vector by a single letter with an arrow on it and its magnitude is denoted by this letter only.

Thus, $\overrightarrow{\boldsymbol{AB}} = \vec{\boldsymbol{a}}$ and $|\overrightarrow{\boldsymbol{AB}}| = |\vec{\boldsymbol{a}}| = \boldsymbol{a}$.

UNIT VECTOR *A vector $\vec{a}$ is called a unit vector if $|\vec{a}| = 1$ and it is denoted by $\hat{a}$.*

EQUAL VECTORS *Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude and the same direction regardless of the positions of their initial points.*

NEGATIVE OF A VECTOR *A vector having the same magnitude as that of a given vector $\vec{a}$ and the direction opposite to that of $\vec{a}$ is called the negative of $\vec{a}$, to be denoted by $-\vec{a}$.*

Thus, if $\overrightarrow{\boldsymbol{AB}} = \vec{\boldsymbol{a}}$ then $\overrightarrow{\boldsymbol{BA}} = -\vec{\boldsymbol{a}}$.

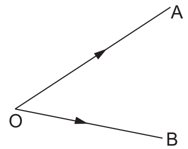
ZERO OR NULL VECTOR *A vector whose initial and terminal points coincide is called a zero vector, denoted by $\vec{0}$.*

Clearly, the magnitude of a zero vector is 0 but it cannot be assigned a definite direction.

Thus, $\overrightarrow{AA} = \vec{0}$.

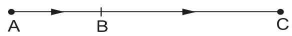
COINITIAL VECTORS *Two or more vectors having the same initial point are called coinitial vectors.*

In the given figure, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the two coinitial vectors having the same initial point O.



COLLINEAR VECTORS *Vectors having the same or parallel supports are known as collinear vectors.*

In the given figure $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ are collinear vectors.



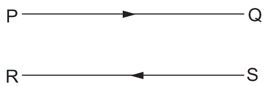
LIKE VECTORS *Collinear vectors having the same direction are called like vectors.*

Thus, $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ shown above are like vectors.

UNLIKE VECTORS *Collinear vectors having opposite directions are known as unlike vectors.*

In the given figure, $PQ \parallel RS$.

$\therefore$ $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ are unlike vectors.



FREE VECTORS *If the initial point of a vector is not specified then it is said to be a free vector.*

LOCALISED VECTORS *A vector drawn parallel to a given vector through a specified point as the initial point is called a localised vector.*

COPLANAR VECTORS *Three or more nonzero vectors lying in the same plane or parallel to the same plane are said to be coplanar, otherwise they are called noncoplanar.*

POSITION VECTOR OF A POINT *Let O be the origin and let A be a point such that $\overrightarrow{OA} = \vec{a}$, then we say that the position vector of A is $\vec{a}$.*

SOLVED EXAMPLES

EXAMPLE 1 *Classify the following measures as scalars and vectors:*

(i) 40 seconds *(ii)* 100 m^2 *(iii)* 30 gm/cm^3

(iv) 60 km/hr *(v)* 56 m/s *towards south*

SOLUTION (i) 40 seconds represents time, which is scalar.

(ii) 100 m^2 represents an area, which is scalar.

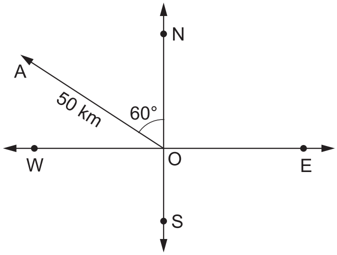
(iii) 30 gm/cm^3 represents density, which is scalar.

(iv) 60 km/hr represents speed, which is scalar.

(v) 56 m/s towards south represents velocity, which is a vector.

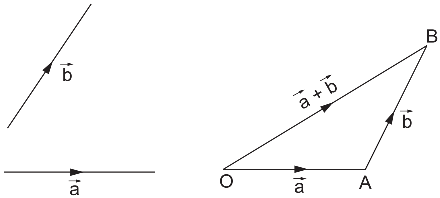
EXAMPLE 2 *Represent graphically a displacement of* 50 km, 60° *west of north.*

SOLUTION The vector $\overrightarrow{OA}$ given below represents a displacement of 50 km, 60° west of north.



VECTOR ADDITION

Let $\vec{a}$ and $\vec{b}$ be any two vectors. Take any point O and draw segments $\overrightarrow{OA}$ and $\overrightarrow{AB}$ such that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$. Join OB. Then, $\overrightarrow{OB}$ is called the ***sum,*** or ***resultant*** of $\vec{a}$ and $\vec{b}$.

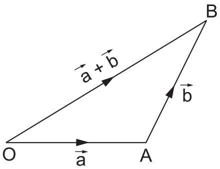


$\therefore\ (\vec{a} + \vec{b}) = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}.$

TRIANGLE LAW OF ADDITION OF VECTORS

In a $\triangle OAB$, if $\overrightarrow{OA}$ and $\overrightarrow{AB}$ represent $\vec{a}$ and $\vec{b}$ respectively, then $\overrightarrow{OB}$ represents $(\vec{a} + \vec{b})$.

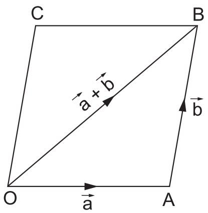
This is known as *Triangle Law of Addition of Vectors.*



PARALLELOGRAM LAW OF ADDITION OF VECTORS

In a ||gm OABC, if $\overrightarrow{OA}$ and $\overrightarrow{AB}$ represent $\vec{a}$ and $\vec{b}$ respectively, then $\overrightarrow{OB}$ represents $(\vec{a} + \vec{b})$.

This is known as *Parallelogram Law of Addition of Vectors.*

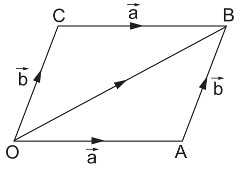


Laws of Addition of Vectors

THEOREM 1 (Commutative Law) *Vector addition is commutative,*

i.e., $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

PROOF Let $\vec{a}$ and $\vec{b}$ be the given vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{AB}$ respectively. Complete the parallelogram $OABC$.



Then, $\overrightarrow{OC}=\overrightarrow{AB}=\vec{b}$

and $\overrightarrow{CB}=\overrightarrow{OA}=\vec{a}$.

$\therefore \quad \vec{a}+\vec{b}=\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$

and $\vec{b}+\vec{a}=\overrightarrow{OC}+\overrightarrow{CB}=\overrightarrow{OB}$.

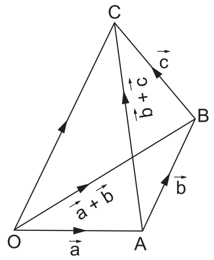
Hence, $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

THEOREM 2 (Associative Law) *Vector addition is associative,*

i.e., $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$.

PROOF Let $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{AB}=\vec{b}$ and $\overrightarrow{BC}=\vec{c}$.

Join OB, OC and AC.



$(\vec{a}+\vec{b})+\vec{c}=(\overrightarrow{OA}+\overrightarrow{AB})+\overrightarrow{BC}$

$=(\overrightarrow{OB}+\overrightarrow{BC})$ $[\because \overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}]$

$=\overrightarrow{OC}$

$\vec{a}+(\vec{b}+\vec{c})=\overrightarrow{OA}+(\overrightarrow{AB}+\overrightarrow{BC})$

$=\overrightarrow{OA}+\overrightarrow{AC}$ $[\because \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}]$

$=\overrightarrow{OC}$

$\therefore \quad (\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$.

THEOREM 3 (Existence of Additive Identity) *For any vector* $\vec{a}$, *prove that*

$$\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}.$$

PROOF Let $\overrightarrow{OA}=\vec{a}$. Then, $\vec{a}+\vec{0}=\overrightarrow{OA}+\overrightarrow{AA}=\overrightarrow{OA}=\vec{a}$

and, $\vec{0}+\vec{a}=\overrightarrow{OO}+\overrightarrow{OA}=\overrightarrow{OA}=\vec{a}$.

$\therefore \quad \vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$.

REMARK The vector $\vec{0}$ is called the *additive identity* for vectors.

THEOREM 4 (Existence of Additive Inverse) *For any vector* $\vec{a}$, *prove that*

$$\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}.$$

PROOF Let $\overrightarrow{OA} = \vec{a}$. Then, $\overrightarrow{AO} = -\vec{a}$.

$\therefore \quad \vec{a} + (-\vec{a}) = \overrightarrow{OA} + \overrightarrow{AO} = \overrightarrow{OO} = \vec{0}$

and, $(-\vec{a}) + \vec{a} = \overrightarrow{AO} + \overrightarrow{OA} = \overrightarrow{AA} = \vec{0}$.

Hence, $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$.

REMARK The vector $-\vec{a}$ is called the *additive inverse* of $\vec{a}$.

DIFFERENCE OF TWO VECTORS *For any two vectors* $\vec{a}$ *and* $\vec{b}$, *we define*

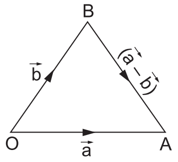
$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}).$$

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Then, $\overrightarrow{BO} = -\vec{b}$.

$\therefore \quad (\vec{a} - \vec{b}) = \vec{a} + (-\vec{b})$

$= \overrightarrow{OA} + \overrightarrow{BO} = \overrightarrow{BO} + \overrightarrow{OA} = \overrightarrow{BA}$

$\therefore \quad (\overrightarrow{OA} - \overrightarrow{OB}) = \overrightarrow{BA}$.



In a similar way, we can prove that

$(\overrightarrow{OB} - \overrightarrow{OA}) = \overrightarrow{AB}$.

AN IMPORTANT REMARK:

(i) $\overrightarrow{AB}$ = (*position vector of B*) – (*position vector of A*)

(ii) $\overrightarrow{BA}$ = (*position* vector of *A*) – (*position vector of B*)

Scalar Multiplication of a Vector

The scalar multiple of $\vec{a}$ *by a scalar k is the vector* $k\vec{a}$ *such that*

(i) $|k\vec{a}| = |k|\,|\vec{a}|$,

(ii) direction of $k\vec{a}$ is the same as that of $\vec{a}$, when $k > 0$ and opposite to that of $\vec{a}$ when $k < 0$.

Examples (i) $5\vec{a}$ is the vector whose magnitude is 5 times the magnitude of $\vec{a}$ and whose direction is the same as that of $\vec{a}$.

(ii) $-2\vec{a}$ is the vector whose magnitude is 2 times the magnitude of $\vec{a}$ and whose direction is opposite to that of $\vec{a}$.

Components of a Vector

Let O be the origin and let $P(x, y, z)$ be any point in space. Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along the x-axis, y-axis and z-axis respectively. Let the position vector of P be $\vec{r}$.

Then, $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$.

This form of a vector is called its ***component form***.

Here x, y, z are called the ***scalar components*** of $\vec{r}$ and $x\hat{i}, y\hat{j}, z\hat{k}$ are called its ***vector components.***

Also, $|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$.

Direction Ratios and Direction Cosines of a Vector

Consider a vector $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$.

Then, the numbers a, b, c are called the ***direction ratios*** of $\vec{r}$.

Direction cosines of $\vec{r}$ are given by

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad \frac{b}{\sqrt{a^2 + b^2 + c^2}} \quad \text{and} \quad \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

NOTE If l, m, n are the direction cosines of a vector then we always have $(l^2 + m^2 + n^2) = 1$.

REMARK If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space then direction ratios of $\overrightarrow{AB}$ are $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$ and direction cosines of $\overrightarrow{AB}$ are

$$\frac{(x_2 - x_1)}{r}, \frac{(y_2 - y_1)}{r}, \frac{(z_2 - z_1)}{r},$$

where $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

SOLVED EXAMPLES

EXAMPLE 1 *Let $\vec{a} = a_1\hat{i} + 3\hat{j} + a_3\hat{k}$ and $\vec{b} = 2\hat{i} + b_2\hat{j} + \hat{k}$. If $\vec{a} = \vec{b}$, find the values of a_1, b_2 and a_3.*

SOLUTION $\vec{a} = \vec{b} \Leftrightarrow a_1\hat{i} + 3\hat{j} + a_3\hat{k} = 2\hat{i} + b_2\hat{j} + \hat{k}$

$\Leftrightarrow a_1 = 2, \; b_2 = 3, \; a_3 = 1.$

EXAMPLE 2 *Let* $\vec{a} = 3\hat{i} + 2\hat{j}$ *and* $\vec{b} = 2\hat{i} + 3\hat{j}$. *Is* $|\vec{a}| = |\vec{b}|$? *Is* $\vec{a} = \vec{b}$?

SOLUTION We have

$$|\vec{a}| = \sqrt{3^2 + 2^2} = \sqrt{13} \text{ and } |\vec{b}| = \sqrt{2^2 + 3^2} = \sqrt{13}.$$

$$\therefore \quad |\vec{a}| = |\vec{b}|.$$

But, $3\hat{i} + 2\hat{j} \neq 2\hat{i} + 3\hat{j}$ and therefore, $\vec{a} \neq \vec{b}$.

EXAMPLE 3 *Find a unit vector in the direction of the vector* $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$.

SOLUTION $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k} \Rightarrow |\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.

Unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{(\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{14}} = \left(\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}\right).$$

EXAMPLE 4 *Write a vector of magnitude* 15 *units in the direction of the vector* $(\hat{i} - 2\hat{j} + 2\hat{k})$. **[CBSE 2010]**

SOLUTION Let $\vec{a} = (\hat{i} - 2\hat{j} + 2\hat{k})$. Then,

a unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{(\hat{i} - 2\hat{j} + 2\hat{k})}{\sqrt{1^2 + (-2)^2 + 2^2}} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}).$$

$\therefore$ a vector of magnitude 15 in the direction of $\vec{a}$

$$= 15 \times \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}) = (5\hat{i} - 10\hat{j} + 10\hat{k}).$$

EXAMPLE 5 *Find a unit vector parallel to the sum of the vectors* $(\hat{i} + \hat{j} + \hat{k})$ *and* $(2\hat{i} - 3\hat{j} + 5\hat{k})$. **[CBSE 2012]**

SOLUTION Let $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$ and $\vec{b} = (2\hat{i} - 3\hat{j} + 5\hat{k})$. Then,

$$(\vec{a} + \vec{b}) = (\hat{i} + \hat{j} + \hat{k}) + (2\hat{i} - 3\hat{j} + 5\hat{k}) = (3\hat{i} - 2\hat{j} + 6\hat{k}).$$

Required unit vectors parallel to $(\vec{a} + \vec{b})$ are

$$\pm \frac{(3\hat{i} - 2\hat{j} + 6\hat{k})}{\sqrt{3^2 + (-2)^2 + 6^2}} = \pm \frac{1}{7}(3\hat{i} - 2\hat{j} + 6\hat{k}).$$

EXAMPLE 6 *If* $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$, $\vec{b} = (4\hat{i} - 2\hat{j} + 3\hat{k})$ *and* $\vec{c} = (\hat{i} - 2\hat{j} + \hat{k})$, *find a vector of magnitude 6 units which is parallel to the vector* $(2\vec{a} - \vec{b} + 3\vec{c})$. **[CBSE 2011C]**

SOLUTION We have

$$(2\vec{a} - \vec{b} + 3\vec{c}) = 2(\hat{i} + \hat{j} + \hat{k}) - (4\hat{i} - 2\hat{j} + 3\hat{k}) + 3(\hat{i} - 2\hat{j} + \hat{k})$$
$$= (2 - 4 + 3)\hat{i} + (2 + 2 - 6)\hat{j} + (2 - 3 + 3)\hat{k}$$
$$= (\hat{i} - 2\hat{j} + 2\hat{k}).$$

Unit vectors parallel to $(2\vec{a} - \vec{b} + 3\vec{c})$ are

$$\pm\frac{(\hat{i} - 2\hat{j} + 2\hat{k})}{\sqrt{1^2 + (-2)^2 + 2^2}} = \pm\frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}).$$

Required vectors of magnitude 6 units are

$$\pm\left\{6 \times \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k})\right\} = \pm 2(\hat{i} - 2\hat{j} + 2\hat{k}).$$

EXAMPLE 7 *Find a unit vector in the direction of* $\overrightarrow{AB}$, *where* $A(1, 2, 3)$ *and* $B(4, 5, 6)$ *are the given points.*

SOLUTION We have

p.v. of $A = (\hat{i} + 2\hat{j} + 3\hat{k})$ and p.v. of $B = (4\hat{i} + 5\hat{j} + 6\hat{k})$.

$\therefore \quad \overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A)$

$= (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (3\hat{i} + 3\hat{j} + 3\hat{k})$, and

$|\overrightarrow{AB}| = \sqrt{3^2 + 3^2 + 3^2} = \sqrt{27}.$

$\therefore$ unit vector in the direction of $\overrightarrow{AB}$

$$= \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{(3\hat{i} + 3\hat{j} + 3\hat{k})}{\sqrt{27}} = \frac{3(\hat{i} + \hat{j} + \hat{k})}{3\sqrt{3}} = \frac{(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3}}$$
$$= \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right).$$

EXAMPLE 8 *For what value of a, the vectors* $(2\hat{i} - 3\hat{j} + 4\hat{k})$ *and* $(a\hat{i} + 6\hat{j} - 8\hat{k})$ *collinear?* **[CBSE 2011]**

SOLUTION The given vectors are collinear only when

$$(a\hat{i} + 6\hat{j} - 8\hat{k}) = \lambda(2\hat{i} - 3\hat{j} + 4\hat{k})$$

for some nonzero scalar λ.

Now, $a\hat{i} + 6\hat{j} - 8\hat{k} = 2\lambda\hat{i} - 3\lambda\hat{j} + 4\lambda\hat{k}$

$\Leftrightarrow \quad 2\lambda = a, -3\lambda = 6$ and $4\lambda = -8$

$\Leftrightarrow \quad \lambda = \frac{a}{2}$ and $\lambda = -2 \Leftrightarrow \frac{a}{2} = -2 \Leftrightarrow a = -4.$

Hence, the given vectors are collinear when $a = -4$.

EXAMPLE 9 *Show that the points* $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$ *and* $C(7\hat{i} - \hat{k})$ *are collinear.* **[CBSE 2009]**

SOLUTION Clearly, we have

$$\overrightarrow{AB} = (\text{position vector of } B) - (\text{position vector of } A)$$
$$= (\hat{i} + 2\hat{j} + 3\hat{k}) - (-2\hat{i} + 3\hat{j} + 5\hat{k}) = (3\hat{i} - \hat{j} - 2\hat{k}).$$

$$\overrightarrow{BC} = (\text{position vector of } C) - (\text{position vector of } B)$$
$$= (7\hat{i} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (6\hat{i} - 2\hat{j} - 4\hat{k}).$$

$\therefore \quad \overrightarrow{AB} = 2\overrightarrow{BC}$, which shows that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel vectors, having a common end point B.

Hence, the points A, B and C are collinear.

EXAMPLE 10 *Write the direction cosines of the vector* $(-2\hat{i} + \hat{j} - 5\hat{k})$. **[CBSE 2011]**

SOLUTION The given vector is $\vec{a} = (-2\hat{i} + \hat{j} - 5\hat{k})$.

Direction ratios of $\vec{a}$ are $-2, 1, -5$.

$$|\vec{a}| = \sqrt{(-2)^2 + 1^2 + (-5)^2} = \sqrt{30}.$$

Hence, the direction cosines of $\vec{a}$ are $\frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}$.

EXAMPLE 11 *What is the cosine of the angle which the vector* $(\sqrt{2}\hat{i} + \hat{j} + \hat{k})$ *makes with the y-axis?* **[CBSE 2010]**

SOLUTION The given vector is $\vec{a} = (\sqrt{2}\hat{i} + \hat{j} + \hat{k})$.

Direction ratios of $\vec{a}$ are $\sqrt{2}, 1, 1$.

$$|\vec{a}| = \sqrt{(\sqrt{2})^2 + 1^2 + 1^2} = \sqrt{4} = 2.$$

$\therefore$ direction cosines of $\vec{a}$ are $\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2}$.

Let $\vec{a}$ make angle β with the y-axis.

Then, clearly $\cos\beta = \frac{1}{2}$.

EXAMPLE 12 *Find the value of p for which* $p(\hat{i} + \hat{j} + \hat{k})$ *is a unit vector.* **[CBSE 2009]**

SOLUTION $p(\hat{i} + \hat{j} + \hat{k})$ is a unit vector

$$\Leftrightarrow \left|p(\hat{i} + \hat{j} + \hat{k})\right|^2 = 1 \Leftrightarrow (p^2 + p^2 + p^2) = 1$$

$$\Leftrightarrow 3p^2 = 1 \Leftrightarrow p^2 = \frac{1}{3} \Leftrightarrow p = \pm\frac{1}{\sqrt{3}}.$$

EXAMPLE 13 *If* $A(1, 2, -3)$ *and* $B(-1, -2, 1)$ *are the two given points in space then find (i) the direction ratios of* $\overrightarrow{AB}$ *and (ii) the direction cosines of* $\overrightarrow{AB}$. *Express* $\overrightarrow{AB}$ *in terms of* $\hat{i}$, $\hat{j}$ *and* $\hat{k}$.

SOLUTION (i) The direction ratios of $\overrightarrow{AB}$ are

$(-1-1), (-2-2), (1+3)$, i.e., $-2, -4, 4$.

(ii) $|\overrightarrow{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{36} = 6.$

$\therefore$ the direction cosines of $\overrightarrow{AB}$ are

$$\frac{-2}{6}, \frac{-4}{6}, \frac{4}{6}, \text{ i.e., } \frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}.$$

Clearly, $\overrightarrow{AB} = -\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}.$

EXAMPLE 14 *Find the scalar and vector components of the vector with initial point* $A(3, -1, 2)$ *and terminal point* $B(-5, 4, 3)$.

SOLUTION We have

$$\overrightarrow{AB} = (\textit{position vector of } B) - (\textit{position vector of } A)$$
$$= (-5\hat{i} + 4\hat{j} + 3\hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k}) = (-8\hat{i} + 5\hat{j} + \hat{k}).$$

The scalar components of $\overrightarrow{AB}$ are $-8, 5, 1$.

The vector components of $\overrightarrow{AB}$ are $-8\hat{i}, 5\hat{j}, \hat{k}$.

EXAMPLE 15 *Write two different vectors having same magnitude.*

SOLUTION Consider the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{b} = 4\hat{i} + 2\hat{j} + 3\hat{k}$.

Clearly, $\vec{a} \neq \vec{b}$.

But, $|\vec{a}| = \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{29}$ and $|\vec{b}| = \sqrt{4^2 + 2^2 + 3^2} = \sqrt{29}$.

Thus, $|\vec{a}| = |\vec{b}|$ and $\vec{a} \neq \vec{b}$.

EXAMPLE 16 *Write two different vectors having same direction.*

SOLUTION Clearly, $3\overrightarrow{AB}$ and $5\overrightarrow{AB}$ are two different vectors having the same direction.

EXAMPLE 17 *Show that the points with position vectors* $\vec{a} = (3\hat{i} - 4\hat{j} - 4\hat{k})$, $\vec{b} = (2\hat{i} - \hat{j} + \hat{k})$ *and* $\vec{c} = (\hat{i} - 3\hat{j} - 5\hat{k})$ *respectively, form the vertices of a right-angled triangle.*

SOLUTION We have

$$\overrightarrow{AB} = (\text{position vector of } B) - (\text{position vector of } A)$$
$$= (\vec{b} - \vec{a}) = (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k})$$
$$= (-\hat{i} + 3\hat{j} + 5\hat{k}).$$

$$\overrightarrow{BC} = (\text{position vector of } C) - (\text{position vector of } B)$$
$$= (\vec{c} - \vec{b}) = (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$$
$$= (-\hat{i} - 2\hat{j} - 6\hat{k}).$$

$$\overrightarrow{CA} = (\text{position vector of } A) - (\text{position vector of } C)$$
$$= (\vec{a} - \vec{c}) = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k})$$
$$= (2\hat{i} - \hat{j} + \hat{k}).$$

$$\therefore \quad |\overrightarrow{AB}|^2 = \{(-1)^2 + 3^2 + 5^2\} = (1 + 9 + 25) = 35,$$
$$|\overrightarrow{BC}|^2 = \{(-1)^2 + (-2)^2 + (-6)^2\} = (1 + 4 + 36) = 41,$$
$$|\overrightarrow{CA}|^2 = \{2^2 + (-1)^2 + 1^2\} = (4 + 1 + 1) = 6.$$
$$\therefore \quad |AB|^2 + |CA|^2 = |BC|^2 \Rightarrow AB^2 + CA^2 = BC^2.$$

Hence, $\triangle ABC$ is right-angled at A.

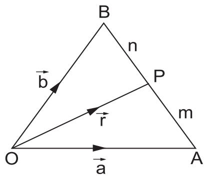
Section Formulae

THEOREM 1 (Section Formula for Internal Division) *Let A and B be two points with position vectors* $\vec{a}$ *and* $\vec{b}$ *respectively and let P be a point dividing AB internally in the ratio m : n. Let* $\overrightarrow{OP} = \vec{r}$. *Then, prove that*

$$\vec{r} = \frac{(m\vec{b} + n\vec{a})}{(m + n)}.$$

PROOF Let O be the origin. Then, $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Let P be a point on AB such that $\frac{AP}{PB} = \frac{m}{n}$ · Then,



$$\frac{AP}{PB} = \frac{m}{n}$$

$$\Rightarrow\ n \cdot AP = m \cdot PB$$

$$\Rightarrow\ n(\overrightarrow{AP}) = m(\overrightarrow{PB})$$

$$\Rightarrow\ n(\overrightarrow{OP} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OP})$$

$$\Rightarrow\ (m+n)\,\overrightarrow{OP} = m\,\overrightarrow{OB} + n\,\overrightarrow{OA} = m\,\vec{b} + n\,\vec{a}$$

$$\Rightarrow\ \overrightarrow{OP} = \frac{m\,\vec{b} + n\,\vec{a}}{(m+n)} \quad \Rightarrow \quad \vec{r} = \frac{(m\,\vec{b} + n\,\vec{a})}{(m+n)}.$$

COROLLARY *The position vector of the midpoint of the join of two points with position vectors* $\vec{a}$ *and* $\vec{b}$ *is* $\frac{1}{2}(\vec{a} + \vec{b})$.

PROOF Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively.
Let P be the midpoint of AB.
Then, P divides AB in the ratio 1 : 1.

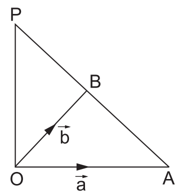
$$\therefore\ \overrightarrow{OP} = \frac{(1 \cdot \vec{b} + 1 \cdot \vec{a})}{(1+1)} = \frac{1}{2}(\vec{a} + \vec{b}).$$

THEOREM 2 (Section Formula for External Division) *Let A and B be two points with position vectors* $\vec{a}$ *and* $\vec{b}$ *respectively and let P be a point dividing AB externally in the ratio m : n. Let* $\overrightarrow{OP} = \vec{r}$. *Then, prove that*

$$\vec{r} = \frac{(m\,\vec{b} - n\,\vec{a})}{(m-n)}.$$

PROOF Let O be the origin. Then, $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.
Let AB be produced to P such that $AP : BP = m : n$.

Now, $\frac{AP}{BP} = \frac{m}{n}$



$$\Rightarrow\ n \cdot AP = m \cdot BP \quad \Rightarrow \quad n \cdot \overrightarrow{AP} = m \cdot \overrightarrow{BP}$$

$$\Rightarrow\ n(\overrightarrow{OP} - \overrightarrow{OA}) = m(\overrightarrow{OP} - \overrightarrow{OB})$$

$$\Rightarrow\ (m-n)\,\overrightarrow{OP} = (m\,\overrightarrow{OB} - n\,\overrightarrow{OA}) = (m\,\vec{b} - n\,\vec{a})$$

$$\Rightarrow\ \overrightarrow{OP} = \frac{(m\,\vec{b} - n\,\vec{a})}{(m-n)} \quad \Rightarrow \quad \vec{r} = \frac{(m\,\vec{b} - n\,\vec{a})}{(m-n)}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the position vector of a point R which divides the line joining the points* $P(\hat{i}+2\hat{j}-\hat{k})$ *and* $Q(-\hat{i}+\hat{j}+\hat{k})$ *in the ratio* $2:1$, *(i) internally and (ii) externally.*

SOLUTION Here $\vec{a}=(\hat{i}+2\hat{j}-\hat{k})$ and $\vec{b}=(-\hat{i}+\hat{j}+\hat{k})$. Also, $m=2, n=1$.

(i) When R divides PQ internally in the ratio 2 : 1; then

$$\text{position vector of } R=\frac{(m\vec{b}+n\vec{a})}{(m+n)}$$

$$=\frac{2(-\hat{i}+\hat{j}+\hat{k})+1\cdot(\hat{i}+2\hat{j}-\hat{k})}{(2+1)}$$

$$=\frac{(-\hat{i}+4\hat{j}+\hat{k})}{3}.$$

(ii) When R divides PQ externally in the ratio 2 : 1; then

$$\text{position vector of } R=\frac{(m\vec{b}-n\vec{a})}{(m-n)}$$

$$=\frac{2(-\hat{i}+\hat{j}+\hat{k})+1\cdot(\hat{i}+2\hat{j}-\hat{k})}{(2-1)}$$

$$=(-3\hat{i}+3\hat{k}).$$

EXAMPLE 2 *P and Q are two points with position vectors* $(3\vec{a}-2\vec{b})$ *and* $(\vec{a}+\vec{b})$ *respectively. Write the position vector of a point R which divides the line segment PQ in the ratio* $2:1$ *externally.* **[CBSE 2013]**

SOLUTION The position vectors of the given points are

$P(3\vec{a}-2\vec{b})$ and $Q(\vec{a}+\vec{b})$.

We have to divide PQ in the ratio 2 : 1 externally at the point R. The position vector of R is

$$\frac{2(\vec{a}+\vec{b})-1\cdot(3\vec{a}-2\vec{b})}{(2-1)}=(-\vec{a}+4\vec{b}).$$

Hence, the position vector of R is $(-\vec{a}+4\vec{b})$.

EXAMPLE 3 *Find the position vector of a point R which divides the line segment joining the points* $A(2,-3,4)$ *and* $B(3,1,-2)$ *externally in the ratio* $3:2$.

SOLUTION The position vector of A is $(2\hat{i} - 3\hat{j} + 4\hat{k})$.

The position vector of B is $(3\hat{i} + \hat{j} - 2\hat{k})$.

Let R divide AB externally in the ratio 3 : 2.

Then, position vector of R

$$= \left(\frac{3\vec{b} - 2\vec{a}}{3-2}\right) = \frac{3(3\hat{i} + \hat{j} - 2\hat{k}) - 2(2\hat{i} - 3\hat{j} + 4\hat{k})}{1}$$

$$= (5\hat{i} + 9\hat{j} - 14\hat{k}).$$

Hence, the position vector of R is $(5\hat{i} + 9\hat{j} - 14\hat{k})$.

EXAMPLE 4 *Find the position vector of the mid-point of the vector joining the points* $P(2, 3, 4)$ *and* $Q(4, 1, -2)$. **[CBSE 2011]**

SOLUTION The position vectors of the given points P and Q are

$\vec{a} = (2\hat{i} + 3\hat{j} + 4\hat{k})$ and $\vec{b} = (4\hat{i} + \hat{j} - 2\hat{k})$ respectively.

EXAMPLE 5 *Show that the three points* $A(1, -2, -8)$, $B(5, 0, -2)$ *and* $C(11, 3, 7)$ *are collinear and find the ratio in which B divides AC.*

SOLUTION The position vectors of A, B and C are $(\hat{i} - 2\hat{j} - 8\hat{k})$, $(5\hat{i} - 2\hat{k})$ and $(11\hat{i} + 3\hat{j} + 7\hat{k})$ respectively.

$\therefore \quad \overrightarrow{AB} = (\textit{position vector of } B) - (\textit{position vector of } A)$

$$= (5\hat{i} - 2\hat{k}) - (\hat{i} - 2\hat{j} + 8\hat{k}) = (4\hat{i} + 2\hat{j} + 6\hat{k}).$$

$\overrightarrow{BC} = (\textit{position vector of } C) - (\textit{position vector of } B)$

$$= (11\hat{i} + 3\hat{j} + 7\hat{k}) - (5\hat{i} - 2\hat{k}) = (6\hat{i} + 3\hat{j} + 9\hat{k}), \text{ and}$$

$\overrightarrow{AC} = (\textit{position vector of } C) - (\textit{position vector of } A)$

$$= (11\hat{i} + 3\hat{j} + 7\hat{k}) - (\hat{i} - 2\hat{j} - 8\hat{k}) = (10\hat{i} + 5\hat{j} + 15\hat{k}).$$

Now, $\overrightarrow{AB} = (4\hat{i} + 2\hat{j} + 6\hat{k}) = 2(2\hat{i} + \hat{j} + 3\hat{k})$

$$= \frac{2}{5}(10\hat{i} + 5\hat{j} + 15\hat{k}) = \frac{2}{5}\overrightarrow{AC}.$$

$\therefore \quad \overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel vectors having same end point A.

Hence, the points A, B and C are collinear.

Also, $\overrightarrow{AB} = (4\hat{i} + 2\hat{j} + 6\hat{k}) = 2(2\hat{i} + \hat{j} + 3\hat{k})$

$$= \frac{2}{3}(6\hat{i} + 3\hat{j} + 9\hat{k}) = \frac{2}{3}\overrightarrow{BC}.$$

$$\therefore \quad \frac{|\overrightarrow{AB}|}{|\overrightarrow{BC}|} = \frac{2}{3}.$$

Hence, B divides AC in the ratio 2 : 3.

EXERCISE 22

1. Write down the magnitude of each of the following vectors:

   (i) $\vec{a} = \hat{i} + 2\hat{j} + 5\hat{k}$ (ii) $\vec{b} = 5\hat{i} - 4\hat{j} - 3\hat{k}$

   (iii) $\vec{c} = \left(\frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$ (iv) $\vec{d} = (\sqrt{2}\hat{i} + \sqrt{3}\hat{j} - \sqrt{5}\hat{k})$

2. Find a unit vector in the direction of the vector:

   (i) $(3\hat{i} + 4\hat{j} - 5\hat{k})$ (ii) $(3\hat{i} - 2\hat{j} + 6\hat{k})$ **[CBSE 2012]**

   (iii) $(\hat{i} + \hat{k})$ (iv) $(2\hat{i} + \hat{j} + 2\hat{k})$ **[CBSE 2009]**

3. If $\vec{a} = (2\hat{i} - 4\hat{j} + 5\hat{k})$ then find the value of λ so that $\lambda\vec{a}$ may be a unit vector.

4. If $\vec{a} = (-\hat{i} + \hat{j} - \hat{k})$ and $\vec{b} = (2\hat{i} - \hat{j} + 2\hat{k})$ then find the unit vector in the direction of $(\vec{a} + \vec{b})$.

5. If $\vec{a} = (3\hat{i} + \hat{j} - 5\hat{k})$ and $\vec{b} = (\hat{i} + 2\hat{j} - \hat{k})$ then find a unit vector in the direction of $(\vec{a} - \vec{b})$.

6. If $\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{b} = (2\hat{i} + 4\hat{j} + 9\hat{k})$ then find a unit vector parallel to $(\vec{a} + \vec{b})$. **[CBSE 2008]**

7. Find a vector of magnitude 9 units in the direction of the vector $(-2\hat{i} + \hat{j} + 2\hat{k})$. **[CBSE 2010]**

8. Find a vector of magnitude 8 units in the direction of the vector $(5\hat{i} - \hat{j} + 2\hat{k})$.

9. Find a vector of magnitude 21 units in the direction of the vector $(2\hat{i} - 3\hat{j} + 6\hat{k})$. **[CBSE 2014]**

10. If $\vec{a} = (\hat{i} - 2\hat{j})$, $\vec{b} = (2\hat{i} - 3\hat{j})$ and $\vec{c} = (2\hat{i} + 3\hat{k})$, find $(\vec{a} + \vec{b} + \vec{c})$. **[CBSE 2012]**

11. If $A(-2, 1, 2)$ and $B(2, -1, 6)$ are two given points, find a unit vector in the direction of $\overrightarrow{AB}$.

12. Find the direction ratios and direction cosines of the vector $\vec{a} = (5\hat{i} - 3\hat{j} + 4\hat{k})$.

13. Find the direction ratios and the direction cosines of the vector joining the points $A(2, 1, -2)$ and $B(3, 5, -4)$.

14. Show that the points A, B and C having position vectors $(\hat{i} + 2\hat{j} + 7\hat{k})$, $(2\hat{i} + 6\hat{j} + 3\hat{k})$ and $(3\hat{i} + 10\hat{j} - 3\hat{k})$ respectively, are collinear.

15. The position vectors of the points A, B and C are $(2\hat{i} + \hat{j} - \hat{k})$, $(3\hat{i} - 2\hat{j} + \hat{k})$ and $(\hat{i} + 4\hat{j} - 3\hat{k})$ respectively. Show that the points A, B and C are collinear.

16. If the position vectors of the vertices A, B and C of a ΔABC be $(\hat{i} + 2\hat{j} + 3\hat{k})$, $(2\hat{i} + 3\hat{j} + \hat{k})$ and $(3\hat{i} + \hat{j} + 2\hat{k})$ respectively, prove that ΔABC is equilateral.

17. Show that the points A, B and C having position vectors $(3\hat{i} - 4\hat{j} - 4\hat{k})$, $(2\hat{i} - \hat{j} + \hat{k})$ and $(\hat{i} - 3\hat{j} - 5\hat{k})$ respectively, form the vertices of a right-angled triangle.

18. Using vector method, show that the points $A(1, -1, 0)$, $B(4, -3, 1)$ and $C(2, -4, 5)$ are the vertices of a right-angled triangle.

19. Find the position vector of the point which divides the join of the points $(2\vec{a} - 3\vec{b})$ and $(3\vec{a} - 2\vec{b})$ (i) internally and (ii) externally in the ratio 2 : 3.

20. The position vectors of two points A and B are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ respectively. Find the position vector of a point C which divides AB externally in the ratio 1 : 2. Also, show that A is the mid-point of the line segment CB. **[CBSE 2010]**

21. Find the position vector of a point R which divides the line joining $A(-2, 1, 3)$ and $B(3, 5, -2)$ in the ratio 2 : 1 (i) internally (ii) externally.

22. Find the position vector of the mid-point of the vector joining the points

$$A(3\hat{i} + 2\hat{j} + 6\hat{k}) \text{ and } B(\hat{i} + 4\hat{j} - 2\hat{k}).$$

23. If $\overrightarrow{AB} = (2\hat{i} + \hat{j} - 3\hat{k})$ and $A(1, 2, -1)$ is the given point, find the coordinates of B.

24. Write a unit vector in the direction of $\overrightarrow{PQ}$, where P and Q are the points $(1, 3, 0)$ and $(4, 5, 6)$ respectively. **[CBSE 2014]**

ANSWERS (EXERCISE 22)

1. (i) $\sqrt{30}$ (ii) $5\sqrt{2}$ (iii) 1 (iv) $\sqrt{10}$

2. (i) $\left(\frac{3}{5\sqrt{2}}\hat{i} + \frac{4}{5\sqrt{2}}\hat{j} - \frac{5}{5\sqrt{2}}\hat{k}\right)$ (ii) $\left(\frac{3}{7}\hat{i} - \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$

(iii) $\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}\right)$ (iv) $\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

3. $\pm\frac{1}{3\sqrt{5}}$

4. $\frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$

5. $\frac{1}{\sqrt{21}}(2\hat{i} - \hat{j} - 4\hat{k})$

6. $\pm\frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

7. $(-6\hat{i} + 3\hat{j} + 6\hat{k})$

8. $\frac{8}{\sqrt{30}}(5\hat{i} - \hat{j} + 2\hat{k})$

9. $(6\hat{i} - 9\hat{j} + 18\hat{k})$

10. $(5\hat{i} - 5\hat{j} + 3\hat{k})$

11. $\left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

12. $(5, -3, 4), \left(\frac{1}{\sqrt{2}}, \frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}\right)$

13. $(1, 4, -2), \left(\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{-2}{\sqrt{21}}\right)$

19. (i) $\frac{12}{5}\vec{a} - \frac{13}{5}\vec{b}$ (ii) $-5\vec{b}$

20. $(3\vec{a} + 5\vec{b})$

21. (i) $\left(\frac{4}{3}\hat{i} + \frac{11}{3}\hat{j} - \frac{1}{3}\hat{k}\right)$ (ii) $(8\hat{i} + 9\hat{j} - 7\hat{k})$

22. $P(2\hat{i} + 3\hat{j} + 2\hat{k})$

23. (3, 3, –4)

24. $\frac{1}{7}(3\hat{i} + 2\hat{j} + 6\hat{k})$

HINTS TO SOME SELECTED QUESTIONS

16. Show that $|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CA}|$.

17. Show that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$ and $|\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2 = |\overrightarrow{BC}|^2$.

$\therefore$ $\triangle ABC$ is right angled at A.

18. The position vectors of A, B, C are $(\hat{i} - \hat{j})$, $(4\hat{i} - 3\hat{j} + \hat{k})$ and $(2\hat{i} - 4\hat{j} + 5\hat{k})$ respectively. Show that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$ and $|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 = |\overrightarrow{CA}|^2$.

$\therefore$ $\triangle ABC$ is right angled at B.

23. Let O be the origin. Then, $\overrightarrow{OA} = (\hat{i} + 2\hat{j} - \hat{k})$.

$\therefore$ $\overrightarrow{AB} = (\overrightarrow{OB} - \overrightarrow{OA}) \Rightarrow \overrightarrow{OB} = (\overrightarrow{AB} + \overrightarrow{OA})$.

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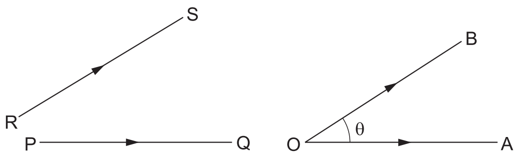
23. SCALAR, OR DOT, PRODUCT OF VECTORS

ANGLE BETWEEN TWO VECTORS Let $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ be two given vectors. Take any point O, and draw $OA \parallel PQ$ and $OB \parallel RS$. Then, $\angle AOB = \theta$ is called the *angle between* $\overrightarrow{PQ}$ *and* $\overrightarrow{RS}$, provided $0 \le \theta \le \pi$.



If $\theta = 0$ or $\theta = \pi$ then $\overrightarrow{PQ} \parallel \overrightarrow{RS}$.

If $\theta = \frac{\pi}{2}$ then $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are called *perpendicular*, or *orthogonal, vectors.*

SCALAR PRODUCT, OR DOT PRODUCT, OF TWO VECTORS Let $\vec{a}$ and $\vec{b}$ be two vectors, and let θ be the angle between them. Then, the *scalar product*, or *dot product*, of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = a\, b \cos\theta$.

Clearly, the scalar product of two vectors is a scalar.

ANGLE BETWEEN TWO VECTORS IN TERMS OF SCALAR PRODUCT Let θ be the angle between two nonzero vectors $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta.$$

$$\therefore \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right).$$

LENGTH OF A VECTOR Let $\vec{a}$ be any vector.

Then, $\vec{a} \cdot \vec{a} = |\vec{a}||\vec{a}| \cos 0° = |\vec{a}|^2$.

$$\therefore \quad |\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}.$$

REMARKS (i) If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, we define $\vec{a} \cdot \vec{b} = 0$.

(ii) If $\vec{a}$ and $\vec{b}$ are like vectors, we have $\theta = 0$.

$\therefore \quad \vec{a} \cdot \vec{b} = a\,b \cos 0° = ab.$

(iii) If $\vec{a}$ and $\vec{b}$ are unlike vectors, we have $\theta = \pi$.

$\therefore \quad \vec{a} \cdot \vec{b} = a\,b \cos \pi = -ab.$

(iv) If $\vec{a}$ and $\vec{b}$ are orthogonal vectors, we have $\theta = \frac{\pi}{2}\cdot$

$\therefore \quad \vec{a} \cdot \vec{b} = a\,b \cos \frac{\pi}{2} = 0.$

SOLVED PROBLEMS

EXAMPLE 1 *Let $\vec{a}$ and $\vec{b}$ be two given vectors such that $|\vec{a}| = 3$, $|\vec{b}| = 4$ and the angle between them is 60°. Find $\vec{a} \cdot \vec{b}$.*

SOLUTION Clearly, we have

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos 60° = \left(3 \times 4 \times \frac{1}{2}\right) = 6.$$

EXAMPLE 2 *Let $\vec{a}$ and $\vec{b}$ be two given vectors such that $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{6}$. Find the angle between $\vec{a}$ and $\vec{b}$.*

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a} \cdot \vec{b} = \sqrt{6} \Rightarrow |\vec{a}|\,|\vec{b}| \cos \theta = \sqrt{6} \Rightarrow (\sqrt{3})(2) \cos \theta = \sqrt{6}$$

$$\Rightarrow \cos \theta = \frac{\sqrt{6}}{2\sqrt{3}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}\cdot$$

Hence, the required angle is $\frac{\pi}{4}\cdot$

EXAMPLE 3 *If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = |\vec{b}| = \sqrt{2}$ and $\vec{a} \cdot \vec{b} = -1$, find the angle between $\vec{a}$ and $\vec{b}$.*

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then, $\vec{a} \cdot \vec{b} = -1$

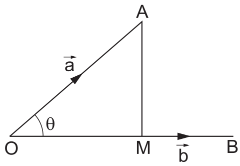
$$\Leftrightarrow |\vec{a}| \cdot |\vec{b}| \cdot \cos \theta = -1 \Leftrightarrow \sqrt{2} \times \sqrt{2} \times \cos \theta = -1$$

$$\Leftrightarrow \cos \theta = \frac{-1}{2} \Leftrightarrow \theta = \frac{2\pi}{3} \qquad [\because \; 0 \le \theta \le 2\pi].$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}\cdot$

PROJECTION

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\angle BOA = \theta$. Draw $AM \perp OB$.



Then, OM is the projection of $\vec{a}$ on $\vec{b}$.

$$OM = (OA)\cos\theta = |\vec{a}|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot$$

$$\therefore \quad \text{projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot$$

PROPERTIES OF SCALAR PRODUCT

THEOREM 1 (Commutative Law) *Prove that* $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

PROOF If $\vec{a}$ or $\vec{b}$ is a zero vector then $\vec{a}\cdot\vec{b} = 0$ and $\vec{b}\cdot\vec{a} = 0$.

So, in this case, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

Now, let $\vec{a}$ and $\vec{b}$ be any two nonzero vectors, and let θ be the angle between them. Then,

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = ab\cos\theta, \quad \text{and}$$

$$\vec{b}\cdot\vec{a} = |\vec{b}||\vec{a}|\cos(-\theta) = ba\cos\theta = ab\cos\theta.$$

So, in this case also, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

Hence, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

THEOREM 2 *Prove that* $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}\perp\vec{b}$.

PROOF Let $\vec{a}\cdot\vec{b} = 0$.

$$\text{Then, } \vec{a}\cdot\vec{b} = 0 \Rightarrow ab\cos\theta = 0 \Rightarrow a = 0 \text{ or } b = 0 \text{ or } \cos\theta = 0$$

$$\Rightarrow a = 0 \text{ or } b = 0 \text{ or } \theta = \frac{\pi}{2}$$

$$\Rightarrow \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a}\perp\vec{b}.$$

Thus, $\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}\perp\vec{b}$.

Conversely, let $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}\perp\vec{b}$.

If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then by definition, $\vec{a}\cdot\vec{b} = 0$.

If $\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = a\, b \cos\frac{\pi}{2} = 0.$

$\therefore \quad \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0.$

Hence, $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$.

THEOREM 3 *For any two vectors $\vec{a}$ and $\vec{b}$, prove that:*

(i) $\vec{a} \cdot (-\vec{b}) = -(\vec{a} \cdot \vec{b}) = (-\vec{a}) \cdot \vec{b}$

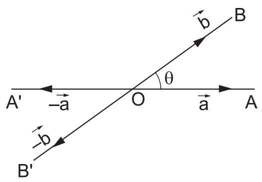
(ii) $(-\vec{a}) \cdot (-\vec{b}) = \vec{a} \cdot \vec{b}$

PROOF Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Produce AO and BO to A' and B' respectively, such that $OA' = OA$ and $OB' = OB$.

Then, $\overrightarrow{OA'} = -\vec{a}$ and $\overrightarrow{OB'} = -\vec{b}$.

Let $\angle AOB = \theta$.

Then, $\angle AOB' = \pi - \theta$.



(i) $\vec{a} \cdot (-\vec{b}) = |\vec{a}|\,|-\vec{b}| \cos(\pi - \theta)$

$= |\vec{a}|\,|\vec{b}| \cdot (-\cos\theta)$

$= -a\, b \cos\theta = -(\vec{a} \cdot \vec{b}).$

$\therefore \quad \vec{a} \cdot (-\vec{b}) = -(\vec{a} \cdot \vec{b}).$

Similarly, $(-\vec{a}) \cdot \vec{b} = -(\vec{a} \cdot \vec{b}).$

Hence, $\vec{a} \cdot (-\vec{b}) = (-\vec{a}) \cdot \vec{b} = -(\vec{a} \cdot \vec{b}).$

(ii) $(-\vec{a}) \cdot (-\vec{b}) = |-\vec{a}|\,|-\vec{b}| \cos \angle A'OB'$

$= |\vec{a}|\,|\vec{b}| \cos\theta \quad [\because \angle A'OB' = \theta]$

$= a\, b \cos\theta = \vec{a} \cdot \vec{b}.$

Hence, $(-\vec{a}) \cdot (-\vec{b}) = (\vec{a} \cdot \vec{b}).$

THEOREM 4 (Distributive Law) *Prove that* $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}.$

PROOF Let O be the origin, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{BC} = \vec{c}$. Then,

$\overrightarrow{OC} = (\overrightarrow{OB} + \overrightarrow{BC}) = (\vec{b} + \vec{c}).$

Draw $BM \perp OA$ and $CN \perp OA$. Then,

$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \overrightarrow{OC} = |\vec{a}|\{(OC)\cos\theta\}$, where $\angle AOC = \theta$

$= a \times ON \quad [\because (OC)\cos\theta = ON]$

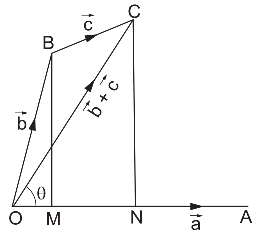
$= a(OM + MN)$

$= a(OM) + a(MN)$

$= a(\text{projection of } \vec{b} \text{ on } \vec{a})$

$\qquad + a(\text{projection of } \vec{c} \text{ on } \vec{a})$

$= (\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}).$



Hence, $\vec{a} \cdot (\vec{b} + \vec{c}) = (\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c})$.

THEOREM 5 (Cauch Schwartz Inequality) *Prove that* $|\vec{a} \cdot \vec{b}| \le |\vec{a}||\vec{b}|$.

PROOF When $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $|\vec{a} \cdot \vec{b}| = 0 = |\vec{a}||\vec{b}|$.

So, let us assume that $\vec{a} \ne \vec{0}$ and $\vec{b} \ne \vec{0}$. Then,

$(\vec{a} \cdot \vec{b}) = |\vec{a}||\vec{b}| \cos\theta \Rightarrow |\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}||\cos\theta|$

$$\Rightarrow \frac{|\vec{a} \cdot \vec{b}|}{|\vec{a}||\vec{b}|} = |\cos\theta| \le 1 \Rightarrow |\vec{a} \cdot \vec{b}| \le |\vec{a}||\vec{b}|.$$

Hence, $|\vec{a} \cdot \vec{b}| \le |\vec{a}||\vec{b}|$.

THEOREM 6 (Triangle Inequality) *Prove that* $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$.

PROOF When $\vec{a} = \vec{0}$, then $|\vec{a}| = 0$.

$\therefore\ |\vec{a} + \vec{b}| = |\vec{0} + \vec{b}| = |\vec{b}|$ and $|\vec{a}| + |\vec{b}| = 0 + |\vec{b}| = |\vec{b}|$.

So, in this case, $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$.

Similarly, when $\vec{b} = \vec{0}$, we have $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$.

Let us consider the case when $\vec{a} \ne \vec{0}$ and $\vec{b} \ne \vec{0}$.

Now, $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})^2$

$= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$

$= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$

$= |\vec{a}|^2 + 2(\vec{a} \cdot \vec{b}) + |\vec{b}|^2$ $[\because\ \vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b}]$

$\le |\vec{a}|^2 + 2|\vec{a} \cdot \vec{b}| + |\vec{b}|^2$ $[\because\ \alpha \le |\alpha|\ \forall\, \alpha \in R]$

$\le |\vec{a}|^2 + 2|\vec{a}| \cdot |\vec{b}| + |\vec{b}|^2 = \{|\vec{a}| + |\vec{b}|\}^2.$

$[\because\ |\vec{a} \cdot \vec{b}| \le |\vec{a}| \cdot |\vec{b}|]$

$\therefore\ |\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$.

ORTHONORMAL VECTOR TRIAD

Let $\hat{i}$, $\hat{j}$, $\hat{k}$ be unit vectors along three mutually perpendicular coordinate axes namely, the x-axis, y-axis and z-axis respectively. Then, these vectors are said to form an orthonormal triad of vectors.

Clearly, we have:

(i) $\hat{i} \cdot \hat{i} = |\hat{i}||\hat{i}| \cos 0° = 1.$

Similarly, $\hat{j} \cdot \hat{j} = 1$ and $\hat{k} \cdot \hat{k} = 1.$

Thus, $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1.$

(ii) $\hat{i} \cdot \hat{j} = |\hat{i}||\hat{j}| \cos \frac{\pi}{2} = 0.$

Similarly, $\hat{i} \cdot \hat{k} = 0$, $\hat{j} \cdot \hat{k} = 0$, $\hat{j} \cdot \hat{i} = 0$, $\hat{k} \cdot \hat{i} = 0$ and $\hat{k} \cdot \hat{j} = 0.$

Hence, $\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{i} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = \hat{k} \cdot \hat{j} = 0.$

SUMMARY

$\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors such that:

(i) $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1.$

(ii) $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = 0$, $\hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = 0$ and $\hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0.$

THEOREM 7 *If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ the prove that $\vec{a} \cdot \vec{b} = (a_1b_1 + a_2b_2 + a_3b_3).$*

PROOF We have

$$\vec{a} \cdot \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1\hat{i} \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{j} \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k} \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= (a_1b_1)(\hat{i} \cdot \hat{i}) + (a_1b_2)(\hat{i} \cdot \hat{j}) + (a_1b_3)(\hat{i} \cdot \hat{k}) + (a_2b_1)(\hat{j} \cdot \hat{i}) + (a_2b_2)(\hat{j} \cdot \hat{j}) + (a_2b_3)(\hat{j} \cdot \hat{k}) + (a_3b_1)(\hat{k} \cdot \hat{i}) + (a_3b_2)(\hat{k} \cdot \hat{j}) + (a_3b_3)(\hat{k} \cdot \hat{k})$$

$$= (a_1b_1 + a_2b_2 + a_3b_3)$$

$[\because \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1, \ \hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{i} = \ldots = 0].$

Hence, $(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = (a_1b_1 + a_2b_2 + a_3b_3).$

CONDITION OF PERPENDICULARITY

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

Then, $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0 \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$.

Thus, $\vec{a} \perp \vec{b} \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$.

ANGLE BETWEEN TWO VECTORS

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, and let θ be the angle between them. Then,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = |\vec{a}||\vec{b}|\cos\theta$$

$$\Leftrightarrow \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\vec{a}||\vec{b}|}$$

$$\Leftrightarrow \theta = \cos^{-1}\left\{\frac{a_1b_1 + a_2b_2 + a_3b_3}{\left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)\left(\sqrt{b_1^2 + b_2^2 + b_3^2}\right)}\right\}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the projection of the vector* $(\hat{i} + 3\hat{j} + 7\hat{k})$ *on the vector* $(2\hat{i} - 3\hat{j} + 6\hat{k})$. **[CBSE 2014]**

SOLUTION Let $\vec{a} = (\hat{i} + 3\hat{j} + 7\hat{k})$ and $\vec{b} = (2\hat{i} - 3\hat{j} + 6\hat{k})$. Then,

$$\text{projection of } \vec{a} \text{ on } \vec{b} = \frac{(\vec{a} \cdot \vec{b})}{|\vec{b}|}$$

$$= \frac{(\hat{i} + 3\hat{j} + 7\hat{k}) \cdot (2\hat{i} - 3\hat{j} + 6\hat{k})}{\sqrt{4 + 9 + 36}}$$

$$= \frac{(2 - 9 + 42)}{\sqrt{49}} = \frac{35}{7} = 5.$$

EXAMPLE 2 *Write the projection of* $(\vec{b} + \vec{c})$ *on* $\vec{a}$, *where* $\vec{a} = (2\hat{i} - 2\hat{j} + \hat{k})$, $\vec{b} = (\hat{i} + 2\hat{j} - 2\hat{k})$ *and* $\vec{c} = (2\hat{i} - \hat{j} + 4\hat{k})$. **[CBSE 2013C]**

SOLUTION We have

$$(\vec{b} + \vec{c}) = (\hat{i} + 2\hat{j} - 2\hat{k}) + (2\hat{i} - \hat{j} + 4\hat{k})$$

$$= (1 + 2)\hat{i} + (2 - 1)\hat{j} + (-2 + 4)\hat{k} = (3\hat{i} + \hat{j} + 2\hat{k}).$$

$\therefore$ projection of $(\vec{b} + \vec{c})$ on $\vec{a} = \dfrac{(\vec{b} + \vec{c}) \cdot \vec{a}}{|\vec{a}|}$

$$= \frac{(3\hat{i} + \hat{j} + 2\hat{k}) \cdot (2\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{2^2 + (-2)^2 + 1^2}}$$

$$= \frac{(6 - 2 + 2)}{\sqrt{9}} = \frac{6}{3} = 2.$$

EXAMPLE 3 *Find λ when the projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = (2\hat{i} + 6\hat{j} + 3\hat{k})$ is 4 units.* **[CBSE 2012]**

SOLUTION Projection of $\vec{a}$ on $\vec{b} = \dfrac{(\vec{a} \cdot \vec{b})}{|\vec{b}|}$

$$= \frac{(\lambda\hat{i} + \hat{j} + 4\hat{k}) \cdot (2\hat{i} + 6\hat{j} + 3\hat{k})}{\sqrt{2^2 + 6^2 + 3^2}}$$

$$= \frac{(2\lambda + 6 + 12)}{\sqrt{4 + 36 + 9}} = \frac{(2\lambda + 18)}{\sqrt{49}} = \frac{2(\lambda + 9)}{7} \cdot$$

$$\therefore \quad \frac{2(\lambda + 9)}{7} = 4 \Rightarrow 2(\lambda + 9) = 28 \Rightarrow \lambda + 9 = 14 \Rightarrow \lambda = 5.$$

Hence, $\lambda = 5$.

EXAMPLE 4 *Write the value of λ so that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ are perpendicular to each other.* **[CBSE 2013C]**

SOLUTION $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$

$$\Leftrightarrow (2\hat{i} + \lambda\hat{j} + \hat{k}) \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 0$$

$$\Leftrightarrow (2 - 2\lambda + 3) = 0 \Leftrightarrow 2\lambda = 5 \Leftrightarrow \lambda = \frac{5}{2} \cdot$$

Hence, $\lambda = \dfrac{5}{2} \cdot$

EXAMPLE 5 *The scalar product of the vector $(\hat{i} + \hat{j} + \hat{k})$ with the unit vector along the sum of the vectors $(2\hat{i} + 4\hat{j} - 5\hat{k})$ and $(\lambda\hat{i} + 2\hat{j} + 3\hat{k})$ is equal to 1. Find the value of λ.* **[CBSE 2009, '14]**

SOLUTION Let $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$, $\vec{b} = (2\hat{i} + 4\hat{j} - 5\hat{k})$ and $\vec{c} = (\lambda\hat{i} + 2\hat{j} + 3\hat{k})$. Then,

$$(\vec{b} + \vec{c}) = (2\hat{i} + 4\hat{j} - 5\hat{k}) + (\lambda\hat{i} + 2\hat{j} + 3\hat{k}) = (2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}.$$

Unit vector along $(\vec{b}+\vec{c}) = \dfrac{(\vec{b}+\vec{c})}{|\vec{b}+\vec{c}|} = \dfrac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}}$

$$= \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}.$$

But, $\dfrac{(\vec{b}+\vec{c})}{|\vec{b}+\vec{c}|}\cdot\vec{a} = 1$ (given).

$\therefore \quad \dfrac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}\cdot(\hat{i}+\hat{j}+\hat{k}) = 1$

$\Rightarrow \quad (2+\lambda)\hat{i}+6\hat{j}-2\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k}) = \sqrt{\lambda^2+4\lambda+44}$

$\Rightarrow \quad (2+\lambda)+6-2 = \sqrt{\lambda^2+4\lambda+44} \Rightarrow (6+\lambda) = \sqrt{\lambda^2+4\lambda+44}$

$\Rightarrow \quad \lambda^2+4\lambda+44 = (6+\lambda)^2 \Rightarrow \lambda^2+4\lambda+44 = 36+\lambda^2+12\lambda$

$\Rightarrow \quad 8\lambda = 8 \Rightarrow \lambda = 1.$

Hence, the required value of λ is 1.

EXAMPLE 6 *Dot products of a vector with the vectors* $(\hat{i}-\hat{j}+\hat{k}), (2\hat{i}+\hat{j}-3\hat{k})$ *and* $(\hat{i}+\hat{j}+\hat{k})$ *are respectively* 4, 0 *and* 2. *Find the vector.* **[CBSE 2013C]**

SOLUTION Let the required vector be $(x\hat{i}+y\hat{j}+z\hat{k})$. Then,

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}-\hat{j}+\hat{k}) = 4 \Rightarrow x-y+z=4$ ... (i)

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}+\hat{j}-3\hat{k}) = 0 \Rightarrow 2x+y-3z=0.$ ... (ii)

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k}) = 2 \Rightarrow x+y+z=2.$ ... (iii)

On subtracting (i) from (iii) we get $2y=-2 \Rightarrow y=-1$.

On adding (i) and (ii), we get $3x-2z=4$ ... (iv)

On adding (i) and (iii), we get $2x+2z=6$ ... (v)

On solving (iv) and (v), we get $x=2$ and $z=1$.

$\therefore \quad x=2, y=-1$ and $z=1$.

Hence, the required vector is $(2\hat{i}-\hat{j}+\hat{k})$.

EXAMPLE 7 *Let* $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}$, $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ *and* $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. *Find a vector* $\vec{p}$ *which is perpendicualr to both* $\vec{a}$ *and* $\vec{b}$ *and* $\vec{p}\cdot\vec{c}=18$.

[CBSE 2012]

SOLUTION Let $\vec{p}=(x\hat{i}+y\hat{j}+z\hat{k})$. Then,

$\vec{p}\perp\vec{a}$, $\vec{p}\perp\vec{b}$ and $\vec{p}\cdot\vec{c}=18$

$\Rightarrow \quad \vec{p}\cdot\vec{a}=0, \vec{p}\cdot\vec{b}=0$ and $\vec{p}\cdot\vec{c}=18$

$$\Rightarrow \begin{cases} (x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+4\hat{j}+2\hat{k})=0 \Rightarrow x+4y+2z=0 & \text{... (i)} \\ (x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-2\hat{j}+7\hat{k})=0 \Rightarrow 3x-2y+7z=0 & \text{... (ii)} \\ (x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}-\hat{j}+4\hat{k})=18 \Rightarrow 2x-y+4z=18 & \text{... (iii)} \end{cases}$$

On solving (i) and (ii) by cross multiplication, we get

$$\frac{x}{(28+4)}=\frac{y}{(6-7)}=\frac{z}{(-2-12)}=k \text{ (say)}$$

$\Rightarrow$ $x=32k$, $y=-k$ and $z=-14k$.

Substituting these values in (iii), we get:

$$64k+k-56k=18 \Rightarrow 9k=18 \Rightarrow k=2.$$

$\therefore$ $x=(32\times 2)=64$, $y=-2$ and $z=(-14)\times 2=-28$.

Hence, the required vector is $(64\hat{i}-2\hat{j}-28\hat{k})$.

EXAMPLE 8 *Find a vector whose magnitude is 3 units and which is perpendicular to each of the vectors* $\vec{a}=3\hat{i}+\hat{j}-4\hat{k}$ *and* $\vec{b}=6\hat{i}+5\hat{j}-2\hat{k}$. **[CBSE 2000C]**

SOLUTION Let the required vector be $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$.

Then, $|\vec{c}|=3 \Leftrightarrow \sqrt{c_1^2+c_2^2+c_3^2}=3 \Leftrightarrow c_1^2+c_2^2+c_3^2=9.$ ... (i)

Also, $\vec{c}\perp\vec{a} \Rightarrow \vec{c}\cdot\vec{a}=0$

$$\Rightarrow (c_1\hat{i}+c_2\hat{j}+c_3\hat{k})\cdot(3\hat{i}+\hat{j}-4\hat{k})=0$$

$$\Rightarrow 3c_1+c_2-4c_3=0. \quad \text{... (ii)}$$

And, $\vec{c}\perp\vec{b} \Rightarrow \vec{c}\cdot\vec{b}=0$

$$\Rightarrow (c_1\hat{i}+c_2\hat{j}+c_3\hat{k})\cdot(6\hat{i}+5\hat{j}-2\hat{k})=0$$

$$\Rightarrow 6c_1+5c_2-2c_3=0. \quad \text{... (iii)}$$

From (ii) and (iii), by cross multiplication, we get

$$\frac{c_1}{(-2+20)}=\frac{c_2}{(-24+6)}=\frac{c_3}{(15-6)}=k \text{ (say)}$$

$$\Rightarrow \frac{c_1}{18}=\frac{c_2}{-18}=\frac{c_3}{9}=k \Rightarrow \frac{c_1}{2}=\frac{c_2}{-2}=\frac{c_3}{1}=k$$

$\Rightarrow$ $c_1=2k$, $c_2=-2k$ and $c_3=k$.

Substituting these values in (i), we get

$$4k^2+4k^2+k^2=9 \Rightarrow k^2=1 \Rightarrow k=1.$$

$\therefore$ $c_1=2$, $c_2=-2$ and $c_3=1$.

Hence, $\vec{c}=(2\hat{i}-2\hat{j}+\hat{k})$ is the required vector.

EXAMPLE 9 *If* $\vec{a}$ *and* $\vec{b}$ *are vectors such that* $|\vec{a}| = 2$, $|\vec{b}| = 3$ *and* $\vec{a} \cdot \vec{b} = 4$, *find* $|\vec{a} - \vec{b}|$.

SOLUTION We have

$$\begin{aligned} |\vec{a} - \vec{b}|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\ &= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} \quad \text{(by distributive law)} \\ &= |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 \quad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}] \\ &= (2^2 - 2 \times 4 + 3^2) = 5. \end{aligned}$$

Hence, $|\vec{a} - \vec{b}| = \sqrt{5}$.

EXAMPLE 10 *If* $\vec{a}$ *makes equal angles with the coordinate axes and has magnitude* 3, *find the angle between* $\vec{a}$ *and each of the three coordinate axes.*

SOLUTION Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and let α be the angle between $\vec{a}$ and each of the coordinate axes.

Then, α is the angle between $\vec{a}$ and each one of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

$$\therefore \quad \cos\alpha = \frac{\vec{a} \cdot \hat{i}}{|\vec{a}||\hat{i}|} = \frac{a_1}{3} \Rightarrow a_1 = 3\cos\alpha \; [\because \vec{a} \cdot \hat{i} = a_1, |\vec{a}| = 3, |\hat{i}| = 1].$$

Similarly, $a_2 = 3\cos\alpha$ and $a_3 = 3\cos\alpha$.

Now, $|\vec{a}| = 3 \Rightarrow |\vec{a}|^2 = 9$

$$\begin{aligned} &\Rightarrow a_1^2 + a_2^2 + a_3^2 = 9 \\ &\Rightarrow 9\cos^2\alpha + 9\cos^2\alpha + 9\cos^2\alpha = 9 \\ &\Rightarrow 27\cos^2\alpha = 9 \Rightarrow \cos^2\alpha = \frac{1}{3} \\ &\Rightarrow \cos\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \end{aligned}$$

Hence, the required angle is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

EXAMPLE 11 *If a unit vector* $\vec{a}$ *makes angles* $\pi/4$ *with* $\hat{i}$, $\pi/3$ *with* $\hat{j}$ *and an acute angle* θ *with* $\hat{k}$ *then find the value of* θ. *Also, find the scalar and vector components of* $\vec{a}$ *along the axes.* **[CBSE 2013]**

SOLUTION Let $\vec{a} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$. Then, $\vec{a}$ being a unit vector, we have $(a_1^2 + a_2^2 + a_3^2) = 1$.

Now, $\vec{a}\cdot\hat{i}=a_1 \Rightarrow |\vec{a}||\hat{i}|\cos\frac{\pi}{4}=a_1 \Rightarrow a_1=\frac{1}{\sqrt{2}}$ $[\because |\vec{a}|=1, |\hat{i}|=1]$

$\vec{a}\cdot\hat{j}=a_2 \Rightarrow |\vec{a}||\hat{j}|\cos\frac{\pi}{3}=a_2 \Rightarrow a_2=\frac{1}{2}$ $[\because |\vec{a}|=1, |\hat{j}|=1]$

$\vec{a}\cdot\hat{k}=a_3 \Rightarrow |\vec{a}||\hat{k}|\cos\theta=a_3 \Rightarrow a_3=\cos\theta$ $[\because |\vec{a}|=1, |\hat{k}|=1]$

Now, $|\vec{a}|=1 \Rightarrow |\vec{a}|^2=1$

$$\Rightarrow a_1^2+a_2^2+a_3^2=1$$

$$\Rightarrow \frac{1}{2}+\frac{1}{4}+\cos^2\theta=1 \quad \left[\because a_1=\frac{1}{\sqrt{2}}, a_2=\frac{1}{2}, a_3=\cos\theta\right]$$

$$\Rightarrow \cos^2\theta=\frac{1}{4} \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}.$$

Hence, the scalar componets of $\vec{a}$ are $\frac{1}{\sqrt{2}}, \frac{1}{2}$ and $\frac{1}{2}$.

And, the vector components of $\vec{a}$ are $\frac{1}{\sqrt{2}}\hat{i}, \frac{1}{2}\hat{j}$ and $\frac{1}{2}\hat{k}$.

EXAMPLE 12 *If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.* **[CBSE 2014]**

SOLUTION Let θ be the angle between the unit vectors $\vec{a}$ and $\vec{b}$.

Since $\vec{a}$ and $\vec{b}$ are unit vectors, we have $|\vec{a}|=1$ and $|\vec{b}|=1$.

Again, since $(\vec{a}+\vec{b})$ is a unit vector, we have

$$|\vec{a}+\vec{b}|^2=1$$

$$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=1 \Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=1$$

$$\Rightarrow |\vec{a}|^2+2(\vec{a}\cdot\vec{b})+|\vec{b}|^2=1 \quad [\because \vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}]$$

$$\Rightarrow 2(\vec{a}\cdot\vec{b})=-1 \Rightarrow \vec{a}\cdot\vec{b}=\frac{-1}{2} \quad [\because |\vec{a}|^2=1, |\vec{b}|^2=1]$$

$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta=\frac{-1}{2} \Rightarrow \cos\theta=\frac{-1}{2} \Rightarrow \theta=\frac{2\pi}{3} \quad [\because |\vec{a}|=1, |\vec{b}|=1].$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}$.

EXAMPLE 13 *If the sum of two unit vectors $\hat{a}$ and $\hat{b}$ is a unit vector, show that the magnitude of their difference is $\sqrt{3}$.* **[CBSE 2012C]**

SOLUTION Let $\hat{a}+\hat{b}=\hat{c}$, where $\hat{c}$ is a unit vector. Then,

$$\hat{a} + \hat{b} = \hat{c} \Rightarrow (\hat{a} + \hat{b}) \cdot (\hat{a} + \hat{b}) = \hat{c} \cdot \hat{c}$$

$$\Rightarrow \hat{a} \cdot \hat{a} + \hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{a} + \hat{b} \cdot \hat{b} = \hat{c} \cdot \hat{c}$$

$$\Rightarrow |\hat{a}|^2 + 2(\hat{a} \cdot \hat{b}) + |\hat{b}|^2 = |\hat{c}|^2 \qquad [\because \hat{b} \cdot \hat{a} = \hat{a} \cdot \hat{b}]$$

$$\Rightarrow 1 + 2(\hat{a} \cdot \hat{b}) + 1 = 1 \qquad [\because |\hat{a}| = |\hat{b}| = |\hat{c}| = 1]$$

$$\Rightarrow 2(\hat{a} \cdot \hat{b}) = -1 \qquad \text{... (i)}$$

Now, $|\hat{a} - \hat{b}|^2 = (\hat{a} - \hat{b}) \cdot (\hat{a} - \hat{b})$

$$= \hat{a} \cdot \hat{a} - \hat{a} \cdot \hat{b} - \hat{b} \cdot \hat{a} + \hat{b} \cdot \hat{b}$$

$$= |\hat{a}|^2 - 2(\hat{a} \cdot \hat{b}) + |\hat{b}|^2 \qquad [\because \hat{b} \cdot \hat{a} = \hat{a} \cdot \hat{b}]$$

$$= 1 - 2(\hat{a} \cdot \hat{b}) + 1 = 1 + 1 + 1 = 3 \qquad \text{[using (i)].}$$

Hence, $|\hat{a} - \hat{b}| = \sqrt{3}$.

EXAMPLE 14 *If* $\vec{a}$, $\vec{b}$ *and* $\vec{c}$ *are three vectors such that* $|\vec{a}| = 5$, $|\vec{b}| = 12$, $|\vec{c}| = 13$ *and* $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, *find the value of* $(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$. **[CBSE 2012]**

SOLUTION $\vec{a} + \vec{b} + \vec{c} = \vec{0} \Rightarrow \vec{a} + \vec{b} = -\vec{c}$

$$\Rightarrow (\vec{a} + \vec{b}) \cdot \vec{c} = (-\vec{c}) \cdot \vec{c} \Rightarrow \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} = -|\vec{c}|^2$$

$$\Rightarrow \vec{c} \cdot \vec{a} + \vec{b} \cdot \vec{c} = -169 \qquad \text{... (i)}$$

$$[\because \vec{a} \cdot \vec{c} = \vec{c} \cdot \vec{a} \text{ and } |\vec{c}| = 13]$$

Again, $\vec{a} + \vec{b} + \vec{c} = \vec{0} \Rightarrow \vec{b} + \vec{c} = -\vec{a}$

$$\Rightarrow (\vec{b} + \vec{c}) \cdot \vec{a} = (-\vec{a}) \cdot \vec{a}$$

$$\Rightarrow \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a} = -|\vec{a}|^2$$

$$\Rightarrow \vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{a} = -25 \qquad \text{... (ii)}$$

$$[\because \vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b} \text{ and } |\vec{a}| = 5]$$

Also, $\vec{a} + \vec{b} + \vec{c} = \vec{0} \Rightarrow \vec{a} + \vec{c} = -\vec{b}$

$$\Rightarrow (\vec{a} + \vec{c}) \cdot \vec{b} = (-\vec{b}) \cdot \vec{b}$$

$$\Rightarrow \vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{b} = -|\vec{b}|^2$$

$$\Rightarrow \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} = -144 \qquad \text{... (iii)}$$

$$[\because \vec{c} \cdot \vec{b} = \vec{b} \cdot \vec{c} \text{ and } |\vec{b}| = 12]$$

Adding the corresponding sides of (i), (ii) and (iii), we get

$$2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=-(169+25+144)$$

$$\Rightarrow\quad (\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=\frac{-338}{2}=-169.$$

Hence, $(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=-169.$

EXAMPLE 15 *If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors such that $|\vec{a}|=3$, $|\vec{b}|=4$ and $|\vec{c}|=5$ and each one of them is perpendicular to the sum of the two then find $|\vec{a}+\vec{b}+\vec{c}|$.* **[CBSE 2011C]**

SOLUTION Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be the given vectors such that

$$\{|\vec{a}|=3,\ |\vec{b}|=4,\ |\vec{c}|=5\}, \quad \text{... (i)}$$

$$\left.\begin{array}{l}\vec{a}\cdot(\vec{b}+\vec{c})=0\\ \vec{b}\cdot(\vec{c}+\vec{a})=0\\ \vec{c}\cdot(\vec{a}+\vec{b})=0\end{array}\right\}. \quad \text{... (ii)}$$

$$\therefore\quad |\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$=\vec{a}\cdot\vec{a}+\vec{a}\cdot(\vec{b}+\vec{c})+\vec{b}\cdot(\vec{c}+\vec{a})+\vec{b}\cdot\vec{b}+\vec{c}\cdot(\vec{a}+\vec{b})+\vec{c}\cdot\vec{c}$$

$$=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2 \quad \text{[using (ii)]}$$

$$=(3^2+4^2+5^2)=(9+16+25)=50$$

Hence, $|\vec{a}+\vec{b}+\vec{c}|=\sqrt{50}=5\sqrt{2}.$

EXAMPLE 16 *If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of the same magnitude, prove that $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined to the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Also find this angle.* **[CBSE 2006C, '11C, '13C]**

SOLUTION It is given that

$$|\vec{a}|=|\vec{b}|=|\vec{c}|=a \text{ (say)} \quad \text{... (i)}$$

Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors, we have

$$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0. \quad \text{... (ii)}$$

Now, $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{c} + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 \text{ [using (ii)]}$$

$$= 3a^2 \text{ [using (i)].}$$

$$\therefore \quad |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}a. \qquad \text{... (iii)}$$

Let $(\vec{a} + \vec{b} + \vec{c})$ make angles α, β and γ with $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

Then $(\vec{a} + \vec{b} + \vec{c})\cdot\vec{a} = |\vec{a} + \vec{b} + \vec{c}||\vec{a}|\cos\alpha$

$$= (\sqrt{3}a \times a \times \cos\alpha) = \sqrt{3}a^2\cos\alpha$$

$$\Rightarrow \quad (\vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} + \vec{c}\cdot\vec{a}) = \sqrt{3}a^2\cos\alpha$$

$$\Rightarrow \quad |\vec{a}|^2 = \sqrt{3}a^2\cos\alpha \Rightarrow a^2 = \sqrt{3}a^2\cos\alpha$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

Similarly, $\beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

$$\therefore \quad \alpha = \beta = \gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

Hence, $(\vec{a} + \vec{b} + \vec{c})$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$ and the required angle is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

EXAMPLE 17 *If* $\vec{a}, \vec{b}, \vec{c}$ *are unit vectors such that* $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ *then find the value of* $(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$.

SOLUTION Since $\vec{a}, \vec{b}, \vec{c}$ are unit vectors, we have

$$|\vec{a}| = 1, |\vec{b}| = 1 \text{ and } |\vec{c}| = 1.$$

Now, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\Rightarrow \quad (\vec{a} + \vec{b} + \vec{c})\cdot(\vec{a} + \vec{b} + \vec{c}) = 0 \qquad [\because \vec{0}\cdot\vec{0} = 0]$$

$$\Rightarrow \quad \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{c} + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$$

$$\Rightarrow \quad (\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = -\frac{3}{2} \quad [\because |\vec{a}|^2 = 1, |\vec{b}|^2 = 1, |\vec{c}|^2 = 1].$$

Hence, $(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = -\frac{3}{2}$.

EXAMPLE 18 *If* $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$, *show that* $\vec{a}=\vec{0}$ *or* $\vec{b}=\vec{c}$ *or* $\vec{a}\perp(\vec{b}-\vec{c})$.

SOLUTION $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c} \Rightarrow \vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}=0 \Rightarrow \vec{a}\cdot(\vec{b}-\vec{c})=0$

$$\Rightarrow \vec{a}=\vec{0} \text{ or } (\vec{b}-\vec{c})=\vec{0} \text{ or } \vec{a}\perp(\vec{b}-\vec{c})$$

$$\Rightarrow \vec{a}=\vec{0} \text{ or } \vec{b}=\vec{c} \text{ or } \vec{a}\perp(\vec{b}-\vec{c}).$$

Hence, $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c} \Rightarrow \vec{a}=\vec{0}$ or $\vec{b}=\vec{c}$ or $\vec{a}\perp(\vec{b}-\vec{c})$.

EXAMPLE 19 *Let* $\vec{a}$ *and* $\vec{b}$ *be two nonzero vectors. Prove that*

$$\vec{a}\perp\vec{b} \Leftrightarrow |\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|.$$

SOLUTION Let $\vec{a}\perp\vec{b}$. Then, $(\vec{a}\cdot\vec{b})=0$. ... (i)

Now, $|\vec{a}+\vec{b}|^2=(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$$=\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$=|\vec{a}|^2+|\vec{b}|^2 \quad \{\because \vec{a}\cdot\vec{b}=0 \text{ and } \vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}=0\}.$$

Also, $|\vec{a}-\vec{b}|^2=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$$=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$=|\vec{a}|^2+|\vec{b}|^2 \quad \{\because \vec{a}\cdot\vec{b}=0 \text{ and } \vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}=0\}.$$

Thus, $|\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2$, and therefore, $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$.

$\therefore \quad \vec{a}\perp\vec{b} \Rightarrow |\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$.

Conversely, suppose that $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$. Then,

$$|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}| \Rightarrow |\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2$$

$$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$$

$$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$
$$=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$\Rightarrow 2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a})=0$$

$$\Rightarrow 4(\vec{a}\cdot\vec{b})=0 \quad [\because \vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}]$$

$$\Rightarrow \vec{a}\cdot\vec{b}=0 \Rightarrow \vec{a}\perp\vec{b}.$$

Thus, $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}| \Rightarrow \vec{a}\perp\vec{b}$.

Hence, $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}| \Leftrightarrow \vec{a}\perp\vec{b}$.

EXAMPLE 20 *Express the vector* $\vec{a} = (5\hat{i} - 2\hat{j} + 5\hat{k})$ *as sum of two vectors such that one is parallel to the vector* $\vec{b} = (3\hat{i} + \hat{k})$ *and the other is perpendicular to* $\vec{b}$. **[CBSE 2005]**

SOLUTION Any vector parallel to $\vec{b}$ is of the form $\lambda\vec{b}$ for some scalar λ.

Let $\vec{a} = \lambda\vec{b} + \vec{c}$, where $\vec{c} \perp \vec{b}$. Then, $\vec{c} = (\vec{a} - \lambda\vec{b}) \perp \vec{b}$

$\Leftrightarrow (\vec{a} - \lambda\vec{b}) \cdot \vec{b} = 0 \Leftrightarrow (\vec{a} \cdot \vec{b}) - \lambda(\vec{b} \cdot \vec{b}) = 0$

$\Leftrightarrow (5\hat{i} - 2\hat{j} + 5\hat{k}) \cdot (3\hat{i} + \hat{k}) - \lambda(3\hat{i} + \hat{k}) \cdot (3\hat{i} + \hat{k}) = 0$

$\Leftrightarrow (15 - 0 + 5) - \lambda(9 + 1) = 0 \Leftrightarrow 10\lambda = 20 \Leftrightarrow \lambda = 2.$

$\therefore \quad \lambda\vec{b} = 2\vec{b} = (6\hat{i} + 2\hat{k}).$

And, $\vec{c} = (\vec{a} - 2\vec{b}) = (5\hat{i} - 2\hat{j} + 5\hat{k}) - 2(3\hat{i} + \hat{k}) = (-\hat{i} - 2\hat{j} + 3\hat{k}).$

Hence, the required vectors are $(6\hat{i} + 2\hat{k})$ and $(-\hat{i} - 2\hat{j} + 3\hat{k})$.

EXAMPLE 21 *Find the values of* λ *for which the angle between the vectors* $\vec{a} = 2\lambda^2\hat{i} + 4\lambda\hat{j} + \hat{k}$ *and* $\vec{b} = 7\hat{i} - 2\hat{j} + \lambda\hat{k}$ *is obtuse.* **[CBSE 2013C]**

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}. \qquad \text{... (i)}$$

Clearly, θ is obtuse $\Leftrightarrow \cos\theta < 0 \Leftrightarrow \vec{a} \cdot \vec{b} < 0$ [from (i)]

$\Leftrightarrow 14\lambda^2 - 8\lambda + \lambda < 0 \Leftrightarrow 14\lambda^2 - 7\lambda < 0$

$\Leftrightarrow 2\lambda^2 - \lambda < 0 \Leftrightarrow \lambda(2\lambda - 1) < 0.$

Now, $\lambda(2\lambda - 1) < 0 \Rightarrow$ either $\{\lambda < 0 \text{ and } (2\lambda - 1) > 0\}$ or $\{\lambda > 0 \text{ and } (2\lambda - 1) < 0\}$

$\Rightarrow \left\{\lambda < 0 \text{ and } \lambda > \frac{1}{2}\right\}$ or $\left\{\lambda > 0 \text{ and } \lambda < \frac{1}{2}\right\}$

$\Rightarrow \left\{\frac{1}{2} < \lambda < 0\right\}$ or $\left\{0 < \lambda < \frac{1}{2}\right\}$

$\Rightarrow 0 < \lambda < \frac{1}{2} \left[\because \frac{1}{2} < \lambda < 0 \text{ is not possible}\right]$

$\Rightarrow \lambda \in \left]0, \frac{1}{2}\right[.$

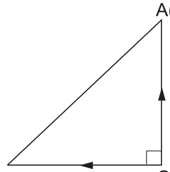
$\therefore$ the required values of λ are all real values in $\left]0, \frac{1}{2}\right[$.

EXAMPLE 22 *Let $A(0, 1, 1)$, $B(3, 1, 5)$ and $C(0, 3, 3)$ be the vertices of a $\triangle ABC$. Using vectors, show that $\triangle ABC$ is right angled at C.*

SOLUTION In order to show that $\overrightarrow{CA} \perp \overrightarrow{CB}$, we must show that $\overrightarrow{CA} \cdot \overrightarrow{CB} = 0$.

Let O be the origin. Then,

$$\overrightarrow{OA} = 0\hat{i} + \hat{j} + \hat{k},\ \overrightarrow{OB} = 3\hat{i} + \hat{j} + 5\hat{k} \text{ and } \overrightarrow{OC} = 0\hat{i} + 3\hat{j} + 3\hat{k}.$$



$\therefore \quad \overrightarrow{CA} = (\overrightarrow{OA} - \overrightarrow{OC})$

$= (0\hat{i} + \hat{j} + \hat{k}) - (0\hat{i} + 3\hat{j} + 3\hat{k})$

$= (0 - 0)\hat{i} + (1 - 3)\hat{j} + (1 - 3)\hat{k}$

$= (0\hat{i} - 2\hat{j} - 2\hat{k})$.

$\therefore \quad \overrightarrow{CB} = (\overrightarrow{OB} - \overrightarrow{OC})$

$= (3\hat{i} + \hat{j} + 5\hat{k}) - (0\hat{i} + 3\hat{j} + 3\hat{k})$

$= (3\hat{i} - 2\hat{j} + 2\hat{k})$.

$\therefore \quad \overrightarrow{CA} \cdot \overrightarrow{CB} = (0\hat{i} - 2\hat{j} - 2\hat{k}) \cdot (3\hat{i} - 2\hat{j} + 2\hat{k})$

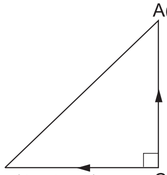
$= [(0 \times 3) + (-2) \times (-2) + (-2) \times 2] = 0$

$\Rightarrow \quad \overrightarrow{CA} \perp \overrightarrow{CB}$.

Hence, $\triangle ABC$ is right angled at C.

EXAMPLE 23 *Show that the points A, B and C having position vectors $(2\hat{i} - \hat{j} + \hat{k})$, $(\hat{i} - 3\hat{j} - 5\hat{k})$ and $(3\hat{i} - 4\hat{j} - 4\hat{k})$ respectively are the vertices of a right-angled triangle. Also, find the remaining angles of the triangle.*

SOLUTION We have



$\overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A)$

$= (-\hat{i} - 2\hat{j} - 6\hat{k})$,

$\overrightarrow{BC} = (\text{p.v. of } C) - (\text{p.v. of } B)$

$= (2\hat{i} - \hat{j} + \hat{k})$ and

$\overrightarrow{CA} = (\text{p.v. of } A) - (\text{p.v. of } C) = (-\hat{i} + 3\hat{j} + 5\hat{k})$.

Clearly, we have $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$.

$\therefore \quad A, B$, and C are the vertices of a triangle.

Now, $\overrightarrow{BC} \cdot \overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + 3\hat{j} + 5\hat{k}) = (-2 - 3 + 5) = 0$.

$\therefore \quad \overrightarrow{BC} \perp \overrightarrow{CA}$ and therefore, $\angle C = 90°$.

Now, $\angle A$ is the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\therefore \quad \overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}|\cos A \Rightarrow \cos A = \dfrac{(\overrightarrow{AB} \cdot \overrightarrow{AC})}{|\overrightarrow{AB}||\overrightarrow{AC}|}$

$$= \frac{(-\hat{i} - 2\hat{j} - 6\hat{k}) \cdot (\hat{i} - 3\hat{j} - 5\hat{k})}{\left\{\sqrt{(-1)^2 + (-2)^2 + (-6)^2}\right\} \cdot \left\{\sqrt{1^2 + (-3)^2 + (-5)^2}\right\}}$$

$[\because \overrightarrow{AC} = -\overrightarrow{CA}]$

$$= \frac{(-1 + 6 + 30)}{\sqrt{41} \times \sqrt{35}} = \frac{35}{\sqrt{41} \times \sqrt{35}} = \frac{\sqrt{35}}{\sqrt{41}} = \sqrt{\frac{35}{41}}$$

$\Rightarrow \quad A = \cos^{-1}\sqrt{\dfrac{35}{41}}.$

Further, $\angle B$ is the angle between $\overrightarrow{BC}$ and $\overrightarrow{BA}$.

$\therefore \quad \overrightarrow{BC} \cdot \overrightarrow{BA} = |\overrightarrow{BC}||\overrightarrow{BA}|\cos B$

$\Rightarrow \quad \cos B = \dfrac{(\overrightarrow{BC} \cdot \overrightarrow{BA})}{|\overrightarrow{BC}||\overrightarrow{BA}|}$

$$= \frac{(2\hat{i} - \hat{j} + \hat{k}) \cdot (\hat{i} + 2\hat{j} + 6\hat{k})}{\left\{\sqrt{2^2 + (-1)^2 + 1^2}\right\} \cdot \left\{\sqrt{1^2 + 2^2 + 6^2}\right\}}$$

$$= \frac{(2 - 2 + 6)}{\sqrt{6} \times \sqrt{41}} = \frac{6}{\sqrt{6} \times \sqrt{41}} = \frac{\sqrt{6}}{\sqrt{41}} = \sqrt{\frac{6}{41}}.$$

$\Rightarrow \quad B = \cos^{-1}\sqrt{\dfrac{6}{41}}.$

$\therefore \quad A = \cos^{-1}\sqrt{\dfrac{35}{41}},\ B = \cos^{-1}\sqrt{\dfrac{6}{41}},$ and $C = 90°$.

EXAMPLE 24 *Let $(\hat{i} + \hat{j} + \hat{k})$, $(2\hat{i} + 5\hat{j})$, $(3\hat{i} + 2\hat{j} - 3\hat{k})$ and $(\hat{i} - 6\hat{j} - \hat{k})$ be the position vectors of points A, B, C, D respectively. Find the angle between AB and CD. Hence, show that AB || CD.* **[CBSE 2008]**

SOLUTION Let the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be θ.

Now, $\overrightarrow{AB}$ = (p.v. of B) – (p.v. of A)

$$= (2\hat{i} + 5\hat{j}) - (\hat{i} + \hat{j} + \hat{k}) = (\hat{i} + 4\hat{j} - \hat{k})$$

and, $\overrightarrow{CD}$ = (p.v. of D) – (p.v. of C)

$$= (\hat{i} - 6\hat{j} - \hat{k}) - (3\hat{i} + 2\hat{j} - 3\hat{k}) = (-2\hat{i} - 8\hat{j} + 2\hat{k}).$$

$\therefore \quad |\overrightarrow{AB}| = \sqrt{1^2 + 4^2 + (-1)^2} = \sqrt{18} = 3\sqrt{2}$

and, $|\overrightarrow{CD}| = \sqrt{(-2)^2 + (-8)^2 + 2^2} = \sqrt{72} = 6\sqrt{2}$.

Now, $\overrightarrow{AB} \cdot \overrightarrow{CD} = (\hat{i} + 4\hat{j} - \hat{k}) \cdot (-2\hat{i} - 8\hat{j} + 2\hat{k})$

$= (-2 - 32 - 2) = -36.$

$\therefore \quad \cos\theta = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|} = \dfrac{-36}{(3\sqrt{2} \times 6\sqrt{2})} = \dfrac{-36}{36} = -1 \Rightarrow \theta = \pi.$

Hence, $AB \parallel CD$.

EXERCISE 23

1. Find $\vec{a} \cdot \vec{b}$ when

   (i) $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 4\hat{j} - 2\hat{k}$

   (ii) $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{j} + 4\hat{k}$

   (iii) $\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{k}$

2. Find the value of λ for which $\vec{a}$ and $\vec{b}$ are perpendicular, where

   (i) $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = (\hat{i} - 2\hat{j} + 3\hat{k})$ **[CBSE 2012C]**

   (ii) $\vec{a} = 3\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = -\lambda\hat{i} + 3\hat{j} + 3\hat{k}$

   (iii) $\vec{a} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$ **[CBSE 2003C]**

   (iv) $\vec{a} = 3\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = -5\hat{j} + \lambda\hat{k}$

3. (i) If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$, show that $(\vec{a} + \vec{b})$ is perpendicular to $(\vec{a} - \vec{b})$. **[CBSE 2002]**

   (ii) If $\vec{a} = (5\hat{i} - \hat{j} - 3\hat{k})$ and $\vec{b} = (\hat{i} + 3\hat{j} - 5\hat{k})$ then show that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal. **[CBSE 2004]**

4. If $\vec{a} = (\hat{i} - \hat{j} + 7\hat{k})$ and $\vec{b} = (5\hat{i} - \hat{j} + \lambda\hat{k})$ then find the value of λ so that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal vectors. **[CBSE 2013]**

5. Show that the vectors

$$\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}), \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) \text{ and } \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

   are mutually perpendicular unit vectors.

6. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$.

   Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and is such that $\vec{d} \cdot \vec{c} = 21$.

7. Let $\vec{a} = (2\hat{i} + 3\hat{j} + 2\hat{k})$ and $\vec{b} = (\hat{i} + 2\hat{j} + \hat{k})$.

   Find the projection of (i) $\vec{a}$ on $\vec{b}$ and (ii) $\vec{b}$ on $\vec{a}$.

8. Find the projection of $(8\hat{i} + \hat{j})$ in the direction of $(\hat{i} + 2\hat{j} - 2\hat{k})$.

9. Write the projection of vector $(\hat{i} + \hat{j} + \hat{k})$ along the vector $\hat{j}$. **[CBSE 2014]**

10. (i) Find the projection of $\vec{a}$ on $\vec{b}$ if $\vec{a} \cdot \vec{b} = 8$ and $\vec{b} = (2\hat{i} + 6\hat{j} + 3\hat{k})$. **[CBSE 2009]**

    (ii) Write the projection of the vector $(\hat{i} + \hat{j})$ on the vector $(\hat{i} - \hat{j})$. **[CBSE 2011]**

11. Find the angle between the vectors $\vec{a}$ and $\vec{b}$, when

    (i) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

    (ii) $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

    (iii) $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{j} + \hat{k}$.

12. If $\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{b} = (3\hat{i} - \hat{j} + 2\hat{k})$ then calculate the angle between $(2\vec{a} + \vec{b})$ and $(\vec{a} + 2\vec{b})$.

13. If $\vec{a}$ is a unit vector such that $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$, find $|\vec{x}|$.

14. Find the angles which the vector $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ makes with the coordinate axes.

15. Show that the vector $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$ is equally inclined to the coordinate axes.

16. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle $\pi/4$ with x-axis, $\pi/2$ with y-axis and an acute angle θ with z-axis. **[CBSE 2014]**

17. Find the angle between $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, if $\vec{a} = (2\hat{i} - \hat{j} + 3\hat{k})$ and $\vec{b} = (3\hat{i} + \hat{j} + 2\hat{k})$. **[CBSE 2006]**

18. Express the vector $\vec{a} = (6\hat{i} - 3\hat{j} - 6\hat{k})$ as sum of two vectors such that one is parallel to the vector $\vec{b} = (\hat{i} + \hat{j} + \hat{k})$ and the other is perpendicular to $\vec{b}$.

19. Prove that $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 \Leftrightarrow \vec{a} \perp \vec{b}$, where $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$.

20. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$. **[CBSE 2008]**

21. Find the angle between $\vec{a}$ and $\vec{b}$, when

    (i) $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = \sqrt{3}$ (ii) $|\vec{a}| = |\vec{b}| = \sqrt{2}$ and $\vec{a} \cdot \vec{b} = -1$.

22. If $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 4$, find $|\vec{a} - \vec{b}|$.

23. If $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$, find $|\vec{a}|$ and $|\vec{b}|$.

24. If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ then prove that:

    (i) $\cos\frac{\theta}{2} = \frac{1}{2}|\hat{a} + \hat{b}|$ (ii) $\tan\frac{\theta}{2} = \frac{|\hat{a} - \hat{b}|}{|\hat{a} + \hat{b}|}$

25. The dot products of a vector with the vectors $(\hat{i} + \hat{j} - 3\hat{k})$, $(\hat{i} + 3\hat{j} - 2\hat{k})$ and $(2\hat{i} + \hat{j} + 4\hat{k})$ are 0, 5 and 8 respectively. Find the vector. **[CBSE 2003]**

26. If $\overrightarrow{AB} = (3\hat{i} - \hat{j} + 2\hat{k})$ and the coordinates of A are (0, –2, –1), find the coordinates of B.

27. If $A(2, 3, 4)$, $B(5, 4, -1)$, $C(3, 6, 2)$ and $D(1, 2, 0)$ be four points, show that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$.

28. Find the value of λ for which the vectors $(2\hat{i} + \lambda\hat{j} + 3\hat{k})$ and $(3\hat{i} + 2\hat{j} - 4\hat{k})$ are perpendicular to each other. **[CBSE 2010]**

29. Show that the vectors $\vec{a} = (3\hat{i} - 2\hat{j} + \hat{k})$, $\vec{b} = (\hat{i} - 3\hat{j} + 5\hat{k})$ and $\vec{c} = (2\hat{i} + \hat{j} - 4\hat{k})$ form a right-angled triangle. **[CBSE 2005]**

30. Three vertices of a triangle are $A(0, -1, -2)$, $B(3, 1, 4)$ and $C(5, 7, 1)$. Show that it is a right-angled triangle. Also, find its other two angles.

31. If the position vectors of the vertices A, B and C of a $\triangle ABC$ be (1, 2, 3), (–1, 0, 0) and (0, 1, 2) respectively then find $\angle ABC$.

32. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $|\vec{a} + \vec{b}| = \sqrt{3}$, find $(2\vec{a} - 5\vec{b}) \cdot (3\vec{a} + \vec{b})$.

33. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}|=|\vec{a}|$ then prove that vector $(2\vec{a}+\vec{b})$ is perpendicular to vector $\vec{b}$. **[CBSE 2013]**

34. If $\vec{a}=(3\hat{i}-\hat{j})$ and $\vec{b}=(2\hat{i}+\hat{j}-3\hat{k})$ then express $\vec{b}$ in the form $\vec{b}=(\vec{b}_1+\vec{b}_2)$, where $\vec{b}_1 \parallel \vec{a}$ and $\vec{b}_2 \perp \vec{a}$. **[CBSE 2013C]**

ANSWERS (EXERCISE 23)

1. (i) 9 (ii) 8 (iii) –7 **2.** (i) $\lambda=\frac{5}{2}$ (ii) $\lambda=3$ (iii) $\lambda=-2$ (iv) $\lambda=-2$

4. $\lambda=\pm 5$ **6.** $\vec{d}=7(\hat{i}-\hat{j}-\hat{k})$ **7.** (i) $\frac{5\sqrt{6}}{3}$ (ii) $\frac{10\sqrt{17}}{17}$

8. $\frac{10}{3}$ **9.** 1 **10.** (i) $\frac{8}{7}$ (ii) 0 **11.** (i) $\cos^{-1}\left(\frac{5}{7}\right)$ (ii) $\cos^{-1}\left(\sqrt{\frac{3}{7}}\right)$ (iii) 120°

12. $\cos^{-1}\left(\frac{31}{50}\right)$ **13.** $|\vec{x}|=3$ **14.** $\cos^{-1}\left(\frac{3}{7}\right)$, $\cos^{-1}\left(\frac{-6}{7}\right)$, $\cos^{-1}\left(\frac{2}{7}\right)$

16. $5(\hat{i}+\hat{k})$ **17.** $\frac{\pi}{2}$ **18.** $\vec{a}=-(\hat{i}+\hat{j}+\hat{k})+(7\hat{i}-2\hat{j}-5\hat{k})$ **20.** $\frac{\pi}{3}$

21. (i) $\frac{\pi}{6}$ (ii) $\frac{2\pi}{3}$ **22.** $\sqrt{5}$ **23.** $|\vec{a}|=8\sqrt{\frac{8}{63}}$ and $|\vec{b}|=\sqrt{\frac{8}{63}}$

25. $(\hat{i}+2\hat{j}+\hat{k})$ **26.** $B(3,-3,1)$ **28.** $\lambda=3$

30. $\angle A=45°$, $\angle B=90°$, $\angle C=45°$ **31.** $\cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$ **32.** $\frac{-11}{2}$

34. $\vec{b}=(\vec{b}_1+\vec{b}_2)$, where $\vec{b}_1=\left(\frac{3}{2}\hat{i}-\frac{1}{2}\hat{j}\right)$ and $\vec{b}_2=\left(\frac{1}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}\right)$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 23)

5. Let the given vectors be $\vec{a}, \vec{b}, \vec{c}$ respectively.

Then, show that $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$, and $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$.

6. Let $\vec{d}=d_1\hat{i}+d_2\hat{j}+d_3\hat{k}$. Then,

$$\vec{d}\cdot\vec{a}=0,\ \vec{d}\cdot\vec{b}=0 \text{ and } \vec{d}\cdot\vec{c}=21$$

$$\Rightarrow \left.\begin{cases}4d_1+5d_2-d_3=0\\ d_1-4d_2+5d_3=0\end{cases}\right\} \text{and } 3d_1+d_2-d_3=21$$

$$\Rightarrow \frac{d_1}{(25-4)}=\frac{d_2}{(-1-20)}=\frac{d_3}{(-16-5)}=k \text{ (say) and } 3d_1+d_2-d_3=21$$

$\Rightarrow \quad (d_1 = 21k, d_2 = -21k, d_3 = -21k)$ and $3d_1 + d_2 - d_3 = 21$

$\Rightarrow \quad 63k - 21k + 21k = 21 \Rightarrow k = \frac{1}{3} \Rightarrow d_1 = 7, d_2 = -7$ and $d_3 = -7$.

11. (iii) Let the required angle be θ. Then,

$$\vec{a} \cdot \vec{b} = (1 \times 0) + (-1) \times 1 + (0 \times 1) = -1$$

$$\Rightarrow |\vec{a}||\vec{b}| \cos\theta = -1$$

$$\Rightarrow (\sqrt{2})(\sqrt{2}) \cos\theta = -1 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = 120°.$$

12. $(2\vec{a} + \vec{b}) \cdot (\vec{a} + 2\vec{b}) = |2\vec{a} + \vec{b}| \cdot |\vec{a} + 2\vec{b}| \cos\theta$. Find θ.

13. $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8 \Rightarrow \vec{x} \cdot \vec{x} - \vec{a} \cdot \vec{a} = 8$ $[\because \vec{x} \cdot \vec{a} = \vec{a} \cdot \vec{x}]$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 8 \Rightarrow |\vec{x}|^2 = (8 + 1) = 9 \Rightarrow |x| = 3.$$

14. $\vec{a} \cdot \hat{i} = |\vec{a}||\hat{i}| \cos\alpha \Rightarrow 7 \times 1 \times \cos\alpha = 3 \Rightarrow \cos\alpha = \frac{3}{7}$.

$\vec{a} \cdot \hat{j} = |\vec{a}||\hat{j}| \cos\beta \Rightarrow 7 \times 1 \times \cos\beta = -6 \Rightarrow \cos\beta = \frac{-6}{7}$.

$\vec{a} \cdot \hat{k} = |\vec{a}||\hat{k}| \cos\gamma \Rightarrow 7 \times 1 \times \cos\gamma = 2 \Rightarrow \cos\gamma = \frac{2}{7}$.

15. $\vec{a} \cdot \hat{i} = |\vec{a}||\hat{i}| \cos\alpha \Rightarrow \sqrt{3} \times 1 \times \cos\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$.

Similarly, $\cos\beta = \frac{1}{\sqrt{3}}$ and $\cos\gamma = \frac{1}{\sqrt{3}}$.

$\therefore \quad \alpha = \beta = \gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

16. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. Then, $|\vec{a}|^2 = 50 \Rightarrow (a_1^2 + a_2^2 + a_3^2) = 50$.

$\vec{a} \cdot \hat{i} = |\vec{a}||\hat{i}| \cos\alpha \Rightarrow a_1 = \sqrt{50} \times 1 \times \cos\frac{\pi}{4} = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5$.

$\vec{a} \cdot \hat{j} = |\vec{a}||\hat{j}| \cos\beta \Rightarrow a_2 = \sqrt{50} \times 1 \times \cos\frac{\pi}{2} = 0$

$\vec{a} \cdot \hat{k} = |\vec{a}||\hat{k}| \cos\gamma \Rightarrow a_3 = \sqrt{50} \times 1 \times \cos\theta$

$(a_1^2 + a_2^2 + a_3^2) = 50 \Rightarrow 25 + 0 + 50\cos^2\theta = 50$

$$\Rightarrow \cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{2}.$$

$\therefore \quad a_3 = \sqrt{50} \times \cos\frac{\pi}{4} = \left(5\sqrt{2} \times \frac{1}{\sqrt{2}}\right) = 5$.

$\therefore \quad \vec{a} = 5(\hat{i} + \hat{k})$.

17. Let $\vec{u} = (\vec{a} + \vec{b}) = (5\hat{i} + \hat{k})$ and $\vec{v} = (\vec{a} - \vec{b}) = (-\hat{i} - 2\hat{j} + 5\hat{k})$.

Then, $|\vec{u}| = \sqrt{5^2 + 1^2} = \sqrt{26}$ and $|\vec{v}| = \sqrt{(-1)^2 + (-2)^2 + 5^2} = \sqrt{30}$.

$$\cos\theta = \frac{(\vec{u} \cdot \vec{v})}{|\vec{u}||\vec{v}|} = \frac{(-5 - 0 + 5)}{\sqrt{26} \cdot \sqrt{30}} = 0 \Rightarrow \theta = \frac{\pi}{2}\cdot$$

18. Let $\vec{a} = \lambda\vec{b} + \vec{c}$, where $\vec{c} \perp \vec{b}$. Then, $\vec{c} = (\vec{a} - \lambda\vec{b}) \perp \vec{b}$.

$$\therefore \quad (\vec{a} - \lambda\vec{b}) \cdot \vec{b} = 0 \Rightarrow (\vec{a} \cdot \vec{b}) - \lambda(\vec{b} \cdot \vec{b}) = 0$$

$$\Rightarrow (6 - 3 - 6) - \lambda(1 + 1 + 1) = 0 \Rightarrow \lambda = \frac{-3}{3} = -1.$$

$$\therefore \quad \vec{a} = -\vec{b} + \vec{c} \Rightarrow \vec{c} = (\vec{a} + \vec{b}).$$

20. $\vec{a} + \vec{b} = -\vec{c} \Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = (-\vec{c}) \cdot (-\vec{c})$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a} \cdot \vec{b} = |\vec{c}|^2 \Rightarrow \vec{a} \cdot \vec{b} = \frac{15}{2}$$

$$\Rightarrow ab\cos\theta = \frac{15}{2} \Rightarrow (3 \times 5)\cos\theta = \frac{15}{2} \Rightarrow \cos\theta = \frac{1}{2}\cdot$$

21. $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\cdot$

22. $|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$

$$= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$$

$$= |\vec{a}|^2 - 2(\vec{a} \cdot \vec{b}) + |\vec{b}|^2 = 5.$$

23. $|\vec{a}|^2 - |\vec{b}|^2 = 8 \Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8 \Rightarrow |\vec{b}|^2 = \dfrac{8}{63}\cdot$

$$\therefore \quad |\vec{b}| = \sqrt{\frac{8}{63}} \text{ and } |\vec{a}| = 8 \cdot \sqrt{\frac{8}{63}}\cdot$$

24. $|\hat{a} + \hat{b}|^2 = (\hat{a} + \hat{b}) \cdot (\hat{a} + \hat{b}) = \hat{a} \cdot \hat{a} + \hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{a} + \hat{b} \cdot \hat{b}$

$$= |\hat{a}|^2 + 2(\hat{a} \cdot \hat{b}) + |\hat{b}|^2 = 1 + 2|\hat{a}||\hat{b}|\cos\theta + 1$$

$$= 2(1 + \cos\theta) = 2 \times 2\cos^2\frac{\theta}{2} = 4\cos^2\frac{\theta}{2}\cdot$$

$$\therefore \quad \cos\frac{\theta}{2} = \frac{1}{2}|\hat{a} + \hat{b}|.$$

Similarly, we have $|\hat{a} - \hat{b}|^2 = 4\sin^2\dfrac{\theta}{2}\cdot$

$$\therefore \quad \sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|.$$

Hence, $\tan\dfrac{\theta}{2} = \dfrac{|\hat{a} - \hat{b}|}{|\hat{a} + \hat{b}|}\cdot$

25. Let the given vectors be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively and let the required vector be $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$.

Then, $\vec{d}\cdot\vec{a} = 0$, $\vec{d}\cdot\vec{b} = 5$ and $\vec{d}\cdot\vec{c} = 8$

$\Rightarrow \quad d_1 + d_2 - 3d_3 = 0$ ... (i), $d_1 + 3d_2 - 2d_3 = 5$ ... (ii) and $2d_1 + d_2 + 4d_3 = 8$. ... (iii)

On subtracting (i) from (ii), we get $2d_2 + d_3 = 5$. ... (iv)

On multiplying (i) by 2 and subtracting (iii) from it, we get $d_2 - 10d_3 = -8$. ... (v)

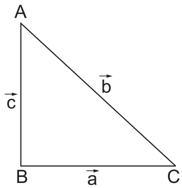
On solving (iv) and (v), we get $d_2 = 2$, $d_3 = 1$ and therefore, $d_1 = 1$.

26. $\overrightarrow{AB} = (\overrightarrow{OB} - \overrightarrow{OA}) \Rightarrow \overrightarrow{OB} = (\overrightarrow{AB} + \overrightarrow{OA})$.

27. $\overrightarrow{AB} = (\overrightarrow{OB} - \overrightarrow{OA}) = (3\hat{i} + \hat{j} - 5\hat{k})$, $\overrightarrow{CD} = (\overrightarrow{OD} - \overrightarrow{OC}) = (-2\hat{i} + 4\hat{j} - 2\hat{k})$.

29. Clearly, $\vec{b} + \vec{c} = \vec{a}$ and $\vec{a}\cdot\vec{c} = 0$.

$\therefore \quad \triangle ABC$ is right angled at B.



30. Let O be the origin. Then,

$\overrightarrow{OA} = (-\hat{j} - 2\hat{k})$, $\overrightarrow{OB} = (3\hat{i} + \hat{j} + 4\hat{k})$ and $\overrightarrow{OC} = (5\hat{i} + 7\hat{j} + \hat{k})$.

$\therefore \quad \overrightarrow{AB} = (\overrightarrow{OB} - \overrightarrow{OA}) = (3\hat{i} + 2\hat{j} + 6\hat{k})$, $\overrightarrow{BC} = (\overrightarrow{OC} - \overrightarrow{OB}) = (2\hat{i} + 6\hat{j} - 3\hat{k})$ and

$\overrightarrow{CA} = (-5\hat{i} - 8\hat{j} - 3\hat{k})$.

Now, $\overrightarrow{AB}\cdot\overrightarrow{BC} = (3\hat{i} + 2\hat{j} + 6\hat{k})\cdot(2\hat{i} + 6\hat{j} - 3\hat{k}) = (6 + 12 - 18) = 0$.

$\therefore \quad \overrightarrow{AB} \perp \overrightarrow{BC}$ and therefore, $\angle B = 90°$.

Hence, $\triangle ABC$ is right angled at B.

Now, $\overrightarrow{AC} = -\overrightarrow{CA} = (5\hat{i} + 8\hat{j} + 3\hat{k})$.

$$\cos A = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{(3\times 5 + 2\times 8 + 6\times 3)}{\left\{\sqrt{3^2 + 2^2 + 6^2}\right\}\left\{\sqrt{5^2 + 8^2 + 3^2}\right\}}$$

$$= \frac{49}{(\sqrt{49})(\sqrt{98})} = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad \angle A = 45°$ and therefore, $\angle C = 180° - (90° + 45°) = 45°$.

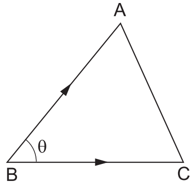
31. Let $\angle ABC = \theta$.

Now, $\overrightarrow{BA} =$ (p.v. of A) – (p.v.of B) $= (2\hat{i} + 2\hat{j} + 3\hat{k})$

and $\overrightarrow{BC} =$ (p.v. of C) – (p.v.of B) $= (\hat{i} + \hat{j} + 2\hat{k})$.

$$\cos\theta = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{(2\times 1 + 2\times 1 + 3\times 2)}{\left\{\sqrt{2^2+2^2+3^2}\right\}\left\{\sqrt{1^2+1^2+2^2}\right\}}$$

$$= \frac{10}{\sqrt{17}\times\sqrt{6}} = \frac{10}{\sqrt{102}}$$



$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right).$$

32. Given: $|\vec{a}| = 1$, $|\vec{b}| = 1$ and $|\vec{a}+\vec{b}| = \sqrt{3}$.

$$\therefore \quad |\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

$$= (\vec{a}\cdot\vec{a}) + 2(\vec{a}\cdot\vec{b}) + (\vec{b}\cdot\vec{b})$$

$$= |\vec{a}|^2 + 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2 = 2 + 2(\vec{a}\cdot\vec{b}).$$

$$\therefore \quad 2[1 + (\vec{a}\cdot\vec{b})] = 3 \;\Rightarrow\; (\vec{a}\cdot\vec{b}) = \frac{1}{2}\cdot$$

Now, find $(2\vec{a} - 5\vec{b})\cdot(3\vec{a}+\vec{b})$.

33. $|\vec{a}+\vec{b}| = |\vec{a}| \Rightarrow |\vec{a}+\vec{b}|^2 = |\vec{a}|^2$

$$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2$$

$$\Rightarrow \vec{a}\cdot\vec{a} + 2(\vec{a}\cdot\vec{b}) + \vec{b}\cdot\vec{b} = |\vec{a}|^2 \quad [\because\; (\vec{b}\cdot\vec{a}) = (\vec{a}\cdot\vec{b})]$$

$$\Rightarrow 2(\vec{a}\cdot\vec{b}) + \vec{b}\cdot\vec{b} = 0 \Rightarrow 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} = 0$$

$$\Rightarrow (2\vec{a}+\vec{b})\cdot\vec{b} = 0.$$

34. Let $\vec{b} = \vec{b}_1 + \vec{b}_2$, where $\vec{b}_1 \parallel \vec{a}$ and $\vec{b}_2 \perp \vec{a}$.

Let $\vec{b}_1 = \lambda(3\hat{i} - \hat{j})$ and let $\vec{b}_2 = x\hat{i} + y\hat{j} + z\hat{k}$. Then,

$$\vec{b}_2 \perp \vec{a} \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k})\cdot(3\hat{i} - \hat{j}) = 0 \Rightarrow 3x - y = 0.$$

$$\therefore \quad \vec{b} = (\vec{b}_1 + \vec{b}_2) \;\Rightarrow (2\hat{i} + \hat{j} - 3\hat{k}) = \lambda(3\hat{i} - \hat{j}) + (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\Rightarrow (2\hat{i} + \hat{j} - 3\hat{k}) = (3\lambda + x)\hat{i} + (y - \lambda)\hat{j} + z\hat{k}$$

$$\Rightarrow z = -3,\; y - \lambda = 1 \text{ and } 3\lambda + x = 2$$

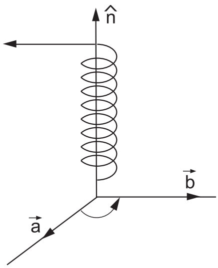
$$\Rightarrow z = -3,\; \lambda = (y-1) \text{ and } 3(y-1) + x = 2$$

$$\Rightarrow z = -3,\; \lambda = (y-1) \text{ and } 3y + x = 5.$$

On solving $3x - y = 0$ and $x + 3y = 5$, we get $x = \frac{1}{2}$ and $y = \frac{3}{2}$, $z = -3$ and $\lambda = \frac{1}{2}\cdot$

$$\therefore \quad \vec{b}_1 = \frac{1}{2}(3\hat{i} - \hat{j}) \text{ and } \vec{b}_2 = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}.$$

24. CROSS, OR VECTOR, PRODUCT OF VECTORS



VECTOR PRODUCT OF TWO VECTORS Let $\vec{a}$ and $\vec{b}$ be two nonzero, nonparallel vectors, and let θ be the angle between them such that $0 < \theta < \pi$.

Then, the vector product of $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin\theta)\,\hat{n},$$

where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}, \hat{n}$ form a right-handed system.

When a right-handed screw is rotated from $\vec{a}$ to $\vec{b}$ and it advances along $\hat{n}$ then the system is said to be right handed.

REMARK 1 If $\vec{a}$ and $\vec{b}$ are parallel or collinear, i.e., when $\theta = 0$ or $\theta = \pi$ then we define, $\vec{a} \times \vec{b} = \vec{0}$.

In particular, $\vec{a} \times \vec{a} = \vec{0}$.

REMARK 2 If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, we define, $\vec{a} \times \vec{b} = \vec{0}$.

REMARK 3 For any vector $\vec{a}$, we have $\vec{a} \times \vec{a} = (|\vec{a}||\vec{a}| \sin 0)\,\hat{n} = 0\,\hat{n} = \vec{0}$.

ANGLE BETWEEN TWO VECTORS Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a} \times \vec{b} = (ab \sin\theta)\,\hat{n}, \text{ where } |\vec{a}| = a \text{ and } |\vec{b}| = b$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = ab \sin\theta$$

$$\Rightarrow \quad \sin\theta = \frac{|\vec{a} \times \vec{b}|}{ab} = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left\{\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right\}.$$

UNIT VECTOR PERPENDICULAR TO TWO GIVEN VECTORS A unit vector $\hat{n}$ perpendicular to each one of $\vec{a}$ and $\vec{b}$ is given by

$$\hat{n} = \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}.$$

Properties of Vector Product

RESULT 1 *Vector product is not commutative.*

In fact, we have $(\vec{b} \times \vec{a}) = -(\vec{a} \times \vec{b})$.

PROOF Let $\vec{a}, \vec{b}, \hat{n}$ form a right-handed system. Then,

$$(\vec{a} \times \vec{b}) = (a\,b \sin\theta)\,\hat{n}. \quad \text{... (i)}$$

Then, $\vec{b}, \vec{a}, -\hat{n}$ form a right-handed system.

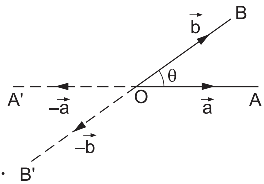
$\therefore \quad (\vec{b} \times \vec{a}) = (b\,a \sin\theta)(-\hat{n})$

$\Rightarrow \quad -(\vec{b} \times \vec{a}) = (a\,b \sin\theta)\,\hat{n}. \quad \text{... (ii)}$

From (i) and (ii), we get $(\vec{a} \times \vec{b}) = -(\vec{b} \times \vec{a})$.

RESULT 2 *For any two vectors $\vec{a}$ and $\vec{b}$, prove that*

$$(-\vec{b}) \times \vec{a} = (\vec{a} \times \vec{b}) = \vec{b} \times (-\vec{a}).$$



PROOF Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then, the angle between $(-\vec{b})$ and $\vec{a}$ is $(\pi - \theta)$.

And, $(-\vec{b}), \vec{a}, \hat{n}$ form a right-handed system.

$$\therefore \quad (-\vec{b}) \times \vec{a} = |-\vec{b}|\,|\vec{a}| \sin(\pi - \theta)\,\hat{n}$$

$$= (|\vec{b}|\,|\vec{a}| \sin\theta)\,\hat{n}$$

$$= (a\,b \sin\theta)\,\hat{n} = (\vec{a} \times \vec{b}).$$

$\therefore \quad (-\vec{b}) \times \vec{a} = (\vec{a} \times \vec{b})$.

Similarly, $\vec{b} \times (-\vec{a}) = (\vec{a} \times \vec{b})$.

Hence, $(-\vec{b}) \times \vec{a} = (\vec{a} \times \vec{b}) = \vec{b} \times (-\vec{a})$.

RESULT 3 *For any scalar m, we have* $(m\vec{a} \times \vec{b}) = m(\vec{a} \times \vec{b}) = \vec{a} \times (m\vec{b})$.

AN IMPORTANT NOTE At a later stage, we shall prove that for any three vectors $\vec{a}, \vec{b}, \vec{c}$, we have $\vec{a} \times \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{b} \times \vec{c}$, i.e., the position of dot and cross can be interchanged. However, we shall use this fact in proving the following theorem.

RESULT 4 (Distributive law) *For any vectors $\vec{a}, \vec{b}, \vec{c}$, prove that*

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}).$$

PROOF Let $\vec{p} = [\vec{a} \times (\vec{b} + \vec{c}) - (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c})]$, and let $\vec{r}$ be any arbitrary vector.

Then,

$$\vec{r}\cdot\vec{p} = \vec{r}\cdot[\vec{a}\times(\vec{b}+\vec{c}) - (\vec{a}\times\vec{b}) - (\vec{a}\times\vec{c})]$$
$$= \vec{r}\cdot[\vec{a}\times(\vec{b}+\vec{c})] - \vec{r}\cdot(\vec{a}\times\vec{b}) - \vec{r}\cdot(\vec{a}\times\vec{c})$$
$$= (\vec{r}\times\vec{a})\cdot(\vec{b}+\vec{c}) - (\vec{r}\times\vec{a})\cdot\vec{b} - (\vec{r}\times\vec{a})\cdot\vec{c}$$
$$= (\vec{r}\times\vec{a})\cdot\vec{b} + (\vec{r}\times\vec{a})\cdot\vec{c} - (\vec{r}\times\vec{a})\cdot\vec{b} - (\vec{r}\times\vec{a})\cdot\vec{c}$$
$$= 0.$$

Now, $\vec{r}\cdot\vec{p} = 0 \Rightarrow \vec{r} = \vec{0}$ or $\vec{p} = \vec{0}$ or $(\vec{r}\perp\vec{p})$.

Since $\vec{r}$ is an arbitrary vector, we may choose it in such a way that $\vec{r} \neq \vec{0}$ and $\vec{r}$ is not perpendicular to $\vec{p}$.

Then, $\vec{p} = \vec{0}$

$$\Rightarrow \vec{a}\times(\vec{b}+\vec{c}) - (\vec{a}\times\vec{b}) - (\vec{a}\times\vec{c}) = \vec{0}$$
$$\Rightarrow \vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}).$$

Hence, $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$.

RESULT 5 *For any three vectors $\vec{a}, \vec{b}, \vec{c}$, prove that*

$$\vec{a}\times(\vec{b}-\vec{c}) = (\vec{a}\times\vec{b}) - (\vec{a}\times\vec{c}).$$

PROOF We have

$$\vec{a}\times(\vec{b}-\vec{c}) = \vec{a}\times[\vec{b}+(-\vec{c})]$$
$$= (\vec{a}\times\vec{b}) + \vec{a}\times(-\vec{c}) \quad \text{[by the distributive law]}$$
$$= (\vec{a}\times\vec{b}) + [-(\vec{a}\times\vec{c})] \quad [\because \vec{a}\times(-\vec{c}) = -(\vec{a}\times\vec{c})]$$
$$= (\vec{a}\times\vec{b}) - (\vec{a}\times\vec{c}).$$

Hence, $\vec{a}\times(\vec{b}-\vec{c}) = (\vec{a}\times\vec{b}) - (\vec{a}\times\vec{c})$.

RESULT 6 *Prove that two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel or collinear if and only if $(\vec{a}\times\vec{b}) = \vec{0}$.*

PROOF Let $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$ and let θ be the angle between them.

Let $\vec{a}$ and $\vec{b}$ be parallel or collinear.

Then, $\theta = 0$ or $\theta = \pi$

$$\Rightarrow \sin\theta = 0 \Rightarrow (\vec{a}\times\vec{b}) = (ab\sin\theta)\hat{n} = 0\hat{n} = \vec{0}.$$

Thus, when $\vec{a}$ and $\vec{b}$ are parallel or collinear, then $(\vec{a}\times\vec{b}) = \vec{0}$.

Again, let $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$ and $(\vec{a} \times \vec{b}) = \vec{0}$. Then,

$$(\vec{a} \times \vec{b}) = \vec{0} \Rightarrow |\vec{a} \times \vec{b}| = 0$$
$$\Rightarrow ab \sin\theta = 0$$
$$\Rightarrow \sin\theta = 0 \quad [a \neq 0 \text{ and } b \neq 0]$$
$$\Rightarrow \theta = 0 \text{ or } \theta = \pi$$
$$\Rightarrow (\vec{a} \,||\, \vec{b}) \text{ or } (\vec{a} \text{ and } \vec{b} \text{ are collinear}).$$

RESULT 7 *Prove that*

$$(\vec{a} \times \vec{b}) = \vec{0} \Rightarrow \vec{a} = \vec{0} \textit{ or } \vec{b} = \vec{0} \textit{ or } \vec{a} \,||\, \vec{b} \textit{ or } \vec{a} \textit{ and } \vec{b} \textit{ are collinear.}$$

PROOF Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a} \times \vec{b} = \vec{0} \Rightarrow (ab \sin\theta)\,\hat{n} = \vec{0}$$
$$\Rightarrow ab \sin\theta = 0$$
$$\Rightarrow a = 0 \text{ or } b = 0 \text{ or } \sin\theta = 0$$
$$\Rightarrow \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \theta = 0 \text{ or } \theta = \pi$$
$$\Rightarrow \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a} \text{ and } \vec{b} \text{ are parallel or collinear.}$$

COROLLARY *Show that* $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ *does not imply that* $\vec{b} = \vec{c}$.

PROOF $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Rightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0}$

$$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$$
$$\Rightarrow \vec{a} = \vec{0} \text{ or } (\vec{b} - \vec{c}) = 0 \text{ or } \vec{a} \,||\, (\vec{b} - \vec{c})$$
$$\Rightarrow \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{c} \text{ or } \vec{a} \,||\, (\vec{b} - \vec{c}).$$

$\therefore$ $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ does not always mean that $\vec{b} = \vec{c}$.

SUMMARY

(i) $(\vec{a} \times \vec{b}) = -(\vec{b} \times \vec{a})$

(ii) $(-\vec{b}) \times \vec{a} = (\vec{a} \times \vec{b}) = \vec{b} \times (-\vec{a})$

(iii) $m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times (m\vec{b})$

(iv) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

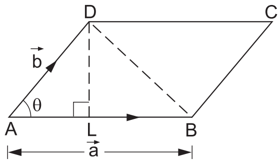
(v) $\vec{a} \times (\vec{b} - \vec{c}) = (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c})$

(vi) $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow (\vec{a} \,||\, \vec{b})$ or ($\vec{a}$ and $\vec{b}$ are collinear), when $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$

Area of a Parallelogram

THEOREM 1 *Two adjacent sides of a ||gm are represented by $\vec{a}$ and $\vec{b}$ respectively. Prove that the area of the ||gm = $|\vec{a} \times \vec{b}|$.*

PROOF Let $ABCD$ be a ||gm. Join BD.



Let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$ and $\angle BAD = \theta$.

Draw $DL \perp AB$. Then,

$$AB = a \text{ and } DL = AD \sin\theta = b \sin\theta.$$

$$\therefore \quad \text{ar}(\triangle ABD) = \frac{1}{2} \times \text{base} \times \text{altitude}$$

$$= \left(\frac{1}{2} \times AB \times DL\right) = \frac{1}{2} a\, b \sin\theta$$

$$= \frac{1}{2} |\vec{a} \times \vec{b}|.$$

$$\therefore \quad \text{ar}(\|\text{gm } ABCD) = 2 \times \text{ar}(\triangle ABD) = |\vec{a} \times \vec{b}|.$$

REMARK $(\vec{a} \times \vec{b})$ is called the vector area of the ||gm.

Area of a Triangle

THEOREM 2 *Prove that the area of $\triangle ABC$, where $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AC} = \vec{b}$, is $\frac{1}{2}|\vec{a} \times \vec{b}|$.*

PROOF In $\triangle ABC$, let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AC} = \vec{b}$ and $\angle BAC = \theta$.

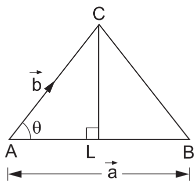
Draw $CL \perp AB$. Then,

$$AB = a \text{ and } CL = (AC) \sin\theta = b \sin\theta$$

and $$\text{ar}(\triangle ABC) = \frac{1}{2} \times AB \times CL$$

$$= \frac{1}{2} a\, b \sin\theta = \frac{1}{2} |\vec{a} \times \vec{b}|.$$

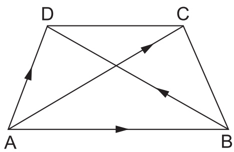
$$\therefore \quad \text{ar}(\triangle ABC) = \frac{1}{2} |\vec{a} \times \vec{b}|.$$



REMARK $\frac{1}{2}(\vec{a} \times \vec{b})$ is called the vector area of $\triangle ABC$.

Area of a Quadrilateral

THEOREM 3 *Prove that the area of a quadrilateral ABCD with diagonals AC and BD is $\frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$.*



PROOF Vector area of quad. $ABCD$

= (vector area of $\triangle ABC$) + (vector area of $\triangle ACD$)

$$= \frac{1}{2}(\overrightarrow{AB} \times \overrightarrow{AC}) + \frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AD})$$

$$= -\frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AB}) + \frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AD})$$

$$= \frac{1}{2} \cdot \{\overrightarrow{AC} \times (\overrightarrow{AD} - \overrightarrow{AB})\} = \frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{BD}).$$

$$\therefore \quad \text{ar(quad. } ABCD) = \frac{1}{2} |\overrightarrow{AC} \times \overrightarrow{BD}|.$$

SUMMARY

(i) ar (||gm $ABCD$) $= |\vec{a} \times \vec{b}|$, where $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$.

(ii) ar (||gm $ABCD$) $= \frac{1}{2} |\vec{d}_1 \times \vec{d}_2|$, where $\vec{d}_1$ and $\vec{d}_2$ are the diagonal vectors.

(iii) ar($\triangle ABC$) $= \frac{1}{2} |\vec{a} \times \vec{b}|$, where $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AC} = \vec{b}$.

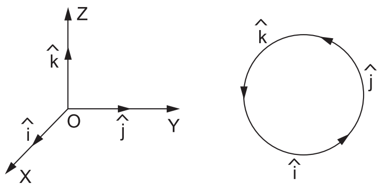
(iv) ar (quad. $ABCD$) $= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{BD}|$, where AC and BD are its diagonals.

VECTOR PRODUCT OF AN ORTHONORMAL VECTOR TRIAD

For mutually perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$, we have

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \hat{0},$$

$$\hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i}, \quad \hat{j} \times \hat{k} = \hat{i} = -\hat{k} \times \hat{j} \text{ and } \hat{k} \times \hat{i} = \hat{j} = -\hat{i} \times \hat{k}.$$



Vector Product in Terms of Components

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ *and* $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. *Then,*

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

PROOF We have

$$\vec{a} \times \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1b_1(\hat{i} \times \hat{i}) + a_1b_2(\hat{i} \times \hat{j}) + a_1b_3(\hat{i} \times \hat{k})$$

$$+ a_2b_1(\hat{j} \times \hat{i}) + a_2b_2(\hat{j} \times \hat{j}) + a_2b_3(\hat{j} \times \hat{k})$$

$$+ a_3b_1(\hat{k} \times \hat{i}) + a_3b_2(\hat{k} \times \hat{j}) + a_3b_3(\hat{k} \times \hat{k})$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$\left[\because \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0,\ \hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}, \text{ and } \hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i} \text{ and } \hat{i} \times \hat{k} = -\hat{j}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

SOLVED EXAMPLES

EXAMPLE 1 *If* $\vec{a} = (3\hat{i} + \hat{j} - 4\hat{k})$ *and* $\vec{b} = (6\hat{i} + 5\hat{j} - 2\hat{k})$, *find* $(\vec{a} \times \vec{b})$ *and* $|\vec{a} \times \vec{b}|$.

SOLUTION We have

$$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -4 \\ 6 & 5 & -2 \end{vmatrix}$$

$$= (-2 + 20)\hat{i} - (-6 + 24)\hat{j} + (15 - 6)\hat{k}$$

$$= (18\hat{i} - 18\hat{j} + 9\hat{k}).$$

$$|\vec{a} \times \vec{b}|^2 = \{(18)^2 + (-18)^2 + 9^2\} = 729$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = \sqrt{729} = 27.$$

EXAMPLE 2 *If* $\vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k})$ *and* $\vec{b} = (2\hat{i} + 3\hat{j} - 5\hat{k})$ *then find* $(\vec{a} \times \vec{b})$ *and verify that* $(\vec{a} \times \vec{b})$ *is perpendicular to each one of* $\vec{a}$ *and* $\vec{b}$.

SOLUTION We have

$$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 3 & -5 \end{vmatrix}$$

$$= (10 - 9)\hat{i} - (-5 - 6)\hat{j} + (3 + 4)\hat{k} = (\hat{i} + 11\hat{j} + 7\hat{k}).$$

Now, $(\vec{a} \times \vec{b}) \cdot \vec{a} = (\hat{i} + 11\hat{j} + 7\hat{k}) \cdot (\hat{i} - 2\hat{j} + 3\hat{k})$.

$\therefore \quad (\vec{a} \times \vec{b}) \cdot \vec{a} = (1 - 22 + 21) = 0.$

$\therefore \quad (\vec{a} \times \vec{b}) \perp \vec{a}.$

And, $(\vec{a} \times \vec{b}) \cdot \vec{b} = (\hat{i} + 11\vec{j} + 7\vec{k}) \cdot (2\hat{i} + 3\hat{j} - 5\hat{k})$

$= (2 + 33 - 35) = 0.$

$\therefore \quad (\vec{a} \times \vec{b}) \perp \vec{b}.$

EXAMPLE 3 *If* $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ *and* $\vec{b} = \hat{j} - \hat{k}$, *find a vector* $\vec{c}$ *such that* $\vec{a} \times \vec{c} = \vec{b}$ *and* $\vec{a} \cdot \vec{c} = 3$. **[CBSE 2013]**

SOLUTION Here $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$.

Let $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then,

$\vec{a} \cdot \vec{c} = 3$ and $\vec{a} \times \vec{c} = \vec{b}$

$\Rightarrow \quad (\hat{i} + \hat{j} + \hat{k}) \cdot (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) = 3$

and $(\hat{i} + \hat{j} + \hat{k}) \times (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) = (\hat{j} - \hat{k})$

$$\Rightarrow \quad c_1 + c_2 + c_3 = 3 \quad \ldots \text{(i)} \quad \text{and} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\hat{j} - \hat{k}) \quad \ldots \text{(ii)}$$

Now, (ii) gives: $(c_3 - c_2)\hat{i} - (c_3 - c_1)\hat{j} + (c_2 - c_1)\hat{k} = (\hat{j} - \hat{k})$

$\Rightarrow \quad c_3 - c_2 = 0,\ c_1 - c_3 = 1$ and $c_2 - c_1 = -1$

$\Rightarrow \quad c_3 = c_2$ and $c_1 - c_2 = 1$.

Putting $c_3 = c_2$ in (i), we get $c_1 + 2c_2 = 3$.

On solving $c_1 + 2c_2 = 3$ and $c_1 - c_2 = 1$, we get $c_2 = \frac{2}{3}$ and $c_1 = \frac{5}{3}\cdot$

$\therefore \quad c_1 = \frac{5}{3},\ c_2 = \frac{2}{3}$ and $c_3 = \frac{2}{3}\cdot$

Hence, $\vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k} \Rightarrow \vec{c} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$.

EXAMPLE 4 *Find a unit vector perpendicular to each one of the vectors* $\vec{a} = (4\hat{i} - \hat{j} + 3\hat{k})$ *and* $\vec{b} = (2\hat{i} + 2\hat{j} - \hat{k})$.

SOLUTION We know that $(\vec{a} \times \vec{b})$ is a vector perpendicular to each one of $\vec{a}$ and $\vec{b}$. So, the required vector is $\frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

$$\text{Now, } (\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ 2 & 2 & -1 \end{vmatrix}$$

$$= (1-6)\hat{i} - (-4-6)\hat{j} + (8+2)\hat{k}$$

$$= (-5\hat{i} + 10\hat{j} + 10\hat{k}).$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-5)^2 + (10)^2 + (10)^2} = \sqrt{225} = 15.$$

$$\text{Hence, the required unit vector} = \frac{(-5\hat{i} + 10\hat{j} + 10\hat{k})}{15}$$

$$= \frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k}).$$

EXAMPLE 5 *Find a vector of magnitude* 15, *which is perpendicular to both the vectors* $(4\hat{i} - \hat{j} + 8\hat{k})$ *and* $(-\hat{j} + \hat{k})$.

SOLUTION Let $\vec{a} = (4\hat{i} - \hat{j} + 8\hat{k})$ and $\vec{b} = (-\hat{j} + \hat{k})$.

A unit vector perpendicular to both $\vec{a}$ and $\vec{b} = \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

$$\text{Now, } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 8 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= (-1+8)\hat{i} - (4-0)\hat{j} + (-4-0)\hat{k}$$

$$= (7\hat{i} - 4\hat{j} - 4\hat{k}).$$

$$\therefore \quad |\vec{a} \times \vec{b}| = \sqrt{7^2 + (-4)^2 + (-4)^2} = \sqrt{81} = 9.$$

So, a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

$$= \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|} = \frac{7\hat{i} - 4\hat{j} - 4\hat{k}}{9}.$$

$$\text{The required vector} = \frac{15(7\hat{i} - 4\hat{j} - 4\hat{k})}{9} = \frac{5}{3}(7\hat{i} - 4\hat{j} - 4\hat{k}).$$

EXAMPLE 6 *Let* $\vec{a} = (\hat{i} + 4\hat{j} + 2\hat{k})$, $\vec{b} = (3\hat{i} - 2\hat{j} + 7\hat{k})$ *and* $\vec{c} = (2\hat{i} - \hat{j} + 4\hat{k})$. *Find a vector* $\vec{d}$ *which is perpendicular to both* $\vec{a}$ *and* $\vec{b}$ *such that* $\vec{c} \cdot \vec{d} = 18$. **[CBSE 2010]**

SOLUTION Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, it follows that $\vec{d}$ is parallel to $(\vec{a} \times \vec{b})$.

Now, $$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$$

$$= (28 + 4)\hat{i} - (7 - 6)\hat{j} + (-2 - 12)\hat{k} = (32\hat{i} - \hat{j} - 14\hat{k}).$$

Since $\vec{d}$ is parallel to $(\vec{a} \times \vec{b})$, we have $\vec{d} = \lambda(\vec{a} \times \vec{b})$ for some scalar λ.

$$\therefore \quad \vec{d} = \lambda(32\hat{i} - \hat{j} - 14\hat{k}) = (32\lambda\hat{i} - \lambda\hat{j} - 14\lambda\hat{k}) \qquad \text{... (i)}$$

Since $\vec{c} \cdot \vec{d} = 18$, we have

$$(2\hat{i} - \hat{j} + 4\hat{k}) \cdot (32\lambda\hat{i} - \lambda\hat{j} - 14\lambda\hat{k}) = 18$$

$$\Rightarrow \quad (64\lambda + \lambda - 56\lambda) = 18 \Rightarrow 9\lambda = 18 \Rightarrow \lambda = 2.$$

Hence, $\vec{d} = (64\hat{i} - 2\hat{j} - 28\hat{k})$ [putting $\lambda = 2$ in (i)].

EXAMPLE 7 *Find a vector of magnitude* 5 *units, perpendicular to each of the vectors* $(\vec{a} + \vec{b})$ *and* $(\vec{a} - \vec{b})$, *where* $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$ *and* $\vec{b} = (\hat{i} + 2\hat{j} + 3\hat{k})$. **[CBSE 2008C]**

SOLUTION We have

$$(\vec{a} + \vec{b}) = (\hat{i} + \hat{j} + \hat{k}) + (\hat{i} + 2\hat{j} + 3\hat{k}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \text{ and}$$

$$(\vec{a} - \vec{b}) = (\hat{i} + \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (-\hat{j} - 2\hat{k}).$$

$$\therefore \quad (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

$$= (-6 + 4)\hat{i} - (-4 - 0)\hat{j} + (-2 - 0)\hat{k}$$

$$= (-2\hat{i} + 4\hat{j} - 2\hat{k}).$$

$$\therefore \quad |(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})| = \sqrt{(-2)^2 + 4^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}.$$

So, the vectors of magnitude 5 units and perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are:

$$\pm\frac{5\{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})\}}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|}=\pm\frac{5(-2\hat{i}+4\hat{j}-2\hat{k})}{2\sqrt{6}}=\pm\frac{5(-\hat{i}+2\hat{j}-\hat{k})}{\sqrt{6}}.$$

EXAMPLE 8 *If* $\vec{a}=4\hat{i}+3\hat{j}+2\hat{k}$ *and* $\vec{b}=3\hat{i}+2\hat{k}$, *find* $|\vec{b}\times 2\vec{a}|$.

SOLUTION We have

$$\vec{b}=(3\hat{i}+2\hat{k}) \text{ and } 2\vec{a}=(8\hat{i}+6\hat{j}+4\hat{k}).$$

$$\therefore\quad (\vec{b}\times 2\vec{a})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 0 & 2\\ 8 & 6 & 4\end{vmatrix}$$

$$=(0-12)\hat{i}+(16-12)\hat{j}+(18-0)\hat{k}$$

$$=(-12\hat{i}+4\hat{j}+18\hat{k}).$$

$$\therefore\quad |\vec{b}\times 2\vec{a}|=|-12\hat{i}+4\hat{j}+18\hat{k}|$$

$$=\sqrt{(-12)^2+4^2+(18)^2}=\sqrt{484}=22.$$

Hence, $|\vec{b}\times 2\vec{a}|=22$.

EXAMPLE 9 *If* $|\vec{a}|=\sqrt{26}$, $|\vec{b}|=7$ *and* $|\vec{a}\times\vec{b}|=35$, *find* $\vec{a}\cdot\vec{b}$.

SOLUTION Given that $|\vec{a}|=\sqrt{26}$ and $|\vec{b}|=7$, and $|\vec{a}\times\vec{b}|=35$.

$$\therefore\quad |\vec{a}\times\vec{b}|=35 \Rightarrow |\vec{a}||\vec{b}|\sin\theta=35$$

$$\Rightarrow \sin\theta=\frac{35}{|\vec{a}||\vec{b}|}=\frac{35}{(\sqrt{26})\times 7}=\frac{5}{\sqrt{26}}.$$

Now, $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-\frac{25}{26}}=\frac{1}{\sqrt{26}}$.

$$\therefore\quad \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta=\left(\sqrt{26}\times 7\times\frac{1}{\sqrt{26}}\right)=7.$$

Hence, $\vec{a}\cdot\vec{b}=7$.

EXAMPLE 10 *If* $|\vec{a}|=2$, $|\vec{b}|=7$ *and* $(\vec{a}\times\vec{b})=(3\hat{i}+2\hat{j}+6\hat{k})$, *find the angle between* $\vec{a}$ *and* $\vec{b}$.

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a} \times \vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$$

$$\Rightarrow \quad |\vec{a}||\vec{b}| \sin\theta = 7 \qquad [\because \ |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta]$$

$$\Rightarrow \quad \sin\theta = \frac{7}{|\vec{a}||\vec{b}|} = \frac{7}{(2 \times 7)} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}.$$

Hence, the required angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$.

EXAMPLE 11 *Find the sine of the angle between the vectors* $\vec{a} = (2\hat{i} - \hat{j} + 3\hat{k})$ *and* $\vec{b} = (\hat{i} + 3\hat{j} + 2\hat{k})$.

SOLUTION We have

$$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= (-2 - 9)\hat{i} - (4 - 3)\hat{j} + (6 + 1)\hat{k}$$

$$= (-11\hat{i} - \hat{j} + 7\hat{k}).$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-11)^2 + (-1)^2 + 7^2} = \sqrt{171} = 3\sqrt{19},$$

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14},$$

$$|\vec{b}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}.$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{3\sqrt{19}}{(\sqrt{14})(\sqrt{14})} = \frac{3}{14}\sqrt{19}.$$

EXAMPLE 12 *If the vectors* $\vec{a}$ *and* $\vec{b}$ *are such that* $|\vec{a}| = 3$, $|\vec{b}| = \frac{2}{3}$ *and* $\vec{a} \times \vec{b}$ *is a unit vector then write the angle between* $\vec{a}$ *and* $\vec{b}$. **[CBSE 2014]**

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$|\vec{a} \times \vec{b}| = 1 \Rightarrow |\vec{a}||\vec{b}| \sin\theta = 1 \Rightarrow \left(3 \times \frac{2}{3}\right) \sin\theta = 1$$

$$\Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = 30°.$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is 30°.

EXAMPLE 13 *Find the area of the parallelogram whose adjacent sides are represented by the vectors* $(3\hat{i} + \hat{j} - 2\hat{k})$ *and* $(\hat{i} - 3\hat{j} + 4\hat{k})$.

SOLUTION Let $\vec{a} = (3\hat{i} + \hat{j} - 2\hat{k})$ and $\vec{b} = (\hat{i} - 3\hat{j} + 4\hat{k})$.

Then, vector area of the ||gm is $(\vec{a} \times \vec{b})$.

$$\text{Now, } (\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$= (4 - 6)\hat{i} - (12 + 2)\hat{j} + (-9 - 1)\hat{k}$$

$$= (-2\hat{i} - 14\hat{j} - 10\hat{k}).$$

Required area $= |\vec{a} \times \vec{b}|$

$$= \sqrt{(-2)^2 + (-14)^2 + (-10)^2} \text{ sq units}$$

$$= \sqrt{300} \text{ sq units} = 10\sqrt{3} \text{ sq units.}$$

EXAMPLE 14 *Find the area of the parallelogram whose diagonals are represented by the vectors* $\vec{d_1} = (2\hat{i} - \hat{j} + \hat{k})$ *and* $\vec{d_2} = (3\hat{i} + 4\hat{j} - \hat{k})$.

SOLUTION Given that $\vec{d_1} = (2\hat{i} - \hat{j} + \hat{k})$ and $\vec{d_2} = (3\hat{i} + 4\hat{j} - \hat{k})$.

Vector area of the ||gm is $\frac{1}{2}(\vec{d_1} \times \vec{d_2})$.

$$\text{Now, } (\vec{d_1} \times \vec{d_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$$= (1 - 4)\hat{i} - (-2 - 3)\hat{j} + (8 + 3)\hat{k}$$

$$= (-3\hat{i} + 5\hat{j} + 11\hat{k}).$$

Required area $= \frac{1}{2}|\vec{d_1} \times \vec{d_2}|$

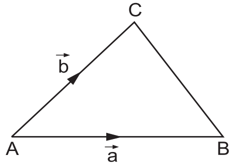
$$= \frac{1}{2}\sqrt{(-3)^2 + 5^2 + (11)^2} \text{ sq units}$$

$$= \frac{1}{2}\sqrt{155} \text{ sq units.}$$

EXAMPLE 15 *Find the area of the triangle whose adjacent sides are determined by the vectors* $\vec{a} = (-2\hat{i} - 5\hat{k})$ *and* $\vec{b} = (\hat{i} - 2\hat{j} - \hat{k})$.

SOLUTION Two adjacent sides of the given triangle are represented by the vectors $\vec{a} = (-2\hat{i} - 5\hat{k})$ and $\vec{b} = (\hat{i} - 2\hat{j} - \hat{k})$.

So, the area of the triangle is $\frac{1}{2}|\vec{a} \times \vec{b}|$.



$$\text{Now, } (\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= (0-10)\hat{i} - (2+5)\hat{j} + (4-0)\hat{k}$$

$$= (-10\hat{i} - 7\hat{j} + 4\hat{k}).$$

$$\therefore \quad \text{required area} = \frac{1}{2}|\vec{a} \times \vec{b}|$$

$$= \frac{1}{2} \cdot \{\sqrt{(-10)^2 + (-7)^2 + 4^2}\} \text{ sq units}$$

$$= \frac{1}{2}\sqrt{165} \text{ sq units.}$$

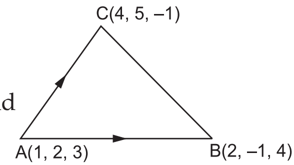
EXAMPLE 16 *Using vectors find the area of* $\triangle ABC$ *whose vertices are* $A(1, 2, 3)$, $B(2, -1, 4)$ *and* $C(4, 5, -1)$. **[CBSE 2013]**

SOLUTION We have

Position vector of $A = (\hat{i} + 2\hat{j} + 3\hat{k})$,

position vector of $B = (2\hat{i} - \hat{j} + 4\hat{k})$ and

position vector of $C = (4\hat{i} + 5\hat{j} - \hat{k})$.



$$\therefore \quad \overrightarrow{AB} = (\text{position vector of } B) - (\text{position vector of } A)$$

$$= (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (\hat{i} - 3\hat{j} + \hat{k}).$$

$$\overrightarrow{AC} = (\text{position vector of } C) - (\text{position vector of } A)$$

$$= (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (3\hat{i} + 3\hat{j} - 4\hat{k}).$$

$$\therefore \quad \text{area of } \triangle ABC = \left|\frac{1}{2}(\overrightarrow{AB} \times \overrightarrow{AC})\right|.$$

$$\text{Now, } \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$$

$$= (12-3)\hat{i} - (-4-3)\hat{j} + (3+9)\hat{k}$$
$$= (9\hat{i} + 7\hat{j} + 12\hat{k}).$$

$\therefore$ area of $\triangle ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= \frac{1}{2}|(9\hat{i} + 7\hat{j} + 12\hat{k})| = \frac{1}{2} \cdot \sqrt{(9)^2 + 7^2 + (12)^2}$$
$$= \frac{1}{2} \cdot \sqrt{(81 + 49 + 144)} = \frac{1}{2}\sqrt{274} \text{ sq units.}$$

Hence, the area of ΔABC is $\frac{1}{2}\sqrt{274}$ sq units.

EXAMPLE 17 *Show that the points whose position vectors are* $(5\hat{i} + 6\hat{j} + 7\hat{k})$, $(7\hat{i} - 8\hat{j} + 9\hat{k})$ *and* $(3\hat{i} + 20\hat{j} + 5\hat{k})$ *are collinear.*

SOLUTION Let the given points be A, B, and C respectively. Then,

$\overrightarrow{AB}$ = (position vector of B) – (position vector of A)

$$= (7\hat{i} - 8\hat{j} + 9\hat{k}) - (5\hat{i} + 6\hat{j} + 7\hat{k}) = (2\hat{i} - 14\hat{j} + 2\hat{k}).$$

And, $\overrightarrow{AC}$ = (position vector of C) – (position vector of A)

$$= (3\hat{i} + 20\hat{j} + 5\hat{k}) - (5\hat{i} + 6\hat{j} + 7\hat{k})$$
$$= (-2\hat{i} + 14\hat{j} - 2\hat{k}).$$

$$\therefore \quad (\overrightarrow{AB} \times \overrightarrow{AC}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -14 & 2 \\ -2 & 14 & -2 \end{vmatrix}$$

$$= 2 \times (-2) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 1 \\ 1 & -7 & 1 \end{vmatrix} = (-4) \times \vec{0} = \vec{0}$$

[$\because$ R_2 and R_3 are identical].

So, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel vectors, having a common end point, A. Hence, the given points A, B and C are collinear.

EXAMPLE 18 *Using vector method, show that the points* $A(2, -1, 3)$, $B(4, 3, 1)$ *and* $C(3, 1, 2)$ *are collinear.*

SOLUTION Clearly, we have

position vector of $A = (2\hat{i} - \hat{j} + 3\hat{k})$, position vector of $B = (4\hat{i} + 3\hat{j} + \hat{k})$, and

position vector of $C = (3\hat{i} + \hat{j} + 2\hat{k})$.

$\therefore \quad \overrightarrow{AB} =$ (position vector of B) − (position vector of A)

$$= (4\hat{i} + 3\hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) = (2\hat{i} + 4\hat{j} - 2\hat{k}).$$

$\overrightarrow{AC} =$ (position vector of C) − (position vector of A)

$$= (3\hat{i} + \hat{j} + 2\hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) = (\hat{i} + 2\hat{j} - \hat{k}).$$

$$\therefore \quad (\overrightarrow{AB} \times \overrightarrow{AC}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -2 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{0} \quad [\because R_2 \text{ and } R_3 \text{ are identical}].$$

Thus, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel vectors, having a common end point, A.

Hence, the given points A, B, and C are collinear.

EXAMPLE 19 *Show that the points having position vectors*

$$(\vec{a} - 2\vec{b} + 3\vec{c}), (-2\vec{a} + 3\vec{b} + 2\vec{c}), (-8\vec{a} + 13\vec{b})$$

are collinear, whatever be $\vec{a}, \vec{b}, \vec{c}$.

SOLUTION Let A, B, C be the given points whose position vectors are $(\vec{a} - 2\vec{b} + 3\vec{c})$, $(-2\vec{a} + 3\vec{b} + 2\vec{c})$ and $(-8\vec{a} + 13\vec{b})$ respectively. Then,

$\overrightarrow{AB} =$ (position vector of B) – (position vector of A)

$$= (-2\vec{a} + 3\vec{b} + 2\vec{c}) - (\vec{a} - 2\vec{b} + 3\vec{c}) = (-3\vec{a} + 5\vec{b} - \vec{c}), \text{ and}$$

$\overrightarrow{AC} =$ (position vector of C) – (position vector of A)

$$= (-8\vec{a} + 13\vec{b}) - (\vec{a} - 2\vec{b} + 3\vec{c}) = (-9\vec{a} + 15\vec{b} - 3\vec{c}).$$

$$\therefore \quad (\overrightarrow{AB} \times \overrightarrow{AC}) = (-3\vec{a} + 5\vec{b} - \vec{c}) \times (-9\vec{a} + 15\vec{b} - 3\vec{c})$$

$$= \vec{p} \times 3\vec{p}, \text{ where } (-3\vec{a} + 5\vec{b} - \vec{c}) = \vec{p}$$

$$= \vec{0} \quad [\because \vec{p} \times \vec{p} = \vec{0}].$$

$\therefore \overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel vectors, having a common end point, A.

Hence, the points A, B and C are collinear.

EXAMPLE 20 *Prove that* $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.

PROOF We have

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$$
$$= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b} \quad \text{[by the distributive law]}$$
$$= \vec{a} \times \vec{b} - \vec{b} \times \vec{a} \qquad [\because \vec{a} \times \vec{a} = \vec{0} \text{ and } \vec{b} \times \vec{b} = \vec{0}]$$
$$= (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{b}) \qquad [\because -\vec{b} \times \vec{a} = (\vec{a} \times \vec{b})]$$
$$= 2(\vec{a} \times \vec{b}).$$

Hence, $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.

EXAMPLE 21 *If* $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ *and* $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, *show that* $(\vec{a} - \vec{d})$ *is parallel to* $(\vec{b} - \vec{c})$, *it being given that* $a \neq d$ *and* $b \neq c$. **[CBSE 2009]**

SOLUTION $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$, and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$

$$\Rightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{c} \times \vec{d} - \vec{b} \times \vec{d}$$
$$\Rightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{c} + \vec{b} \times \vec{d} - \vec{c} \times \vec{d} = \vec{0}$$
$$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) + (\vec{b} - \vec{c}) \times \vec{d} = \vec{0}$$
$$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) - \vec{d} \times (\vec{b} - \vec{c}) = \vec{0}$$
$$\Rightarrow (\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{0}$$
$$\Rightarrow (\vec{a} - \vec{d}) \,||\, (\vec{b} - \vec{c}).$$

Hence, $(\vec{a} - \vec{d})$ is parallel to $(\vec{b} - \vec{c})$.

EXAMPLE 22 *Prove that* $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$.

SOLUTION We have

$$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$$
$$= (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}) + (\vec{b} \times \vec{a}) + (\vec{c} \times \vec{a}) + (\vec{c} \times \vec{b})$$
[by the distributive law]
$$= (\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) - (\vec{a} \times \vec{b}) - (\vec{b} \times \vec{c}) - (\vec{c} \times \vec{a})$$
$$= \vec{0} \quad [\because (\vec{b} \times \vec{a}) = -(\vec{a} \times \vec{b}), (\vec{c} \times \vec{b}) = -(\vec{b} \times \vec{c})$$
$$\text{and } (\vec{a} \times \vec{c}) = -(\vec{c} \times \vec{a})].$$

Hence, $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$.

EXAMPLE 23 *If* $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, *prove that* $(\vec{a} \times \vec{b}) = (\vec{b} \times \vec{c}) = (\vec{c} \times \vec{a})$.

[CBSE 2001, '04C]

SOLUTION $\vec{a} + \vec{b} + \vec{c} = \vec{0} \Rightarrow \vec{a} + \vec{b} = -\vec{c}$

$\Rightarrow (\vec{a} + \vec{b}) \times \vec{b} = (-\vec{c}) \times \vec{b}$

$\Rightarrow (\vec{a} \times \vec{b}) + (\vec{b} \times \vec{b}) = (-\vec{c}) \times \vec{b}$

[by the distributive law]

$\Rightarrow (\vec{a} \times \vec{b}) + \vec{0} = (\vec{b} \times \vec{c})$

[$\because$ $\vec{b} \times \vec{b} = \vec{0}$ and $(-\vec{c}) \times \vec{b} = \vec{b} \times \vec{c}$]

$\Rightarrow \vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ ... (i)

Also, $\vec{a} + \vec{b} + \vec{c} = \vec{0} \Rightarrow \vec{b} + \vec{c} = -\vec{a}$

$\Rightarrow (\vec{b} + \vec{c}) \times \vec{c} = (-\vec{a}) \times \vec{c}$

$\Rightarrow (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{c}) = (-\vec{a}) \times \vec{c}$

[by the distributive law]

$\Rightarrow (\vec{b} \times \vec{c}) + \vec{0} = \vec{c} \times \vec{a}$

[$\because$ $\vec{c} \times \vec{c} = \vec{0}$ and $(-\vec{a}) \times \vec{c} = \vec{c} \times \vec{a}$]

$\Rightarrow \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$ ... (ii)

From (i) and (ii), we get $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

EXAMPLE 24 *Prove that the points A, B, C with position vectors* $\vec{a}, \vec{b}, \vec{c}$ *are collinear if and only if* $(\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) + (\vec{a} \times \vec{b}) = \vec{0}$.

PROOF We have

$\overrightarrow{AB}$ = (position vector of B) − (position vector of A) = $(\vec{b} - \vec{a})$

and $\overrightarrow{BC}$ = (position vector of C) − (position vector of B) = $(\vec{c} - \vec{b})$.

Now, A, B, C are collinear

$\Leftrightarrow$ $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel $\Leftrightarrow$ $(\vec{b} - \vec{a}) \times (\vec{c} - \vec{b}) = \vec{0}$

$\Leftrightarrow (\vec{b} - \vec{a}) \times \vec{c} - (\vec{b} - \vec{a}) \times \vec{b} = \vec{0}$ [by the distributive law]

$\Leftrightarrow \vec{b} \times \vec{c} - \vec{a} \times \vec{c} - \vec{b} \times \vec{b} + \vec{a} \times \vec{b} = \vec{0}$ [by the distributive law]

$\Leftrightarrow (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) + (\vec{a} \times \vec{b}) = \vec{0}$ [$\because$ $\vec{b} \times \vec{b} = \vec{0}$ and $-\vec{a} \times \vec{c} = \vec{c} \times \vec{a}$].

Thus, A, B, C are collinear $\Leftrightarrow (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) + (\vec{a} \times \vec{b}) = \vec{0}$.

EXAMPLE 25 *If* $\vec{a} \cdot \vec{b} = 0$ *and* $\vec{a} \times \vec{b} = \vec{0}$, *prove that* $\vec{a} = \vec{0}$ *or* $\vec{b} = \vec{0}$.

PROOF Let $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$. Then,

$$\vec{a} \cdot \vec{b} = 0 \text{ and } \vec{a} \times \vec{b} = \vec{0}$$

$$\Rightarrow \quad (\vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a} \perp \vec{b}) \text{ and } (\vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a} \,||\, \vec{b})$$

$$\Rightarrow \quad \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0}$$

[$\because$ $\vec{a} \perp \vec{b}$ and $\vec{a} \,||\, \vec{b}$ can never hold simultaneously].

Hence, $(\vec{a} \cdot \vec{b} = 0 \text{ and } \vec{a} \times \vec{b} = \vec{0}) \Rightarrow (\vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0})$.

EXAMPLE 26 *If* $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ *and* $\vec{a} \neq \vec{0}$ *then prove that* $\vec{b} = \vec{c}$.

PROOF $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \neq \vec{0}$

$$\Rightarrow \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = 0 \text{ and } \vec{a} \neq \vec{0} \Rightarrow \vec{a} \cdot (\vec{b} - \vec{c}) = 0 \text{ and } \vec{a} \neq \vec{0}$$

$$\Rightarrow \vec{b} - \vec{c} = \vec{0} \text{ or } \vec{a} \perp (\vec{b} - \vec{c}) \Rightarrow \vec{b} = \vec{c} \text{ or } \vec{a} \perp (\vec{b} - \vec{c}). \quad \text{... (i)}$$

Again, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ and $\vec{a} \neq \vec{0}$

$$\Rightarrow (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) = \vec{0} \text{ and } \vec{a} \neq \vec{0} \Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = \vec{0} \text{ and } \vec{a} \neq \vec{0}$$

$$\Rightarrow (\vec{b} - \vec{c}) = \vec{0} \text{ or } \vec{a} \,||\, (\vec{b} - \vec{c}) \Rightarrow \vec{b} = \vec{c} \text{ or } \vec{a} \,||\, (\vec{b} - \vec{c}). \quad \text{... (ii)}$$

From (i) and (ii), we get $\vec{b} = \vec{c}$

[$\because$ $\vec{a} \perp (\vec{b} - \vec{c})$ and $\vec{a} \,||\, (\vec{b} - \vec{c})$ both cannot hold simultaneously].

Lagrange's Identity

EXAMPLE 27 *Prove that* $|\vec{a} \times \vec{b}|^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}$. **[CBSE 2002C, '04]**

SOLUTION Let θ be the angle betwen $\vec{a}$ and $\vec{b}$. Then,

$$|\vec{a} \times \vec{b}|^2 = (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) = (ab \sin\theta)\hat{n} \cdot (ab \sin\theta)\hat{n}$$

$$= (a^2 b^2 \sin^2\theta)(\hat{n} \cdot \hat{n}) = a^2 b^2 \sin^2\theta$$

$$= a^2 b^2 (1 - \cos^2\theta) = a^2 b^2 - (ab\cos\theta)^2$$

$$= (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}.$$

Hence, $|\vec{a} \times \vec{b}|^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}$.

EXERCISE 24

1. Find $(\vec{a} \times \vec{b})$ and $|\vec{a} \times \vec{b}|$, when
   (i) $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 4\hat{k}$
   (ii) $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$
   (iii) $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$
   (iv) $\vec{a} = 4\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 3\hat{i} + \hat{k}$
   (v) $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$

2. Find λ if $(2\hat{i} + 6\hat{j} + 14\hat{k}) \times (\hat{i} - \lambda\hat{j} + 7\hat{k}) = \vec{0}$. **[CBSE 2010]**

3. If $\vec{a} = (-3\hat{i} + 4\hat{j} - 7\hat{k})$ and $\vec{b} = (6\hat{i} + 2\hat{j} - 3\hat{k})$, find $(\vec{a} \times \vec{b})$.
   Verify that (i) $\vec{a}$ and $(\vec{a} \times \vec{b})$ are perpendicular to each other
   and (ii) $\vec{b}$ and $(\vec{a} \times \vec{b})$ are perpendicular to each other.

4. Find the value of:
   (i) $(\hat{i} \times \hat{j}) \cdot \hat{k} + \hat{i} \cdot \hat{j}$ (ii) $(\hat{k} \times \hat{j}) \cdot \hat{i} + \hat{j} \cdot \hat{k}$ **[CBSE 2012]**
   (iii) $\hat{i} \times (\hat{j} + \hat{k}) + \hat{j} \times (\hat{k} + \hat{i}) + \hat{k} \times (\hat{i} + \hat{j})$ **[CBSE 2014]**

5. Find the unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$ when
   (i) $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$
   (ii) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$
   (iii) $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = -\hat{i} + 3\hat{k}$
   (iv) $\vec{a} = 4\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 4\hat{j} - \hat{k}$.

6. Find the unit vectors perpendicular to the plane of the vectors $\vec{a} = 2\hat{i} - 6\hat{j} - 3\hat{k}$ and $\vec{b} = 4\hat{i} + 3\hat{j} - \hat{k}$.

7. Find a vector of magnitude 6 which is perpendicular to both the vectors $\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$.

8. Find a vector of magnitude 5 units, perpendiuclar to each of the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, where $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$ and $\vec{b} = (\hat{i} + 2\hat{j} + 3\hat{k})$.
   [CBSE 2008C]

9. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and $|\vec{a} \times \vec{b}| = \sqrt{3}$. **[CBSE 2009]**

10. If $\vec{a} = (\hat{i} - \hat{j})$, $\vec{b} = (3\hat{j} - \hat{k})$ and $\vec{c} = (7\hat{i} - \hat{k})$, find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and for which $\vec{c} \cdot \vec{d} = 1$. **[CBSE 2010]**

11. If $\vec{a} = (4\hat{i} + 5\hat{j} - \hat{k})$, $\vec{b} = (\hat{i} - 4\hat{j} + 5\hat{k})$, and $\vec{c} = (3\hat{i} + \hat{j} - \hat{k})$, find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and for which $\vec{c} \cdot \vec{d} = 21$.

12. Prove that $|\vec{a} \times \vec{b}| = (\vec{a} \cdot \vec{b}) \tan\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

13. Write the value of p for which $\vec{a} = (3\hat{i} + 2\hat{j} + 9\hat{k})$ and $\vec{b} = (\hat{i} + p\hat{j} + 3\hat{k})$ are parallel vectors. **[CBSE 2009]**

14. Verify that $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$, when
    (i) $\vec{a} = \hat{i} - \hat{j} - 3\hat{k}$, $\vec{b} = 4\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 2\hat{k}$
    (ii) $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} + \hat{k}$.

15. Find the area of the parallelogram whose adjacent sides are represented by the vectors
    (i) $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = -3\hat{i} - 2\hat{j} + \hat{k}$
    (ii) $\vec{a} = (3\hat{i} + \hat{j} + 4\hat{k})$ and $\vec{b} = (\hat{i} - \hat{j} + \hat{k})$
    (iii) $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} - \hat{j}$
    (iv) $\vec{a} = 2\hat{i}$ and $\vec{b} = 3\hat{j}$.

16. Find the area of the parallelogram whose diagonals are represented by the vectors
    (i) $\vec{d}_1 = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{d}_2 = \hat{i} - 3\hat{j} + 4\hat{k}$ **[CBSE 2004]**
    (ii) $\vec{d}_1 = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{d}_2 = 3\hat{i} + 4\hat{j} - \hat{k}$
    (iii) $\vec{d}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{d}_2 = -\hat{i} + 2\hat{j}$.

17. Find the area of the triangle whose two adjacent sides are determined by the vectors
    (i) $\vec{a} = -2\hat{i} - 5\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} - \hat{k}$
    (ii) $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = -5\hat{i} + 7\hat{j}$.

18. Using vectors, find the area of $\triangle ABC$ whose vertices are
    (i) $A(1, 1, 2)$, $B(2, 3, 5)$ and $C(1, 5, 5)$ **[CBSE 2011]**
    (ii) $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$ **[CBSE 2013]**
    (iii) $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$
    (iv) $A(1, -1, 2)$, $B(2, 1, -1)$ and $C(3, -1, 2)$.

19. Using vector method, show that the given points A, B, C are collinear:
    (i) $A(3, -5, 1)$, $B(-1, 0, 8)$ and $C(7, -10, -6)$
    (ii) $A(6, -7, -1)$, $B(2, -3, 1)$ and $C(4, -5, 0)$.

20. Show that the points A, B, C with position vectors $(3\hat{i} - 2\hat{j} + 4\hat{k})$, $(\hat{i} + \hat{j} + \hat{k})$ and $(-\hat{i} + 4\hat{j} - 2\hat{k})$ respectively are collinear.

21. Show that the points having position vectors $\vec{a}$, $\vec{b}$, $(\vec{c} = 3\vec{a} - 2\vec{b})$ are collinear, whatever be $\vec{a}, \vec{b}, \vec{c}$.

22. Show that the points having position vectors $(-2\vec{a} + 3\vec{b} + 5\vec{c})$, $(\vec{a} + 2\vec{b} + 3\vec{c})$ and $(7\vec{a} - \vec{c})$ are collinear, whatever be $\vec{a}, \vec{b}, \vec{c}$.

23. Find a unit vector perpendicular to the plane ABC, where the points A, B, C are $(3, -1, 2)$, $(1, -1, -3)$ and $(4, -3, 1)$ respectively.

24. If $\vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k})$ and $\vec{b} = (\hat{i} - 3\hat{k})$ then find $|\vec{b} \times 2\vec{a}|$.

25. If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, find $\vec{a} \cdot \vec{b}$.

26. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $(\vec{a} \times \vec{b}) = (3\hat{i} + 2\hat{j} + 6\hat{k})$, find the angle between $\vec{a}$ and $\vec{b}$.

ANSWERS (EXERCISE 24)

1. (i) $\vec{a} \times \vec{b} = (-2\hat{i} + 8\hat{j} + 5\hat{k})$ and $|\vec{a} \times \vec{b}| = \sqrt{93}$

(ii) $\vec{a} \times \vec{b} = (-17\hat{i} + 13\hat{j} + 7\hat{k})$ and $|\vec{a} \times \vec{b}| = 13\sqrt{3}$

(iii) $\vec{a} \times \vec{b} = 19(\hat{j} + \hat{k})$ and $|\vec{a} \times \vec{b}| = 19\sqrt{2}$

(iv) $\vec{a} \times \vec{b} = (\hat{i} - 10\hat{j} - 3\hat{k})$ and $|\vec{a} \times \vec{b}| = \sqrt{110}$

(v) $\vec{a} \times \vec{b} = (4\hat{i} - 3\hat{j} - \hat{k})$ and $|\vec{a} \times \vec{b}| = \sqrt{26}$

2. $\lambda = -3$ **3.** $(2\hat{i} - 51\hat{j} - 30\hat{k})$ **4.** (i) 1 (ii) –1 (iii) 0

5. (i) $\pm \frac{1}{5\sqrt{3}}(5\hat{i} - \hat{j} + 7\hat{k})$ (ii) $\pm \frac{1}{\sqrt{3}}(-\hat{i} + \hat{j} + \hat{k})$

(iii) $\pm \frac{1}{\sqrt{91}}(9\hat{i} - \hat{j} + 3\hat{k})$ (iv) $\pm \frac{1}{\sqrt{209}}(2\hat{i} + 3\hat{j} + 14\hat{k})$

6. $\pm \frac{1}{7}(3\hat{i} - 2\hat{j} + 6\hat{k})$ **7.** $\pm 2(-\hat{i} + 2\hat{j} + 2\hat{k})$ **8.** $\pm \frac{5(-\hat{i} + 2\hat{j} - \hat{k})}{\sqrt{6}}$

9. $\frac{\pi}{3}$ **10.** $\vec{d} = \frac{1}{4}(\hat{i} + \hat{j} + 3\hat{k})$ **11.** $\vec{d} = 7(\hat{i} - \hat{j} - \hat{k})$ **13.** $p = \frac{2}{3}$

15. (i) $6\sqrt{5}$ sq units (ii) $\sqrt{42}$ sq units
(iii) $3\sqrt{3}$ sq units (iv) 6 sq units

16. (i) $5\sqrt{3}$ sq units (ii) $\frac{1}{2}\sqrt{155}$ sq units (iii) $\frac{1}{2}\sqrt{21}$ sq units

17. (i) $\frac{1}{2}\sqrt{165}$ sq units (ii) $\frac{41}{2}$ sq units

18. (i) $\frac{1}{2}\sqrt{61}$ sq units (ii) $\frac{1}{2}\sqrt{274}$ sq units (iii) $\frac{1}{2}\sqrt{165}$ sq units
(iv) $\sqrt{13}$ sq units

23. $\frac{-\hat{i} - 2\hat{j} + 4\hat{k}}{\sqrt{21}}$ **24.** $4\sqrt{29}$ **25.** 6 **26.** $\frac{\pi}{6}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 24)

4. (i) $(\hat{i} \times \hat{j}) \cdot \hat{k} + \hat{i} \cdot \hat{j} = \hat{k} \cdot \hat{k} + 0 = (1 + 0) = 1$ $[\because \hat{i} \times \hat{j} = \hat{k}$ and $\hat{i} \cdot \hat{j} = 0]$.

(ii) $(\hat{k} \times \hat{j}) \cdot \hat{i} + \hat{j} \cdot \hat{k} = -\hat{i} \cdot \hat{i} + 0 = (-1 + 0) = -1.$

5. Required vector $= \frac{\pm(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

6. Required vector $= \frac{\pm(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

7. Required vector $= \frac{\pm 6(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

8. Required vector $= \frac{\pm 5[(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})]}{|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})|}$.

9. $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta = (1 \times 2 \times \sin\theta)$

$\therefore \; 2\sin\theta = \sqrt{3} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$.

10. Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, so it is parallel to $(\vec{a} \times \vec{b})$.

$\therefore \; \vec{d} = \lambda(\vec{a} \times \vec{b}) = \lambda(\hat{i} + \hat{j} + 3\hat{k}) = (\lambda\hat{i} + \lambda\hat{j} + 3\lambda\hat{k})$

$$\vec{c}\cdot\vec{d}=1 \Rightarrow (7\hat{i}-\hat{k})\cdot(\lambda\hat{i}+\lambda\hat{j}+3\lambda\hat{k})=1$$

$$\Rightarrow 7\lambda-3\lambda=1 \Rightarrow 4\lambda=1 \Rightarrow \lambda=\frac{1}{4}\cdot$$

$$\therefore \quad \vec{d}=\frac{1}{4}(\hat{i}+\hat{j}+3\hat{k}).$$

12. $(\vec{a}\times\vec{b})=|\vec{a}||\vec{b}|\sin\theta\cdot\hat{n}$

$\Rightarrow |\vec{a}\times\vec{b}|=ab\sin\theta$, where $|\vec{a}|=a$ and $|\vec{b}|=b$

$$=(ab\cos\theta)(\tan\theta)=(\vec{a}\cdot\vec{b})\tan\theta.$$

13. $\vec{a}$ and $\vec{b}$ are parallel vectors $\Leftrightarrow \vec{a}\times\vec{b}=\vec{0}$.

15. (iv) Area $=|\vec{a}\times\vec{b}|=|2\hat{i}\times3\hat{j}|=|6(\hat{i}\times\hat{j})|=|6\hat{k}|=6 \quad [\because (\hat{i}\times\hat{j})=\hat{k}]$.

16. Area of a ||gm $=\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$.

17. Area of a triangle $=\frac{1}{2}|\vec{a}\times\vec{b}|$, where $\vec{a}$ and $\vec{b}$ are adjacent sides.

23. Required vector $=\dfrac{(\overrightarrow{AB}\times\overrightarrow{AC})}{|\overrightarrow{AB}\times\overrightarrow{AC}|}\cdot$

25. $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta \Rightarrow 2\times5\times\sin\theta=8 \Rightarrow \sin\theta=\frac{4}{5}$

$$\Rightarrow \cos\theta=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}\cdot$$

$$\therefore \quad \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta=\left(2\times5\times\frac{3}{5}\right)=6.$$

26. $\vec{a}\times\vec{b}=(3\hat{i}+2\hat{j}+6\hat{k})$

$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{3^2+2^2+6^2}=7 \Rightarrow |\vec{a}||\vec{b}|\sin\theta=7$

$\Rightarrow \sin\theta=\dfrac{7}{|\vec{a}||\vec{b}|}=\dfrac{7}{(2\times7)}=\dfrac{1}{2} \Rightarrow \theta=30°.$

25. PRODUCT OF THREE VECTORS

SCALAR TRIPLE PRODUCT *The scalar triple product of three vectors* $\vec{a}, \vec{b}, \vec{c}$ *is defined as* $(\vec{a} \times \vec{b}) \cdot \vec{c}$, *and it is denoted by* $[\vec{a}\ \vec{b}\ \vec{c}]$.

Thus $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

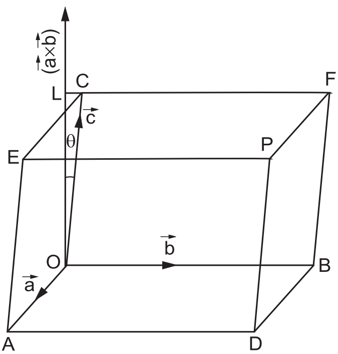
Since $(\vec{a} \times \vec{b})$ is a vector, $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is a scalar.

Consequently, $[\vec{a}\ \vec{b}\ \vec{c}]$ is a scalar quantity.

Some Theorems on Scalar Triple Product

THEOREM 1 (Geometrical interpretation) $(\vec{a} \times \vec{b}) \cdot \vec{c}$ *represents the volume of a parallelepiped whose coterminous edges are represented by* $\vec{a}, \vec{b}, \vec{c}$.

PROOF Let us consider a parallelepiped having coterminous edges OA, OB and OC respectively.



Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$.

Then, $(\vec{a} \times \vec{b})$ is a vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

Let θ be the angle between $(\vec{a} \times \vec{b})$ and $\vec{c}$.

$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

$= |\vec{a} \times \vec{b}|\, |\vec{c}| \cos\theta$

= (area of ||gm $OADB$)[projection of $\vec{c}$ along $(\vec{a} \times \vec{b})$]

= (area of ||gm $OADB$) · OL

= (area of the base of the parallelepiped) · (height)

= volume of the parallelepiped with coterminous edges $\vec{a}, \vec{b}, \vec{c}$.

Hence $[\vec{a}\ \vec{b}\ \vec{c}]$ represents the volume of the parallelepiped with coterminous edges $\vec{a}, \vec{b}, \vec{c}$ forming a right-handed system.

THEOREM 2 *For any three vectors* $\vec{a}, \vec{b}, \vec{c}$ *prove that* $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$.

PROOF Let $\vec{a}, \vec{b}, \vec{c}$ form a right-handed system, representing the coterminous edges of a parallelepiped of volume V.

Then, $V = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

Again, $\vec{b}, \vec{c}, \vec{a}$ form a right-handed system, representing the coterminous edges of the same parallelepiped.

$\therefore \quad V = (\vec{b} \times \vec{c}) \cdot \vec{a} = \vec{a} \cdot (\vec{b} \times \vec{c}) \quad [\because \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}]$.

Hence, $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$.

THEOREM 3 *The scalar triple product of three vectors remains unchanged so long as their cyclic order remains unchanged,*

i.e., $\quad (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$.

PROOF Let $\vec{a}, \vec{b}, \vec{c}$ form a right-handed system, representing the coterminous edges of a rectangular parallelepiped of volume V.

Then, $V = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

Clearly, $\vec{b}, \vec{c}, \vec{a}$ as well as $\vec{c}, \vec{a}, \vec{b}$ form a right-handed system, representing the coterminous edges of the same parallelepiped.

$\therefore \quad V = (\vec{b} \times \vec{c}) \cdot \vec{a}$ and $V = (\vec{c} \times \vec{a}) \cdot \vec{b}$.

Thus, $\quad (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b} \quad$ [each equal to V].

Hence, $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$.

THEOREM 4 *The scalar triple product changes in sign but not in magnitude when the cyclic order of vectors is changed.*

PROOF For any three vectors $\vec{a}, \vec{b}, \vec{c}$, we know that

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}].$$

$\therefore \quad [\vec{c}\ \vec{b}\ \vec{a}] = (\vec{c} \times \vec{b}) \cdot \vec{a} = -(\vec{b} \times \vec{c}) \cdot \vec{a} = -[\vec{b}\ \vec{c}\ \vec{a}] = -[\vec{a}\ \vec{b}\ \vec{c}]$.

And, $[\vec{a}\ \vec{c}\ \vec{b}] = (\vec{a} \times \vec{c}) \cdot \vec{b} = -(\vec{c} \times \vec{a}) \cdot \vec{b} = -[\vec{c}\ \vec{a}\ \vec{b}] = -[\vec{a}\ \vec{b}\ \vec{c}]$.

$\therefore \quad [\vec{c}\ \vec{b}\ \vec{a}] = -[\vec{a}\ \vec{b}\ \vec{c}]$ and $[\vec{a}\ \vec{c}\ \vec{b}] = -[\vec{a}\ \vec{b}\ \vec{c}]$.

THEOREM 5 *The scalar triple product vanishes if any two of its vectors are equal, i.e.,* $[\vec{a}\ \vec{a}\ \vec{b}] = 0$, $[\vec{a}\ \vec{b}\ \vec{a}] = 0$ *and* $[\vec{b}\ \vec{a}\ \vec{a}] = 0$.

PROOF We have $\quad [\vec{a}\ \vec{a}\ \vec{b}] = (\vec{a} \times \vec{a}) \cdot \vec{b} = \vec{0} \cdot \vec{b} = 0$;

$$[\vec{a}\ \vec{b}\ \vec{a}] = [\vec{a}\ \vec{a}\ \vec{b}] \quad \text{[cyclic-ordered property]}$$

$$= (\vec{a} \times \vec{a}) \cdot \vec{b} = \vec{0} \cdot \vec{b} = 0;$$

and $[\vec{b}\ \vec{a}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{a}]$ [cyclic-ordered property]

$$= 0.$$

Hence, $[\vec{a}\ \vec{a}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{a}] = [\vec{b}\ \vec{a}\ \vec{a}] = 0.$

THEOREM 6 *The scalar triple product vanishes if any two of its vectors are parallel or collinear.*

PROOF Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $\vec{a} \parallel \vec{b}$, or $\vec{a}$ and $\vec{b}$ are collinear.

Then, $\vec{a} = m\vec{b}$ for some scalar m.

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = [m\vec{b}\ \vec{b}\ \vec{c}] = (m\vec{b} \times \vec{b}) \cdot \vec{c} = \vec{0} \cdot \vec{c} = 0.$$

THEOREM 7 (Scalar triple product in terms of components)

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ *and* $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

Then, $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

PROOF We have $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}.$$

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

THEOREM 8 *For any three vectors* $\vec{a}$, $\vec{b}$ *and* $\vec{c}$, *prove that*

$$[\vec{a} + \vec{b},\ \vec{b} + \vec{c},\ \vec{c} + \vec{a}] = 2[\vec{a},\ \vec{b},\ \vec{c}]$$ **[CBSE 2014]**

PROOF We have

$$[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}]$$

$$=(\vec{a}+\vec{b})\cdot[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$$

$$=(\vec{a}+\vec{b})\cdot[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}] \text{ [by the distributive law]}$$

$$=(\vec{a}+\vec{b})\cdot[(\vec{b}\times\vec{c})-(\vec{a}\times\vec{b})+(\vec{c}\times\vec{a})]$$

$$[\because \vec{c}\times\vec{c}=\vec{0}, \text{ and } \vec{b}\times\vec{a}=-\vec{a}\times\vec{b}]$$

$$=\vec{a}\cdot(\vec{b}\times\vec{c})-\vec{a}\cdot(\vec{a}\times\vec{b})+\vec{a}\cdot(\vec{c}\times\vec{a})+\vec{b}\cdot(\vec{b}\times\vec{c})$$

$$-\vec{b}\cdot(\vec{a}\times\vec{b})+\vec{b}\cdot(\vec{c}\times\vec{a}) \text{ [by the distributive law]}$$

$$=[\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{a}\ \vec{b}]+[\vec{a}\ \vec{c}\ \vec{a}]+[\vec{b}\ \vec{b}\ \vec{c}]-[\vec{b}\ \vec{a}\ \vec{b}]+[\vec{b}\ \vec{c}\ \vec{a}]$$

$$=[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}]$$

[∵ scalar triple product with two equal vectors is 0]

$$=2[\vec{a}\ \vec{b}\ \vec{c}] \ \{\because [\vec{b}\ \vec{c}\ \vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]\}.$$

Hence, $[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}]=2[\vec{a}\ \vec{b}\ \vec{c}]$.

THEOREM 9 *The necessary and sufficient condition for three nonzero, noncollinear vectors* $\vec{a}, \vec{b}, \vec{c}$ *to be coplanar is that* $[\vec{a}\ \vec{b}\ \vec{c}]=0$.

PROOF Let $\vec{a}, \vec{b}, \vec{c}$ be three nonzero, noncollinear and coplanar vectors. Then,

$\vec{b}\times\vec{c}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$

$\Rightarrow \quad (\vec{b}\times\vec{c})\perp\vec{a} \qquad [\because \vec{a}, \vec{b}, \vec{c}$ are coplanar]

$\Rightarrow \quad \vec{a}\cdot(\vec{b}\times\vec{c})=0 \Rightarrow [\vec{a}\ \vec{b}\ \vec{c}]=0.$

Thus, whenever $\vec{a}, \vec{b}, \vec{c}$ are coplanar, we have $[\vec{a}\ \vec{b}\ \vec{c}]=0$.

Conversely, let $\vec{a}, \vec{b}, \vec{c}$ be three nonzero, noncollinear vectors such that $[\vec{a}\ \vec{b}\ \vec{c}]=\vec{0}$. Then,

$$[\vec{a}\ \vec{b}\ \vec{c}]=0 \Rightarrow \vec{a}\cdot(\vec{b}\times\vec{c})=0$$

$$\Rightarrow \vec{a}=\vec{0}, \text{ or } (\vec{b}\times\vec{c})=\vec{0}, \text{ or } (\vec{b}\times\vec{c})\perp\vec{a}$$

$$\Rightarrow (\vec{b}\times\vec{c})=\vec{0}, \text{ or } (\vec{b}\times\vec{c})\perp\vec{a} \qquad [\because \vec{a}\neq\vec{0}]$$

$$\Rightarrow (\vec{b}\times\vec{c})\perp\vec{a}$$

$[\because \vec{b}\neq\vec{0}, \ \vec{c}\neq\vec{0}$, and $\vec{b}, \vec{c}$ are noncollinear $\Rightarrow \vec{b}\times\vec{c}\neq\vec{0}]$.

Thus, $(\vec{b} \times \vec{c})$ is perpendicular to $\vec{a}$.

But, $(\vec{b} \times \vec{c})$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

$\therefore$ $\vec{a}$ must lie in the plane of $\vec{b}$ and $\vec{c}$.

Hence $\vec{a}, \vec{b}, \vec{c}$ must be coplanar.

NOTE: $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow$ $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

THEOREM 10 *For any three vectors* $\vec{a}, \vec{b}, \vec{c}$ *show that the vectors* $(\vec{a} - \vec{b}), (\vec{b} - \vec{c}), (\vec{c} - \vec{a})$ *are coplanar.* **[CBSE 2001]**

PROOF We know that

$(\vec{a} - \vec{b}), (\vec{b} - \vec{c}), (\vec{c} - \vec{a})$ are coplanar

$\Leftrightarrow$ $[\vec{a} - \vec{b}\quad \vec{b} - \vec{c}\quad \vec{c} - \vec{a}] = 0$

Now, $[\vec{a} - \vec{b}\quad \vec{b} - \vec{c}\quad \vec{c} - \vec{a}]$

$$= (\vec{a} - \vec{b}) \cdot [(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})]$$

$$= (\vec{a} - \vec{b}) \cdot [\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{c} \times \vec{c} + \vec{c} \times \vec{a}] \text{ [by the distributive law]}$$

$$= (\vec{a} - \vec{b}) \cdot [\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}] \quad [\because -\vec{b} \times \vec{a} = \vec{a} \times \vec{b}, \text{ and } \vec{c} \times \vec{c} = \vec{0}]$$

$$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{a} \times \vec{b}) + \vec{a} \cdot [\vec{c} \times \vec{a}] - \vec{b} \cdot (\vec{b} \times \vec{c}) - \vec{b} \cdot (\vec{a} \times \vec{b}) - \vec{b} \cdot (\vec{c} \times \vec{a})$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{a}\ \vec{b}] + [\vec{a}\ \vec{c}\ \vec{a}] - [\vec{b}\ \vec{b}\ \vec{c}] - [\vec{b}\ \vec{a}\ \vec{b}] - [\vec{b}\ \vec{c}\ \vec{a}]$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}] \quad [\because \text{scalar triple product with two equal vectors is } 0]$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{c}] = 0 \quad \{\because [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]\}.$$

Hence, $(\vec{a} - \vec{b}), (\vec{b} - \vec{c}), (\vec{c} - \vec{a})$ are coplanar.

THEOREM 11 *Show that the vectors* $\vec{a}, \vec{b}, \vec{c}$ *are coplanar if and only if* $(\vec{a} + \vec{b}), (\vec{b} + \vec{c}), (\vec{c} + \vec{a})$ *are coplanar.* **[CBSE 2013C, '14]**

PROOF $(\vec{a} + \vec{b}), (\vec{b} + \vec{c}), (\vec{c} + \vec{a})$ are coplanar

$$\Leftrightarrow [\vec{a} + \vec{b}\quad \vec{b} + \vec{c}\quad \vec{c} + \vec{a}] = 0$$

$$\Leftrightarrow (\vec{a} + \vec{b}) \cdot \{(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})\} = 0$$

$$\Leftrightarrow (\vec{a} + \vec{b}) \cdot \{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}\} = 0$$

$$\Leftrightarrow (\vec{a}+\vec{b})\cdot\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\}=0 \qquad [\because \vec{c}\times\vec{c}=\vec{0}\,]$$

$$\Leftrightarrow \vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})$$

$$+\vec{b}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{c}\times\vec{a})=0$$

$$\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}\,]+[\vec{a}\ \vec{b}\ \vec{a}\,]+[\vec{a}\ \vec{c}\ \vec{a}\,]+[\vec{b}\ \vec{b}\ \vec{c}\,]$$

$$+[\vec{b}\ \vec{b}\ \vec{a}\,]+[\vec{b}\ \vec{c}\ \vec{a}\,]=0$$

$$\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}\,]+[\vec{b}\ \vec{c}\ \vec{a}\,]=0$$

[∵ scalar triple product with two equal vectors is 0]

$$\Leftrightarrow 2[\vec{a}\ \vec{b}\ \vec{c}\,]=0 \qquad \{\because [\vec{b}\ \vec{c}\ \vec{a}\,]=[\vec{a}\ \vec{b}\ \vec{c}\,]\}$$

$$\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}\,]=0$$

$$\Leftrightarrow \vec{a},\ \vec{b},\ \vec{c} \text{ are coplanar.}$$

Hence, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar if and only if $(\vec{a}+\vec{b})$, $(\vec{b}+\vec{c})$ and $(\vec{c}+\vec{a})$ are coplanar.

<u>THEOREM 12</u> *For any three vectors* $\vec{a}, \vec{b}, \vec{c}$, *prove that* $[\vec{a}\quad \vec{b}+\vec{c}\quad \vec{a}+\vec{b}+\vec{c}\,]=0.$

PROOF We have

$$[\vec{a}\quad \vec{b}+\vec{c}\quad \vec{a}+\vec{b}+\vec{c}\,]$$

$$=\{\vec{a}\times(\vec{b}+\vec{c})\}\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$=\{(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})\}\cdot(\vec{a}+\vec{b}+\vec{c}) \quad \text{[by the distributive law]}$$

$$=(\vec{a}\times\vec{b})\cdot\vec{a}+(\vec{a}\times\vec{b})\cdot\vec{b}+(\vec{a}\times\vec{b})\cdot\vec{c}+(\vec{a}\times\vec{c})\cdot\vec{a}$$

$$+(\vec{a}\times\vec{c})\cdot\vec{b}+(\vec{a}\times\vec{c})\cdot\vec{c} \quad \text{[by the distributive law]}$$

$$=[\vec{a}\ \vec{b}\ \vec{a}\,]+[\vec{a}\ \vec{b}\ \vec{b}\,]+[\vec{a}\ \vec{b}\ \vec{c}\,]+[\vec{a}\ \vec{c}\ \vec{a}\,]$$

$$+[\vec{a}\ \vec{c}\ \vec{b}\,]+[\vec{a}\ \vec{c}\ \vec{c}\,]$$

$$=[\vec{a}\ \vec{b}\ \vec{c}\,]+[\vec{a}\ \vec{c}\ \vec{b}\,]$$

[∵ scalar triple product with two equal vectors is 0]

$$=[\vec{a}\ \vec{b}\ \vec{c}\,]-[\vec{a}\ \vec{b}\ \vec{c}\,]=0 \quad \{\because [\vec{a}\ \vec{c}\ \vec{b}\,]=-[\vec{a}\ \vec{b}\ \vec{c}\,]\}.$$

Hence, $[\vec{a}\quad \vec{b}+\vec{c}\quad \vec{a}+\vec{b}+\vec{c}\,]=0.$

<u>THEOREM 13</u> *If* $\vec{a}, \vec{b}, \vec{c}$ *are the position vectors of points A, B, C, prove that* $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$ *is a vector perpendicular to the plane of triangle ABC.* **[CBSE 2001C]**

PROOF In order to prove the required result, we have to show that $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$ is perpendicular to each of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

We have, $\overrightarrow{AB}=(\vec{b}-\vec{a})$, $\overrightarrow{BC}=(\vec{c}-\vec{b})$ and $\overrightarrow{CA}=(\vec{a}-\vec{c})$.

Now, $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})\cdot(\vec{b}-\vec{a})$

$$=(\vec{a}\times\vec{b})\cdot(\vec{b}-\vec{a})+(\vec{b}\times\vec{c})\cdot(\vec{b}-\vec{a})+(\vec{c}\times\vec{a})\cdot(\vec{b}-\vec{a})$$

$$=(\vec{a}\times\vec{b})\cdot\vec{b}-(\vec{a}\times\vec{b})\cdot\vec{a}+(\vec{b}\times\vec{c})\cdot\vec{b}-(\vec{b}\times\vec{c})\cdot\vec{a}+(\vec{c}\times\vec{a})\cdot\vec{b}-(\vec{c}\times\vec{a})\cdot\vec{a}$$

$$=[\vec{a}\ \vec{b}\ \vec{b}]-[\vec{a}\ \vec{b}\ \vec{a}]+[\vec{b}\ \vec{c}\ \vec{b}]-[\vec{b}\ \vec{c}\ \vec{a}]+[\vec{c}\ \vec{a}\ \vec{b}]-[\vec{c}\ \vec{a}\ \vec{a}]$$

$$=[\vec{c}\ \vec{a}\ \vec{b}]-[\vec{b}\ \vec{c}\ \vec{a}]$$

[∵ scalar triple product with two equal vectors is 0]

$$=0\quad (\because\ [\vec{c}\ \vec{a}\ \vec{b}]=[\vec{b}\ \vec{c}\ \vec{a}]).$$

Similarly, $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})\cdot(\vec{c}-\vec{b})=0.$

And, $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})\cdot(\vec{a}-\vec{c})=0.$

Thus, $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$ is perpendicular to each one of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$, and therefore, it is perpendicular to the plane of $\triangle ABC$.

SOLVED EXAMPLES

EXAMPLE 1 *Prove that* $[\hat{i}\ \hat{j}\ \hat{k}]=1$, *and* $[\hat{i}\ \hat{k}\ \hat{j}]=-1$.

SOLUTION We have

$$[\hat{i}\ \hat{j}\ \hat{k}]=(\hat{i}\times\hat{j})\cdot\hat{k}=\hat{k}\cdot\hat{k}=1 \text{ and}$$

$$[\hat{i}\ \hat{k}\ \hat{j}]=(\hat{i}\times\hat{k})\cdot\hat{j}=-\hat{j}\cdot\hat{j}=-1.$$

EXAMPLE 2 *If* $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$, *and* $\vec{c}=-3\hat{i}+\hat{j}+2\hat{k}$, *find* $[\vec{a}\ \vec{b}\ \vec{c}]$.

SOLUTION We have

$$[\vec{a}\ \vec{b}\ \vec{c}]=\begin{vmatrix}2 & 1 & 3\\ -1 & 2 & 1\\ -3 & 1 & 2\end{vmatrix}=\begin{vmatrix}0 & 5 & 5\\ -1 & 2 & 1\\ 0 & -5 & -1\end{vmatrix}$$

$[R_1\to R_1+2R_2,\ R_3\to R_3-3R_2]$

$$=-(-1)\cdot[-5+25]=20.$$

EXAMPLE 3 *Find the volume of the parallelepiped whose coterminous edges are represented by the vectors*

$$\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k},\ \ \vec{b} = \hat{i} - \hat{j} + 2\hat{k} \text{ and } \vec{c} = 2\hat{i} + \hat{j} - \hat{k}.$$

[CBSE 2000C]

SOLUTION We have

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & -3 \\ 1 & -1 & 2 \\ 0 & 3 & -5 \end{vmatrix}$$

$[R_1 \to R_1 - 2R_2, \text{ and } R_3 \to R_3 - 2R_2]$

$$= (-1) \cdot (5 + 9) = -14.$$

$\therefore$ volume of the parallelepiped

$$= |[\vec{a}\ \vec{b}\ \vec{c}]| = |-14| = 14 \text{ cubic units.}$$

EXAMPLE 4 *Show that the vectors*

$$\hat{i} - 3\hat{j} + 4\hat{k},\ 2\hat{i} - \hat{j} + 2\hat{k} \text{ and } 4\hat{i} - 7\hat{j} + 10\hat{k} \text{ are coplanar.}$$

SOLUTION Let $\vec{a} = \hat{i} - 3\hat{j} + 4\hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = 4\hat{i} - 7\hat{j} + 10\hat{k}$.

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & -3 & 4 \\ 2 & -1 & 2 \\ 4 & -7 & 10 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 5 & -6 \\ 0 & 5 & -6 \end{vmatrix} \left| \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix} \right|$$

$$= (-30 + 30) = 0.$$

Hence, the given vectors are coplanar.

EXAMPLE 5 *Find the value of* λ *so that the vectors*

$$\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k},\ \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k} \text{ and } \vec{c} = \hat{j} + \lambda\hat{k} \text{ are coplanar.}$$

SOLUTION The given vectors will be coplanar if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

$$\text{Now, } [\vec{a}\ \vec{b}\ \vec{c}] = 0 \Leftrightarrow \begin{vmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 0 & 1 & \lambda \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 0 & -7 & 7 \\ 1 & 2 & -3 \\ 0 & 1 & \lambda \end{vmatrix} = 0$$

$[R_1 \to R_1 - 2R_2]$

$$\Leftrightarrow (-1)(-7\lambda - 7) = 0 \Leftrightarrow 7\lambda + 7 = 0 \Leftrightarrow \lambda = -1.$$

Hence, the given vectors are coplanar when $\lambda = -1$.

EXAMPLE 6 *Show that the four points with position vectors* $(4\hat{i} + 5\hat{j} + \hat{k}),\ (-\hat{j} - \hat{k}),$ $(3\hat{i} + 9\hat{j} + 4\hat{k})$ *and* $4(-\hat{i} + \hat{j} + \hat{k})$ *are coplanar.* **[CBSE 2014]**

SOLUTION Let the given points be A, B, C, D respectively.

Points A, B, C, D are coplanar $\Leftrightarrow \overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar

$$\Leftrightarrow [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0.$$

Now, $\overrightarrow{AB}$ = (p.v. of B) − (p.v. of A)

$= (-\hat{j} - \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = (-4\hat{i} - 6\hat{j} - 2\hat{k})$

$\overrightarrow{AC}$ = (p.v. of C) − (p.v. of A)

$= (3\hat{i} + 9\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = (-\hat{i} + 4\hat{j} + 3\hat{k})$

$\overrightarrow{AD}$ = (p.v. of D) − (p.v. of A)

$= (-4\hat{i} + 4\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = (-8\hat{i} - \hat{j} + 3\hat{k})$.

$$\therefore \quad [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -22 & -14 \\ -1 & 4 & 3 \\ 0 & -21 & -33 \end{vmatrix} \quad \left\{ \begin{matrix} R_1 \to R_1 - 4R_2 \\ R_3 \to R_3 - 8R_2 \end{matrix} \right\}$$

$$= -(-1)[462 - 462] = 0.$$

$\therefore$ $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar.

Hence, the points A, B, C, D are coplanar.

EXAMPLE 7 *Find the value of* λ *so that the four points with position vectors*

$$(-6\hat{i} + 3\hat{j} + 2\hat{k}), (3\hat{i} + \lambda\hat{j} + 4\hat{k}), (5\hat{i} + 7\hat{j} + 3\hat{k})$$

and $(-13\hat{i} + 17\hat{j} - 2\hat{k})$ *are coplanar.* **[CBSE 2000]**

SOLUTION Let the given points be A, B, C, D respectively. Then,

$\overrightarrow{AB}$ = (p.v. of B) − (p.v. of A)

$= (3\hat{i} + \lambda\hat{j} + 4\hat{k}) - (-6\hat{i} + 3\hat{j} + 2\hat{k})$

$= 9\hat{i} + (\lambda - 3)\hat{j} + 2\hat{k}$

$\overrightarrow{AC}$ = (p.v. of C) − (p.v. of A)

$= (5\hat{i} + 7\hat{j} + 3\hat{k}) - (-6\hat{i} + 3\hat{j} + 2\hat{k}) = (11\hat{i} + 4\hat{j} + \hat{k})$

$\overrightarrow{AD}$ = (p.v. of D) − (p.v. of A)

$= (-13\hat{i} + 17\hat{j} - \hat{k}) - (-6\hat{i} + 3\hat{j} + 2\hat{k})$

$= (-7\hat{i} + 14\hat{j} - 3\hat{k})$

Now, A, B, C, D are coplanar

$$\Leftrightarrow [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0 \Leftrightarrow \begin{vmatrix} 9 & \lambda - 3 & 2 \\ 11 & 4 & 1 \\ -7 & 14 & -3 \end{vmatrix} = 0$$

$\Leftrightarrow\ 9(-12-14)-(\lambda-3)(-33+7)+2(154+28)=0$

$\Leftrightarrow\ -234+26\lambda-78+364=0 \Leftrightarrow\ 26\lambda=-52 \Leftrightarrow\ \lambda=-2.$

Hence, the required value of λ is -2.

EXAMPLE 8 *Show that the points $A(-1, 4, -3)$, $B(3, 2, -5)$, $C(-3, 8, -5)$ and $D(-3, 2, 1)$ are coplanar.*

SOLUTION Clearly, the position vectors of A, B, C, D are $(-\hat{i}+4\hat{j}-3\hat{k})$, $(3\hat{i}+2\hat{j}-5\hat{k})$, $(-3\hat{i}+8\hat{j}-5\hat{k})$ and $(-3\hat{i}+2\hat{j}+\hat{k})$ respectively.

$\therefore\ \overrightarrow{AB}=$ (p.v. of B) $-$ (p.v. of A)

$=(3\hat{i}+2\hat{j}-5\hat{k})-(-\hat{i}+4\hat{j}-3\hat{k})=(4\hat{i}-2\hat{j}-2\hat{k})$

$\overrightarrow{AC}=$ (p.v. of C) $-$ (p.v. of A)

$=(-3\hat{i}+8\hat{j}-5\hat{k})-(-\hat{i}+4\hat{j}-3\hat{k})=(-2\hat{i}+4\hat{j}-2\hat{k})$

$\overrightarrow{AD}=$ (p.v. of D) $-$ (p.v. of A)

$=(-3\hat{i}+2\hat{j}+\hat{k})-(-\hat{i}+4\hat{j}-3\hat{k})=(-2\hat{i}-2\hat{j}+4\hat{k}).$

$$\therefore\ [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]=\begin{vmatrix}4 & -2 & -2\\ -2 & 4 & -2\\ -2 & -2 & 4\end{vmatrix}=2\times(-2)\times(-2)\cdot\begin{vmatrix}2 & -1 & -1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{vmatrix}$$

$$=8\cdot\begin{vmatrix}0 & -3 & 3\\ 0 & -3 & 3\\ 1 & 1 & -2\end{vmatrix}$$

$[R_1 \to R_1-2R_3,\ R_2 \to R_2-R_3]$

$=(8\times 0)=0$ [$\because$ R_1 and R_2 are identical]

$\Rightarrow\ \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar

$\Rightarrow$ the points A, B, C, D are coplanar.

EXERCISE 25A

1. Prove that

(i) $[\hat{i}\ \hat{j}\ \hat{k}]=[\hat{j}\ \hat{k}\ \hat{i}]=[\hat{k}\ \hat{j}\ \hat{i}]=1$

(ii) $[\hat{i}\ \hat{k}\ \hat{j}]=[\hat{k}\ \hat{j}\ \hat{i}]=[\hat{j}\ \hat{i}\ \hat{k}]=1$

2. Find $[\vec{a}\ \vec{b}\ \vec{c}]$, when

(i) $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$

(ii) $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$

(iii) $\vec{a} = 2\hat{i} - 3\hat{j}$, $\vec{b} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} - \hat{k}$

3. Find the volume of the parallelepiped whose coterminous edges are represented by the vectors

(i) $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$

(ii) $\vec{a} = -3\hat{i} + 7\hat{j} + 5\hat{k}$, $\vec{b} = -5\hat{i} + 7\hat{j} - 3\hat{k}$, $\vec{c} = 7\hat{i} - 5\hat{j} - 3\hat{k}$

(iii) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{c} = \hat{j} + \hat{k}$

(iv) $\vec{a} = 6\hat{i}$, $\vec{b} = 2\hat{j}$, $\vec{c} = 5\hat{i}$

4. Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, when

(i) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$

(ii) $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = 7\hat{j} + 3\hat{k}$

(iii) $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} - 4\hat{j} + 7\hat{k}$

5. Find the value of λ for which the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, where

(i) $\vec{a} = (2\hat{i} - \hat{j} + \hat{k})$, $\vec{b} = (\hat{i} + 2\hat{j} + 3\hat{k})$ and $\vec{c} = (3\hat{i} + \lambda\hat{j} + 5\hat{k})$ **[CBSE 2004]**

(ii) $\vec{a} = \lambda\hat{i} - 10\hat{j} - 5\hat{k}$, $\vec{b} = -7\hat{i} - 5\hat{j}$ and $\vec{c} = \hat{i} - 4\hat{j} - 3\hat{k}$

(iii) $\vec{a} = \hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{i} - \hat{j} + \lambda\hat{k}$ **[CBSE 2004C]**

6. If $\vec{a} = (2\hat{i} - \hat{j} + \hat{k})$, $\vec{b} = (\hat{i} - 3\hat{j} - 5\hat{k})$ and $\vec{c} = (3\hat{i} - 4\hat{j} - \hat{k})$, find $[\vec{a}\ \vec{b}\ \vec{c}]$ and interpret the result.

7. The volume of the parallelepiped whose edges are $(-12\hat{i} + \lambda\hat{k})$, $(3\hat{j} - \hat{k})$ and $(2\hat{i} + \hat{j} - 15\hat{k})$ is 546 cubic units. Find the value of λ. **[CBSE 2004]**

8. Show that the vectors $\vec{a} = (\hat{i} + 3\hat{j} + \hat{k})$, $\vec{b} = (2\hat{i} - \hat{j} - \hat{k})$ and $\vec{c} = (7\hat{j} + 3\hat{k})$ are parallel to the same plane. {**HINT:** Show that $[\vec{a}\ \vec{b}\ \vec{c}] = 0$}

9. If the vectors $(a\hat{i} + a\hat{j} + c\hat{k})$, $(\hat{i} + \hat{k})$ and $(c\hat{i} + c\hat{j} + b\hat{k})$ be coplanar, show that $c^2 = ab$. **[CBSE 2005]**

10. Show that the four points with position vectors $(4\hat{i} + 8\hat{j} + 12\hat{k})$, $(2\hat{i} + 4\hat{j} + 6\hat{k})$, $(3\hat{i} + 5\hat{j} + 4\hat{k})$ and $(5\hat{i} + 8\hat{j} + 5\hat{k})$ are coplanar.

11. Show that the four points with position vectors $(6\hat{i} - 7\hat{j})$, $(16\hat{i} - 19\hat{j} - 4\hat{k})$, $(3\hat{j} - 6\hat{k})$ and $(2\hat{i} - 5\hat{j} + 10\hat{k})$ are coplanar.

[CBSE 2004, '05]

12. Find the value of λ for which the four points with position vectors $(\hat{i} + 2\hat{j} + 3\hat{k})$, $(3\hat{i} - \hat{j} + 2\hat{k})$, $(-2\hat{i} + \lambda\hat{j} + \hat{k})$ and $(6\hat{i} - 4\hat{j} + 2\hat{k})$ are coplanar. **[CBSE 2000]**

13. Find the value of λ for which the four points with position vectors $(-\hat{j} + \hat{k})$, $(2\hat{i} - \hat{j} - \hat{k})$, $(\hat{i} + \lambda\hat{j} + \hat{k})$ and $(3\hat{j} + 3\hat{k})$ are coplanar. **[CBSE 2000]**

14. Using vector method, show that the points $A(4, 5, 1)$, $B(0, -1, -1)$, $C(3, 9, 4)$ and $D(-4, 4, 4)$ are coplanar.

15. Find the value of λ for which the points $A(3, 2, 1)$, $B(4, \lambda, 5)$, $C(4, 2, -2)$ and $D(6, 5, -1)$ are coplanar. **[CBSE 2004C]**

ANSWERS (EXERCISE 25A)

2. (i) -10 (ii) -7 (iii) 4

3. (i) 4 cubic units (ii) 264 cubic units (iii) 12 cubic units (iv) 60 cubic units

5. (i) $\lambda = -4$ (ii) $\lambda = -3$ (iii) $\lambda = 1$ **6.** 0, the given vectors are coplanar

7. $\lambda = -3$ **12.** $\lambda = 3$ **13.** $\lambda = 1$ **15.** $\lambda = 5$

EXERCISE 25B

Very-Short-Answer Questions

1. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors then $x + y + z = ?$ **[CBSE 2013]**

2. Write a unit vector in the direction of the sum of the vectors $\vec{a} = (2\hat{i} + 2\hat{j} - 5\hat{k})$ and $\vec{b} = (2\hat{i} + \hat{j} - 7\hat{k})$. **[CBSE 2014]**

3. Write the value of λ so that the vectors $\vec{a} = (2\hat{i} + \lambda\hat{j} + \hat{k})$ and $\vec{b} = (\hat{i} - 2\hat{j} + 3\hat{k})$ are perpendicular to each other. **[CBSE 2013C]**

4. Find the value of p for which the vectors $\vec{a} = (3\hat{i} + 2\hat{j} + 9\hat{k})$ and $\vec{b} = (\hat{i} - 2p\hat{j} + 3\hat{k})$ are parallel. **[CBSE 2014]**

5. Find the value of λ when the projection of $\vec{a} = (\lambda\hat{i} + \hat{j} + 4\hat{k})$ on $\vec{b} = (2\hat{i} + 6\hat{j} + 3\hat{k})$ is 4 units. **[CBSE 2012]**

6. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors such that $|\vec{a} + \vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$. **[CBSE 2014]**

7. If $\vec{a}$ is a unit vector such that $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 15$, find $|\vec{x}|$. **[CBSE 2013]**

8. Find the sum of the vectors $\vec{a} = (\hat{i} - 3\hat{k})$, $\vec{b} = (2\hat{j} - \hat{k})$ and $\vec{c} = (2\hat{i} - 3\hat{j} + 2\hat{k})$. **[CBSE 2012]**

9. Find the sum of the vectors $\vec{a} = (\hat{i} - 2\hat{j})$, $\vec{b} = (2\hat{i} - 3\hat{j})$ and $\vec{c} = (2\hat{i} + 3\hat{k})$. **[CBSE 2012]**

10. Write the projection of the vector $(\hat{i} + \hat{j} + \hat{k})$ along the vector $\hat{j}$. **[CBSE 2014]**

11. Write the projection of the vector $(7\hat{i} + \hat{j} - 4\hat{k})$ on the vector $(2\hat{i} + 6\hat{j} + 3\hat{k})$. **[CBSE 2013C]**

12. Find $\vec{a} \cdot (\vec{b} \times \vec{c})$ when $\vec{a} = (2\hat{i} + \hat{j} + 3\hat{k})$, $\vec{b} = (-\hat{i} + 2\hat{j} + \hat{k})$ and $\vec{c} = (3\hat{i} + \hat{j} + 2\hat{k})$. **[CBSE 2014]**

13. Find a vector in the direction of $(2\hat{i} - 3\hat{j} + 6\hat{k})$ which has magnitude 21 units. **[CBSE 2014]**

14. If $\vec{a} = (2\hat{i} + 2\hat{j} + 3\hat{k})$, $\vec{b} = (-\hat{i} + 2\hat{j} + \hat{k})$ and $\vec{c} = (3\hat{i} + \hat{j})$ are such that $(\vec{a} + \lambda\vec{b})$ is perpendicular to $\vec{c}$ then find the value of λ. **[CBSE 2009C]**

15. Write a vector of magnitude 15 units in the direction of vector $(\hat{i} - 2\hat{j} + 2\hat{k})$. **[CBSE 2010]**

16. If $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$, $\vec{b} = (4\hat{i} - 2\hat{j} + 3\hat{k})$ and $\vec{c} = (\hat{i} - 2\hat{j} + \hat{k})$, find a vector of magnitude 6 units which is parallel to the vector $(2\vec{a} - \vec{b} + 3\vec{c})$. **[CBSE 2010]**

17. Write the projection of the vector $(\hat{i} - \hat{j})$ on the vector $(\hat{i} + \hat{j})$. **[CBSE 2011]**

18. Write the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a} \cdot \vec{b} = \sqrt{6}$. **[CBSE 2011]**

19. If $\vec{a} = (\hat{i} - 7\hat{j} + 7\hat{k})$ and $\vec{b} = (3\hat{i} - 2\hat{j} + 2\hat{k})$ then find $|\vec{a} \times \vec{b}|$. **[CBSE 2008C]**

20. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively, when $|\vec{a} \times \vec{b}| = \sqrt{3}$. **[CBSE 2009C]**

21. What conclusion can you draw about vectors $\vec{a}$ and $\vec{b}$ when $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a} \cdot \vec{b} = 0$?

22. Find the value of λ when the vectors $\vec{a} = (\hat{i} + \lambda\hat{j} + 3\hat{k})$ and $\vec{b} = (3\hat{i} + 2\hat{j} + 9\hat{k})$ are parallel.

23. Write the value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$.

24. Find the volume of the parallelepiped whose edges are represented by the vectors $\vec{a} = (2\hat{i} - 3\hat{j} + 4\hat{k})$, $\vec{b} = (\hat{i} + 2\hat{j} - \hat{k})$ and $\vec{c} = (3\hat{i} - 2\hat{j} + 2\hat{k})$.

25. If $\vec{a} = (-2\hat{i} - 2\hat{j} + 4\hat{k})$, $\vec{b} = (-2\hat{i} + 4\hat{j} - 2\hat{k})$ and $\vec{c} = (4\hat{i} - 2\hat{j} - 2\hat{k})$ then prove that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

26. If $\vec{a} = (2\hat{i} + 6\hat{j} + 27\hat{k})$ and $\vec{b} = (\hat{i} + \lambda\hat{j} + \mu\hat{k})$ are such that $\vec{a} \times \vec{b} = \vec{0}$ then find the values of λ and μ.

27. If θ is the angle between $\vec{a}$ and $\vec{b}$, and $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$ then what is the value of θ?

28. When does $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$ hold?

29. Find the direction cosines of a vector which is equally inclined to the x-axis, y-axis and z-axis.

30. If $P(1, 5, 4)$ and $Q(4, 1, -2)$ be the position vectors of two points P and Q, find the direction ratios of $\overrightarrow{PQ}$.

31. Find the direction cosines of the vector $\vec{a} = (\hat{i} + 2\hat{j} + 3\hat{k})$.

32. If $\hat{a}$ and $\hat{b}$ are unit vectors such that $(\hat{a} + \hat{b})$ is a unit vector, what is the angle between $\hat{a}$ and $\hat{b}$?

ANSWERS (EXERCISE 25B)

1. 0 **2.** $\frac{1}{13}(4\hat{i} + 3\hat{j} - 12\hat{k})$ **3.** $\lambda = \frac{5}{2}$ **4.** $p = \frac{-1}{3}$ **5.** $\lambda = 5$ **6.** $|\vec{b}| = 12$

7. $|\vec{x}| = 4$ **8.** $(3\hat{i} - \hat{j} - 2\hat{k})$ **9.** $(5\hat{i} - 5\hat{j} + 3\hat{k})$ **10.** 1 **11.** $\frac{8}{7}$ **12.** -10

13. $(6\hat{i} - 9\hat{j} + 18\hat{k})$ **14.** $\lambda = 8$ **15.** $(5\hat{i} - 10\hat{j} + 10\hat{k})$ **16.** $(2\hat{i} - 4\hat{j} + 4\hat{k})$

17. 0 **18.** $\frac{\pi}{4}$ **19.** $19\sqrt{2}$ **20.** $\frac{\pi}{3}$ **21.** $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ **22.** $\lambda = \frac{2}{3}$

23. 1 **24.** 7 cubic untis **26.** $\lambda = 3, \mu = \frac{27}{2}$ **27.** $\frac{\pi}{4}$

28. $\vec{a}$ and $\vec{b}$ are parallel or collinear **29.** $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ **30.** 3, –4, –6

31. $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ **32.** $\frac{2\pi}{3}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 25B)

1. $\vec{a} = \vec{b} \Rightarrow (x\hat{i} + 2\hat{j} - z\hat{k}) - (3\hat{i} - y\hat{j} + \hat{k})$

$$\Rightarrow x = 3, -y = 2 \text{ and } -z = 1 \Rightarrow x = 3, y = -2 \text{ and } z = -1$$

$$\Rightarrow (x + y + z) = [3 + (-2) + (-1)] = 0.$$

2. Required unit vector $= \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|} = \frac{(4\hat{i} + 3\hat{j} - 12\hat{k})}{\sqrt{16 + 9 + 144}} = \frac{1}{13}(4\hat{i} + 3\hat{j} - 12\hat{k}).$

3. $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0 \Leftrightarrow (2 - 2\lambda + 3) = 0 \Leftrightarrow 2\lambda = 5 \Leftrightarrow \lambda = \frac{5}{2}.$

4. $\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$

$$\Leftrightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 9 \\ 1 & -2p & 3 \end{vmatrix} = 0 \Leftrightarrow (6 + 18p)\hat{i} - (9 - 9)\hat{j} + (-6p - 2)\hat{k} = \vec{0}$$

$$\Leftrightarrow (6 + 18p)\hat{i} + 0 \cdot \hat{j} + (-6p - 2)\hat{k} = \hat{0}$$

$$\Leftrightarrow 6 + 18p = 0 \text{ and } -6p - 2 = 0 \Leftrightarrow p = \frac{-1}{3}.$$

5. Clearly, $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = 4 \Rightarrow \frac{(2\lambda + 6 + 12)}{\sqrt{4 + 36 + 9}} = 4 \Rightarrow (2\lambda + 18) = 28 \Rightarrow 2\lambda = 10 \Rightarrow \lambda = 5.$

6. Since $\vec{a}$ and $\vec{b}$ are perpendicular vectors, we have $\vec{a} \cdot \vec{b} = 0.$

Now, $|\vec{a} + \vec{b}|^2 = 169 \Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = 169$

$$\Rightarrow \vec{a} \cdot \vec{a} + 2\,\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = 169$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 = 169 \quad [\because \ \vec{a}\cdot\vec{b} = 0]$$

$$\Rightarrow 25 + |\vec{b}|^2 = 169 \Rightarrow |\vec{b}|^2 = 144 \Rightarrow |\vec{b}| = 12.$$

7. $(\vec{x} - \vec{a})\cdot(\vec{x} + \vec{a}) = 15 \Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 15$

$$\Rightarrow |\vec{x}|^2 - 1 = 15 \Rightarrow |\vec{x}|^2 = 16 \Rightarrow |\vec{x}| = 4 \ \{\because \ |\vec{a}| = 1\}.$$

8. $(\vec{a} + \vec{b} + \vec{c}) = (1 + 0 + 2)\hat{i} + (0 + 2 - 3)\hat{j} + [(-3) + (-1) + 2]\hat{k}$

$$= (3\hat{i} - \hat{j} - 2\hat{k}).$$

9. $(\vec{a} + \vec{b} + \vec{c}) = (1 + 2 + 2)\hat{i} + (-2 - 3 + 0)\hat{j} + 3\hat{k} = (5\hat{i} - 5\hat{j} + 3\hat{k}).$

10. Required projection $= \dfrac{(\hat{i} + \hat{j} + \hat{k})\cdot\hat{j}}{|\hat{j}|} = \dfrac{(0 + 1 + 0)}{1} = 1.$

11. Required projection $= \dfrac{(7\hat{i} + \hat{j} - 4\hat{k})\cdot(2\hat{i} + 6\hat{j} + 3\hat{k})}{|2\hat{i} + 6\hat{j} + 3\hat{k}|}$

$$= \frac{(14 + 6 - 12)}{\sqrt{2^2 + 6^2 + 3^2}} = \frac{8}{\sqrt{49}} = \frac{8}{7}\cdot$$

12. $(\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = (3\hat{i} + 5\hat{j} - 7\hat{k})$

$$\therefore \quad \vec{a}\cdot(\vec{b} \times \vec{c}) = (2\hat{i} + \hat{j} + 3\hat{k})\cdot(3\hat{i} + 5\hat{j} - 7\hat{k}) = (6 + 5 - 21) = -10.$$

13. Required vector $= \dfrac{21(2\hat{i} - 3\hat{j} + 6\hat{k})}{\sqrt{2^2 + (-3)^2 + 6^2}} = \dfrac{21(2\hat{i} - 3\hat{j} + 6\hat{k})}{\sqrt{49}}$

$$= 3(2\hat{i} - 3\hat{j} + 6\hat{k}) = (6\hat{i} - 9\hat{j} + 18\hat{k}).$$

14. $(\vec{a} + \lambda\vec{b}) = (2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}$

$(\vec{a} + \lambda\vec{b})\cdot\vec{c} = 0 \Rightarrow [(2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}]\cdot(3\hat{i} + \hat{j}) = 0$

$$\Rightarrow 3(2 - \lambda) + 1\cdot(2 + 2\lambda) = 0 \Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0 \Rightarrow \lambda = 8.$$

15. Required vector $= \dfrac{15(\hat{i} - 2\hat{j} + 2\hat{k})}{\sqrt{1^2(-2)^2 + 2^2}} = 5(\hat{i} - 2\hat{j} + 2\hat{k}) = (5\hat{i} - 10\hat{j} + 10\hat{k}).$

16. $(2\vec{a} - \vec{b} + 3\vec{c}) = (2\hat{i} + 2\hat{j} + 2\hat{k}) - (4\hat{i} - 2\hat{j} + 3\hat{k}) + (3\hat{i} - 6\hat{j} + 3\hat{k})$

$$= (\hat{i} - 2\hat{j} + 2\hat{k}).$$

$\therefore$ required vector $= \dfrac{6(\hat{i} - 2\hat{j} + 2\hat{k})}{\sqrt{1^2 + (-2)^2 + 2^2}} = \dfrac{6(\hat{i} - 2\hat{j} + 2\hat{k})}{3} = (2\hat{i} - 4\hat{j} + 4\hat{k})$

17. Required projection $= \dfrac{(\hat{i} - \hat{j})\cdot(\hat{i} + \hat{j})}{|\hat{i} + \hat{j}|} = \dfrac{(\hat{i}\cdot\hat{i} + \hat{i}\cdot\hat{j} - \hat{j}\cdot\hat{i} - \hat{j}\cdot\hat{j})}{\sqrt{1^2 + 1^2}}$

$$= \frac{(1 + 0 - 0 - 1)}{\sqrt{2}} = 0.$$

18. Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}\times 2} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}\times\sqrt{2}\times\sqrt{3}}{6}$$

$$= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}\cdot$$

19. $(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix} = (-14 + 14)\hat{i} - (2 - 21)\hat{j} + (-2 + 21)\hat{k}$

$$= (19\hat{j} + 19\hat{k})$$

$\therefore$ $|\vec{a} \times \vec{b}| = \sqrt{0^2 + (19)^2 + (19)^2} = \sqrt{2 \times (19)^2} = 19\sqrt{2}.$

20. $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta \Rightarrow 1 \times 2 \times \sin\theta = \sqrt{3} \Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = \dfrac{\pi}{3}\cdot$

21. $(\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a}\cdot\vec{b} = 0)$ only when $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

22. Clearly, $\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$.

$$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \lambda & 3 \\ 3 & 2 & 9 \end{vmatrix} = (9\lambda - 6)\hat{i} - 0\hat{j} + (2 - 3\lambda)\hat{k}$$

$\therefore$ $(\vec{a} \times \vec{b}) = \vec{0} \Leftrightarrow 9\lambda - 6 = 0$ and $2 - 3\lambda = 0 \Leftrightarrow \lambda = \dfrac{2}{3}\cdot$

23. Clearly, $(\hat{j} \times \hat{k}) = \hat{i}$, $(\hat{i} \times \hat{k}) = -\hat{j}$ and $(\hat{i} \times \hat{j}) = \hat{k}$.

$\therefore$ $\hat{i}\cdot(\hat{j} \times \hat{k}) + \hat{j}\cdot(\hat{i} \times \hat{k}) + \hat{k}\cdot(\hat{i} \times \hat{j}) = (\hat{i}\cdot\hat{i} - \hat{j}\cdot\hat{j} + \hat{k}\cdot\hat{k})$

$$= (1 - 1 + 1) = 1.$$

24. $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 2(4 - 1) + 3(2 + 3) + 4(-1 - 6)$

$$= (6 + 15 - 28) = -7.$$

$\therefore$ volume of the parallelepiped $= |-7| = 7$ cubic units.

25. $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} -2 & -2 & 4 \\ -2 & 4 & -2 \\ 4 & -2 & -2 \end{vmatrix} = (-2)(-8-4) + 2(4+8) + 4(4-16)$

$$= (24 + 24 - 48) = 0.$$

Hence $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

26. $(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = (6\mu - 27\lambda)\,\hat{i} - (2\mu - 27)\,\hat{j} + (2\lambda - 6)\,\hat{k}.$

$\therefore \quad (\vec{a} \times \vec{b}) = \vec{0} \Leftrightarrow 2\lambda - 6 = 0$ and $2\mu - 27 = 0 \Leftrightarrow \lambda = 3$ and $\mu = \frac{27}{2}\cdot$

27. $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}| \Rightarrow |\vec{a}|\,|\vec{b}|\cos\theta = |\vec{a}|\,|\vec{b}|\sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}\cdot$

28. $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}| \Rightarrow |\vec{a} + \vec{b}|^2 = \{|\vec{a}| + |\vec{b}|\}^2$

$$\Rightarrow (\vec{a} + \vec{b})\cdot(\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|$$

$$\Rightarrow \vec{a}\cdot\vec{a} + 2\,\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|$$

$$\Rightarrow |\vec{a}|^2 + 2\,\vec{a}\cdot\vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\,|\vec{a}||\vec{b}|$$

$$\Rightarrow 2(\vec{a}\cdot\vec{b}) = 2(|\vec{a}||\vec{b}|) \Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$$

$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}| \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0°.$$

$\therefore$ either $\vec{a}$ and $\vec{b}$ are parallel or collinear.

29. $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1 \Rightarrow 3\cos^2\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}\cdot$

So, the direction cosines of the vector are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\cdot$

30. D. R.'s of $\overrightarrow{PQ}$ are $(4-1), (1-5), (-2-4)$, i.e., 3, –4, –6.

31. Direction ratios of $\vec{a}$ are 1, 2, 3.

And, $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.

$\therefore$ direction cosines of $\vec{a}$ are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\cdot$

32. $(\hat{a} + \hat{b})\cdot(\hat{a} + \hat{b}) = 1 \Rightarrow a^2 + b^2 + 2\,\hat{a}\cdot\hat{b} = 1$

$$\Rightarrow 1 + 1 + 2\,\hat{a}\cdot\hat{b} = 1 \Rightarrow 2\,\hat{a}\cdot\hat{b} = -1$$

$$\Rightarrow \hat{a}\cdot\hat{b} = \frac{-1}{2} \Rightarrow |\hat{a}|\,|\hat{b}|\cos\theta = \frac{-1}{2}$$

$$\Rightarrow \cos\theta = \frac{-1}{2} \Rightarrow \theta = \frac{2\pi}{3}\cdot$$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following.

1. A unit vector in the direction of the vector $\vec{a} = (2\hat{i} - 3\hat{j} + 6\hat{k})$ is

(a) $\left(\hat{i} - \frac{3}{2}\hat{j} + 3\hat{k}\right)$ (b) $\left(\frac{2}{5}\hat{i} - \frac{3}{5}\hat{j} + \frac{6}{5}\hat{k}\right)$

(c) $\left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$ (d) none of these

2. The direction cosines of the vector $\vec{a} = (-2\hat{i} + \hat{j} - 5\hat{k})$ are

(a) $-2, 1, -5$ (b) $\frac{1}{3}, \frac{-1}{6}, \frac{-5}{6}$

(c) $\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{5}{\sqrt{30}}$ (d) $\frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}$

3. If $A(1, 2, -3)$ and $B(-1, -2, 1)$ are the end points of a vector $\overrightarrow{AB}$ then the direction cosines of $\overrightarrow{AB}$ are

(a) $-2, -4, 4$ (b) $\frac{-1}{2}, -1, 1$ (c) $\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}$ (d) none of these

4. If a vector makes angles α, β and γ with the x-axis, y-axis and z-axis respectviely then the value of $(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)$ is

(a) 1 (b) 2 (c) 0 (d) 3

5. The vector $(\cos\alpha \cos\beta)\hat{i} + (\cos\alpha \sin\beta)\hat{j} + (\sin\alpha)\hat{k}$ is a

(a) null vector (b) unit vector

(c) a constant vector (d) none of these

6. What is the angle which the vector $(\hat{i} + \hat{j} + \sqrt{2}\hat{k})$ makes with the z-axis?

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{2\pi}{3}$

7. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{6}$ then the angle between $\vec{a}$ and $\vec{b}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{2\pi}{3}$

8. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = |\vec{b}| = \sqrt{2}$ and $\vec{a} \cdot \vec{b} = -1$ then the angle between $\vec{a}$ and $\vec{b}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

9. The angle between the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$ is

(a) $\cos^{-1}\frac{5}{7}$ (b) $\cos^{-1}\frac{3}{5}$ (c) $\cos^{-1}\frac{3}{\sqrt{14}}$ (d) none of these

10. If $\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{b} = (3\hat{i} - \hat{j} + 2\hat{k})$ then the angle between $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

11. If $\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{b} = (3\hat{i} - \hat{j} + 2\hat{k})$ then the angle between $(2\vec{a} + \vec{b})$ and $(\vec{a} + 2\vec{b})$ is

(a) $\cos^{-1}\left(\frac{21}{40}\right)$ (b) $\cos^{-1}\left(\frac{31}{50}\right)$ (c) $\cos^{-1}\left(\frac{11}{30}\right)$ (d) none of these

12. If $\vec{a} = (2\hat{i} + 4\hat{j} - \hat{k})$ and $\vec{b} = (3\hat{i} - 2\hat{j} + \lambda\hat{k})$ be such that $\vec{a} \perp \vec{b}$ then $\lambda = ?$

(a) 2 (b) –2 (c) 3 (d) –3

13. What is the projection of $\vec{a} = (2\hat{i} - \hat{j} + \hat{k})$ on $\vec{b} = (\hat{i} - 2\hat{j} + \hat{k})$?

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{4}{\sqrt{5}}$ (c) $\frac{5}{\sqrt{6}}$ (d) none of these

14. If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, then

(a) $|\vec{a}| = |\vec{b}|$ (b) $\vec{a} \parallel \vec{b}$ (c) $\vec{a} \perp \vec{b}$ (d) none of these

15. If $\vec{a}$ and $\vec{b}$ are mutually perpendicular unit vectors then $(3\vec{a} + 2\vec{b}) \cdot (5\vec{a} - 6\vec{b}) = ?$

(a) 3 (b) 5 (c) 6 (d) 12

16. If the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \lambda\hat{j} - 3\hat{k}$ are perpendicular to each other then $\lambda = ?$

(a) –3 (b) –6 (c) –9 (d) –1

17. If θ is the angle between two unit vectors $\hat{a}$ and $\hat{b}$ then $\frac{1}{2}|\hat{a} - \hat{b}| = ?$

(a) $\cos\frac{\theta}{2}$ (b) $\sin\frac{\theta}{2}$ (c) $\tan\frac{\theta}{2}$ (d) none of these

18. If $\vec{a} = (\hat{i} - \hat{j} + 2\hat{k})$ and $\vec{b} = (2\hat{i} + 3\hat{j} - 4\hat{k})$ then $|\vec{a} \times \vec{b}| = ?$

(a) $\sqrt{174}$ (b) $\sqrt{87}$ (c) $\sqrt{93}$ (d) none of these

19. If $\vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k})$ and $\vec{b} = (\hat{i} - 3\hat{k})$ then $|\vec{b} \times 2\vec{a}| = ?$

(a) $10\sqrt{3}$ (b) $5\sqrt{17}$ (c) $4\sqrt{19}$ (d) $2\sqrt{23}$

20. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $(\vec{a} \times \vec{b}) = (3\hat{i} + 2\hat{j} + 6\hat{k})$ then the angle between $\vec{a}$ and $\vec{b}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{3\pi}{4}$

21. If $|\vec{a}| = \sqrt{26}$, $|\vec{b}| = 7$ and $|\vec{a} \times \vec{b}| = 35$ then $\vec{a} \cdot \vec{b} = ?$

(a) 5 (b) 7 (c) 13 (d) 12

22. Two adjacent sides of a ||gm are represented by the vectors $\vec{a} = (3\hat{i} + \hat{j} + 4\hat{k})$ and $\vec{b} = (\hat{i} - \hat{j} + \hat{k})$. The area of the ||gm is

(a) $\sqrt{42}$ sq units (b) 6 sq units
(c) $\sqrt{35}$ sq units (d) none of these

23. The diagonals of a ||gm are represented by the vectors $\vec{d_1} = (3\hat{i} + \hat{j} - 2\hat{k})$ and $\vec{d_2} = (\hat{i} - 3\hat{j} + 4\hat{k})$. The area of the ||gm is

(a) $7\sqrt{3}$ sq units (b) $5\sqrt{3}$ sq units
(c) $3\sqrt{5}$ sq units (d) none of these

24. Two adjacent sides of a triangle are represented by the vectors $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = -5\hat{i} + 7\hat{j}$. The area of the triangle is

(a) 41 sq units (b) 37 sq units (c) $\frac{41}{2}$ sq units (d) none of these

25. The unit vector normal to the plane containing $\vec{a} = (\hat{i} - \hat{j} - \hat{k})$ and $\vec{b} = (\hat{i} + \hat{j} + \hat{k})$ is

(a) $(\hat{j} - \hat{k})$ (b) $(-\hat{j} + \hat{k})$ (c) $\frac{1}{\sqrt{2}}(-\hat{j} + \hat{k})$ (d) $\frac{1}{\sqrt{2}}(-\hat{i} + \hat{k})$

26. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ then $(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = ?$

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) $\frac{3}{2}$ (d) $\frac{-3}{2}$

27. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular unit vectors then $|\vec{a} + \vec{b} + \vec{c}| = ?$

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

28. $[\hat{i}\ \hat{j}\ \hat{k}] = ?$

(a) 0 (b) 1 (c) 2 (d) 3

29. If $\vec{a} = (2\hat{i} - 3\hat{j} + 4\hat{k})$, $\vec{b} = (\hat{i} + 2\hat{j} - \hat{k})$ and $\vec{c} = (3\hat{i} - \hat{j} - 2\hat{k})$ be the coterminous edges of a parallelepiped then its volume is

(a) 21 cubic units (b) 14 cubic units

(c) 7 cubic units (d) none of these

30. If the volume of a parallelepiped having $\vec{a} = (5\hat{i} - 4\hat{j} + \hat{k})$, $\vec{b} = (4\hat{i} + 3\hat{j} + \lambda\hat{k})$ and $\vec{c} = (\hat{i} - 2\hat{j} + 7\hat{k})$ as coterrminous edges, is 216 cubic units then the value of λ is

(a) $\frac{5}{3}$ (b) $\frac{4}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

31. It is given that the vectors $\vec{a} = (2\hat{i} - 2\hat{k})$, $\vec{b} = \hat{i} + (\lambda + 1)\hat{j}$ and $\vec{c} = (4\hat{i} + 2\hat{k})$ are coplanar. Then, the value of λ is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) 2 (d) 1

32. Which of the following is meaningless?

(a) $\vec{a} \cdot (\vec{b} \times \vec{c})$ (b) $\vec{a} \times (\vec{b} \cdot \vec{c})$ (c) $(\vec{a} \times \vec{b}) \cdot \vec{c}$ (d) none of these

33. $\vec{a} \cdot (\vec{a} \times \vec{b}) = ?$

(a) 0 (b) 1 (c) $a^2 b$ (d) meaningless

34. For any three vectors $\vec{a}, \vec{b}, \vec{c}$ the value of $[\vec{a} - \vec{b}\ \ \vec{b} - \vec{c}\ \ \vec{c} - \vec{a}]$ is

(a) $2[\vec{a}\ \vec{b}\ \vec{c}]$ (b) 1 (c) 0 (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (c)	**2.** (d)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (a)	**7.** (c)
8. (d)	**9.** (a)	**10.** (c)	**11.** (b)	**12.** (b)	**13.** (c)	**14.** (c)
15. (a)	**16.** (c)	**17.** (b)	**18.** (c)	**19.** (c)	**20.** (a)	**21.** (b)
22. (a)	**23.** (b)	**24.** (c)	**25.** (c)	**26.** (d)	**27.** (c)	**28.** (b)
29. (c)	**30.** (a)	**31.** (d)	**32.** (b)	**33.** (a)	**34.** (c)	

HINTS TO THE GIVEN OBJECTIVE QUESTIONS

1. $|\vec{a}| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7.$

$\therefore$ required unit vector is $\left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$.

2. $|\vec{a}| = \sqrt{(-2)^2 + 1^2 + (-5)^2} = \sqrt{30}.$

$\therefore$ direction cosines of $\vec{a}$ are $\frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}$.

3. $\overrightarrow{AB}$ = (p.v. of B) – (p.v. of A)

$= (-\hat{i} - 2\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} - 3\hat{k}) = (-2\hat{i} - 4\hat{j} + 4\hat{k}).$

$|\overrightarrow{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{36} = 6.$

$\therefore$ direction cosines of $\overrightarrow{AB}$ are $\frac{-2}{6}, \frac{-4}{6}, \frac{4}{6}$, i.e., $\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}$.

4. We have $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

$\Rightarrow (\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 2.$

5. Let $\vec{a} = (\cos\alpha\cos\beta)\hat{i} + (\cos\alpha\sin\beta)\hat{j} + (\sin\alpha)\hat{k}$. Then,

$|\vec{a}|^2 = (\cos\alpha\cos\beta)^2 + (\cos\alpha\sin\beta)^2 + (\sin\alpha)^2$

$= \cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha$

$= \cos^2\alpha(\cos^2\beta + \sin^2\beta) + \sin^2\alpha$

$= (\cos^2\alpha + \sin^2\alpha) = 1.$

Hence, the given vector $\vec{a}$ is a unit vector.

6. $\vec{a} = (\hat{i} + \hat{j} + \sqrt{2}\hat{k}) \Rightarrow |\vec{a}| = \sqrt{1^2 + 1^2 + (\sqrt{2})^2} = \sqrt{4} = 2.$

Direction ratios of $\vec{a}$ are 1, 1, $\sqrt{2}$.

Direction cosines of $\vec{a}$ are $\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}$, i.e., $\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}$.

$\therefore \cos\gamma = \frac{1}{\sqrt{2}} \Rightarrow \gamma = \frac{\pi}{4}$.

7. $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}\times 2} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$.

8. $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-1}{\sqrt{2}\times\sqrt{2}} = \frac{-1}{2} \Rightarrow \theta = \frac{2\pi}{3}$.

9. $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{[1\times 3 + (-2)\times(-2) + 3\times 1]}{[\sqrt{1^2 + (-2)^2 + 3^2}][\sqrt{3^2 + (-2)^2 + 1^2}]} = \frac{10}{14} = \frac{5}{7} \Rightarrow \theta = \cos^{-1}\frac{5}{7}$.

10. $(\vec{a} + \vec{b}) = (4\hat{i} + \hat{j} - \hat{k})$ and $(\vec{a} - \vec{b}) = (-2\hat{i} + 3\hat{j} - 5\hat{k})$.

$\therefore \cos\theta = \frac{(\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b})}{|\vec{a} + \vec{b}|\cdot|\vec{a} - \vec{b}|} = \frac{-8 + 3 + 5}{\sqrt{18}\times\sqrt{38}} = 0 \Rightarrow \theta = \frac{\pi}{2}$.

11. $(2\vec{a}+\vec{b})=(5\hat{i}+3\hat{j}-4\hat{k})$ and $(\vec{a}+2\vec{b})=(7\hat{i}+0\hat{j}+\hat{k})$.

$\therefore\quad \cos\theta=\dfrac{(5\times 7+3\times 0-4\times 1)}{\sqrt{50}\times\sqrt{50}}=\dfrac{31}{50}\Rightarrow\theta=\cos^{-1}\left(\dfrac{31}{50}\right).$

12. $\vec{a}\perp\vec{b}\Rightarrow\vec{a}\cdot\vec{b}=0\Rightarrow 2\times 3+4\times(-2)+(-1)\times\lambda=0$

$\Rightarrow\lambda=(6-8)=-2.$

13. Projection of $\vec{a}$ on $\vec{b}=\dfrac{(\vec{a}\cdot\vec{b})}{|\vec{a}|}=\dfrac{2\times 1+(-1)\times(-2)+1\times 1}{\sqrt{1^2+(-2)^2+1^2}}=\dfrac{5}{\sqrt{6}}\cdot$

14. $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|\Rightarrow|\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2$

$\Rightarrow(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$\Rightarrow 4(\vec{a}\cdot\vec{b})=0\Rightarrow\vec{a}\cdot\vec{b}=0\Rightarrow\vec{a}\perp\vec{b}.$

15. Given $a^2=b^2=1$ and $\vec{a}\cdot\vec{b}=0$.

$\therefore\quad(3\vec{a}+2\vec{b})\cdot(5\vec{a}-6\vec{b})=15a^2-12b^2-8\vec{a}\cdot\vec{b}$

$=(15\times 1)-(12\times 1)-(8\times 0)$

$=(15-12-0)=3.$

16. $\vec{a}\perp\vec{b}\Rightarrow\vec{a}\cdot\vec{b}=0$

$\Rightarrow 3\times 1+1\times\lambda+(-2)\times(-3)=0\Rightarrow\lambda=-9.$

17. $|\hat{a}-\hat{b}|^2=(\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})=\hat{a}\cdot\hat{a}+\hat{b}\cdot\hat{b}-2\hat{a}\cdot\hat{b}$

$=(1+1-2\times 1\times 1\times\cos\theta)=2(1-\cos\theta)=4\sin^2\dfrac{\theta}{2}\cdot$

$\therefore\quad\sin^2\dfrac{\theta}{2}=\dfrac{1}{4}\cdot|\hat{a}-\hat{b}|^2\Rightarrow\dfrac{1}{2}|\hat{a}-\hat{b}|=\sin\dfrac{\theta}{2}\cdot$

18. $(\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-1&2\\2&3&-4\end{vmatrix}=(4-6)\hat{i}-(-4-4)\hat{j}+(3+2)\hat{k}$

$=(-2)\hat{i}+8\hat{j}+5\hat{k}.$

$\therefore\quad|\vec{a}\times\vec{b}|^2=\{(-2)^2+8^2+5^2\}=(4+64+25)=93\Rightarrow|\vec{a}\times\vec{b}|=\sqrt{93}.$

19. $\vec{b}=(\hat{i}+0\hat{j}-3\hat{k})$ and $2\vec{a}=(2\hat{i}-4\hat{j}+6\hat{k})$.

$\therefore\quad(\vec{b}\times 2\vec{a})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&0&-3\\2&-4&6\end{vmatrix}=(0-12)\hat{i}-(6+6)\hat{j}+(-4-0)\hat{k}$

$=(-12\hat{i}-12\hat{j}-4\hat{k}).$

$\therefore \quad |\vec{b} \times 2\vec{a}|^2 = \{(-12)^2 + (-12)^2 + (-4)^2\} = (144 + 144 + 16) = 304.$

Hence, $|\vec{b} \times 2\vec{a}| = \sqrt{304} = 4\sqrt{19}.$

20. $|\vec{a} \times \vec{b}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$

$\Rightarrow \quad |\vec{a}||\vec{b}|\sin\theta = 7 \Rightarrow \sin\theta = \dfrac{7}{2 \times 7} = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}.$

21. $|\vec{a} \times \vec{b}| = 35 \Rightarrow |\vec{a}|\,|\vec{b}|\sin\theta = 35 \Rightarrow \sin\theta = \dfrac{35}{\sqrt{26} \times 7} = \dfrac{5}{\sqrt{26}}.$

$\therefore \quad \cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \dfrac{25}{26}} = \dfrac{1}{\sqrt{26}}.$

$\therefore \quad \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = \sqrt{26} \times 7 \times \dfrac{1}{\sqrt{26}} = 7.$

22. $(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = (1+4)\hat{i} - (3-4)\hat{j} + (-3-1)\hat{k} = 5\hat{i} + \hat{j} - 4\hat{k}.$

Required area $= |\vec{a} \times \vec{b}| = \sqrt{25 + 1 + 16} = \sqrt{42}$ sq units.

23. $(\vec{d_1} \times \vec{d_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (4-6)\hat{i} - (12+2)\hat{j} + (-9-1)\hat{k}$

$= (-2\hat{i} - 14\hat{j} - 10\hat{k})$

$\Rightarrow \quad |\vec{d_1} \times \vec{d_2}| = \sqrt{(-2)^2 + (-14)^2 + (-10)^2} = \sqrt{300} = 10\sqrt{3}$

$\Rightarrow$ area of the ||gm $= \dfrac{1}{2}|\vec{d_1} \times \vec{d_2}| = 5\sqrt{3}$ sq units.

24. $(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ -5 & 7 & 0 \end{vmatrix} = 0\hat{i} - 0\hat{j} + (21 + 20)\hat{k}$

$\Rightarrow \triangle = \dfrac{1}{2}|\vec{a} \times \vec{b}| = \dfrac{41}{2}$ sq units.

25. $(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = (-2\hat{j} + 2\hat{k})$

$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\Rightarrow$ required vector $= \dfrac{2(-\hat{j} + \hat{k})}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}}(-\hat{j} + \hat{k}).$

26. $(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow \quad (\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=\frac{-3}{2} \quad [\because |\vec{a}|=|\vec{b}|=|\vec{c}|=1].$

27. Given $a^2=b^2=c^2=1$ and $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0.$

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$=\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{c}+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

$=a^2+b^2+c^2+2\times 0=(1+1+1+0)=3.$

Hence, $|\vec{a}+\vec{b}+\vec{c}|=\sqrt{3}.$

28. $[\hat{i}\ \ \hat{j}\ \ \hat{k}]=(\hat{i}\times\hat{j})\cdot\hat{k}=\hat{k}\cdot\hat{k}=1.$

29. We have

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}2 & -3 & 4\\ 1 & 2 & -1\\ 3 & -1 & 2\end{vmatrix}=2(4-1)+3(2+3)+4(-1-6)$$

$=(6+15-28)=-7.$

$\therefore$ volume of the parallelopiped

$=|(\vec{a}\times\vec{b})\cdot\vec{c}|=|-7|=7$ cubic units.

30. $$(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}5 & -4 & 1\\ 4 & 3 & \lambda\\ 1 & -2 & 7\end{vmatrix}=5(21+2\lambda)+4(28-\lambda)+1\cdot(-8-3)$$

$=(105+112-11)+(10\lambda-4\lambda)=(206+6\lambda).$

$\therefore \quad 206+6\lambda=216 \Rightarrow 6\lambda=10 \Rightarrow \lambda=\frac{10}{6}=\frac{5}{3}\cdot$

31. Since $\vec{a},\vec{b},\vec{c}$ are coplanar, we must have $[\vec{a}\ \ \vec{b}\ \ \vec{c}]=0.$

Now, $$[\vec{a}\ \ \vec{b}\ \ \vec{c}]=\begin{vmatrix}2 & 0 & -2\\ 1 & \lambda+1 & 0\\ 0 & 4 & 2\end{vmatrix}$$

$=2(2\lambda+2-0)-2(4-0)$

$=4\lambda+4-8=4\lambda-4.$

$\therefore \quad [\vec{a}\ \ \vec{b}\ \ \vec{c}]=0 \Leftrightarrow 4\lambda-4=0 \Leftrightarrow 4\lambda=4 \Leftrightarrow \lambda=1.$

32. Clearly, $\vec{a}\times(\vec{b}\cdot\vec{c})$ is meaningless.

33. $\vec{a}\cdot(\vec{a}\times\vec{b})=(\vec{a}\times\vec{a})\cdot\vec{b}=\vec{0}\cdot\vec{b}=0.$

34. We have proved in the text that $[\vec{a}-\vec{b}\ \ \ \vec{b}-\vec{c}\ \ \ \vec{c}-\vec{a}]=0.$

26. FUNDAMENTAL CONCEPTS OF 3-DIMENSIONAL GEOMETRY

This chapter consists of some important concepts of three-dimensional geometry. Though we have studied these in Class 11, yet we shall review here these fundamental concepts for ready reference.

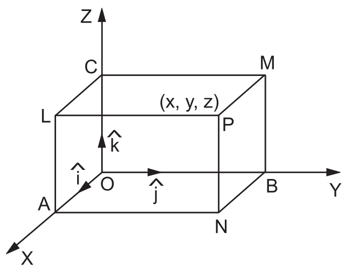
Coordinates of a Point in Space

Let O be the origin, and let OX, OY and OZ be three mutually perpendicular lines, taken as the x-axis, y-axis and z-axis respectively in such a way that they form a right-handed system.

The planes YOZ, ZOX and XOY are respectively known as the yz-plane, the zx-plane and the xy-plane.

These planes, known as the coordinate planes, divide the space into eight parts called *octants.*

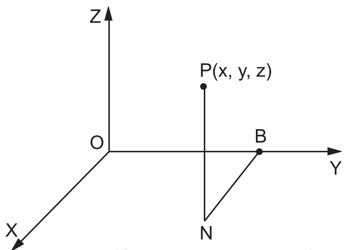
Let P be a point in space. Through P, draw three planes $PLAN$, $PNBM$ and $PLCM$ parallel to the yz-plane, the zx-plane and the xy-plane respectively, and meeting the x-axis, y-axis and z-axis at the points A, B, C respectively. Complete the parallelepiped whose coterminous edges are OA, OB and OC.



Let $OA = x$, $OB = y$ and $OC = z$. We say that the coordinates of P are (x, y, z).

It is clear from the figure alongside that

(i) x = distance of P from the yz-plane
(ii) y = distance of P from the zx-plane
(iii) z = distance of P from the xy-plane.



(OB = x, BN = y, NP = z)

Also, we can say that

(i) the equation of the xy-plane is $z = 0$
(ii) the equation of the xz-plane is $y = 0$
(iii) the equation of the yz-plane is $x = 0$.

POSITION VECTOR OF A POINT IN SPACE Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along OX, OY and OZ respectively.

If P(a, b, c) is a point in space, we say that the position vector (or, *p.v.) of P is* $(a\hat{i} + b\hat{j} + c\hat{k})$.

Some Results on Points in Space

1. DISTANCE BETWEEN TWO POINTS The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

2. SECTION FORMULAE

(i) If $P(x, y, z)$ divides the join of $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $m : n$ then

$$x = \frac{(mx_2 + nx_1)}{(m+n)},\ y = \frac{(my_2 + ny_1)}{(m+n)},\ z = \frac{(mz_2 + nz_1)}{(m+n)}.$$

(ii) The mid-point of the line joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

(iii) The centroid of $\triangle ABC$ with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

Some Results on Lines in Space

1. DIRECTION COSINES OF A LINE If a line makes angles α, β, γ with the x-axis, y-axis and z-axis respectively then

$$l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$$

are called the direction cosines (or, d.c.'s) of the line.

We always have $l^2 + m^2 + n^2 = 1$.

REMARKS (i) d.c.'s of the x-axis are 1, 0, 0.
(ii) d.c.'s of the y-axis are 0, 1, 0.
(iii) d.c.'s of the z-axis are 0, 0, 1.

2. DIRECTION RATIOS OF A LINE Any three numbers a, b, c, proportional to the direction cosines l, m, n respectively of a line, are called the direction ratios of the line.

Clearly, we have $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.

Some Important Facts

(i) If a, b, c are the direction ratios of a line then its direction cosines are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

(ii) If $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ then the direction ratios of $\vec{r}$ are a, b, c.

(iii) Let PQ be a line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Then, the direction ratios of the line PQ are $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$.

3. ANGLE BETWEEN TWO LINES If θ is the angle between two lines L_1 and L_2 whose d.c.'s are l_1, m_1, n_1, and l_2, m_2, n_2 then the following hold true.

(i) $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

(ii) $\sin\theta = \sqrt{\Sigma(m_1 n_2 - m_2 n_1)^2}$

(iii) lines L_1 and L_2 are perpendicular $\Leftrightarrow l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

(iv) lines L_1 and L_2 are parallel $\Leftrightarrow \dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$.

REMARKS In fact, there are two angles θ and $(\pi - \theta)$ between two lines.

SOLVED EXAMPLES

EXAMPLE 1 *Find the direction cosines of a line whose direction ratios are* 2, – 6, 3.

SOLUTION Here, $a = 2$, $b = -6$, $c = 3$.

$$\therefore \quad \sqrt{a^2 + b^2 + c^2} = \sqrt{2^2 + (-6)^2 + 3^2}$$
$$= \sqrt{4 + 36 + 9} = \sqrt{49} = 7.$$

Hence, the direction cosines of the given line are $\frac{2}{7}, \frac{-6}{7}, \frac{3}{7}$.

EXAMPLE 2 *Find the direction cosines of each of the following vectors:*

(i) $2\hat{i} + \hat{j} - 2\hat{k}$ *(ii)* $-\hat{i} - \hat{k}$ *(iii)* $-\hat{j}$

SOLUTION (i) Direction ratios of the vector $(2\hat{i} + \hat{j} - 2\hat{k})$ are 2, 1, –2.

And, $\sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$.

$\therefore$ d.c.'s of the given vector are $\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$.

(ii) Direction ratios of the vector $(-\hat{i} - \hat{k})$ are –1, 0, –1.

And, $\sqrt{(-1)^2 + 0^2 + (-1)^2} = \sqrt{1 + 0 + 1} = \sqrt{2}$.

$\therefore$ d.c.'s of the given vector are $\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}$.

(iii) Direction ratios of the vector $-\hat{j}$ are 0, –1, 0.

And, $\sqrt{0^2 + (-1)^2 + 0^2} = \sqrt{1} = 1$.

$\therefore$ d.c.'s of the given vector are 0, –1, 0.

EXAMPLE 3 *Find the direction cosines of a line which makes angles* $90°, 135°$ *and* $45°$ *with the positive directions of the x-, y-, and z-axis respectively.*

SOLUTION The direction cosines of the given line are

$$\cos 90°, \cos 135°, \cos 45°, \text{ i.e., } 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}.$$

EXAMPLE 4 *If a line makes angles* α, β *and* γ *with the coordinate axes, prove that* $(\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 2.$

SOLUTION Let the direction cosines of the given line be l, m, n. Then,

$$l = \cos\alpha, \quad m = \cos\beta \text{ and } n = \cos\gamma.$$

$$\therefore \quad (l^2 + m^2 + n^2) = 1 \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$$

Hence, $(\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 2.$

EXAMPLE 5 *If a line makes angles* α, β *and* γ *with the coordinate axes, prove that* $(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) = -1.$

SOLUTION Let the direction cosines of the given line be l, m, n. Then,

$$l = \cos\alpha, \quad m = \cos\beta \text{ and } n = \cos\gamma.$$

$$\therefore \quad (l^2 + m^2 + n^2) = 1 \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow 2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma = 2$$

$$\Rightarrow (1 + \cos 2\alpha) + (1 + \cos 2\beta) + (1 + \cos 2\gamma) = 2$$

$$\Rightarrow (\cos 2\alpha + \cos 2\beta + \cos 2\gamma) = -1.$$

Hence, $(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) = -1.$

EXAMPLE 6 *Find the direction cosines of a line that makes equal angles with the coordinate axes.* **[CBSE 2011]**

SOLUTION Let the given line make angle α with each of the coordinate axes and let the direction cosines of this line be l, m, n. Then,

$$l = m = n = \cos\alpha.$$

$$\therefore \quad (l^2 + m^2 + n^2) = 1 \Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$

$$\Rightarrow 3\cos^2\alpha = 1 \Rightarrow \cos^2 = \frac{1}{3}$$

$$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}.$$

Hence, the d.c.'s of the given line are

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \text{ or } \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}.$$

EXAMPLE 7 *A line makes angle 60° and 45° with the positive direction of x-axis and y-axis respectively. What acute angle does it make with the z-axis?*

SOLUTION Let the given line make an angle γ with the z-axis. Then,

$$\cos^2 60° + \cos^2 45° + \cos^2 \gamma = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2 \gamma = 1$$

$$\Rightarrow \left(\frac{1}{4} + \frac{1}{2}\right) + \cos^2 \gamma = 1 \Rightarrow \cos^2 \gamma = \left(1 - \frac{3}{4}\right) = \frac{1}{4}$$

$$\Rightarrow \cos \gamma = \pm \frac{1}{2}$$

$$\Rightarrow \gamma = 60° \quad \text{or} \quad 120°.$$

Hence, the acute angle which the given line makes with the z-axis is 60°.

EXAMPLE 8 *Find the direction cosines of the vector* $\vec{r} = (6\hat{i} + 2\hat{j} - 3\hat{k})$.

SOLUTION The direction ratios of $\vec{r}$ are 6, 2, –3.

And, $\sqrt{6^2 + 2^2 + (-3)^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7.$

Hence, the direction cosines of $\vec{r}$ are $\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}$.

EXAMPLE 9 *Find the direction cosines of the line segment joining the points A(7, –5, 9) and B(5, – 3, 8).*

SOLUTION D.r.'s of AB are $(5 - 7), (-3 + 5), (8 - 9)$, i.e., –2, 2, –1.

$$|AB| = \sqrt{(-2)^2 + 2^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3.$$

Hence, the d.c.'s of the line AB are $\frac{-2}{3}, \frac{2}{3}, \frac{-1}{3}$.

EXAMPLE 10 *Find the angles made by the vector* $\vec{r} = (\hat{i} + \hat{j} - \hat{k})$ *with the coordinate axes.*

SOLUTION D.r.'s of the given vector $\vec{r}$ are 1, 1, –1.

$$|\vec{r}| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}.$$

$\therefore$ d.c.'s of $\vec{r}$ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$.

Let $\vec{r}$ make angles α, β and γ with the x-axis, y-axis and z-axis respectively. Then, d.c.'s of the x-axis are 1, 0, 0.

$\therefore \quad \cos\alpha = \left\{\left(\frac{1}{\sqrt{3}} \times 1\right) + \left(\frac{1}{\sqrt{3}} \times 0\right) + \left(\frac{1}{\sqrt{3}} \times 0\right)\right\} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$

Similarly, $\cos\beta = \frac{1}{\sqrt{3}} \Rightarrow \beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$

And, $\cos\gamma = \frac{-1}{\sqrt{3}} \Rightarrow \gamma = \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right).$

Hence, the angles made by the given vector $\vec{r}$ with the x-axis, y-axis and z-axis are $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\cos^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ respectively.

EXAMPLE 11 *Find the angle between the lines whose direction ratios are* 3, 2, – 6 *and* 1, 2, 2.

SOLUTION Let the given lines be L_1 and L_2 respectively. Then,

D.r.'s of L_1 are 3, 2, –6.

And, $\sqrt{3^2 + 2^2 + (-6)^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7.$

$\therefore \quad$ d.c.'s of L_1 are $\frac{3}{7}, \frac{2}{7}, \frac{-6}{7}.$

D.r.'s of L_2 are 1, 2, 2.

And, $\sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3.$

$\therefore \quad$ d.c.'s of L_2 are $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}.$

Let θ be the angle between L_1 and L_2. Then,

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

$$= \left|\left(\frac{3}{7} \times \frac{1}{3}\right) + \left(\frac{2}{7} \times \frac{2}{3}\right) + \left(\frac{-6}{7} \times \frac{2}{3}\right)\right|$$

$$= \left|\frac{1}{7} + \frac{4}{21} - \frac{4}{7}\right| = \left|\frac{-5}{21}\right| = \frac{5}{21}.$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{5}{21}\right).$$

EXAMPLE 12 *Find the angle between the vectors* $\vec{r}_1 = (4\hat{i} - 3\hat{j} + 5\hat{k})$ *and* $\vec{r}_2 = (3\hat{i} + 4\hat{j} + 5\hat{k})$.

SOLUTION Let θ be the angle between $\vec{r}_1$ and $\vec{r}_2$. Then,

$$\cos\theta = \frac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|} = \frac{(4\hat{i} - 3\hat{j} + 5\hat{k}) \cdot (3\hat{i} + 4\hat{j} + 5\hat{k})}{\{\sqrt{4^2 + (-3)^2 + 5^2}\}\{\sqrt{3^2 + 4^2 + 5^2}\}}$$

$$= \frac{(12-12+25)}{\{\sqrt{16+9+25}\}\{\sqrt{9+16+25}\}} = \frac{25}{\{\sqrt{50}\times\sqrt{50}\}} = \frac{25}{50} = \frac{1}{2}\cdot$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}\cdot$$

Hence, the angle between the given vectors is $\frac{\pi}{3}\cdot$

EXAMPLE 13 *Find the direction cosines of the line which is perpendicular to each of the lines having direction ratios* $1, -2, -2$ *and* $0, 2, 1$ *respectively.*

SOLUTION Let the direction ratios of the required line be a, b, c. Then, it being perpendicular to each of the lines having d.r.'s 1, –2, –2 and 0, 2, 1 we have

$$a - 2b - 2c = 0 \quad \text{... (i)}$$

$$0a + 2b + c = 0. \quad \text{... (ii)}$$

On solving (i) and (ii) by cross multiplication, we have

$$\frac{a}{(-2+4)} = \frac{b}{(0-1)} = \frac{c}{(2-0)} \Rightarrow \frac{a}{2} = \frac{b}{-1} = \frac{c}{2}\cdot$$

$\therefore$ d.r.'s of the required line are 2, –1, 2.

And, $\sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$.

$\therefore$ d.c.'s of the required line are $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\cdot$

EXAMPLE 14 *If* $A(8, 2, 0)$, $B(4, 6, -7)$, $C(-3, 1, 2)$ *and* $D(-9, -2, 4)$ *are four given point then find the angle between* $\overrightarrow{AB}$ *and* $\overrightarrow{CD}$.

SOLUTION D.r.'s of $\overrightarrow{AB}$ are $(4-8), (6-2), (-7-0)$, i.e., –4, 4, –7.

$|\overrightarrow{AB}| = \sqrt{(-4)^2 + 4^2 + (-7)^2} = \sqrt{16+16+49} = \sqrt{81} = 9$.

$\therefore$ d.c.'s of $\overrightarrow{AB}$ are $\frac{-4}{9}, \frac{4}{9}, \frac{-7}{9}\cdot$

D.r.'s of $\overrightarrow{CD}$ are $(-9+3), (-2-1), (4-2)$, i.e., –6, –3, 2.

$|\overrightarrow{CD}| = \sqrt{(-6)^2 + (-3)^2 + 2^2} = \sqrt{36+9+4} = \sqrt{49} = 7$.

$\therefore$ d.c.'s of $\overrightarrow{CD}$ are $\frac{-6}{7}, \frac{-3}{7}, \frac{2}{7}\cdot$

Let θ be the acute angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Then,

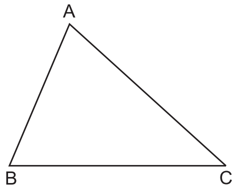
$$\cos\theta = \left|\left(\frac{-4}{9}\right)\times\left(\frac{-6}{7}\right) + \left(\frac{4}{9}\right)\times\left(\frac{-3}{7}\right) + \left(\frac{-7}{9}\right)\times\left(\frac{2}{7}\right)\right|$$

$$= \left|\frac{24}{63} - \frac{12}{63} - \frac{14}{63}\right| = \left|\frac{-2}{63}\right| = \frac{2}{63}.$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{2}{63}\right).$$

Hence, the required angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is $\cos^{-1}\left(\frac{2}{63}\right)$.

EXAMPLE 15 *Find the angles of* $\triangle ABC$ *whose vertices are* $A(-1, 3, 2)$, $B(2, 3, 5)$ *and* $C(3, 5, -2)$.



SOLUTION D.r.'s of AB are $(2+1), (3-3), (5-2)$, i.e. 3, 0, 3.

D.r.'s of AC are $(3+1), (5-3), (-2-2)$, i.e., 4, 2, –4.

$$\therefore \quad \cos A = \frac{\{3 \times 4 + 0 \times 2 + 3 \times (-4)\}}{\left\{\sqrt{3^2 + 0^2 + 3^2}\right\} \cdot \left\{\sqrt{4^2 + 2^2 + (-4)^2}\right\}} = 0.$$

$$\therefore \quad \angle A = \frac{\pi}{2}.$$

D.r.'s of BA are $(-1-2), (3-3), (2-5)$, i.e., –3, 0, –3.

D.r.'s of BC are $(3-2), (5-3), (-2-5)$, i.e., 1, 2, –7.

$$\therefore \quad \cos B = \frac{\{(-3) \times 1 + 0 \times 2 + (-3) \times (-7)\}}{\{\sqrt{(-3)^2 + 0^2 + (-3)^2}\} \cdot \{\sqrt{1^2 + 2^2 + (-7)^2}\}}$$

$$= \frac{18}{\sqrt{18} \times \sqrt{54}} = \frac{1}{\sqrt{3}}.$$

$$\therefore \quad \angle B = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

D.r.'s of CB are $(2-3), (3-5), (5+2)$, i.e., –1, –2, 7.

D.r.'s of CA are $(-1-3), (3-5), (2+2)$, i.e., –4, –2, 4.

$$\therefore \quad \cos C = \frac{\{(-1) \times (-4) + (-2) \times (-2) + 7 \times 4\}}{\{\sqrt{(-1)^2 + (-2)^2 + 7^2}\} \cdot \{\sqrt{(-4)^2 + (-2)^2 + 4^2}\}}$$

$$= \frac{36}{\sqrt{54} \times \sqrt{36}} = \frac{\sqrt{36}}{\sqrt{54}} = \sqrt{\frac{36}{54}} = \sqrt{\frac{2}{3}}.$$

$$\therefore \quad C = \cos^{-1}\left(\sqrt{\frac{2}{5}}\right).$$

Hence, $\angle A = \frac{\pi}{2}$, $\angle B = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\angle C = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$.

EXAMPLE 16 *Show that the points $A(2, 3, -4)$, $B(1, -2, 3)$ and $C(3, 8, -11)$ are collinear.*

SOLUTION D.r.'s of the line AB are $(1-2), (-2-3), (3+4)$, i.e., $-1, -5, 7$.

D.r.'s of the AC are $(3-2), (8-3), (-11+4)$, i.e., $1, 5, -7$, i.e., $-1, -5, 7$.

$\therefore$ d.r.'s of AB and AC are the same and so they are parallel.

But, the lines AB and AC have a common point A.

Hence, the points A, B and C are collinear.

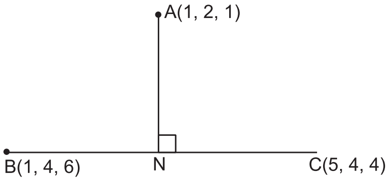
EXAMPLE 17 *Find the coordinates of the foot of the perpendicular drawn from the point $A(1, 2, 1)$ to the line joining $B(1, 4, 6)$ and $C(5, 4, 4)$.*

SOLUTION Draw $AN \perp BC$.

Let N divide BC in the ratio $k:1$.

Then, the coordinates of N are

$$\left(\frac{5k+1}{k+1}, \frac{4k+4}{k+1}, \frac{4k+6}{k+1}\right).$$



$\therefore$ d.r.'s of AN are

$$\left(\frac{5k+1}{k+1}-1, \frac{4k+4}{k+1}-2, \frac{4k+6}{k+1}-1\right), \text{ i.e., } \left(\frac{4k}{k+1}, \frac{2k+2}{k+1}, \frac{3k+5}{k+1}\right)$$

And, the d.r.'s of BC are $(5-1), (4-4), (4-6)$, i.e., $4, 0, -2$.

Since $AN \perp BC$, we have

$$\left(4\times\frac{4k}{k+1}\right)+\left(0\times\frac{2k+2}{k+1}\right)-\left(2\times\frac{3k+5}{k+1}\right)=0$$

$\Rightarrow$ $16k+0-6k-10=0 \Rightarrow 10k=10 \Rightarrow k=1.$

Putting $k=1$, we get the coordinates of N as $N(3, 4, 5)$.

EXAMPLE 18 *If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction ratios of two lines then show that the direction ratios of a line which is perpendicular to each of the given lines are: $(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1), (l_1m_2 - l_2m_1)$.*

SOLUTION Let a, b, c be the direction ratios of the required line.

Then, this line being perpendicular to each of the given two lines, we have:

$$l_1a + m_1b + n_1c = 0 \quad \text{... (i)}$$

$$l_2a + m_2b + n_2c = 0. \quad \text{... (ii)}$$

On solving (i) and (ii) for a, b, c by cross multiplication, we have

$$\frac{a}{(m_1n_2 - m_2n_1)} = \frac{b}{(n_1l_2 - n_2l_1)} = \frac{c}{(l_1m_2 - l_2m_1)}\cdot$$

Hence, the d.r.'s. of the required line are

$$(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1), (l_1m_2 - l_2m_1).$$

EXAMPLE 19 *Find the angle between the two lines whose direction cosines are connected by the relations* $l + m + n = 0$ *and* $l^2 + m^2 - n^2 = 0$.

SOLUTION The given relations are

$$l + m + n = 0 \qquad \text{... (i)}$$

$$l^2 + m^2 - n^2 = 0. \qquad \text{... (ii)}$$

From (i), we get $l = -(m + n)$.

Putting $l = -(m + n)$ in (ii), we get

$$\{-(m+n)\}^2 + m^2 - n^2 = 0 \Rightarrow 2m^2 + 2mn = 0$$

$$\Rightarrow 2m(m+n) = 0$$

$$\Rightarrow m = 0 \text{ or } (m+n) = 0$$

$$\Rightarrow m = 0 \text{ or } m = -n.$$

Putting $m = 0$ in (i), we get $l = -n$.

Putting $m = -n$ in (i), we get $l = 0$.

$\therefore$ d.r.'s of these lines are $-n, 0, n$ and $0, -n, n$, i.e., –1, 0, 1 and 0, –1, 1.

If θ is the acute angle between these lines, we have

$$\cos\theta = \frac{|(-1)\times 0 + 0\times(-1) + 1\times 1|}{\left\{\sqrt{(-1)^2 + 0^2 + 1^2}\right\}\cdot\left\{\sqrt{0^2 + (-1)^2 + 1^2}\right\}} = \frac{1}{(\sqrt{2}\times\sqrt{2})} = \frac{1}{2}\cdot$$

$$\therefore \quad \theta = \frac{\pi}{3}\cdot$$

Hence, the angle between the given lines is $\frac{\pi}{3}\cdot$

EXAMPLE 20 *Find the direction cosines of the lines which are connected by the relations* $l - 5m + 3n = 0$ *and* $7l^2 + 5m^2 - 3n^2 = 0$.

SOLUTION The given equations are

$$l - 5m + 3n = 0 \qquad \text{... (i)}$$

$$7l^2 + 5m^2 - 3n^2 = 0. \qquad \text{... (ii)}$$

Putting $l = (5m - 3n)$ from (i) in (ii), we get

$$7(5m - 3n)^2 + 5m^2 - 3n^2 = 0$$

$$\Rightarrow \quad 6m^2 - 7mn + 2n^2 = 0$$

$$\Rightarrow \quad 6\left(\frac{m}{n}\right)^2 - 7\left(\frac{m}{n}\right) + 2 = 0 \Rightarrow 6p^2 - 7p + 2 = 0, \text{ where } \frac{m}{n} = p$$

$$\Rightarrow \quad (3p-2)(2p-1) = 0$$

$$\Rightarrow \quad p = \frac{2}{3} \text{ or } p = \frac{1}{2} \quad \Rightarrow \quad \frac{m}{n} = \frac{2}{3} \text{ or } \frac{m}{n} = \frac{1}{2}.$$

Now, $\frac{m}{n} = \frac{2}{3} \Rightarrow \frac{m}{2} = \frac{n}{3} = \frac{5m-3n}{5\times 2 - 3\times 3} = \frac{l}{1}$

$$\Rightarrow \quad \frac{l}{1} = \frac{m}{2} = \frac{n}{3} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1^2+2^2+3^2}} = \frac{1}{\sqrt{14}}$$

$$\Rightarrow \quad l = \frac{1}{\sqrt{14}}, m = \frac{2}{\sqrt{14}}, n = \frac{3}{\sqrt{14}}.$$

Again, $\frac{m}{n} = \frac{1}{2} \Rightarrow \frac{m}{1} = \frac{n}{2} = \frac{5m-3n}{5\times 1 - 3\times 2} = \frac{l}{-1}$

$$\Rightarrow \frac{l}{-1} = \frac{m}{1} = \frac{n}{2} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{(-1)^2+1^2+2^2}} = \frac{1}{\sqrt{6}}$$

$$\Rightarrow l = \frac{-1}{\sqrt{6}}, m = \frac{1}{\sqrt{6}}, n = \frac{2}{\sqrt{6}}.$$

Hence, the direction cosines of the lines are

$$\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) \text{ and } \left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right).$$

EXAMPLE 21 *If a variable line in two adjacent positions has direction cosines (l, m, n) and $(l+\delta l, m+\delta m, n+\delta n)$, show that the small angle $\delta\theta$ between the two positions is given by*

$$(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2.$$

SOLUTION Clearly, we have

$$l^2 + m^2 + n^2 = 1 \quad \ldots \text{(i)}$$

and $(l+\delta l)^2 + (m+\delta m)^2 + (n+\delta n)^2 = 1.$ … (ii)

Subtracting (i) from (ii), we get

$$(l+\delta l)^2 + (m+\delta m)^2 + (n+\delta n)^2 - (l^2+m^2+n^2) = 0$$

$$\Rightarrow \quad (\delta l)^2 + (\delta m)^2 + (\delta n)^2 = -2(l\cdot\delta l + m\cdot\delta m + n\cdot\delta n) \quad \ldots \text{(iii)}$$

$$\therefore \quad \cos\delta\theta = l\cdot(l+\delta l) + m\cdot(m+\delta m) + n\cdot(n+\delta n)$$

$$= (l^2+m^2+n^2) + (l\cdot\delta l + m\cdot\delta m + n\cdot\delta n)$$

$$= 1 - \frac{1}{2}\{(\delta l)^2 + (\delta m)^2 + (\delta n)^2\} \quad \text{[using (i) and (iii)]}.$$

$$\therefore \quad (\delta l)^2 + (\delta m)^2 + (\delta n)^2 = 2(1 - \cos \delta\theta)$$

$$= 4 \sin^2 \frac{\delta\theta}{2} = 4 \cdot \left(\frac{\delta\theta}{2}\right)^2$$

$$\left[\because \frac{\delta\theta}{2} \text{ being small, } \sin \frac{\delta\theta}{2} = \frac{\delta\theta}{2}\right]$$

$$= (\delta\theta)^2.$$

Hence, $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

EXAMPLE 22 *Prove that the straight lines whose direction cosines are given by the relations* $al + bm + cn = 0$ *and* $fmn + gnl + hlm = 0$ *are*

perpendicular to each other if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$, *and*

parallel if $a^2f^2 + b^2g^2 + c^2h^2 - 2bcgh - 2cahf - 2abfg = 0$.

SOLUTION The given equations are

$$al + bm + cn = 0 \quad \ldots \text{(i)}$$

$$fmn + gnl + hlm = 0. \quad \ldots \text{(ii)}$$

Putting $n = \frac{-(al + bm)}{c}$ from (i) in (ii), we get

$$fm \cdot \left\{\frac{-(al + bm)}{c}\right\} + gl \cdot \left\{\frac{-(al + bm)}{c}\right\} + hlm = 0$$

$$\Rightarrow \quad agl^2 + (af + bg - ch)\, lm + bfm^2 = 0$$

$$\Rightarrow \quad ag\left(\frac{l}{m}\right)^2 + (af + bg - ch) \cdot \left(\frac{l}{m}\right) + bf = 0 \quad \ldots \text{(iii)}$$

Now, equation (iii), being a quadratic equation in $\left(\frac{l}{m}\right)$, will have two roots, say $\left(\frac{l_1}{m_1}\right)$ and $\left(\frac{l_2}{m_2}\right)$.

$$\therefore \quad \frac{l_1}{m_1} \times \frac{l_2}{m_2} = \frac{bf}{ag} \Rightarrow \frac{l_1 l_2}{bf} = \frac{m_1 m_2}{ag}$$

$$\Rightarrow \frac{l_1 l_2}{\left(\frac{f}{a}\right)} = \frac{m_1 m_2}{\left(\frac{g}{b}\right)} = \frac{n_1 n_2}{\left(\frac{h}{c}\right)} = k \quad \text{[by symmetry].}$$

$$\therefore \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = k\left(\frac{f}{a} + \frac{g}{b} + \frac{h}{c}\right).$$

Thus, the given lines will be perpendicular to each other

$$\Leftrightarrow \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \Leftrightarrow \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0.$$

The lines will be parallel only when the roots of (iii) are equal.

$\therefore \quad (af + bg - ch)^2 - 4abgf = 0$

$\Leftrightarrow \quad a^2f^2 + b^2g^2 + c^2h^2 - 2bcgh - 2cahf - 2abfg = 0.$

EXAMPLE 23 *Show that the straight lines whose direction cosines are given by the equations* $al + bm + cn = 0$ *and* $ul^2 + vm^2 + wn^2 = 0$ *are mutually perpendicular if* $a^2(v + w) + b^2(u + w) + c^2(u + v) = 0$, *and parallel if* $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0.$

SOLUTION The given equations are

$$al + bm + cn = 0 \quad \ldots \text{(i)}$$

$$ul^2 + vm^2 + wn^2 = 0. \quad \ldots \text{(ii)}$$

Putting $l = \frac{-(bm + cn)}{a}$ from (i) in (ii), we get

$$\frac{u(bm + cn)^2}{a^2} + vm^2 + wn^2 = 0$$

$$\Rightarrow \quad (b^2u + a^2v)m^2 + 2ubcmn + (c^2u + a^2w)n^2 = 0$$

$$\Rightarrow \quad (b^2u + a^2v)\left(\frac{m}{n}\right)^2 + 2ubc\left(\frac{m}{n}\right) + (c^2u + a^2w) = 0. \quad \ldots \text{(iii)}$$

Let $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$ be the roots of (iii).

Then, $\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{c^2u + a^2w}{b^2u + a^2v}$

$$\Rightarrow \quad \frac{m_1m_2}{c^2u + a^2w} = \frac{n_1n_2}{b^2u + a^2v} = \frac{l_1l_2}{b^2w + c^2v} = k \quad \text{[by symmetry]}.$$

$$\therefore \quad l_1l_2 + m_1m_2 + n_1n_2 = k(b^2w + c^2v + c^2u + a^2w + b^2u + a^2v).$$

The given lines are mutually perpendicular

$\Leftrightarrow \quad l_1l_2 + m_1m_2 + n_1n_2 = 0$

$\Leftrightarrow \quad a^2(v + w) + b^2(w + u) + c^2(u + v) = 0.$

For the given lines to be parallel, the direction cosines must be equal.

$\therefore$ the roots of (iii) must be equal.

$$\therefore \quad 4u^2b^2c^2 - 4(b^2u + a^2v)(c^2u + a^2w) = 0 \Leftrightarrow \frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0.$$

EXAMPLE 24 *If the edges of a rectangular parallelepiped are a, b, c, prove that the angles between the four diagonals are given by* $\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right)$.

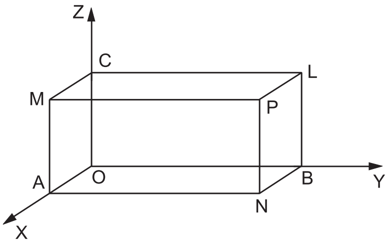
SOLUTION Let OA, OB, OC be the coterminous edges of the parallelepiped, taken along the axes in such a way that $OA = a$, $OB = b$ and $OC = c$. Then, the coordinates of the vertices are

$O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$,

$C(0, 0, c)$, $P(a, b, c)$, $L(0, b, c)$,

$M(a, 0, c)$ and $N(a, b, 0)$.

The direction ratios of the diagonals OP, AL, BM and CN are (a, b, c), $(-a, b, c)$, $(a, -b, c)$ and $(a, b, -c)$ respectively.



$\therefore$ the direction cosines of OP, AL, BM and CN are

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right),$$

$$\left(\frac{-a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right),$$

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{-b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right),$$

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{-c}{\sqrt{a^2+b^2+c^2}}\right) \text{ respectively.}$$

Let θ_1 be the angle between OP and AL. Then,

$$\cos\theta_1 = \frac{(-a^2+b^2+c^2)}{(a^2+b^2+c^2)} \quad \text{or} \quad \theta_1 = \cos^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right).$$

Again, let θ_2 be the angle between OP and BM. Then,

$$\cos\theta_2 = \frac{(a^2-b^2+c^2)}{(a^2+b^2+c^2)} \quad \text{or} \quad \theta_2 = \cos^{-1}\left(\frac{a^2-b^2+c^2}{a^2+b^2+c^2}\right).$$

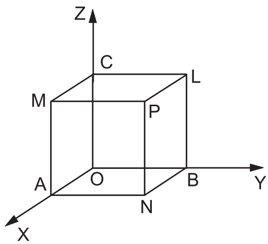
Similarly, the angles between the other pairs of diagonals can be obtained.

Clearly, the angles between the four diagonals can be given by

$$\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right).$$

EXAMPLE 25 *Show that the angle between any two diagonals of a cube is* $\cos^{-1}\left(\frac{1}{3}\right)$.

SOLUTION Let OA, OB, OC be the coterminous edges of a cube, taken along the axes in such a way that $OA = OB = OC = a$.



Then, the coordinates of the vertices of the cube are

$O(0, 0, 0)$, $A(a, 0, 0,)$, $B(0, a, 0)$, $C(0, 0, a)$, $P(a, a, a,)$, $L(0, a, a)$, $M(a, 0, a)$ and $N(a, a, 0)$.

The direction ratios of the diagonals OP, AL, BM and CN are $(a, a, a), (-a, a, a), (a, -a, a)$ and $(a, a, -a)$ respectively.

Thus, direction cosines of OP, AL, BM and CN are

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$ respectively.

If θ_1 be the angle between OP and AL then

$$\cos\theta_1 = \left\{\frac{1}{\sqrt{3}}\cdot\left(\frac{-1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right\} = \frac{1}{3}$$

$$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{1}{3}\right).$$

Similarly, the angle between each one of the other pairs is $\cos^{-1}\left(\frac{1}{3}\right)$.

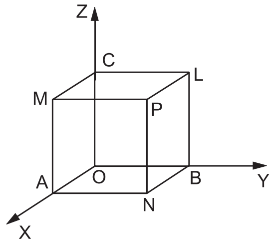
Hence, the angle between any two diagonals of the cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

EXAMPLE 26 *A line makes angles* $\alpha, \beta, \gamma, \delta$ *with the four diagonals of a cube. Prove that*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}. \qquad \textbf{[CBSE 2007]}$$

SOLUTION Let OA, OB, OC be the coterminous edges of a cube, taken along the axes in such a way that $OA = OB = OC = a$.

Then, the coordinates of the vertices of the cube are $O(0, 0, 0)$, $A(a, 0, 0)$,
$B(0, a, 0)$, $C(0, 0, a)$, $P(a, a, a)$,
$L(0, a, a)$, $M(a, 0, a)$ and $N(a, a, 0)$.



The direction ratios of the diagonals OP, AL, BM and CN are
$(a, a, a), (-a, a, a), (a, -a, a)$ and $(a, a, -a)$ respectively.

$\therefore$ the direction cosines of OP, AL, BM and CN are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ and}$$

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \text{ respectively.}$$

Let (l, m, n) be the direction cosines of a line which makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of the cube. Then,

$$\cos\alpha = \left(l \cdot \frac{1}{\sqrt{3}} + m \cdot \frac{1}{\sqrt{3}} + n \cdot \frac{1}{\sqrt{3}}\right) = \frac{(l+m+n)}{\sqrt{3}},$$

$$\cos\beta = \left\{l \cdot \left(\frac{-1}{\sqrt{3}}\right) + m \cdot \frac{1}{\sqrt{3}} + n \cdot \frac{1}{\sqrt{3}}\right\} = \frac{(-l+m+n)}{\sqrt{3}},$$

$$\cos\gamma = \left\{l \cdot \frac{1}{\sqrt{3}} + m \cdot \left(\frac{-1}{\sqrt{3}}\right) + n \cdot \frac{1}{\sqrt{3}}\right\} = \frac{(l-m+n)}{\sqrt{3}},$$

$$\cos\delta = \left\{l \cdot \frac{1}{\sqrt{3}} + m \cdot \frac{1}{\sqrt{3}} + n \cdot \left(\frac{-1}{\sqrt{3}}\right)\right\} = \frac{(l+m-n)}{\sqrt{3}}.$$

On squaring and adding, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3} \cdot \{(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2\}$$

$$= \frac{4}{3}.$$

EXERCISE 26

1. Find the direction cosines of a line segment whose direction ratios are:
   (i) 2, –6, 3 (ii) 2, –1, –2 (iii) –9, 6, –2
2. Find the direction ratios and the direction cosines of the line segment joining the points:
   (i) $A(1, 0, 0)$ and $B(0, 1, 1)$ **[CBSE 2011]**
   (ii) $A(5, 6, -3)$ and $B(1, -6, 3)$
   (iii) $A(-5, 7, -9)$ and $B(-3, 4, -6)$

3. Show that the line joining the points $A(1, -1, 2)$ and $B(3, 4, -2)$ is perpendicular to the line joining the points $C(0, 3, 2)$ and $D(3, 5, 6)$.

4. Show that the line segment joining the origin to the point $A(2, 1, 1)$ is perpendicular to the line segment joining the points $B(3, 5, -1)$ and $C(4, 3, -1)$.

5. Find the value of p for which the line through the points $A(4, 1, 2)$ and $B(5, p, 0)$ is perpendicular to the line through the points $C(2, 1, 1)$ and $D(3, 3, -1)$.

6. If O be the origin and $P(2, 3, 4)$ and $Q(1, -2, 1)$ be any two points, show that $OP \perp OQ$.

7. Show that the line segment joining the points $A(1, 2, 3)$ and $B(4, 5, 7)$ is parallel to the line segment joining the points $C(-4, 3, -6)$ and $D(2, 9, 2)$.

8. If the line segment joining the points $A(7, p, 2)$ and $B(q, -2, 5)$ be parallel to the line segment joining the points $C(2, -3, 5)$ and $D(-6, -15, 11)$, find the values of p and q.

9. Show that the points $A(2, 3, 4)$, $B(-1, -2, 1)$ and $C(5, 8, 7)$ are collinear.

10. Show that the points $A(-2, 4, 7)$, $B(3, -6, -8)$ and $C(1, -2, -2)$ are collinear.

11. Find the value of p for which the points $A(-1, 3, 2)$, $B(-4, 2, -2)$, and $C(5, 5, p)$ are collinear.

12. Find the angle between the two lines whose direction cosines are:
$$\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \text{ and } \frac{3}{7}, \frac{2}{7}, \frac{6}{7}.$$

13. Find the angle between the two lines whose direction ratios are:
$$a, b, c \text{ and } (b-c), (c-a), (a-b).$$

14. Find the angle between the lines whose direction ratios are:
$$2, -3, 4 \text{ and } 1, 2, 1.$$

15. Find the angle between the lines whose direction ratios are:
$$1, 1, 2 \text{ and } (\sqrt{3}-1), (-\sqrt{3}-1), 4.$$

16. Find the angle between the vectors $\vec{r}_1 = (3\hat{i} - 2\hat{j} + \hat{k})$ and $\vec{r}_2 = (4\hat{i} + 5\hat{j} + 7\hat{k})$.

17. Find the angles made by the following vectors with the coordinate axes:

    (i) $(\hat{i} - \hat{j} + \hat{k})$ (ii) $(\hat{j} - \hat{k})$ (iii) $(\hat{i} - 4\hat{j} + 8\hat{k})$

18. Find the coordinates of the foot of the perpendicular drawn from the point $A(1, 8, 4)$ to the line joining the points $B(0, -1, 3)$ and $C(2, -3, -1)$.

[CBSE 2004, '05]

ANSWERS (EXERCISE 26)

1. (i) $\frac{2}{7}, \frac{-6}{7}, \frac{3}{7}$ (ii) $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$ (iii) $\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$

2. (i) d.r.'s are –1, 1, 1 and d.c.'s are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

(ii) d.r.'s are 2, 6, –3 and d.c.'s are $\frac{2}{7}, \frac{6}{7}, \frac{-3}{7}$.

(iii) d.r.'s are 2, –3, 3 and d.c.'s are $\frac{2}{\sqrt{22}}, \frac{-3}{\sqrt{22}}, \frac{3}{\sqrt{22}}$.

5. $p = \frac{-3}{2}$ **8.** $p = 4, q = 3$ **11.** $p = 10$ **12.** $\cos^{-1}\left(\frac{-8}{21}\right)$

13. $\frac{\pi}{2}$ **14.** $\frac{\pi}{2}$ **15.** $\frac{\pi}{3}$ **16.** $\cos^{-1}\left(\frac{3}{2\sqrt{35}}\right)$

17. (i) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right), \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right), \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (ii) $\frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}$

(iii) $\cos^{-1}\left(\frac{1}{9}\right), \cos^{-1}\left(\frac{4}{9}\right), \cos^{-1}\left(\frac{8}{9}\right)$

18. $\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right)$

HINTS TO SOME SELECTED QUESTIONS

1. (ii) D.r.'s of AB are $(1-5), (-6-6), (3+3)$, i.e., –4, –12, 6, i.e., 2, 6, –3.

$$\sqrt{2^2 + 6^2 + (-3)^2} = \sqrt{4 + 36 + 9} = \sqrt{49} = 7.$$

$\therefore$ d.c.'s of AB are $\frac{2}{7}, \frac{6}{7}, \frac{-3}{7}$.

7. $a_1 = (4-1) = 3$, $b_1 = (5-2) = 3$, $c_1 = (7-3) = 4$.

$a_2 = (2+4) = 6$, $b_2 = (9-3) = 6$, $c_2 = (2+6) = 8$.

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{2}$. Hence, $AB \parallel CD$.

8. $a_1 = (q-7)$, $b_1 = (-2-p)$, $c_1 = (5-2) = 3$.

$a_2 = (-6-2) = -8$, $b_2 = (-15+3) = -12$, $c_2 = (11-5) = 6$.

Since $AB \parallel CD$, we have $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

$\therefore \frac{q-7}{-8} = \frac{-2-p}{-12} = \frac{3}{6} = \frac{1}{2}$. Find p and q.

9. D.r.'s of AB are $(-1-2), (-2,-3), (1-4)$, i.e., –3, –5, –3, i.e., 3, 5, 3.

D.r.'s. of AC are $(5-2), (8-3), (7-4)$, i.e., 3, 5, 3.

$\therefore$ $AB \parallel AC$ and both have a common end point A.

Hence, A, B and C are collinear.

10. D.r.'s of AB are $(3+2), (-6-4), (-8-7)$, i.e., $5, -10, -15$, i.e., $1, -2, -3$.

D.r.'s of AC are $(1+2), (-2-4), (-2-7)$, i.e., $3, -6, -9$, i.e., $1, -2, -3$.

$\therefore \quad AB \parallel AC$ and both have a common end point A.

Hence, A, B and C are collinear.

11. D.r.'s of AB are $(-4+1), (2-3), (-2-2)$, i.e., $-3, -1, -4$, i.e., $3, 1, 4$.

D.r.'s of AC are $(5+1), (5-3), (p-2)$, i.e., $6, 2, p-2$.

Since A, B and C are collinear, we must have $AB \parallel AC$.

$$\therefore \quad \frac{3}{6} = \frac{1}{2} = \frac{4}{p-2} \Rightarrow p-2 = 8 \Rightarrow p = 10.$$

12. $$\cos\theta = \left[\left(\frac{2}{3}\times\frac{3}{7}\right) + \left(\frac{-1}{3}\times\frac{2}{7}\right) + \left(\frac{-2}{3}\times\frac{6}{7}\right)\right] = \left[\frac{6}{21} - \frac{2}{21} - \frac{12}{21}\right] = \frac{(6-14)}{21} = \frac{-8}{21}\cdot$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{-8}{21}\right).$$

13. $$\cos\theta = \frac{a(b-c) + b(c-a) + c(a-b)}{\sqrt{\Sigma a^2}\sqrt{\Sigma(b-c)^2}} = 0 \Rightarrow \theta = \frac{\pi}{2}\cdot$$

14. $\sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{4+9+16} = \sqrt{29}$ and $\sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$.

$$\therefore \quad \cos\theta = \left(\frac{2}{\sqrt{29}}\times\frac{1}{\sqrt{6}}\right) + \left(\frac{-3}{\sqrt{29}}\times\frac{2}{\sqrt{6}}\right) + \left(\frac{4}{\sqrt{29}}\times\frac{1}{\sqrt{6}}\right)$$

$$= \frac{2}{\sqrt{174}} - \frac{6}{\sqrt{174}} + \frac{4}{\sqrt{174}} = \frac{0}{\sqrt{174}} = 0.$$

$$\therefore \quad \theta = \frac{\pi}{2}\cdot$$

15. $\sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$ and $\sqrt{(\sqrt{3}-1)^2 + (-\sqrt{3}-1)^2 + 4^2} = \sqrt{24}$.

$$\therefore \quad \cos\theta = \left\{\frac{1}{\sqrt{6}}\times\frac{(\sqrt{3}-1)}{\sqrt{24}} + \frac{1}{\sqrt{6}}\times\frac{(-\sqrt{3}-1)}{\sqrt{24}} + \frac{2}{\sqrt{6}}\times\frac{4}{\sqrt{24}}\right\}$$

$$= \frac{(\sqrt{3}-1)}{12} + \frac{(-\sqrt{3}-1)}{12} + \frac{8}{12} = \frac{6}{12} = \frac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{3}\cdot$$

16. $$\cos\theta = \frac{\vec{r}_1\cdot\vec{r}_2}{|\vec{r}_1||\vec{r}_2|} = \frac{(3\hat{i} - 2\hat{j} + \hat{k})\cdot(4\hat{i} + 5\hat{j} + 7\hat{k})}{\left\{\sqrt{3^2 + (-2)^2 + 1^2}\right\}\left\{\sqrt{4^2 + 5^2 + 7^2}\right\}} = \frac{3}{2\sqrt{35}}\cdot$$

17. (i) Let $\vec{a} = \hat{i} - \hat{j} + \hat{k}$.

$$\cos\alpha = \frac{(\hat{i} - \hat{j} + \hat{k})\cdot\hat{i}}{|\hat{i} - \hat{j} + \hat{k}||\hat{i}|} = \frac{1}{\sqrt{3}}, \ \cos\beta = \frac{(\hat{i} - \hat{j} + \hat{k})\cdot\hat{j}}{|\hat{i} - \hat{j} + \hat{k}||\hat{j}|} = \frac{-1}{\sqrt{3}}$$

and $$\cos\gamma = \frac{(\hat{i} - \hat{j} + \hat{k})\cdot\hat{k}}{|\hat{i} - \hat{j} + \hat{k}||\hat{k}|} = \frac{1}{\sqrt{3}}\cdot$$

27. STRAIGHT LINE IN SPACE

Equation of a Line Passing through a Given Point and Parallel to a Given Vector

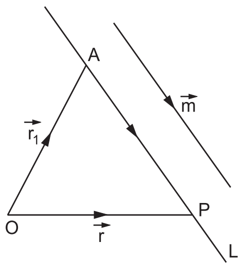
Vector Form

THEOREM 1 *The vector equation of a straight line passing through a given point with position vector $\vec{r_1}$ and parallel to a given vector $\vec{m}$ is $\vec{r} = \vec{r_1} + \lambda \vec{m}$, where λ is a scalar.*

PROOF Let L be the line, passing through a given point A with position vector $\vec{r_1}$ and parallel to a given vector $\vec{m}$.

Let O be the origin. Then, $\overrightarrow{OA} = \vec{r_1}$.

Let P be an arbitrary point on L, and let the position vector of P be $\vec{r}$.



Then, $\overrightarrow{OP} = \vec{r}$.

Clearly, $\overrightarrow{AP} \parallel \vec{m}$

$\Rightarrow \quad \overrightarrow{AP} = \lambda \vec{m}$, for some scalar λ

$\Rightarrow \quad$ (p.v. of P) – (p.v. of A) = $\lambda \vec{m}$

$\Rightarrow \quad \overrightarrow{OP} - \overrightarrow{OA} = \lambda \vec{m}$

$\Rightarrow \quad \vec{r} - \vec{r_1} = \lambda \vec{m}$

$\Rightarrow \quad \vec{r} = \vec{r_1} + \lambda \vec{m}$. $\qquad$ ... (i)

Clearly, every point on the line L satisfies (i), and for any value of λ, (i) gives the position vector of a point P on the line.

Hence, $\vec{r} = \vec{r_1} + \lambda \vec{m}$ is the desired equation.

COROLLARY *The vector equation of a straight line passing through the origin and parallel to a given vector $\vec{m}$ is $\vec{r} = \lambda \vec{m}$.*

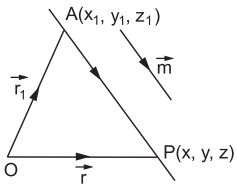
PROOF Taking $\vec{r_1} = \vec{0}$ in (i), we get the desired equation, $\vec{r} = \lambda \vec{m}$.

Cartesian Form

THEOREM 2 *The equations of a straight line with direction ratios a, b, c, and passing through a point $A(x_1, y_1, z_1)$ are*

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

PROOF We know that the vector equation of a straight line passing through a fixed point $A(x_1, y_1, z_1)$ with position vector $\vec{r_1}$ and parallel to a given vector $\vec{m}$ is given by



$\vec{r} = \vec{r_1} + \lambda \vec{m}$ ... (i), where λ is a scalar.

Let $P(x, y, z)$ be the given point on line with position vector $\vec{r}$.

Then, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$.

Since the direction ratios of the given line are a, b, c, and this line is parallel to $\vec{m}$, the direction ratios of $\vec{m}$ are a, b, c.

$\therefore \quad \vec{m} = a\hat{i} + b\hat{j} + c\hat{k}$.

Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{r_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{m} = a\hat{i} + b\hat{j} + c\hat{k}$ in (i), we get the equation of the line as

$$(x\hat{i} + y\hat{j} + z\hat{k}) = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) = (x_1 + \lambda a)\hat{i} + (y_1 + \lambda b)\hat{j} + (z_1 + \lambda c)\hat{k}$$

$$\Rightarrow \quad x = x_1 + \lambda a, \; y = y_1 + \lambda b \text{ and } z = z_1 + \lambda c$$

$$\Rightarrow \quad \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda.$$

Hence, the equations of a line having direction ratios a, b, c and passing through $A(x_1, y_1, z_1)$ are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

COROLLARY *The equations of a line having direction cosines l, m, n and passing through (x_1, y_1, z_1) are*

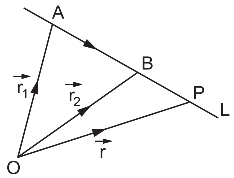
$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

PROOF Since the direction cosines of a line are proportional to the direction ratios of the line, the result follows.

Equation of a Line Passing through Two Given Points

Vector Form

THEOREM 3 *The vector equation of a straight line passing through two points with position vectors $\vec{r_1}$ and $\vec{r_2}$ is given by $\vec{r} = \vec{r_1} + \lambda(\vec{r_2} - \vec{r_1})$.*



PROOF Let L be the given line, passing through two given points A and B with position vectors $\vec{r_1}$ and $\vec{r_2}$ respectively.

Let O be the origin.

Then, $\overrightarrow{OA} = \vec{r_1}$ and $\overrightarrow{OB} = \vec{r_2}$.

$\therefore \quad \overrightarrow{AB} = $ (p.v. of B) − (p.v. of A) $= (\vec{r_2} - \vec{r_1})$.

Let P be an arbitrary point on L, having the position vector $\vec{r}$.

Then, $\overrightarrow{OP} = \vec{r}$.

$\therefore \overrightarrow{AP} = $ (p.v. of P) − (p.v. of A)

$= (\overrightarrow{OP} - \overrightarrow{OA}) = (\vec{r} - \vec{r_1})$.

Since $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear vectors, we have

$\overrightarrow{AP} = \lambda(\overrightarrow{AB})$, for some scalar λ

$\Rightarrow \quad (\vec{r} - \vec{r_1}) = \lambda(\vec{r_2} - \vec{r_1})$

$\Rightarrow \quad \vec{r} = \vec{r_1} + \lambda(\vec{r_2} - \vec{r_1})$.

Hence, the vector equation of a line L, passing through two given points A and B with position vectors $\vec{r_1}$ and $\vec{r_2}$, is given by

$$\vec{r} = \vec{r_1} + \lambda(\vec{r_2} - \vec{r_1}).$$

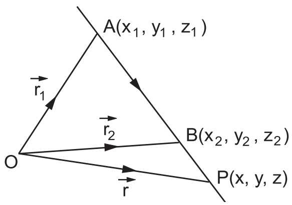
Cartesian Form

THEOREM 4 *The equations of a line passing through two given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are given by*

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

PROOF We know that the vector equation of a line passing through two points A and B with position vectors $\vec{r_1}$ and $\vec{r_2}$ is given by

$$\vec{r} = \vec{r_1} + \lambda(\vec{r_2} - \vec{r_1}). \quad \text{... (i)}$$



Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the points on the given line with position vectors $\vec{r_1}$ and $\vec{r_2}$ respectively.

Let $P(x, y, z)$ be an arbitrary point on this line with position vector $\vec{r}$.

Then, $\vec{r_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{r_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Substituting these values in (i), we get

$$x\hat{i} + y\hat{j} + z\hat{k} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$$

$\Rightarrow \quad (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$

$\quad = \lambda[(x_2 - x_1)\hat{i} + (y_2 - y_1\hat{j} + (z_2 - z_1)\hat{k}]$

$\Rightarrow \quad (x - x_1) = \lambda(x_2 - x_1), (y - y_1) = \lambda(y_2 - y_1)$ and $(z - z_1) = \lambda(z_2 - z_1)$

$\Rightarrow \quad \dfrac{(x - x_1)}{(x_2 - x_1)} = \dfrac{(y - y_1)}{(y_2 - y_1)} = \dfrac{(z - z_1)}{(z_2 - z_1)} (= \lambda),$

which are the required equations.

SUMMARY

1. (i) If a line passes through a point with p.v. $\vec{r_1}$ and it is parallel to $\vec{m}$ then its vector equation is:

$$\vec{r} = \vec{r_1} + \lambda\vec{m}.$$

(ii) If a line passes through a point $A(x_1, y_1 . z_1)$ and it has d.r.'s a, b, c then its Cartesian equations are:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

2. (i) If a line passes through two points having p.v.'s $\vec{r_1}$ and $\vec{r_2}$, then its vector equation is:

$$\vec{r} = \vec{r_1} + \lambda(\vec{r_2} - \vec{r_1}).$$

(ii) If a line passes through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ then its Cartesian equations are:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the vector equation of a line which is parallel to the vector* $(2\hat{i} - \hat{j} + 3\hat{k})$ *and which passes through the point* $(5, -2, 4)$. *Also find its Cartesian equations.* **[CBSE 2007]**

SOLUTION *Vector equation of the given line:*

The given line passes through the point $A(5, -2, 4)$ and it is parallel to the vector $\vec{m} = (2\hat{i} - \hat{j} + 3\hat{k})$.

The position vector of A is given by $\vec{r_1} = (5\hat{i} - 2\hat{j} + 4\hat{k})$.

Hence, the vector equation of the given line is

$$\vec{r} = \vec{r_1} + \lambda\vec{m} \Rightarrow \vec{r} = (5\hat{i} - 2\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k}). \quad \text{... (i)}$$

Cartesisan equation of the given line:

Taking $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$, equation (i) becomes:

$$(x\hat{i} + y\hat{j} + z\hat{k}) = (5\hat{i} - 2\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) = (5 + 2\lambda)\hat{i} - (2 + \lambda)\hat{j} + (4 + 3\lambda)\hat{k}$$

$$\Rightarrow \quad x = 5 + 2\lambda,\ y = -(2 + \lambda) \text{ and } z = 4 + 3\lambda$$

$$\Rightarrow \quad \frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3} = \lambda.$$

Hence, $\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}$ are the required equations of the given line in Cartesian form.

EXAMPLE 2 *The Cartesian equations of a line are* $3x - 3 = 2y + 1 = 5 - 6z$.

(a) Write these equations in standard form and find the direction ratios of the given line.

(b) Write the equation of the given line in vector form.

SOLUTION (a) The given equations may be written as

$$3(x-1) = 2\left(y + \frac{1}{2}\right) = -6\left(z - \frac{5}{6}\right)$$

$$\Rightarrow \quad \frac{(x-1)}{\left(\frac{1}{3}\right)} = \frac{\left(y + \frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = \frac{\left(z - \frac{5}{6}\right)}{\left(\frac{-1}{6}\right)}$$

$$\Rightarrow \quad \frac{(x-1)}{2} = \frac{\left(y + \frac{1}{2}\right)}{3} = \frac{\left(z + \frac{5}{6}\right)}{-1} \quad \text{[on dividing each by 6].}$$

This is the standard form of the given equations in Cartesian form. Clearly, its direction ratios are 2, 3, –1.

(b) The given line passes through the point $A\left(1, \frac{-1}{2}, \frac{5}{6}\right)$ and it is parallel to the vector $\vec{m} = (2\hat{i} + 3\hat{j} - \hat{k})$.

The position vector of the point A is $\vec{r_1} = \left(\hat{i} - \frac{1}{2}\hat{j} + \frac{5}{6}\hat{k}\right)$.

So, the vector equation of the given line is

$$\vec{r} = \vec{r_1} + \lambda \vec{m} \Rightarrow \vec{r} = \left(\hat{i} - \frac{1}{2}\hat{j} + \frac{5}{6}\hat{k}\right) + \lambda(2\hat{i} + 3\hat{j} - \hat{k}).$$

EXAMPLE 3 *The Cartesian equations of a line are* $6x - 2 = 3y + 1 = 2z - 2$.

(a) Write these equations in standard form and find the direction cosines of the given line.

(b) Write down the Cartesian and vector equations of a line, passing through $(2, -1, 1)$ *and parallel to the given line.* **[CBSE 2013C]**

SOLUTION (a) The given equations may be written as

$$6\left(x - \frac{1}{3}\right) = 3\left(y + \frac{1}{3}\right) = 2(z - 1)$$

$$\Rightarrow \frac{\left(x - \frac{1}{3}\right)}{\left(\frac{1}{6}\right)} = \frac{\left(y + \frac{1}{3}\right)}{\left(\frac{1}{3}\right)} = \frac{(z - 1)}{\left(\frac{1}{2}\right)}$$

$$\Rightarrow \frac{\left(x - \frac{1}{3}\right)}{1} = \frac{\left(y + \frac{1}{3}\right)}{2} = \frac{(z - 1)}{3} \quad \text{[on dividing each by 6]}$$

This is the standard form of the given equations in Cartesian form. The direction ratios of the given are 1, 2, 3.

Also, $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.

So, the direction cosines of the given line are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$.

(b) The required line passes through the point $A(2, -1, 1)$ and it is parallel to the line having direction ratios 1,2, 3.

The equations of this line in Cartesian form are given by

$$\frac{x - 2}{1} = \frac{y + 1}{2} = \frac{z - 1}{3}. \quad \text{... (i)}$$

The position vector of the point A is $\vec{r_1} = (2\hat{i} - \hat{j} + \hat{k})$.

The required line is parallel to the vector $\vec{m} = (\hat{i} + 2\hat{j} + 3\hat{k})$.

So, its equation in vector form is

$$\vec{r} = \vec{r_1} + \lambda \vec{m} \Rightarrow \vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k}). \quad \text{... (ii)}$$

Thus, the Cartesian and vector forms of equations of the desired line are given by (i) and (ii) respectively.

EXAMPLE 4 *Find the vector and Cartesian equations of the line passing through the point* (1, 2, – 4) *and perpendicular to each of the lines*

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \text{ and } \frac{x-15}{3}=\frac{y+29}{8}=\frac{z-5}{-5}\cdot \qquad \textbf{[CBSE 2012]}$$

SOLUTION The given lines are

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \qquad \text{... (i)}$$

and $$\frac{x-15}{3}=\frac{y+29}{8}=\frac{z-5}{-5}\cdot \qquad \text{... (ii)}$$

Let a, b, c be the direction ratios of the required line.

Then, it being perpendicular to each of the lines (i) and (ii), we have $3a-16b+7c=0$ and $3a+8b-5c=0$.

On solving these equations by cross multiplication, we get

$$\frac{a}{(80-56)}=\frac{b}{(21+15)}=\frac{c}{(24+48)}$$

$$\Rightarrow \frac{a}{24}=\frac{b}{36}=\frac{c}{72} \Rightarrow \frac{a}{2}=\frac{b}{3}=\frac{c}{6}\cdot$$

Thus, the desired line has direction ratios 2, 3, 6.

So, we have to find the equations of a line passing through the point $A(1, 2, -4)$ and having 2, 3, 6 as its direction ratios.

So, the required equations in Cartesian form are

$$\frac{(x-1)}{2}=\frac{(y-2)}{3}=\frac{(z+4)}{6}\cdot$$

The position vector of point A is $\vec{r_1}=(\hat{i}+2\hat{j}-4\hat{k})$.

Also, the required line has direction ratios 2, 3, 6 and so it is parallel to the vector $\vec{m}=(2\hat{i}+3\hat{j}+6\hat{k})$.

So, its equation in vector form is

$$\vec{r}=\vec{r_1}+\lambda\vec{m} \Rightarrow \vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k}).$$

EXAMPLE 5 *Find the equation of a line passing through the point* $P(2,-1,3)$ *and perpendicular to the lines*

$$\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$$

and $$\vec{r}=(2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k}) \qquad \textbf{[CBSE 2012]}$$

SOLUTION The given lines are

$\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ ... (i), where $\vec{a_1}=(\hat{i}+\hat{j}-\hat{k})$ and $\vec{b_1}=(2\hat{i}-2\hat{j}+\hat{k})$ and

$\vec{r}=\vec{a_2}+\lambda\vec{b_2}$ ... (ii), where $\vec{a_2}=(2\hat{i}-\hat{j}-3\hat{k})$ and $\vec{b_2}=(\hat{i}+2\hat{j}+2\hat{k})$.

The required line is perpendicular to (i) as well as (ii).

Also (i) is parallel to $\vec{b_1}$ and (ii) is parallel to $\vec{b_2}$.

So, the required line is perpendicular to both $\vec{b_1}$ and $\vec{b_2}$.

Consequently, this line must be parallel to $(\vec{b_1} \times \vec{b_2})$.

Now, $(\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = (-6\hat{i} - 3\hat{j} + 6\hat{k})$.

So, we have to find the equation of a line passing through the point $P(2, -1, 3)$ and parallel to $(\vec{b_1} \times \vec{b_2})$.

Hence, the required equation is

$$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + t(-6\hat{i} - 3\hat{j} + 6\hat{k}),$$

where t is an arbitrary constant.

EXAMPLE 6 *Find the vector equation of the line passing through the point* $A(2, -1, 1)$, *and parallel to the line joining the points* $B(-1, 4, 1)$ *and* $C(1, 2, 2)$. *Also, find the Cartesian equations of the line.* **[CBSE 2003]**

SOLUTION *Vector equation of the given line*:

The p.v. of $B = (-\hat{i} + 4\hat{j} + \hat{k})$ and p.v. of $C = (\hat{i} + 2\hat{j} + 2\hat{k})$.

$\therefore\quad \overrightarrow{BC} = (\text{p.v. of } C) - (\text{p.v. of } B)$

$\qquad = (\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + 4\hat{j} + \hat{k}) = (2\hat{i} - 2\hat{j} + \hat{k})$.

The p.v. of A is $\vec{r_1} = 2\hat{i} - \hat{j} + \hat{k}$.

$\therefore$ the vector equation of the given line is

$\vec{r} = \vec{r_1} + \lambda(\overrightarrow{BC})$

$\Leftrightarrow\quad \vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ ... (i)

Cartesian equations of the given line:

Taking $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, equation (i) becomes

$(x\hat{i} + y\hat{j} + z\hat{k}) = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$

$\Leftrightarrow\quad (x\hat{i} + y\hat{j} + z\hat{k}) = (2 + 2\lambda)\hat{i} + (-1 - 2\lambda)\hat{j} + (1 + \lambda)\hat{k}$

$\Leftrightarrow\quad x = 2 + 2\lambda,\ y = -1 - 2\lambda$ and $z = 1 + \lambda$

$$\Leftrightarrow \quad \frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1} = \lambda.$$

Hence, $\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$ are the required equations of the given line in the Cartesian form.

EXAMPLE 7 *Find the vector and Cartesian equations of the line passing through the points $A(2, -1, 4)$ and $B(1, 1, -2)$.*

SOLUTION *Vector equation of the given line*:

Let the position vectors of A and B be $\vec{r_1}$ and $\vec{r_2}$ respectively. Then,

$$\vec{r_1} = 2\hat{i} - \hat{j} + 4\hat{k} \text{ and } \vec{r_2} = \hat{i} + \hat{j} - 2\hat{k}.$$

$$\therefore \quad (\vec{r_2} - \vec{r_1}) = (\hat{i} + \hat{j} - 2\hat{k}) - (2\hat{i} - \hat{j} + 4\hat{k}) = (-\hat{i} + 2\hat{j} - 6\hat{k}).$$

$\therefore$ the vector equation of the line AB is

$$\vec{r} = \vec{r_1} + \lambda(\vec{r_2} - \vec{r_1}) \text{ for some scalar, } \lambda,$$

i.e., $\vec{r} = (2\vec{i} - \vec{j} + 4\vec{k}) + \lambda(-\vec{i} + 2\vec{j} - 6\vec{k}).$ ... (i)

Cartesian equations of the given line:

Taking $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, equation (i) becomes

$$(x\hat{i} + y\hat{j} + z\hat{k}) = (2\hat{i} - \hat{j} + 4\hat{k}) + \lambda(-\hat{i} + 2\hat{j} - 6\hat{k})$$

$$\Leftrightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) = (2-\lambda)\hat{i} + (2\lambda - 1)\hat{j} + (4 - 6\lambda)\hat{k}$$

$$\Leftrightarrow \quad x = 2 - \lambda,\ y = 2\lambda - 1 \text{ and } z = 4 - 6\lambda$$

$$\Leftrightarrow \quad \frac{x-2}{-1} = \frac{y+1}{2} = \frac{z-4}{-6} = \lambda.$$

Hence, $\frac{x-2}{-1} = \frac{y+1}{2} = \frac{z-4}{-6}$ are the Cartesian equations of the given line.

EXAMPLE 8 *Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also find their point of intersection.* **[CBSE 2014]**

SOLUTION The given lines are

$$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} = \lambda \text{ (say)} \quad \text{... (1)}$$

and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5} = \mu$ (say). ... (2)

The general point on (1) is $P(3\lambda - 1, 5\lambda - 3, 7\lambda - 5)$.

The general point on (2) is $Q(\mu + 2,\ 3\mu + 4, 5\mu + 6)$.

The given lines will intersect only when they have a common point. This happens when P and Q coincide for some particular values of λ and μ.

So, the given lines will intersect only when

$$3\lambda - 1 = \mu + 2,\ 5\lambda - 3 = 3\mu + 4 \text{ and } 7\lambda - 5 = 5\mu + 6$$

$$\Rightarrow \quad 3\lambda - \mu = 3 \ \ldots \text{(i)},\ 5\lambda - 3\mu = 7 \ \ldots \text{(ii) and } 7\lambda - 5\mu = 11. \qquad \ldots \text{(iii)}$$

On solving (i) and (ii), we get $\lambda = \frac{1}{2}$ and $\mu = \frac{-3}{2}$.

Clearly, these values of λ and μ also satisfy (iii).

Hence, the given lines intersect.

Putting $\lambda = \frac{1}{2}$ in P or $\mu = \frac{-3}{2}$ in Q, we get the point of intersection of the given lines as $\left(\frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}\right)$.

EXAMPLE 9 *Show that the lines*

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5} \ \textit{and}\ \frac{x+2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$$

do not intersect. **[CBSE 2002]**

SOLUTION The given lines are

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5} = \lambda \text{ (say)} \qquad \ldots (1)$$

$$\frac{x+2}{4} = \frac{y-1}{3} = \frac{z+1}{-2} = \mu \text{ (say)}. \qquad \ldots (2)$$

The general point on (1) is $P(3\lambda + 1, 2\lambda - 1, 5\lambda + 1)$.

The general point on (2) is $Q(4\mu - 2,\ 3\mu + 1, -2\mu - 1)$.

If possible, let the given lines intersect.

Then, P and Q coincide for some particular values of λ and μ.

In that case, we have

$$3\lambda + 1 = 4\mu - 2,\ 2\lambda - 1 = 3\mu + 1 \text{ and } 5\lambda + 1 = -2\mu - 1$$

$$\Leftrightarrow \quad 3\lambda - 4\mu = -3 \ \ldots \text{(i)},\ 2\lambda - 3\mu = 2 \ \ldots \text{(ii)},\ 5\lambda + 2\mu = -2 \ \ldots \text{(iii)}.$$

Solving (i) and (ii), we get $\lambda = -17$ and $\mu = -12$.

However, these values of λ and μ do not satisfy (iii).

Hence, the given lines do not intersect.

EXAMPLE 10 *Show that the lines*

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) \ \textit{and}\ \vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

intersect each other. Find their point of intersection. **[CBSE 2013, '14]**

SOLUTION The given lines are

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) \quad \ldots (1)$$

$$\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k}). \quad \ldots (2)$$

These lines will intersect if for some particular values of λ and μ, the values of $\vec{r}$ given by (1) and (2) are the same.

So, we must have

$$(\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

$\Rightarrow$ $(1 + 3\lambda)\hat{i} + (1 - \lambda)\hat{j} - \hat{k} = (4 + 2\mu)\hat{i} + (3\mu - 1)\hat{k}$

$\Rightarrow$ $1 + 3\lambda = 4 + 2\mu, 1 - \lambda = 0$ and $3\mu - 1 = -1$

$\Rightarrow$ $3\lambda - 2\mu = 3$ ... (i), $\lambda = 1$ ... (ii) and $\mu = 0$ ... (iii).

Clearly, $\lambda = 1$ and $\mu = 0$ also satisfy (i).

Hence, the given lines intersect.

Putting $\lambda = 1$ in (1), we get $\vec{r} = (4\hat{i} + 0\hat{j} - \hat{k})$.

Hence, the point of intersection of the given lines is (4, 0, –1).

EXAMPLE 11 *Show that the lines*

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \text{ and } \vec{r} = (\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + 2\hat{k})$$

do not intersect.

SOLUTION The given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \quad \ldots (1)$$

$$\vec{r} = (\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + 2\hat{k}) \quad \ldots (2)$$

These lines will intersect if for some particular values of λ and μ, the values of $\vec{r}$ given by (1) and (2) are the same.

This happens, when

$$(\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) = (\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + 2\hat{k})$$

$\Rightarrow$ $(1 + \lambda)\hat{i} + (2 - \lambda)\hat{j} + (1 + \lambda)\hat{k} = (1 + \mu)\hat{i} + (1 - \mu)\hat{j} + (1 + 2\mu)\hat{k}$

$\Rightarrow$ $1 + \lambda = 1 + \mu$, $2 - \lambda = 1 - \mu$ and $1 + \lambda = 1 + 2\mu$

$\Rightarrow$ $\lambda - \mu = 0$ … (i), $\lambda - \mu = 1$ … (ii), $\lambda - 2\mu = 0$ … (iii).

From (ii) and (iii), we get $\lambda = 2$ and $\mu = 1$.

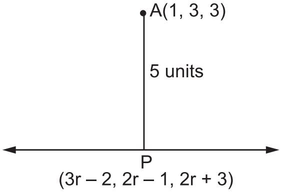
And, these values of λ and μ do not satisfy (i).

Hence, the given lines do not intersect.

EXAMPLE 12 *Find the points on the line* $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ *at a distance of* 5 *units from the point* (1, 3, 3). **[CBSE 2010]**

SOLUTION The given line is

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = r \text{ (say)}. \quad \text{... (i)}$$



The general point on this line is

$P(3r-2, 2r-1, 2r+3)$.

The given point is $A(1, 3, 3)$.

Now, $PA = 5 \Rightarrow (PA)^2 = 25$

$$\Rightarrow (3r-2-1)^2 + (2r-1-3)^2 + (2r+3-3)^2 = 25$$
$$\Rightarrow (3r-3)^2 + (2r-4)^2 + (2r)^2 = 25$$
$$\Rightarrow 17r^2 - 34r = 0 \Rightarrow 17r(r-2) = 0$$
$$\Rightarrow r = 0 \text{ or } r = 2.$$

$r = 0 \Rightarrow$ the required point is $P(-2, -1, 3)$.

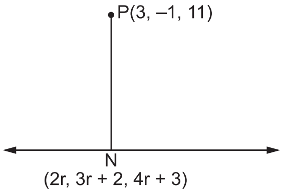
$r = 2 \Rightarrow$ the required point is $P(4, 3, 7)$.

Hence, the rquired points are $(-2, -1, 3)$ and $(4, 3, 7)$.

EXAMPLE 13 *Find the equations of the perpendicular from the point* $(3, -1, 11)$ *to the line* $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. *Also find the coordinates of the foot of the perpendicular and the length of the perpendicular.* **[CBSE 2011C]**

SOLUTION The given line is

$$\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r \text{ (say)}. \quad \text{... (i)}$$



The general point on this line is

$(2r, 3r+2, 4r+3)$.

Let N be the foot of the perpendicular drawn from the point $P(3, -1, 11)$ on the given line.

Then, this point is $N(2r, 3r+2, 4r+3)$ for some fixed value of r.

D.r.'s of PN are $(2r-3), (3r+3), (4r-8)$. ... (ii)

D.r.'s of the given line (i) are 2, 3, 4.

Since PN is perpendicular to the given line (i), we have

$$2(2r-3) + 3(3r+3) + 4(4r-8) = 0 \Rightarrow 29r = 29 \Rightarrow r = 1.$$

$\therefore$ D.r.'s of PN are $-1, 6, -4$ [putting $r = 1$ in (ii)].

So, the required equations of perpendicular PN are

$$\frac{x-3}{-1} = \frac{y+1}{6} = \frac{z-11}{-4}.$$

Now, the coordinates of N are $N(2, 5, 7)$ [*Putting* $r = 1$]

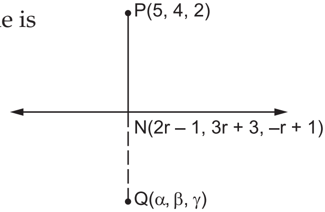
Length of perpendicular PN

$$= \sqrt{(2-3)^2 + (5+1)^2 + (7-11)^2} = \sqrt{(-1)^2 + 6^2 + (-4)^2}$$
$$= \sqrt{1 + 36 + 16} = \sqrt{53}.$$

EXAMPLE 14 *Find the coordinates of the foot of perpendicular and the length of the perpendicular drawn from the point* $P(5, 4, 2)$ *to the line* $\vec{r} = (-\hat{i} + 3\hat{j} + \hat{k}) + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$. *Also find the image of P in this line.* **[CBSE 2012]**

SOLUTION The vector equation of the given line is

$\vec{r} = (-\hat{i} + 3\hat{j} + \hat{k}) + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$.



Clearly, it passes through the point $(-1, 3, 1)$ and it has direction ratios 2, 3, –1.

So, its Cartesian equations are

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} = r \text{ (say)}. \quad \text{... (i)}$$

The general point on this line is $(2r-1, 3r+3, -r+1)$.

Let N be the foot of the perpendicular drawn from the point $P(5, 4, 2)$ on the given line.

Then, this point is $N(2r-1, 3r+3, -r+1)$ for some fixed value of r.

D.r.'s of PN are $(2r-6, 3r-1, -r-1)$.

D.r.'s of the given line are 2, 3, –1.

Since PN is perpendicular to the given line (i), we have

$$2(2r-6) + 3(3r-1) - 1\cdot(-r-1) = 0 \Rightarrow 14r = 14 \Rightarrow r = 1.$$

So, the point N is given by $N(1, 6, 0)$.

Hence, the foot of the perpendicular from the given point $P(5, 4, 2)$ on the given line is $N(1, 6, 0)$.

Let $Q(\alpha, \beta, \gamma)$ be the image of $P(5, 4, 2)$ in the given line.

Then, $N(1, 6, 0)$ is the midpoint of PQ.

$$\therefore \quad \frac{5+\alpha}{2} = 1, \frac{4+\beta}{2} = 6 \text{ and } \frac{2+\gamma}{2} = 0 \Rightarrow \alpha = -3, \beta = 8 \text{ and } \gamma = -2.$$

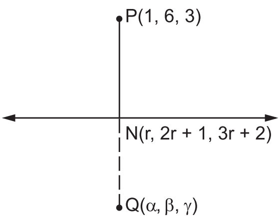
Hence, the image of $P(5, 4, 2)$ in the given line is $Q(-3, 8, -2)$.

EXAMPLE 15 *Find the image of the point* $P(1, 6, 3)$ *in the line* $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

[CBSE 2003C, '08C, '10C]

SOLUTION The equations of the given line are

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = r \text{ (say)}. \quad \text{... (i)}$$



The general point on this line is

$(r, 2r+1, 3r+2)$.

Let N be the foot of the perpendicular drawn from the point $P(1, 6, 3)$ on the given line.

Then, this point is $N(r, 2r+1, 3r+2)$ for some fixed value of r.

D.r.'s of PN are $(r-1, 2r-5, 3r-1)$.

D.r.'s of the given line are 1, 2, 3.

Since, PN is perpendicular to the given line (i), we have

$$1 \cdot (r-1) + 2(2r-5) + 3(3r-1) = 0 \Rightarrow 14r = 14 \Rightarrow r = 1.$$

So, the point N is given by $N(1, 3, 5)$.

Let $Q(\alpha, \beta, \gamma)$ be the image of $P(1, 6, 3)$ in the given line.

Then, N is the midpoint of PQ.

$$\therefore \quad \frac{\alpha+1}{2} = 1, \frac{\beta+6}{2} = 3, \frac{\gamma+3}{2} = 5 \Rightarrow \alpha = 1, \beta = 0 \text{ and } \gamma = 7.$$

Hence, the image of the point $P(1, 6, 3)$ in the given line is $Q(1, 0, 7)$.

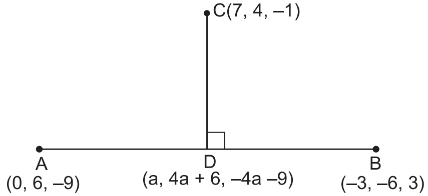
EXAMPLE 16 *Find the equations of the line passing through the points* $A(0, 6, -9)$ *and* $B(-3, -6, 3)$. *If D is the foot of the perpendicular drawn from a point* $C(7, 4, -1)$ *on the line AB then find the coordinates of D and the equations of the line CD.* **[CBSE 2010C]**

SOLUTION The equations of line AB are

$$\frac{x-0}{-3-0} = \frac{y-6}{-6-6} = \frac{z+9}{3+9} \left[\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \right]$$

$$\Rightarrow \quad \frac{x}{-3} = \frac{y-6}{-12} = \frac{z+9}{12}$$

$$\Rightarrow \quad \frac{x}{1} = \frac{y-6}{4} = \frac{z+9}{-4} = a \text{ (say)}. \qquad \text{... (i)}$$



Thus, the required equations of line AB are given by (i).

From (i), we get $x = a$, $y = 4a + 6$ and $z = -4a - 9$.

So, the coordinates of D are $D(a, 4a+6, -4a-9)$ for some particular value of a.

D.r.'s of AB are 1, 4, –4.

D.r.'s of CD are $(a-7), (4a+2), (-4a-8)$.

Since $AB \perp CD$, we have

$$1 \cdot (a-7) + 4(4a+2) - 4(-4a-8) = 0 \Rightarrow 33a = -33 \Rightarrow a = -1.$$

Putting $a = -1$, we get the coordinates of D as $D(-1, 2, -5)$.

The equations of the line CD are

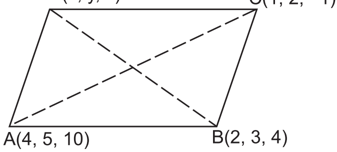
$$\frac{x-7}{-1-7}=\frac{y-4}{2-4}=\frac{z+1}{-5+1} \Rightarrow \frac{x-7}{-8}=\frac{y-4}{-2}=\frac{z+1}{-4}$$

$$\Rightarrow \frac{x-7}{4}=\frac{y-4}{1}=\frac{z+1}{2}.$$

EXAMPLE 17 *The points $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$ are three vertices of a parallelogram ABCD. Find the vector equations of the sides AB and BC and also find the coordinates of point D.* **[CBSE 2010]**

SOLUTION D.r.'s of side AB are $(2-4), (3-5), (4-10)$, i.e., $-2, -2, -6$.

Thus, side AB passes through the point $A(4, 5, 10)$ and it has D.r.'s $-2, -2, -6$.



So, the vector equation of side AB is

$$\vec{r}=(4\hat{i}+5\hat{j}+10\hat{k})+\lambda(-2\hat{i}-2\hat{j}-6\hat{k}).$$

D.r.'s. of side BC are $(1-2)$, $(2-3)$, $(-1-4)$, i.e., $-1, -1, -5$.

Thus, side BC passes through the point $B(2, 3, 4)$ and it has d.r.'s $-1, -1, -5$.

So, the vector equation of side BC is

$$\vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})+\mu(-\hat{i}-3\hat{j}-5\hat{k}).$$

Let the coordinates of D be $D(x, y, z)$.

We know that midpoint of BD coincides with the midpoint of AC.

$$\therefore \quad \frac{2+x}{2}=\frac{4+1}{2}, \ \frac{3+y}{2}=\frac{5+2}{2} \text{ and } \frac{4+z}{2}=\frac{10+(-1)}{2}$$

$\Rightarrow \quad 2+x=5,\ 3+y=7$ and $4+z=9$

$\Rightarrow \quad x=3,\ y=4$ and $z=5$.

$\therefore \quad$ coordinates of D are $D(3, 4, 5)$.

EXERCISE 27A

1. A line passes through the point (3, 4, 5) and is parallel to the vector $(2\hat{i}+2\hat{j}-3\hat{k})$. Find the equations of the line in the vector as well as Cartesian forms.
2. A line passes through the point (2, 1, –3) and is parallel to the vector $(\hat{i}-2\hat{j}+3\hat{k})$. Find the equations of the line in vector and Cartesian forms.

3. Find the vector equation of the line passing through the point with position vector $(2\hat{i} + \hat{j} - 5\hat{k})$ and parallel to the vector $(\hat{i} + 3\hat{j} - \hat{k})$. Deduce the Cartesian equations of the line.

4. A line is drawn in the direction of $(\hat{i} + \hat{j} - 2\hat{k})$ and it passes through a point with position vector $(2\hat{i} - \hat{j} - 4\hat{k})$. Find the equations of the line in the vector as well as Cartesian forms.

5. The Cartesian equations of a line are
$$\frac{x-3}{2} = \frac{y+2}{-5} = \frac{z-6}{4}.$$
Find the vector equation of the line.

6. The Cartesian equations of a line are $3x + 1 = 6y - 2 = 1 - z$. Find the fixed point through which it passes, its direction ratios and also its vector equation. **[CBSE 2004]**

7. Find the Cartesian equations of the line which passes through the point $(1, 3, -2)$ and is parallel to the line given by
$$\frac{x+1}{3} = \frac{y-4}{5} = \frac{z+3}{-6}.$$
Also, find the vector form of the equations so obtained.

8. Find the equations of the line passing through the point $(1, -2, 3)$ and parallel to the line $\frac{x-6}{3} = \frac{y-2}{-4} = \frac{z+7}{5}$.

   Also find the vector form of this equation so obtained.

9. Find the Cartesian and vector equations of a line which passes through the point $(1, 2, 3)$ and is parallel to the line
$$\frac{-x-2}{1} = \frac{y+3}{7} = \frac{2z-6}{3}.$$ **[CBSE 2004, '07]**

10. Find the equations of the line passing through the point $(-1, 3, -2)$ and perpendicular to each of the lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$. **[CBSE 2005, '12]**

11. Find the Cartesian and vector equations of the line passing through the point $(1, 2, -4)$ and perpendicular to each of the lines
$$\frac{x-8}{8} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y+29}{8} = \frac{z-5}{-5}.$$ **[CBSE 2012]**

12. Prove that the lines
$$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} \text{ and } \frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$$
intersect each other and find the point of their intersection.

13. Show that the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-4}{5}=\frac{y-1}{2}=z$$ **[CBSE 2004, '05]**

intersect each other. Also, find the point of their intersection.

14. Show that the lines

$$\frac{x-1}{2}=\frac{y+1}{3}=z \text{ and } \frac{x+1}{5}=\frac{y-2}{1},\ z=2$$

do not intersect each other.

15. Find the coordinates of the foot of the perpendicular drawn from the point (1, 2, 3) to the line

$$\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}.$$ **[CBSE 2003C]**

Also, find the length of the perpendicular from the given point to the line.

16. Find the length and the foot of the perpendicular drawn from the point (2, −1, 5) to the line

$$\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}.$$ **[CBSE 2008, '09C]**

17. Find the vector and Cartesian equations of the line passing through the points $A(3, 4, -6)$ and $B(5, -2, 7)$.

18. Find the vector and Cartesian equations of the line passing through the points $A(2, -3, 0)$ and $B(-2, 4, 3)$.

19. Find the vector and Cartesian equations of the line joining the points whose position vectors are $(\hat{i}-2\hat{j}+\hat{k})$ and $(\hat{i}+3\hat{j}-2\hat{k})$.

20. Find the vector equation of a line passing through the point $A(3, -2, 1)$ and parallel to the line joining the points $B(-2, 4, 2)$ and $C(2, 3, 3)$. Also, find the Cartesian equations of the line.

21. Find the vector equation of a line passing through the point having the position vector $(\hat{i}+2\hat{j}-3\hat{k})$ and parallel to the line joining the points with position vectors $(\hat{i}-\hat{j}+5\hat{k})$ and $(2\hat{i}+3\hat{j}-4\hat{k})$. Also, find the Cartesian equivalents of this equation.

22. Find the coordinates of the foot of the perpendicular drawn from the point $A(1, 2, 1)$ to the line joining the points $B(1, 4, 6)$ and $C(5, 4, 4)$.

23. Find the coordinates of the foot of the perpendicular drawn from the point $A(1, 8, 4)$ to the line joining the points $B(0, -1, 3)$ and $C(2, -3, -1)$. **[CBSE 2005]**

24. Find the image of the point (0, 2, 3) in the line $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$.

25. Find the image of the point (5, 9, 3) in the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$.

26. Find the image of the point (2, –1, 5) in the line

$$\vec{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k}).$$

ANSWERS (EXERCISE 27A)

1. $\vec{r} = (3\hat{i} + 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 2\hat{j} - 3\hat{k}),\ \frac{x-3}{2} = \frac{y-4}{2} = \frac{z-5}{-3}$

2. $\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 3\hat{k}),\ \frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+3}{3}$

3. $\vec{r} = (2\hat{i} + \hat{j} - 5\hat{k}) + \lambda(\hat{i} + 3\hat{j} - \hat{k}),\ \frac{x-2}{1} = \frac{y-1}{3} = \frac{z+5}{-1}$

4. $\vec{r} = (2\hat{i} - \hat{j} + 4\hat{k}) + \lambda(\hat{i} + \hat{j} - 2\hat{k}),\ \frac{x-2}{1} = \frac{y+1}{1} = \frac{z-4}{-2}$

5. $\vec{r} = (3\hat{i} - 2\hat{j} + 6\hat{k}) + \lambda(2\hat{i} - 5\hat{j} + 4\hat{k})$

6. $\left(\frac{-1}{3}, \frac{1}{3}, 1\right)$, d.r.'s are (2, 1, – 6), $\vec{r} = \left(\frac{-1}{3}\hat{i} + \frac{1}{3}\hat{j} + \hat{k}\right) + \lambda(2\hat{i} + \hat{j} - 6\hat{k})$

7. $\frac{x-1}{3} = \frac{y-3}{5} = \frac{z+2}{-6},\ \vec{r} = (\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(3\hat{i} + 5\hat{j} - 6\hat{k})$

8. $\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z-3}{5},\ \vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 4\hat{j} + 5\hat{k})$

9. $\frac{x-1}{-2} = \frac{y-2}{14} = \frac{z-3}{3},\ \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-2\hat{i} + 14\hat{j} + 3\hat{k})$

10. $\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4},\ \vec{r} = (-\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k})$

11. $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6},\ \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

12. (5, –7, 6)

13. (–1, –1, –1)

15. (3, 5, 9), 7 units

16. $\sqrt{14}$ units, (1, 2, 3)

17. $\vec{r} = (3\hat{i} + 4\hat{j} - 6\hat{k}) + \lambda(2\hat{i} - 6\hat{j} + 13\hat{k}),\ \frac{x-3}{2} = \frac{y-4}{-6} = \frac{z+6}{13}$

18. $\vec{r} = (2\hat{i} - 3\hat{j}) + \lambda(-4\hat{i} + 7\hat{j} + 3\hat{k}),\ \frac{x-2}{-4} = \frac{y+3}{7} = \frac{z}{3}$

19. $\vec{r} = (\hat{i} - 2\hat{j} + \hat{k}) + \lambda(5\hat{j} - 3\hat{k}),\ \frac{x-1}{0} = \frac{y+2}{5} = \frac{z-1}{-3}$

20. $\vec{r} = (3\hat{i} - 2\hat{j} + \hat{k}) + \lambda(4\hat{i} - \hat{j} + \hat{k}),\ \frac{x-3}{4} = \frac{y+2}{1} = \frac{z-1}{1}$

21. $\vec{r} = (\hat{i} + 2\hat{j} - 3\hat{k}) + \lambda(\hat{i} + 4\hat{j} - 9\hat{k}),\ \frac{x-1}{1} = \frac{y-2}{4} = \frac{z+3}{-9}$

22. (3, 4, 5) **23.** $\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right)$ **24.** (4, 4, –5)

25. (1, 1, 11) **26.** (0, 5, 1)

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 27A)

9. The given equation may be written as

$$\frac{x+2}{-1} = \frac{y+3}{7} = \frac{z-3}{\frac{3}{2}} \Rightarrow \frac{x+2}{-2} = \frac{y+3}{14} = \frac{z-3}{3}.$$

10. Let the required equations be $\frac{x+1}{a} = \frac{y-3}{b} = \frac{z+2}{c}$. Then,

$$a + 2b + 3c = 0 \qquad \text{... (i)}$$

$$-3a + 2b + 5c = 0 \qquad \text{... (ii)}$$

On solving (i) and (ii) by cross multiplication, we get

$$\frac{a}{(10-6)} = \frac{b}{(-9-5)} = \frac{c}{(2+6)} \Rightarrow \frac{a}{4} = \frac{b}{-14} = \frac{c}{8} \Rightarrow \frac{a}{2} = \frac{b}{-7} = \frac{c}{4}$$

Hence, the required equations are $\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$.

14. The second equation is $\frac{x+1}{5} = \frac{y-2}{1} = \frac{z-2}{0}$.

17. The given line passes through the point $A(3, 4, -6)$ and it has d.r.'s $(5-3), (-2-4), (7+6)$, i.e., 2, –6, 13.

$\therefore$ its vector equation is $\vec{r} = (3\hat{i} + 4\hat{j} - 6\hat{k}) + \lambda(2\hat{i} - 6\hat{j} + 13\hat{k})$.

Its Cartesian equations are $\frac{(x-3)}{2} = \frac{(y-4)}{-6} = \frac{(z+6)}{13}$.

19. The given line passes through the point $A(1, -2, 1)$ and it has d.r.'s $(1-1), (3+2), (-2-1)$, i.e., 0, 5, –3.

So, its vector equation is $\vec{r} = (\hat{i} - 2\hat{j} + \hat{k}) + \lambda(5\hat{j} - 3\hat{k})$.

And, its Cartesian equations are $\frac{x-1}{0} = \frac{y+2}{5} = \frac{z-1}{-3}$.

20. The given line passes through the point $A(3, -2, 1)$ and it has d.r.'s $(2+2), (3-4), (3-2)$, i.e., 4, –1, 1.

21. The required line passes through the point, $A(1, 2, -3)$ and it is parallel to the line joining $B(1, -1, 5)$ and $C(2, 3, -4)$.

$\therefore$ its d.r.'s are $(2-1), (3+1), (-4-5)$, i.e., 1, 4, –9.

23. The equations of BC are $\frac{x-0}{0-2} = \frac{y+1}{-1+3} = \frac{z-3}{3+1} \Rightarrow \frac{x}{-2} = \frac{y+1}{2} = \frac{z-3}{4}$

$$\Rightarrow \frac{x}{-2} = \frac{y+1}{2} = \frac{z-3}{4} \Rightarrow \frac{x}{-1} = \frac{y+1}{1} = \frac{z-3}{2} = r \text{ (say)}.$$

Any general point on BC is $(-r, r-1, 2r+3)$.

Let $AP \perp BC$. Then $P(-r, r-1, 2r+3)$ for some fixed value of r.

d.r.'s of BC are –1, 1, 2.

d.r.'s of AP are $(-r-1), (r-1-8), (2r+3-4)$, i.e., $(-r-1), (r-9), (2r-1)$.

Since $AP \perp BC$, we have $-1\cdot(-r-1) + 1\cdot(r-9) + 2(2r-1) = 0$.

$\therefore \quad 6r = 10 \Rightarrow r = \frac{5}{3}$.

Hence, the coordinates of P are $\left(\frac{-5}{3}, \frac{5}{3} - 1, \frac{10}{3} + 3\right)$, i.e., $\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right)$.

26. $x\hat{i} + y\hat{j} + z\hat{k} = (11 + 10\lambda)\hat{i} - (2 + 4\lambda)\hat{j} - (8 + 11\lambda)\hat{k}$

$\Leftrightarrow x = 11 + 10\lambda, y = -(2 + 4\lambda)$ and $z = -(8 + 11\lambda)$

$\Leftrightarrow \frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}$ are the equations of the given line.

Collinearity of Three Given Points

When the coordinates of Three Points are Given

THEOREM 1 *The condition for three given points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ to be collinear is that*

$$\frac{x_3 - x_1}{x_2 - x_1} = \frac{y_3 - y_1}{y_2 - y_1} = \frac{z_3 - z_1}{z_2 - z_1}.$$

PROOF The equations of the line AB are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \qquad \dots \text{(i)}$$

Clearly, A, B, C will be collinear only when C lies on the line AB. This happens when $C(x_3, y_3, z_3)$ lies on (i).

$$\therefore \quad \frac{x_3 - x_1}{x_2 - x_1} = \frac{y_3 - y_1}{y_2 - y_1} = \frac{z_3 - z_1}{z_2 - z_1}.$$

When Position Vectors of Three Points are Given

THEOREM 2 *Three points A, B and C with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively are collinear if and only if there exist scalars d_1, d_2, d_3 not all zero such that*

$$d_1\vec{a} + d_2\vec{b} + d_3\vec{c} = \vec{0} \text{ and } d_1 + d_2 + d_3 = 0.$$

PROOF Let A, B and C be three collinear points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

Then, the vector equation of line $\overrightarrow{AB}$ is

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \text{ for some scalar } \lambda$$

$\Rightarrow \quad \vec{c} = \vec{a} + \lambda(\vec{b} - \vec{a}) \qquad [\because A, B$ and C being collinear, C lies on $AB]$

$\Rightarrow \quad (1-\lambda)\vec{a} + \lambda\vec{b} - \vec{c} = \vec{0}$

$\Rightarrow \quad d_1\vec{a} + d_2\vec{b} + d_3\vec{c} = \vec{0}$, where $d_3 = -1 \neq 0$ and

$$d_1 + d_2 + d_3 = (1-\lambda) + \lambda - 1 = 0.$$

Conversely, let A, B, and C be the given points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively and let d_1, d_2, d_3 be scalars, not all zero such that $d_1\vec{a} + d_2\vec{b} + d_3\vec{c} = \vec{0}$, and $d_1 + d_2 + d_3 = 0$.

Let $d_3 \neq 0$. Then,

$$d_1\vec{a} + d_2\vec{b} + d_3\vec{c} = 0, \text{ and } d_1 + d_2 + d_3 = 0$$

$$\Rightarrow \quad \left(\frac{d_1}{d_3}\right)\vec{a} + \left(\frac{d_2}{d_3}\right)\vec{b} + \vec{c} = \vec{0} \text{ and } \left(\frac{d_1}{d_3}\right) + \left(\frac{d_2}{d_3}\right) + 1 = 0$$

$$\Rightarrow \quad \left(\frac{d_1}{d_3}\right)\vec{a} - \lambda\vec{b} + \vec{c} = \vec{0} \text{ and } \left(\frac{d_1}{d_3}\right) - \lambda + 1 = 0, \text{ where} \left(\frac{d_2}{d_3}\right) = -\lambda$$

$$\Rightarrow \quad (\lambda - 1)\vec{a} - \lambda\vec{b} + \vec{c} = \vec{0} \quad \left[\because \left(\frac{d_1}{d_3}\right) = (\lambda - 1)\right]$$

$\Rightarrow \quad \vec{c} = (1-\lambda)\vec{a} + \lambda\vec{b}$

$\Rightarrow$ the point C lies on the line $\overrightarrow{AB}$

$\Rightarrow$ the points A, B and C are collinear.

SOLVED EXAMPLES

EXAMPLE 1 *Show that the points $A(2, 0, 3)$, $B(3, 2, -1)$ and $C(1, -2, -5)$ are collinear.*

SOLUTION The equations of the line AB are

$$\frac{x-2}{3-2} = \frac{y-0}{2-0} = \frac{z-3}{-1-3} \Rightarrow \frac{x-2}{1} = \frac{y}{2} = \frac{z-3}{-4}. \qquad \text{... (i)}$$

Putting $x = 1$, $y = -2$ and $z = -5$ in (i), we get

$$\frac{1-2}{1} = \frac{-2}{2} = \frac{-5-3}{-4}, \text{ which is clearly true.}$$

Thus, the point $C(1, -2, -5)$ satisfies the equation of line AB.

$\therefore$ C lies on line AB.

Hence, the given points A, B and C are collinear.

EXAMPLE 2 *Find the value of λ for which the points $A(-1, 3, 2)$, $B(-4, 2, -2)$ and $C(5, 5, \lambda)$ are collinear.*

SOLUTION The equations of the line AB are

$$\frac{x+1}{-4+1} = \frac{y-3}{2-3} = \frac{z-2}{-2-2}$$

$$\Rightarrow \quad \frac{x+1}{-3} = \frac{y-3}{-1} = \frac{z-2}{-4}. \qquad \text{... (i)}$$

Since the points A, B and C are collinear, so the point $C(5, 5, \lambda)$ lies on (i).

$$\therefore \quad \frac{5+1}{-3} = \frac{5-3}{-1} = \frac{\lambda-2}{-4} \Rightarrow \frac{\lambda-2}{-4} = -2 \Rightarrow \lambda - 2 = 8 \Rightarrow \lambda = 10.$$

Hence, the required value of λ is 10.

EXAMPLE 3 *Show that the points whose position vectors are* $(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $(\hat{i} + 2\hat{j} + 3\hat{k})$ *and* $(7\hat{i} - \hat{k})$ *are collinear.*

SOLUTION The coordinates of the given points are $A(-2, 3, 5)$, $B(1, 2, 3)$ and $C(7, 0, -1)$.

The equations of line AB are

$$\frac{x+2}{1+2} = \frac{y-3}{2-3} = \frac{z-5}{3-5} \Rightarrow \frac{x+2}{3} = \frac{y-3}{-1} = \frac{z-5}{-2}. \qquad \text{... (i)}$$

Putting $x = 7$, $y = 0$ and $z = -1$ in (i), we get

$$\frac{7+2}{3} = \frac{0-3}{-1} = \frac{-1-5}{-2}, \text{ which is clearly true.}$$

Thus, the point $C(7, 0, -1)$ satisfies the equation of line AB.

$\therefore$ C lies on line AB.

Hence, the given points A, B and C are collinear.

EXAMPLE 4 *Using the vector method, find the values of* λ *and* μ *for which the points* $A(3, \lambda, \mu)$, $B(2, 0, -3)$ *and* $C(1, -2, -5)$ *are collinear.*

SOLUTION Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vectors of the given points A, B and C respectively. Then,

$$\vec{a} = 3\hat{i} + \lambda\hat{j} + \mu\hat{k}, \quad \vec{b} = 2\hat{i} - 3\hat{k} \text{ and } \vec{c} = \hat{i} - 2\hat{j} - 5\hat{k}.$$

Now, the vector equation of the line BC is given by

$$\vec{r} = \vec{b} + \alpha(\vec{c} - \vec{b}) \text{ for some scalar } \alpha$$

$$\Rightarrow \quad \vec{r} = (1-\alpha)\vec{b} + \alpha\vec{c}$$

$$\Rightarrow \quad \vec{r} = (1-\alpha)(2\hat{i} - 3\hat{k}) + \alpha(\hat{i} - 2\hat{j} - 5\hat{k})$$

$$\Rightarrow \quad \vec{r} = (2 - 2\alpha + \alpha)\hat{i} - 2\alpha\hat{j} + (-3 + 3\alpha - 5\alpha)\hat{k}$$

$$\Rightarrow \quad \vec{r} = (2-\alpha)\hat{i} - 2\alpha\hat{j} + (-3-2\alpha)\hat{k}.$$

If this line passes through the point A then we must have

$$3\hat{i} + \lambda\hat{j} + \mu\hat{k} = (2-\alpha)\hat{i} - 2\alpha\hat{j} + (-3-2\alpha)\hat{k}$$

$\Leftrightarrow$ $2-\alpha = 3$, $-2\alpha = \lambda$ and $-3-2\alpha = \mu$

$\Leftrightarrow$ $\alpha = -1$, $\lambda = 2$ and $\mu = (-3+2) = -1$.

Hence, $\lambda = 2$ and $\mu = -1$.

EXERCISE 27B

1. Show that the points $A(2, 1, 3)$, $B(5, 0, 5)$ and $C(-4, 3, -1)$ are collinear.
2. Show that the points $A(2, 3, -4)$, $B(1, -2, 3)$ and $C(3, 8, -11)$ are collinear. **[CBSE 2006]**
3. Find the value of λ for which the points $A(2, 5, 1)$, $B(1, 2, -1)$ and $C(3, \lambda, 3)$ are collinear.
4. Find the values of λ and μ so that the points $A(3, 2, -4)$, $B(9, 8, -10)$ and $C(\lambda, \mu, -6)$ are collinear.
5. Find the values of λ and μ so that the points $A(-1, 4, -2)$, $B(\lambda, \mu, 1)$ and $C(0, 2, -1)$ are collinear.
6. The position vectors of three points A, B and C are $(-4\hat{i} + 2\hat{j} - 3\hat{k})$, $(\hat{i} + 3\hat{j} - 2\hat{k})$ and $(-9\hat{i} + \hat{j} - 4\hat{k})$ respectively. Show that the points A, B and C are collinear.

ANSWERS (EXERCISE 27B)

3. $\lambda = 8$ **4.** $\lambda = 5$, $\mu = 4$ **5.** $\lambda = 2$, $\mu = -2$

Angle between Two Lines

(i) When the Given Lines are in Vector Form

Let the equations of the given lines in vector form be $\vec{r} = \vec{r_1} + \lambda\vec{m_1}$ and $\vec{r} = \vec{r_2} + \mu\vec{m_2}$, where λ and μ are scalars.

Let θ be the angle between these lines.

Since the given lines are parallel to $\vec{m_1}$ and $\vec{m_2}$ respectively, the angle between the given lines must be equal to the angle between $\vec{m_1}$ and $\vec{m_2}$.

$$\therefore \quad \cos\theta = \frac{|\vec{m_1}\cdot\vec{m_2}|}{|\vec{m_1}|\cdot|\vec{m_2}|}.$$

(ii) When the Given Lines are in Cartesian Form

Let the Cartesian equations of two given lines be

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}.$$

Then, the direction ratios of these lines are a_1, b_1, c_1, and a_2, b_2, c_2 respectively.

Let θ be the angle between these lines. Then,

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{(\sqrt{a_1^2 + b_1^2 + c_1^2})(\sqrt{a_2^2 + b_2^2 + c_2^2})}.$$

SUMMARY

(i) If θ is the angle between the lines $\vec{r} = \vec{r_1} + \lambda\vec{m_1}$ and $\vec{r} = \vec{r_2} + \mu\vec{m_2}$ then

$$\cos\theta = \frac{|\vec{m_1}\cdot\vec{m_2}|}{|\vec{m_1}|\cdot|\vec{m_2}|}.$$

REMARK: $\theta = \frac{\pi}{2} \Leftrightarrow m_1 \cdot m_2 = 0.$

(ii) If θ is the angle between the lines whose Cartesian equations are

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \text{ then}$$

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{(\sqrt{a_1^2 + b_1^2 + c_1^2})(\sqrt{a_2^2 + b_2^2 + c_2^2})}.$$

REMARK: $\theta = \frac{\pi}{2} \Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0.$

SOLVED EXAMPLES

EXAMPLE 1 *Find the angle between the lines*

$$\vec{r} = (2\hat{i} - 5\hat{j} + \hat{k}) + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k}) \textit{ and}$$

$$\vec{r} = (7\hat{i} - 6\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k}).$$ **[CBSE 2014]**

SOLUTION The given lines are of the form

$$\vec{r} = \vec{r_1} + \lambda\vec{m_1} \text{ and } \vec{r} = \vec{r_2} + \lambda\vec{m_2}, \text{ where}$$

$$\vec{m_1} = (3\hat{i} + 2\hat{j} + 6\hat{k}) \text{ and } \vec{m_2} = (\hat{i} + 2\hat{j} + 2\hat{k}).$$

Let θ be the angle between the given lines. Then,

$$\cos\theta = \frac{|\vec{m_1}\cdot\vec{m_2}|}{|\vec{m_1}||\vec{m_2}|}$$

$$= \frac{|(3\hat{i}+2\hat{j}+6\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})|}{(\sqrt{3^2+2^2+6^2})\cdot(\sqrt{1^2+2^2+2^2})} = \frac{(3+4+12)}{(\sqrt{49})(\sqrt{9})}$$

$$= \frac{19}{(7\times 3)} = \frac{19}{21}.$$

$\therefore \quad \theta = \cos^{-1}\left(\frac{19}{21}\right).$

Hence, the angle between the given lines is $\cos^{-1}\left(\frac{19}{21}\right)$.

EXAMPLE 2 *Find the angle between the lines*

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}. \quad \textbf{[CBSE 2011]}$$

SOLUTION The given equations in standard form are

$$\frac{x-2}{2} = \frac{y-1}{7} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{y-4}{2} = \frac{z-5}{4}.$$

Here $a_1 = 2$, $b_1 = 7$, $c_1 = -3$ and $a_2 = -1$, $b_2 = 2$, $c_2 = 4$.

Let θ be the angle between the given lines. Then,

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{(\sqrt{a_1^2+b_1^2+c_1^2})(\sqrt{a_2^2+b_2^2+c_2^2})}$$

$$= \frac{|2\times(-1)+7\times 2+(-3)\times 4|}{\left\{\sqrt{2^2+7^2+(-3)^2}\right\}\left\{\sqrt{(-1)^2+2^2+4^2}\right\}}$$

$$= \frac{|(-2)+14+(-12)|}{(\sqrt{62})(\sqrt{21})} = 0.$$

$\therefore \quad \theta = \frac{\pi}{2}.$

Hence, the angle between the given lines is $\frac{\pi}{2}$.

EXAMPLE 3 *Find the value of λ so that the following lines are perpendicular to each other:*

$$\frac{x-5}{5\lambda+2} = \frac{2-y}{5} = \frac{1-z}{-1} \text{ and } \frac{x}{1} = \frac{2y+1}{4\lambda} = \frac{1-z}{-3}. \quad \textbf{[CBSE 2009]}$$

SOLUTION The given equations in standard form are

$$\frac{x-5}{5\lambda+2} = \frac{y-2}{-5} = \frac{z-1}{1} \text{ and } \frac{x}{1} = \frac{y+\frac{1}{2}}{2\lambda} = \frac{z-1}{3}.$$

Here $a_1 = 5\lambda + 2$, $b_1 = -5$, $c_1 = 1$ and $a_2 = 1$, $b_2 = 2\lambda$ and $c_2 = 3$.

Since the given lines are perpendicular to each other, we must have $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

$\therefore \quad (5\lambda + 2) \times 1 + (-5) \times (2\lambda) + 1 \times 3 = 0$

$\Rightarrow \quad 5\lambda + 2 - 10\lambda + 3 = 0 \Rightarrow 5\lambda = 5 \Rightarrow \lambda = 1.$

Hence, the required value of λ is 1.

EXAMPLE 4 *Find the angle between the lines*

$$\frac{x+1}{1} = \frac{2y-3}{3} = \frac{z-6}{2} \textit{ and } \frac{x-4}{3} = \frac{y+3}{-2},\ z = 5.$$

SOLUTION The given equations may be written as

$$\frac{x+1}{1} = \frac{y-\frac{3}{2}}{\frac{3}{2}} = \frac{z-6}{2} \Rightarrow \frac{x+1}{2} = \frac{y-\frac{3}{2}}{3} = \frac{z-6}{4} \qquad \dots \text{(i)}$$

and $\dfrac{x-4}{3} = \dfrac{y+3}{-2} = \dfrac{z-5}{0}$. $\qquad \dots$ (ii)

Here $a_1 = 2$, $b_1 = 3$, $c_1 = 4$ and $a_2 = 3$, $b_2 = -2$, $c_2 = 0$.

Let θ be the angle between the given lines. Then,

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\left(\sqrt{a_1^2 + b_1^2 + c_1^2}\right)\left(\sqrt{a_2^2 + b_2^2 + c_2^2}\right)}$$

$$= \frac{|(2\times 3) + 3\times(-2) + 4\times 0|}{(\sqrt{4+9+16})(\sqrt{9+4+0})} = \frac{0}{(\sqrt{29}\times\sqrt{13})} = 0.$$

$\therefore \quad \theta = \dfrac{\pi}{2}$.

Hence, the angle between the given lines is $\dfrac{\pi}{2}$.

EXAMPLE 5 *Find the angle between the lines*

$$\frac{5-x}{3} = \frac{y+3}{-4},\ z = 7 \textit{ and } \frac{x}{1} = \frac{1-y}{2} = \frac{z-6}{2}.$$

SOLUTION The given lines in standard form are

$$\frac{x-5}{-3} = \frac{y+3}{-4} = \frac{z-7}{0} \text{ and } \frac{x}{1} = \frac{y-1}{-2} = \frac{z-6}{2}$$

Here $a_1 = -3$, $b_1 = -4$, $c_1 = 0$ and $a_2 = 1$, $b_2 = -2$, $c_2 = 2$.

Let θ be the angle between the given lines. Then,

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\left\{\sqrt{a_1^2 + b_1^2 + c_1^2}\right\}\left\{\sqrt{a_2^2 + b_2^2 + c_2^2}\right\}}$$

$$= \frac{|(-3)\times 1 + (-4)\times(-2) + 0\times 2|}{\{\sqrt{9+16+0}\}\{\sqrt{1+4+4}\}} = \frac{5}{5\times 3} = \frac{1}{3}.$$

$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{3}\right).$

Hence, the angle between the given lines is $\cos^{-1}\left(\frac{1}{3}\right)$.

EXAMPLE 6 *Find the value of k so that the lines*

$$\frac{1-x}{3} = \frac{7y-14}{2k} = \frac{z-3}{2} \text{ and } \frac{7-7x}{3k} = \frac{5-y}{1} = \frac{6-z}{5}$$

are at right angles.

SOLUTION The given equation in standard form are

$$\frac{x-1}{-3} = \frac{y-2}{\left(\frac{2k}{7}\right)} = \frac{z-3}{2} \Rightarrow \frac{x-1}{-21} = \frac{y-2}{2k} = \frac{z-3}{14} \qquad \text{... (i)}$$

$$\frac{x-1}{\left(\frac{-3k}{7}\right)} = \frac{y-5}{-1} = \frac{z-6}{-5} \Rightarrow \frac{x-1}{-3k} = \frac{y-5}{-7} = \frac{z-6}{-35}. \qquad \text{... (ii)}$$

Here, $a_1 = -21$, $b_1 = 2k$, $c_1 = 14$ and $a_2 = -3k$, $b_2 = -7$, $c_2 = -35$.

Since the given lines are at right angles, we have

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\Rightarrow (-21) \times (-3k) + (2k) \times (-7) + 14 \times (-35) = 0$$

$$\Rightarrow 63k - 14k - 490 = 0 \Rightarrow 49k = 490 \Rightarrow k = 10.$$

Hence, $k = 10$.

EXAMPLE 7 *Prove that the lines $x = ay + b, z = cy + d$, and $x = a'y + b', z = c'y + d'$ are perpendicular if $aa' + cc' + 1 = 0$.*

SOLUTION The equations of the first line are

$$x = ay + b, z = cy + d$$

$$\Leftrightarrow \frac{x-b}{a} = y, \frac{z-d}{c} = y$$

$$\Leftrightarrow \frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c}. \qquad \text{... (i)}$$

Similarly, the equations of the second line are

$$\frac{x-b'}{a'} = \frac{y}{1} = \frac{z-d'}{c'}. \qquad \text{... (ii)}$$

The given lines are perpendicular to each other

$$\Leftrightarrow aa' + 1 \times 1 + cc' = 0$$

$$\Leftrightarrow aa' + cc' + 1 = 0.$$

EXAMPLE 8 *Find the angle between the two lines, one of which has direction ratios 2, 2, 1, and the other is obtained by joining the points (3, 1, 4) and (7, 2, 12).*

SOLUTION Let L_1 and L_2 be the given lines.

Then, d.r.'s of L_1 are 2, 2, 1.

D.r.'s of L_2 are (7 – 3), (2 – 1), (12 – 4), i.e., 4, 1, 8.

Let θ be the angle between the given lines. Then,

$$\cos\theta = \frac{|(2\times 4 + 2\times 1 + 1\times 8)|}{(\sqrt{2^2+2^2+1^2})(\sqrt{4^2+1^2+8^2})} = \frac{18}{27} = \frac{2}{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{3}\right).$$

Hence, the angle between the given lines is $\cos^{-1}\left(\frac{2}{3}\right)$.

EXERCISE 27C

Find the angle between each of the following pairs of lines:

1. $\vec{r} = (3\hat{i} + \hat{j} - 2\hat{k}) + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - 5\hat{k}) + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$
2. $\vec{r} = (3\hat{i} - 4\hat{j} + 2\hat{k}) + \lambda(\hat{i} + 3\hat{k})$ and $\vec{r} = 5\hat{i} + \mu(-\hat{i} + \hat{j} + \hat{k})$
3. $\vec{r} = (\hat{i} - 2\hat{j}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\vec{r} = 3\hat{k} + \mu(\hat{i} + 2\hat{j} - 2\hat{k})$

Find the angle between each of the following pairs of lines:

4. $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$ and $\frac{x+3}{3} = \frac{y-2}{5} = \frac{z+5}{4}$
5. $\frac{x-4}{3} = \frac{y+1}{4} = \frac{z-6}{5}$ and $\frac{x-5}{1} = \frac{2y+5}{-2} = \frac{z-3}{1}$
6. $\frac{3-x}{-2} = \frac{y+5}{1} = \frac{1-z}{3}$ and $\frac{x}{3} = \frac{1-y}{-2} = \frac{z+2}{-1}$
7. $\frac{x}{1} = \frac{z}{-1}$, $y = 0$ and $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$
8. $\frac{5-x}{3} = \frac{y+3}{-2}$, $z = 5$ and $\frac{x-1}{1} = \frac{1-y}{3} = \frac{z-5}{2}$
9. Show that the lines $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z-2}{4}$ and $\frac{x+2}{2} = \frac{y-4}{4} = \frac{z+5}{2}$ are perpendicular to each other.
10. If the lines $\frac{x-1}{-3} = \frac{y-2}{2\lambda} = \frac{z-3}{2}$ and $\frac{x-1}{3\lambda} = \frac{y-1}{1} = \frac{6-z}{5}$ are perpendicular to each other then find the value of λ. **[CBSE 2009]**
11. Show that the lines $x = -y = 2z$ and $x + 2 = 2y - 1 = -z + 1$ are perpendicular to each other.

HINT: The given lines are $\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$ and $\frac{x+2}{2} = \frac{y-\frac{1}{2}}{1} = \frac{z-1}{-2}$.

12. Find the angle between two lines whose direction ratios are

(i) 2, 1, 2 and 4, 8, 1 (ii) 5, –12, 13 and –3, 4, 5

(iii) 1, 1, 2 and $(\sqrt{3}-1), (-\sqrt{3}-1), 4$ (iv) a, b, c and $(b-c), (c-a), (a-b)$

13. If $A(1, 2, 3)$, $B(4, 5, 7)$, $C(-4, 3, -6)$ and $D(2, 9, 2)$ are four given points then find the angle between the lines AB and CD.

ANSWERS (EXERCISE 27C)

1. $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$ **2.** $\cos^{-1}\left(\frac{\sqrt{30}}{15}\right)$ **3.** $\cos^{-1}\left(\frac{-4}{9}\right)$ **4.** $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$

5. $\cos^{-1}\left(\frac{2\sqrt{6}}{15}\right)$ **6.** $\cos^{-1}\left(\frac{11}{14}\right)$ **7.** $\cos^{-1}\left(\frac{1}{5}\right)$ **8.** $\cos^{-1}\left(\frac{3}{\sqrt{182}}\right)$

10. $\lambda = \frac{-10}{7}$ **12.** (i) $\cos^{-1}\left(\frac{2}{3}\right)$ (ii) $\cos^{-1}\left(\frac{1}{65}\right)$ (iii) $\frac{\pi}{3}$ (iv) $\frac{\pi}{2}$ **13.** 0°

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 27C)

5. Given lines in standard form are:

$$\frac{x-4}{3} = \frac{y+1}{4} = \frac{z-6}{5} \text{ and } \frac{x-5}{1} = \frac{y+\frac{5}{2}}{-1} = \frac{z-3}{1}.$$

6. Given lines in standard form are:

$$\frac{x-3}{2} = \frac{y+5}{1} = \frac{z-1}{-3} \text{ and } \frac{x}{3} = \frac{y-1}{2} = \frac{z+2}{-1}.$$

7. Given lines in standard form are:

$$\frac{x}{1} = \frac{y-0}{0} = \frac{z}{-1} \text{ and } \frac{x}{3} = \frac{y}{4} = \frac{z}{5}.$$

8. Given lines in standard form are:

$$\frac{x-5}{-3} = \frac{y+3}{-2} = \frac{z-5}{0} \text{ and } \frac{x-1}{1} = \frac{y-1}{-3} = \frac{z-5}{2}.$$

9. Given lines in standard form are:

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1} \text{ and } \frac{x+2}{2} = \frac{y-\frac{1}{2}}{1} = \frac{z-1}{-2}.$$

13. $\overrightarrow{AB}$ = (p.v. of B) – (p.v. of A) $= (4\hat{i} + 5\hat{j} + 7\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (3\hat{i} + 3\hat{j} + 4\hat{k})$

$\overrightarrow{CD}$ = (p.v. of D) – (p.v. of C) $= (2\hat{i} + 9\hat{j} + 2\hat{k}) - (-4\hat{i} + 3\hat{j} - 6\hat{k}) = (6\hat{i} + 6\hat{j} + 8\hat{k})$

$$\cos\theta = \frac{|\overrightarrow{AB}\cdot\overrightarrow{CD}|}{|\overrightarrow{AB}||\overrightarrow{CD}|} = \frac{(3\times 6 + 3\times 6 + 4\times 8)}{(\sqrt{9+9+16})(\sqrt{36+36+64})}$$

$$= \frac{68}{\sqrt{34}\times\sqrt{136}} = \frac{68}{34\times 2} = 1.$$

Shortest Distance between Two Lines

COPLANAR LINES *Two lines lying in the same plane are called coplanar lines.*

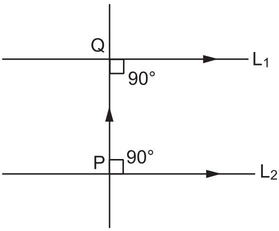
Coplanar lines are either parallel or intersecting.

SKEW LINES *Two lines in space which are not coplanar are called skew lines.*

Skew lines are neither parallel nor intersecting.

LINE OF SHORTEST DISTANCE BETWEEN TWO SKEW LINES If L_1 and L_2 are two skew lines then there is a unique line which is perpendicular to both the lines L_1 and L_2. This line is called the *line of shortest distance* between L_1 and L_2.

SHORTEST DISTANCE BETWEEN TWO SKEW LINES The length of the line segment $\overrightarrow{PQ}$, intercepted by two skew lines L_1 and L_2 on the common perpendicular to both the lines, is called the *shortest distance* (SD) between L_1 and L_2.



REMARK If two lines in space intersect at a point then the shortest distance between them is zero.

To Find the Shortest Distance between Two Skew Lines

Vector Form

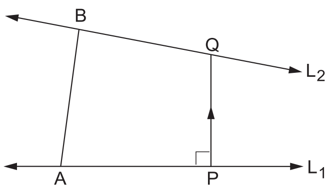
THEOREM *The shortest distance between two skew lines* $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ *and* $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ *is given by*

$$d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|.$$

PROOF Let L_1 and L_2 be two skew lines whose vector equations are respectively

$$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \qquad \text{... (i)}$$

and $$\vec{r} = \vec{a}_2 + \mu \vec{b}_2. \qquad \text{... (ii)}$$



Then, L_1 is parallel to $\vec{b}_1$ and passes through a point A, whose position vector is $\vec{a}_1$.

And, L_2 is parallel to $\vec{b_2}$ and passes through a point B, whose position vector is $\vec{a_2}$.

Let $\overrightarrow{PQ}$ be the shortest-distance vector between L_1 and L_2.

Then, $\overrightarrow{PQ} \perp \vec{b_1}$ and $\overrightarrow{PQ} \perp \vec{b_2}$.

$\therefore \quad \overrightarrow{PQ} \,||\, (\vec{b_1} \times \vec{b_2})$.

$\therefore \quad PQ = |\text{projection of } \overrightarrow{AB} \text{ along } (\vec{b_1} \times \vec{b_2})|$

$$= \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|.$$

CONDITION FOR TWO GIVEN LINES TO INTERSECT Suppose that the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ intersect. Then, the shortest distance between them is zero.

$$\therefore \quad [(\vec{a_2} - \vec{a_1})\ \vec{b_1}\ \ \vec{b_2}] = 0.$$

REMARK Two lines intersect only when the shortest distance between them is zero.

SOLVED EXAMPLES

EXAMPLE 1 *Find the shortest distance between the lines whose vector equations are*

$$\vec{r} = (6\hat{i} + 2\hat{j} + 2\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$$

and $\vec{r} = (-4\hat{i} - \hat{k}) + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$. **[CBSE 2013C]**

SOLUTION Comparing the given equations with the standard equations $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda \vec{b_2}$, we have

$$\vec{a_1} = (6\hat{i} + 2\hat{j} + 2\hat{k}),\ \vec{b_1} = (\hat{i} - 2\hat{j} + 2\hat{k}),$$

$$\vec{a_2} = (-4\hat{i} - \hat{k}) \text{ and } \vec{b_2} = (3\hat{i} - 2\hat{j} - 2\hat{k}).$$

$$\therefore \quad (\vec{a_2} - \vec{a_1}) = (-4\hat{i} - \hat{k}) - (6\hat{i} + 2\hat{j} + 2\hat{k}) = (-10\hat{i} - 2\hat{j} - 3\hat{k})$$

$$\text{and, } (\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = (4+4)\hat{i} - (-2-6)\hat{j} + (-2+6)\hat{k}$$

$$= (8\hat{i} + 8\hat{j} + 4\hat{k}).$$

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{8^2 + 8^2 + 4^2} = \sqrt{64 + 64 + 16}$

$= \sqrt{144} = 12.$

$\therefore \quad \text{SD} = \left| \dfrac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$

$= \left| \dfrac{(-10\hat{i} - 2\hat{j} - 3\hat{k}) \cdot (8\hat{i} + 8\hat{j} + 4\hat{k})}{12} \right| = \left| \dfrac{-80 - 16 - 12}{12} \right|$

$= \left| \dfrac{-108}{12} \right| = |-9| = 9$ units.

EXAMPLE 2 *Find the shortest distance between the lines L_1 and L_2 whose vector equations are given below:*

$L_1 : \vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$

$L_2 : \vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{i} + 2\hat{k})$ **[CBSE 2008C, '14]**

SOLUTION Comparing the given equations with the standard equations $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$, we have

$\vec{a_1} = (\hat{i} + \hat{j}),\ \vec{b_1} = (2\hat{i} - \hat{j} + \hat{k})$

$\vec{a_2} = (2\hat{i} + \hat{j} - \hat{k})$ and $\vec{b_2} = (3\hat{i} - 5\hat{j} + 2\hat{k})$

$\therefore \quad (\vec{a_2} - \vec{a_1}) = (2\hat{i} + \hat{j} - \hat{k}) - (\hat{i} + \hat{j}) = (\hat{i} - \hat{k})$

and, $(\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = (-2 + 5)\hat{i} - (4 - 3)\hat{j} + (-10 + 3)\hat{k}$

$= (3\hat{i} - \hat{j} - 7\hat{k}).$

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{3^2 + (-1)^2 + (-7)^2} = \sqrt{59}.$

$\therefore \quad \text{SD} = \left| \dfrac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| = \dfrac{|(\hat{i} - \hat{k}) \cdot (3\hat{i} - \hat{j} - 7\hat{k})|}{\sqrt{59}}$

$= \dfrac{|3 - 0 + 7|}{\sqrt{59}} = \dfrac{10\sqrt{59}}{59}$ units.

EXAMPLE 3 *Find the shortest distance between the lines whose vector equations are*

$\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k}$, *and*

$\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}.$ **[CBSE 2011]**

SOLUTION The given equations can be written as

$$\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}), \text{ and}$$

$$\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}).$$

Comparing the given equations with the standard equations $\vec{r} = \vec{a_1} + t\,\vec{b_1}$ and $\vec{r} = \vec{a_2} + s\,\vec{b_2}$, we get

$$\vec{a_1} = (\hat{i} - 2\hat{j} + 3\hat{k}),\ \vec{b_1} = (-\hat{i} + \hat{j} - 2\hat{k})$$

$$\vec{a_2} = (\hat{i} - \hat{j} - \hat{k}) \text{ and } \vec{b_2} = (\hat{i} + 2\hat{j} - 2\hat{k}).$$

$$\therefore\quad (\vec{a_2} - \vec{a_1}) = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = (\hat{j} - 4\hat{k})$$

$$\text{and, } (\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= (-2 + 4)\hat{i} - (2 + 2)\hat{j} + (-2 - 1)\hat{k}$$

$$= (2\hat{i} - 4\hat{j} - 3\hat{k}).$$

$$\therefore\quad |\vec{b_1} \times \vec{b_2}| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}.$$

$$\therefore\quad \text{SD} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$= \frac{|(\hat{j} - 4\hat{k}) \cdot (2\hat{i} - 4\hat{j} - 3\hat{k})|}{\sqrt{29}} = \frac{|0 - 4 + 12|}{\sqrt{29}}$$

$$= \frac{8\sqrt{29}}{29} \text{ units.}$$

EXAMPLE 4 *Show that the lines*

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) \textit{ and } \vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

intersect. Find their point of intersection. **[CBSE 2014]**

SOLUTION Comparing the given equations with the standard equations $\vec{r} = \vec{a_1} + \lambda\,\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\,\vec{b_2}$, we get

$$\vec{a_1} = (\hat{i} + \hat{j} - \hat{k}),\ \vec{b_1} = (3\hat{i} - \hat{j}),$$

$$\vec{a_2} = (4\hat{i} - \hat{k}) \text{ and } \vec{b_2} = (2\hat{i} + 3\hat{k}).$$

$$\therefore\quad (\vec{a_2} - \vec{a_1}) = (4\hat{i} - \hat{k}) - (\hat{i} + \hat{j} - \hat{k}) = (3\hat{i} - \hat{j}).$$

$$\text{And, } (\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = (-3-0)\hat{i} - (9-0)\hat{j} + (0+2)\hat{k}$$

$$= (-3\hat{i} - 9\hat{j} + 2\hat{k}).$$

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-3)^2 + (-9)^2 + 2^2} = \sqrt{94}.$

$$\therefore \quad \text{SD} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$= \frac{|(3\hat{i} - \hat{j}) \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k})|}{\sqrt{94}}$$

$$= \frac{|-9 + 9 + 0|}{\sqrt{94}} = 0.$$

Thus, the shortest distance between the given lines is 0.

Hence, the given lines intersect.

Thus, for some particular values of λ and μ, we have

$$(\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

$\Rightarrow \quad (1 + 3\lambda)\hat{i} + (1 - \lambda)\hat{j} - \hat{k} = (4 + 2\mu)\hat{i} + (3\mu - 1)\hat{k}$

$\Rightarrow \quad 1 + 3\lambda = 4 + 2\mu,\ 1 - \lambda = 0$ and $3\mu - 1 = -1$

$\Rightarrow \quad \lambda = 1$ and $\mu = 0$.

Thus, the position vector of the point of intersection of the given lines is given by

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + (3\hat{i} - \hat{j}) \quad [\text{putting } \lambda = 1], \text{ i.e., } \vec{r} = (4\hat{i} - \hat{k}).$$

Hence, the point of intersection of the given lines is $P(4, 0, -1)$.

EXAMPLE 5 *Show that the lines*

$$\vec{r} = (\hat{i} - \hat{j}) + \lambda(2\hat{i} + \hat{k}) \text{ and } \vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} + \hat{j} - \hat{k})$$

do not intersect. **[CBSE 2012C]**

SOLUTION Comparing the given equations with the standard equations $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$, we get

$$\vec{a_1} = (\hat{i} - \hat{j}), \quad \vec{b_1} = (2\hat{i} + \hat{k})$$

$$\vec{a_2} = (2\hat{i} - \hat{j}) \text{ and } \vec{b_2} = (\hat{i} + \hat{j} - \hat{k}).$$

$\therefore \quad (\vec{a_2} - \vec{a_1}) = (2\hat{i} - \hat{j}) - (\hat{i} - \hat{j}) = \hat{i}.$

$$\text{And, } (\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = (0-1)\hat{i} - (-2-1)\hat{j} + (2-0)\hat{k}$$

$$= (-\hat{i} + 3\hat{j} + 2\hat{k}).$$

$$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}.$$

$$\therefore \quad \text{SD} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$= \left| \frac{\hat{i} \cdot (-\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{14}} \right| = \frac{|-1|}{\sqrt{14}}$$

$$= \frac{1 \times \sqrt{14}}{14} = \frac{\sqrt{14}}{14} \neq 0.$$

Since the shortest distance between the given lines is not zero, the given lines do not intersect.

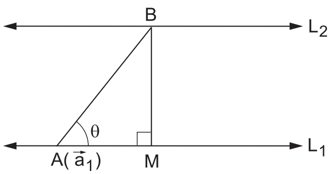
DISTANCE BETWEEN PARALLEL LINES Let L_1 and L_2 be two parallel lines. Then, these lines are clearly coplanar.

Let the equations of these lines be

$$\vec{r} = \vec{a_1} + \lambda \vec{b} \qquad \text{... (i)}$$

$$\vec{r} = \vec{a_2} + \mu \vec{b}. \qquad \text{... (ii)}$$

Let A be a point on L_1 with position vector $\vec{a_1}$ and let B be a point on L_2 with position vector $\vec{a_2}$.



Draw $BM \perp L_1$. Then,

distance between L_1 and $L_2 = |\overrightarrow{BM}|$.

Let θ be the angle between $\overrightarrow{AB}$ and $\vec{b}$. Then,

$$(\vec{b} \times \overrightarrow{AB}) = \{|\vec{b}|\,|\overrightarrow{AB}| \cdot \sin\theta\}\hat{n},$$

where $\hat{n}$ is a unit vector, perpendicular to the plane of L_1 and L_2.

$$\therefore \quad \vec{b} \times (\vec{a_2} - \vec{a_1}) = \{|\vec{b}|\,|\overrightarrow{AB}| \cdot \sin\theta\}\,\hat{n}$$

$$\Rightarrow \quad \vec{b} \times (\vec{a_2} - \vec{a_1}) = |\vec{b}|(BM)\,\vec{n} \qquad \{\because (AB)\sin\theta = BM\}$$

$$\Rightarrow \quad |\vec{b} \times (\vec{a_2} - \vec{a_1})| = |\vec{b}|\,|\overrightarrow{BM}| \cdot 1 \qquad [\because |\hat{n}| = 1]$$

$$\Rightarrow \quad |\overrightarrow{BM}| = \left| \frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right|.$$

EXAMPLE 6 *Find the shortest distance between the lines L_1 and L_2, given by*

$$\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k}) \text{ and } \vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(4\hat{i} - 2\hat{j} + 2\hat{k}).$$

SOLUTION The given lines are

$$L_1 : \vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k}) \qquad \text{... (i)}$$

$$L_2 : \vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + 2\mu(2\hat{i} - \hat{j} + \hat{k}). \qquad \text{... (ii)}$$

These equations are of the form:

$$\vec{r} = \vec{a_1} + \lambda\vec{b} \text{ and } \vec{r} = \vec{a_2} + 2\mu\vec{b} = \vec{a_2} + \mu'\vec{b}, \text{ where}$$

$$\vec{a_1} = (\hat{i} + \hat{j}),\ \vec{a_2} = (2\hat{i} + \hat{j} - \hat{k}),\ \vec{b} = (2\hat{i} - \hat{j} + \hat{k}) \text{ and } \mu' = 2\mu.$$

Clearly, the given lines are parallel.

Now, $(\vec{a_2} - \vec{a_1}) = (2\hat{i} + \hat{j} - \hat{k}) - (\hat{i} + \hat{j}) = (\hat{i} - \hat{k})$

$$\therefore \quad \{\vec{b} \times (\vec{a_2} - \vec{a_1})\} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= (1 - 0)\hat{i} - (-2 - 1)\hat{j} + (0 + 1)\hat{k}$$

$$= (\hat{i} + 3\hat{j} + \hat{k})$$

$$\Rightarrow \quad |\vec{b} \times (\vec{a_2} - \vec{a_1})| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$$

$$\text{and } |\vec{b}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$\Rightarrow$ shortest distance between L_1 and L_2

= distance between L_1 and L_2

$$= \frac{|\vec{b} \times (\vec{a_2} - \vec{a_1})|}{|\vec{b}|} = \frac{\sqrt{11}}{\sqrt{6}}$$

$$= \left(\frac{\sqrt{11}}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}\right) = \frac{\sqrt{66}}{6} \text{ units.}$$

EXAMPLE 7 *Write the vector equations of the following lines and hence find the shortest distance between them:*

$$\frac{x+1}{2} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \qquad \textbf{[CBSE 2008, '14]}$$

SOLUTION The given lines are

$$\frac{x+1}{2} = \frac{y+1}{-6} = \frac{z+1}{1} \quad \text{... (i)}$$

and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$. ... (ii)

The line (i) passes through the point (–1, –1, –1) and has d.r.'s. 7, –6, 1.

So, its vector equation is

$$\vec{r} = \vec{a_1} + \lambda \vec{b_1}, \quad \text{... (iii)}$$

where $\vec{a_1} = (-\hat{i} - \hat{j} - \hat{k})$ and $\vec{b_1} = 7\hat{i} - 6\hat{j} + \hat{k}$.

The line (ii) passes through the point (3, 5, 7) and has d.r.'s. 1, –2, 1.
So, its vector equation is

$$\vec{r} = \vec{a_2} + \lambda \vec{b_2}, \quad \text{... (iv)}$$

where $\vec{a_2} = (3\hat{i} + 5\hat{j} + 7\hat{k})$ and $\vec{b_2} = \hat{i} - 2\hat{j} + \hat{k}$.

$\therefore \quad (\vec{a_2} - \vec{a_1}) = (3\hat{i} + 5\hat{j} + 7\hat{k}) - (-\hat{i} - \hat{j} - \hat{k}) = (4\hat{i} + 6\hat{j} + 8\hat{k})$

and $(\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (-6+2)\hat{i} - (7-1)\hat{j} + (-14+6)\hat{k}$

$= (-4\hat{i} - 6\hat{j} - 8\hat{k})$.

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-4)^2 + (-6)^2 + (-8)^2} = \sqrt{16 + 36 + 64} = \sqrt{116}$.

$\therefore \quad \text{SD} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| = \left| \frac{(4\hat{i} + 6\hat{j} + 8\hat{k}) \cdot (-4\hat{i} - 6\hat{j} - 8\hat{k})}{\sqrt{116}} \right|$

$= \frac{|(-16-36-64)|}{\sqrt{116}} = \frac{|-116|}{\sqrt{116}} = \frac{116}{\sqrt{116}} \times \frac{\sqrt{116}}{\sqrt{116}}$

$= \sqrt{116} = \sqrt{4 \times 29} = 2\sqrt{29}$ units.

EXERCISE 27D

In problems* 1–8, *find the shortest distance between the given lines.

1. $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$,
$\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$. **[CBSE 2007, '08C, '09, '14]**

2. $\vec{r} = (-4\hat{i} + 4\hat{j} + \hat{k}) + \lambda(\hat{i} + \hat{j} - \hat{k})$,
   $\vec{r} = (-3\hat{i} - 8\hat{j} - 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 3\hat{k})$.
3. $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$,
   $\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$. **[CBSE 2006, '11]**
4. $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$,
   $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$. **[CBSE 2008, '11]**
5. $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$,
   $\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(-2\hat{i} + 3\hat{j} + 8\hat{k})$.
6. $\vec{r} = (6\hat{i} + 3\hat{k}) + \lambda(2\hat{i} - \hat{j} + 4\hat{k})$,
   $\vec{r} = (-9\hat{i} + \hat{j} - 10\hat{k}) + \mu(4\hat{i} + \hat{j} + 6\hat{k})$.
7. $\vec{r} = (3 - t)\hat{i} + (4 + 2t)\hat{j} + (t - 2)\hat{k}$,
   $\vec{r} = (1 + s)\hat{i} + (3s - 7)\hat{j} + (2s - 2)\hat{k}$.
8. $\vec{r} = (\lambda - 1)\hat{i} + (\lambda + 1)\hat{j} - (\lambda + 1)\hat{k}$,
   $\vec{r} = (1 - \mu)\hat{i} + (2\mu - 1)\hat{j} + (\mu + 2)\hat{k}$.
9. Compute the shortest distance between the lines
   $$\vec{r} = (\hat{i} - \hat{j}) + \lambda(2\hat{i} - \hat{k}) \text{ and } \vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} - \hat{j} - \hat{k}).$$
   Determine whether these lines intersect or not. **[CBSE 2012C]**
10. Show that the lines
    $$\vec{r} = (3\hat{i} - 15\hat{j} + 9\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 5\hat{k}), \text{ and}$$
    $$\vec{r} = (-\hat{i} + \hat{j} + 9\hat{k}) + \mu(2\hat{i} + \hat{j} - 3\hat{k})$$
    do not intersect.
11. Show that the lines
    $\vec{r} = (2\hat{i} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$ and $\vec{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 4\hat{k})$
    intersect.
    Also, find their point of intersection.
12. Show that the lines
    $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$ and $\vec{r} = (4\hat{i} + \hat{j}) + \mu(5\hat{i} + 2\hat{j} + \hat{k})$
    intersect.
    Also, find their point of intersection.

13. Find the shortest distance between the lines L_1 and L_2 whose vector equations are **[CBSE 2014]**

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}) \text{ and}$$
$$\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 6\hat{k}).$$

HINT: The given lines are parallel.

14. Find the distance between the parallel lines L_1 and L_2 whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}), \text{ and } \vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k}).$$

15. Find the vector equation of a line passing through the point (2, 3, 2) and parallel to the line $\vec{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$. Also, find the distance between these lines.

HINT: The given line is $L_1 : \vec{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$.

The required line is $L_2 : \vec{r} = (2\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(2\hat{i} - 3\hat{j} + 6\hat{k})$.

Now, find the distance between the parallel lines L_1 and L_2.

16. Write the vector equation of each of the following lines and hence determine the distance between them: **[CBSE 2010]**

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6} \text{ and } \frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}.$$

HINT: The given lines are $L_1 : \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

$L_2 : \vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + 2\mu(2\hat{i} + 3\hat{j} + 6\hat{k})$.

Now, find the distance between the parallel lines L_1 and L_2.

17. Write the vector equations of the following lines and hence find the shortest distancebetween them: **[CBSE 2010C]**

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-5}{5}.$$

Find the shortest distance between the lines given below:

18. $\frac{x-1}{-1} = \frac{y+2}{1} = \frac{z-3}{-2}$ and $\frac{x-1}{1} = \frac{y+1}{2} = \frac{z+1}{-2}$.

19. $\frac{x-12}{-9} = \frac{y-1}{4} = \frac{z-5}{2}$ and $\frac{x-23}{-6} = \frac{y-19}{-4} = \frac{z-25}{3}$.

HINT: Change the given equations in vector form.

ANSWERS (EXERCISE 27D)

1. $\frac{10}{\sqrt{59}}$ units **2.** $\sqrt{62}$ units **3.** $\frac{3\sqrt{19}}{19}$ units **4.** $\frac{3\sqrt{2}}{2}$ units

5. $\frac{14\sqrt{241}}{241}$ units **6.** $\sqrt{38}$ units **7.** $\sqrt{35}$ units **8.** $\frac{5\sqrt{2}}{2}$ units

9. SD $= \dfrac{\sqrt{14}}{14} \neq 0$, the given lines do not intersect **11.** (2, 6, 3)

12. $(-1, -1, -1)$ **13.** $\dfrac{\sqrt{293}}{7}$ units **14.** $\sqrt{26}$ units

15. $L_2 : \vec{r} = (2\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(2\hat{i} - 3\hat{j} + 6\hat{k})$, $\dfrac{\sqrt{580}}{7}$ units

16. $\left.\begin{array}{l} L_1 : \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}) \\ L_2 : \vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + 2\mu(2\hat{i} + 3\hat{j} + 6\hat{k}) \end{array}\right\}$, SD $= \dfrac{\sqrt{293}}{7}$ units

17. $\left.\begin{array}{l} L_1 : \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) \\ L_2 : \vec{r} = (2\hat{i} + 3\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 4\hat{j} + 5\hat{k}) \end{array}\right\}$, SD $= \dfrac{\sqrt{6}}{6}$ units

18. $\dfrac{8\sqrt{29}}{29}$ units **19.** 26 units

Shortest Distance between Two Skew Lines in the Cartesian Form

The shortest distance between the skew lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \text{ and } \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

is given by

$$\text{SD} = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{D}},$$

where $D = \{(a_1 b_2 - a_2 b_1)^2 + (b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2\}$.

CONDITION FOR TWO GIVEN LINES TO INTERSECT Let L_1 and L_2 be the given lines whose equations are

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \text{ and } \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}.$$

1. L_1 and L_2 intersect $\Leftrightarrow$ SD between them is 0

$$\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

2. L_1 and L_2 do not intersect $\Leftrightarrow$ they are skew lines

$$\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \neq 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the shortest distance between the lines*

$$\frac{x+3}{-4}=\frac{y-6}{3}=\frac{z}{2} \text{ and } \frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}.$$

SOLUTION Comparing the given equations with

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}, \text{ we get}$$

$(x_1=-3,\ y_1=6,\ z_1=0),\ (x_2=-2,\ y_2=0,\ z_2=7),$

$(a_1=-4,\ b_1=3,\ c_1=2)$ and $(a_2=-4,\ b_2=1,\ c_2=1).$

Now, $D=(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2$

$$=(-4+12)^2+(3-2)^2+(-8+4)^2$$

$$=(64+1+16)=81.$$

$$\therefore \quad SD=\frac{1}{\sqrt{D}}\cdot\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$=\frac{1}{\sqrt{81}}\cdot\begin{vmatrix} -2+3 & 0-6 & 7-0 \\ -4 & 3 & 2 \\ -4 & 1 & 1 \end{vmatrix}=\frac{1}{9}\cdot\begin{vmatrix} 1 & -6 & 7 \\ -4 & 3 & 2 \\ -4 & 1 & 1 \end{vmatrix}$$

$$=\frac{1}{9}\cdot\{1\cdot(3-2)+6\cdot(-4+8)+7\cdot(-4+12)\}$$

$$=\frac{81}{9}=9 \text{ units.}$$

Hence, the shortest distance between the given lines is 9 units.

EXAMPLE 2 *Find the length and the equations of the line of shortest distance between the lines*

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \text{ and } \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}.$$ **[CBSE 2008, '14]**

SOLUTION The equations of the given lines are

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}=\lambda \text{ (say)} \qquad \ldots \text{(i)}$$

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}=\mu \text{ (say).} \qquad \ldots \text{(ii)}$$

Any point on (i) is $P(7\lambda-1,\ -6\lambda-1,\ \lambda-1)$.

Any point on (ii) is $Q(\mu+3,\ -2\mu+5,\ \mu+7)$.

The direction ratios of PQ are $(\mu-7\lambda+4,\ -2\mu+6\lambda+6,\ \mu-\lambda+8)$.

Now, PQ will be the shortest distance between (i) and (ii) only when PQ is perpendicular to both (i) and (ii).

$$\therefore \begin{cases} 7(\mu - 7\lambda + 4) - 6(-2\mu + 6\lambda + 6) + 1\cdot(\mu - \lambda + 8) = 0 \\ \text{and} \\ 1\cdot(\mu - 7\lambda + 4) - 2(-2\mu + 6\lambda + 6) + 1\cdot(\mu - \lambda + 8) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 20\mu - 86\lambda = 0 \\ 6\mu - 20\lambda = 0 \end{cases} \Rightarrow \begin{cases} 10\mu - 43\lambda \\ 3\mu - 10\lambda \end{cases} \Rightarrow \lambda = 0 \text{ and } \mu = 0.$$

$\therefore$ PQ will be the line of shortest distance when $\lambda = 0$ and $\mu = 0$.

Putting $\lambda = 0$ and $\mu = 0$, we get the points $P(-1, -1, -1)$ and $Q(3, 5, 7)$.

$$\therefore \quad \text{SD} = |PQ| = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2}$$
$$= \sqrt{(4)^2 + (6)^2 + (8)^2} = \sqrt{16 + 36 + 64}$$
$$= \sqrt{116} = 2\sqrt{29} \text{ units.}$$

Clearly, the equation of the line of shortest distance is the equation of line PQ given by

$$\frac{x-3}{3+1} = \frac{y-5}{5+1} = \frac{z-7}{7+1}$$
$$\Leftrightarrow \frac{x-3}{4} = \frac{y-5}{6} = \frac{z-7}{8}$$
$$\Leftrightarrow \frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}.$$

Hence, the equation of the line of shortest distance is

$$\frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}.$$

EXAMPLE 3 *Find the length and the equations of the line of shortest distance between the lines*

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

SOLUTION The equations of the given lines are

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} = \lambda \text{ (say)} \qquad \text{... (i)}$$
$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} = \mu \text{ (say)} \qquad \text{... (iii)}$$

Any point on the line (i) is $P(3\lambda + 8, -16\lambda - 9, 7\lambda + 10)$.

Any point on the line (ii) is $Q(3\mu + 15, 8\mu + 29, -5\mu + 5)$.

The direction ratios of PQ are

$$(3\mu - 3\lambda + 7, 8\mu + 16\lambda + 38, -5\mu - 7\lambda - 5).$$

Now, PQ will be the shortest distance between (i) and (ii) only when PQ is perpendicular to each one of (i) and (ii).

$$\therefore \begin{cases} 3(3\mu - 3\lambda + 7) - 16(8\mu + 16\lambda + 38) + 7(-5\mu - 7\lambda - 5) = 0 \\ \text{and} \\ 3(3\mu - 3\lambda + 7) + 8(8\mu + 16\lambda + 38) - 5(-5\mu - 7\lambda - 5) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 314\lambda + 154\mu + 622 = 0 \\ 154\lambda + 98\mu + 350 = 0 \end{cases} \Rightarrow \begin{cases} 157\lambda + 77\mu + 311 = 0 & \text{...(iii)} \\ 77\lambda + 49\mu + 175 = 0. & \text{...(iv)} \end{cases}$$

On multiplying (iii) by 7 and (iv) by 11 and subtracting, we get

$$(1099\lambda - 847\lambda) + (2177 - 1925) = 0 \Rightarrow 252\lambda = -252 \Rightarrow \lambda = -1.$$

Putting $\lambda = -1$ in (iv), we get

$$49\mu + (175 - 77) = 0 \Rightarrow 49\mu + 98 = 0 \Rightarrow 49\mu = -98 \Rightarrow \mu = -2.$$

Now, $\lambda = -1$ gives $P(5, 7, 3)$ and $\mu = -2$ gives $Q(9, 13, 15)$.

$$\therefore \quad \text{SD} = |PQ| = \sqrt{(9-5)^2 + (13-7)^2 + (15-3)^2}$$

$$= \sqrt{16 + 36 + 144} = \sqrt{196} = 14 \text{ units.}$$

Equation of the line of shortest distance is the equation of PQ, given by

$$\frac{x-5}{9-5} = \frac{y-7}{13-7} = \frac{z-3}{15-3}$$

$$\Rightarrow \frac{x-5}{4} = \frac{y-7}{6} = \frac{z-3}{12} \Rightarrow \frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}.$$

Hence, the equation of the line of shortest distance is

$$\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}.$$

EXAMPLE 4 *Show that the lines*

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-4}{5} = \frac{y-1}{2} = z$$

intersect each other. Find their point of intersection. **[CBSE 2004]**

SOLUTION The equations of the given lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda \text{ (say)} \quad \text{... (i)}$$

$$\frac{x-4}{5} = \frac{y-1}{2} = \frac{z-0}{1} = \mu. \quad \text{... (ii)}$$

Any point on the line (i) is $P(2\lambda + 1, 3\lambda + 2, 4\lambda + 3)$.

Any point on the line (ii) is $Q(5\mu + 4, 2\mu + 1, \mu)$.

If the lines (i) and (ii) intesect then P and Q must coincide for some particular values of λ and μ.

This gives

$$2\lambda + 1 = 5\mu + 4, \; 3\lambda + 2 = 2\mu + 1 \text{ and } 4\lambda + 3 = \mu$$

$$\Rightarrow \begin{cases} 2\lambda - 5\mu = 3 & \text{... (i)} \\ 3\lambda - 2\mu = -1 & \text{... (ii)} \\ 4\lambda - \mu = -3. & \text{... (iii)} \end{cases}$$

On solving (i) and (ii), we get $\lambda = -1$ and $\mu = -1$.

These values of λ and μ also satisfy (iii).

Hence, the given lines intersect.

Putting $\lambda = -1$, we get $P(-1, -1, -1)$.

Note that putting $\mu = -1$, we get $Q(-1, -1, -1)$.

Hence, the point of intersection of the given lines is $(-1, -1, -1)$.

EXAMPLE 5 *Show that the lines*

$$\frac{x-1}{2} = \frac{y+1}{3} = z \text{ and } \frac{x+1}{5} = \frac{y-2}{2}, z = 2.$$

do not intersect each other. **[CBSE 2010]**

SOLUTION The equations of the given lines are

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-0}{1} = \lambda \text{ (say)} \qquad \text{... (i)}$$

$$\frac{x+1}{5} = \frac{y-2}{1} = \frac{z-2}{0} = \mu \text{ (say)} \qquad \text{... (ii)}$$

Any point on the line (i) is $P(2\lambda + 1, 3\lambda - 1, \lambda)$.

Any point on the line (ii) is $Q(5\mu - 1, \mu + 2, 2)$.

If the lines (i) and (ii) intersect, then P and Q must coincide for some particular values of λ and μ.

This gives

$$2\lambda + 1 = 5\mu - 1,\ 3\lambda - 1 = \mu + 2 \text{ and } \lambda = 2$$

$$\Rightarrow \begin{cases} 2\lambda - 5\mu = -2 & \text{... (iii)} \\ 3\lambda - \mu = 3 & \text{... (iv)} \\ \lambda = -3. & \text{... (v)} \end{cases}$$

Putting $\lambda = 2$ in (iv), we get $\mu = 3$.

Clearly, $\lambda = 2$ and $\mu = 3$ do not satisfy (iii).

Hence, the given lines do not intersect each other.

EXERCISE 27E

Find the length and the equations of the line of shortest distance between the lines given by

1. $\frac{x-3}{3} = \frac{y-8}{-1} = z - 3$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.
2. $\frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1}$ and $\frac{x-1}{1} = \frac{y+7}{3} = \frac{z+2}{2}$.
3. $\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$ and $\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}$.

4. $\frac{x-6}{3}=\frac{y-7}{-1}=\frac{z-4}{1}$ and $\frac{x}{-3}=\frac{y+9}{2}=\frac{z-2}{4}$.

5. Show that the lines

$$\frac{x}{1}=\frac{y-2}{2}=\frac{z+3}{3} \text{ and } \frac{x-2}{2}=\frac{y-6}{3}=\frac{z-3}{4}$$

intersect and find their point of intersection.

6. Show that the lines

$$\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5} \text{ and } \frac{x-2}{2}=\frac{y-1}{3}=\frac{z+1}{-2}$$

do not intersect each other.

ANSWERS (EXERCISE 27E)

1. $3\sqrt{30}$ units, $\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$ **2.** $\sqrt{35}$ units, $\frac{x-4}{-1}=\frac{y-2}{-3}=\frac{z+3}{5}$

3. $4\sqrt{3}$ units, $x=y=z$ **4.** $3\sqrt{30}$ units, $\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$

5. (2, 6, 3)

EXERCISE 27F

Very-Short-Answer Questions

1. If a line has direction ratios 2, –1, –2 then what are its direction cosines? **[CBSE 2012]**

2. Find the direction cosines of the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$. **[CBSE 2013C]**

3. If the equations of a line are $\frac{3-x}{-3}=\frac{y+2}{-2}=\frac{z+2}{6}$, find the direction cosines of a line parallel to the given line. **[CBSE 2012]**

4. Write the equations of a line parallel to the line $\frac{x-2}{-3}=\frac{y+3}{2}=\frac{z+5}{6}$ and passing through the point (1, –2, 3). **[CBSE 2009C]**

5. Find the Cartesian equations of the line which passes through the point (–2, 4, –5) and which is parallel to the line $\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6}$. **[CBSE 2013]**

6. Write the vector equation of a line whose Cartesian equations are $\frac{x-5}{3}=\frac{y+4}{7}=\frac{6-z}{2}$. **[CBSE 2010, '11]**

7. The Cartesian equations of a line are $\frac{3-x}{5}=\frac{y+4}{7}=\frac{2z-6}{4}$. Write the vector equation of the line. **[CBSE 2014]**

8. Write the vector equation of a line passing through the point (1, –1, 2) and parallel to the line whose equations are $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z+1}{-2}\cdot$ **[CBSE 2013C]**

9. If $P(1, 5, 4)$ and $Q(4, 1, -2)$ be two given points, find the direction ratios of PQ. **[CBSE 2008]**

10. The equations of a line are $\frac{4-x}{2}=\frac{y+3}{2}=\frac{z+2}{1}\cdot$ Find the direction cosines of a line parallel to this line. **[CBSE 2013]**

11. The Cartesian equations of a line are $\frac{x-1}{2}=\frac{y+2}{3}=\frac{5-z}{1}\cdot$ Find its vector equation.

12. Find the vector equation of a line passing through the point (1, 2, 3) and parallel to the vector $(3\hat{i}+2\hat{j}-2\hat{k})$.

13. The vector equation of a line is $\vec{r}=(2\hat{i}+\hat{j}-4\hat{k})+\lambda(\hat{i}-\hat{j}-\hat{k})$. Find its Cartesian equation.

14. Find the Cartesian equation of a line which passes through the point (–2, 4, –5) and which is parallel to the line $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}\cdot$

15. Find the Cartesian equation of a line which passes through the point having position vector $(2\hat{i}-\hat{j}+4\hat{k})$ and is in the direction of the vector $(\hat{i}+2\hat{j}-\hat{k})$.

16. Find the angle between the lines $\vec{r}=(2\hat{i}-5\hat{j}+\hat{k})+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$ and $\vec{r}=(7\hat{i}-6\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$.

17. Find the angle between the lines $\frac{x+3}{3}=\frac{y-1}{5}=\frac{z+3}{4}$ and $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}\cdot$

18. Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are at right angles.

19. The direction ratios of a line are 2, 6, –9. What are its direction cosines?

20. A line makes angles 90°, 135° and 45° with the positive directions of x-axis, y-axis and z-axis respectively. What are the direction cosines of the line?

21. What are the direction cosines of the y-axis?

22. What are the direction cosines of the vector $(2\hat{i}+\hat{j}-2\hat{k})$?

23. What is the angle between the vector $\vec{r}=(4\hat{i}+8\hat{j}+\hat{k})$ and the x-axis?

ANSWERS (EXERCISE 27F)

1. $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$ **2.** $\frac{-2}{7}, \frac{6}{7}, \frac{-3}{7}$ **3.** $\frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$

4. $\frac{x-1}{-3} = \frac{y+2}{2} = \frac{z-3}{6}$ **5.** $\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$

6. $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} - 2\hat{k})$

7. $\vec{r} = (3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-5\hat{i} + 7\hat{j} + 2\hat{k})$

8. $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 2\hat{k})$ **9.** 3, –4, –6

10. $\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}$ **11.** $\vec{r} = (\hat{i} - 2\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$

12. $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$ **13.** $\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z+4}{-1}$

14. $\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$ **15.** $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$ **16.** $\cos^{-1}\left(\frac{19}{21}\right)$

17. $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$ **19.** $\frac{2}{11}, \frac{6}{11}, \frac{-9}{11}$ **20.** $0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ **21.** 0, 1, 0

22. $\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$ **23.** $\cos^{-1}\left(\frac{4}{9}\right)$

HINTS TO THE GIVEN QUESTIONS (EXERCISE 27F)

1. D.r.'s of the given line are 2, –1, –2 and $\sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$.

   ∴ d.c.'s of the given line are $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$.

2. The given equations are $\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3}$.

   ∴ d.r.'s of the given line are –2, 6, –3 and $\sqrt{(-2)^2 + 6^2 + (-3)^2} = \sqrt{49} = 7$.

   ∴ d.c.'s of the given line are $\frac{-2}{7}, \frac{6}{7}, \frac{-3}{7}$.

3. The given equations are $\frac{x-3}{3} = \frac{y+2}{-2} = \frac{z+2}{6}$.

   ∴ d.r.'s of this line are 3, –2, 6.

   ∴ d.r.'s of a line parallel to this line are 3, –2, 6 and $\sqrt{3^2 + (-2)^2 + 6^2} = \sqrt{49} = 7$.

   ∴ d.c.'s of a line parallel to the given line are $\frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$.

4. D.r.'s of the given line are –3, 2, 6.

   D.r.'s of a line parallel to the given line are –3, 2, 6.

   Required equations of the line are $\frac{x-1}{-3} = \frac{y+2}{2} = \frac{z-3}{6}$.

5. The equations of the given line are $\frac{x+3}{3} = \frac{y-4}{-5} = \frac{z+8}{6}$.

D.r.'s of this line are 3, –5, 6.

D.r.'s of a line parallel to this line are 3, –5, 6.

Required equations of a line through (–2, 4, –5) are $\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$.

6. The given equations of the line are $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{-2}$.

Clearly, this line passes through the point (5, –4, 6) and it has d.r.'s 3, 7, –2.

$\therefore$ its vector equation is $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} - 2\hat{k})$.

7. The given equations of the line are $\frac{x-3}{-5} = \frac{y-(-4)}{7} = \frac{z-3}{2}$.

Clearly, this line passes through the point (3, –4, 3) and it has d.r.'s –5, 7, 2.

$\therefore$ its vector equation is $\vec{r} = (3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-5\hat{i} + 7\hat{j} + 2\hat{k})$.

8. Clearly, the requried line passes through the point (1, –1, 2) and its d.r.'s are 1, 2, –2.

Hence, its vector equation is $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 2\hat{k})$.

9. D.r.'s of PQ are $(4-1), (1-5), (-2-4)$, i.e., 3, –4, –6.

10. The given equations are $\frac{x-4}{-2} = \frac{y+3}{2} = \frac{z+2}{1}$.

The d.r.'s of this line are –2, 2, 1.

The d.r.'s of a line parallel to this line are –2, 2, 1 and $\sqrt{(-2)^2 + 2^2 + 1^2} = \sqrt{9} = 3$.

The d.c.'s of a line parallel to the given line are $\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}$.

11. The given equations are $\frac{x-1}{2} = \frac{y-(-2)}{3} = \frac{z-5}{-1}$.

Clearly, this line passes thorugh the point (1, –2, 5) and it has d.r.'s 2, 3, –1.

So, its vector equation is $\vec{r} = (\hat{i} - 2\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$.

12. Clearly, the required equation is $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$.

13. The given line passes through the point (2, 1, –4) and it is parallel to a line whose direction ratios are 1, –1, –1.

So, its Cartesian equation is $\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z+4}{-1}$.

14. Clearly, the line passes through the point (–2, –4, –5) and it has direction ratios 3, 5, 6.

So, its equation is $\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$.

15. The required line passes through the point (2, –1, 4) and it has direction ratios 1, 2, –1.

$\therefore$ its equation is $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$.

16. The angle between the lines and $\vec{r} = a_1 + \lambda b_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ is given by

$$\cos\theta = \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}.$$

$$\therefore \quad \cos\theta = \frac{|(3\hat{i} + 2\hat{j} + 6\hat{k})\cdot(\hat{i} + 2\hat{j} + 2\hat{k})|}{\{\sqrt{3^2 + 2^2 + 6^2}\}\{\sqrt{1^2 + 2^2 + 2^2}\}} = \frac{(3 + 4 + 12)}{(7 \times 3)} = \frac{19}{21}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{19}{21}\right).$$

17. Here $(a_1) = 3, b_1 = 5, c_1 = 4)$ and $(a_2 = 1, b_2 = 1, c_2 = 2)$.

$$\therefore \quad \cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\{\sqrt{a_1^2 + b_1^2 + c_1^2}\}\{\sqrt{a_2^2 + b_2^2 + c_2^2}\}}$$

$$= \frac{|(3 \times 1) + (5 \times 1) + (4 \times 2)|}{\{\sqrt{3^2 + 5^2 + 4^2}\}\{\sqrt{1^2 + 1^2 + 2^2}\}}$$

$$= \frac{16}{(\sqrt{50} \times \sqrt{6})} = \frac{16}{10\sqrt{3}} = \frac{8}{5\sqrt{3}}.$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{8}{5\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\right) = \cos^{-1}\left(\frac{8\sqrt{3}}{15}\right).$$

18. Here $(a_1 = 7, b_1 = -5, c_1 = 1)$ and $(a_2 = 1, b_2 = 2, c_2 = 3)$.

$$(a_1a_2 + b_1b_2 + c_1c_2) = (7 \times 1) + (-5) \times 2 + (1 \times 3) = 0.$$

Hence, the given lines are at right angles.

19. We have $\sqrt{2^2 + 6^2 + (-9)^2} = \sqrt{121} = 11.$

$\therefore$ d.c.'s of the given line are $\frac{2}{11}, \frac{6}{11}, \frac{-9}{11}$.

20. D.c.'s of the line are cos 90°, cos 135° and cos 45°, i.e., $\left(0, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right)$.

21. Clearly, the y-axis makes an angle of 90°, 0°, 90° with the x-axis, y-axis and z-axis respectively.

So, its d.c.'s are cos 90°, cos 0°, cos 90°, i.e., 0, 1, 0.

22. D.r.'s of the given vector are 2, 1, –2 and $\sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$.

$\therefore$ d.c.'s of the given vector are $\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$.

23. D.r.'s of the given vector are 4, 8, 1 and $\sqrt{4^2 + 8^2 + 1^2} = \sqrt{81} = 9$.

$\therefore$ d.c.'s of the given vector are $\frac{4}{9}, \frac{8}{9}, \frac{1}{9}$.

Let α be the angle between the given vectors and the x-axis.

Then, $\cos\alpha = \frac{4}{9} \Rightarrow \alpha = \cos^{-1}\left(\frac{4}{9}\right)$.

OBJECTIVE QUESTIONS

Multiple-Choice Questions

1. The direction ratios of two lines are 3, 2, –6 and 1, 2, 2 respectively. The acute angle between these lines is

(a) $\cos^{-1}\left(\frac{5}{18}\right)$ (b) $\cos^{-1}\left(\frac{3}{20}\right)$

(c) $\cos^{-1}\left(\frac{5}{21}\right)$ (d) $\cos^{-1}\left(\frac{8}{21}\right)$

2. The direction ratios of two lines are a, b, c and $(b-c), (c-a), (a-b)$ respectively. The angle between these lines is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

3. The angle between the lines $\frac{x-2}{2}=\frac{y-1}{7}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{2}=\frac{z-5}{4}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\cos^{-1}\left(\frac{3}{8}\right)$

4. If the lines $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$ are perpendicular to each other then $k=?$

(a) $\frac{-5}{7}$ (b) $\frac{5}{7}$ (c) $\frac{10}{7}$ (d) $\frac{-10}{7}$

5. A line passes through the points $A(2, -1, 4)$ and $B(1, 2, -2)$. The equations of the line AB are

(a) $\frac{x-2}{-1}=\frac{y+1}{2}=\frac{z-4}{-6}$ (b) $\frac{x+2}{-1}=\frac{y+1}{2}=\frac{z-4}{6}$

(c) $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z-4}{6}$ (d) none of these

6. The angle between the lines $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x-5}{4}=\frac{y-2}{1}=\frac{z-3}{8}$ is

(a) $\cos^{-1}\left(\frac{3}{4}\right)$ (b) $\cos^{-1}\left(\frac{5}{6}\right)$ (c) $\cos^{-1}\left(\frac{2}{3}\right)$ (d) $\frac{\pi}{3}$

7. The angle between the lines

$\vec{r}=(3\hat{i}+\hat{j}-2\hat{k})+\lambda(\hat{i}-\hat{j}-2\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j}-5\hat{k})+\mu(3\hat{i}-5\hat{j}-4\hat{k})$

is

(a) $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$ (b) $\cos^{-1}\left(\frac{6\sqrt{2}}{5}\right)$ (c) $\cos^{-1}\left(\frac{5\sqrt{3}}{8}\right)$ (d) $\cos^{-1}\left(\frac{5\sqrt{2}}{6}\right)$

8. A line is perpendicular to two lines having direction ratios 1, –2, –2 and 0, 2, 1. The direction cosines of the line are

(a) $\frac{-2}{3}, \frac{1}{3}, \frac{2}{3}$ (b) $\frac{2}{3}, \frac{1}{3}, \frac{-1}{3}$ (c) $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$ (d) none of these

9. A line passes through the point $A(5, -2. 4)$ and it is parallel to the vector $(2\hat{i} - \hat{j} + 3\hat{k})$. The vector equation of the line is

(a) $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(5\hat{i} - 2\hat{j} + 4\hat{k})$

(b) $\vec{r} = (5\hat{i} - 2\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$

(c) $\vec{r} \cdot (5\hat{i} - 2\hat{j} + 4\hat{k}) = \sqrt{14}$

(d) none of these

10. The Cartesian equations of a line are $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-5}{-1}$. Its vector equation is

(a) $\vec{r} = (-\hat{i} + 2\hat{j} - 5\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$

(b) $\vec{r} = (2\hat{i} + 3\hat{j} - \hat{k}) + \lambda(\hat{i} - 2\hat{j} + 5\hat{k})$

(c) $\vec{r} = (\hat{i} - 2\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - 4\hat{k})$

(d) none of these

11. A line passes through the point $A(-2, 4, -5)$ and is parallel to the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$. The vector equation of the line is

(a) $\vec{r} = (-3\hat{i} + 4\hat{j} - 8\hat{k}) + \lambda(-2\hat{i} + 4\hat{j} - 5\hat{k})$

(b) $\vec{r} = (-2\hat{i} + 4\hat{j} - 5\hat{k}) + \lambda(-3\hat{i} + 5\hat{j} + 6\hat{k})$

(c) $\vec{r} = (3\hat{i} + 5\hat{j} + 6\hat{k}) + \lambda(-2\hat{i} + 4\hat{j} - 5\hat{k})$

(d) none of these

12. The coordinates of the point where the line through the points $A(5, 1, 6)$ and $B(3, 4, 1)$ crosses the yz-plane is

(a) $(0, 17, -13)$ (b) $\left(0, \frac{-17}{2}, \frac{13}{2}\right)$

(c) $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$ (d) none of these

13. The vector equation of the x-axis is given by

(a) $\vec{r} = \hat{i}$ (b) $\vec{r} = \hat{j} + \hat{k}$ (c) $\vec{r} = \lambda\hat{i}$ (d) none of these

14. The Cartesian equations of a line are $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z-3}{-2}$. What is its vector equation?

(a) $\vec{r} = (2\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$

(b) $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - 2\hat{k})$

(c) $\vec{r} = (2\hat{i} + 3\hat{j} - 2\hat{k})$

(d) none of these

15. The angle between two lines having direction ratios 1, 1, 2 and $(\sqrt{3}-1)$, $(-\sqrt{3}-1)$, 4 is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

16. The straight line $\frac{x-2}{3}=\frac{y-3}{1}=\frac{z+1}{0}$ is

(a) parallel to the x-axis (b) parallel to the y-axis

(c) parallel to the z-axis (d) perpendicular to the z-axis

17. If a line makes angles α, β and γ with the x-axis, y-axis and z-axis respectively then $(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)$ =?

(a) 1 (b) 3 (c) 2 (d) $\frac{3}{2}$

18. If (a_1, b_1, c_1) and (a_2, b_2, c_2) be the direction ratios of two parallel lines then

(a) $a_1 = a_2, b_1 = b_2, c_1 = c_2$ (b) $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

(c) $a_1^2+b_1^2+c_1^2 = a_2^2+b_2^2+c_2^2$ (d) $a_1a_2+b_1b_2+c_1c_2=0$

19. If the points $A(-1, 3, 2)$, $B(-4, 2, -2)$ and $C(5, 5, \lambda)$ are collinear then the value of λ is

(a) 5 (b) 7 (c) 8 (d) 10

ANSWERS (OBJECTIVE QUESTIONS)

1. (c)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (a)	**6.** (c)	**7.** (a)	**8.** (c)
9. (b)	**10.** (c)	**11.** (b)	**12.** (c)	**13.** (c)	**14.** (b)	**15.** (c)	**16.** (d)
17. (c)	**18.** (b)	**19.** (d)					

HINTS TO THE GIVEN OBJECTIVE QUESTIONS

1. (c) $\cos\theta = \frac{|(3\times 1)+(2\times 2)+(-6)\times 2|}{\{\sqrt{3^2+2^2+(-6)^2}\}\{\sqrt{1^2+2^2+2^2}\}} = \frac{|-5|}{(\sqrt{49}\times\sqrt{49})} = \frac{5}{(7\times 3)} = \frac{5}{21}$

$= \frac{1-51}{(\sqrt{49}\times\sqrt{9})} = \frac{5}{(7\times 3)} = \frac{5}{21}$

$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{5}{21}\right)$.

2. (b) $\cos\theta = \frac{a(b-c)+b(c-a)+c(a-b)}{\{\sqrt{a^2+b^2+c^2}\}\{\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}\}} = 0 \Rightarrow \theta = \frac{\pi}{2}$.

3. (c) The d.r.'s of these lines are 2, 7, –3 and –1, 2, 4.

$\therefore \quad \cos\theta = \frac{|2\times(-1)+7\times 2+(-3)\times 4|}{\{\sqrt{2^2+7^2+(-3)^2}\}\{\sqrt{(-1)^2+2^2+4^2}\}} = 0 \Rightarrow \theta = \frac{\pi}{2}$.

4. (d) Since the given lines are perpendicular to each other, we have

$$(-3)(3k) + (2k \times 1) + 2 \times (-5) = 0 \Rightarrow 7k = -10 \Rightarrow k = \frac{-10}{7}.$$

5. (a) The given line passes through the point $A(2, -1, 4)$ and its direction ratios are $(1-2), (1+1), (-2-4)$, i.e., –1, 2, –6.

$\therefore$ required equations of the line are $\frac{x-2}{-1} = \frac{y+1}{2} = \frac{x-4}{-6}$.

6. (c) The d.r.'s of the given lines are 2, 2, 1 and 4, 1, 8.

$$\therefore \quad \cos\theta = \frac{|(2\times 4) + (2\times 1) + (1\times 8)|}{\{\sqrt{2^2+2^2+1^2}\}\{\sqrt{4^2+1^2+8^2}\}} = \frac{18}{(\sqrt{9}\times\sqrt{81})} = \frac{18}{(3\times 9)} = \frac{2}{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{3}\right).$$

7. (a) The given lines are $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$.

$$\therefore \quad \cos\theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1||\vec{b}_2|} = \frac{|(\hat{i} - \hat{j} - 2\hat{k})\cdot(3\hat{i} - 5\hat{j} - 4\hat{k})|}{\{\sqrt{1^2+(-1)^2+(-2)^2}\}\{\sqrt{3^2+(-5)^2+(-4)^2}\}}$$

$$= \frac{(3+5+8)}{(\sqrt{6}\times\sqrt{50})} = \frac{16}{10\sqrt{3}} = \frac{8}{5}\times\frac{\sqrt{3}}{3} = \frac{8}{15}\sqrt{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{8\sqrt{3}}{15}\right).$$

8. (c) Let the direction ratios of the required line be a, b, c. Then,

$$a - 2b - 2c = 0 \quad \text{... (i)}$$

$$0a + 2b + c = 0. \quad \text{... (ii)}$$

On solving (i) and (ii) by cross multiplication, we get

$$\frac{a}{(-2+4)} = \frac{b}{(0-1)} = \frac{c}{(2-0)} \Rightarrow \frac{a}{2} = \frac{b}{-1} = \frac{c}{2}.$$

$\therefore$ d.r.'s of the required line are 2, –1, 2 and $\sqrt{2^2+(-1)^2+2^2} = \sqrt{9} = 3$.

$\therefore$ d.c.'s of the required line are $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$.

9. (b) Clearly, the required vector equation of the line is

$$\vec{r} = (5\hat{i} - 2\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k}).$$

10. (c) The line passes through the point (1, –2, 5) and it has direction ratios 2, 3, –1.

So, its vector equation is $\vec{r} = (\hat{i} - 2\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$.

11. (b) The required line passes through the point $(-2\hat{i} + 4\hat{j} + 5\hat{k})$ and it is parallel to the vector $(3\hat{i} + 5\hat{j} + 6\hat{k})$.

Hence, the required equation of the line is

$$\vec{r} = (-2\hat{i} + 4\hat{j} - 5\hat{k}) + \lambda(3\hat{i} + 5\hat{j} + 6\hat{k}).$$

12. (c) Equation of the line AB is

$$\frac{x-5}{5-3}=\frac{y-1}{1-4}=\frac{z-6}{6-1} \Rightarrow \frac{x-5}{2}=\frac{y-1}{-3}=\frac{z-6}{5}=\lambda.$$

Any point on this line is $(2\lambda+5, -3\lambda+1, 5\lambda+6)$.

If this point lies on the yz-plane, we have $2\lambda+5=0 \Rightarrow \lambda=\frac{-5}{2}\cdot$

$\therefore$ the required point is $\left(0, \frac{15}{2}+1, \frac{-25}{2}+6\right)$, i.e., $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$.

13. (c) Note that the Cartesian equation of x-axis is $\frac{x-0}{1}=\frac{y-0}{0}=\frac{z-0}{0}$ and its vector equation is $\vec{r}=\lambda\hat{i}$.

14. (b) The line passes through the point $(2\hat{i}-\hat{j}+3\hat{k})$ and it is parallel to the vector $(2\hat{i}+3\hat{j}-2\hat{k})$.

So, its vector equation is $\vec{r}=(2\hat{i}-\hat{j}+3\hat{k})+\lambda(2\hat{i}+3\hat{j}-2\hat{k})$.

15. (c) $\cos\theta=\dfrac{1\times(\sqrt{3}-1)+1\times(-\sqrt{3}-1)+2\times 4}{\{\sqrt{1^2+1^2+2^2}\}\{\sqrt{(\sqrt{3}-1)^2+(-\sqrt{3}-1)^2+4^2}\}}$

$$=\frac{(\sqrt{3}-1-\sqrt{3}-1+8)}{(\sqrt{6}\times\sqrt{24})}=\frac{\sqrt{6}}{\sqrt{24}}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}\cdot$$

16. (c) If a line is perpendicular to the z-axis then $\cos\gamma=\cos\frac{\pi}{2}=0$.

So, the given line is perpendicular to the z-axis.

17. (c) We have $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$.

$\therefore \quad (1-\sin^2\alpha)+(1-\sin^2\beta)+(1-\sin^2\gamma)=1$

$\Rightarrow \quad \sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$.

18. (b) If the given lines are parallel, then $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}\cdot$

19. (c) Equations of the line AB are

$$\frac{x+1}{-4+1}=\frac{y-3}{2-3}=\frac{z-2}{-2-2} \Rightarrow \frac{x+1}{-3}=\frac{y-3}{-1}=\frac{z-2}{-4}$$

$$\Rightarrow \frac{x+1}{3}=\frac{y-3}{1}=\frac{z-2}{4}\cdot$$

If the point $C(5, 5, \lambda)$ lies on AB, we have

$$\frac{5+1}{3}=\frac{5-3}{1}=\frac{\lambda-2}{4} \Rightarrow \frac{\lambda-2}{4}=2 \Rightarrow \lambda=(8+2)=10.$$

28. THE PLANE

PLANE *A plane is a surface such that a line segment joining any two points on it lies wholly on it.*

NORMAL TO A PLANE *A straight line which is perpendicular to every line lying on a plane is called a normal to the plane.*

All the normals to a plane are parallel to each other.

General Equation of a Plane in the Cartesian Form

THEOREM 1 *Every equation $ax + by + cz + d = 0$ of the first degree in x, y, z always represents a plane. Also, a, b, c are the direction ratios of the normal to this plane.*

PROOF Let us consider a surface represented by the equation

$$ax + by + cz + d = 0. \quad \text{... (i)}$$

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points on the surface represented by (i). Then,

$$ax_1 + by_1 + cz_1 + d = 0 \quad \text{... (ii)}$$

and, $ax_2 + by_2 + cz_2 + d = 0.$ ... (iii)

Multiplying (iii) by λ and adding it to (ii), we get

$$a(\lambda x_2 + x_1) + b(\lambda y_2 + y_1) + c(\lambda z_2 + z_1) + d(\lambda + 1) = 0$$

$$\Rightarrow \quad a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c\left(\frac{\lambda z_2 + z_1}{\lambda + 1}\right) + d = 0$$

$$\Rightarrow \quad \left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1}\right) \text{ lies on surface (i), when } \lambda \neq -1.$$

But, these are the general coordinates of a point which divides AB in the ratio $\lambda : 1$.

Since λ may take any real value other than –1, it follows that every point of AB lies on (i).

Hence, $ax + by + cz + d = 0$ represents a plane.

To Show that a, b, c are the Direction Ratios of the Normal to the Plane

Subtracting (ii) from (iii), we get

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$$

$\Rightarrow$ a line with direction ratios a, b, c is perpendicular to an arbitrary line AB taken on plane (i) [$\because$ $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$ are d.r.'s of AB]

$\Rightarrow$ a line with d.r.'s a, b, c is perpendicular to the plane (i)

$\Rightarrow$ a, b, c are the direction ratios of the normal to the plane (i).

Hence, $ax + by + cz + d = 0$ represents a plane, and a, b, c are the direction ratios of the normal to this plane.

Equation of a Plane Passing through a Given Point

THEOREM 2 *The equation of a plane passing through a point $P(x_1, y_1, z_1)$ is* $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, *where a, b, c are constants.*

PROOF The general equation of a plane is

$$ax + by + cz + d = 0. \quad \ldots \text{(i)}$$

If this plane passes through the point $P(x_1, y_1, z_1)$ then

$$ax_1 + by_1 + cz_1 + d = 0. \quad \ldots \text{(ii)}$$

Subtracting (ii) from (i), we get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

This is the general equation of a plane passing through the point $P(x_1, y_1, z_1)$.

Equation of a Plane in the Intercept Form

THEOREM 3 *If a plane makes intercepts of lengths a, b, c with the x-axis, y-axis and z-axis respectively, the equation of the plane is* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

[CBSE 2004C]

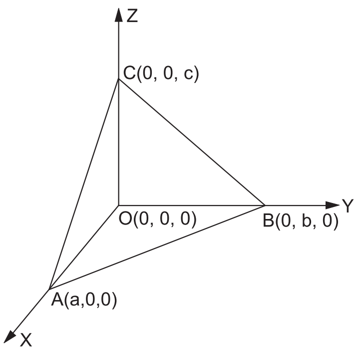
PROOF Let O be the origin, and let the plane meet the coordinate axes at A, B, C respectively such that

$$OA = a, OB = b \text{ and } OC = c.$$

So, the coordinates of these points are $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Let the equation of the given plane be

$$Ax + By + Cz + D = 0. \quad \ldots \text{(i)}$$



Since the given plane does not pass through $O(0, 0, 0)$, $D \neq 0$.

Also, since (i) passes through $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$, we have

$$Aa + D = 0 \Rightarrow A = \frac{-D}{a},$$

$$Bb + D = 0 \Rightarrow B = \frac{-D}{b},$$

$$Cc + D = 0 \Rightarrow C = \frac{-D}{c}.$$

Putting these values in (i), we get

$$\frac{-Dx}{a} - \frac{Dy}{b} - \frac{Dz}{c} + D = 0$$

$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ [on dividing throughout by $-D$].

This is the required equation of the plane in the intercept form.

Equation of a Plane Passing through Three Given Points

Suppose that a plane passes through the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$and $C(x_3, y_3, z_3)$.

We know that the equation of a plane passing through the point $A(x_1, y_1, z_1)$ is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0. \quad \text{... (i)}$$

If this plane passes through the points $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, we have

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \quad \text{... (ii)}$$

and $a(x_3 - x_1) + b(y_3 - y_1) + c(z_3 - z_1) = 0.$ ... (iii)

On eliminating a, b, c from (i), (ii) and (iii), we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

This is the required equation of the plane passing through three points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and (x_3, y_3, z_3).

SUMMARY OF THE RESULTS

1. The general equation of a plane is $ax + by + cz + d = 0$. The d.r.'s of the normal to the plane are a, b, c.
2. The equation of the plane passing through the point $P(x_1, y_1, z_1)$ is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.
3. If a plane makes intercepts a, b and c with the x-axis, y-axis and z-axis respectively, then its equation is
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \cdot$$
4. The equation of a plane passing through three points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by
$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the equation of the plane passing through the points* $A(2, 5, -3)$, $B(-2, -3, 5)$ *and* $C(5, 3, -3)$.

SOLUTION The equation of the plane passing through the point $A(2, 5, -3)$ is given by

$$a(x-2) + b(y-5) + c(z+3) = 0. \quad \text{... (i)}$$

If this plane passes through the points $B(-2, -3, 5)$ and $C(5, 3, -3)$, then we have

$$a(-2-2) + b(-3-5) + c(5+3) = 0 \Rightarrow -4a - 8b + 8c = 0 \quad \text{... (ii)}$$

$$a(5-2) + b(3-5) + c(-3+3) = 0 \Rightarrow 3a - 2b + 0c = 0. \quad \text{... (iii)}$$

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(0+16)} = \frac{b}{(24-0)} = \frac{c}{(8+24)} \Rightarrow \frac{a}{16} = \frac{b}{24} = \frac{c}{32}$$

$$\Rightarrow \frac{a}{2} = \frac{b}{3} = \frac{c}{4} = k \text{ (say)} \Rightarrow a = 2k, b = 3k \text{ and } c = 4k.$$

Putting these values of a, b and c in (i), we get

$$2k(x-2) + 3k(y-5) + 4k(z+3) = 0$$

$$\Rightarrow 2(x-2) + 3(y-5) + 4(z+3) = 0$$

$$\Rightarrow 2x + 3y + 4z - 7 = 0.$$

Hence, the required equation of the plane is $2x + 3y + 4z - 7 = 0$.

Alternate method

We know that equation of the plane passing through the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

Here,

$(x_1 = 2, y_1 = 5, z_1 = -3)$, $(x_2 = -2, y_2 = -3, z_2 = 5)$ and $(x_3 = 5, y_3 = 3, z_3 = -3)$.

$\therefore$ the required equation of the plane is

$$\begin{vmatrix} x-2 & y-5 & z+3 \\ -2-2 & -3-5 & 5+3 \\ 5-2 & 3-5 & -3+3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-2 & y-5 & z+3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0.$$

$$\Rightarrow (x-2)(0+16) - (y-5)(0-24) + (z+3)(8+24) = 0$$

$$\Rightarrow 16(x-2) + 24(y-5) + 32(z+3) = 0 \Rightarrow 16x + 24y + 32z - 56 = 0$$

$$\Rightarrow 2x + 3y + 4z - 7 = 0.$$

Hence, the required equation of the plane is $2x + 3y + 4z - 7 = 0$.

EXAMPLE 2 *Show that the four points* $A(1,-1,1)$, $B(2,3,1)$, $C(1,2,3)$ *and* $D(0,-2,3)$ *are coplanar. Find the equation of the plane containing them.*

SOLUTION We know that the equation of the plane passing through the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

Here,

$(x_1 = 1, y_1 = -1, z_1 = 1), (x_2 = 2, y_2 = 3, z_2 = 1)$ and $(x_3 = 1, y_3 = 2, z_3 = 3)$.

$\therefore$ the equation of the plane passing through the points A, B and C is given by

$$\begin{vmatrix} x-1 & y+1 & z-1 \\ 2-1 & 3+1 & 1-1 \\ 1-1 & 2+1 & 3-1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-1 & y+1 & z-1 \\ 1 & 4 & 0 \\ 0 & 3 & 2 \end{vmatrix} = 0$$

$\Rightarrow$ $(x-1)(8-0)-(y+1)(2-0)+(z-1)(3-0)=0$

$\Rightarrow$ $8(x-1)-2(y+1)+3(z-1)=0 \Rightarrow 8x-2y+3z=13.$ ... (i)

Putting $x=0$, $y=-2$ and $z=3$ in (i), we get

$$\text{LHS} = (8\times 0) - 2\times(-2) + 3\times 3 = (0+4+9) = 13 = \text{RHS}.$$

Thus, the point $D(0,-2,3)$ lies on the plane (i).

Hence, the given four points are coplanar and the equation of the plane containing them is $8x = 2y + 3z = 13$.

EXAMPLE 3 *Find the equation of the plane which cuts off intercepts* 3, 6 *and* -4 *from the axes of coordinates.*

SOLUTION We know that the equation of a plane which cuts off intercepts a, b, c from the x-axis, y-axis and z-axis respectively, is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

Here $a=3$, $b=6$ and $c=-4$.

Hence, the required equation of the plane is

$$\frac{x}{3}+\frac{y}{6}+\frac{z}{-4}=1 \Rightarrow 4x+2y-3z=12.$$

EXAMPLE 4 *Reduce the equation of the plane* $2x-3y+z=6$ *to intercept form and find its intercepts on the coordinate axes.*

SOLUTION We have

$$2x-3y+z=6 \Rightarrow \frac{2x}{6}-\frac{3y}{6}+\frac{z}{6}=1 \Rightarrow \frac{x}{3}+\frac{y}{-2}+\frac{z}{6}=1.$$

Hence, the equation of the given plane in intercept form is $\frac{x}{3}+\frac{y}{-2}+\frac{z}{6}=1.$

The intercepts of the given plane on x-axis, y-axis and z-axis are 3, –2 and 6 respectively.

EXAMPLE 5 *A plane meets the coordinate axes in A, B, C such that the centroid of* $\triangle ABC$ *is* (p, q, r). *Show that the equation of the plane is* $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$.

SOLUTION Let the required equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. ... (i)

Then, clearly the plane meets the coordinate axis at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Since the centroid of $\triangle ABC$ is $G(p, q, r)$, we have

$$\frac{a+0+0}{3}=p,\ \frac{0+b+0}{3}=q \text{ and } \frac{0+0+c}{3}=r$$

$\Rightarrow$ $a = 3p$, $b = 3q$ and $c = 3r$.

Putting these values of a, b, c in (i) we get

$$\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r}=1 \Rightarrow \frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3.$$

Hence, the required equation of the plane is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$.

EXAMPLE 6 *A variable plane moves in such a way that the sum of the reciprocals of its intercepts on the coordinate axes is constant. Show that the plane passes through a fixed point.*

SOLUTION Let the equation of the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \qquad \dots \text{(i)}$$

Then, it makes intercepts a, b, c with the coordinate axes.

$\therefore$ $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=k$, where k is a constant (given)

$$\Rightarrow \frac{1}{ka}+\frac{1}{kb}+\frac{1}{kc}=1 \Rightarrow \frac{1}{a}\left(\frac{1}{k}\right)+\frac{1}{b}\left(\frac{1}{k}\right)+\frac{1}{c}\left(\frac{1}{k}\right)=1$$

$\Rightarrow$ $\left(\frac{1}{k}, \frac{1}{k}, \frac{1}{k}\right)$ satisfies (i).

Hence, the given plane passes through a fixed point $\left(\frac{1}{k}, \frac{1}{k}, \frac{1}{k}\right)$.

EXERCISE 28A

1. Find the equation of the plane passing through each group of points:
   (i) $A(2, 2, -1)$, $B(3, 4, 2)$ and $C(7, 0, 6)$
   (ii) $A(0, -1, -1)$, $B(4, 5, 1)$ and $C(3, 9, 4)$ **[CBSE 2006]**
   (iii) $A(-2, 6, -6)$, $B(-3, 10, -9)$ and $C(-5, 0, -6)$.
2. Show that the four points $A(3, 2, -5)$, $B(-1, 4, -3)$, $C(-3, 8, -5)$ and $D(-3, 2, 1)$ are coplanar. Find the equation of the plane containing them.

3. Show that the four points $A(0, -1, 0)$, $B(2, 1, -1)$, $C(1, 1, 1)$ and $D(3, 3, 0)$ are coplanar. Find the equation of the plane containing them.
4. Write the equation of the plane whose intercepts on the coordinate axes are 2, – 4 and 5 respectively.
5. Reduce the equation of the plane $4x - 3y + 2z = 12$ to the intercept form, and hence find the intercepts made by the plane with the coordinate axes.
6. Find the equation of the plane which passes through the point (2, –3, 7) and makes equal intercepts on the coordinate axes.
7. A plane meets the coordinate axes at A, B and C respectively such that the centroid of $\triangle ABC$ is $(1, -2, 3)$. Find the equation of the plane.
8. Find the Cartesian and vector equations of a plane passing through the point (1, 2, 3) and perpendicular to a line with direction ratios 2, 3, –4.
[CBSE 2005C]
9. If O be the origin and $P(1, 2, -3)$ be a given point, then find the equation of the plane passing through P and perpendicular to OP.

ANSWERS (EXERCISE 28A)

1. (i) $5x + 2y - 3z = 17$ (ii) $5x - 7y + 11z + 4 = 0$ (iii) $2x - y - 2z = 2$

2. $x + y + z = 0$ **3.** $4x - 3y + 2z = 3$ **4.** $10x - 5y + 4z = 20$

5. $\frac{x}{3} + \frac{y}{-4} + \frac{z}{6} = 1$, Intercepts on the axes are 3, –4, 6.

6. $x + y + z = 6$ **7.** $\frac{x}{3} + \frac{y}{-6} + \frac{z}{9} = 1$

8. $2x + 3y - 4z + 4 = 0$, $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) + 4 = 0$

9. $x + 2y - 3z - 14 = 0$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28A)

5. $4x - 3y + 2z = 12 \Rightarrow \frac{x}{3} + \frac{y}{-4} + \frac{z}{6} = 1$ [on dividing each term by 12].

6. Let it make intercept a on each of the coordinate axes.

   Then, its equation is $\frac{x}{a} + \frac{y}{a} + \frac{z}{a} = 1 \Rightarrow x + y + z = a$.

   Putting $x = 2$, $y = -3$, $z = 7$, we get $a = 2 + (-3) + 7 = 6$.

   So, the required equation of the plane is $x + y + z = 6$.

7. Let the plane meet the coordinate axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$. Then, $\frac{a}{3} = 1, \frac{b}{3} = -2, \frac{c}{3} = 3 \Rightarrow a = 3, b = -6, c = 9$.

   $\therefore$ required equation of the plane is $\frac{x}{3} + \frac{y}{-6} + \frac{z}{9} = 1$.

8. Required equation is $2(x - 1) + 3(y - 2) - 4(z - 3) = 0 \Rightarrow 2x + 3y - 4z + 4 = 0$.

Its vector equation is

$$(x\hat{i} + y\hat{j} + z\hat{k})\cdot(2\hat{i} + 3\hat{j} - 4\hat{k}) + 4 = 0, \text{ i.e., } \vec{r}\cdot(2\hat{i} + 3\hat{j} - 4\hat{k}) + 4 = 0.$$

9. Let the required equation of the plane passing through the point $A(1, 2, -3)$ be

$$a(x-1) + b(y-2) + c(z+3) = 0 \qquad \text{... (i)}$$

D.r.'s of OP are $(1-0), (2-0), (-3-0)$, i.e., $1, 2, -3$.

Since the plane is perpendicular to OP, so the normal to the plane is parallel to OP.

$\therefore \quad \frac{a}{1} = \frac{b}{2} = \frac{c}{-3} = k \text{ (say)} \Rightarrow a = k,\ b = 2k \text{ and } c = -3k.$

$\therefore$ required equation of the plane is $k(x-1) + 2k(y-2) - 3k(z+3) = 0$

$\Rightarrow \quad (x-1) + 2(y-2) - 3(z+3) = 0.$

$\Rightarrow \quad (x + 2y - 3z) + (-1 - 4 - 9) = 0$

$\Rightarrow \quad x + 2y - 3z - 14 = 0.$

EQUATIONS OF A PLANE IN VARIOUS FORMS

Equation of a Plane in the Normal Form

Vector Form

THEOREM 1 *If $\hat{n}$ is a unit vector normal to a given plane, directed from the origin to the plane, and p is the length of the perpendicular drawn from the origin to the plane then the vector equation of the plane is $\vec{r}\cdot\hat{n} = p$.*

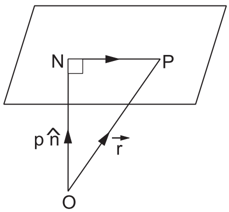
PROOF Let O be the origin, and let ON be the perpendicular drawn from O to the given plane. Let $ON = p$.

Let $\hat{n}$ be a unit vector along $\overrightarrow{ON}$.

Then, $\overrightarrow{ON} = p\hat{n}$.

Let P be an arbitrary point on the plane, and let the position vector of P be $\vec{r}$.

Then, $\overrightarrow{OP} = \vec{r}$.



Since $\overrightarrow{NP}$ lies on the plane, $\overrightarrow{NP}$ is perpendicular to $\hat{n}$.

$\therefore \quad \overrightarrow{NP}\cdot\hat{n} = 0$

$\Rightarrow \quad (\overrightarrow{OP} - \overrightarrow{ON})\cdot\hat{n} = 0 \Rightarrow (\vec{r} - p\hat{n})\cdot\hat{n} = 0$

$\Rightarrow \quad \vec{r}\cdot\hat{n} - p\,\hat{n}\cdot\hat{n} = 0 \Rightarrow \vec{r}\cdot\hat{n} = p.$

Hence, the required equation of the plane is $\vec{r}\cdot\vec{n} = p$.

REMARK The equation of a plane which is at a distance p from the origin and which is perpendicular to $\hat{n}$ is $\vec{r}\cdot\hat{n} = p$.

COROLLARY *If $\vec{n}$ is a vector normal to a given plane then $\vec{r}\cdot\hat{n} = q$ represents a plane.*

PROOF $\vec{r}\cdot\vec{n} = q \Rightarrow \vec{r}\cdot\dfrac{\vec{n}}{|\vec{n}|} = \dfrac{q}{|\vec{n}|} \Rightarrow \vec{r}\cdot\hat{n} = p$, where $\dfrac{q}{|\vec{n}|} = p$.

But, $\vec{r}\cdot\hat{n} = p$ represents a plane.

Hence, $\vec{r}\cdot\vec{n} = q$ also represents a plane.

Cartesian Forms

(i) Normal Form:

Let l, m, n be the d.c.'s of $\hat{n}$ and $P(x, y, z)$ be the given point.

Then, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $n = l\hat{i} + m\hat{j} + n\hat{k}$.

On substituting these values in $\vec{r}\cdot\hat{n} = p$, we get

$(x\hat{i} + y\hat{j} + z\hat{k})\cdot(l\hat{i} + m\hat{j} + n\hat{k}) = p$

$\Rightarrow \quad lx + my + nz = p.$

This is known as the *normal form* of the Cartesian equation of a plane.

(ii) General Form:

Let a, b, c be the d.r.'s of $\vec{n}$ and $P(x, y, z)$ be the given point.

Then, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$.

On substituting these values $\vec{r}\cdot\vec{n} = q$, we get

$(x\hat{i} + y\hat{j} + z\hat{k})\cdot(a\hat{i} + b\hat{j} + c\hat{k}) = q$

$\Rightarrow \quad ax + by + cz = q \quad \Rightarrow \quad ax + by + cz + d = 0$, where $d = -q$.

The equation $ax + by + cz + d = 0$ is called the *General Form* of the Cartesian equation of a plane.

Equation of a Plane Passing through a Given Point and Perpendicular to a Given Vector

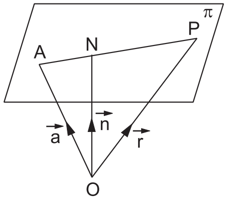
Vector Form

THEOREM 1 *The vector equation of a plane passing through a point A with position vector $\vec{a}$ and perpendicular to a given vector $\vec{n}$ is $(\vec{r} - \vec{a})\cdot\vec{n} = 0$.*

PROOF Let O be the origin, and let π be the given plane.

Let A be a given point on the plane with position vector $\vec{a}$.

Let P be an arbitrary point on the plane with position vector $\vec{r}$.



Then, $\overrightarrow{OA} = \vec{a}$, and $\overrightarrow{OP} = \vec{r}$.

$\therefore \quad \overrightarrow{AP} = (\overrightarrow{OP} - \overrightarrow{OA}) = (\vec{r} - \vec{a})$.

Let $\overrightarrow{ON} = \vec{n}$ be normal to the plane.

Now, $\overrightarrow{AP}$ lies in the plane and $\vec{n}$ is normal to the plane.

$\therefore \quad \overrightarrow{AP} \perp \vec{n} \Rightarrow \overrightarrow{AP} \cdot \vec{n} = 0 \Rightarrow (\vec{r} - \vec{a}) \cdot \vec{n} = 0.$

Hence, the equation of the plane passing through the point with position vector $\vec{a}$ and perpendicular to $\vec{n}$ is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.

COROLLARY *The vector equation of the plane passing through the origin and perpendicular to* $\vec{n}$ *is* $\vec{r} \cdot \vec{n} = 0$.

Cartesian Form

THEOREM 2 *The Cartesian equation of a plane passing through a point* $A(x_1, y_1, z_1)$ *and perpendicular to a line having d.r.'s a, b, c is*

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

PROOF Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$.

Substituting these values in $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$, we get

$$\{(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\} \cdot (A\hat{i} + B\hat{j} + C\hat{k}) = 0$$

$$\Rightarrow \quad a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

This is known as the equation of a plane in ***one-point form.***

SUMMARY

1. (i) Let a plane be at a distance p form the origin and let $\hat{n}$ be a unit vector perpendicular to the plane.
   Then, equation of the plane is $\vec{r} \cdot \hat{n} = p$.

   (ii) The Cartesian form of equation of this plane is $lx + my + nz = p$, where l, m, n are the d.c.'s of normal to the plane.

   (iii) The general equation of a plane is $\vec{r} \cdot \vec{n} = q$.

   (iv) Its Cartesian form is $ax + by + cz + d = 0$, where a, b, c are the d.r.'s of $\vec{n}$.

2. (i) The vector equation of a plane passing through a point A with position vector $\vec{a}$ and perpendicular to $\vec{n}$ is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.

   (ii) The Cartesian equation of a plane passing through a point $A(x_1, y_1, z_1)$ and perpendicular to a line having d.r.'s a, b, c is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the vector equation of a plane which is at a distance of* 5 *units from the origin and which has* $\hat{j}$ *as the unit vector normal to it.*

SOLUTION Clearly, the required equation of the plane is $\vec{r} \cdot \hat{j} = 5$.

EXAMPLE 2 *Find the vector equation of a plane which is at a distance of* 6 *units from the origin and which is normal to the vector* $(\hat{i} + 2\hat{j} - 2\hat{k})$.

SOLUTION Here $\vec{n} = (\hat{i} + 2\hat{j} - 2\hat{k})$ and $p = 6$.

$$\therefore \quad \hat{n} = \frac{(\hat{i} + 2\hat{j} - 2\hat{k})}{\sqrt{1^2 + 2^2 + (-2)^2}} = \left(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}\right).$$

So, the vector equation of the plane is

$$\vec{r} \cdot \hat{n} = p \Rightarrow \vec{r} \cdot \left(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}\right) = 6$$

$$\Rightarrow \vec{r} = (\hat{i} + 2\hat{j} - 2\hat{k}) = 18.$$

Hence, the required vector equation of the plane is

$$\vec{r} \cdot (\hat{i} + 2\hat{j} - 2\hat{k}) = 18.$$

EXAMPLE 3 *Find the vector equation of a plane passing through a point having position vector* $(2\hat{i} - \hat{j} + \hat{k})$ *and perpendicular to the vector* $(4\hat{i} + 2\hat{j} - 3\hat{k})$. *Also reduce it to Cartesian form.*

SOLUTION Here $\vec{a} = (2\hat{i} - \hat{j} + \hat{k})$ and $\vec{n} = (4\hat{i} + 2\hat{j} - 3\hat{k})$.

Clearly, the required vector equation is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (4\hat{i} + 2\hat{j} - 3\hat{k}) = (2\hat{i} - \hat{j} + \hat{k}) \cdot (4\hat{i} + 2\hat{j} - 3\hat{k})$$

$$= (8 - 2 - 3) = 3.$$

Hence, the vector equation of the given plane is

$$\vec{r} \cdot (4\hat{i} + 2\hat{j} - 3\hat{k}) = 3 \qquad \text{... (i)}$$

Reduction to Cartesian Form:

Putting $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$, we get

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (4\hat{i} + 2\hat{j} - 3\hat{k}) = 3 \Rightarrow 4x + 2y - 3z = 3.$$

Hence, the equation of the given plane in Cartesian form is $4x + 2y - 3z = 3$.

EXAMPLE 4 *Find a unit vector normal to the plane* $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) + 14 = 0$.

SOLUTION The equation of the given plane is

$$\vec{r}\cdot(2\hat{i}-3\hat{j}+6\hat{k})+14=0$$

$$\Leftrightarrow \quad \vec{r}\cdot(2\hat{i}-3\hat{j}+6\hat{k})=-14 \;\Rightarrow\; \vec{r}\cdot(-2\hat{i}+3\hat{j}-6\hat{k})=14$$

$$\Rightarrow \quad \vec{r}\cdot\vec{n}=14, \text{ where } \vec{n}=(-2\hat{i}+3\hat{j}-6\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot\frac{\vec{n}}{|\vec{n}|}=\frac{14}{|\vec{n}|}, \text{ where } |\vec{n}|=\sqrt{(-2)^2+3^2+(-6)^2}=7$$

$$\Rightarrow \quad \vec{r}\cdot\frac{(-2\hat{i}+3\hat{j}-6\hat{k})}{7}=\frac{14}{7} \Rightarrow \vec{r}\cdot\left(-\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)=2.$$

Hence, the unit vector normal to the given plane is

$$\hat{n}=\left(-\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right).$$

EXAMPLE 5 *Find the direction cosines of the perpendicular from the origin to the plane* $\vec{r}\cdot(6\hat{i}-3\hat{j}-2\hat{k})+3=0.$

SOLUTION Clearly, we have to find the direction cosines of the normal to the given plane.

The given equation may be written as

$$\vec{r}\cdot(6\hat{i}-3\hat{j}-2\hat{k})=-3 \Rightarrow \vec{r}\cdot(-6\hat{i}+3\hat{j}+2\hat{k})=3$$

$$\Rightarrow \quad \vec{r}\cdot\vec{n}=3, \text{ where } \vec{n}=(-6\hat{i}+3\hat{j}+2\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot\frac{\vec{n}}{|\vec{n}|}=\frac{3}{|\vec{n}|}, \text{ where } |\vec{n}|=\sqrt{(-6)^2+3^2+2^2}=7$$

$$\Rightarrow \quad \vec{r}\cdot\frac{(-6\hat{i}+3\hat{j}+2\hat{k})}{7}=\frac{3}{7} \Rightarrow \vec{r}\cdot\left(-\frac{6}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{2}{7}\hat{k}\right)=\frac{3}{7}\cdot$$

Hence, the direction cosines of the normal to the given plane are

$$\left(-\frac{6}{7},\frac{3}{7},\frac{2}{7}\right).$$

EXAMPLE 6 *Find the Cartesian equation of a plane whose vector equation is* $\vec{r}\cdot(2\hat{i}+5\hat{j}-4\hat{k})=3.$

SOLUTION We have

$$\vec{r}\cdot(2\hat{i}+5\hat{j}-4\hat{k})=3 \Leftrightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}+5\hat{j}-4\hat{k})=3$$

$$\Leftrightarrow \quad 2x+5y-4z=3.$$

Hence, the required equation is $2x+5y-4z=3$.

EXAMPLE 7 *Find the vectors equation of a plane whose Cartesian equation is* $2x - 3y + 4z + 6 = 0$.

Find the direction cosines of the normal to the plane and its distance from the origin.

SOLUTION The given equation of the plane is

$$2x - 3y + 4z = -6 \Rightarrow -2x + 3y - 4z = 6. \quad \text{... (i)}$$

If $\vec{r}$ is the position vector of any point $P(x, y, z)$ on plane (i), then we may write it as

$$(x\hat{i} + y\hat{j} + z\hat{k})\cdot(-2\hat{i} + 3\hat{j} - 4\hat{k}) = 6$$

$$\Rightarrow \vec{r}\cdot(-2\hat{i} + 3\hat{j} - 4\hat{k}) = 6$$

$$\Rightarrow \vec{r}\cdot\vec{n} = 6, \text{ where } \vec{n} = (-2\hat{i} + 3\hat{j} - 4\hat{k})$$

$$\Rightarrow \vec{r}\cdot\frac{\vec{n}}{|\vec{n}|} = \frac{6}{|\vec{n}|}, \text{ where } |\vec{n}| = \sqrt{(-2)^2 + 3^2 + (-4)^2} = \sqrt{29}$$

$$\Rightarrow \vec{r}\cdot\left(\frac{-2}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{4}{\sqrt{29}}\hat{k}\right) = \frac{6}{\sqrt{29}}\cdot$$

$\therefore$ d.c.'s of normal to the given plane are $\frac{-2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{-4}{\sqrt{29}}$, and its distance from the origin is $\frac{6}{\sqrt{29}}\cdot$

EXAMPLE 8 *Find the vector equation of the plane whose Cartesian equation is* $5y + 8 = 0$.

Find the direction cosines of the normal to the plane and its distance from the origin.

SOLUTION The given equation of the plane is

$$5y = -8 \Rightarrow -5y = 8 \Rightarrow 0\cdot x - 5y + 0z = 8. \quad \text{... (i)}$$

Let $\vec{r}$ be the position vector of any point $P(x, y, z)$ on plane (i). Then, the above equation may be written as

$$\vec{r}\cdot(0\hat{i} - 5\hat{j} + 0\hat{k}) = 8 \Rightarrow \vec{r}\cdot\vec{n} = 8, \text{ where } \vec{n} = 0\hat{i} - 5\hat{j} + 0\hat{k}$$

$$\Rightarrow \vec{r}\cdot\frac{\vec{n}}{|\vec{n}|} = \frac{8}{|\vec{n}|}, \text{ where } |\vec{n}| = \sqrt{0^2 + (-5)^2 + 0^2} = 5$$

$$\Rightarrow \vec{r}\cdot\left(\frac{0}{5}\hat{i} - \frac{5}{5}\hat{j} + \frac{0}{5}\hat{k}\right) = \frac{8}{5} \Rightarrow \vec{r}\cdot(0\hat{i} - \hat{j} + 0\hat{k}) = \frac{8}{5}\cdot$$

$\therefore$ d.c.'s of the normal to the plane are 0, –1, 0 and its distance from the origin is $\frac{8}{5}$ units.

EXAMPLE 9 *Find the Cartesian form of the equation of the plane*

$$\vec{r} = (s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}.$$

SOLUTION We have

$$\vec{r} = (s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}$$

$\Leftrightarrow$ $(x\hat{i} + y\hat{j} + z\hat{k}) = (s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}$

$\Leftrightarrow$ $x = s - 2t$, $y = 3 - t$ and $z = 2s + t$

$\Leftrightarrow$ $x - 2y = (s - 6)$ and $y + z = (3 + 2s)$ [eliminating t]

$\Leftrightarrow$ $x - 2y + 6 = \frac{1}{2}(y + z - 3)$ [equating the values of s]

$\Leftrightarrow$ $2x - 4y + 12 = y + z - 3$ $\Leftrightarrow$ $2x - 5y - z + 15 = 0.$

This is the required Cartesian form of the equation of the given plane.

EXAMPLE 10 *Find the vector and Cartesian equations of the plane which passes through the point* $(5, 2, -4)$ *and perpendicular to the line with direction ratios* $2, 3, -1$.

SOLUTION The plane passes through the point $A(5, 2, -4)$ whose position vector is $\vec{a} = (5\hat{i} + 2\hat{j} - 4\hat{k})$ and the normal vector $\vec{n}$ perpendicular to the plane is $\vec{n} = (2\hat{i} + 3\hat{j} - \hat{k})$.

So, the vector equation of the plane is

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$\Rightarrow$ $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = (5\hat{i} + 2\hat{j} - 4\hat{k}) \cdot (2\hat{i} + 3\hat{j} - \hat{k})$

$$= (10 + 6 + 4) = 20.$$

Hence, the required vector equation of the plane is

$$\vec{r} = (2\hat{i} + 3\hat{j} - \hat{k}) = 20. \quad \text{... (i)}$$

Cartesian Form:

Taking $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$, equation (i) may be written as

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 20 \Rightarrow 2x + 3y - z = 20.$$

Hence, the required Cartesian equation is $2x + 3y - z = 20$.

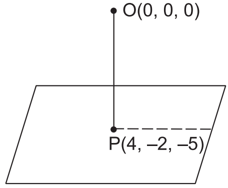
EXAMPLE 11 *The foot of the perpendicular drawn from the origin to a plane is* $(4, -2, -5)$. *Find the equation of the plane in (i) vector form, (ii) Cartesian form.* **[CBSE 2007]**

SOLUTION Let $O(0, 0, 0)$ be the origin and $P(4, -2, -5)$ be the foot of the perpendicular drawn from O to the given plane.

(i) Clearly, the required plane passes through the point P and it is perpendicular to OP.

$\therefore \quad \vec{a} = (4\hat{i} - 2\hat{j} - 5\hat{k})$ and $\vec{n} = \overrightarrow{OP}\,(4\hat{i} - 2\hat{j} - 5\hat{k})$.

So, the required equation of the plane is



$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} = (4\hat{i} - 2\hat{j} - 5\hat{k})$$

$$= (4\hat{i} - 2\hat{j} - 5\hat{k}) \cdot (4\hat{i} - 2\hat{j} - 5\hat{k})$$

$$= (16 + 4 + 25) = 45.$$

Hence, the vector equation of the plane is

$$\vec{r} \cdot (4\hat{i} - 2\hat{j} - 5\hat{k}) = 45 \qquad \text{... (i)}$$

(ii) Putting $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$ in (i), we get

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (4\hat{i} - 2\hat{j} - 5\hat{k}) = 45$$

$\Rightarrow \quad 4x - 2y - 5z = 45$, which is the required equation of the plane in Cartesian form.

EXAMPLE 12 *Find the coordinates of the foot of the perpendicular drawn from the origin to the plane* $3y + 4z - 6 = 0$.

SOLUTION The equation of the given plane is

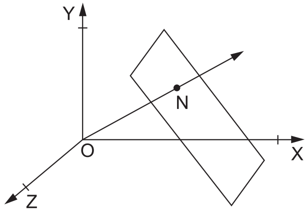
$$0x + 3y + 4z - 6 = 0. \qquad \text{... (i)}$$

D.r.'s of normal to the given plane are 0, 3, 4.

The equation of the line through the origin O and perpendicular to the plane (i) is given by

$$\frac{x}{0} = \frac{y}{3} = \frac{z}{4} = \lambda \text{ (say)}. \qquad \text{... (ii)}$$

The general point on (ii) is given by $N(0, 3\lambda, 4\lambda)$.



If this point lies on plane (i), we have

$$(0 \times 0) + (3 \times 3\lambda) + (4 \times 4\lambda) = 6 \Rightarrow 25\lambda = 6 \Rightarrow \lambda = \frac{6}{25}.$$

Hence, the foot of the perpendicular drawn from the origin to the given plane is $N\left(0, \frac{18}{20}, \frac{24}{25}\right)$.

EXAMPLE 13 *Find the coordinates of the point where the line* $\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4}$ *meets the plane* $x + y + 4z = 6$. **[CBSE 2006, '08]**

SOLUTION The equation of the given line is

$$\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4} = \lambda \text{ (say)}. \qquad \text{... (i)}$$

The equation of the given plane is

$$x + y + 4z = 6 \qquad \text{... (ii)}$$

The general point on the line (i) is $(2\lambda - 1, 3\lambda - 2, 4\lambda - 3)$.

Let the given line (i) meet the given plane (ii) at the point $P(2\lambda - 1, 3\lambda - 2, 4\lambda - 3)$ for some value of λ.

Then, $(2\lambda - 1) + (3\lambda - 2) + 4(4\lambda - 3) = 6 \Rightarrow 21\lambda = 21 \Rightarrow \lambda = 1$.

Hence, the required point of intersection of the given line (i) and the given plane (ii) is $P(2 \times 1 - 1, 3 \times 1 - 2, 4 \times 1 - 3)$, i.e. $P(1, 1, 1)$.

EXAMPLE 14 *Find the coordinates of the point where the line through the points $A(3, 4, 1)$ and $B(5, 1, 6)$ crosses the XY-plane.* **[CBSE 2012]**

SOLUTION The equation of the line through the points $A(3, 4, 1)$ and $B(5, 1, 6)$ is given by

$$\frac{x-3}{(5-3)} = \frac{y-4}{(1-4)} = \frac{z-1}{(6-1)} \Rightarrow \frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5}. \qquad \text{... (i)}$$

The equation of the XY-plane is $z = 0$.

Since the line (i) meets the XY-plane, we have

$$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{0-1}{5} \Rightarrow \frac{x-3}{2} = \frac{-1}{5} \text{ and } \frac{y-4}{-3} = \frac{-1}{5}$$

$$\Rightarrow \quad x = \left(3 - \frac{2}{5}\right) = \frac{13}{5},\ y = \left(4 + \frac{3}{5}\right) = \frac{23}{5} \text{ and } z = 0.$$

Hence, the given line crosses the XY-plane at the point

$$P\left(\frac{13}{5}, \frac{23}{5}, 0\right).$$

EXAMPLE 15 *Find the distance of the point $P(-1, -5, -10)$ from the point of intersection of the line joining the points $A(2, -1, 2)$ and $B(5, 3, 4)$ with the plane $x - y + z = 5$.* **[CBSE 2014]**

SOLUTION The equation of the given plane is

$$x - y + z = 5 \qquad \text{... (i)}$$

The equation of the line AB is

$$\frac{x-2}{(5-2)} = \frac{y+1}{(3+1)} = \frac{z-2}{(4-2)} \Rightarrow \frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = \lambda \text{ (say)}. \qquad \text{... (ii)}$$

For some value of λ, let the point $Q(3\lambda + 2, 4\lambda - 1, 2\lambda + 2)$ be the point of intersection of the plane (i) and the line (ii).

Then, $(3\lambda + 2) - (4\lambda - 1) + (2\lambda + 2) = 5 \Rightarrow \lambda = 0$.

So, the required point Q is $Q(2, -1, 2)$.

$$\therefore \quad PQ = \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$$

$$= \sqrt{3^2 + 4^2 + (12)^2} = \sqrt{9 + 16 + 144}$$

$$= \sqrt{169} = 13 \text{ units.}$$

Hence, the required distance is 13 units.

EXAMPLE 16 *Find the coordinates of the point where the line through* $(3, -4, -5)$ *and* $(2, -3, 1)$ *crosses the plane passing through the points* $(2, 2, 1), (3, 0, 1)$ *and* $(4, -1, 0)$. **[CBSE 2013]**

SOLUTION We know that the equation of a plane passing through the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

$\therefore$ the equation of the plane passing through the points $A(2, 2, 1)$, $B(3, 0, 1)$ and $C(4, -1, 0)$ is given by

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 1-1 \\ 4-2 & -1-2 & 0-1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(2-0)-(y-2)(-1-0)+(z-1)(-3+4)=0$$

$$\Rightarrow 2(x-2)+1\cdot(y-2)+1\cdot(z-1)=0 \Rightarrow 2x+y+z=7. \qquad \text{... (i)}$$

The equation of the line passing through the points $P(3, -4, -5)$ and $Q(2, -3, 1)$ is given by

$$\frac{x-3}{(2-3)} = \frac{y+4}{(-3+4)} = \frac{z+5}{(1+5)} \Rightarrow \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda \text{ (say)}.$$

Any general point on line PQ is given as $(-\lambda+3, \lambda-4, 6\lambda-5)$.

For some value of λ, let the point $R(-\lambda+3, \lambda-4, 6\lambda-5)$ lie on the plane (i). Then,

$$2(-\lambda+3)+(\lambda-4)+(6\lambda-5)=7 \Rightarrow 5\lambda=10 \Rightarrow \lambda=2.$$

Putting $\lambda = 2$, we get the point of intersection of the given line and the given plane as $R(-2+3, 2-4, 12-5)$, i.e., $R(1, -2, 7)$.

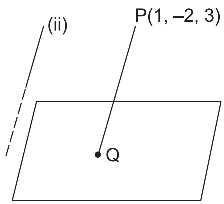
EXAMPLE 17 *Find the distance of the point* $(1, -2, 3)$ *from the plane* $x-y+z=5$ *measured parallel to the line* $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+2}{-6}$. **[CBSE 2013C]**

SOLUTION The equation of the given plane is

$$x-y+z=5 \qquad \text{.... (i)}$$

The equation of the given line is

$$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+2}{-6}. \qquad \text{... (ii)}$$



The equation of the line passing through the point $P(1, -2, 3)$ and parallel to line (ii) is given by

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = \lambda \text{ (say)}. \qquad \text{... (iii)}$$

The general point on line (iii) is $(2\lambda+1, 3\lambda-2, 6\lambda+3)$.

For some value of λ, let the point $Q(2\lambda+1,\ 3\lambda-2,\ -6\lambda+3)$ lie on plane (i). Then, we have

$$(2\lambda+1)-(3\lambda-2)+(-6\lambda+3)=5 \Rightarrow -7\lambda=-1 \Rightarrow \lambda=\frac{1}{7}.$$

So, the coordinates of Q are

$$Q\left(\frac{2}{7}+1,\ \frac{3}{7}-2,\frac{-6}{7}+3\right), \text{ i.e., } Q\left(\frac{9}{7},\frac{-11}{7},\frac{15}{7}\right).$$

$$\text{Distance } PQ=\sqrt{\left(\frac{9}{7}-1\right)^2+\left(\frac{-11}{7}+2\right)^2+\left(\frac{15}{7}-3\right)^2}$$

$$=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{-6}{7}\right)^2}=\frac{1}{7}\sqrt{4+9+36}$$

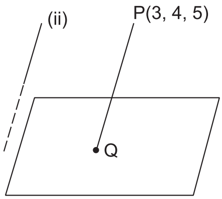
$$=\frac{1}{7}\times\sqrt{49}=\left(\frac{1}{7}\times 7\right)=1 \text{ unit.}$$

Hence, the required distance is 1 unit.

EXAMPLE 18 *Find the distance of the point* $(3, 4, 5)$ *from the plane* $x+y+z=2$, *measured parallel to the line* $2x=y=z$. **[CBSE 2012C]**

SOLUTION The equation of the given plane is

$$x+y+z=2. \quad \text{... (i)}$$



The equation of the given line is

$$2x=y=z \Rightarrow \frac{x}{1}=\frac{y}{2}=\frac{z}{2}. \quad \text{... (ii)}$$

The equation of the line passing through the point $P(3, 4, 5)$ and parallel to the given line (ii) is

$$\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}=\lambda \text{ (say).} \quad \text{... (iii)}$$

The general point on line (iii) is $(\lambda+3,\ 2\lambda+4,\ 2\lambda+5)$.

For some value of λ, let the point $Q(\lambda+3,\ 2\lambda+4,\ 2\lambda+5)$ lie on the plane (i). Then,

$$(\lambda+3)+(2\lambda+4)+(2\lambda+5)=2$$

$\Rightarrow$ $5\lambda=-10 \Rightarrow \lambda=-2.$

$\therefore$ coordinates of Q are $Q(-2+3,\ -4+4,\ -4+5)$, i.e., $Q(1, 0, 1)$.

$$\text{Distance } PQ=\sqrt{(3-1)^2+(4-0)^2+(5-1)^2}$$

$$=\sqrt{2^2+4^2+4^2}=\sqrt{4+16+16}$$

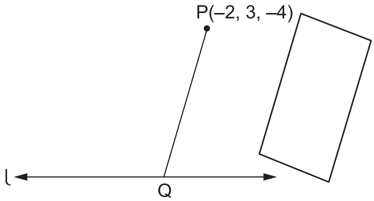
$$=\sqrt{36}=6 \text{ units.}$$

Hence, the required distance is 6 units.

EXAMPLE 19 *Find the distance of the point* $(-2, 3, -4)$ *from the line* $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$, *measured parallel to the plane* $4x + 12y - 3z + 1 = 0$. **[CBSE 2008]**

SOLUTION The equation of the given plane is

$$4x + 12y - 3z + 1 = 0. \quad \text{... (i)}$$



Let $P(-2, 3, -4)$ be the given point.

Let l be the given line whose equation is

$$\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5} = \lambda \text{ (say)}. \quad \text{... (ii)}$$

The general point on this line is $\left(3\lambda - 2, \frac{4\lambda-3}{2}, \frac{5\lambda-4}{3}\right)$.

For some value of λ, let the point $Q\left(3\lambda - 2, \frac{4\lambda-3}{2}, \frac{5\lambda-4}{3}\right)$ lie on the line (ii) such that PQ is parallel to the given plane (i).

D.r.'s of PQ are $(3\lambda - 2 + 2)$, $\left(\frac{4\lambda-3}{2} - 3\right)$, $\left(\frac{5\lambda-4}{3} + 4\right)$,

i.e., $3\lambda, \frac{4\lambda-9}{2}, \frac{5\lambda+8}{3}$.

D.r.'s of the normal to the given plane are 4, 12, –3.

Now, PQ is parallel to the given plane (i).

$\Rightarrow$ PQ is perpendicular to the normal to the the plane (i).

$\Rightarrow$ $(4 \times 3\lambda) + 12 \times \frac{(4\lambda - 9)}{2} - 3 \times \frac{(5\lambda - 8)}{3} = 0$

$\Rightarrow$ $12\lambda + (24\lambda - 54) - (5\lambda - 8) = 0 \Rightarrow 31\lambda = 62 \Rightarrow \lambda = 2.$

$\therefore$ coordinates of Q are $Q\left(4, \frac{5}{2}, 2\right)$

$$\Rightarrow PQ = \sqrt{(4+2)^2 + \left(\frac{5}{2} - 3\right)^2 + (2+4)^2} = \sqrt{6^2 + \left(\frac{-1}{2}\right)^2 + 6^2}$$

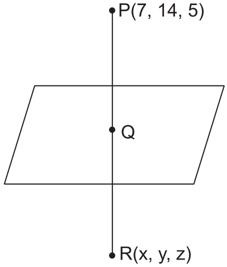
$$= \sqrt{72 + \frac{1}{4}} = \sqrt{\frac{289}{4}} = \frac{17}{2} \text{ units} = 8.5 \text{ units}.$$

Hence, the required distance of the given point from the given line is 8.5 units.

EXAMPLE 20 *Find the length and the foot of the perpendicular from the point $P(7, 14, 5)$ to the plane $2x + 4y - z = 2$. Also, find the image of the point P in the plane.* **[CBSE 2012]**

SOLUTION The equation of the given plane is

$$2x + 4y - z = 2 \qquad \text{... (i)}$$



The d.r.'s of the normal to this plane are 2, 4, –1.

The equation of the line passing through the point $P(7, 14, 5)$ and perpendicular to the given plane (i) is given by

$$\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1} = \lambda \text{ (say)}$$

A general point on this line is $(2\lambda + 7,\ 4\lambda + 14,\ -\lambda + 5)$.

For some value of λ, let the point $Q(2\lambda + 7,\ 4\lambda + 14,\ -\lambda + 5)$ lie on the plane (i). Then,

$$2(2\lambda + 7) + 4(4\lambda + 14) - (-\lambda + 5) = 2 \Rightarrow 21\lambda = -63 \Rightarrow \lambda = -3.$$

$\therefore$ the coordinates of the foot of the perpendicular PQ are

$$Q[2 \times (-3) + 7,\ 4 \times (-3) + 14,\ -(-3) + 5], \text{ i.e., } Q(1, 2, 8).$$

$$\therefore \quad PQ = \sqrt{(7-1)^2 + (14-2)^2 + (5-8)^2}$$

$$= \sqrt{6^2 + (12)^2 + (-3)^2} = \sqrt{36 + 144 + 9}$$

$$= \sqrt{189} = 3\sqrt{21} \text{ units.}$$

Let $R(x, y, z)$ be the image of P in the plane (i).

Then, Q is the midpoint of PR.

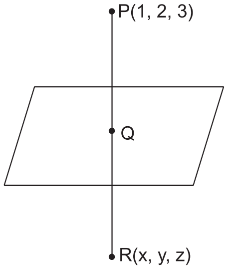
$$\therefore \quad \frac{7+x}{2} = 1,\ \frac{14+y}{2} = 2 \text{ and } \frac{5+z}{2} = 8 \Rightarrow x = -5,\ y = -10, z = 11.$$

Hence, the image of $P(7, 14, 5)$ in the given plane is $R(-5, -10, 11)$.

EXAMPLE 21 *Find the image of the point* $(1, 2, 3)$ *in the plane* $x + 2y + 4z = 38$.

SOLUTION The equation of the given plane is

$$x + 2y + 4z = 38. \qquad \text{... (i)}$$



The d.r.'s of the normal to this plane are 1, 2, 4.

The equation of the line passing through the point $P(1, 2, 3)$ and perpendicular to the given plane (i) is given by

$$\frac{(x-1)}{1} = \frac{(y-2)}{2} = \frac{(z-3)}{4} = \lambda \text{ (say).}$$

A general point on this line is $(\lambda + 1,\ 2\lambda + 2,\ 4\lambda + 3)$.

For some value of λ, let the point $Q(\lambda + 1,\ 2\lambda + 2,\ 4\lambda + 3)$ lie on the plane (i). Then,

$$(\lambda+1)+2(2\lambda+2)+4(4\lambda+3)=38 \Rightarrow 21\lambda=21 \Rightarrow \lambda=1.$$

$\therefore$ the coordinates of the foot of the perpendicular PQ are

$Q(1+1,\ 2+2,\ 4+3)$, i.e., $Q(2,\ 4,\ 7)$.

Let $R(x, y, z)$ be the image of $P(1, 2, 3)$ in the given plane.

Then, Q is the midpoint of PR.

$\therefore \quad \dfrac{1+x}{2}=2,\ \dfrac{2+y}{2}=4$ and $\dfrac{3+z}{2}=7 \Rightarrow x=3,\ y=6,\ z=11.$

Hence, the image of $P(1, 2, 3)$ in the given plane is $R(3, 6, 11)$.

EXERCISE 28B

1. Find the vector and Cartesian equations of a plane which is at a distance of 5 units from the origin and which has $\hat{k}$ as the unit vector normal to it.
2. Find the vector and Cartesian equations of a plane which is at a distance of 7 units from the origin and whose normal vector from the origin is $(3\hat{i}+5\hat{j}-6\hat{k})$.
3. Find the vector and Cartesian equations of a plane which is at a distance of $\dfrac{6}{\sqrt{29}}$ from the origin and whose normal vector from the origin is $(2\hat{i}-3\hat{j}+4\hat{k})$.
4. Find the vector and Cartesian equations of a plane which is at a distance of 6 units from the origin and which has a normal with direction ratios 2, –1, –2.
5. Find the vector and Cartesian equations of a plane which passes through the point (1, 4, 6) and normal vector to the plane is $(\hat{i}-2\hat{j}+\hat{k})$.
6. Find the length of perpendicular from the origin to the plane $\vec{r}\cdot(3\hat{i}-12\hat{j}-4\hat{k})+39=0$. Also write the unit normal vector from the origin to the plane.
7. Find the Cartesian equation of the plane whose vector equation is $\vec{r}\cdot(3\hat{i}+5\hat{j}-9\hat{k})=8$.
8. Find the vector equation of a plane whose Cartesian equation is $5x-7y+2z+4=0$.
9. Find a unit vector normal to the plane $x-2y+2z=6$.
10. Find the direction cosines of the normal to the plane $3x-6y+2z=7$.
11. For each of the following planes, find the direction cosines of the normal to the plane and the distance of the plane from the origin:
    (i) $2x+3y-z=5$ (ii) $z=3$ (iii) $3y+5=0$

12. Find the vector and Cartesian equations of the plane passing through the point (2, –1, 1) and perpendicular to the line having direction ratios 4, 2, –3.
13. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane (i) $2x + 3y + 4z - 12 = 0$ (ii) $5y + 8 = 0$.
14. Find the length and the foot of perpendicular drawn from the point (2, 3, 7) to the plane $3x - y - z = 7$.
15. Find the length and the foot of the perpendicular drawn from the point (1, 1, 2) to the plane $\vec{r} \cdot (2\hat{i} - 2\hat{j} + 4\hat{k}) + 5 = 0$.
16. From the point $P(1, 2, 4)$, a perpendicular is drawn on the plane $2x + y - 2z + 3 = 0$. Find the equation, the length and the coordinates of the foot of the perpendicular. **[CBSE 2008]**
17. Find the coordinates of the foot of the perpendicular and the perpendicular distance from the point $P(3, 2, 1)$ to the plane $2x - y + z + 1 = 0$.

    Find also the image of the point P in the plane. **[CBSE 2010]**
18. Find the coordinates of the image of the point $P(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$. **[CBSE 2008]**
19. Find the point where the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ meets the plane $2x + 4y - z = 1$.
20. Find the coordinates of the point where the line through (3, –4, –5) and (2, – 3, 1) crosses the plane $2x + y + z = 7$.
21. Find the distance of the point (2, 3, 4) from the plane $3x + 2y + 2z + 5 = 0$, measured parallel to the line $\frac{x+3}{3} = \frac{y-2}{6} = \frac{z}{2}$. **[CBSE 2007, '09C]**
22. Find the distance of the point (0, –3, 2) from the plane $x + 2y - z = 1$, measured parallel to the line $\frac{x+1}{3} = \frac{y+1}{2} = \frac{z}{3}$.
23. Find the equation of the line passing through the point $P(4, 6, 2)$ and the point of intersection of the line $\frac{x-1}{3} = \frac{y}{2} = \frac{z+1}{7}$ and the plane $x + y - z = 8$. **[CBSE 2008]**
24. Show that the distance of the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ from the point $(-1, -5, -10)$ is 13 units.
25. Find the distance of the point $A(-1, -5, -10)$ from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$. **[CBSE 2011, '14]**

    HINT: Convert the equations of the line and the plane to Cartesian form.

26. Prove that the normals to the planes $4x+11y+2z+3=0$ and $3x-2y+5z=8$ are perpendicular to each other.

27. Show that the line $\vec{r}=(2\hat{i}-2\hat{j}+3\hat{k})+\lambda(\hat{i}-\hat{j}+4\hat{k})$ is parallel to the plane $\vec{r}\cdot(\hat{i}+5\hat{j}+\hat{k})=7$.

28. Find the equation of a plane which is at a distance of $3\sqrt{3}$ units from the origin and the normal to which is equally inclined to the coordinate axes.

29. A vector $\vec{n}$ of magnitude 8 units is inclined to the x-axis at 45°, y-axis at 60° and an acute angle with the z-axis. If a plane passes through a point $(\sqrt{2}, -1, 1)$ and is normal to $\vec{n}$, find its equation in vector form.

30. Find the vector equation of a line passing through the point $(2\hat{i}-3\hat{j}-5\hat{k})$ and perpendicular to the plane $\vec{r}\cdot(6\hat{i}-3\hat{j}+5\hat{k})+2=0$.
Also, find the point of intersection of this line and the plane. **[CBSE 2006C]**

ANSWERS (EXERCISE 28B)

1. $\vec{r}\cdot\hat{k}=5, z=5$ **2.** $\vec{r}\cdot(3\hat{i}+5\hat{j}-6\hat{k})=7\sqrt{70},\ 3x+5y-6z=7\sqrt{70}$

3. $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=6,\ 2x-3y+4z=6$

4. $\vec{r}\cdot(2\hat{i}-\hat{j}-2\hat{k})=18,\ 2x-y-2z=18$

5. $\vec{r}\cdot(-\hat{i}+2\hat{j}-\hat{k})=1,\ x-2y+z+1=0$

6. $p=3, \hat{n}=\left(\frac{-3}{13}\hat{i}+\frac{12}{13}\hat{j}+\frac{4}{13}\hat{k}\right)$ **7.** $3x+5y+9z=8$

8. $\vec{r}\cdot(-5\hat{i}+7\hat{j}-2\hat{k})=4$ **9.** $\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$ **10.** $\frac{3}{7}, \frac{-6}{7}, \frac{2}{7}$

11. (i) $\left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right), \frac{5}{\sqrt{14}}$ (ii) $(0, 0, 1), 3$ (iii) $(0, -1, 0), \frac{5}{3}$

12. $\vec{r}\cdot(4\hat{i}+2\hat{j}-3\hat{k})=3,\ 4x+2y-3z=3$

13. (i) $\left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$ (ii) $\left(0, \frac{-8}{5}, 0\right)$ **14.** $\sqrt{11}$ units, $(5, 2, 6)$

15. $\frac{13\sqrt{6}}{12}$ units, $\left(\frac{-1}{12}, \frac{25}{12}, \frac{-1}{6}\right)$ **16.** $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-4}{-2}; \frac{1}{3}$ unit; $\left(\frac{11}{9}, \frac{19}{9}, \frac{34}{9}\right)$

17. $(1, 3, 0), \sqrt{6}$ units, $(-1, 4, -1)$ **18.** $(-3, 5, 2)$ **19.** $(3, -1, 1)$ **20.** $(1, -2, 7)$

21. 7 units **22.** 10 units **23.** $\frac{x-4}{1}=\frac{y-6}{2}=\frac{z-2}{2}$ **25.** 13 units

28. $x+y+z=9$ **29.** $\vec{r}\cdot(\sqrt{2}\hat{i}+\hat{j}+\hat{k})=2$

30. $\vec{r}\cdot(2\hat{i}-3\hat{j}-5\hat{k})+\lambda(6\hat{i}-3\hat{j}+5\hat{k}), \left(\frac{76}{35}, \frac{-108}{35}, \frac{-34}{4}\right)$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28B)

6. Given equation in standard form is $\vec{r}\cdot(-3\hat{i}+12\hat{j}+4\hat{k})=39$.

   This is $\vec{r}\cdot\vec{n}=p$, where $\vec{n}=(-3\hat{i}+12\hat{j}+4\hat{k})$ and $p=39$.

   $$|\vec{n}|=\sqrt{(-3)^2+(12)^2+4^2}=\sqrt{9+144+16}=\sqrt{169}=13.$$

   $$\therefore \quad \vec{r}\cdot\frac{\vec{n}}{|\vec{n}|}=\frac{p}{|\vec{n}|}=\frac{39}{13}=3.$$

   $$\vec{r}\cdot\left(\frac{-3}{13}\hat{i}+\frac{12}{13}\hat{j}+\frac{4}{13}\hat{k}\right)=3.$$

   Hence, $p=3$ and $\hat{n}=\frac{-3}{13}\hat{i}+\frac{12}{13}\hat{j}+\frac{4}{13}\hat{k}$.

9. Given equation in vector form is $\vec{r}\cdot(\hat{i}-2\hat{j}+2\hat{k})=6$.

   Here, $|\hat{i}-2\hat{j}+2\hat{k}|=\sqrt{1^2+(-2)^2+2^2}=\sqrt{9}=3$.

   $$\vec{r}\cdot\left(\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)=\frac{6}{3}=2.$$

   Required unit vector $=\left(\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$.

10. Given equation in vector form is $\vec{r}\cdot(3\hat{i}-6\hat{j}+2\hat{k})=7$.

   $$|3\hat{i}-6\hat{j}+2\hat{k}|=\sqrt{3^2+(-6)^2+2^2}=\sqrt{49}=7.$$

   $$\therefore \quad \vec{r}\cdot\left(\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}\right)=1.$$

   $\therefore$ d.c.'s of the normal to the plane are $\frac{3}{7}, \frac{-6}{7}, \frac{2}{7}$.

11. (i) Given equation in vector form is $\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})=5$.

   $$|2\hat{i}+3\hat{j}-\hat{k}|=\sqrt{2^2+3^2+(-1)^2}=\sqrt{14}.$$

   $$\therefore \quad \vec{r}\cdot\left(\frac{2}{\sqrt{14}}\hat{i}+\frac{3}{\sqrt{14}}\hat{j}-\frac{1}{\sqrt{14}}\hat{k}\right)=\frac{5}{\sqrt{14}}.$$

d.c.'s of $\hat{n}$ are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}$ and $p = \frac{5}{\sqrt{14}}\cdot$

(ii) Given equation in vector form is $\vec{r}\cdot(0\hat{i} + 0\hat{j} + 1\hat{k}) = 3$.

This is $\vec{r}\cdot\hat{n} = p$.

D.c.'s of $\hat{n}$ are 0, 0, 1 and $p = 3$.

(iii) Given equation in vector form is $\vec{r}\cdot(0\hat{i} - 1\cdot\hat{j} + 0\cdot\hat{k}) = \frac{5}{3}\left[\because -y = \frac{5}{3}\right]$.

Clearly, $|0\hat{i} - 1\cdot\hat{j} + 0\cdot\hat{k}| = \sqrt{0^2 + (-1)^2 + 0^2} = 1$.

$\therefore \quad \vec{r}\cdot\hat{n} = p$, where $\hat{n} = (0\hat{i} - 1\cdot\hat{j} + 0\hat{k})$ and $p = \frac{5}{3}\cdot$

D.c.'s of $\hat{n}$ are 0, –1, 0 and $p = \frac{5}{3}\cdot$

12. Let the equation of the plane be $a(x-2) + b(y+1) + c(z-1) = 0$, where $a = 4$, $b = 2$ and $c = -3$.

So, the required equation is

$$4(x-2) + 2(y+1) - 3(z-1) = 0 \Rightarrow 4x + 2y - 3z = 3.$$

In vector form, we have $\vec{r}\cdot(4\hat{i} + 2\hat{j} - 3\hat{k}) = 3$.

13. (i) Given equation is $2x + 3y + 4z = 12$. ... (i)

D.r.'s of normal to the plane are 2, 3, 4.

Equation of the line through $O(0, 0, 0)$ and perpendicular to plane (i) is

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{4} = \lambda \text{ (say)}. \quad \text{... (ii)}$$

A general point on (ii) is $N(2\lambda, 3\lambda, 4\lambda)$.

If this point lies on plane (i), then

$$(2\times 2\lambda) + (3\times 3\lambda) + (4\times 4\lambda) = 12 \Rightarrow 29N = 12 \Rightarrow \lambda = \frac{12}{29}\cdot$$

Hence, the required point is $\left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$.

(ii) The given plane is $0x - 5y + 0z = 8$. ... (i)

D.r.'s of normal to the plane are 0, –5, 0.

Equation of the line through $O(0, 0, 0)$ and perpendicular to plane (i) are

$$\frac{x}{0} = \frac{y}{-5} = \frac{z}{0} = \lambda \text{ (say)}. \quad \text{... (ii)}$$

A general point on this line is $N(0, -5\lambda, 0)$.

If this point lies on plane (i), we have

$$-5\times(-5\lambda) = 8 \Rightarrow 25\lambda = 8 \Rightarrow \lambda = \frac{8}{25}\cdot$$

Hence, the required point is $N\left(0, \frac{-8}{5}, 0\right)$.

21. Equation of a line through $A(2, 3, 4)$ and parallel to the given line is

$$\frac{x-2}{3} = \frac{y-3}{6} = \frac{z-4}{2} = \lambda \text{ (say)}. \qquad \text{... (i)}$$

A general point on line (i) is $N(3\lambda + 2, 6\lambda + 3, 2\lambda + 4)$.

For some value of λ, it lies on the plane $3x + 2y + 2z + 5 = 0$.

$\therefore \quad 3(3\lambda + 2) + 2(6\lambda + 3) + 2(2\lambda + 4) + 5 = 0 \Rightarrow 25\lambda = -25 \Rightarrow \lambda = -1.$

$\therefore$ the coordinates of N are $(-3 + 2, -6 + 3, -2 + 4)$, i.e., $(-1, -3, 2)$.

Now, find $AN = 7$ units.

24. Given line is $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = \lambda$ (say)

A general point on this line is $P(3\lambda + 2, 4\lambda - 1, 12\lambda + 2)$.

If this point lies on the plane $x - y + z = 5$, then

$$(3\lambda + 2) - (4\lambda - 1) + (12\lambda + 2) = 5 \Rightarrow 11\lambda = 0 \Rightarrow \lambda = 0.$$

$\therefore$ point P is $P(2, -1, 2)$ and the other point is $Q(-1, -5, -10)$

Find $PQ = 13$ units.

25. The Cartesian equation of the plane is $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = \lambda$ (say).

A general point on this line is $P(3\lambda + 2, 4\lambda - 1, 2\lambda + 2)$.

The Cartesian equation of the given plan is $x - y + z = 5$.

If this point lies on the given plane, we have

$$(3\lambda + 2) - (4\lambda - 1) + (2\lambda + 2) = 5 \Rightarrow \lambda = 0.$$

$\therefore$ point P is $P(2, -1, 2)$ and the other point is $Q(-1, -5, -10)$.

$\therefore$ $PQ = 13$ units.

26. $\vec{n_1} = (4\hat{i} + 11\hat{j} + 2\hat{k})$ and $\vec{n_2} = (3\hat{i} - 2\hat{j} + 5\hat{k})$.

$\therefore \quad \vec{n_1} \cdot \vec{n_2} = (4 \times 3) + 11 \times (-2) + 2 \times 5 = 0.$

Hence, $\vec{n_1} \perp \vec{n_2}$.

27. D.r.'s of the line are 1, –1, 4.

D.r.'s of normal to the plane are 1, 5, 1.

$\therefore \quad (1 \times 1) + (-1) \times 5 + (4 \times 1) = 0.$

So, the given line is perpendicular to the normal of the given plane.

Hence, the given line is parallel to the given plane.

28. Let the required equation of the plane be $\vec{r} \cdot \hat{n} = p$, where $p = 3\sqrt{3}$.

Let $\hat{n} = (\cos\alpha)\hat{i} + (\cos\alpha)\hat{j} + (\cos\alpha)\hat{k}$, where α is acute.

Then, $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1 \Rightarrow 3\cos^2\alpha = 1 \Rightarrow \cos^2\alpha = \frac{1}{3}$

$$\Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}.$$

$\therefore$ the required equation is

$$\vec{r} \cdot \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k} \right) = 3\sqrt{3}.$$

Hence, $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=9 \Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})=9$

$$\Rightarrow x+y+z=9.$$

29. We know that $\hat{n}=\dfrac{\vec{n}}{|\vec{n}|}=(l\hat{i}+m\hat{j}+n\hat{k})$.

Here $l=\cos 45°=\dfrac{1}{\sqrt{2}}$, $m=\cos 60°=\dfrac{1}{2}$ and let $n=\cos\gamma$. Then,

$$l^2+m^2+n^2=1 \Rightarrow \frac{1}{2}+\frac{1}{4}+\cos^2\gamma=1$$

$$\Rightarrow \cos^2\gamma=\frac{1}{4}$$

$$\Rightarrow \cos\gamma=\frac{1}{2}\cdot$$

$$\therefore \quad \vec{n}=|\vec{n}|\,(l\hat{i}+m\hat{j}+n\hat{k}) \Rightarrow \vec{n}=8\left(\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{2}\hat{j}+\frac{1}{2}\hat{k}\right)=(4\sqrt{2}\,\hat{i}+4\hat{j}+4\hat{k}).$$

So, the equation of the plane is

$$\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n} \Rightarrow \vec{r}\cdot(4\sqrt{2}\,\hat{i}+4\hat{j}+4\hat{k})=(\sqrt{2}\,\hat{i}-\hat{j}+\hat{k})\cdot(4\sqrt{2}\,\hat{i}+4\hat{j}+4\hat{k})$$

$$\Rightarrow \vec{r}\cdot(4\sqrt{2}\,\hat{i}+4\hat{j}+4\hat{k})=(8-4+4)=8$$

$$\Rightarrow \vec{r}\cdot(\sqrt{2}\,\hat{i}+\hat{j}+\hat{k})=2.$$

30. Clearly, the required line passes through the point $(2\hat{i}-3\hat{j}-5\hat{k})$ and is parallel to the normal of the given plane, which is $(6\hat{i}-3\hat{j}+5\hat{k})$.

So, the required vector equation is

$$\vec{r}=(2\hat{i}-3\hat{j}-5\hat{k})+\lambda(6\hat{i}-3\hat{j}+5\hat{k}).$$

The Cartesian equation of the line is $\dfrac{x-2}{6}=\dfrac{y+3}{-3}=\dfrac{z+5}{5}=k.$

A general point on this line is $P(6k+2,-3k-3,5k-5)$.

For some particular value of k, let the line cut the plane $6x-3y+5z+2=0$.

Then, $6(6k+2)-3(-3k-3)+5(5k-5)+2=0$

$\Rightarrow \quad (36k+9k+25k)=2 \Rightarrow 70k=2$

$\Rightarrow \quad k=\dfrac{1}{35}\cdot$

$\therefore$ required point of intersection of the line and plane is

$$P\left(\frac{6}{35}+2,\frac{-3}{35}-3,\frac{1}{7}-5\right), \text{ i.e., } P\left(\frac{76}{35},\frac{-108}{35},\frac{-34}{7}\right).$$

DISTANCE OF A POINT FROM A PLANE

1. VECTOR FORM

(i) Let p be the length of perpendicular drawn from a point P with position vector $\vec{a}$ to the plane $\vec{r} \cdot \vec{n} = q$. Then,

$$p = \frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|}.$$

(ii) Let p be the length of perpendicular drawn from the origin to the plane $\vec{r} \cdot \vec{n} = q$. Then,

$$p = \frac{|q|}{|\vec{n}|}.$$

2. CARTESIAN FORM

(i) Let p be the length of perpendicular drawn from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$. Then,

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

(ii) Let p be the length of perpendicular drawn from the origin to the plane $ax + by + cz + d = 0$. Then,

$$p = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}.$$

3. DISTANCE BETWEEN PARALLEL PLANES

Let $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ be two parallel planes.
Let $P(x_1, y_1, z_1)$ be any point on the first plane.
Then, the length of perpendicular from $P(x_1, y_1, z_1)$ to the other plane is the required distance between these planes.

SOLVED EXAMPLES

EXAMPLE 1 *Find the distance of the point* $(\hat{i} + 2\hat{j} - 3\hat{k})$ *from the plane* $\vec{r} \cdot (2\hat{i} - 5\hat{j} - \hat{k}) = 4$.

SOLUTION We know that the perpendicular distance of a point with position vector $\vec{a}$, from the plane $\vec{r} \cdot \vec{n} = q$ is given by $p = \frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|}$.

Here $\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$, $\vec{n} = (2\hat{i} - 5\hat{j} - \hat{k})$ and $q = 4$.

Hence, the required distance is given by

$$p = \frac{|(\hat{i} + 2\hat{j} - 3\hat{k}) \cdot (2\hat{i} - 5\hat{j} - \hat{k}) - 4|}{\sqrt{2^2 + (-5)^2 + (-1)^2}}$$

$$= \frac{|2 - 10 + 3 - 4|}{\sqrt{30}} = \frac{|-9|}{\sqrt{30}} = \frac{9}{\sqrt{30}} \times \frac{\sqrt{30}}{\sqrt{30}}$$

$$= \frac{9}{30}\sqrt{30} = \frac{3}{10}\sqrt{30} \text{ units.}$$

Hence, the required distance is $\frac{3}{10}\sqrt{30}$ units.

EXAMPLE 2 *Find the distance of the point* (2, 3, 4) *from the plane*

$$\vec{r} \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) + 11 = 0.$$

SOLUTION We know that the perpendicular distance of a point with position vector $\vec{a}$, from the plane $\vec{r} \cdot \vec{n} = q$ is given by

$$p = \frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|}.$$

Here $\vec{a} = (2\hat{i} + 3\hat{j} + 4\hat{k})$, $\vec{n} = (3\hat{i} - 6\hat{j} + 2\hat{k})$ and $q = -11$.

∴ the required distance is given by

$$p = \frac{|(2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) - (-11)|}{\sqrt{3^2 + (-6)^2 + 2^2}}$$

$$= \frac{|(2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) + 11|}{\sqrt{9 + 36 + 4}}$$

$$= \frac{|(2 \times 3) + 3 \times (-6) + (4 \times 2) + 11|}{\sqrt{49}}$$

$$= \frac{|6 - 18 + 8 + 11|}{7} = \frac{7}{7} = 1 \text{ unit.}$$

Hence, the required distance is 1 unit.

EXAMPLE 3 *Find the distance of the point* (2, 3, −5) *from the plane* $x + 2y - 2z = 9$.

SOLUTION Required distance

= Length of perpendicular from (2, 3, –5) to the plane

$x + 2y - 2z - 9 = 0$

$$= \frac{|2 + 2 \times 3 - 2 \times (-5) - 9|}{\sqrt{1^2 + 2^2 + (-2)^2}} = \frac{|2 + 6 + 10 - 9|}{\sqrt{9}}$$

$$= \frac{9}{3} = 3 \text{ units.}$$

Hence, the required distance is 3 units.

EXAMPLE 4 *If a plane has intercepts a, b and c on the coordinate axes and is at a distance of p untis from the origin, prove that* $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}\cdot$

SOLUTION We know that the equation of the plane which makes intercepts a, b and c on the axes is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0. \quad \text{... (i)}$$

It is given that p is the length of perpendicular drawn from $O(0, 0, 0)$ to the plane (i).

$$\therefore \quad p=\frac{|0+0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow \quad \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}=\frac{1}{p} \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}\cdot$$

Hence $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}\cdot$

EXAMPLE 5 *Find the distance of the point* $P(6, 5, 9)$ *from the plane determined by the points* $A(3, -1, 2)$, $B(5, 2, 4)$ *and* $C(-1, -1, 6)$. **[CBSE 2009, '10, '12]**

SOLUTION Here

$$(x_1=3, y_1=-1, z_1=2), (x_2=5, y_2=2, z_2=4)$$

and $(x_3=-1, y_3=-1, z_3=6)$.

The equation of the plane ABC is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-3 & y+1 & z-2 \\ 5-3 & 2+1 & 4-2 \\ -1-3 & -1+1 & 6-2 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix}=0$$

$\Rightarrow \quad (x-3)(12-0)-(y+1)(8+8)+(z-2)(0+12)=0$

$\Rightarrow \quad 12(x-3)-16(y+1)+12(z-2)=0$

$\Rightarrow \quad 12x-16y+12z-76=0 \Rightarrow 3x-4y+3z-19=0.$ ... (i)

Distance of the point $P(6, 5, 9)$ from the plane $3x-4y+3z-19=0$

= length of perpendicular from $P(6, 5, 9)$ to the plane

$3x-4y+3z-19=0$

$$=\frac{|(3\times 6)-(4\times 5)+(3\times 9)-19|}{\sqrt{3^2+(-4)^2+3^2}}=\frac{|18-20+27-19|}{\sqrt{34}}$$

$$= \frac{6}{\sqrt{34}} \times \frac{\sqrt{34}}{\sqrt{34}} = \frac{6}{34}\sqrt{34} = \frac{3}{17}\sqrt{34} \text{ units.}$$

Hence, the required distance is $\frac{3}{17}\sqrt{34}$ units.

EXAMPLE 6 *Find the distance between the parallel planes* $2x - y + 3z + 4 = 0$ *and* $6x - 3y + 9z - 3 = 0$.

SOLUTION Let $P(x_1\ y_1, z_1)$ be any point on the plane $2x - y + 3z + 4 = 0$.

Then, $2x_1 - y_1 + 3z_1 = -4$. ... (i)

Length of perpendicular from $P(x_1, y_1, z_1)$ to the plane $6x - 3y + 9z = 0$ is given by

$$p = \frac{|6x_1 - 3y_1 + 9z_1 - 3|}{\sqrt{6^2 + (-3)^2 + 9^2}} = \frac{|3(2x_1 - y_1 + 3z_1) - 3|}{\sqrt{126}}$$

$$= \frac{|3 \times (-4) - 3|}{3\sqrt{14}} \text{ [using (i)]}$$

$$= \frac{|-15|}{3\sqrt{14}} = \frac{15}{3\sqrt{14}} \times \frac{\sqrt{14}}{\sqrt{14}} = \frac{5}{14}\sqrt{14} \text{ units.}$$

Hence, the distance between the given planes is $\frac{5}{14}\sqrt{14}$ units.

EXAMPLE 7 *Find the distance between the parallel planes* $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 5$ *and* $\vec{r} \cdot (6\hat{i} - 9\hat{j} + 18\hat{k}) + 20 = 0$.

SOLUTION Taking $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$, the Cartesian equations of the given planes are

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 5$$

$\Rightarrow \quad 2x - 3y + 6z - 5 = 0$ ... (i)

and $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (6\hat{i} - 9\hat{j} + 18\hat{k}) + 20 = 0$

$\Rightarrow \quad 6x - 9y + 18z + 20 = 0$ ... (ii)

Let $P(x_1, y_1, z_1)$ be any point on (i). Then, $2x_1 - 3y_1 + 6z_1 = 5$ ... (iii)

Distance between the given planes

= length of the perpendicular from $P(x_1, y_1, z_1)$ to the plane (ii)

$$= \frac{|6x_1 - 9y_1 + 18z_1 + 20|}{\sqrt{6^2 + (-9)^2 + (18)^2}} = \frac{|3(2x_1 - 3y_1 + 6z_1) + 20|}{\sqrt{36 + 81 + 324}}$$

$$= \frac{|(3 \times 5) + 20|}{\sqrt{441}} \text{ [using (iii)]}$$

$$= \frac{35}{21} = \frac{5}{3} \text{ units.}$$

Hence, the distance between the given planes is $\frac{5}{3}$ units.

EXAMPLE 8 *Find the equations of planes parallel to the plane* $x - 2y + 2z = 3$ *which are at a unit distance from the point* (1, 2, 3).

SOLUTION The equation of the given plane is $x - 2y + 2z - 3 = 0$.

Let the required plane parallel to the given plane be

$x - 2y + 2z + \lambda = 0$ for some real number λ ... (i)

Now, the plane (i) is at a unit distance from the point (1, 2, 3).

$$\therefore \quad \frac{|1-(2\times 2)+(2\times 3)+\lambda|}{1^2+(-2)^2+2^2} = 1 \Rightarrow |\lambda+3| = 3$$

$$\Rightarrow \lambda + 3 = 3 \text{ or } -(\lambda+3) = 3$$

$$\Rightarrow \lambda = 0 \text{ or } \lambda = -6.$$

Hence, the equations of the required planes are

$x - 2y + 2z = 0$ and $x - 2y + 2z - 6 = 0$.

EXAMPLE 9 *Find the equation of the plane passing through the point* (2, – 3, 5) *and parallel to the plane* $3x - 7y - 2z = 5$. *Also, find the distance between the two planes.*

SOLUTION The given plane is $3x - 7y - 2z - 5 = 0$. ... (i)

Let the required plane parallel to the given plane be

$3x - 7y - 2z + \lambda = 0$ for some real number λ. ... (ii)

Since the plane (ii) passes through the point (2, –3, 5), we have

$$(3\times 2) - 7\times(-3) - (2\times 5) + \lambda = 0 \Rightarrow (6+21-10) + \lambda = 0$$

$$\Rightarrow \lambda = -17.$$

$\therefore$ the equation of the required plane is $3x - 7y - 2z - 17 = 0$. ... (iii)

Now, the planes (i) and (ii) are parallel and the point (2, –3, 5) lies on (iii).

$\therefore$ distance between the two planes

= distance between the point (2, –3, 5) and the plane (i)

$$= \frac{|(3\times 2) - 7\times(-3) - (2\times 5) - 5|}{\sqrt{3^2+(-7)^2+(-2)^2}}$$

$$= \frac{|6+21-10-5|}{\sqrt{9+49+4}}$$

$$= \frac{12}{\sqrt{62}} \times \frac{\sqrt{62}}{\sqrt{62}} = \frac{12}{62}\sqrt{62}$$

$$= \frac{6}{31}\sqrt{62} \text{ units.}$$

Hence, the distance between the two planes is $\frac{6}{31}\sqrt{62}$ units.

EXAMPLE 10 *Find the Cartesian as well as the vector equations of the planes through the intersection of the planes* $\vec{r}\cdot(2\hat{i}+6\hat{j})+12=0$ *and* $\vec{r}\cdot(3\hat{i}-\hat{j}+4\hat{k})=0$ *which are at a unit distance from the origin.* **[CBSE 2005]**

SOLUTION The equations of the given planes are

$$\vec{r}\cdot(2\hat{i}+6\hat{j})+12=0$$

$$\Rightarrow\quad (x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}+6\hat{j})+12=0$$

$$\Rightarrow\quad 2x+6y+12=0 \;\Rightarrow\; x+3y+6=0 \qquad \text{... (i)}$$

and, $\vec{r}\cdot(3\hat{i}-\hat{j}+4\hat{k})=0$

$$\Rightarrow\quad (x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-\hat{j}+4\hat{k})=0$$

$$\Rightarrow\quad 3x-y+4z=0. \qquad \text{... (ii)}$$

Equation of any plane through the intersection of these planes is

$$(x+3y+4)+\lambda(3x-y+4z)=0 \text{ for some real number } \lambda$$

$$\Rightarrow\quad (1+3\lambda)x+(3-\lambda)y+4\lambda z+6=0 \qquad \text{... (iii)}$$

Now, plane (iii) is at a unit distance from $O(0, 0, 0)$.

$$\therefore\quad \frac{|0+0+0+6|}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(4\lambda)^2}}=1$$

$$\Rightarrow\quad (1+3\lambda)^2+(3-\lambda)^2+(4\lambda)^2=36$$

$$\Rightarrow\quad (1+9\lambda^2+6\lambda)+(9+\lambda^2-6\lambda)+16\lambda^2=36$$

$$\Rightarrow\quad 26\lambda^2=26 \;\Rightarrow\; \lambda^2=1 \;\Rightarrow\; \lambda=1 \text{ or } \lambda=-1.$$

Putting $\lambda=1$ in (iii), we get

$$4x+2y+4z+6=0 \;\Rightarrow\; 2x+y+2z+3=0.$$

Putting $\lambda=-1$ in (iii) we get

$$-2x+4y-4z+6=0 \;\Rightarrow\; x-2y+2z-3=0.$$

Hence, the required equations of the planes in Cartesian form are

$$2x+y+2z+3=0 \text{ and } x-2y+2z-3=0.$$

The vector forms of these equations are

$$\vec{r}\cdot(2\hat{i}+\hat{j}+2\hat{k})+3=0 \text{ and } \vec{r}\cdot(\hat{i}-2\hat{j}+2\hat{k})=3.$$

EXAMPLE 11 *If the points* $(1, 1, p)$ *and* $(-3, 0, 1)$ *be equidistant from the plane* $\vec{r}\cdot(3\hat{i}+4\hat{j}-12\hat{k})+13=0$, *find the values of p.*

SOLUTION The equation of the given plane in Cartesian form is

$$x\hat{i}+y\hat{j}+z\hat{k}\cdot(3\hat{i}+4\hat{j}-12\hat{k})+13=0$$

$$\Rightarrow\quad 3x+4y-12z+13=0. \qquad \text{... (i)}$$

It is being given that the distances of the plane (i) from each of the points $A(1, 1, p)$ and $B(-3, 0, 1)$ are equal.

$$\therefore \quad \frac{|(3\times1)+(4\times1)-(12\times p)+13|}{\sqrt{3^2+4^2+(-12)^2}} = \frac{|(3\times(-3))+(4\times0)-(12\times1)+13|}{\sqrt{3^2+4^2+(-12)^2}}$$

$$|20-12p| = |-8| \Rightarrow |20-12p| = 8$$

$$\Rightarrow (20-12p) = 8 \text{ or } -(20-12p) = 8$$

$$\Rightarrow 12p = 2 \text{ or } 12p = 28$$

$$\Rightarrow p = 1 \text{ or } p = \frac{7}{3}.$$

Hence, $p = 1$ or $p = \frac{7}{3}$.

EXAMPLE 12 *Find the equation of the plane mid-parallel to the planes* $2x-2y+z+3=0$ *and* $2x-2y+z+9=0$.

SOLUTION Let the required equation of the plane be $2x-2y+z+k=0$. This plane equidistant from each of the given planes. Let $P(x_1, y_1, z_1)$ be any point on the plane $2x-2y+z+k=0$.

Then, $2x_1-2y_1+z_1+k=0 \Rightarrow 2x_1-2y_1+z_1=-k$. ... (i)

$\therefore$ $P(x_1, y_1, z_1)$ is equidistant from the planes $2x-2y+z+3=0$ and $2x-2y+z+9=0$.

$$\therefore \quad \frac{|2x_1-2y_1+z_1+3|}{\sqrt{2^2+(-2)^2+1^2}} = \frac{|2x_1-2y_1+z_1+9|}{\sqrt{2^2+(-2)^2+1^2}}$$

$$\Rightarrow |(2x_1-2y_1+z_1)+3| = |(2x_1-2y_1+z_1)+9|$$

$$\Rightarrow |-k+3| = |-k+9| \text{ [using (i)]}$$

$$\Rightarrow (-k+3) = (-k+9) \text{ or } -(-k+3) = -k+9$$

$$\Rightarrow k-3 = -k+9 \Rightarrow 2k = 12 \Rightarrow k = 6 \text{ } [\because (-k+3) \neq (-k+9)].$$

Hence, the required equation of the plane is $2x-2y+z+6=0$.

EXAMPLE 13 *A variable plane which remains at a constant distance* $3p$ *from the origin cuts the coordinate axes at A, B, C. Show that the locus of the centroid of* $\triangle ABC$ *is* $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

SOLUTION Let the equation of the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \qquad \text{... (i)}$$

This plane meets the x-axis, y-axis and z-axis at the points $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ respectively. Let (α, β, γ) be the coordinates of the centroid of $\triangle ABC$.

Then, $\alpha = \frac{a+0+0}{3}$, $\beta = \frac{0+b+0}{3}$ and $\gamma = \frac{0+0+c}{3}$

$$\Rightarrow \alpha = \frac{a}{3}, \ \beta = \frac{b}{3} \text{ and } \gamma = \frac{c}{3} \Rightarrow a = 3\alpha, b = 3\beta \text{ and } c = 3\gamma. \qquad \text{... (ii)}$$

$\therefore$ $3p$ = length of the perpendicular from (0, 0, 0) to the plane (i)

$$\Rightarrow \quad 3p = \frac{\left|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1\right|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow \quad \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} = \frac{1}{3p}$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{1}{9p^2} \Rightarrow \frac{1}{9\alpha^2}+\frac{1}{9\beta^2}+\frac{1}{9\gamma^2} = \frac{1}{9p^2} \text{ [using (ii)]}$$

$$\Rightarrow \quad \alpha^{-2}+\beta^{-2}+\gamma^{-2} = p^{-2}.$$

Hence, the required locus is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

EXAMPLE 14 *A variable plane is at a constant distance p from the origin and meets the coordinate axes in A, B, C. Show that the locus of the centroid of the tetrahedron OABC is* $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$.

SOLUTION Let the equation of the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \qquad \ldots\text{(i)}$$

This plane meets the x-axis, y-axis and z-axis at the points $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ respectively.

Let (α, β, γ) be the coordinates of the centroid of the tetrahedron $OABC$.

Then, $\alpha = \frac{0+a+0+0}{4} = \frac{a}{4}$, $\beta = \frac{0+0+b+0}{4} = \frac{b}{4}$ and

$$\gamma = \frac{0+0+0+c}{4} = \frac{c}{4} \Rightarrow a = 4\alpha,\ b = 4\beta,\ c = 4\gamma \qquad \ldots\text{(ii)}$$

$\therefore \quad p$ = distance of the plane (i) from (0, 0, 0)

$$\Rightarrow \quad p = \frac{\left|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1\right|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow \quad \frac{1}{p} = \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} = \frac{1}{p^2} = \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{16\alpha^2}+\frac{1}{16\beta^2}+\frac{1}{16\gamma^2} \text{ [using (ii)]}$$

$$\Rightarrow \quad \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{16}{p^2} \Rightarrow \alpha^{-2}+\beta^{-2}+\gamma^{-2} = 16p^{-2}.$$

Hence, the required locus is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$.

EXERCISE 28C

1. Find the distance of the point $(2\hat{i} - \hat{j} - 4\hat{k})$ from the plane $\vec{r} \cdot (3\hat{i} - 4\hat{j} + 12\hat{k}) = 9$.
2. Find the distance of the point $(\hat{i} + 2\hat{j} + 5\hat{k})$ from the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) + 17 = 0$.
3. Find the distance of the point $(3, 4, 5)$ from the plane $\vec{r} \cdot (2\hat{i} - 5\hat{j} + 3\hat{k}) = 13$.
4. Find the distance of the point $(1, 1, 2)$ from the plane $\vec{r} \cdot (2\hat{i} - 2\hat{j} + 4\hat{k}) + 5 = 0$.
5. Find the distance of the point (2, 1, 0) from the plane $2x + y + 2z + 5 = 0$.
6. Find the distance of the point (2, 1, –1) from the plane $x - 2y + 4z = 9$.
7. Show that the point $(1, 2, 1)$ is equidistant from the planes $\vec{r} \cdot (\hat{i} + 2\hat{j} - 2\hat{k}) = 5$ and $\vec{r} \cdot (2\hat{i} - 2\hat{j} + \hat{k}) + 3 = 0$.
8. Show that the points (–3, 0, 1) and (1, 1, 1) are equidistant from the plane $3x + 4y - 12z + 13 = 0$.
9. Find the distance between the parallel planes $2x + 3y + 4z = 4$ and $4x + 6y + 8z = 12$.
10. Find the distance between the parallel planes $x + 2y - 2z + 4 = 0$ and $x + 2y - 2z - 8 = 0$.
11. Find the equations of the planes parallel to the plane $x - 2y + 2z - 3 = 0$, each one of which is at a unit distance from the point (1, 1, 1).
12. Find the equation of the plane parallel to the plane $2x - 3y + 5z + 7 = 0$ and passing through the point (3, 4, –1). Also, find the distance between the two planes.
13. Find the equation of the plane mid-parallel to the planes $2x - 3y + 6z + 21 = 0$ and $2x - 3y + 6z - 14 = 0$.

ANSWERS (EXERCISE 28C)

1. $\frac{47}{13}$ units **2.** $\frac{25\sqrt{3}}{3}$ units **3.** $\frac{6}{19}\sqrt{38}$ units **4.** $\frac{13}{12}\sqrt{6}$ units

5. $\frac{10}{3}$ units **6.** $\frac{13}{21}\sqrt{21}$ units **9.** $\frac{2}{29}\sqrt{29}$ units **10.** $\frac{5}{14}\sqrt{14}$ units

11. $x - 2y + 2z + 2 = 0$, $x - 2y + 2z - 4 = 0$

12. $2x - 3y + 5z + 11 = 0$, $\frac{2}{19}\sqrt{38}$ units **13.** $4x - 6y + 12z + 7 = 0$

Equation of a Plane Parallel to a Given Plane

VECTOR FORM

Any plane parallel to $\vec{r} \cdot \vec{n} = q_1$ is given by $\vec{r} \cdot \vec{n} = q_2$, where the constant q_2 is determined by a given condition.

CARTESIAN FORM

Let $ax + by + cz + d = 0$ be a given plane.

Then, any plane parallel to this plane is of the form $ax + by + cz + k = 0$, where k is determined by a given condition.

DISTANCE BETWEEN TWO PARALLEL PLANES

Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_1x + b_1y + c_1z + d_2 = 0$ be any two parallel planes. Then, we take a point $P(x_1, y_1, z_1)$ on any of these planes and find the length of perpendicular drawn from $P(x_1, y_1, z_1)$ to the other plane.

SOLVED EXAMPLES

EXAMPLE 1 *Find the vector equation of a plane which is parallel to the plane* $\vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 5$ *and passes through the point whose position vector is* $(\hat{i} + \hat{j} + \hat{k})$.

SOLUTION Any plane parallel to the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 5$ is given by

$$\vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = q \text{ for some constant } q. \quad \text{... (i)}$$

Since it passes through a point having position vector $(\hat{i} + \hat{j} + \hat{k})$, we have

$$(\hat{i} + \hat{j} + \hat{k}) \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = q \Rightarrow q = (2 - 1 + 2) = 3.$$

Putting $q = 3$ in (i), we get the required equation of the plane as

$$\vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 3.$$

EXAMPLE 2 *Find the vector equation of a plane which is parallel to the plane* $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$ *and passes through the point* $(3, 4, -1)$.

SOLUTION Any plane parallel to the plane $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$ is given by $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 5\hat{k}) = q$ for some constant q. ... (i)

Since it passes through a point $A(3, 4, -1)$ whose position vector is $(3\hat{i} + 4\hat{j} - \hat{k})$, we have

$$(3\hat{i} + 4\hat{j} - \hat{k})\cdot(2\hat{i} - 3\hat{j} + 5\hat{k}) = q \Rightarrow q = (6 - 12 - 5) = -11.$$

Putting $q = -11$ in (i), we get the required equation of the plane as

$$\vec{r}\cdot(2\hat{i} - 3\hat{j} + 5\hat{k}) = -11 \Rightarrow \vec{r}\cdot(2\hat{i} - 3\hat{j} + 5\hat{k}) + 11 = 0.$$

EXAMPLE 3 *Find the equation of the plane which is parallel to the plane* $2x - 3y + z + 8 = 0$ *and which passes through the point* $(-1, 1, 2)$.

SOLUTION Any plane parallel to the given plane is

$2x - 3y + z + k = 0$ for some constant k. ... (i)

If it passes through the point $A(-1, 1, 2)$, then

$$2\times(-1) - 3\times 1 + 2 + k = 0$$

$$\Rightarrow (-2 - 3 + 2) + k = 0 \Rightarrow -3 + k = 0 \Rightarrow k = 3.$$

Hence, the required equation of the plane is $2x - 3y + z + 3 = 0$.

EXAMPLE 4 *Find the distance between the planes* $2x - y + 3z + 4 = 0$ *and* $6x - 3y + 9z - 3 = 0$.

SOLUTION Clearly, we have $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

$\therefore$ the given planes are parallel.

Let $P(x_1, y_1, z_1)$ be any point on the plane $2x - y + 3z + 4 = 0$.

Then, $2x_1 - y_1 + 3z_1 + 4 = 0 \Rightarrow 2x_1 - y_1 + 3z_1 = -4$. ... (i)

Let p be the distance of the point $P(x_1, y_1, z_1)$ from the plane $6x - 3y + 9z - 3 = 0$. Then,

$$p = \frac{|6x_1 - 3y_1 + 9z_1 - 3|}{\sqrt{6^2 + (-3)^2 + 9^2}} = \frac{|3(2x_1 - y_1 + 3z_1) - 3|}{\sqrt{36 + 9 + 81}}$$

$$= \frac{|3\times(-4) - 3|}{\sqrt{126}} \text{ [using (i)]}$$

$$= \frac{|-15|}{3\sqrt{14}} = \frac{15}{3\sqrt{14}} = \frac{5}{\sqrt{14}} \text{ units.}$$

Hence, the distance between the given planes is $\frac{5}{\sqrt{14}}$ units.

EXERCISE 28D

1. Show that the planes $2x - y + 6z = 5$ and $5x - 2.5y + 15z = 12$ are parallel.
2. Find the vector equation of the plane through the point $(3\hat{i} + 4\hat{j} - \hat{k})$ and parallel to the plane $\vec{r}\cdot(2\hat{i} - 3\hat{j} + 5\hat{k}) + 5 = 0$.

3. Find the vector equation of the plane passing through the point (a, b, c) and parallel to the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$.

4. Find the vector equation of the plane passing through the point (1, 1, 1) and parallel to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=5$.

5. Find the equation of the plane passing through the point (1, 4, –2) and parallel to the plane $2x-y+3z+7=0$.

6. Find the equation of the plane passing through the origin and parallel to the plane $5x-3y+7z+13=0$.

7. Find the equation of the plane passing through the point (–1, 0, 7) and parallel to the plane $3x-5y+4z=11$.

8. Find the equations of planes parallel to the plane $x-2y+2z=3$ which are at a unit distance from the point (1, 2, 3).

9. Find the distance between the planes $x+2y+3z+7=0$ and $2x+4y+6z+7=0$.

ANSWERS (EXERCISE 28D)

2. $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+11=0$ **3.** $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=a+b+c$

4. $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=3$ **5.** $2x-y+3z+8=0$

6. $5x-3y+7z=0$ **7.** $3x-5y+4z=25$

8. $x-2y+2z=0$ or $x-2y+2z-6=0$ **9.** $\frac{7}{\sqrt{56}}$ units

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28D)

1. The given planes are in the form $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$, where $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$. Hence, the given planess are parallel.

8. Let the required plane be $x-2y+2z+k=0$ for some constants k.

Then, its distance from the point $P(1, 2, 3)$ is

$$\frac{|1-2\times 2+2\times 3+k|}{\sqrt{1^2+(-2)^2+2^2}}=\frac{|3+k|}{3}=1 \Rightarrow |3+k|=3$$

$\Rightarrow \quad 3+k=3$ or $3+k=-3 \Rightarrow k=0$ or $k=-6$.

Hence, the required equations are $x-2y+2z=0$ or $x-2y+2z-6=0$.

9. Let $P(x_1, y_1\ z_1)$ be any point on the plane $x+2y+3z+7=0$.

Then, $x_1+2y_1+3z_1=-7$. ... (i)

$$\therefore \quad p=\frac{|2x_1+4y_1+6z_1+7|}{\sqrt{2^2+4^2+6^2}}=\frac{|2(x_1+2y_1+3z_1)+7|}{\sqrt{56}}$$

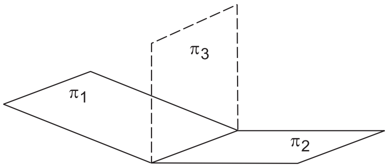
$$=\frac{|2\times(-7)+7|}{\sqrt{56}}=\frac{7}{\sqrt{56}} \text{ units.}$$

PLANE PASSING THROUGH THE INTERSECTION OF TWO PLANES

THEOREM 1 *The equation of a plane passing through the intersection of two planes* $\vec{r}\cdot\vec{n_1} = q_1$ *and* $\vec{r}\cdot\vec{n_2} = q_2$ *is given by* $\vec{r}\cdot(\vec{n_1} + \lambda\vec{n_2}) = (q_1 + \lambda q_2)$.

PROOF Let π_1 and π_2 be the two given planes with equations $\vec{r}\cdot\vec{n_1} = q_1$ and $\vec{r}\cdot\vec{n_2} = q_2$ respectivley.

Let $\vec{t}$ be the position vector of any point on their line of intersection.



Then, it must satisfy both the equations.

$\therefore \quad \vec{t}\cdot\vec{n_1} = q_1$ and $\vec{t}\cdot\vec{n_2} = q_2$

$\Rightarrow \quad \vec{t}\cdot(\vec{n_1} + \lambda\vec{n_2}) = (q_1 + \lambda q_2)$ for every real value λ.

Since $\vec{t}$ is arbitrary, the required equation of a plane π_3 passing through the intersection of the given planes, is given by

$$\vec{r}\cdot(\vec{n_1} + \lambda\vec{n_2}) = (q_1 + \lambda q_2).$$

THEOREM 2 *The equation of a plane passing through the intersection of two planes* $a_1x + b_1y + c_1z = d_1$ *and* $a_2x + b_2y + c_2z = d_2$ *is given by*

$$(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_z) = 0.$$

PROOF Taking $\vec{n_1} = (a_1\hat{i} + b_1\hat{j} + c_1\hat{k})$, $\vec{n_2} = (a_2\hat{i} + b_2\hat{j} + c_3\hat{k})$, $q_1 = d_1$, $q_2 = d_2$ and $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$, we may write the above vector equation as

$$(x\hat{i} + y\hat{j} + z\hat{k})\cdot[(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) + \lambda(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}) = (d_1 + \lambda d_2)]$$

$\Rightarrow \quad (a_1x + b_1y + c_1z = d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0,$

which is the required Cartesian form of the equation of the plane passing through the intersection of the given planes for each value of λ.

SUMMARY

1. The vector equation of a plane passing through the intersection of two planes $\vec{r}\cdot\vec{n_1} = q_1$ and $\vec{r}\cdot\vec{n_2} = q_2$ is given by
$$\vec{r}\cdot(\vec{n_1} + \lambda\vec{n_2}) = (q_1 + \lambda q_2).$$
2. The equation of a plane passing through the intersection of two planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is given by $(a_1x + b_1y + c_1z - d_1 + \lambda(a_2x + b_2y + c_2z - d_2) = 0$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the equation of the plane through the intersection of the planes* $3x - y + 2z = 4$ *and* $x + y + z = 2$ *and passing through the point* $(2, 2, 1)$.

SOLUTION Let the required equation of the plane be

$$(3x - y + 2z - 4) + \lambda(x + y + z - 2) = 0 \text{ for some real value of } \lambda$$

$$\Rightarrow \quad (3 + \lambda)x + (-1 + \lambda)y + (2 + \lambda)z - (4 + 2\lambda) = 0. \qquad \text{... (i)}$$

It is given that the point $A(2, 2, 1)$ lies on (i).

$$\therefore \quad (3 + \lambda) \times 2 + (-1 + \lambda) \times 2 + (2 + \lambda) \times 1 - (4 + 2\lambda) = 0$$

$$\Rightarrow \quad (6 + 2\lambda) + (-2 + 2\lambda) + (2 + \lambda) - 4 - 2\lambda = 0$$

$$\Rightarrow \quad 3\lambda = -2 \Rightarrow \lambda = \frac{-2}{3}.$$

Putting $\lambda = \frac{-2}{3}$ in (i), we get

$$\left(3 - \frac{2}{3}\right)x + \left(-1 - \frac{2}{3}\right)y + \left(2 - \frac{2}{3}\right)z - 4 + \frac{4}{3} = 0$$

$$\Rightarrow \quad \frac{7x}{3} - \frac{5y}{3} + \frac{4z}{3} - \frac{8}{3} = 0 \Rightarrow 7x - 5y + 4z - 8 = 0.$$

Hence, the required equation of the plane is $7x - 5y + 4z - 8 = 0$.

EXAMPLE 2 *Find the equation of the plane through the line of intersection of the planes* $x + y + z = 1$ *and* $2x + 3y + 4z = 5$, *which is perpendicular to the plane* $x - y + z = 0$. *Also find the distance of the plane so obtained from the origin.* **[CBSE 2014]**

SOLUTION Let the required equation of the plane be

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$$

for some real value of λ

$$\Rightarrow \quad (1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z - (1 + 5\lambda) = 0. \qquad \text{... (i)}$$

Since this plane is perpendicular to the plane $x - y + z = 0$, we have

$$(1 + 2\lambda) \times 1 + (1 + 3\lambda) \times (-1) + (1 + 4\lambda) \times 1 = 0$$

$$\Rightarrow \quad (1 + 2\lambda) - (1 + 3\lambda) + (1 + 4\lambda) = 0 \Rightarrow 3\lambda = -1 \Rightarrow \lambda = \frac{-1}{3}.$$

Putting $\lambda = \frac{-1}{3}$ in (i), we get

$$\left(1 - \frac{2}{3}\right)x + (1 - 1)y + \left(1 - \frac{4}{3}\right)z - 1 + \frac{5}{3} = 0$$

$$\Rightarrow \quad \frac{1}{3}x - \frac{1}{3}z + \frac{2}{3} = 0 \Rightarrow x - z + 2 = 0.$$

Hence, the required equation of the plane is $x - z + 2 = 0$.

Distance of the plane from the origin

= Length of perpendicular from the origin to the plane

$$= \frac{|0 - 0 + 2|}{\sqrt{1^2 + 0^2 + (-1)^2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

EXAMPLE 3 *Find the equation of the plane passing through the line of intersection of the planes* $2x + y - z = 3$ *and* $5x - 3y + 4z + 9 = 0$ *and parallel to the line* $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$. **[CBSE 2011]**

SOLUTION Let the required equation of the plane be

$$(2x + y - z - 3) + \lambda(5x - 3y + 4z + 9) = 0$$

$$\Rightarrow \quad (2+5\lambda)x + (1-3\lambda)y + (-1+4\lambda)z - 3 + 9\lambda = 0. \qquad \text{... (i)}$$

The plane given by (i) is parallel to the line

$$\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}. \qquad \text{... (ii)}$$

So, the normal to the plane (i) is perpendicular to the line (ii).

$$\therefore \quad 2(2+5\lambda) + 4(1-3\lambda) + 5(-1+4\lambda) = 0$$

$$\Rightarrow \quad (4+10\lambda) + (4-12\lambda) + (-5+20\lambda) = 0$$

$$\Rightarrow \quad 18\lambda = -3 \Rightarrow \lambda = \frac{-3}{18} \Rightarrow \lambda = \frac{-1}{6}.$$

Putting $\lambda = \frac{-1}{6}$ in (i), we get

$$\left(2 - \frac{5}{6}\right)x + \left(1 + \frac{3}{6}\right)y + \left(-1 - \frac{4}{6}z\right) - 9 \times \frac{1}{6} = 0$$

$$\Rightarrow \quad \frac{7x}{6} + \frac{9y}{6} - \frac{10z}{6} - \frac{27}{6} = 0$$

$$\Rightarrow \quad 7x + 9y - 10z - 27 = 0.$$

Hence, the required equation of the plane is $7x + 9y - 10z - 27 = 0$.

EXAMPLE 4 *Find the equation of the plane passing through the intersection of the planes* $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ *and* $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ *and parallel to the x-axis.* **[CBSE 2011]**

SOLUTION Converting the given equations of the planes in Cartesian form, we get

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1 \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$$

$$\Rightarrow x + y + z - 1 = 0, \qquad \text{... (i)}$$

$$\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0 \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$$

$$\Rightarrow 2x + 3y - z + 4 = 0. \qquad \text{... (ii)}$$

Now, the equation of a plane passing through the intersection of the planes (i) and (ii) is given by

$$(x + y + z - 1) + \lambda(2x + 3y - z + 4) = 0 \text{ for some real value of } \lambda$$

$$\Rightarrow \quad (1+2\lambda)x + (1+3\lambda)y + (1-\lambda)z + (-1+4\lambda) = 0. \qquad \text{... (iii)}$$

D.r.'s of the normal to the plane are $(1+2\lambda), (1+3\lambda), (1-\lambda)$.

D.r.'s of the x-axis are 1, 0, 0.

Plane (iii) is parallel to the x-axis means that the normal to this plane is perpendicular to the x-axis.

$$\therefore \quad 1\times(1+2\lambda)+0\times(1+3\lambda)+0\times(1-\lambda)=0 \Rightarrow 1+2\lambda=0 \Rightarrow \lambda=\frac{-1}{2}\cdot$$

Putting $\lambda=\frac{-1}{2}$ in (iii), we get the required equation of the plane as

$$(1-1)x+\left(1-\frac{3}{2}\right)y+\left(1+\frac{1}{2}\right)z+(-1-2)=0$$

$$\Rightarrow \quad \frac{-y}{2}+\frac{3z}{2}-3=0 \Rightarrow y-3z+6=0.$$

Hence, the required equation of the plane is $y-3z+6=0$.

EXAMPLE 5 *Find the equation of the plane passing through the line of intersection of the planes* $\vec{r}\cdot(\hat{i}+3\hat{j})-6=0$ *and* $\vec{r}\cdot(3\hat{i}-\hat{j}-4\hat{k})=0$, *whose perpendicular distance from the origin is unity.* **[CBSE 2013]**

SOLUTION Taking $\vec{r}=(x\hat{i}+y\hat{j}+z\hat{k})$, the equations of the given planes are

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+3\hat{j})-6=0 \Rightarrow x+3y-6=0, \quad \text{... (i)}$$

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-\hat{j}-4\hat{k})=0 \Rightarrow 3x-y-4z=0. \quad \text{... (ii)}$$

The equation of any plane passing through the line of intersection of the given planes is given by

$(x+3y-6)+\lambda(3x-y-4z)$ for some real number λ.

$$\Rightarrow \quad (1+3\lambda)x+(3-\lambda)y-4\lambda z-6=0. \quad \text{... (iii)}$$

Length of perpendicular from origin to plane (iii) is given as 1.

$$\therefore \quad \frac{|0+0-0-6|}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}=1$$

$$\Rightarrow \quad (1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2=36$$

$$\Rightarrow \quad 26\lambda^2=26 \Rightarrow \lambda^2=1 \Rightarrow \lambda=\pm 1.$$

Putting $\lambda=1$ in (iii), we get

$$4x+2y-4z-6=0 \Rightarrow 2x+y-2z-3=0.$$

Putting $\lambda=-1$ in (iii), we get

$$-2x+4y+4z-6=0 \Rightarrow x-2y-2z+3=0.$$

Hence, the required equations of the plane are

$2x+y-2z-3=0$ and $x-2y-2z+3=0$.

Note: The vector equations of these planes are

$$\vec{r}\cdot(2\hat{i}+\hat{j}-2\hat{k})=3 \text{ and } \vec{r}\cdot(\hat{i}-2\hat{j}-2\hat{k})+3=0.$$

EXAMPLE 6 *Find the vector equation of the plane passing through the intersection of the planes* $\vec{r}\cdot(2\hat{i}+2\hat{j}-3\hat{k})=7$ *and* $\vec{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})=9$ *and passing through the point* (2, 1, 3).

SOLUTION The equations of the given planes are $\vec{r}\cdot(2\hat{i}+2\hat{j}-3\hat{k})=7$ and $\vec{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})=9$.

These are of the form $\vec{r}\cdot\vec{n_1}=q_1$ and $\vec{r}\cdot\vec{n_2}=q_2$, where

$$\vec{n_1}=(2\hat{i}+2\hat{j}-3\hat{k}), \vec{n_2}=(2\hat{i}+5\hat{j}+3\hat{k}), q_1=7 \text{ and } q_2=9.$$

The equation of the plane through the intersection of the given planes is given by

$$\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2})=(q_1+\lambda q_2) \text{ for some real number } \lambda$$

$$\Rightarrow \quad \vec{r}\cdot[(2\hat{i}+2\hat{j}-3\hat{k})+\lambda(2\hat{i}+5\hat{j}+3\hat{k})](7+9\lambda)$$

$$\Rightarrow \quad \vec{r}\cdot[(2+2\lambda)\hat{i}\ (2+5\lambda)\hat{j}+(-3+3\lambda)\hat{k}]=(7+9\lambda). \qquad \text{... (i)}$$

If the plane (i) passes through the point $A(2, 1, 3)$, then

$$\vec{r}=(2\hat{i}+\hat{j}+3\hat{k}) \text{ must satisfy it.}$$

$$\therefore \quad (2\hat{i}+\hat{j}+3\hat{k})\cdot[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(-3+3\lambda)\hat{k}]=(7+9\lambda)$$

$$\Rightarrow \quad 2(2+2\lambda)+1\cdot(2+5\lambda)+3(-3+3\lambda)=7+9\lambda$$

$$\Rightarrow \quad (4+4\lambda)+(2+5\lambda)+(-9+9\lambda)=7+9\lambda \Rightarrow 9\lambda=10 \Rightarrow \lambda=\frac{10}{9}\cdot$$

Putting $\lambda=\frac{10}{9}$ in (i), we get the required equation of the plane as

$$\vec{r}\cdot\left[\left(2+\frac{20}{9}\right)\hat{i}+\left(2+\frac{50}{9}\right)\hat{j}+\left(-3+\frac{30}{9}\right)\hat{k}\right]=(7+10)$$

$$\Rightarrow \quad \vec{r}\cdot(38\hat{i}+68\hat{j}+3\hat{k})=153.$$

Hence, the required equation of the plane is

$$\vec{r}\cdot(38\hat{i}+68\hat{j}+3\hat{k})=153.$$

EXAMPLE 7 *Find the vector equation of the plane through the line of intersection of the planes* $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0$ *and* $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$ *and perpendicular to the plane* $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$. **[CBSE 2011]**

SOLUTION The equations of the given planes are

$$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=4 \text{ and } \vec{r}\cdot(-2\hat{i}-\hat{j}+\hat{k})=5.$$

These equations are of the form $\vec{r}\cdot\vec{n_1}=q_1$ and $\vec{r}\cdot\vec{n_2}=q_2$, where

$$\vec{n_1}=(\hat{i}+2\hat{j}+3\hat{k}),\ \vec{n_2}=(-2\hat{i}-\hat{j}+\hat{k}),\ q_1=4 \text{ and } q_2=5.$$

The equation of the plane through the intersection of the given planes is given by

$$\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2})=(q_1+\lambda q_2) \text{ for some real number } \lambda$$

$$\Rightarrow \quad \vec{r}\cdot[(\hat{i}+2\hat{j}+3\hat{k})+\lambda(-2\hat{i}-\hat{j}+\hat{k})]=(4+5\lambda)$$

$$\Rightarrow \quad \vec{r}\cdot[(1-2\lambda)\hat{i}+(2-\lambda)\hat{j}+(3+\lambda)\hat{k}]=(4+5\lambda) \qquad \text{... (i)}$$

This plane (i) is perpendicular to the plane $\vec{r}\cdot(-5\hat{i}-3\hat{j}+6\hat{k})=8$, so we have

$$(1-2\lambda)\times(-5)+(2-\lambda)\times(-3)+(3+\lambda)\times 6=0$$

$$\Rightarrow \quad (-5+10\lambda)+(-6+3\lambda)+(18+6\lambda)=0$$

$$\Rightarrow \quad 19\lambda=-7 \ \Rightarrow \ \lambda=\frac{-7}{19}\cdot$$

Putting $\lambda=\frac{-7}{19}$ in (i), we get

$$\vec{r}\cdot\left[\left(1+\frac{14}{19}\right)\hat{i}+\left(2+\frac{7}{19}\right)\hat{j}+\left(3-\frac{7}{19}\right)\hat{k}\right]=\left(4-\frac{35}{19}\right)$$

$$\Rightarrow \quad \vec{r}\cdot(33\hat{i}+45\hat{j}+50\hat{k})=41.$$

Hence, the required equation of the plane is

$$\vec{r}\cdot(33\hat{i}+45\hat{j}+50\hat{k})=41.$$

EXERCISE 28E

1. Find the equation of the plane through the line of intersection of the planes $x+y+z=6$ and $2x+3y+4z+5=0$, and passing through the point (1, 1, 1).
2. Find the equation of the plane through the line of intersection of the planes $x-3y+z+6=0$ and $x+2y+3z+5=0$, and passing through the origin.
3. Find the equation of the plane passing through the intersection of the planes $2x+3y-z+1=0$ and $x+y-2z+3=0$, and perpendicular to the plane $3x-y-2z-4=0$.
4. Find the equation of the plane passing through the line of intersection of the planes $2x-y=0$ and $3z-y=0$, and perpendicular to the plane $4x+5y-3z=9$.

5. Find the equation of the plane passing through the intersection of the planes $x-2y+z=1$ and $2x+y+z=8$, and parallel to the line with direction ratios 1, 2, 1. Also, find the perpendicular distance of (1, 1, 1) from the plane. **[CBSE 2005]**

6. Find the equation of the plane passing through the line of intersection of the planes $x+2y+3z-5=0$ and $3x-2y-z+1=0$ and cutting off equal intercepts on the x-axis and z-axis.

7. Find the equation of the plane through the intersection of the planes $3x-4y+5z=10$ and $2x+2y-3z=4$ and parallel to the line $x=2y=3z$.

HINT: The equations of the given line are $\frac{x}{6}=\frac{y}{3}=\frac{z}{2}$.

8. Find the vector equation of the plane through the intersection of the planes $\vec{r}\cdot(\hat{i}+3\hat{j}-\hat{k})=0$ and $\vec{r}\cdot(\hat{j}+2\hat{k})=0$, and passing through the point (2, 1, –1).

9. Find the vector equation of the plane through the point (1, 1, 1), and passing through the intersection of the planes

$$\vec{r}\cdot(\hat{i}-\hat{j}+3\hat{k})+1=0 \text{ and } \vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})-5=0.$$

10. Find the vector equation of the plane passing through the intersection of the planes

$$\vec{r}\cdot(2\hat{i}-7\hat{j}+4\hat{k})=3 \text{ and } \vec{r}\cdot(3\hat{i}-5\hat{j}+4\hat{k})+11=0,$$

and passing through the point (–2, 1, 3). **[CBSE 2006]**

11. Find the equation of the plane through the line of intersection of the planes $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1$ and $\vec{r}\cdot(\hat{i}-\hat{j})+4=0$ and perpendicular to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+8=0$.

12. Find the Cartesian and vector equations of the planes through the line of intersection of the planes $\vec{r}\cdot(\hat{i}-\hat{j})+6=0$ and $\vec{r}\cdot(3\hat{i}+3\hat{j}-4\hat{k})=0$, which are at a unit distance from the origin.

ANSWERS (EXERCISE 28E)

1. $20x+23y+26z=69$

2. $x+27y+13z=0$

3. $7x+13y+4z=9$

4. $28x-17y+9z=0$

5. $9x-8y+7z=21$, $\frac{13}{\sqrt{194}}$ units

6. $5x+2y+5z-9=0$

7. $x-20y+27z=14$

8. $\vec{r}\cdot(\hat{i}+9\hat{j}+11\hat{k})=0$

9. $\vec{r}\cdot(11\hat{i}+\hat{j}+5\hat{k})-17=0$

10. $\vec{r}\cdot(15\hat{i}-47\hat{j}+28\hat{k})=7$

11. $\vec{r}\cdot(-5\hat{i}+2\hat{j}+12\hat{k})=47$

12. $2x + y - 2z + 3 = 0,\ x + 2y - 2z - 3 = 0$, and

$\vec{r} \cdot (2\hat{i} + \hat{j} - 2\hat{k}) + 3 = 0,\ \vec{r} \cdot (\hat{i} + 2\hat{j} - 2\hat{k}) - 3 = 0$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28E)

5. Let the required plane be

$$(x - 2y + z - 1) + \lambda(2x + y + z - 8) = 0$$

$$\Leftrightarrow \quad (1 + 2\lambda)x + (\lambda - 2)y + (1 + \lambda)z - (1 + 8\lambda) = 0. \qquad \ldots \text{(i)}$$

The d.r.'s of the normal to this plane are

$$(1 + 2\lambda), (\lambda - 2), (1 + \lambda).$$

The normal to the plane (i) is perpendicular to the line with d.r.'s 1, 2, 1.

$$\therefore \quad (1 + 2\lambda) + 2(\lambda - 2) + (1 + \lambda) = 0 \Rightarrow \lambda = \frac{2}{5}\cdot$$

6. Let the required plane be

$$(x + 2y + 3z - 5) + \lambda(3x - 2y - z + 1) = 0.$$

Then,

$$(1 + 3\lambda)x + (2 - 2\lambda)y + (3 - \lambda)z - 5 + \lambda = 0.$$

$$(1 + 3\lambda)x + (2 - 2\lambda)y + (3 - \lambda)z = 5 - \lambda$$

$$\Rightarrow \quad \frac{(1 + 3\lambda)}{(5 - \lambda)}x + \frac{(2 - 2\lambda)}{(5 - \lambda)}y + \frac{(3 - \lambda)}{(5 - \lambda)}z = 1$$

$$\Rightarrow \quad \frac{x}{\left(\frac{5 - \lambda}{1 + 3\lambda}\right)} + \frac{y}{\left(\frac{5 - \lambda}{2 - 2\lambda}\right)} + \frac{z}{\left(\frac{5 - \lambda}{3 - \lambda}\right)} = 1.$$

Since the intercepts on the x-axis and z-axis are equal, we have

$$\frac{5 - \lambda}{1 + 3\lambda} = \frac{5 - \lambda}{3 - \lambda} \Rightarrow 3 - \lambda = 1 + 3\lambda$$

$$\Rightarrow 4\lambda = 2 \Rightarrow \lambda = \frac{1}{2}\cdot$$

7. Let the required plane be

$$(3x - 4y + 5z - 10) + \lambda(2x + 2y - 3z - 4) = 0$$

$$\Rightarrow \quad (3 + 2\lambda)x + (2\lambda - 4)y + (5 - 3\lambda)z - 10 - 4\lambda = 0. \qquad \ldots \text{(i)}$$

D.r's of normal to the plane are $(3 + 2\lambda), (2\lambda - 4), (5 - 3\lambda)$.

Given line is $\frac{x}{6} = \frac{y}{3} = \frac{z}{2}\cdot$

D.r.'s of the line are 6, 3, 2.

$\therefore$ This line is perpendicular to the plane.

$$\therefore \quad 6(3 + 2\lambda) + 3(2\lambda - 4) + 2(5 - 3\lambda) = 0 \Rightarrow 12\lambda = -16 \Rightarrow \lambda = \frac{-4}{3}\cdot$$

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12. The equations of the given planes are

$$(x\hat{i} + y\hat{j} + z\hat{k})\cdot(\hat{i} - \hat{j}) + 6 = 0 \text{ and } (x\hat{i} + y\hat{j} + z\hat{k})\cdot(3\hat{i} + 3\hat{j} - 4\hat{k}) = 0$$

$\Rightarrow$ $x - y + 6 = 0$ and $3x + 3y - 4z = 0$.

Any plane through their intersection is

$$(x - y + 6) + \lambda(3x + 3y - 4z) = 0 \Rightarrow (1 + 3\lambda)x + (3\lambda - 1)y - 4\lambda z + 6 = 0. \quad \ldots \text{(i)}$$

$$\therefore \frac{6}{\sqrt{(1+3\lambda)^2 + (3\lambda - 1)^2 + (-4\lambda)^2}} = 1 \Rightarrow 34\lambda^2 + 2 = 36 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

So, the required planes are

$2x + y - 2z + 3 = 0$ and $x + 2y - 2z - 3 = 0$.

In vector form, they are

$$\vec{r}\cdot(2\hat{i} + \hat{j} - 2\hat{k}) + 3 = 0 \text{ and } \vec{r}\cdot(\hat{i} + 2\hat{j} - 2\hat{k}) - 3 = 0.$$

Equation of a Plane Passing through Three Noncollinear Points

Vector Form

THEOREM 1 *The vector equation of a plane passing through three noncollinear points with position vectors* $\vec{a}, \vec{b}, \vec{c}$ *is*

$$(\vec{r} - \vec{a})\cdot[(\vec{b} - \vec{a})\times(\vec{c} - \vec{a})] = 0.$$

PROOF Let A, B, C be three given noncollinear points having position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively.

Let P be an arbitrary point on the plane passing through the points A, B, C, and let $\vec{r}$ be the position vector of P. Then,

$$\overrightarrow{AP} = (\text{p.v. of } P) - (\text{p.v. of } A) = (\vec{r} - \vec{a}),$$

$$\overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A) = (\vec{b} - \vec{a}),$$

$$\overrightarrow{AC} = (\text{p.v. of } C) - (\text{p.v. of } A) = (\vec{c} - \vec{a}).$$

Since the points A, B, C, P lie on the plane, the vectors $\overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}$ are coplanar.

So, the scalar triple product of these vectors is 0.

$\therefore$ $[\overrightarrow{AP}\ \overrightarrow{AB}\ \overrightarrow{AC}] = 0$

$\Rightarrow$ $\overrightarrow{AP}\cdot(\overrightarrow{AB}\times\overrightarrow{AC}) = 0$

$\Rightarrow$ $(\vec{r} - \vec{a})\cdot[(\vec{b} - \vec{a})\times(\vec{c} - \vec{a})] = 0,$

which is the required equation of the plane.

Cartesian Form

THEOREM 2 *The equation of a plane passing through three given noncollinear points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by*

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

PROOF Let $P(x, y, z)$ be an arbitrary point on the given plane. Then,

$$\overrightarrow{AP} = (\text{p.v. of } P) - (\text{p.v. of } A)$$
$$= (x\hat{i} + y\hat{j} + z\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$
$$= (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}.$$

Similarly, we have

$$\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

and $\overrightarrow{AC} = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}.$

Since $\overrightarrow{AP}$, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are coplanar, we have

$$[\overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}] = 0.$$

Hence, the required equation of the plane passing through three given points A, B, C is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

SUMMARY

(i) The vector equation of a plane passing through three noncollinear points having position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is given by

$$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0.$$

(ii) The Cartesian equation of a plane passing through three noncollinear points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the vector equation of a plane passing through the points $A(2, 5, -3)$, $B(-2, -3, 5)$ and $C(5, 3, -3)$.*

SOLUTION The position vectors of the given points A, B, and C are $\vec{a} = (2\hat{i} + 5\hat{j} - 3\hat{k})$, $\vec{b} = (-2\hat{i} - 3\hat{j} + 5\hat{k})$ and $\vec{c} = (5\hat{i} + 3\hat{j} - 3\hat{k})$ respectively.

Then, the vector equation of the required plane is

$$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0.$$

Now $(\vec{b} - \vec{a}) = (-2-2)\hat{i} + (-3-5)\hat{j} + (5+3)\hat{k} = (-4\hat{i} - 8\hat{j} + 8\hat{k})$.

and $(\vec{c} - \vec{a}) = (5-2)\hat{i} + (3-5)\hat{j} + (-3+3)\hat{k} = (3\hat{i} - 2\hat{j} + 0\hat{k})$.

$$\therefore \quad (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}$$

$$= (0+16)\hat{i} - (0-24)\hat{j} + (8+24)\hat{k}$$

$$= (16\hat{i} + 24\hat{j} + 32\hat{k}).$$

$\therefore$ the required vector equation is

$$(\vec{r} - \vec{a}) \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) = 0$$

$$\Rightarrow \quad \vec{r} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) = \vec{a} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k})$$

$$= (2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (16\hat{i} + 24\hat{j} + 32\hat{k})$$

$$= (32 + 120 - 96) = 56.$$

Hence, the required vector equation of the plane is

$$\vec{r} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) = 56 \Rightarrow \vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 7.$$

EXAMPLE 2 *Find the equation of the plane passing through the points $A(1, 1, 0)$, $B(1, 2, 1)$ and $C(-2, 2, -1)$.*

SOLUTION Here

$$(x_1 = 1, y_1 = 1, z_1 = 0), (x_2 = 1, y_2 = 2, z_2 = 1)$$

and $(x_3 = -2, y_3 = 2, z_3 = -1)$.

Hence, the required equation of the plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z-0 \\ 1-1 & 2-1 & 1-0 \\ -2-1 & 2-1 & -1-0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-1)(-1-1)-(y-1)(0+3)+z(0+3)=0$

$\Rightarrow \quad (-2)(x-1)-3(y-1)+3z=0$

$\Rightarrow \quad -2x-3y+3z+5=0 \Rightarrow 2x+3y-3z-5=0.$

Hence, the required equation of the plane is $2x+3y-3z-5=0$.

ANGLE BETWEEN TWO PLANES

The angle between two given planes is the angle between their normals.

If θ is an angle between two planes, then $(180° - \theta)$ is also an angle between them.

An Important Note: We shall take the acute angle as the angle between two planes.

Vector Form

Let θ be the angle between two planes whose vector equations are

$$\vec{r} \cdot \vec{n_1} = q_1 \text{ and } \vec{r} \cdot \vec{n_2} = q_2.$$

Then, θ is the angle between their normals $\vec{n_1}$ and $\vec{n_2}$.

$$\therefore \quad \cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}|\,|\vec{n_2}|}.$$

Two Important Results

1. *Condition for two planes to be perpendicular to each other*

   Two planes $\vec{r} \cdot \vec{n_1} = q_1$ and $\vec{r} \cdot \vec{n_2} = q_2$ are perpendicular to each other

   $$\Leftrightarrow \vec{n_1} \perp \vec{n_2} \Leftrightarrow \vec{n_1} \cdot \vec{n_2} = 0.$$

2. *Condition for two planes to be parallel to each other*

   Two planes $\vec{r} \cdot \vec{n_1} = q_1$ and $\vec{r} \cdot \vec{n_2} = q_2$ are parallel to each other

$\Leftrightarrow \vec{n_1} \parallel \vec{n_2}$

$\Leftrightarrow \vec{n_1} = \lambda \vec{n_2}$ for some scalar λ.

REMARK 1 Any plane parallel to $\vec{r} \cdot \vec{n} = q$ is $\vec{r} \cdot \vec{n} = q_1$.

REMARK 2 The equation of the plane parallel to the plane $\vec{r} \cdot \vec{n} = q$ and passing through the point with position vector $\vec{a}$ is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.

Cartesian Form

Let θ be the acute angle between the planes:

$$a_1x + b_1y + c_1z + d_1 = 0 \text{ and } a_2x + b_2y + c_2z + d_2 = 0.$$

Then, θ is the angle between their normals.

D.r.'s of the normals to the given planes are

a_1, b_1, c_1 and a_2, b_2, c_2.

$$\therefore \quad \cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\{\sqrt{a_1^2 + b_1^2 + c_1^2}\}\{\sqrt{a_2^2 + b_2^2 + c_2^2}\}}.$$

Two Important Results

1. *Condition for two planes to be perpendicular to each other*
   Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular to each other
   $\Leftrightarrow$ their normals are perpendicular to each other
   $\Leftrightarrow$ lines with direction ratios a_1, b_1, c_1, and a_2, b_2, c_2 are perpendicular to each other
   $\Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$.
2. *Condition for two planes to be parallel to each other*
   Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel to each other
   $\Leftrightarrow$ their normals are parallel to each other
   $\Leftrightarrow$ lines with direction ratios a_1, b_1, c_1, and a_2, b_2, c_2 are parallel to each other
   $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

REMARK 1 The equation of any plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$.

REMARK 2 The equation of a plane passing through (x_1, y_1, z_1) and parallel to the plane $ax + by + cz + d = 0$ is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

REMARK 3 The equation of any plane parallel to the xy-plane is $z = \lambda$.

This plane is perpendicular to the z-axis.

The equation of any plane parallel to the xz-plane is $y = \lambda$.

xz-plane is perpendiuclar to the *y*-axis.

The equation of any plane parallel to the *yz*-plane is $x = \lambda$.

yz-plane is perpendicular to the *x*-axis.

SUMMARY OF THE RESULTS

1. The acute angle θ between the planes $\vec{r} \cdot \vec{n_1} = q_1$ and $\vec{r} \cdot \vec{n_2} = q_2$ is given by

$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}|\,|\vec{n_2}|}.$$

(i) Two planes $\vec{r} \cdot \vec{n_1} = q_1$ and $\vec{r} \cdot \vec{n_2} = q_2$ are perpendicular to each other $\Leftrightarrow \vec{n_1} \cdot \vec{n_2} = 0$.

(ii) Two planes $\vec{r} \cdot \vec{n_1} = q_1$ and $\vec{r} \cdot \vec{n_2} = q_2$ are perpendicular to each other $\Leftrightarrow \vec{n_1} = \lambda \vec{n_2}$ for some scalar λ.

(iii) Any plane parallel to the plane $\vec{r} \cdot \vec{n} = q$ is $\vec{r} \cdot \vec{n} = q_1$.

(iv) Any plane parallel to $\vec{r} \cdot \vec{n} = q$ and passing through a point with p.v. $\vec{a}$ is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.

2. The acute angle θ between the planes

$$a_1x + b_1y + c_1z + d_1 = 0 \text{ and } a_2x + b_2y + c_2z + d_2 = 0$$

is given by

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\{\sqrt{a_1^2 + b_1^2 + c_1^2}\}\,\{\sqrt{a_2^2 + b_2^2 + c_2^2}\}}.$$

(i) Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular $\Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$.

(ii) Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel $\Leftrightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

(iii) The equation of a plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$.

(iv) The equation of a plane passing through the point (x_1, y_1, z_1) and parallel to the plane $ax + by + cz + d = 0$ is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

(v) Any plane parallel to the *yz*-plane is $x = \lambda$.

Any plane parallel to the *xz*-plane is $y = \lambda$.

Any plane parallel to the *xy*-plane is $z = \lambda$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the angle between the planes*

$$\vec{r} \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 5 \text{ and } \vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 8.$$

SOLUTION We know that the angle between the planes $\vec{r} \cdot \vec{n_1} = q_1$ and $\vec{r} \cdot \vec{n_2} = q_2$ is given by

$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}.$$

Here, $\vec{n_1} = (\hat{i} + \hat{j} + 2\hat{k})$ and $\vec{n_2} = (2\hat{i} - \hat{j} + \hat{k})$.

$$\therefore \quad \cos\theta = \frac{|(\hat{i} + \hat{j} + 2\hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k})|}{|\hat{i} + \hat{j} + 2\hat{k}||2\hat{i} - \hat{j} + \hat{k}|}$$

$$= \frac{|1 \times 2 + 1 \times (-1) + 2 \times 1|}{\{\sqrt{1^2 + 1^2 + 2^2}\}\{\sqrt{2^2 + (-1)^2 + 1^2}\}} = \frac{3}{6} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}.$$

Hence, the angle between the given planes is $\left(\frac{\pi}{3}\right)$.

EXAMPLE 2 *Find the angle between the planes whose vector equations are*

$$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5 \text{ and } \vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3.$$

SOLUTION We know that the acute angle θ between the planes $\vec{r} \cdot \vec{n_1} = q_1$ and $\vec{r} \cdot \vec{n_2} = q_2$ is given by

$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}.$$

Here, $\vec{n_1} = (2\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{n_2} = (3\hat{i} - 3\hat{j} + 5\hat{k})$.

$$\therefore \quad \cos\theta = \frac{|2\hat{i} + 2\hat{j} - 3\hat{k}) \cdot (3\hat{i} - 3\hat{j} + 5\hat{k})|}{|2\hat{i} + 2\hat{j} - 3\hat{k}||3\hat{i} - 3\hat{j} + 5\hat{k}|}$$

$$= \frac{|2 \times 3 + 2 \times (-3) + (-3) \times 5|}{\{\sqrt{2^2 + 2^2 + (-3)^2}\}\{\sqrt{3^2 + (-3)^2 + 5^2}\}}$$

$$= \frac{|-15|}{(\sqrt{17})(\sqrt{43})} = \frac{15}{\sqrt{731}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{15}{\sqrt{731}}\right).$$

Hence, the angle between the given planes is $\cos^{-1}\left(\frac{15}{\sqrt{731}}\right)$.

EXAMPLE 3 *Find the value of λ for which the planes*

$$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 13 \text{ and } \vec{r}\cdot(\lambda\hat{i}+2\hat{j}-7\hat{k}) = 9$$

are perpendicular to each other.

SOLUTION We know that the planes $\vec{r}\cdot\vec{n_1} = q_1$ and $\vec{r}\cdot\vec{n_2} = q_2$ are perpendicular to each other only when $\vec{n_1}\cdot\vec{n_2} = 0$.

Here, $\vec{n_1} = (\hat{i} + 2\hat{j} + 3\hat{k})$ and $\vec{n_2} = (\lambda\hat{i} + 2\hat{j} - 7\hat{k})$.

∴ the given planes are perpendicular to each other

$$\Leftrightarrow \quad \vec{n_1}\cdot\vec{n_2} = 0$$

$$\Leftrightarrow \quad (\hat{i}+2\hat{j}+3\hat{k})\cdot(\lambda\hat{i}+2\hat{j}-7\hat{k}) = 0$$

$$\Leftrightarrow \quad 1\times\lambda + 2\times 2 + 3\times(-7) = 0 \Leftrightarrow \lambda = 17.$$

Hence, the required value of λ is 17.

EXAMPLE 4 *Find the angle between the planes whose Cartesian equations are* $7x + 5y + 6z + 30 = 0$ *and* $3x - y - 10z + 4 = 0$.

SOLUTION We know that the acute angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\{\sqrt{a_1^2 + b_1^2 + c_1^2}\}\ \{\sqrt{a_2^2 + b_2^2 + c_2^2}\}}.$$

Here $a_1 = 7$, $b_1 = 5$, $c_1 = 6$ and $a_2 = 3$, $b_2 = -1$. $c_2 = -10$.

$$\therefore \quad \cos\theta = \frac{|\ 7\times 3 + 5\times(-1) + 6(-10)\,|}{\{\sqrt{7^2 + 5^2 = 6^2}\}\ \{\sqrt{3^2 + (-1)^2 + (-10)^2}\}}$$

$$= \frac{|21 - 5 - 60|}{\{\sqrt{49 + 25 + 36}\}\{\sqrt{9 + 1 + 100}\}}$$

$$= \frac{44}{\{\sqrt{110}\times\sqrt{110}\}} = \frac{44}{110} = \frac{2}{5}.$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{5}\right).$$

Hence, the acute angle between the given planes is $\cos^{-1}\left(\frac{2}{5}\right)$.

EXAMPLE 5 *Find the value of* λ *for which the planes* $2x - 4y + 3z = 7$ *and* $x + 2y + \lambda z = 18$ *are perpendicular to each other.*

SOLUTION The equations of the given planes are

$$2x - 4y + 3z - 7 = 0 \text{ and } x + 2y + \lambda z - 18 = 0.$$

They are of the form

$$a_1x + b_1y + c_1z + d_1 = 0 \text{ and } a_2x + b_2y + c_2z + d_2 = 0$$

where $a_1 = 2, b_1 = -4, c_1 = 3, d_1 = -7$, and

$$a_2 = 1, b_2 = 2, c_2 = \lambda, d_2 = -18.$$

The given planes are perpendicular to each other

$\Leftrightarrow \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Leftrightarrow \quad 2 \times 1 + (-4) \times 2 + 3 \times \lambda = 0$

$\Leftrightarrow \quad 3\lambda = 6 \Leftrightarrow \lambda = 2.$

Hence, the required value of λ is 2.

EXAMPLE 6 *Find the equation of the plane passing through the point* $(-1, 3, 2)$ *and perpendicular to each of the planes* $x + 2y + 3z = 5$ *and* $3x + 3y + z = 0$. **[CBSE 2009, '11]**

SOLUTION Any plane passing through the point (–1, 3, 2) is given by

$$a(x+1) + b(y-3) + c(z-2) = 0. \quad \text{... (i)}$$

Now, (i) being perpendicular to each of the planes

$$x + 2y + 3z - 5 = 0 \text{ and } 3x + 3y + z = 0,$$

we have,

$$(a \times 1) + (b \times 2) + (c \times 3) = 0 \Rightarrow a + 2b + 3c = 0 \quad \text{... (ii)}$$

$$(a \times 3) + (b \times 3) + (c \times 1) = 0 \Rightarrow 3a + 3b + c = 0. \quad \text{... (iii)}$$

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(2-9)} = \frac{b}{(9-1)} = \frac{c}{(3-6)}$$

$\Rightarrow \quad \frac{a}{-7} = \frac{b}{8} = \frac{c}{-3} \Rightarrow \frac{a}{7} = \frac{b}{-8} = \frac{c}{3} = \lambda$ (say)

$\Rightarrow \quad a = 7\lambda, b = -8\lambda$ and $c = 3\lambda$.

Putting $a = 7\lambda, b = -8\lambda$ and $c = 3\lambda$ in (i), we get

$$7\lambda(x+1) - 8\lambda(y-3) + 3\lambda(z-2) = 0$$

$\Rightarrow \quad 7(x+1) - 8(y-3) + 3(z-2) = 0$

$\Rightarrow \quad 7x - 8y + 3z + 25 = 0,$

which is the required equation of the plane.

EXAMPLE 7 *Find the vector equation of the plane through the points* $(2, 1, -1)$ *and* $(-1, 3, 4)$ *and perpendicular to the plane* $x - 2y + 4z = 10$. **[CBSE 2013]**

SOLUTION The general equation of a plane passing through the point $A(2, 1, -1)$ is given by

$$a(x-2)+b(y-1)+c(z+1)=0. \quad \text{... (i)}$$

If the point $B(-1, 3, 4)$ lies on plane (i), then we have

$$a(-1-2)+b(3-1)+c(4+1)=0$$

$$\Rightarrow \quad -3a+2b+5c=0. \quad \text{... (ii)}$$

If the plane (i) is perpendicular to the plane $x-2y+4z=10$, then we have

$$(1\times a)-(2\times b)+(4\times c)=0$$

$$\Rightarrow \quad a-2b+4c=0. \quad \text{... (iii)}$$

On solving (ii) and (iii) by cross multiplication, we have

$$\frac{a}{(8+10)}=\frac{b}{(5+12)}=\frac{c}{(6-2)}$$

$$\Rightarrow \quad \frac{a}{18}=\frac{b}{17}=\frac{c}{4}=\lambda \text{ (say)}$$

$$\Rightarrow \quad a=18\lambda,\ b=17\lambda \text{ and } c=4\lambda.$$

Substituting these values of a, b and c in (i), we get

$$18\lambda(x-2)+17\lambda(y-1)+4\lambda(z+1)=0$$

$$\Rightarrow \quad 18(x-2)+17(y-1)+4(z+1)=0$$

$$\Rightarrow \quad 18x+17y+4z=49.$$

Required equation of the plane in vector form is given by

$$\vec{r}\cdot(18\hat{i}+17\hat{j}+4\hat{k})=49.$$

EXAMPLE 8 *Find the equation of the plane passing through the point* $(1, 3, 2)$ *and parallel to the plane* $3x-2y+2z+33=0$.

SOLUTION The equation of a plane parallel to the given plane is of the form

$$3x-2y+2z=k \text{ for some scalar } k. \quad \text{... (i)}$$

Since it passes through the point $A(1, 3, 2)$, we have

$$(3\times 1)-(2\times 3)+(2\times 2)=k \Rightarrow k=(3-6+4)=1.$$

Hence, the required equation of the plane is

$$3x-2y+2z-1=0.$$

EXERCISE 28F

1. Find the acute angle between the following planes:

(i) $\vec{r}\cdot(\hat{i}+\hat{j}-2\hat{k})=5$ and $\vec{r}\cdot(2\hat{i}+2\hat{j}-\hat{k})=9$

(ii) $\vec{r}\cdot(\hat{i}+2\hat{j}-\hat{k})=6$ and $\vec{r}\cdot(2\hat{i}-\hat{j}-\hat{k})+3=0$

(iii) $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1$ and $\vec{r}\cdot(-\hat{i}+\hat{j})=4$

(iv) $\vec{r}\cdot(2\hat{i}-3\hat{j}+6\hat{k})=8$ and $\vec{r}\cdot(3\hat{i}+4\hat{j}-12\hat{k})+7=0$

2. Show that the following planes are at right angles:

   (i) $\vec{r}\cdot(4\hat{i}-7\hat{j}-8\hat{k})=5$ and $\vec{r}\cdot(3\hat{i}-4\hat{j}+5\hat{k})+10=0$

   (ii) $\vec{r}\cdot(2\hat{i}+6\hat{j}+6\hat{k})=13$ and $\vec{r}\cdot(3\hat{i}+4\hat{j}-5\hat{k})+7=0$

3. Find the value of λ for which the given planes are perpendicular to each other:

   (i) $\vec{r}\cdot(2\hat{i}-\hat{j}-\lambda\hat{k})=7$ and $\vec{r}\cdot(3\hat{i}+2\hat{j}+2\hat{k})=9$

   (ii) $\vec{r}\cdot(\lambda\hat{i}+2\hat{j}+3\hat{k})=5$ and $\vec{r}\cdot(\hat{i}+2\hat{j}-7\hat{k})+11=0$

4. Find the acute angle between the following planes:

   (i) $2x-y+z=5$ and $x+y+2z=7$

   (ii) $x+2y+2z=3$ and $2x-3y+6z=8$

   (iii) $x+y-z=4$ and $x+2y+z=9$

   (iv) $x+y-2z=6$ and $2x-2y+z=11$

5. Show that each of the following pairs of planes are at right angles:

   (i) $3x+4y-5z=7$ and $2x+6y+6z+7=0$

   (ii) $x-2y+4z=10$ and $18x+17y+4z=49$

6. Prove that the plane $2x+3y-4z=9$ is perpendicular to each of the planes $x+2y+2z-7=0$ and $5x+6y+7z=23$.

7. Show that the planes $2x-2y+4z+5=0$ and $3x-3y+6z-1=0$ are parallel.

8. Find the value of λ for which the planes $x-4y+\lambda z+3=0$ and $2x+2y+3z=5$ are perpendicular to each other.

9. Write the equation of the plane passing through the origin and parallel to the plane $5x-3y+7z+11=0$.

10. Find the equation of the plane passing through the point (a, b, c) and parallel to the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$..

11. Find the equation of the plane passing through the point (1, –2, 7) and parallel to the plane $5x+4y-11z=6$.

12. Find the equation of the plane passing through the point $A(-1,-1,2)$ and perpendicular to each of the planes $3x+2y-3z=1$ and $5x-4y+z=5$. **[CBSE 2008]**

13. Find the equation of the plane passing through the origin and perpendicular to each of the planes $x+2y-z=1$ and $3x-4y+z=5$. **[CBSE 2004]**

14. Find the equation of the plane that contains the point $A(1,-1,2)$ and is perpendicular to both the planes $2x+3y-2z=5$ and $x+2y-3z=8$. Hence, find the distance of the point $P(-2,5,5)$ from the plane obtained above. **[CBSE 2014]**

15. Find the equation of the plane passing through the points $A(1, -1, 2)$ and $B(2, -2, 2)$ and perpendicular to the plane $6x - 2y + 2z = 9$. **[CBSE 2005]**

16. Find the equation of the plane passing through the points $A(-1, 1, 1)$ and $B(1, -1, 1)$ and perpendicular to the plane $x + 2y + 2z = 5$.

17. Find the equation of the plane through the points $A(3, 4, 2)$ and $B(7, 0, 6)$ and perpendicular to the plane $2x - 5y = 15$. **[CBSE 2012]**

HINT: The given plane is $2x - 5y + 0z = 15$.

18. Find the equation of the plane through the points $A(2, 1, -1)$ and $B(-1, 3, 4)$ and perpendicular to the plane $x - 2y + 4z = 10$. Also, show that the plane thus obtained contains the line

$$\vec{r} = (-\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - 2\hat{j} - 5\hat{k}).$$ **[CBSE 2012]**

ANSWERS (EXERCISE 28F)

1. (i) $\cos^{-1}\left(\frac{\sqrt{6}}{3}\right)$ (ii) $\cos^{-1}\left(\frac{1}{6}\right)$ (iii) $\cos^{-1}\left(\frac{5}{\sqrt{58}}\right)$ (iv) $\cos^{-1}\left(\frac{6}{7}\right)$

3. (i) $\lambda = -2$ (ii) $\lambda = 17$

4. (i) $\frac{\pi}{3}$ (ii) $\cos^{-1}\left(\frac{8}{21}\right)$ (iii) $\cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$ (iv) $\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$ **8.** $\lambda = 2$

9. $5x - 3y + 7z = 0$ **10.** $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c$ **11.** $5x + 4y - 11z + 80 = 0$

12. $5x + 9y + 11z = 8$ **13.** $x + 2y + 5z = 0$ **14.** $5x - 4y - z - 7 = 0, \sqrt{42}$ units

15. $x + y - 2z + 4 = 0$ **16.** $2x + 2y - 3z + 3 = 0$ **17.** $5x + 2y - 3z - 17 = 0$

18. $18x + 17y + 4z = 49$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28F)

10. We know that any plane parallel to $\vec{r} \cdot \vec{n} = q$ is $\vec{r} \cdot \vec{n} = q_1$.

Let the required plane parallel to the given plane be $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = q_1$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = q_1$

$\Rightarrow (a\hat{i} + b\hat{j} + c\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = q_1$

$\Rightarrow q_1 = (a + b + c)$.

Hence, the required equation is $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = (a + b + c)$.

14. Any plane through $A(1, -1, 2)$ is given by

$$a(x - 1) + b(y + 1) + c(z - 2) = 0. \quad \text{... (i)}$$

Since it is perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$, we have

$2a + 3b - 2c = 0$ ... (ii)

$a + 2b - 3c = 0.$ ... (iii)

On solving (ii) and (iii) by cross multiplication, we have

$$\frac{a}{(-9+4)} = \frac{b}{(-2+6)} = \frac{c}{(4-3)} = \lambda \Rightarrow a = -5\lambda, b = 4\lambda, c = \lambda.$$

Putting these value in (i), we get the required equation as

$$-5\lambda(x-1) + 4\lambda(y+1) + \lambda(z-2) = 0$$

$$\Rightarrow 5(x-1) - 4(y+1) - (z-2) = 0 \Rightarrow 5x - 4y - z - 7 = 0.$$

Distance of the point $P(-2, 5, 5)$ from this plane is given by

$$d = \frac{|5\times(-2) - 4\times 5 - 5 - 7|}{\sqrt{5^2 + (-4)^2 + (-1)^2}} = \frac{|-42|}{\sqrt{42}} = \frac{42}{\sqrt{42}} = \sqrt{42} \text{ units.}$$

18. Obtain the required equation of the plane as $18x + 17y + 4z = 49$. ... (A)

The given line is $\vec{r} = (3\lambda - 1)\hat{i} + (3 - 2\lambda)\hat{j} + (4 - 5\lambda)\hat{k}$.

Coordinates of any point on this line are $(3\lambda - 1, 3 - 2\lambda, 4 - 5\lambda)$.

These coordinates satisfy (A), as shown below:

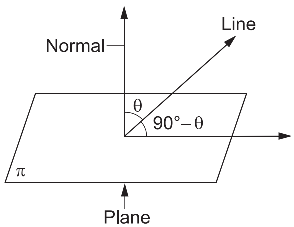
LHS $= 18(3\lambda - 1) + 17(3 - 2\lambda) + 4(4 - 5\lambda) = 49 =$ RHS.

So, $(3\lambda - 1, 3 - 2\lambda, 4 - 5\lambda)$ lies on plane (A).

Hence, the plane (A) obtained above contains the given line.

ANGLE BETWEEN A LINE AND A PLANE

The angle between a line and a plane is the complement of the angle between the line and normal to the plane.



Vector Form

THEOREM 1 *If ϕ is the angle between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r}\cdot\vec{n} = q$, then prove that*

$$\sin\phi = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}.$$

PROOF We know that the line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to $\vec{b}$.

And, the plane $\vec{r}\cdot\vec{n} = q$ is normal to $\vec{n}$.

Let θ be the angle between the line and the normal to the plane. Then,

$$\cos\theta = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}.$$

Let ϕ be the angle between the line and the plane

Then, $\phi = 90° - \theta \Rightarrow \theta = 90° - \phi$.

$$\therefore \quad \cos(90° - \phi) = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|} \Rightarrow \sin\phi = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|}.$$

Two Important Results

1. *Condition for a given line to be perpendicular to a given plane*

The line $\vec{r} = \vec{a} + \lambda\vec{b}$ is perpendicular to the plane $\vec{r} \cdot \vec{n} = q$

$\Leftrightarrow$ $\vec{b}$ is perpendicular to the plane $\vec{r} \cdot \vec{n} = q$

$\Leftrightarrow$ $\vec{b}$ is parallel to the normal $\vec{n}$ to the plane

$\Leftrightarrow$ $\vec{b} = t\vec{n}$, for some scalar t.

2. *Condition for a given line to be parallel to a given plane*

The line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = q$

$\Leftrightarrow$ $\vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = q$

$\Leftrightarrow$ $\vec{b}$ is perpendicular to the normal $\vec{n}$ to the plane

$\Leftrightarrow$ $\vec{b} \cdot \vec{n} = 0$.

Cartesian Form

THEOREM 2 *If ϕ is the angle between the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and the plane $a_2x + b_2y + c_2z + d = 0$, then*

$$\sin\phi = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\left\{\sqrt{a_1^2 + b_1^2 + c_1^2}\right\}\left\{\sqrt{a_2^2 + b_2^2 + c_2^2}\right\}}.$$

PROOF The direction ratios of the given line are a_1, b_1, c_1.

So, the given line is parallel to $\vec{b} = (a_1\hat{i} + b_1\hat{j} + c_1\hat{k})$.

The normal to the given plane is parallel to $\vec{n} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.

Let ϕ be the angle between the line and the plane. Then,

$$\sin\phi = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|}$$

$$= \frac{|(a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}) \cdot (a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k})|}{|(a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}| \, |(a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}|}$$

$$= \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\left\{\sqrt{a_1^2 + b_1^2 + c_1^2}\right\}\left\{\sqrt{a_2^2 + b_2^2 + c_2^2}\right\}}.$$

Important Results

1. *Condition for the given line to be perpendicular to the given plane*

The line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is perpendicular to the plane

$a_2x + b_2y + c_2z + d = 0$

$\Leftrightarrow$ the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is parallel to the normal to the plane

$a_2x + b_2y + c_2z + d = 0$

$\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$

2. *Condition for the given line to be parallel to the given plane*

The line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is parallel to the plane

$a_2x + b_2y + c_2z + d = 0$

$\Leftrightarrow$ the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is perpendicular to the normal to the plane $a_2x + b_2y + c_2z + d = 0$

$\Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0.$

3. *Distance between a line and a plane, parallel to each other*

If the line $\vec{r} = \vec{a} + \lambda \vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = q$ then the distance between them is $\frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|}$, which is the same as the distance of a point from the plane.

4. *Length of perpendicular from $P(x_1, y_1, z_1)$ to the plane $ax + by + cz = d$ is*

$$\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

SUMMARY

1. If ϕ is the angle between the line $\vec{r} = \vec{a} + \lambda \vec{b}$ and the plane $\vec{r} \cdot \vec{n} = q$ then

$$\sin \phi = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}|\,|\vec{n}|}.$$

2. The line $\vec{r} = \vec{a} + \lambda \vec{b}$ is **perpendicular** to the plane $\vec{r} \cdot \vec{n} = q$ only when $\vec{b} = t\vec{n}$ for some scalar t.

3. (i) The line $\vec{r} = \vec{a} + \lambda \vec{b}$ is **parallel** to the plane $\vec{r} \cdot \vec{n} = q$ only when $\vec{b} \cdot \vec{n} = 0$.

   (ii) If the line $\vec{r} = \vec{a} + \lambda \vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = q$ then the distance between them is

$$\frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|}.$$

4. If ϕ is the angle between the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and the plane $a_2x + b_2y + c_2z + d = 0$ then

$$\sin \phi = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{(\sqrt{a_1^2 + b_1^2 + c_1^2})(\sqrt{a_2^2 + b_2^2 + c_2^2})}.$$

5. The line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is **perpendicular** to the plane $a_2x + b_2y + c_2z + d = 0$ only if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

6. The line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is **parallel** to the plane $a_2x + b_2y + c_2z + d = 0$ only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the angle between the line* $\vec{r} = (\hat{i} + \hat{j} - 3\hat{k}) + \lambda(2\hat{i} + 2\hat{j} + \hat{k})$ *and the plane* $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 5$.

SOLUTION We know that the angle ϕ between the line $\vec{r} = \vec{a} + \lambda \vec{b}$ and the plane $\vec{r} \cdot \vec{n} = q$ is given by

$$\sin\phi = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|}.$$

Here, $\vec{b} = (2\hat{i} + 2\hat{j} + \hat{k})$ and $\vec{n} = (6\hat{i} - 3\hat{j} + 2\hat{k})$.

$$\therefore \quad \sin\phi = \frac{|(2\hat{i} + 2\hat{j} + \hat{k}) \cdot (6\hat{i} - 3\hat{j} + 2\hat{k})|}{\left\{\sqrt{2^2 + 2^2 + 1^2}\right\}\left\{\sqrt{6^2 + (-3)^2 + 2^2}\right\}}$$

$$= \frac{|(12 - 6 + 2)|}{\{\sqrt{9} \times \sqrt{49}\}} = \frac{8}{3 \times 7} = \frac{8}{21}$$

$$\Rightarrow \quad \phi = \sin^{-1}\left(\frac{8}{21}\right).$$

Hence, the angle between the given line and the given plane is $\sin^{-1}\left(\frac{8}{21}\right)$.

EXAMPLE 2 *Find the value of m for which the line* $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$ *is parallel to the plane* $\vec{r} \cdot (3\hat{i} - 2\hat{j} + m\hat{k}) = 12$.

SOLUTION We know that the line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = q$ only when $\vec{b} \cdot \vec{n} = 0$.

Here, $\vec{b} = (2\hat{i} + \hat{j} + 2\hat{k})$ and $\vec{n} = (3\hat{i} - 2\hat{j} + m\hat{k})$.

So, the given line is parallel to the given plane

$\Leftrightarrow \quad \vec{b} \cdot \vec{n} = 0$

$\Leftrightarrow \quad (2\hat{i} + \hat{j} + 2\hat{k}) \cdot (3\hat{i} - 2\hat{j} + m\hat{k}) = 0$

$\Leftrightarrow \quad (2 \times 3) + 1 \times (-2) + 2 \times m = 0$

$\Leftrightarrow \quad 2m = -4 \Leftrightarrow m = -2.$

Hence, the required value of m is -2.

EXAMPLE 3 *Show that the line* $\vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ *is parallel to the plane* $\vec{r} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$.
Also, find the distance between the given line and the given plane.
[CBSE 2010]

SOLUTION The given line is in the form $\vec{r} = \vec{a} + \lambda \vec{b}$ and the given plane is in the form $\vec{r} \cdot \vec{n} = q$, where

$$\vec{a} = (2\hat{i} - 2\hat{j} + 3\hat{k}),\ \vec{b} = (\hat{i} - \hat{j} + 4\hat{k}),\ \vec{n} = (\hat{i} + 5\hat{j} + \hat{k}) \text{ and } q = 5.$$

We know that the line $\vec{r} = \vec{a} + \lambda \vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = q$ only when $\vec{b} \cdot \vec{n} = 0$.

Here, $\vec{b} \cdot \vec{n} = (\hat{i} - \hat{j} + 4\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k})$

$$= (1 \times 1) + (-1) \times 5 + 4 \times 1 = 0.$$

Hence, the given line is parallel to the given plane.

Distance between the given line and the given plane

$$= \frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|} = \frac{|(2\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k}) - 5|}{|\hat{i} + 5\hat{j} + \hat{k}|}$$

$$= \frac{|(2 \times 1) + (-2) \times 5 + (3 \times 1) - 5|}{\sqrt{1^2 + 5^2 + 1^2}}$$

$$= \frac{|2 - 10 + 3 - 5|}{\sqrt{27}} = \frac{10}{3\sqrt{3}} \text{ units.}$$

Hence, the distance between the given line and the given plane is $\frac{10}{3\sqrt{3}}$ units.

EXAMPLE 4 *Find the vector equation of a line passing through the point* $A(1, -1, 2)$ *and perpendicular to the plane* $\vec{r} \cdot (2\hat{i} - \hat{j} + 3\hat{k}) = 5$.

SOLUTION The position vector of the given point is $\vec{a} = (\hat{i} - \hat{j} + 2\hat{k})$.

The given plane is $\vec{r} \cdot \vec{n} = q$; where $\vec{n} = (2\hat{i} - \hat{j} + 3\hat{k})$ and $q = 5$.

Since the required line is perpendicular to the given plane, so it must be parallel to its normal $\vec{n}$.

Thus, we need a line which passes through a point having position vector $\vec{a}$ and which is parallel to vector $\vec{n}$.

So, its vector equation is

$$\vec{r} = \vec{a} + \lambda \vec{n}, \text{ i.e., } \vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$$

for some scalar λ.

EXAMPLE 5 *Find the angle between the line* $\frac{x-2}{-1}=\frac{y+3}{2}=\frac{z+4}{3}$ *and the plane* $2x-3y+z=5$.

SOLUTION The given line is $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$, where $a_1=-1, b_1=2, c_1=3$.

The given plane is $a_2x+b_2y+c_2z+d=0$, where $a_2=2$, $b_2=-3$, $c_2=1, d=-5$.

Let the required angle be ϕ. Then

$$\sin\phi=\frac{|a_1a_2+b_1b_2+c_1c_2|}{\left\{\sqrt{a_1^2+b_1^2+c_1^2}\right\}\cdot\left\{\sqrt{a_2^2+b_2^2+c_2^2}\right\}}$$

$$=\frac{|(-1)\times 2+2\times(-3)+3\times 1|}{\left\{\sqrt{(-1)^2+2^2+3^2}\right\}\cdot\left\{\sqrt{2^2+(-3)^2+1^2}\right\}}$$

$$=\frac{|-2-6+3|}{\{\sqrt{14}\times\sqrt{14}\}}=\frac{|-5|}{14}=\frac{5}{14}$$

$$\Rightarrow \quad \phi=\sin^{-1}\left(\frac{5}{14}\right).$$

Hence, the angle between the given line and and the given plane is $\sin^{-1}\left(\frac{5}{14}\right)$.

EXAMPLE 6 *Find the equation of the line passing through the point* $(3, 0, 1)$ *and parallel to the planes* $x+2y=0$ *and* $3y-z=0$. **[CBSE 2012]**

SOLUTION Let the direction ratios of the required line be a, b, c.

Then, its equation is given by

$$\frac{x-3}{a}=\frac{y-0}{b}=\frac{z-1}{c}. \quad \text{... (i)}$$

Since the line (i) is parallel to each of the planes $x+2y+0z=0$ and $0x+3y-z=0$, so it must be perpendicular to the normal of each of these planes.

$\therefore \quad a\times 1+b\times 2+c\times 0=0 \Rightarrow a+2b+0c=0$ … (ii)

and $a\times 0+b\times 3+c\times(-1)=0 \Rightarrow 0a+3b-c=0$ … (iii)

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(-2-0)}=\frac{b}{(0+1)}=\frac{c}{(3-0)}$$

$$\Rightarrow \quad \frac{a}{-2}=\frac{b}{1}=\frac{c}{3}=\lambda \text{ (say)} \Rightarrow a=-2\lambda,\ b=\lambda \text{ and } c=3\lambda.$$

Putting these values of a, b, c in (i), we get

$$\frac{x-3}{-2\lambda}=\frac{y-0}{\lambda}=\frac{z-1}{3\lambda} \Rightarrow \frac{x-3}{-2}=\frac{y}{1}=\frac{z-1}{3}.$$

Hence, the required equation of the line is $\frac{x-3}{-2}=\frac{y}{1}=\frac{z-1}{3}$.

EXAMPLE 7 *Find the equation of the plane passing through the point* $A(1, 2, 1)$ *and perpendicular to the line joining the points* $P(1, 4, 2)$ *and* $Q(2, 3, 5)$. *Also find the distance of this plane from the line* $\frac{x+3}{2}=\frac{y-5}{-1}=\frac{z-7}{-1}$. **[CBSE 2010C, '11]**

SOLUTION The general equation of a plane passing through the point $A(1, 2, 1)$ is given by

$$a(x-1)+b(y-2)+c(z-1)=0. \quad \text{... (i)}$$

D.r.'s of noraml to the plane (i) are a, b, c.

D.r.'s of line PQ are $(2-1), (3-4), (5-2)$, i.e., $1, -1, 3$.

Since the required plane is perpendicular to line PQ, so the normal to this plane must be parallel to PQ.

$$\therefore \quad \frac{a}{1}=\frac{b}{-1}=\frac{c}{3}=\lambda \text{ (say)} \Rightarrow a=\lambda,\ b=-\lambda \text{ and } c=3\lambda.$$

Putting these values in (i), we get the required equation of the plane as $\lambda(x-1)-\lambda(y-2)+3\lambda(z-1)=0$

$$\Rightarrow \quad (x-1)-(y-2)+3(z-1)=0 \Rightarrow x-y+3z=2 \quad \text{... (ii)}$$

Hence, the required equation of the plane is $x-y+3z=2$.

The given line is $\frac{x+3}{2}=\frac{y-5}{-1}=\frac{z-7}{-1}$. ... (iii)

This line passes through the point $(-3, 5, 7)$.

Distance between the plane (ii) and the line (iii)

= Distance of any point on this line from the plane

= Length of perpendicular from $(-3, 5, 7)$ on plane (ii)

$$=\frac{|-3-5+21-2|}{\sqrt{1^2+(-1)^2+3^2}}=\frac{11}{\sqrt{11}}=\sqrt{11} \text{ units.}$$

Hence, the distance between the desired plane and the given line is $\sqrt{11}$ units.

EXAMPLE 8 *Find the equation of the plane passing through the points* $(0, 0, 0)$ *and* $(3, -1, 2)$ *and parallel to the line* $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$. **[CBSE 2011]**

SOLUTION The general equation of the plane passing through the point $(3, -1, 2)$ is given by

$$a(x-3)+b(y+1)+c(z-2)=0. \quad \text{... (i)}$$

If this plane passes through the point $(0, 0, 0)$, we have

$$a(0-3)+b(0+1)+c(0-2)=0 \Rightarrow -3a+b-2c=0. \quad \text{... (ii)}$$

D.r.'s of the normal to the plane (i) are a, b, c.

D.r.'s of the line $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ are 1, –4, 7.

The required plane is parallel to the given line only when the normal to this plane is perpendicular to this line.

$\therefore \quad (a \times 1) + b \times (-4) + c \times 7 = 0 \Rightarrow a - 4b + 7c = 0.$ ... (iii)

On solving (ii) and (iii) by cross multiplication, we get:

$$\frac{a}{(7-8)} = \frac{b}{(-2+21)} = \frac{c}{(12-1)}$$

$$\Rightarrow \frac{a}{-1} = \frac{b}{19} = \frac{c}{11} = \lambda \text{(say)} \Rightarrow a = -\lambda,\ b = 19\lambda \text{ and } c = 11\lambda.$$

Putting these values of a, b, c in (i), we get:

$$-\lambda(x-3) + 19\lambda(y+1) + 11\lambda(z-2) = 0$$

$$\Rightarrow (x-3) - 19(y+1) - 11(z-2) = 0$$

$$\Rightarrow x - 19y - 11z = 0.$$

Hence, the required equation of the plane is $x - 19y - 11z = 0$.

EXAMPLE 9 *Find the equation of the plane which passes through the point* (4, −1, 2) *and which is parallel to each of the lines* $\frac{x-1}{1} = \frac{y-3}{2} = \frac{z-4}{3}$ *and* $\frac{x+2}{3} = \frac{y-2}{-1} = \frac{z+1}{2}$.

SOLUTION The general equation of a plane passing through the point $A(4, -1, 2)$ is given by

$$a(x-4) + b(y+1) + c(z-2) = 0 \quad \text{... (i)}$$

This plane will be parallel to each of the given lines only when the normal to the plane is perpendicular to each of the given lines.

$\therefore \quad (1 \times a) + (2 \times b) + (3 \times c) = 0 \Rightarrow a + 2b + 3c = 0.$ ... (ii)

$(3 \times a) + (-1) \times b + (2 \times c) = 0 \Rightarrow 3a - b + 2c = 0$ ... (iii)

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(4+3)} = \frac{b}{(9-2)} = \frac{c}{(-1-6)}$$

$$\Rightarrow \frac{a}{7} = \frac{b}{7} = \frac{c}{-7} \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{-1} = \lambda \text{ (say)}$$

$$\Rightarrow a = \lambda,\ b = \lambda \text{ and } c = -\lambda.$$

Putting these values of a, b, c in (i), we get

$$\lambda(x-4) + \lambda(y+1) - \lambda(z-2) = 0$$

$$\Rightarrow (x-4) + (y+1) - (z-2) = 0 \Rightarrow x + y - z = 1.$$

Hence, the required equation of the plane is $x + y - z = 1$.

EXAMPLE 10 *Find the equation of the plane passing through* (2, 3, – 4) *and* (1, −1, 3) *and parallel to the x-axis.*

SOLUTION The general equation of the plane passing through the point $A(2, 3, -4)$ is given by

$$a(x-2)+b(y-3)+c(z+4)=0. \qquad \text{... (i)}$$

Since it passes through the point $B(1, -1, 3)$, we have

$$a(1-2)+b(-1-3)+c(3+4)=0 \Rightarrow -a-4b+7c=0. \qquad \text{... (ii)}$$

If this plane is parallel to x-axis, then the normal to the plane is perpendicular to the x-axis.

D.r.'s of normal to plane (i) are a, b, c and d.r.'s of the x-axis are 1, 0, 0.

$\therefore \quad a \times 1 + b \times 0 + c \times 0 = 0 \Rightarrow a = 0.$

Putting $a = 0$ in (ii), we get $7c - 4b = 0$.

Let $b = \lambda$. Then, $7c = 4\lambda \Rightarrow c = \frac{4\lambda}{7}\cdot$

Putting $a = 0$, $b = \lambda$ and $c = \frac{4\lambda}{7}$ in (i), we get

$$0 \times (x-2) + \lambda(y-3) + \frac{4\lambda}{7}(z+4) = 0$$

$$\Rightarrow \quad (y-3) + \frac{4}{7}(z+4) = 0 \Rightarrow 7y - 21 + 4z + 16 = 0$$

$$\Rightarrow \quad 7y + 4z - 5 = 0.$$

Hence, the required equation of the plane is $7y + 4z - 5 = 0$.

EXAMPLE 11 *Find the equation of the plane passing through the point $(0, 7, -7)$ and containing the line* $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}\cdot$

SOLUTION The general equation of the plane passing through the point $(0, 7, -7)$ is given by

$$a(x-0)+b(y-7)+c(z+7)=0. \qquad \text{... (i)}$$

If (i) contains the given line, then it must pass through the point $(-1, 3, -2)$ and must be parallel to the given line.

If (i) passes through the point $(-1, 3, -2)$, we have

$$a(-1-0)+b(3-7)+c(-2+7)=0 \Rightarrow a+4b-5c=0. \qquad \text{... (ii)}$$

If (i) is parallel to the given line, then its normal should be perpendicular to this line.

$$\therefore \quad (-3)a + 2b + 1 \times c = 0 \Rightarrow -3a + 2b + c = 0. \qquad \text{... (iii)}$$

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(4+10)} = \frac{b}{(15-1)} = \frac{c}{(2+12)} \Rightarrow \frac{a}{14} = \frac{b}{14} = \frac{c}{14}$$

$$\Rightarrow \quad \frac{a}{1} = \frac{b}{1} = \frac{c}{1} = \lambda \text{ (say)}.$$

Then $a = \lambda$, $b = \lambda$ and $c = \lambda$.

Putting $a = \lambda, b = \lambda$ and $c = \lambda$ in, we get

$$\lambda x + \lambda(y-7) + \lambda(z+7) = 0 \Rightarrow x + (y-7) + (z+7) = 0$$

$$\Rightarrow \quad x + y + z = 0.$$

Hence, the required equation of the plane is $x + y + z = 0$.

EXAMPLE 12 *Find the equation of the plane passing through the line of intersection of the planes* $2x + y - z = 3$ *and* $5x - 3y + 4z + 9 = 0$ *and parallel to the line* $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$. **[CBSE 2011]**

SOLUTION The equation of a plane passing through the intersection of the given planes is given by

$$(2x + y - z - 3) + \lambda(5x - 3y + 4z + 9) = 0 \text{ for some real number } \lambda$$

$$\Rightarrow \quad (2+5\lambda)x + (1-3\lambda)y + (4\lambda-1)z + (9\lambda-3) = 0 \qquad \text{... (i)}$$

If this plane is parallel to the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$, then the normal to the plane is perpendicular to this line.

$$\therefore \quad 2(2+5\lambda) + 4(1-3\lambda) + 5(4\lambda-1) = 0$$

$$\Rightarrow \quad (10\lambda - 12\lambda + 20\lambda) + (4 + 4 - 5) = 0$$

$$\Rightarrow \quad 18\lambda = -3 \Rightarrow \lambda = \frac{-3}{18} = \frac{-1}{6}.$$

Putting $\lambda = \frac{-1}{6}$ in (i), we get

$$\left(2 - \frac{5}{6}\right)x + \left(1 + \frac{3}{6}\right)y + \left(\frac{-4}{6} - 1\right)z + \left(\frac{-9}{6} - 3\right) = 0$$

$$\Rightarrow \quad 7x + 9y - 10z - 27 = 0.$$

Hence, the required equation of the plane is $7x + 9y - 10z - 27 = 0$.

EXAMPLE 13 *Show that the equation* $by + cz + d = 0$ *represents a plane parallel to the x-axis. Find the equation of a plane which is parallel to the x-axis and passes through the points* $A(2, 3, 1)$ *and* $B(4. -5, 3)$.

SOLUTION The given equation is $0 \cdot x + by + cz + d = 0$, which is of the form $ax + by + cz + d = 0$.

Hence, the given equation represents a plane.

D.r.'s of the normal to this plane are 0, b, c.

D.r.'s of the x-axis are 1, 0, 0.

Now $0 \times 1 + b \times 0 + c \times 0 = 0$.

This shows that the given plane is parallel to the x-axis.

Thus, the equation of a plane parallel to the x-axis is

$$by + cz + d = 0. \qquad \text{... (i)}$$

If it passes through the points $A(2, 3, 1)$ and $B(4, -5, 3)$, we have

$$3b + c + d = 0 \qquad \text{... (ii)}$$

$$-5b + 3c + d = 0 \qquad \text{... (ii)}$$

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{b}{(1-3)} = \frac{c}{(-5-3)} = \frac{d}{(9+5)}$$

$$\Rightarrow \quad \frac{b}{-2} = \frac{c}{-8} = \frac{d}{14} \Rightarrow \frac{b}{1} = \frac{c}{4} = \frac{d}{-7} = k \text{ (say)}$$

$\Rightarrow \quad b = k,\ c = 4k$ and $d = -7k$.

Putting these values in (i), we get

$$ky + 4kz - 7k = 0 \Rightarrow y + 4z - 7 = 0.$$

Hence, the required equation of the plane is $y + 4z - 7 = 0$.

EXERCISE 28G

1. Find the angle between the line $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 4$.
2. Find the angle between the line $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} - \hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 3$.
3. Find the angle between the line $\vec{r} = (3\hat{i} + \hat{k}) + \lambda(\hat{j} + \hat{k})$ and the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 1$.
4. Find the angle between the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{2}$ and the plane $3x + 4y + z + 5 = 0$.
5. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $10x + 2y - 11z = 3$.
6. Find the angle between the line joining the points $A(3, -4, -2)$ and $B(12, 2, 0)$ and the plane $3x - y + z = 1$.
7. If the plane $2x - 3y - 6z = 13$ makes an angle $\sin^{-1}(\lambda)$ with the x-axis, then find the value of λ.
8. Show that the line $\vec{r} = (2\hat{i} + 5\hat{j} + 7\hat{k}) + \lambda(\hat{i} + 3\hat{j} + 4\hat{k})$ is parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 7$.
   Also, find the distance between them.
9. Find the value of m for which the line $\vec{r} = (\hat{i} + 2\hat{k}) + \lambda(2\hat{i} - m\hat{j} - 3\hat{k})$ is parallel to the plane $\vec{r} \cdot (m\hat{i} + 3\hat{j} + \hat{k}) = 4$.

10. Find the vector equation of a line passing through the origin and perpendicular to the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 3$.

11. Find the vector equation of the line passing through the point with position vector $(\hat{i} - 2\hat{j} + 5\hat{k})$ and perpendicular to the plane $\vec{r} \cdot (2\hat{i} - 3\hat{j} - \hat{k}) = 0$.

12. Show that the equation $ax + by + d = 0$ represents a plane parallel to the z-axis. Hence, find the equation of a plane which is parallel to the z-axis and passes through the points $A(2, -3, 1)$ and $B(-4, 7, 6)$.

13. Find the equation of the plane passing through the points (1, 2, 3) and (0, −1, 0) and parallel to the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{-3}$. **[CBSE 2006]**

14. Find the equation of a plane passing through the point (2, −1, 5), perpendicular to the plane $x + 2y - 3z = 7$ and parallel to the line $\frac{x+5}{3} = \frac{y+1}{-1} = \frac{z-2}{1}$.

15. Find the equation of the plane passing through the intersection of the planes $4x - y + z = 10$ and $x + y - z = 4$ and parallel to the line with direction ratios 2, 1, 1. Find also the perpendicular distance of (1, 1, 1) from this plane.

ANSWERS (EXERCISE 28G)

1. $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ **2.** $\sin^{-1}\left(\frac{4}{\sqrt{42}}\right)$

3. $\sin^{-1}\left(\frac{1}{3\sqrt{2}}\right)$ **4.** $\sin^{-1}\left(\frac{7}{2\sqrt{91}}\right)$

5. $\sin^{-1}\left(\frac{8}{21}\right)$ **6.** $\sin^{-1}\left(\frac{23}{11\sqrt{11}}\right)$

7. $\frac{2}{7}$ **8.** $\frac{7}{\sqrt{3}}$ units

9. $m = -3$ **10.** $\vec{r} = \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$ for some scalar λ

11. $\vec{r} = (\hat{i} - 2\hat{j} + 5\hat{k}) + \lambda(2\hat{i} - 3\hat{j} - \hat{k})$ for some scalar λ

12. $5x + 3y - 1 = 0$ **13.** $6x - 3y + z = 3$

14. $x + 10y + 7z - 27 = 0$ **15.** $5y - 5z - 6 = 0$, $\frac{3\sqrt{2}}{5}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28G)

6. Equation of line AB is $\frac{x-3}{(12-3)} = \frac{y+4}{(2+4)} = \frac{z+2}{(0+2)}$, i.e., $\frac{x-3}{9} = \frac{y+4}{6} = \frac{z+2}{2}$.

7. D.r.'s of the x-axis are 1, 0, 0 and d.r.'s of normal to the plane are 2, –3, –6.

   Let ϕ be the angle between the x-axis and the given plane. Then,

$$\sin\phi = \frac{|1\times 2 + 0\times(-3) + 0\times(-6)|}{\left\{\sqrt{1^2+0^2+0^2}\right\}\left\{\sqrt{2^2+(-3)^2+(-6)^2}\right\}} = \frac{2}{7} \Rightarrow \phi = \sin^{-1}\left(\frac{2}{7}\right).$$

   Hence, $\lambda = \frac{2}{7}$.

8. We know that a line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to the plane $\vec{r}\cdot\vec{n} = q$ only when this line is perpendicular to the normal to the plane.

   So, we must have $\vec{b}\cdot\vec{n} = 0$.

   Here, $(\vec{b}\cdot\vec{n}) = (\hat{i} + 3\hat{j} + 4\hat{k})\cdot(\hat{i} + \hat{j} - \hat{k}) = (1 + 3 - 4) = 0$.

   Hence, the given line is parallel to the given plane.

   Required distance between the line and the plane

$$= \frac{|\vec{a}\cdot\vec{n} - q|}{|\vec{n}|} = \frac{|(2\hat{i} + 5\hat{j} + 7\hat{k})\cdot(\hat{i} + \hat{j} - \hat{k}) - 7|}{|\hat{i} + \hat{j} - \hat{k}|}$$

$$= \frac{|2+5-7-7|}{\sqrt{1^2+1^2+(-1)^2}} = \frac{|-7|}{\sqrt{3}} = \frac{7}{\sqrt{3}} \text{ units.}$$

12. The given equation is $ax + by + 0\cdot z + d = 0$ which is of the form $ax + by + cz + d = 0$. Therefore, it represents a plane.

   D.r.'s of normal to the plane are a, b, 0.

   D.r.'s of the z-axis are 0, 0, 1.

   Now, $a\times 0 + b\times 0 + 0\times 1 = 0$.

   This shows that the given plane is parallel to the z-axis.

   Let the required plane be $ax + by + d = 0$. ... (i)

   Since it passes through the points $A(2, -3, 1)$ and $B(-4, 7, 6)$, we have

   $2a - 3b + d = 0$ ... (ii)

   $-4a + 7b + d = 0$ ... (iii)

   On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(-3-7)} = \frac{b}{(-4-2)} = \frac{c}{(14-12)}$$

$$\Rightarrow \frac{a}{-10} = \frac{b}{-6} = \frac{c}{2} \Rightarrow \frac{a}{5} = \frac{b}{3} = \frac{c}{-1} = k \text{ (say)}.$$

   $\therefore$ $a = 5k$, $b = 3k$ and $c = -k$.

   Putting these values in (i), we get

$$5kx + 3ky - k = 0 \Rightarrow 5x + 3y - 1 = 0,$$

   which is the required equation of the plane.

13. Any plane through the point (1, 2, 3) is

$$a(x-1)+b(y-2)+c(z-3)=0. \quad \text{... (i)}$$

Since it passes through the point (0, –1, 0), we have

$$a(0-1)+b(-1-2)+c(0-3)=0 \Rightarrow a+3b+3c=0 \quad \text{... (ii)}$$

This plane is parallel to the line $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z}{-3}$, so

$$2a+3b-3c=0 \quad \text{... (iii)}$$

On solving (ii) and (iii), we get

$$\frac{a}{(9+9)}=\frac{b}{(-3-6)}=\frac{c}{(6-3)} \Rightarrow \frac{a}{18}=\frac{b}{-9}=\frac{c}{3} \Rightarrow \frac{a}{6}=\frac{b}{-3}=\frac{c}{1}=\lambda \text{ (say)}.$$

Hence, the required equation of the plane is

$$6\lambda(x-1)-3\lambda(y-2)+\lambda(z-3)=0 \Rightarrow 6x-3y+z=3.$$

Equation of a Plane Passing through a Given Point and Parallel to Two Given Lines

Vector Form

THEOREM 1 *The vector equation of a plane passing through a given point with position vector* $\vec{a}$ *and parallel to two given vectors* $\vec{b}$ *and* $\vec{c}$ *is* $(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0$.

PROOF The required plane is parallel to vectors $\vec{b}$ and $\vec{c}$.

So, the vector $\vec{n}=(\vec{b}\times\vec{c})$ is perpendicular to this plane.

Thus, we have to find the equation of a plane passing through the point with position vector $\vec{a}$ and perpendicular to the vector $\vec{n}$.

So, its equation is

$$(\vec{r}-\vec{a})\cdot\vec{n}=0, \text{ i.e., } (\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0.$$

Hence, the required equation of the plane is $(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0$.

Cartesian Form

THEOREM 2 *The equation of the plane passing through a given point* $A(x_1, y_1, z_1)$ *and parallel to two given lines having direction ratios* b_1, b_2, b_3 *and* c_1, c_2, c_3 *is*

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$$

PROOF Let us consider a plane passing through a given point $A(x_1, y_1, z_1)$ and parallel to two given lines having direction ratios b_1, b_2, b_3 and c_1, c_2, c_3.

Let $P(x, y, z)$ be an arbitrary point on the plane. Then,

$$\vec{AP} = (\text{p.v. of } P) - (\text{p.v. of } A)$$

$$= (x\hat{i} + y\hat{j} + z\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}.$$

It is given that the plane is parallel to two lines having direction ratios b_1, b_2, b_3 and c_1, c_2, c_3.

So, the given plane is parallel to each of the vectors

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}.$$

$\therefore \vec{AP}, \vec{b}$ and $\vec{c}$ are coplanar, and therefore their scalar triple product must be zero.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0,$$

which is the required equation.

SUMMARY OF THE RESULTS

(i) The vector equation of a plane passing through a point having position vector $\vec{a}$ and parallel to vectors $\vec{b}$ and $\vec{c}$, is given by

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$$

(ii) The Cartesian equation of a plane passing through a point $A(x_1, y_1, z_1)$ and parallel to two nonparallel lines having d.r.'s b_1, b_2, b_3 and c_1, c_2, c_3 is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the vector and Cartesian equations of the plane passing through the point* (1, 2, – 4) *and parallel to the lines*

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) - \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$\text{and } \vec{r} = (\hat{i} - 3\hat{j} + 5\hat{k}) + \mu(\hat{i} + \hat{j} - \hat{k}).$$

SOLUTION We know that the vector equation of a plane passing through a point $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is given by $(\vec{r} \cdot \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0.$

Here $\vec{a} = (\hat{i} + 2\hat{j} - 4\hat{k})$, $\vec{b} = (2\hat{i} + 3\hat{j} + 6\hat{k})$

and $\vec{c} = (\hat{i} + \hat{j} - \hat{k})$.

$$\therefore \quad \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= (-3-6)\hat{i} - (-2-6)\hat{j} + (2-3)\hat{k} = (-9\hat{i} + 8\hat{j} - \hat{k}).$$

So, the required vector equation is

$$\vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$\Rightarrow \quad \vec{r} \cdot (-9\hat{i} + 8\hat{j} - \hat{k}) = (\hat{i} + 2\hat{j} - 4\hat{k}) \cdot (-9\hat{i} + 8\hat{j} - \hat{k})$$

$$= (-9 + 16 + 4) = 11$$

$$\Rightarrow \quad \vec{r} \cdot (-9\hat{i} + 8\hat{j} - \hat{k}) - 11 = 0$$

$$\Rightarrow \quad \vec{r} \cdot (9\hat{i} - 8\hat{j} + \hat{k}) + 11 = 0.$$

Hence, the required vector equation of the plane is

$$\vec{r} \cdot (9\hat{i} - 8\hat{j} + \hat{k}) + 11 = 0.$$

Clearly, the required Cartesian equation of the plane is

$$9x - 8y + z + 11 = 0.$$

EXAMPLE 2 *Find the Cartesian and vector equation of the plane passing through the point* (2, 0, −1) *and parallel to the lines*

$$\frac{x}{-3} = \frac{y-2}{4} = z + 1 \text{ and } x - 4 = \frac{1-y}{2} = 2z.$$

SOLUTION The given lines are

$$\frac{x}{-3} = \frac{y-2}{4} = \frac{z+1}{1} \text{ and } \frac{x-4}{1} = \frac{y-1}{-2} = \frac{z}{\frac{1}{2}},$$

i.e., $\dfrac{x}{-3} = \dfrac{y-2}{4} = \dfrac{z+1}{1}$ and $\dfrac{x-4}{2} = \dfrac{y-1}{-4} = \dfrac{z}{1}$.

So, the required plane passes through the point $A(2, 0, -1)$ and it is parallel to the lines having direction ratios −3, 4, 1 and 2, −4, 1.

Here $(x_1 = 2, y_1 = 0, z_1 = -1), (a_1 = -3, b_1 = 4, c_1 = 1)$

and $(a_2 = 2, b_2 = -4, c_2 = 1)$.

The required equation of the plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-0 & z+1 \\ -3 & 4 & 1 \\ 2 & -4 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-2)(4+4) - y(-3-2) + (z+1)(12-8) = 0$

$\Rightarrow \quad 8(x-2) + 5y + 4(z+1) = 0$

$\Rightarrow \quad 8x + 5y + 4z - 12 = 0.$

Hence, the required Cartesian equation of the plane is $8x + 5y + 4z = 12$ and its corresponding vector equation is $\vec{r} \cdot (8\hat{i} + 5\hat{j} + 4\hat{k}) = 12.$

EXERCISE 28H

1. Find the vector and Cartesian equations of the plane passing through the origin and parallel to the vectors $(\hat{i} + \hat{j} - \hat{k})$ and $(3\hat{i} - \hat{k})$.
2. Find the vector and Cartesian equations of the plane passing through the point (3, –1, 2) and parallel to the lines $\vec{r} = (-\hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 5\hat{j} - \hat{k})$ and $\vec{r} = (\hat{i} - 3\hat{j} + \hat{k}) + \mu(-5\hat{i} + 4\hat{j})$.
3. Find the vector equation of a plane passing through the point (1, 2, 3) and parallel to the lines whose direction ratios are 1, –1, –2, and –1, 0, 2.
4. Find the Cartesian and vector equations of a plane passing through the point (1, 2, –4) and parallel to the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+1}{6}$ and $\frac{x-1}{1} = \frac{y+3}{1} = \frac{z}{-1}$.
5. Find the vector equation of the plane passing through the point $(3\hat{i} + 4\hat{j} + 2\hat{k})$ and parallel to the vectors $(\hat{i} + 2\hat{j} + 3\hat{k})$ and $(\hat{i} - \hat{j} + \hat{k})$.

ANSWERS (EXERCISE 28H)

1. $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 0,\ x + 2y + 3z = 0$

2. $\vec{r} \cdot (4\hat{i} + 5\hat{j} - 17\hat{k}) + 27 = 0,\ 4x + 5y - 17z + 27 = 0$

3. $\vec{r} \cdot (2\hat{i} + \hat{k}) = 5$ **4.** $9x - 8y + z + 11 = 0$, $\vec{r} \cdot (9\hat{i} - 8\hat{j} + \hat{k}) + 11 = 0$

5. $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$

Condition for the Coplanarity of Two Lines

Vector Form

THEOREM 1 *The condition for two lines* $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ *and* $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ *to be coplanar is that*

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0, \text{ i.e., } [\vec{a_2} - \vec{a_1} \quad \vec{b_1} \quad \vec{b_2}] = 0.$$

Also, the equation of the plane containing both these lines is

$$(\vec{r} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0 \quad or \quad (\vec{r} - \vec{a_2}) \cdot (\vec{b_1} \times \vec{b_2}) = 0.$$

PROOF The equations of the given lines are

$$\vec{r} = \vec{a_1} + \lambda \vec{b_1} \qquad \ldots \text{(i)}$$

$$\vec{r} = \vec{a_2} + \mu \vec{b_2}. \qquad \ldots \text{(ii)}$$

The line (i) passes through a point A with position vector $\vec{a_1}$ and is parallel to $\vec{b_1}$.

The line (ii) passes through a point B with position vector $\vec{a_2}$ and is parallel to $\vec{b_2}$.

Now, $\overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A) = (\vec{a_2} - \vec{a_1})$.

$\therefore$ the given lines are coplanar

$\Leftrightarrow$ $\overrightarrow{AB}$, $\vec{b_1}$, $\vec{b_2}$ are coplanar

$\Leftrightarrow$ $\overrightarrow{AB} \cdot (\vec{b_1} \times \vec{b_2}) = 0$

$\Leftrightarrow$ $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0.$

Equation of the Plane Containing Both the Lines

If the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ are coplanar then their common plane is perpendicular to the vector $(\vec{b_1} \times \vec{b_2})$, and this plane passes through each of the points $\vec{a_1}$ and $\vec{a_2}$.

So, its equation is

$$(\vec{r} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0, \text{ or } (\vec{r} - \vec{a_2}) \cdot (\vec{b_1} \times \vec{b_2}) = 0.$$

Cartesian Form

THEOREM 2 *The lines*

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}, \quad \textit{and} \quad \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

$$\textit{are coplanar} \Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

And, the equation of the plane containing both these lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \; \textit{or} \; \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

PROOF The given lines are

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \qquad \ldots \text{(i)}$$

$$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}. \qquad \ldots \text{(ii)}$$

The line (i) passes through a point $A(x_1, y_1, z_1)$ and is parallel to the vector $\vec{u_1} = a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}$.

Also, the line (ii) passes through a point $B(x_2, y_2, z_2)$ and is parallel to the vector $\vec{u_2} = a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}$.

Now, $\overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A)$

$$= (x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}) - (x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k})$$

$$= (x_2 - x_1) \hat{i} + (y_2 - y_1) \hat{j} + (z_2 - z_1) \hat{k}.$$

$\therefore$ lines (i) and (ii) are coplanar

$\Leftrightarrow$ there is a plane which passes through the points A and B, and which is parallel to each of $\vec{u_1}$ and $\vec{u_2}$

$\Leftrightarrow$ $\overrightarrow{AB}, \vec{u_1}, \vec{u_2}$ are parallel to the same plane

$\Leftrightarrow$ $\overrightarrow{AB}, \vec{u_1}, \vec{u_2}$ are coplanar

$\Leftrightarrow$ $[\overrightarrow{AB}, \vec{u_1}, \vec{u_2}] = 0$

$$\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Equation of the Plane Containing Both the Lines

If the lines (i) and (ii) are coplanar then their common plane is the plane containing the line (i) and parallel to the line (ii) or it is the plane containing the line (ii) and parallel to the line (i).

Hence, its equation is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

SUMMARY

1. (i) Two lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda\vec{b_2}$ are coplanar only when $(\vec{a_2} - \vec{a_1})\cdot(\vec{b_1} \times \vec{b_2}) = 0$.

   (ii) If two lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda\vec{b_2}$ are coplanar, then the equation of the plane containing both of these lines is given by

$$\{(\vec{r} - \vec{a_1})\cdot(\vec{b_1} \times \vec{b_2}) = 0\} \text{ or } \{(\vec{r} - \vec{a_2})\cdot(\vec{b_1} \times \vec{b_2}) = 0\}.$$

2. (i) The lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ are coplanar only when $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$

   (ii) The equation of the common plane is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *Show that the lines*

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) \text{ and } \vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

are coplanar. Also, find the equation of the plane containing both these lines. **[CBSE 2013C]**

SOLUTION We know that the lines $\vec{r} = \vec{a_1} + \lambda\,\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\,\vec{b_2}$ are coplanar

$$\Leftrightarrow \; (\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0.$$

Here, $\vec{a}_1 = \hat{i} + \hat{j} - \hat{k}$, $\vec{b}_1 = 3\hat{i} - \hat{j}$, $\vec{a}_2 = 4\hat{i} - \hat{k}$ and $\vec{b}_2 = 2\hat{i} + 3\hat{k}$.

$\therefore \quad (\vec{a}_2 - \vec{a}_1) = (4\hat{i} - \hat{k}) - (\hat{i} + \hat{j} - \hat{k}) = (3\hat{i} - \hat{j})$.

$$(\vec{b}_1 \times \vec{b}_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}$$

$$= (-3 - 0)\hat{i} - (9 - 0)\hat{j} + (0 + 2)\hat{k} = -3\hat{i} - 9\hat{j} + 2\hat{k}.$$

$\therefore \quad (\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = (3\hat{i} - \hat{j}) \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k})$

$$= [3 \times (-3) + (-1) \times (-9) + 0 \times 2] = 0.$$

Hence, the given lines are coplanar.

The equation of the plane containing both the given lines is given by

$$(\vec{r} - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0 \Leftrightarrow \vec{r} \cdot (\vec{b}_1 \times \vec{b}_2) = \vec{a}_1 \cdot (\vec{b}_1 \times \vec{b}_2)$$

$$\Leftrightarrow \; \vec{r} \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k}) = (\hat{i} + \hat{j} - \hat{k}) \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k})$$

$$\Leftrightarrow \; \vec{r} \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k}) = (-3 - 9 - 2) \Leftrightarrow \vec{r} \cdot (3\hat{i} + 9\hat{j} - 2\hat{k}) = 14.$$

Hence, the required equation of the plane is

$$\vec{r} \cdot (3\hat{i} + 9\hat{j} - 2\hat{k}) = 14.$$

EXAMPLE 2 *Show that the lines* $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$ *and* $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$ *are coplanar. Also, find the equation of the plane containing these lines.*

[CBSE 2013C]

SOLUTION We know that the lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}, \text{ and } \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

are coplanar only when $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$

Here, $x_1 = -3$, $y_1 = 1$, $z_1 = 5$ and $x_2 = -1$, $y_2 = 2$, $z_2 = 5$.

$$a_1 = -3,\; b_1 = 1,\; c_1 = 5 \text{ and } a_2 = -1,\; b_2 = 2,\; c_2 = 5$$

$$\therefore \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} -1+3 & 2-1 & 5-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ -1 & 2 & 5 \end{vmatrix} \; [R_2 \to (R_2 - R_3)]$$

$$= 5(-2+2) = (5 \times 0) = 0 \text{ [expanding by } c_3].$$

Hence, the given lines are coplanar.

Equation of the plane containing both these lines is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x+3 & y-1 & z-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

$\Leftrightarrow \quad (x+3)(5-10) - (y-1)(-15+5) + (z-5)(-6+1) = 0$

$\Leftrightarrow \quad (-5)(x+3) + 10(y-1) - 5(z-5) = 0$

$\Leftrightarrow \quad -5x + 10y - 5z = 0 \;\Leftrightarrow\; x - 2y + z = 0.$

Hence, the required equation plane is $x - 2y + z = 0$.

EXAMPLE 3 *Show that the lines*

$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}, \textit{ and } \frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$$

are coplanar. Also find the equation of the plane containing them.

SOLUTION The given lines are

$$\frac{x-(a-d)}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-(a+d)}{\alpha+\delta} \qquad \ldots \text{(i)}$$

$$\frac{x-(b-c)}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-(b+c)}{\beta+\gamma}. \qquad \ldots \text{(ii)}$$

We know that the lines

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}, \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

are coplanar $\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$

Here, $x_1 = (a-d),\ y_1 = a,\ z_1 = (a+d);$

$x_2 = (b-c),\ y_2 = b,\ z_2 = (b+c);$

$a_1 = (\alpha-\delta),\ b_1 = \alpha,\ c_1 = (\alpha+\delta);$ and

$a_2 = (\beta-\gamma),\ b_2 = \beta,\ c_2 = (\beta+\gamma).$

$$\therefore \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} (b-c)-(a-d) & b-a & (b+c)-(a+d) \\ \alpha - \delta & \alpha & \alpha + \delta \\ \beta - \gamma & \beta & \beta + \gamma \end{vmatrix}$$

$$= \begin{vmatrix} 2(b-a) & b-a & b+c-a+d \\ 2\alpha & \alpha & \alpha + \delta \\ 2\beta & \beta & \beta + \gamma \end{vmatrix} \quad \{C_1 \to C_1 + C_3\}$$

$= 0$ [$\because$ C_1 and C_2 are proportional].

Hence, the given lines are coplanar.

Equation of the plane containing the given lines is given by

$$\begin{vmatrix} x-(a-d) & y-a & z-(a+d) \\ \alpha - \delta & \alpha & \alpha + \delta \\ \beta - \gamma & \beta & \beta + \gamma \end{vmatrix} = 0.$$

Applying $C_1 \to C_1 + C_3 - 2C_2$, we get

$$\begin{vmatrix} x-z-2y & y-a & z-(a+d) \\ 0 & \alpha & \alpha + \delta \\ 0 & \beta & \beta + \gamma \end{vmatrix} = 0$$

$\Rightarrow \quad (x+z-2y) - [\alpha(\beta+\gamma) - \beta(\alpha+\delta)] = 0$

$\Rightarrow \quad (x+z-2y)(\alpha\gamma - \beta\delta) = 0 \Rightarrow x+z-2y = 0.$

Hence, the required equation of the plane is $x+z-2y = 0$.

PLANE CONTAINING PARALLEL LINES

EXAMPLE 4 *Find the equation of the plane which contains the two parallel lines* $\frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1}$, *and* $\frac{x+1}{3} = \frac{y-2}{2} = \frac{z}{1}$.

SOLUTION Clearly, the plane which contains the two given parallel lines must pass through the points $A(3, -4, 1)$ and $B(-1, 2, 0)$, and it must be parallel to the line having direction ratios 3, 2, 1.

Any plane passing through the point $A(3, -4, 1)$ is given by

$$a(x-3) + b(y+4) + c(z-1) = 0. \quad \ldots \text{(i)}$$

If this plane passes through the point $B(-1, 2, 0)$, we have

$$a(-1-3) + b(2+4) + c(0-1) = 0 \Rightarrow 4a - 6b + c = 0 \quad \ldots \text{(ii)}$$

If the plane (ii) is parallel to the line having direction ratios 3, 2, 1 then

$$3a + 2b + c = 0. \quad \ldots \text{(iii)}$$

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(-6-2)} = \frac{b}{(3-4)} = \frac{c}{(8+18)} \Rightarrow \frac{a}{-8} = \frac{b}{-1} = \frac{c}{26} = \lambda \text{ (say)}$$

$$\Rightarrow a = -8\lambda,\ b = -\lambda \text{ and } c = 26\lambda.$$

Putting these values of a, b and c in (i), we get

$$-8\lambda(x-3) - \lambda(y+4) + 26\lambda(z-1) = 0$$

$$\Rightarrow 8(x-3) + (y+4) - 26(z-1) = 0 \Rightarrow 8x + y - 26z + 6 = 0.$$

Hence, the required equation of the plane is $8x + y - 26z + 6 = 0$.

EXERCISE 28I

1. Show that the lines

$$\vec{r} = (2\hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k}) \text{ and } \vec{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 4\hat{k})$$

are coplanar.

Also find the equation of the plane containing these lines.

2. Find the vector and Cartesian forms of the equations of the plane containing the two lines

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and $\vec{r} = (9\hat{i} + 5\hat{j} - \hat{k}) + \mu(-2\hat{i} + 3\hat{j} + 8\hat{k})$.

3. Find the vector and Cartesian equations of a plane containing the two lines

$$\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 5\hat{k})$$

and $\vec{r} = (3\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(3\hat{i} - 2\hat{j} + 5\hat{k})$.

Also show that the line $\vec{r} = (2\hat{i} + 5\hat{j} + 2\hat{k}) + p(3\hat{i} - 2\hat{j} + 5\hat{k})$ lies in the plane. **[CBSE 2011C]**

4. Prove that the lines $\frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3}$ and $\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$ are coplanar.

Also find the equation of the plane containing these lines.

5. Prove that the lines $\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7}$ and $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ are coplanar. Also find the equation of the plane containing these lines.

6. Show that the lines $\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$ are coplanar. Find the equation of the plane containing these lines. **[CBSE 2014]**

7. Show that the lines $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ are coplanar.

Find the equation of the plane containing these lines.

8. Show that the lines $\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z}{-1}$ and $\frac{x-4}{3}=\frac{y-1}{-2}=\frac{z-1}{-1}$ are coplanar. Also find the equation of the plane containing these lines.

9. Find the equation of the plane which contains two parallel lines given by $\frac{x-3}{1}=\frac{y+2}{-4}=\frac{z}{5}$ and $\frac{x-4}{1}=\frac{y-3}{-4}=\frac{z-2}{5}$.

ANSWERS (EXERCISE 28I)

1. $\vec{r}\cdot(\hat{i}-2\hat{j}+\hat{k})+7=0$

2. $\vec{r}\cdot(6\hat{i}-28\hat{j}+12\hat{k})+98=0,\ 6x-28y+12z+98=0$

3. $\vec{r}\cdot(10\hat{i}+5\hat{j}-4\hat{k})=37,\ 10x+5y-4z=37$

4. $x-2y+z+7=0$

5. $x-2y+z=0$

6. $17x-47y-24z+172=0$

7. $x+y+z=0$

8. $2x-5y-16z+13=0$

9. $11x-y-3z=35$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28I)

3. The given lines are $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\lambda\vec{b_2}$, where

$\vec{a_1}=(2\hat{i}+\hat{j}-3\hat{k}),\ \vec{a_2}=(3\hat{i}+3\hat{j}+2\hat{k}),$

$\vec{b_1}=(\hat{i}+2\hat{j}+5\hat{k}),\ \vec{b_2}=(3\hat{i}-2\hat{j}+5\hat{k}).$

$$\therefore\quad (\vec{b_1}\times\vec{b_2})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 5\\ 3 & -2 & 5\end{vmatrix}=(10+10)\hat{i}-(5-15)\hat{j}+(-2-6)\hat{k}$$

$$=(20\hat{i}+10\hat{j}-8\hat{k}).$$

Vector equation of the required plane is

$$\vec{r}\cdot(\vec{b_1}\times\vec{b_2})=\vec{a_1}\cdot(\vec{b_1}\times\vec{b_2})$$

$$\Rightarrow\ \vec{r}\cdot(20\hat{i}+10\hat{j}-8\hat{k})=(2\hat{i}+\hat{j}-3\hat{k})\cdot(20\hat{i}+10\hat{j}-8\hat{k})$$

$$=(40+10+24)=74$$

$$\Rightarrow\ \vec{r}\cdot(10\hat{i}+5\hat{j}-4\hat{k})=37. \quad \text{... (i)}$$

The Cartesian equation is

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(10\hat{i}+5\hat{j}-4\hat{k})=37\ \Rightarrow\ 10x+5y-4z=37. \quad \text{... (ii)}$$

The third line is $\vec{r}=(2\hat{i}+5\hat{j}+2\hat{k})+p(3\hat{i}-2\hat{j}+5\hat{k})$. ... (iii)

Now, the line (iii) will lie in the plane (ii) if (2, 5, 2) lies on (ii) and $(3\hat{i} - 2\hat{j} + 5\hat{k})$ is perpendicular to the normal of (ii).

Now, $10 \times 2 + 5 \times 5 - 4 \times 2 = 37$ shows that (2, 5, 2) lies on (ii).

Also $10 \times 3 + 5 \times (-2) - 4 \times 5 = 0$ shows that $(3\hat{i} - 2\hat{j} + 5\hat{k})$ is perpendicular to the normal of (ii).

Hence, the line (iii) lies in plane (ii).

6. The given equations are: $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$.

EXERCISE 28J

Very-Small-Answer Questions

1. Find the direction ratios of the normal to the plane $x + 2y - 3z = 5$.
2. Find the direction cosines of the normal to the plane $2x + 3y - z = 4$.
3. Find the direction cosines of the normal to the plane $y = 3$.
4. Find the direction cosines of the normal to the plane $3x + 4 = 0$.
5. Write the equation of the plane parallel to XY-plane and passing through the point (4, –2, 3).
6. Write the equation of the plane parallel to YZ-plane and passing through the point (–3, 2, 0).
7. Write the general equation of a plane parallel to the x-axis.
8. Write the intercept cut off by the plane $2x + y - z = 5$ on the x-axis. **[CBSE 2011]**
9. Write the intercepts made by the plane $4x - 3y + 2z = 12$ on the coordinate axes.
10. Reduce the equation $2x - 3y + 5z + 4 = 0$ to intercept form and find the intercepts made by it on the coordinate axes.
11. Find the equation of a plane passing through the points $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.
12. Write the value of k for which the planes $2x - 5y + kz = 4$ and $x + 2y - z = 6$ are perpendicular to each other.
13. Find the angle between the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$.
14. Find the angle between the planes $\vec{r} \cdot (\hat{i} + \hat{j}) = 1$ and $\vec{r} \cdot (\hat{i} + \hat{k}) = 3$.
15. Find the angle between the planes $\vec{r} \cdot (3\hat{i} - 4\hat{j} + 5\hat{k}) = 0$ and $\vec{r} \cdot (2\hat{i} - \hat{j} - 2\hat{k}) = 7$.
16. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $10x + 2y - 11z = 3$.

17. Find the angle between the line $\vec{r} = (\hat{i} + \hat{j} - 2\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 4$.

18. Find the value of λ such that the line $\frac{x-2}{6} = \frac{y-1}{\lambda} = \frac{z+5}{4}$ is perpendicular to the plane $3x - y - 2z = 7$. **[CBSE 2010C]**

19. Write the equation of the plane passing through the point (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$. **[CBSE 2014]**

20. Find the length of perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$. **[CBSE 2013]**

21. Find the direction cosines of the perpendicular from the origin to the plane $\vec{r} \cdot (6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$.

22. Show that the line $\vec{r} = (4\hat{i} - 7\hat{k}) + \lambda(4\hat{i} - 2\hat{j} + 3\hat{k})$ is parallel to the plane $\vec{r} \cdot (5\hat{i} + 4\hat{j} - 4\hat{k}) = 7$.

23. Find the length of perpendicular from the origin to the plane $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) + 14 = 0$.

24. Find the value of λ for which the line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{\lambda}$ is parallel to the plane $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 4$.

25. Write the angle between the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+3}{-2}$ and the plane $x + y + 4 = 0$.

26. Write the equation of a plane passing through the point (2, –1, 1) and parallel to the plane $3x + 2y - z = 7$.

ANSWERS (EXERCISE 28J)

1. 1, 2, –3 **2.** $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}$ **3.** 0, 1, 0 **4.** –1, 0, 0

5. $z = 3$ **6.** $x = -3$ **7.** $by + cz + d = 0$ **8.** $\frac{5}{2}$

9. 3, –4, 6 **10.** $-2, \frac{4}{3}, \frac{-4}{5}$ **11.** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ **12.** $k = -8$

13. $\cos^{-1}\left(\frac{4}{21}\right)$ 14. $\frac{\pi}{3}$ **15.** $\frac{\pi}{2}$ **16.** $\sin^{-1}\left(\frac{8}{21}\right)$

17. $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ **18.** $\lambda = -2$ **19.** $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) + a + b + c$

20. 3 units **21.** $\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}$ **23.** 2 units **24.** $\lambda = \frac{-13}{4}$

25. $\frac{\pi}{4}$ **26.** $3x + 2y - z = 3$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28J)

1. The direction ratios of the normal to the plane $x + 2y - 3z = 5$ are 1, 2, –3.
2. The given plane is $2x + 3y - z = 4$.

   Direction ratios of the normal to the given plane are 2, 3, –1 and $\sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{14}$.

   Hence, the required direction cosines are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}$.
3. Direction ratios of the normal to the plane are 0, 1, 0 and $\sqrt{0^2 + 1^2 + 0^2} = 1$.

   Hence, the required direction cosines are 0, 1, 0.
4. $3x + 4 = 0 \Rightarrow -x = \frac{4}{3}$.

   Direction ratios of the normal to this plane are –1, 0, 0 and $\sqrt{(-1)^2 + 0^2 + 0^2} = 1$.

   Hence, the required direction cosines are –1, 0, 0.
5. Any plane parallel to XY-plane is $z = k$.

   Since it passes through (4, –2, 3), we have $3 = k$.

   Hence, the required equation of the plane is $z = 3$.
6. Any plane parallel to YZ-plane is $x = k$.

   Since it passes through (–3, 2, 0), we have $-3 = k$.

   Hence, the required equation of the plane is $x = -3$.
7. Let the required equation of the plane be $ax + by + cz + d = 0$.

   The d.r.'s of this plane are a, b, c.

   The d.r.'s of the x-axis are 1, 0, 0.

   Normal of the required plane is perpendicular to the x-axis.

   $\therefore\ (a \times 1) + (b \times 0) + (c \times 0) = 0 \Rightarrow a = 0$.

   Hence, the required equation is $by + cz + d = 0$.
8. $2x + y - z = 5 \Rightarrow \frac{x}{\left(\frac{5}{2}\right)} + \frac{y}{5} + \frac{z}{-5} = 1$.

   $\therefore$ intercept cut off by the given plane on the x-axis is $\frac{5}{2}$.
9. $4x - 3y + 2z = 12 \Rightarrow \frac{4x}{12} + \frac{(-3y)}{12} + \frac{2z}{12} = 1 \Rightarrow \frac{x}{3} + \frac{y}{-4} + \frac{z}{6} = 1$.

   Hence, the required intercepts are 3, –4, 6.
10. The given equation may be written as $-2x + 3y - 5z = 4$

$$\Rightarrow \frac{(-2x)}{4} + \frac{3y}{4} + \frac{(-5z)}{4} = 1 \Rightarrow \frac{x}{-2} + \frac{y}{\frac{4}{3}} + \frac{z}{\frac{-4}{5}} = 1.$$

$\therefore$ the required intercepts are $-2, \frac{4}{3}, \frac{-4}{5}$.

11. Clearly, the required equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

12. Clearly, the normals of the given planes are perpendicular to each other.

$\therefore \quad (2 \times 1) + (-5) \times 2 + k \times (-1) = 0 \Rightarrow k = (2 - 10) = -8.$

13. D.r.'s of normals to the given planes are 2, 1, –2 and 3, –6, –2.

$$\therefore \quad \cos\theta = \frac{|(2 \times 3) + 1 \times (-6) + (-2) \times (-2)|}{\left\{\sqrt{2^2 + 1^2 + (-2)^2}\right\}\left\{\sqrt{3^2 + (-6)^2 + (-2)^2}\right\}}$$

$$= \frac{4}{(\sqrt{9})(\sqrt{49})} = \frac{4}{(3 \times 7)} = \frac{4}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{4}{21}\right).$$

14. Given planes are $x + 1 = 1$ and $x + z = 3$.

The d.r.'s of normals to these planes are 1, 1, 0 and 1, 0, 1.

$$\therefore \quad \cos\theta = \frac{|(1 \times 1) + (1 \times 0) + (0 \times 1)|}{\left\{\sqrt{1^2 + 1^2 + 0^2}\right\}\left\{\sqrt{1^2 + 0^2 + 1^2}\right\}}$$

$$= \frac{1}{(\sqrt{2} \times \sqrt{2})} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

15. The given planes are $3x - 4y + 5z = 0$ and $2x - y - 2z = 7$.

The d.r.'s of normals to these planes are 3, –4, 5 and 2, –1, –2.

$$\therefore \quad \cos\theta = \frac{|(3 \times 2) + (-4) \times (-1) + 5 \times (-2)|}{\left\{\sqrt{3^2 + (-4)^2 + 5^2}\right\}\left\{\sqrt{2^2 + (-1)^2 + (-2)^2}\right\}} = 0 \Rightarrow \theta = \frac{\pi}{2}.$$

16. D.r.'s of the given line are 2, 3, 6.

D.r.'s of the normal to the given plane are 10, 2, –11.

$$\therefore \quad \sin\theta = \frac{|(2 \times 10) + (3 \times 2) + 6 \times (-11)|}{\left\{\sqrt{2^2 + 3^2 + 6^2}\right\}\left\{\sqrt{(10)^2 + 2^2 + (-11)^2}\right\}}$$

$$= \frac{40}{\{\sqrt{49}\} \times \{\sqrt{225}\}} = \frac{40}{(7 \times 15)} = \frac{8}{21} \Rightarrow \theta = \sin^{-1}\left(\frac{8}{21}\right).$$

17. Given line is $\vec{r} = \vec{a} + \lambda \vec{b}$ and given plane is $\vec{r} \cdot \vec{n} = p$.

$$\sin\theta = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|} = \frac{|(\hat{i} - \hat{j} + \hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k})|}{\left\{\sqrt{1^2 + (-1)^2 + 1^2}\right\}\left\{\sqrt{2^2 + (-1)^2 + 1^2}\right\}}$$

$$= \frac{|(2 \times 1) + (-1) \times (-1) + 1 \times 1|}{(\sqrt{3} \times \sqrt{6})}$$

$$= \frac{4}{\sqrt{18}} = \left(\frac{4}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \right) = \frac{2\sqrt{2}}{3} \Rightarrow \theta = \sin^{-1}\left(\frac{2\sqrt{2}}{3} \right).$$

18. D.r.'s of the given line are 6, λ, – 4.

D.r.'s of normal to the given plane are 3, –1, –2.

Given line is parallel to the normal of the plane.

$\therefore \quad \frac{6}{3} = \frac{\lambda}{-1} = \frac{-4}{-2} \Rightarrow \lambda = -2.$

19. Given plane is $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 2 \quad \Rightarrow \quad x + y + z = 2.$

Let the required plane be $x + y + z = k$, where k is a constant.

Since it passes through (a, b, c), we have $k = (a + b + c)$.

So, the required plane is $x + y + z = a + b + c$.

In vector form, it is given by $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c.$

20. We have $p = \frac{|2 \times 0 - 3 \times 0 + 6 \times 0 + 21|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{21}{\sqrt{49}} = \frac{21}{7} = 3$ units.

21. The given equation is $\vec{r} \cdot (-6\hat{i} + 3\hat{j} + 2\hat{k}) = 1.$

D.r.'s of normal to the plane are –6, 3, 2 and $\sqrt{(-6)^2 + 3^2 + 2^2} = \sqrt{49} = 7.$

$\therefore$ d.c.'s of normal to the plane are $\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}$.

22. Given line is $\vec{r} = \vec{a} + \lambda \vec{b}$, where $\vec{b} = (4\hat{i} - 2\hat{j} + 3\hat{k})$.

D.r.'s of the line are 4, –2, 3.

Given plane is $\vec{r} \cdot \vec{n} = q$, where $\vec{n} = (5\hat{i} + 4\hat{j} - 4\hat{k})$.

D.r.'s of the normal to the given plane are 5, 4, –4.

So, the given line will be parallel to the given plane when this line is perpendicular to the normal to the plane.

Hence, we must have $(4 \times 5) + (-2 \times 4) + 3 \times (-4) = 0$, which is true.

23. We have $\vec{r} \cdot (-2\hat{i} + 3\hat{j} - 6\hat{k}) = 14.$

Here $\vec{n} = (-2\hat{i} + 3\hat{j} - 6\hat{k})$ and $|\vec{n}| = \sqrt{(-2)^2 + 3^2 + (-6)^2} = 7.$

$\therefore \quad \vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{14}{|\vec{n}|} \Rightarrow \vec{r} \cdot \hat{n} = \frac{14}{7} = 2.$

Hence, the length of perpendicular from origin to the given plane is 2 units.

24. Clearly, the given line must be perpendicular to the normal to the given plane.

D.r.'s of the given line are 2, 3, λ.

D.r.'s of the normal to the given plane are 2, 3, 4.

$\therefore \quad (2 \times 2) + (3 \times 3) + (\lambda \times 4) = 0 \Rightarrow 4\lambda = -13 \Rightarrow \lambda = \frac{-13}{4}.$

25. D.r.'s of the given line are 2, 1, –2.

D.r.'s of the normal to the given plane are 1, 1, 0.

$$\therefore \quad \sin\theta = \frac{(2 \times 1) + (1 \times 1) + (-2) \times 0}{\left\{\sqrt{2^2 + 1^2 + (-2)^2}\right\}\left\{\sqrt{1^2 + 1^2 + 0^2}\right\}} = \frac{(2+1+0)}{(\sqrt{9})(\sqrt{2})}$$

$$= \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}.$$

26. Let the required equation of the plane be $a(x-2) + b(y+1) + c(z-1) = 0$.

Here $a = 3$, $b = 2$ and $c = -1$.

So, the required equation of the plane is

$$3(x-2) + 2(y+1) - 1\cdot(z-1) = 0 \Rightarrow 3x + 2y - z = 3.$$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. The direction cosines of the perpendicular from the origin to the plane $\vec{r}\cdot(6\hat{i} - 3\hat{j} + 2\hat{k}) + 1 = 0$ are

   (a) $\frac{6}{7}, \frac{3}{7}, \frac{-2}{7}$ (b) $\frac{6}{7}, \frac{-3}{7}, \frac{2}{7}$ (c) $\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}$ (d) none of these

2. The direction cosines of the normal to the plane $5y + 4 = 0$ are

   (a) $0, \frac{-4}{5}, 0$ (b) 0, 1, 0 (c) 0, –1, 0 (d) none of these

3. The length of perpendicular from the origin to the plane $\vec{r}\cdot(3\hat{i} - 4\hat{j} - 12\hat{k}) + 39 = 0$ is

   (a) 3 units (b) $\frac{13}{5}$ units (c) $\frac{5}{3}$ units (d) none of these

4. The equation of a plane passing through the point $A(2, -3, 7)$ and making equal intercepts on the axes, is

   (a) $x + y + z = 3$ (b) $x + y + z = 6$ (c) $x + y + z = 9$ (d) $x + y + z = 4$

5. A plane cuts off intercepts 3, –4, 6 on the coordinate axes. The length of perpendicular from the origin to this plane is

   (a) $\frac{5}{\sqrt{29}}$ units (b) $\frac{8}{\sqrt{29}}$ units (c) $\frac{6}{\sqrt{29}}$ units (d) $\frac{12}{\sqrt{29}}$ units

6. If the line $\frac{x+1}{3} = \frac{y-2}{4} = \frac{z+6}{5}$ is parallel to the plane $2x - 3y + kz = 0$, then the value of k is

   (a) $\frac{5}{6}$ (b) $\frac{6}{5}$ (c) $\frac{3}{4}$ (d) $\frac{4}{5}$

7. If O is the origin and $P(1, 2, -3)$ is a given point, then the equation of the plane through P and perpendicular to OP is
   (a) $x + 2y - 3z = 14$
   (b) $x - 2y + 3z = 12$
   (c) $x - 2y - 3z = 14$
   (d) none of these
8. If the line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$, then the value of k is
   (a) -7 (b) 7 (c) 4 (d) -4
9. The plane $2x + 3y + 4z = 12$ meets the coordinate axes in A, B and C. The centroid of $\triangle ABC$ is
   (a) $(2, 3, 4)$ (b) $(6, 4, 3)$ (c) $\left(2, \frac{4}{3}, 1\right)$ (d) none of these
10. If a plane meets the coordinate axes in A, B and C such that the centroid of $\triangle ABC$ is $(1, 2, 4)$, then the equation of the plane is
    (a) $x + 2y + 4z = 6$
    (b) $4x + 2y + z = 12$
    (c) $x + 2y + 4z = 7$
    (d) $4x + 2y + z = 7$
11. The equation of a plane through the point $A(1, 0, -1)$ and perpendicular to the line $\frac{x+1}{2} = \frac{y+3}{4} = \frac{z+7}{-3}$ is
    (a) $2x + 4y - 3z = 3$
    (b) $2x - 4y + 3z = 5$
    (c) $2x + 4y - 3z = 5$
    (d) $x + 3y + 7z = -6$
12. The line $\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-3}{4}$ meets the plane $2x + 3y - z = 14$ in the point
    (a) $(2, 5, 7)$ (b) $(3, 5, 7)$ (c) $(5, 7, 3)$ (d) $(6, 5, 3)$
13. The equation of the plane passing through the points $A(2, 2, 1)$ and $B(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 1$, is
    (a) $x + 2y - 3z + 5 = 0$
    (b) $2x - 3y + 4z - 6 = 0$
    (c) $4x + 5y - 6z + 3 = 0$
    (d) $3x + 4y - 5z - 9 = 0$
14. The equation of the plane passing through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and passing through the point $A(2, 2, 1)$ is given by
    (a) $7x + 5y - 4z - 8 = 0$
    (b) $7x - 5y + 4z - 8 = 0$
    (c) $5x - 7y + 4z - 8 = 0$
    (d) $5x + 7y - 4z + 8 = 0$
15. The equation of the plane passing through the points $A(0, -1, 0)$, $B(2, 1, -1)$ and $C(1, 1, 1)$ is given by
    (a) $4x + 3y - 2z - 3 = 0$
    (b) $4x - 3y + 2z + 3 = 0$
    (c) $4x - 3y + 2z - 3 = 0$
    (d) none of these
16. If the plane $2x - y + z = 0$ is parallel to the line $\frac{2x-1}{2} = \frac{2-y}{2} = \frac{z+1}{a}$, then the value of a is
    (a) -4 (b) -2 (c) 4 (d) 2

17. The angle between the line $\frac{x+1}{1}=\frac{y}{2}=\frac{z-1}{1}$ and a normal to the plane $x-y+z=0$ is

(a) 0° (b) 30° (c) 45° (d) 90°

18. The point of intersection of the line $\frac{x-1}{3}=\frac{y+2}{4}=\frac{z-3}{-2}$ and the plane $2x-y+3z-1=0$, is

(a) (–10, 10, 3) (b) (10, 10, –3) (c) (10, –10, 3) (d) (10, –10, –3)

19. The equation of a plane passing through the points $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ is given by

(a) $ax+by+cz=0$ (b) $ax+by+cz=1$

(c) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ (d) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

20. If θ is the angle between the planes $2x-y+2z=3$ and $6x-2y+3z=5$, then $\cos\theta=$?

(a) $\frac{11}{20}$ (b) $\frac{12}{23}$ (c) $\frac{17}{25}$ (d) $\frac{20}{21}$

21. The angle between the planes $2x-y+z=6$ and $x+y+2z=7$, is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

22. The angle between the planes $\vec{r}\cdot(3\hat{i}-6\hat{j}+2\hat{k})=4$ and $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=3$, is

(a) $\cos^{-1}\left(\frac{16}{21}\right)$ (b) $\cos^{-1}\left(\frac{4}{21}\right)$ (c) $\cos^{-1}\left(\frac{3}{4}\right)$ (d) $\cos^{-1}\left(\frac{1}{4}\right)$

23. The equation of the plane through the points $A(2, 3, 1)$ and $B(4, -5, 3)$, parallel to the x-axis, is

(a) $x+y-3z=2$ (b) $y+4z=7$ (c) $y+3z=6$ (d) $x+5y-3z=4$

24. A variable plane moves so that the sum of the reciporcals of its intercepts on the coordinate axes is (1/2). Then, the plane passes through the point

(a) (0, 0, 0) (b) (1, 1, 1) (c) $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ (d) (2, 2, 2)

25. The equation of a plane which is perpendicular to $(2\hat{i}-3\hat{j}+\hat{k})$ and at a distance of 5 units from the origin is

(a) $2x-3y+z=5$ (b) $2x-3y+z=5\sqrt{14}$

(c) $\frac{x}{2}-\frac{y}{3}+\frac{z}{1}=5$ (d) $\frac{x}{2}-\frac{y}{3}+\frac{z}{1}=\frac{5}{\sqrt{14}}$

26. The equation of the plane passing through the point $A(2, 3, 4)$ and parallel to the plane $5x - 6y + 7z = 3$, is
 (a) $5x - 6y + 7z = 20$ (b) $7x - 6y + 5z = 72$
 (c) $20x - 18y + 14z = 11$ (d) $10x - 18y + 28z = 13$
27. The foot of the perpendicular from the point $A(7, 14, 5)$ to the plane $2x + 4y - z = 2$ is
 (a) (3, 1, 8) (b) (1, 2, 8) (c) (3, –3, 5) (d) (5, –3, –4)
28. The equation of the plane which makes with the coordinate axes, a triangle with centroid (α, β, γ) is given by
 (a) $\alpha x + \beta y + \gamma z = 1$ (b) $\alpha x + \beta y + \gamma z = 3$
 (c) $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$ (d) $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$
29. The intercepts made by the plane $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) = 12$ are
 (a) 2, –3, 4 (b) 2, –3, –6 (c) –6, –4, 3 (d) –6, 4, 3
30. The angle between the line $\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z+4}{-3}$ and the plane $2x - 3y + z = 5$ is
 (a) $\cos^{-1}\left(\frac{5}{14}\right)$ (b) $\sin^{-1}\left(\frac{5}{14}\right)$ (c) $\cos^{-1}\left(\frac{3}{7}\right)$ (d) $\sin^{-1}\left(\frac{3}{7}\right)$
31. The angle between the line $\vec{r} \cdot (\hat{i} + \hat{j} - 3\hat{k}) + \lambda(2\hat{i} + 2\hat{j} + \hat{k})$ and the plane $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 5$, is
 (a) $\cos^{-1}\left(\frac{8}{21}\right)$ (b) $\cos^{-1}\left(\frac{5}{21}\right)$ (c) $\sin^{-1}\left(\frac{5}{21}\right)$ (d) $\sin^{-1}\left(\frac{8}{21}\right)$
32. The distance of the point $(\hat{i} + 2\hat{j} + 5\hat{k})$ from the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) + 17 = 0$, is
 (a) $\frac{25}{\sqrt{2}}$ units (b) $\frac{25}{\sqrt{3}}$ units (c) $25\sqrt{2}$ units (d) $25\sqrt{3}$ units
33. The distance between the parallel planes $2x - 3y + 6z = 5$ and $6x - 9y + 18z + 20 = 0$, is
 (a) $\frac{5}{3}$ units (b) $5\sqrt{3}$ units (c) $\frac{8}{5}$ units (d) $8\sqrt{5}$ units
34. The distance between the planes $x + 2y - 2z + 1 = 0$ and $2x + 4y - 4z + 5 = 0$, is
 (a) 4 units (b) 2 units (c) $\frac{1}{2}$ units (d) $\frac{1}{4}$ units
35. The image of the point $P(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$, is
 (a) (3, –5, 2) (b) (3, 5, –2) (c) (3, 5, 2) (d) (–3, 5, 2)

ANSWERS (OBJECTIVE QUESTIONS)

1. (c)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (d)	**6.** (b)	**7.** (a)	**8** (b)	**9.** (c)	**10.** (b)
11. (c)	**12.** (b)	**13.** (d)	**14.** (b)	**15.** (c)	**16.** (a)	**17.** (d)	**18.** (b)	**19.** (d)	**20.** (d)
21. (c)	**22.** (a)	**23.** (b)	**24.** (d)	**25.** (b)	**26.** (a)	**27.** (b)	**28.** (d)	**29.** (c)	**30.** (b)
31. (d)	**32.** (b)	**33.** (a)	**34.** (c)	**35.** (d)					

HINTS TO GIVEN OBJECTIVE QUESTIONS

1. (c) Given plane is $\vec{r}\cdot(-6\hat{i}+3\hat{j}+2\hat{k})=1.$

$\therefore\ \vec{r}\cdot\vec{n}=p.$ where $\vec{n}=(-6\hat{i}+3\hat{j}+2\hat{k}).$

D.r.'s of the normal to the given plane are –6, 3, 2 and

$\sqrt{(-6)^2+3^2+2^2}=\sqrt{49}=7.$

$\therefore$ d.c.'s of the normal to the given plane are $\frac{-6}{7},\frac{3}{7},\frac{2}{7}\cdot$

2. (c) $5y+4=0 \Rightarrow -5y=4 \Rightarrow -y=\frac{4}{5} \Rightarrow 0x-1\cdot y+0\cdot z=\frac{4}{5}\cdot$

D.r.'s of the normal to this plane are 0, –1, 0 and $\sqrt{0^2+(-1)^2+0^2}=\sqrt{1}=1.$

$\therefore$ d.c.'s of the normal to the given plane are 0, –1, 0.

3. (a) Given plane is $\vec{r}\cdot(-3\hat{i}+4\hat{j}+12\hat{k})=39.$

$\vec{r}\cdot\vec{n}=39 \Rightarrow \vec{r}\cdot\frac{\vec{n}}{|\vec{n}|}=\frac{39}{|\vec{n}|} \Rightarrow \vec{r}\cdot\hat{n}=\frac{39}{|\vec{n}|}\cdot$

And, $|\vec{n}|=\sqrt{(-3)^2+4^2+(12)^2}=\sqrt{169}=13.$

$\therefore\ \vec{r}\cdot\hat{n}=\frac{39}{13}=3$ units. Hence, $p=3$ units.

4. (b) Let the required equation of the plane be $\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1$, i.e., $x+y+z=a.$

Since, it passes through the point $A(2,-3,7)$, we have $2+(-3)+7=a \Rightarrow a=6.$
Hence, the required equation of the plane is $x+y+z=6.$

5. (d) The given plane is $\frac{x}{3}+\frac{y}{-4}+\frac{z}{6}=1 \Rightarrow 4x-3y+2z=12$

$$\therefore \quad p=\frac{|4\times 0-3\times 0+2\times 0-12|}{\sqrt{4^2+(-3)^2+2^2}}=\frac{12}{\sqrt{29}}\text{ units.}$$

6. (b) D.r.'s of the given line are 3, 4, 5.
D.r.'s of the normal to the given plane are 2, –3, k.
Since the given line is perpendicular to the normal of the plane, we have

$$(3\times 2)+4\times(-3)+5\times k=0 \Rightarrow 5k=6 \Rightarrow k=\frac{6}{5}\cdot$$

7. (a) Let the required equation of the plane through $P(1, 2, -3)$ be

$$a(x-1)+b(y-2)+c(z+3)=0. \qquad \text{... (i)}$$

D.r.'s of OP are $(1-0), (2-0), (-3, 0)$, i.e., 1, 2, –3.

$\therefore \quad a=1, b=2, c=-3.$

Hence, the required equation of the plane is

$$1(x-1)+2(y-2)-3(z+3)=0 \Rightarrow x+2y-3z=14.$$

8. (b) Clearly, the given line passes through the point $(4, 2, k)$.

Since, the given line lies in the plane $2x-4y+z=7$, so the above point lies in this plane.

$\therefore \quad (2\times 4)-(4\times 2)+k=7 \Rightarrow k=7.$

9. (c) The given equation is $\frac{x}{6}+\frac{y}{4}+\frac{c}{3}=1.$

This plane meets the coordinate axes in $A(6, 0, 0)$, $B(0, 4, 0)$ and $C(0, 0, 3)$.

$\therefore \quad$ centroid of $\triangle ABC$ is $G\left(\frac{6+0+0}{3}, \frac{0+4+0}{3}, \frac{0+0+3}{3}\right)$, i.e., $G\left(2, \frac{4}{3}, 1\right)$.

10. (b) Let the required equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

Then, it meets the coordinate axes in $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$.

$\therefore \quad$ centroid of $\triangle ABC$ is $G\left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right)$, i.e., $G\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$.

$\therefore \quad \left(\frac{a}{3}=1, \frac{b}{3}=2, \frac{c}{3}=4\right) \Rightarrow a=3, b=6, c=12.$

Hence, the required equation of the plane is

$$\frac{x}{3}+\frac{y}{6}+\frac{z}{12}=1 \Rightarrow 4x+2y+z=12.$$

11. (c) D.r.'s of the given line are 2, 4, –3.

Let the required equation of plane through (1, 0, –1) be

$$a(x-1)+b(y-0)+c(z+1)=0.$$

Since the line is perpendicular to the plane, so it is parallel to the normal to the plane.

$\therefore \quad a=2, b=4$ and $c=-3.$

Hence, the required equation of the plane is

$$2(x-1)+4(y-0)-3(z+1)=0 \Rightarrow 2x+4y-3z=5.$$

12. (b) Let $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$ (say).

A general point on this line is $(2\lambda+1, 3\lambda+2, 4\lambda+3)$.

For some value of λ, let the given line meet the plane $2x+3y-z=14$ at a point $P(2\lambda+1, 3\lambda+2, 4\lambda+3)$. Then,

$$2(2\lambda+1)+3(3\lambda+2)-(4\lambda+3)=14 \Rightarrow 9\lambda=9 \Rightarrow \lambda=1.$$

So, the required point is $P(2+1, 3+2, 4+3)$, i.e., $P(3, 5, 7)$.

13. (d) The equation of a plane passing through the point $A(2, 2, 1)$ is

$$a(x-2)+b(y-2)+c(z-1)=0. \quad \text{... (i)}$$

Since it passes through the point $B(9, 3, 6)$, we have

$$a(9-2)+b(3-2)+c(6-1)=0 \Rightarrow 7a+b+5c=0. \quad \text{... (ii)}$$

Also, it being perpendicular to the plane $2x+6y+6z=1$, we have

$$2a+6b+6c=0 \Rightarrow a+3b+3c=0. \quad \text{... (iii)}$$

On solving (ii) and (iii) by cross multiplication, we have

$$\frac{a}{(3-15)}=\frac{b}{(5-21)}=\frac{c}{(21-1)} \Rightarrow \frac{a}{-12}=\frac{b}{-16}=\frac{c}{20} \Rightarrow \frac{a}{3}=\frac{b}{4}=\frac{c}{-5}=k$$

$\Rightarrow \quad a=3k,\ b=4k$ and $c=-5k$

Substituting these values in (i), we get

$$3k(x-2)+4k(y-2)-5k(z-1)=0$$

$$\Rightarrow \quad 3(x-2)+4(y-2)-5(z-1)=0 \Rightarrow 3x+4y-5z-9=0.$$

14. (b) Let the required equation of the plane be

$$(3x-y+2z-4)+\lambda(x+y+z-2)=0$$

$$\Rightarrow \quad (3+\lambda)x+(\lambda-1)y+(2+\lambda)z-(4+2\lambda)=0. \quad \text{... (ii)}$$

Since it passes through the point $A(2, 2, 1)$, we have

$$(3+\lambda)\times 2+(\lambda-1)\times 2+(2+\lambda)\times 1-(4+2\lambda)=0$$

$$\Rightarrow \quad (6+2\lambda)+(2\lambda-2)+(2+\lambda)-(4+2\lambda)=0$$

$$\Rightarrow \quad 3\lambda+2=0 \Rightarrow \lambda=\frac{-2}{3}.$$

So, the required equation of the plane is

$$\left(3-\frac{2}{3}\right)x+\left(\frac{-2}{3}-1\right)y+\left(2-\frac{2}{3}\right)z-\left(4-\frac{4}{3}\right)=0$$

$$\Rightarrow \quad \frac{7x}{3}-\frac{5y}{3}+\frac{4z}{3}-\frac{8}{3}=0 \Rightarrow 7x-5y+4z-8=0.$$

15. (c) Here $(x_1=0, y_1=-1, z_1=0), (x_2=2, y_2=1, z_2=-1)$ and $(x_3=1, y_3=1, z_3=1)$.

So, the required equation of the plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

$$\Rightarrow \quad \begin{vmatrix} x-0 & y+1 & z-0 \\ 2-0 & 1+1 & -1-0 \\ 1-0 & 1+1 & 1-0 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} x & y+1 & z \\ 2 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad x(2+2)-(y+1)(2+1)+z(4-2)=0$$

$$\Rightarrow \quad 4x-3(y+1)+2z=0 \Rightarrow 4x-3y+2z-3=0.$$

16. (a) The given line is $\dfrac{x-\frac{1}{2}}{1}=\dfrac{y-2}{-2}=\dfrac{z-(-1)}{a}.$

The d.r.'s of the line are $1, -2, a$.

The given plane is $2x-y+z=0$.

The d.r.'s of the normal to this plane are 2, –1, 1.

Since the given plane is parallel to the given line, so the normal to this plane is perpendicular to the given line.

$\therefore \quad (2 \times 1) - 1 \times (-2) + 1 \times a = 0 \Rightarrow a = -4$

17. (d) D.r.'s of the given line are 1, 2, 1.

D.r.'s of the normal to the given plane are 1, –1, 1.

$$\therefore \quad \cos\theta = \frac{|(1 \times 1) + 2 \times (-1) + (1 \times 1)|}{\left\{\sqrt{1^2 + 2^2 + 1^2}\right\}\left\{\sqrt{1^2 + (-1)^2 + 1^2}\right\}} = 0 \Rightarrow \theta = 90°.$$

18. (b) The given line is $\frac{x-1}{3} = \frac{y+2}{4} = \frac{z-3}{-2} = \lambda$ (say).

A general point on this line is $(3\lambda + 1,\ 4\lambda - 2,\ -2\lambda + 3)$.

For some value of λ, let the point $P(3\lambda + 1,\ 4\lambda - 2,\ -2\lambda + 3)$ lie on the plane $2x - y + 3z - 1 = 0$. Then,

$$2(3\lambda + 1) - (4\lambda - 2) + 3(-2\lambda + 3) - 1 = 0$$

$$\Rightarrow \quad 4\lambda = 12 \Rightarrow \lambda = 3.$$

So, the required point is $P(9 + 1, 12 - 2, -6 + 3)$, i.e., $P(10, 10, -3)$.

19. (d) The required equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

20. (d) Angle between the planes means the angle between their normals.

$$\therefore \quad \cos\theta = \frac{|(2 \times 6) + (-1) \times (-2) + 2 \times 3|}{\left\{\sqrt{2^2 + (-1)^2 + 2^2}\right\}\left\{\sqrt{6^2 + (-2)^2 + 3^2}\right\}} = \frac{20}{(\sqrt{9})(\sqrt{49})} = \frac{20}{(3 \times 7)} = \frac{20}{21}.$$

21. (c) Angle between the planes means the angle between their normals.

$$\therefore \quad \cos\theta = \frac{|(2 \times 1) + (-1) \times 1 + (1 \times 2)|}{\left\{\sqrt{2^2 + (-1)^2 + 1^2}\right\}\left\{\sqrt{1^2 + 1^2 + 2^2}\right\}}$$

$$= \frac{3}{(\sqrt{6} \times \sqrt{6})} = \frac{3}{6} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

22. (a) The given planes are $3x - 6y + 2z = 4$ and $2x - y + 2z = 3$.

$$\therefore \quad \cos\theta = \frac{|(3 \times 2) + (-6) \times (-1) + (2 \times 2)|}{\left\{\sqrt{3^2 + (-6)^2 + 2^2}\right\}\left\{\sqrt{2^2 + (-1)^2 + 2^2}\right\}}$$

$$= \frac{16}{(\sqrt{49})(\sqrt{9})} = \frac{16}{(7 \times 3)} = \frac{16}{21}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{16}{21}\right).$$

23. (b) Let the required equation be $by + cz + d = 0$. ... (i)

Since it passes through the points $A(2, 3, 1)$ and $B(4, -5, 3)$, we have

$3b + c + d = 0$ ... (ii)

$-5b + 3c + d = 0.$ ... (iii)

On solving (ii) and (iii) by cross multiplication, we have

$$\frac{b}{(1-3)} = \frac{c}{(-5-3)} = \frac{d}{(9+5)}$$

$\Rightarrow \quad \vec{r} \cdot \vec{n} = 5\sqrt{14}.$

$\Rightarrow \quad \frac{b}{1} = \frac{c}{4} = \frac{d}{-7} = k$ (say).

Then, $b = k$, $c = 4k$ and $d = -7k$.

Substituting these values in (i), we get the required equation as

$$ky + 4kz - 7k = 0 \Rightarrow y + 4z - 7 = 0 \Rightarrow y + 4z = 7.$$

24. (d) Let the required equation of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

Then, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2} \Rightarrow \frac{2}{a} + \frac{2}{b} + \frac{2}{c} = 1.$

It means that the plane passes through the point (2, 2, 2).

25. (b) We have $|\vec{n}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}.$

$\therefore \quad \vec{r} \cdot \hat{n} = 5 \Rightarrow \vec{r} \cdot \frac{\vec{n}}{\sqrt{14}} = 5$

$\Rightarrow \vec{r} \cdot \vec{n} = 5\sqrt{14}.$

$\therefore \quad (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} - 3\hat{j} + \hat{k}) = 5\sqrt{14} \Rightarrow 2x - 3y + z = 5\sqrt{14}.$

26. (a) Let the required equation of the plane be $5x - 6y + 7z = k$.

Since it passes through the point $A(2, 3, 4)$, we have

$$(5 \times 2) - 6 \times 3 + 7 \times 4 = k \Rightarrow k = (10 - 18 + 28) = 20.$$

Hence, the required equation of the plane is $5x - 6y + 7z = 20$.

27. (b) Let N be the foot of the perpendicular drawn from the point $A(7, 14, 5)$ and perpendicular to the plane $2x + 4y - z = 2$.

Then, the equation of the line PN is $\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1} = \lambda$ (say).

Let the coordinates of N be $N(2\lambda + 7, 4\lambda + 14, -\lambda + 5)$.

Since N lies on the plane $2x + 4y - z = 2$, so

$$2(2\lambda + 7) + 4(4\lambda + 14) - (-\lambda + 5) = 2 \Rightarrow 21\lambda = -63 \Rightarrow \lambda = -3.$$

$\therefore$ required foot of the perpendicular is

$$N(-6 + 7, -12 + 14, 3 + 5), \text{ i.e., } N(1, 2, 8).$$

28. (d) Let the equation of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

Then, it meets the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

$\therefore$ centroid of $\triangle ABC$ is $G\left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right)$, i.e., $G\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$.

$\therefore \left(\frac{a}{3} = \alpha, \frac{b}{3} = \beta \text{ and } \frac{c}{3} = \gamma\right) \Rightarrow a = 3\alpha,\ b = 3\beta \text{ and } c = 3\gamma.$

$\therefore$ the required equation of the plane is

$$\frac{x}{3\alpha} + \frac{y}{3\beta} + \frac{z}{3\gamma} = 1 \Rightarrow \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3.$$

29. (c) The given plane is $2x - 3y + 4z = 12 \Rightarrow \frac{x}{6} + \frac{y}{-4} + \frac{z}{3} = 1.$

$\therefore$ required intercepts are 6, –4, 3.

30. (b) D.r.'s of the given line are 1, –2, –3.

D.r.'s of the noraml to the given plane are 2, –3, 1.

$$\therefore \sin\theta = \frac{|(1\times 2) + (-2)\times(-3) + (-3)\times 1|}{\left\{\sqrt{1^2 + (-2)^2 + (-3)^2}\right\}\left\{\sqrt{2^2 + (-3)^2 + 1^2}\right\}}$$

$$= \frac{5}{14} \Rightarrow \theta = \sin^{-1}\left(\frac{5}{14}\right).$$

31. (d) $\sin\theta = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}$.

Here $\vec{b} = (2\hat{i} + 2\hat{j} + \hat{k})$ and $\vec{n} = (6\hat{i} - 3\hat{j} + 2\hat{k})$.

$$\therefore \sin\theta = \frac{|(2\hat{i} \times 2\hat{j} + \hat{k})\cdot(6\hat{i} - 3\hat{j} + 2\hat{k})|}{\left\{\sqrt{2^2 + 2^2 + 1^2}\right\}\left\{\sqrt{6^2 + (-3)^2 + 2^2}\right\}}$$

$$= \frac{|12 - 6 + 2|}{(\sqrt{9})(\sqrt{49})} = \frac{8}{(3\times 7)} = \frac{8}{21} \Rightarrow \theta = \sin^{-1}\left(\frac{8}{21}\right).$$

32. (b) We know that the perpendicular distance of a point having position vector $\vec{a}$ from the plane $\vec{r}\cdot\vec{n} = q$ is given by

$$p = \frac{|\vec{a}\cdot\vec{n} - q|}{|\vec{n}|}.$$

Here $\vec{a} = (\hat{i} + 2\hat{j} + 5\hat{k})$, $\vec{n} = (-\hat{i} - \hat{j} - \hat{k})$ and $q = 17$.

$$\therefore \quad p = \frac{|(\hat{i} + 2\hat{j} + 5\hat{k})\cdot(-\hat{i} - \hat{j} - \hat{k}) - 17|}{|-\hat{i} - \hat{j} - \hat{k}|}$$

$$= \frac{|(-1) + (-2) + (-5) + (-17)|}{\sqrt{(-1)^2 + (-1)^2 + (-1)^2}} = \frac{25}{\sqrt{3}} \text{ units.}$$

33. (a) The given parallel planes are

$$2x - 3y + 6z = 5 \quad \text{... (i)}$$

$$6x - 9y + 18z + 20 = 0. \quad \text{... (ii)}$$

Let $P(x_1, y_1, z_1)$ be a point on (i). Then, $2x_1 - 3y_1 + 6z_1 = 5$. ... (iii)

Distance between the given planes

= Length of perpendicular from $P(x_1, y_1, z_1)$ on (ii)

$$= \frac{|6x_1 - 9y_1 + 18z_1 + 20|}{\sqrt{6^2 + (-9)^2 + (18)^2}} = \frac{|3(2x_1 - 3y_1 + 6z_1) + 20|}{\sqrt{441}}$$

$$= \frac{|(3 \times 5) + 20|}{21} = \frac{35}{21} \text{ units} = \frac{5}{3} \text{ units} \quad \text{[using (iii)]}$$

34. (c) Clearly the given planes are parallel. These are

$$x + 2y - 2z + 1 = 0 \quad \text{... (i)}$$

$$2x + 4y - 4z + 5 = 0. \quad \text{... (ii)}$$

Let $P(x_1, y_1, z_1)$ be a point on (i). Then, $x_1 + 2y_1 - 2z_1 = -1$.

$$\therefore \quad p = \frac{|2x_1 + 4y_1 - 4z_1 + 5|}{\sqrt{2^2 + 4^2 + (-4)^2}}$$

$$= \frac{|2(x_1 + 2y_1 - 2z_1) + 5|}{\sqrt{36}}$$

$$= \frac{|2 \times (-1) + 5|}{6} = \frac{3}{6} = \frac{1}{2} \text{ units.}$$

35. (d) The equation of a line through the point $P(1, 3, 4)$ and perpendicular to the plane $2x - y + z + 3 = 0$ is

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = k \text{ (say)}$$

A general point on this line is $(2k + 1, -k + 3, k + 4)$.

For some value of k let $N(2k + 1, -k + 3, k + 4)$ be a point of the line lying on the plane. Then,

$$2(2k + 1) - (-k + 3) + (k + 4) + 3 = 0 \Rightarrow 6k = -6 \Rightarrow k = -1.$$

$\therefore$ coordinates of N are $N(-2 + 1, 1 + 4, -1 + 4)$, i.e., $N(-1, 4, 3)$.

Let $Q(\alpha, \beta, \gamma)$ be the image of P in the given plane. Then

$$\frac{1 + \alpha}{2} = -1, \frac{3 + \beta}{2} = 4 \text{ and } \frac{4 + \gamma}{2} = 3. \text{ So, } \alpha = -3, \beta = 5 \text{ and } \gamma = 2.$$

Hence, the required image of $P(1, 3, 4)$ in the given plane is $Q(-3, 5, 2)$.

29. PROBABILITY

CONDITIONAL PROBABILITY AND PROBABILITY OF INDEPENDENT EVENTS

The concept of probability and various results on it were discussed in class XI. In this chapter, we shall be dealing with problems based on conditional probability and probability of independent events.

CONDITIONAL PROBABILITY

Let A and B be the two events associated with the same random experiment. Then, the probability of occurrence of A under the condition B has already occurred and $P(B) \neq 0$, is called conditional probability, denoted by $P(A/B)$.

We define:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \text{ where } P(B) \neq 0$$

$$\text{and } P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}, \text{ where } P(A) \neq 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *A die is rolled. If the outcome is an odd number, what is the probability that it is prime?*

SOLUTION When a die is rolled, the sample space is given by

$S = \{1, 2, 3, 4, 5, 6\}$.

Let A = event of getting a prime number, and

B = event of getting an odd number.

Then, $A = \{2, 3, 5\}$, $B = \{1, 3, 5\}$ and $A \cap B = \{3, 5\}$.

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}, \; P(B) = \frac{n(B)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

$$\text{and } P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{6} = \frac{1}{3}.$$

Suppose B has already occurred and then A occurs.

So, we have to find $P(A/B)$.

Now, $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{(1/3)}{(1/2)} = \left(\dfrac{1}{3} \times \dfrac{2}{1}\right) = \dfrac{2}{3}\cdot$

$P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{(1/3)}{(1/2)} = \left(\dfrac{1}{3} \times \dfrac{2}{1}\right) = \dfrac{2}{3}\cdot$

Hence, the required probability is $\dfrac{2}{3}\cdot$

EXAMPLE 2 *Ten cards numbered* 1 *to* 10 *are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than* 3, *what is the probability that it is an even number?*

SOLUTION Clearly, the sample space is $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Let A = event that the number on the drawn card is even, and

B = event that the number on the drawn card is more than 3.

Then $A = \{2, 4, 6, 8, 10\}$, $B = \{4, 5, 6, 7, 8, 9, 10\}$

and $A \cap B = \{4, 6, 8, 10\}$.

$\therefore\ P(A) = \dfrac{n(A)}{n(S)} = \dfrac{5}{10} = \dfrac{1}{2}$, $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{7}{10}$ and

$P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{4}{10} = \dfrac{2}{5}\cdot$

Suppose B has already occurred and then A occurs.

So, we have to find $P(A/B)$.

Now, $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{(2/5)}{(7/10)} = \left(\dfrac{2}{5} \times \dfrac{10}{7}\right) = \dfrac{4}{7}\cdot$

Hence, the required probability is $\dfrac{4}{7}\cdot$

EXAMPLE 3 *A die is thrown twice and the sum of the numbers appearing is observed to be* 6. *What is the conditional probability that the number* 4 *has appeared at least once?*

SOLUTION We know that when a die is thrown twice, then the sample space has 36 possible outcomes.

Let A = event that 4 appears at least once, and

B = event that the sum of the numbers appearing is 6.

Then,

$A = \{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (1, 4), (2, 4), (3, 4), (5, 4), (6, 4)\}$

and $B = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$.

$\therefore\ A \cap B = \{(2, 4), (4, 2)\}$.

So, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{11}{36}$, $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{5}{36}$

and $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{36} = \frac{1}{18}\cdot$

Suppose B has already occurred and then A occurs.

So, we have to find $P(A/B)$.

Now, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{(1/18)}{(5/36)} = \left(\frac{1}{18} \times \frac{36}{5}\right) = \frac{2}{5}\cdot$

Hence, the required probability is $\frac{2}{5}\cdot$

EXAMPLE 4 *Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (i) the youngest child is a girl, (ii) at least one of the children is a girl?*

SOLUTION We may write the sample space as

$S = \{G_1G_2, G_1B_2, B_1G_2, B_1B_2\}$, where the youngest child appears later.

(i) Let A = event that both the children are girls, and

B = event that the youngest child is a girl.

Then, $A = \{G_1G_2\}$, $B = \{G_1G_2, B_1G_2\}$ and $A \cap B = \{G_1G_2\}$.

$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{1}{4}, \quad P(B) = \frac{n(B)}{n(S)} = \frac{2}{4} = \frac{1}{2}$

and $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{4}\cdot$

Suppose B has already occurred and then A occurs.

So, we have to find $P(A/B)$.

Now, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{(1/4)}{(1/2)} = \left(\frac{1}{4} \times \frac{2}{1}\right) = \frac{1}{2}\cdot$

Hence, the required probability is $\frac{1}{2}\cdot$

(ii) Let A = event that both the children are girls, and

E = event that at least one of the children is a girl.

Then, $A = \{G_1G_2\}$, $E = \{G_1B_2, B_1G_2, G_1G_2\}$ and $A \cap E = \{G_1G_2\}$.

$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{1}{4}, \quad P(E) = \frac{n(E)}{n(S)} = \frac{3}{4}$

and $P(A \cap E) = \frac{n(A \cap E)}{n(S)} = \frac{1}{4}\cdot$

Suppose E has already occurred and then A occurs.

So, we have to find $P(A/E)$.

Now, $P(A/E) = \frac{P(A \cap E)}{P(E)} = \frac{(1/4)}{(3/4)} = \left(\frac{1}{4} \times \frac{4}{3}\right) = \frac{1}{3}\cdot$

Hence, the required probability is $\frac{1}{3}\cdot$

EXAMPLE 5 *An instructor has a question bank consisting of* 300 *easy true/false questions;* 200 *difficult true/false questions;* 500 *easy multiple-choice questions and* 400 *difficult multiple-choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple-choice question?*

SOLUTION Clearly, the sample space consists of 1400 questions.

$\therefore \quad n(S) = 1400.$

Let A = event of selecting an easy question, and

B = event of selecting a multiple-choice question.

Then, $A \cap B$ = event of selecting an easy multiple-choice question.

$\therefore \quad n(A) = (300 + 500) = 800,\ n(B) = (500 + 400) = 900$

and $n(A \cap B) = 500.$

So, $P(A) = \frac{n(A)}{n(S)} = \frac{800}{1400} = \frac{4}{7},\ P(B) = \frac{n(B)}{n(S)} = \frac{900}{1400} = \frac{9}{14}$

and $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{500}{1400} = \frac{5}{14}\cdot$

Suppose B has already occurred and then A occurs.

Thus we have to find $P(A/B)$.

Now, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{(5/14)}{(9/14)} = \left(\frac{5}{14} \times \frac{14}{9}\right) = \frac{5}{9}\cdot$

Hence, the required probability is $\frac{5}{9}\cdot$

EXAMPLE 6 *Two numbers are selected at random from the integers* 1 *through* 9. *If the sum is even, find the probability that both the numbers are odd.*

SOLUTION Out of the numbers from 1 to 9, there are 5 odd numbers and 4 even numbers.

Let A = event of choosing two odd numbers, and

B = event of choosing two numbers whose sum is even.

Then, $n(A)$ = number of ways of choosing 2 odd numbers out of 5 $= {}^5C_2.$

$n(B)$ = number of ways of choosing 2 numbers whose sum is even

$= ({}^4C_2 + {}^5C_2)$ [2 out of 4 even and 2 out of 5 odd].

$n(A \cap B)$ = number of ways of choosing 2 odd numbers out of 5 $= {}^5C_2.$

Suppose B has already occurred and then A occurs.

Then, we have to find $P(A/B)$.

Now, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)} = \frac{{}^5C_2}{{}^4C_2 + {}^5C_2}$

$$= \frac{5 \times 4}{2} \times \frac{2}{(4 \times 3 + 5 \times 4)} = \frac{20}{32} = \frac{5}{8}.$$

Hence, the required probability is $\frac{5}{8}$.

Properties of Conditional Probability

THEOREM 1 *Let A and B be the events of a sample space S of an experiment. Then, prove that* $P(S/B) = P(B/B) = 1$.

PROOF We know that:

$$P(S/B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1 \;\; [\because \; S \cap B = B].$$

And, $P(B/B) = \frac{P(B \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$

Hence, $P(S/B) = P(B/B) = 1.$

THEOREM 2 *Let A and B be the two events of a sample space S and let E be an event such that* $P(E) \neq 0$. *Then, prove that*

$$P[(A \cup B)/E] = P(A/E) + P(B/E) - P[(A \cap B)/E].$$

PROOF We have

$$P[(A \cup B)/E] = \frac{P[(A \cup B) \cap E]}{P(E)}$$

$$= \frac{P[(A \cap E) \cup (B \cap E)]}{P(E)}$$

$$= \frac{P(A \cap E) + P(B \cap E) - P[(A \cap E) \cap (B \cap E)]}{P(E)}$$

$$= \frac{P(A \cap E) + P(B \cap E) - P(A \cap B \cap E)}{P(E)}$$

$$= \frac{P(A \cap E)}{P(E)} + \frac{P(B \cap E)}{P(E)} - \frac{P[(A \cap B) \cap E]}{P(E)}$$

$$= P(A/E) + P(B/E) - P[(A \cap B)/E].$$

Hence, $P[(A \cup B)/E] = P(A/E) + P(B/E) - [(A \cap B)/E].$

COROLLARY *If A and B are disjoint events, prove that*

$$P[(A \cup B)/E] = P(A/E) + P(B/E).$$

PROOF For any events A and B, we have

$$P[(A \cup B)/E] = P(A/E) + P(B/E) - P[(A \cap B)/E].$$

If A and B are disjoint, then $P[(A \cap B)/E] = 0$.

Hence, in this case, we have

$$P[(A \cup B)/E] = P(A/E) + P(B/E).$$

THEOREM 3 *For any events A and B of a sample space S, prove that* $P(\overline{A}/B) = 1 - P(A/B)$, *where* $\overline{A}$ *denotes 'not A'.*

PROOF We know that

$$P(S/B) = 1 \Rightarrow P[(A \cup \overline{A})/B] = 1 \quad [\because S = A \cup \overline{A}]$$

$$\Rightarrow P(A/B) + P(\overline{A}/B) = 1 \quad [\because A \cap \overline{A} = \varphi]$$

$$\Rightarrow P(\overline{A}/B) = 1 - P(A/B).$$

Hence, $P(\overline{A}/B) = 1 - P(A/B)$.

EXAMPLE 7 *If A and B are the two events such that* $P(A) = \frac{3}{5}$, $P(B) = \frac{7}{10}$ *and* $P(A \cup B) = \frac{9}{10}$, *then find (i)* $P(A \cap B)$ *(ii)* $P(A/B)$ *(iii)* $P(B/A)$.

SOLUTION (i) We know that

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= \left(\frac{3}{5} + \frac{7}{10} - \frac{9}{10}\right) = \frac{(6+7-9)}{10} = \frac{4}{10} = \frac{2}{5}.$$

(ii) $P(A/B) = \dfrac{(A \cap B)}{P(B)} = \dfrac{(2/5)}{(7/10)} = \left(\dfrac{2}{5} \times \dfrac{10}{7}\right) = \dfrac{4}{7}.$

(iii) $P(B/A) = \dfrac{(B \cap A)}{P(A)} = \dfrac{P(A \cap B)}{P(A)} = \left(\dfrac{2/5}{3/5}\right) = \left(\dfrac{2}{5} \times \dfrac{5}{3}\right) = \dfrac{2}{3}.$

EXAMPLE 8 *Evaluate* $P(A \cup B)$, *if* $2P(A) = P(B) = \frac{6}{13}$ *and* $P(A/B) = \frac{1}{3}$.

SOLUTION $2P(A) = P(B) = \dfrac{6}{13} \Rightarrow P(A) = \dfrac{3}{13}$ and $P(B) = \dfrac{6}{13}$.

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A \cap B) = P(A/B) \cdot P(B) = \left(\frac{1}{3} \times \frac{6}{13}\right) = \frac{2}{13}.$$

So, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \left(\frac{3}{13} + \frac{6}{13} - \frac{2}{13}\right) = \frac{(3+6-2)}{13} = \frac{7}{13}.$$

Hence, $P(A \cup B) = \dfrac{7}{13}$.

EXAMPLE 9 *Let A and B be the events such that* $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ *and* $P(A \cap B) = \frac{1}{5}$.

Find: (i) $P(A/B)$ *(ii)* $P(B/A)$ *(iii)* $P(A \cup B)$ *(iv)* $P(\bar{B}/\bar{A})$

SOLUTION We have:

(i) $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{(1/5)}{(1/4)} = \left(\frac{1}{5} \times \frac{4}{1}\right) = \frac{4}{5}$.

(ii) $P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} = \frac{(1/5)}{(1/3)} = \left(\frac{1}{5} \times \frac{3}{1}\right) = \frac{3}{5}$.

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \left(\frac{1}{3} + \frac{1}{4} - \frac{1}{5}\right) = \frac{(20 + 15 - 12)}{60} = \frac{23}{60}.$$

(iv) $P(\bar{B}/\bar{A}) = \frac{P(\bar{B} \cap \bar{A})}{P(\bar{A})} = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{A})} = \frac{P(\overline{A \cup B})}{P(\bar{A})}$

$$= \frac{1 - P(A \cup B)}{1 - P(A)} = \frac{\left(1 - \frac{23}{60}\right)}{\left(1 - \frac{1}{3}\right)} = \frac{\left(\frac{37}{60}\right)}{\left(\frac{2}{3}\right)} = \left(\frac{37}{60} \times \frac{3}{2}\right) = \frac{37}{40}.$$

EXERCISE 29A

1. Let A and B be the events such that
   $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$.
   Find (i) $P(A/B)$ (ii) $P(B/A)$ (iii) $P(A \cup B)$ (iv) $P(\bar{B}/\bar{A})$.

2. Let A and B be the events such that
   $P(A) = \frac{5}{11}$, $P(B) = \frac{6}{11}$ and $P(A \cup B) = \frac{7}{11}$.
   Find (i) $P(A \cap B)$ (ii) $P(A/B)$ (iii) $P(B/A)$ (iv) $P(\bar{A}/\bar{B})$.

3. Let A and B be the events such that
   $P(A) = \frac{3}{10}$, $P(B) = \frac{1}{2}$ and $P(B/A) = \frac{2}{5}$.
   Find (i) $P(A \cap B)$ (ii) $P(A \cup B)$ (iii) $P(A/B)$.

4. Let A and B be the events such that
   $2P(A) = P(B) = \frac{5}{13}$ and $P(A/B) = \frac{2}{5}$.
   Find (i) $P(A \cap B)$ (ii) $P(A \cup B)$.

5. A die is rolled. If the outcome is an even number, what is the probability that it is a number greater than 2?

6. A coin is tossed twice. If the outcome is at most one tail, what is the probability that both head and tail have appeared?
7. Three coins are tossed simultaneously. Find the probability that all coins show heads if at least one of the coins shows a head.
8. Two unbiased dice are thrown. Find the probability that the sum of the numbers appearing is 8 or greater, if 4 appears on the first die.
9. A die is thrown twice and the sum of the numbers appearing is observed to be 8. What is the conditional probability that the number 5 has appeared at least once?
10. Two dice were thrown and it is known that the numbers which come up were different. Find the probability that the sum of the two numbers was 5.
11. A coin is tossed and then a die is thrown. Find the probability of obtaining a 6, given that a head came up.
12. A couple has 2 children. Find the probability that both are boys if it is known that (i) one of the children is a boy, and (ii) the elder child is a boy.
13. In a class, 40% students study mathematics; 25% study biology and 15% study both mathematics and biology. One student is selected at random. Find the probability that
    (i) he studies mathematics if it is known that he studies biology
    (ii) he studies biology if it is known that he studies mathematics.
14. The probability that a student selected at random from a class will pass in Hindi is $\frac{4}{5}$ and the probability that he passes in Hindi and English is $\frac{1}{2}\cdot$ What is the probability that he will pass in English if it is known that he has passed in Hindi?
15. The probability that a certain person will buy a shirt is 0.2, the probability that he will buy a coat is 0.3 and the probability that he will buy a shirt given that he buys a coat is 0.4. Find the probability that he will buy both a shirt and a coat.
16. In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.
    (i) Find the probability that he reads neither Hindi nor English newspaper.
    (ii) If he reads Hindi newspaper, what is the probability that he reads English newspaper?
    (iii) If he reads English newspaper, what is the probability that he reads Hindi newspaper?
17. Two integers are selected at random from integers 1 through 11. If the sum is even, find the probability that both the numbers selected are odd.

ANSWERS (EXERCISE 29A)

1. (i) $\frac{4}{9}$ (ii) $\frac{4}{7}$ (iii) $\frac{12}{13}$ (iv) $\frac{1}{6}$ **2.** (i) $\frac{4}{11}$ (ii) $\frac{2}{3}$ (iii) $\frac{4}{5}$ (iv) $\frac{4}{5}$

3. (i) $\frac{3}{25}$ (ii) $\frac{17}{25}$ (iii) $\frac{6}{25}$ **4.** (i) $\frac{2}{13}$ (ii) $\frac{11}{26}$ **5.** $\frac{2}{3}$ **6.** $\frac{2}{3}$ **7.** $\frac{1}{7}$

8. $\frac{1}{2}$ **9.** $\frac{2}{5}$ **10.** $\frac{2}{15}$ **11.** $\frac{1}{6}$ **12.** (i) $\frac{1}{3}$ (ii) $\frac{1}{2}$ **13.** (i) $\frac{3}{5}$ (ii) $\frac{3}{8}$

14. $\frac{5}{8}$ **15.** 0.12 **16.** (i) $\frac{1}{5}$ (ii) $\frac{1}{3}$ (iii) $\frac{1}{2}$ **17.** $\frac{3}{5}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 29A)

8. Let A = event of getting the sum 8 or greater, and
B = event of getting a 4 on the first die.

$\therefore\ A = \{(2,6),(3,5),(3,6),(4,4),(4,5),(4,6),(5,3),(5,4),(5,5),(5,6),$
$(6,2),(6,3),(6,4),(6,5),(6,6)\}$

$B = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$.

$(A \cap B) = \{(4,4),(4,5),(4,6)\}$.

$\therefore\ n(A) = 15, n(B) = 6$ and $n(A \cap B) = 3$.

Hence, the required probability $= P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{3}{6} = \frac{1}{2}$.

10. Let A = event that the sum of two numbers appeared is 5, and
B = event that the two dice show different numbers.

Then, $A = \{(2,3),(3,2),(1,4),(4,1)\}$.

Thus, $n(A) = 4, n(B) = 30, A \cap B = A$ and so $n(A \cap B) = n(A) = 4$.

$\therefore$ the required probability $= P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{4}{30} = \frac{2}{15}$.

11. Let A = event that the die shows a 6, and
B = event that a head comes up.

Then, $A = \{(H,6),(T,6)\}$

and $B = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6)\}$.

Then, $P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{6}$.

13. Let M = event of studying mathematics, and
B = event of studying biology.

Then $P(M) = \frac{40}{100} = \frac{2}{5}$, $P(B) = \frac{25}{100} = \frac{1}{4}$ and $P(M \cap B) = \frac{15}{100} = \frac{3}{20}$.

(i) $P(M/B) = \frac{P(M \cap B)}{P(B)}$.

(ii) $P(B/M) = \frac{P(M \cap B)}{P(M)}$.

14. Given, $P(H) = \frac{4}{5}$ and $P(H \cap E) = \frac{1}{2}$.

$\therefore \quad P(E/H) = \frac{P(E \cap H)}{P(H)} = \frac{P(H \cap E)}{P(H)}$.

15. $P(S) = 0.2, P(C) = 0.3, P(S/C) = 0.4$.

$\therefore \quad P(S/C) = \frac{P(S \cap C)}{P(C)} \Rightarrow P(S \cap C) = P(C) \cdot P(S/C)$.

17. Let A = event of choosing both odd numbers,

B = event that sum of chosen numbers is even.

In integers from 1 to 11, there are 5 even and 6 odd integers.

$$P(A) = \frac{{}^6C_2}{{}^{11}C_2} = \frac{3}{11},\ P(B) = \left(\frac{{}^6C_2 + {}^5C_2}{{}^{11}C_2} \right) = \frac{5}{11}, P(A \cap B) = \frac{{}^6C_2}{{}^{11}C_2} = \frac{3}{11}.$$

$\therefore$ required probability $= P(A/B) = \frac{P(A \cap B)}{P(B)}$.

Multiplication Theorem on Probability

Let A and B be the two events associated with a sample space S. Then, the simultaneous occurrence of two events A and B is denoted by $(A \cap B)$ and also written as AB.

MULTIPLICATION THEOREM *Let A and B be the two events associated with a sample space S. Then, prove that*

$$P(AB) = P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B),$$

provided $P(A) \neq 0$ *and* $P(B) \neq 0$.

PROOF For any events A and B, we have

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \text{ where } P(A) \neq 0.$$

$$\therefore \quad P(A \cap B) = P(B) \cdot P(A/B). \qquad \text{... (i)}$$

Again, $P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$, where $P(B) \neq 0$.

$$\therefore \quad P(A \cap B) = P(A) \cdot P(B/A). \qquad \text{... (ii)}$$

From (i) and (ii), we get

$$P(A \cap B) = P(B) \cdot P(A/B) = P(A) \cdot P(A/B), \text{ where } P(A) \neq 0.$$

Multiplication Rule for Three Events

For any three events A, B, C of the same sample space, we have

$$P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P[C/(A \cap B)]$$
$$= P(A) \cdot P(B/A) \cdot (C/AB).$$

This rule can be extended for four or more events.

SOLVED EXAMPLES

EXAMPLE 1 *An urn contains 8 white and 4 red balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are white?*

SOLUTION Let A and B denote respectively the events that first and second balls both drawn are white.

Then, we have to find $P(A \cap B)$.

Now, $P(A) = P$ (white ball in the first draw) $= \frac{8}{12}\cdot$

After the occurrence of event A, we are left with 7 white and 4 red balls.

The probability of drawing second white ball, given that the first ball drawn is white, is clearly the conditional probability of occurrence of B, given that A has occurred.

$\therefore \quad P(B/A) = \frac{7}{11}\cdot$

By multiplication rule of probability, we have

$$P(A \cap B) = P(A)\cdot P(B/A) = \left(\frac{8}{12} \times \frac{7}{11}\right) = \frac{14}{33}\cdot$$

Hence, the required probability is $\frac{14}{33}\cdot$

EXAMPLE 2 *Three cards are drawn successively without replacement from a pack of 52 well-shuffled cards. What is the probability that first two cards are queens and the third card drawn is a king?*

SOLUTION Let Q denote the event that the card drawn is a queen and K be the event that the card drawn is a king. Then, we have to find $P(QQK)$.

Probability of drawing first queen is $P(Q) = \frac{4}{52}\cdot$

Now, there are 3 queens in remaining 51 cards.

Let $P(Q/Q)$ be the probability of getting the second queen with the condition that one queen has already been drawn.

$\therefore \quad P(Q/Q) = \frac{3}{51}\cdot$

Lastly, $P(K/QQ)$ is probability of third drawn card to be a king, with the condition that two queens have already been drawn.

Now, there are 4 kings in remaining 50 cards.

$\therefore \quad P(K/QQ) = \frac{4}{50}\cdot$

By multiplication law of probability, we have

$$\begin{aligned} P(QQK) &= P(Q \cap Q \cap K) \\ &= P(Q) \cdot P(Q/Q) \cdot P(K/QQ) \\ &= \left(\frac{4}{52} \times \frac{3}{51} \times \frac{4}{50}\right) = \left(\frac{1}{13} \times \frac{1}{17} \times \frac{2}{25}\right) = \frac{2}{5525}. \end{aligned}$$

Hence, the required probability is $\frac{2}{5525}$.

Independent Events

Two events A and B are said to be independent if

$P(A/B) = P(A)$, where $P(B) \neq 0$

and $P(B/A) = P(B)$, where $P(A) \neq 0$.

i.e., the event A does not depend on the occurence of event B and vice versa.

Condition for Independence of Two Events

For any two events A and B, we have

$P(A \cap B) = P(A) \cdot P(B/A)$. ... (i)

If A and B are independent, we have $P(B/A) = P(B)$.

$\therefore$ (i) becomes $\boldsymbol{P(A \cap B) = P(A) \times P(B)}$.

Thus, *two events A and B associated with the same random experiment are said to be independent if* $P(A \cap B) = P(A) \times P(B)$.

NOTE: Two events A and B are said to be dependent if they are not independent, i.e., if $P(A \cap B) \neq P(A) \times P(B)$.

Difference between Two Mutually Exclusive and Independent Events

Two events A and B are said to be mutually exclusive if $A \cap B = \varphi$ and in this case $P(A \cap B) = P(\varphi) = 0$.

Also we know that two events A and B are independent if $P(A \cap B) = P(A) \times P(B)$.

Clearly, two independent events with nonzero probabilities cannot be mutually exclusive.

Also, two mutually exclusive events with nonzero probabilities cannot be mutually independent.

Three events A, B and C are said to be mutually independent, if

$P(A \cap B) = P(A) \times P(B)$, $P(A \cap C) = P(A) \times P(C)$, $P(B \cap C) = P(B) \times P(C)$

and $P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$.

If at least one of the above is not true for three given events A, B and C, then we say that these events are not independent.

EXAMPLE 3 *Let E_1 and E_2 be two events such that $P(E_1) = 0.3$, $P(E_1 \cup E_2) = 0.4$ and $P(E_2) = x$. Find the value of x such that*
(i) E_1 and E_2 are mutually exclusive,
(ii) E_1 and E_2 are independent.

SOLUTION (i) Let E_1 and E_2 be mutually exclusive. Then, $E_1 \cap E_2 = \varphi$.

$\therefore\ P(E_1 \cup E_2) = P(E_1) + P(E_2)$

$\Rightarrow\ 0.4 = 0.3 + x$

$\Rightarrow\ x = 0.1.$

Thus, when E_1 and E_2 mutually exclusive, then $x = 0.1$.

(ii) Let E_1 and E_2 be two independent events. Then,

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2) = 0.3 \times x = 0.3x.$$

$\therefore\ P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$\Rightarrow\ 0.4 = 0.3 + x - 0.3x$

$\Rightarrow\ 0.7x = 0.1$

$\Rightarrow\ x = \dfrac{0.1}{0.7} = \dfrac{1}{7}.$

Thus, when E_1 and E_2 are independent, then $x = \dfrac{1}{7}$.

EXAMPLE 4 *Let E_1 and E_2 are the two independent events such that $P(E_1) = 0.35$ and $P(E_1 \cup E_2) = 0.60$, find $P(E_2)$.*

SOLUTION Let $P(E_2) = x$.

Then, E_1 and E_2 being independent events, we have

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2) = 0.35 \times x = 0.35x.$$

Now, $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$\Rightarrow\ 0.60 = 0.35 + x - 0.35x$

$\Rightarrow\ 0.65x = 0.25$

$\Rightarrow\ x = \dfrac{0.25}{0.65} = \dfrac{25}{65} = \dfrac{5}{13}.$

Hence, $P(E_2) = \dfrac{5}{13}$.

EXAMPLE 5 *A coin is tossed thrice. Let the event E be 'the first throw results in a head', and the event F be 'the last throw results in a tail'. Find whether the events E and F are independent.*

SOLUTION When a coin is tossed three times, the sample space is given by

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}.$$

Now, E = event that the first throw results in a head.

$\therefore\ E = \{HHH, HHT, HTH, HTT\}.$

And, F = event that the last throw results in a tail.

$\therefore\ F = \{HHT, THT, HTT, TTT\}.$

So, $(E \cap F) = \{HHT, HTT\}$.

Clearly, $n(E) = 4$, $n(F) = 4$, $n(E \cap F) = 2$ and $n(S) = 8$.

$\therefore \quad P(E) = \dfrac{n(E)}{n(S)} = \dfrac{4}{8} = \dfrac{1}{2}$, $P(F) = \dfrac{n(F)}{n(S)} = \dfrac{4}{8} = \dfrac{1}{2}$

and $P(E \cap F) = \dfrac{n(E \cap F)}{n(S)} = \dfrac{2}{8} = \dfrac{1}{4}$.

Thus, $P(E \cap F) = P(E) \times P(F)$.

Hence, E and F are independent events.

EXAMPLE 6 *An unbiased die is tossed twice. Find the probability of getting a* 4, 5 *or* 6 *on the first toss and a* 1, 2, 3 *or* 4 *on the second toss.*

SOLUTION In each case, the sample space is given by $S = \{1, 2, 3, 4, 5, 6\}$.

Let E = event of getting a 4, 5 or 6 on the first toss.

And, F = event of getting a 1, 2, 3 or 4 on the second toss.

Then, $P(E) = \dfrac{3}{6} = \dfrac{1}{2}$ and $P(F) = \dfrac{4}{6} = \dfrac{2}{3}$.

Clearly, E and F are independent events.

$\therefore$ required probability $= P(E \cap F) = P(E) \times P(F)$

[$\because$ E and F are independent]

$$= \left(\frac{1}{2} \times \frac{2}{3}\right) = \frac{1}{3}.$$

EXAMPLE 7 *Ramesh appears for an interview for two posts, A and B, for which the selection is independent. The probability for his selection for Post A is* (1/6) *and for Post B, it is* (1/7). *Find the probability that Ramesh is selected for at least one post.* **[CBSE 2001]**

SOLUTION Let E_1 = event that Ramesh is selected for the post A,

and E_2 = event that Ramesh is selected for the post B.

Then, $P(E_1) = \dfrac{1}{6}$ and $P(E_2) = \dfrac{1}{7}$.

Clearly, E_1 and E_2 are independent events.

$\therefore \quad P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \left(\dfrac{1}{6} \times \dfrac{1}{7}\right) = \dfrac{1}{42}$.

$\therefore \quad P$(Ramesh is selected for at least one post)

$$= P(E_1 \cup E_2)$$
$$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$$
$$= \left(\frac{1}{6} + \frac{1}{7} - \frac{1}{42}\right) = \frac{12}{42} = \frac{2}{7}.$$

Hence, the required probability is $\dfrac{2}{7}$.

EXAMPLE 8 A *can solve* 90% *of the problems given in a book, and* B *can solve* 70%. *What is the probability that at least one of them will solve a problem selected at random from the book?*

SOLUTION Let E_1 = event that A solves the problem,

and E_2 = event that B solves the problem.

Then, $P(E_1) = \frac{90}{100} = \frac{9}{10}$ and $P(E_2) = \frac{70}{100} = \frac{7}{10}$.

Clearly, E_1 and E_2 are independent events.

$\therefore \quad P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \left(\frac{9}{10} \times \frac{7}{10}\right) = \frac{63}{100}$.

$\therefore \quad P$(at least one of them will solve the problem)

$= P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= \left(\frac{9}{10} + \frac{7}{10} - \frac{63}{100}\right) = \frac{(90 + 70 - 63)}{100} = \frac{97}{100}$.

Hence, the required probability is 0.97.

EXAMPLE 9 *The probability that* A *hits a target is* (1/3) *and the probability that* B *hits it is* (2/5). *What is the probability that the target will be hit if both A and* B *shoot at it?*

SOLUTION Let E_1 = event that A hits the target,

and E_2 = event that B hits the target.

Then, $P(E_1) = \frac{1}{3}$ and $P(E_2) = \frac{2}{5}$.

Clearly, E_1 and E_2 are independent events.

$\therefore \quad P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \left(\frac{1}{3} \times \frac{2}{5}\right) = \frac{2}{15}$.

$\therefore \quad P$(target is hit) = P(A hits or B hits)

$= P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= \left(\frac{1}{3} + \frac{2}{5} - \frac{2}{15}\right) = \frac{9}{15} = \frac{3}{5}$.

Hence, the required probability is $\frac{3}{5}$.

EXAMPLE 10 A *and* B *appear for an interview for two posts. The probability of* A*'s selection is* (1/3) *and that of* B*'s selection is* (2/5). *Find the probability that only one of them will be selected.*

SOLUTION Let E_1 = event that A is selected,

and E_2 = event that B is selected.

Then, $P(E_1) = \frac{1}{3}$ and $P(E_2) = \frac{2}{5}$

$\Rightarrow \quad P(\overline{E}_1) = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$ and $P(\overline{E}_2) = \left(1 - \frac{2}{5}\right) = \frac{3}{5}$.

$\therefore$ P(event that only one of them is selected)

$= P[(E_1$ and not $E_2)$ or $(E_2$ and not $E_1)]$

$= P[(E_1 \cap \overline{E}_2)$ or $(E_2 \cap \overline{E}_1)]$

$= P(E_1 \cap \overline{E}_2) + P(E_2 \cap \overline{E}_1) \qquad [\because \ (E_1 \cap \overline{E}_2) \cap (E_2 \cap \overline{E}_1) = \phi]$

$= P(E_1) \cdot P(\overline{E}_2) + P(E_2) \cdot P(\overline{E}_1)$

$\left[\because E_1 \text{ and } \overline{E}_2 \text{ are independent, and } E_2 \text{ and } \overline{E}_1 \text{ are independent}\right]$

$$= \left(\frac{1}{3} \times \frac{3}{5}\right) + \left(\frac{2}{5} \times \frac{2}{3}\right) = \left(\frac{1}{5} + \frac{4}{15}\right) = \frac{7}{15}.$$

EXAMPLE 11 A *speaks the truth in* 60% *of the cases, and* B *in* 90% *of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact?* **[CBSE 2001]**

SOLUTION Let E_1 = event that A speaks the truth,

and E_2 = event that B speaks the truth.

Then, $\overline{E}_1$ = event that A tells a lie,

and $\overline{E}_2$ = event that B tells a lie.

Clearly, E_1 and E_2 are independent events.

Also, $(E_1$ and $\overline{E}_2)$ as well as $(\overline{E}_1$ and $E_2)$ are independent.

Now, $P(E_1) = \frac{60}{100} = \frac{3}{5}$; $P(E_2) = \frac{90}{100} = \frac{9}{10}$;

$$P(\overline{E}_1) = \left(1 - \frac{3}{5}\right) = \frac{2}{5} \text{ and } P(\overline{E}_2) = \left(1 - \frac{9}{10}\right) = \frac{1}{10}.$$

$\therefore$ P(A and B contradict each other)

$= P[$(A speaks the truth and B tells a lie)

or (A tells a lie and B speaks the truth)$]$

$= P[(E_1 \cap \overline{E}_2) \cup (\overline{E}_1 \cap E_2)]$

$= P(E_1 \cap \overline{E}_2) + P(\overline{E}_1 \cap E_2) \ [\because (E_1 \cap \overline{E}_2) \cap (\overline{E}_1 \cap E_2) = \phi]$

$= \{P(E_1) \times P(\overline{E}_2)\} + \{P(\overline{E}_1) \times P(E_2)\}$

$$= \left(\frac{3}{5} \times \frac{1}{10}\right) + \left(\frac{2}{5} \times \frac{9}{10}\right) = \left(\frac{3}{50} + \frac{18}{50}\right) = \frac{21}{50}.$$

Percentage of cases in which A and B contradict each other

$$= \left(\frac{21}{50} \times 100\right)\% = 42\%.$$

EXAMPLE 12 *The probabilities of a specific problem being solved independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that*

(i) the problem is solved

(ii) exactly one of them solves the problem.

SOLUTION Let E_1 = event that A solves the problem,

and E_2 = event that B solves the problem.

Then, $P(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{3}$

$$\Rightarrow \ P(\overline{E}_1) = \left(1 - \frac{1}{2}\right) = \frac{1}{2} \text{ and } P(\overline{E}_2) = \left(1 - \frac{1}{3}\right) = \frac{2}{3}.$$

Clearly, E_1 and E_2 are independent events.

$$\therefore \ P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \left(\frac{1}{2} \times \frac{1}{3}\right) = \frac{1}{6}.$$

(i) P(the problem is solved)

= P(at least one of A and B solves the problem)

$= P(E_1 \text{ or } E_2) = P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{6}\right) = \frac{4}{6} = \frac{2}{3}.$

(ii) P(exactly one of them solves the problem)

$= P[(E_1 \text{ and not } E_2) \text{ or } (E_2 \text{ and not } E_1)]$

$= P(E_1 \text{ and not } E_2) + P(E_2 \text{ and not } E_1)$

$= P(E_1 \cap \overline{E}_2) + P(E_2 \cap \overline{E}_1)$

$= P(E_1) \times P(\overline{E}_2) + P(E_2) \times P(\overline{E}_1)$

$= \left(\frac{1}{2} \times \frac{2}{3}\right) + \left(\frac{1}{3} \times \frac{1}{2}\right) = \left(\frac{1}{3} + \frac{1}{6}\right) = \frac{3}{6} = \frac{1}{2}.$

EXAMPLE 13 *Amit and Nisha appear for an interview for two vacancies in a company. The probability of Amit's selection is $\frac{1}{5}$ and that of Nisha's selection is $\frac{1}{6}$. What is the probability that*

(i) both of them are selected?

(ii) only one of them is selected?

(iii) none of them is selected?

SOLUTION Let E_1 = event that Amit is selected,

and E_2 = event that Nisha is selected.

Then, $P(E_1) = \frac{1}{5}$ and $P(E_2) = \frac{1}{6}$.

Clearly, E_1 and E_2 are independent events.

(i) $P(\text{both are selected}) = P(E_1 \cap E_2)$

$= P(E_1) \times P(E_2)$

[$\because$ E_1 and E_2 are independent]

$= \left(\frac{1}{5} \times \frac{1}{6}\right) = \frac{1}{30}\cdot$

(ii) $P(\text{only one of them is selected})$

$= P[(E_1 \text{ and not } E_2) \text{ or } (E_2 \text{ and not } E_1)]$

$= P(E_1 \text{ and not } E_2) + P(E_2 \text{ and not } E_1)$

$= P(E_1) \cdot P(\text{not } E_2) + P(E_2) \cdot P(\text{not } E_1)$

$= P(E_1) \cdot [1 - P(E_2)] + P(E_2) \cdot [1 - P(E_1)]$

$= \frac{1}{5} \cdot \left(1 - \frac{1}{6}\right) + \frac{1}{6} \cdot \left(1 - \frac{1}{5}\right) = \left(\frac{1}{5} \times \frac{5}{6}\right) + \left(\frac{1}{6} \times \frac{4}{5}\right)$

$= \left(\frac{1}{6} + \frac{2}{15}\right) = \frac{9}{30} = \frac{3}{10}\cdot$

(iii) P (none of them is selected)

$= P(\text{not } E_1 \text{ and not } E_2)$

$= P(\text{not } E_1) \text{ and } P(\text{not } E_2)$

$= [1 - P(E_1)] \cdot [1 - P(E_2)]$

$= \left(1 - \frac{1}{5}\right) \cdot \left(1 - \frac{1}{6}\right) = \left(\frac{4}{5} \times \frac{5}{6}\right) = \frac{2}{3}\cdot$

EXAMPLE 14 *Three groups of children contain* 3 *girls and* 1 *boy;* 2 *girls and* 2 *boys; and* 1 *girl and* 3 *boys. One child is selected at random from each group. Find the chance that the three children selected comprise* 1 *girl and* 2 *boys.*

SOLUTION Let G_1, G_2, G_3 be the events of selecting a girl from the first, second and third group respectively, and let B_1, B_2, B_3 be the events of selecting a boy from the first, second and third group respectively. Then,

$$P(G_1) = \frac{3}{4};\ P(G_2) = \frac{2}{4} = \frac{1}{2};\ P(G_3) = \frac{1}{4}\cdot$$

$$P(B_1) = \frac{1}{4};\ P(B_2) = \frac{2}{4} = \frac{1}{2} \text{ and } P(B_3) = \frac{3}{4}\cdot$$

$\therefore$ $P(\text{selecting 1 girl and 2 boys})$

$= P[(G_1B_2B_3) \text{ or } (B_1G_2B_3) \text{ or } (B_1B_2G_3)]$

$= P(G_1B_2B_3) + P(B_1G_2B_3) + P(B_1B_2G_3)$

$= \{P(G_1) \times P(B_2) \times P(B_3)\} + \{P(B_1) \times P(G_2) \times P(B_3)\}$

$+ \{P(B_1) \times P(B_2) \times P(G_3)\}$

$$= \left(\frac{3}{4} \times \frac{1}{2} \times \frac{3}{4}\right) + \left(\frac{1}{4} \times \frac{1}{2} \times \frac{3}{4}\right) + \left(\frac{1}{4} \times \frac{1}{2} \times \frac{1}{4}\right) = \left(\frac{9}{32} + \frac{3}{32} + \frac{1}{32}\right) = \frac{13}{32}\cdot$$

Hence, the chances of selecting 1 girl and 2 boys are $\frac{13}{32}$.

EXAMPLE 15 *A problem is given to three students whose chances of solving it are $1/3$, $2/7$ and $3/8$. What is the probability that the problem will be solved?*

SOLUTION Let the three students be named A, B, and C respectively. Let E_1, E_2, E_3 be the events that the problem is solved by A, B, C respectively. Then,

$$P(E_1) = \frac{1}{3},\ P(E_2) = \frac{2}{7},\ P(E_3) = \frac{3}{8}.$$

$$\therefore\quad P(\overline{E}_1) = \left(1 - \frac{1}{3}\right) = \frac{2}{3};\ P(\overline{E}_2) = \left(1 - \frac{2}{7}\right) = \frac{5}{7} \text{ and}$$

$$P(\overline{E}_3) = \left(1 - \frac{3}{8}\right) = \frac{5}{8}.$$

$\therefore$ P(none solves the problem)

$$= P[(\text{not } E_1) \text{ and } (\text{not } E_2) \text{ and } (\text{not } E_3)]$$
$$= P(\overline{E}_1 \cap \overline{E}_2 \cap \overline{E}_3)$$
$$= P(\overline{E}_1) \times P(\overline{E}_2) \times P(\overline{E}_3) \qquad [\because\ \overline{E}_1, \overline{E}_2, \overline{E}_3 \text{ are independent}]$$
$$= \left(\frac{2}{3} \times \frac{5}{7} \times \frac{5}{8}\right) = \frac{25}{84}.$$

$\therefore$ P(that the problem is solved)

$$= 1 - P(\text{none solves the problem})$$
$$= \left(1 - \frac{25}{84}\right) = \frac{59}{84}.$$

Hence, the required probability is $\frac{59}{84}$.

EXAMPLE 16 *A problem in mathematics is given to three students whose chances of solving it correctly are $1/2$, $1/3$ and $1/4$ respectively. What is the probability that only one of them solves it correctly?* **[CBSE 1999C]**

SOLUTION Let A, B, C be the given students and let E_1, E_2 and E_3 be the events that the problem is solved by A, B, C respectively. Then, $\overline{E}_1$, $\overline{E}_2$ and $\overline{E}_3$ are the events that the given problem is not solved by A, B, C respectively. Then,

$$P(E_1) = \frac{1}{2};\ P(E_2) = \frac{1}{3};\ P(E_3) = \frac{1}{4};$$

$$P(\overline{E}_1) = \left(1 - \frac{1}{2}\right) = \frac{1}{2};\ P(\overline{E}_2) = \left(1 - \frac{1}{3}\right) = \frac{2}{3} \text{ and } P(\overline{E}_3) = \left(1 - \frac{1}{4}\right) = \frac{3}{4}.$$

P(exactly one of them solves the problem)

$$= P[(E_1 \cap \overline{E}_2 \cap \overline{E}_3) \text{ or } (\overline{E}_1 \cap E_2 \cap \overline{E}_3) \text{ or } (\overline{E}_1 \cap \overline{E}_2 \cap E_3)]$$
$$= P(E_1 \cap \overline{E}_2 \cap \overline{E}_3) + P(\overline{E}_1 \cap E_2 \cap \overline{E}_3) + P(\overline{E}_1 \cap \overline{E}_2 \cap E_3)$$

$$= \{P(E_1) \times P(\overline{E}_2) \times P(\overline{E}_3)\} + \{P(\overline{E}_1) \times P(E_2) \times P(\overline{E}_3)\} + \{P(\overline{E}_1) \times P(\overline{E}_2) \times P(E_3)\}$$

$$= \left(\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}\right) + \left(\frac{1}{2} \times \frac{1}{3} \times \frac{3}{4}\right) + \left(\frac{1}{2} \times \frac{2}{3} \times \frac{1}{4}\right)$$

$$= \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{12}\right) = \frac{11}{24}.$$

Hence, the required probability is $\frac{11}{24}$.

EXAMPLE 17 *Three critics review a book. For the three critics, the odds in favour of the book are* (5 : 2), (4 : 3) *and* (3 : 4) *respectively. Find the probability that the majority is in favour of the book.*

SOLUTION Let A, B, C denote the events that the book be favoured by the first, second and third critic respectively. Then,

$$P(A) = \frac{5}{7};\ P(B) = \frac{4}{7};\ P(C) = \frac{3}{7};$$

$$P(\overline{A}) = \left(1 - \frac{5}{7}\right) = \frac{2}{7};\ P(\overline{B}) = \left(1 - \frac{4}{7}\right) = \frac{3}{7} \text{ and } P(\overline{C}) = \left(1 - \frac{3}{7}\right) = \frac{4}{7}.$$

Required probability

$= P$(2 critics favour the book or 3 critics favour the book)

$= P$(2 critics favour the book) + P(3 critics favour the book)

$= P$[{A and B and not C} or {A and C and not B}

or {B and C and not A}] + P(A and B and C)

$$= P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(\overline{A} \cap B \cap C) + P(A \cap B \cap C)$$

$$= \{P(A) \times P(B) \times P(\overline{C})\} + \{P(A) \times P(\overline{B}) \times P(C)\} + \{P(\overline{A}) \times P(B) \times P(C)\} + \{P(A) \times P(B) \times P(C)\}$$

$$= \left(\frac{5}{7} \times \frac{4}{7} \times \frac{4}{7}\right) + \left(\frac{5}{7} \times \frac{3}{7} \times \frac{3}{7}\right) + \left(\frac{2}{7} \times \frac{4}{7} \times \frac{3}{7}\right) + \left(\frac{5}{7} \times \frac{4}{7} \times \frac{3}{7}\right)$$

$$= \left(\frac{80}{343} + \frac{45}{343} + \frac{24}{343} + \frac{60}{343}\right) = \frac{209}{343}.$$

Hence, the required probability is $\frac{209}{343}$.

EXAMPLE 18 *The odds against a man who is* 45 *years old, living till he is* 70 *are* 7 : 5, *and the odds against his wife who is now* 36, *living till she is* 61 *are* 5 : 3. *Find the probability that*

(i) the couple will be alive 25 *years hence*

(ii) at least one of them will be alive 25 *years hence.* **[CBSE 2007]**

SOLUTION Let E_1 = event that the husband will be alive 25 years hence, and

E_2 = event that the wife will be alive 25 years hence.

Then, $P(E_1) = \frac{5}{12}$ and $P(E_2) = \frac{3}{8}$.

$\therefore \quad P(\overline{E}_1) = \left(1 - \frac{5}{12}\right) = \frac{7}{12}$ and $P(\overline{E}_2) = \left(1 - \frac{3}{8}\right) = \frac{5}{8}$.

Clearly, E_1 and E_2 are independent events.

(i) P(the couple will be alive 25 years hence)

$= P(E_1 \text{ and } E_2) = P(E_1 \cap E_2)$

$= P(E_1) \cdot P(E_2) = \left(\frac{5}{12} \times \frac{3}{8}\right) = \frac{5}{32}$.

(ii) P(at least one of them will be alive 25 years hence)

$= 1 - P$ (none will be alive 25 years hence)

$= 1 - P[(\text{not } E_1) \text{ and } (\text{not } E_2)]$

$= 1 - P(\overline{E}_1 \cap \overline{E}_2)$

$= 1 - [P(\overline{E}_1) \cdot P(\overline{E}_2)]$ [$\because$ $\overline{E}_1$ and $\overline{E}_2$ are independent]

$= 1 - \left(\frac{7}{12} \times \frac{5}{8}\right) = \left(1 - \frac{35}{96}\right) = \frac{61}{96}$.

EXAMPLE 19 *A, B and C shoot to hit a target. If* A *hits the target* 4 *times in* 5 *trials;* B *hits it* 3 *times in* 4 *trials and* C *hits it* 2 *times in* 3 *trials, what is the probability that the target is hit by at least* 2 *persons?*

SOLUTION Let E_1, E_2 and E_3 be the events that A hits the target, B hits the target and C hits the target respectively. Then,

$P(E_1) = \frac{4}{5}$, $P(E_2) = \frac{3}{4}$, $P(E_3) = \frac{2}{3}$;

$P(\overline{E}_1) = \left(1 - \frac{4}{5}\right) = \frac{1}{5}$, $P(\overline{E}_2) = \left(1 - \frac{3}{4}\right) = \frac{1}{4}$ and $P(\overline{E}_3) = \left(1 - \frac{2}{3}\right) = \frac{1}{3}$.

Case I A, B, C *all hit the target*

In this case, P(A, B and C all hit the target)

$= P(E_1 \text{ and } E_2 \text{ and } E_3)$

$= P(E_1) \cdot P(E_2) \cdot P(E_3)$ [$\because$ E_1, E_2, E_3 are independent]

$= \left(\frac{4}{5} \times \frac{3}{4} \times \frac{2}{3}\right) = \frac{2}{5}$.

Case II A *and* B *hit but not* C

In this case, P(A and B hit but not C)

$= P(E_1 \text{ and } E_2 \text{ and not } E_3)$

$= P(E_1 \cap E_2 \cap \overline{E}_3)$

$= P(E_1) \cdot P(E_2) \cdot P(\overline{E}_3)$ [$\because$ $E_1, E_2, \overline{E}_3$ are independent]

$= \left(\frac{4}{5} \times \frac{3}{4} \times \frac{1}{3}\right) = \frac{1}{5}.$

Case III A *and* C *both hit but not* B

In this case, P(A and C hit but not B)

$= P(E_1$ and E_3 and $\overline{E}_2)$

$= P(E_1) \cdot P(E_3) \cdot P(\overline{E}_2)$ [$\because$ $E_1, E_3, \overline{E}_2$ are independent]

$= \left(\frac{4}{5} \times \frac{2}{3} \times \frac{1}{4}\right) = \frac{2}{15}.$

Case IV B *and* C *both hit but not* A

In this case, P(B and C hit but not A)

$= P(E_2$ and E_3 and $\overline{E}_1)$

$= P(E_2) \cdot P(E_3) \cdot P(\overline{E}_1)$ [$\because$ $E_2, E_3, \overline{E}_1$ are independent]

$= \left(\frac{3}{4} \times \frac{2}{3} \times \frac{1}{5}\right) = \frac{1}{10}.$

Clearly, all these are mutually exclusive.

Hence, required probability $= \left(\frac{2}{5} + \frac{1}{5} + \frac{2}{15} + \frac{1}{10}\right) = \frac{5}{6}.$

EXERCISE 29B

1. A bag contains 17 tickets, numbered from 1 to 17. A ticket is drawn and then another ticket is drawn without replacing the first one. Find the probability that both the tickets may show even numbers.
2. Two marbles are drawn successively from a box containing 3 black and 4 white marbles. Find the probability that both the marbles are black, if the first marble is not replaced before the second draw.
3. A card is drawn from a well-shuffled deck of 52 cards and without replacing this card, a second card is drawn. Find the probability that the first card is a club and the second card is a spade.
4. There is a box containing 30 bulbs of which 5 are defective. If two bulbs are chosen at random from the box in succession without replacing the first, what is the probability that both the bulbs chosen are defective?
5. A bag contains 10 white and 15 black balls. Two balls are drawn in succession without replacement. What is the probability that the first ball is white and the second is black?
6. An urn contains 5 white and 8 black balls. Two successive drawings of 3 balls at a time are made such that the balls drawn in the first draw are not replaced before the second draw. Find the probability that the first draw gives 3 white balls and the second draw gives 3 black balls.

7. Let E_1 and E_2 be the events such that $P(E_1) = \frac{1}{3}$ and $P(E_2) = \frac{3}{5}\cdot$

   Find:
   (i) $P(E_1 \cup E_2)$, when E_1 and E_2 are mutually exclusive,
   (ii) $P(E_1 \cap E_2)$, when E_1 and E_2 are independent.

8. If E_1 and E_2 are the two events such that $P(E_1) = \frac{1}{4}$, $P(E_2) = \frac{1}{3}$ and $P(E_1 \cup E_2) = \frac{1}{2}$, show that E_1 and E_2 are independent events.

9. If E_1 and E_2 are independent events such that $P(E_1) = 0.3$ and $P(E_2) = 0.4$, find (i) $P(E_1 \cap E_2)$ (ii) $P(E_1 \cap \bar{E}_2)$ (iii) $P(\bar{E}_1 \cap \bar{E}_2)$ (iv) $P(\bar{E}_1 \cap E_2)$.

10. Let A and B be the events such that
    $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\text{not } A \text{ or not } B) = \frac{1}{4}\cdot$
    State whether A and B are
    (i) mutually exclusive, (ii) independent.

11. Kamal and Vimal appeared for an interview for two vacancies. The probability of Kamal's selection is $1/3$ and that of Vimal's selection is $1/5\cdot$ Find the probability that only one of them will be selected.

12. Arun and Ved appeared for an interview for two vacancies. The probability of Arun's selection is $1/4$ and that of Ved's rejection is $2/3\cdot$ Find the probability that at least one of them will be selected.

13. A and B appear for an interview for two vacancies in the same post. The probability of A's selection is $1/6$ and that of B's selection is $1/4\cdot$ Find the probability that
    (i) both of them are selected
    (ii) only one of them is selected
    (iii) none is selected
    (iv) at least one of them is selected.

14. Given the probability that A can solve a problem is $2/3$, and the probability that B can solve the same problem is $3/5$, find the probability that
    (i) at least one of A and B will solve the problem
    (ii) none of the two will solve the problem.

15. A problem is given to three students whose chances of solving it are $1/4$, $1/5$ and $1/6$ respectively. Find the probability that the problem is solved.

16. The probabilities of A, B, C solving a problem are $1/3$, $1/4$ and $1/6$ respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them will solve it.

17. A can hit a target 4 times in 5 shots, B can hit 3 times in 4 shots, and C can hit 2 times in 3 shots. Calculate the probability that
    (i) A, B and C all hit the target
    (ii) B and C hit and A does not hit the target.

18. Neelam has offered physics, chemistry and mathematics in Class XII. She estimates that her probabilities of receiving a grade A in these courses are 0.2, 0.3 and 0.9 respectively. Find the probabilities that Neelam receives
(i) all A grades (ii) no A grade (iii) exactly 2 A grades.

19. An article manufactured by a company consists of two parts X and Y. In the process of manufacture of part X, 8 out of 100 parts may be defective. Similarly, 5 out of 100 parts of Y may be defective. Calculate the probability that the assembled product will not be defective.

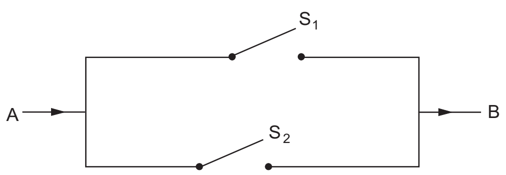
20. A town has two fire-extinguishing engines, functioning independently. The probability of availability of each engine when needed is 0.95. What is the probability that
(i) neither of them is available when needed?
(ii) an engine is available when needed?

21. A machine operates only when all of its three components function. The probabilities of the failures of the first, second and third components are 0.14, 0.10 and 0.05 respectively. What is the probability that the machine will fail? **[CBSE 2005C]**

22. An anti-aircraft gun can take a maximum of 4 shots at an enemy plane moving away from it. The probabilities of hitting the plane at the first, second, third and fourth shots are 0.4, 0.3, 0.2 and 0.1 respectively. What is the probability that at least one shot hits the plane?

23. Let S_1 and S_2 be the two switches and let their probabilities of working be given by $P(S_1) = 4/5$ and $P(S_2) = 9/10 \cdot$ Find the probability that the current flows from the terminal A to terminal B when S_1 and S_2 are installed in series, shown as follows:

24. Let S_1 and S_2 be two the switches and let their probabilities of working be given by $P(S_1) = 2/3$ and $P(S_2) = 3/4$. Find the probability that the current flows from terminal A to terminal B, when S_1 and S_2 are installed in parallel, as shown below:



25. A coin is tossed. If a head comes up, a die is thrown but if a tail comes up, the coin is tossed again. Find the probability of obtaining
(i) two tails (ii) a head and the number 6
(iii) a head and an even number.

ANSWERS (EXERCISE 29B)

1. $\frac{7}{34}$ **2.** $\frac{1}{7}$ **3.** $\frac{13}{204}$ **4.** $\frac{2}{87}$ **5.** $\frac{1}{4}$ **6.** $\frac{7}{429}$ **7.** (i) $\frac{14}{15}$ (ii) $\frac{1}{5}$

9. (i) 0.12 (ii) 0.58 (iii) 0.42 (iv) 0.28 **10.** (i) No (ii) No **11.** $\frac{2}{5}$ **12.** $\frac{1}{2}$

13. (i) $\frac{1}{24}$ (ii) $\frac{1}{3}$ (iii) $\frac{5}{8}$ (iv) $\frac{3}{8}$ **14.** (i) $\frac{13}{15}$ (ii) $\frac{2}{15}$ **15.** $\frac{1}{2}$ **16.** $\frac{31}{72}$

17. (i) $\frac{2}{5}$ (ii) $\frac{1}{10}$ **18.** (i) 0.054 (ii) 0.056 (iii) 0.348 **19.** $\frac{437}{500}$

20. (i) $\frac{1}{400}$ (ii) $\frac{19}{200}$ **21.** 0.2647 **22.** 0.6976 **23.** $\frac{18}{25}$ **24.** $\frac{11}{12}$

25. (i) $\frac{1}{8}$ (ii) $\frac{1}{8}$ (iii) $\frac{3}{8}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 29B)

6. Required probability $=\left\{\frac{{}^5C_3}{{}^{13}C_3}\times\frac{{}^8C_3}{{}^{10}C_3}\right\}=\left(\frac{5\times4\times3}{13\times12\times11}\times\frac{8\times7\times6}{10\times9\times8}\right)$

10. $P(\text{not } A \text{ or not } B)=P(\overline{A} \text{ or } \overline{B})=P(\overline{A}\cup\overline{B})$

$$=P(\overline{A\cap B})=1-P(A\cap B)$$

$\Rightarrow\ 1-P(A\cap B)=\frac{1}{4}\Rightarrow\ P(A\cap B)=\frac{3}{4}\cdot$

(i) Since $P(A\cap B)\neq 0$, so A and B are not mutually exclusive.

(ii) $P(A)\times P(B)=\left(\frac{1}{2}\times\frac{7}{12}\right)=\frac{7}{24}\neq P(A\cap B)$

$\Rightarrow$ A and B are not independent.

11. Let E_1 = event that Kamal is selected,
and E_2 = event that Vimal is selected. Then,

$P(E_1)=\frac{1}{3}, P(E_2)=\frac{1}{5}, P(\overline{E}_1)=\left(1-\frac{1}{3}\right)=\frac{2}{3}$ and $P(\overline{E}_2)=\left(1-\frac{1}{5}\right)=\frac{4}{5}\cdot$

$\therefore$ required probability

$=P[(E_1 \text{ and not } E_2) \text{ or } (E_2 \text{ and not } E_1)]$

$=P[(E_1 \text{ and } \overline{E}_2) \text{ or } (E_2 \text{ and } \overline{E}_1)]$

$=P(E_1\cap\overline{E}_2)+P(E_2\cap\overline{E}_1)$

$=\{P(E_1)\times P(\overline{E}_2)\}+\{P(E_2)\times P(\overline{E}_1)\}.$

12. Let E_1 = event that Arun is selected,
and E_2 = event that Ved is selected.

Then, $P(E_1)=\frac{1}{4}$ and $P(E_2)=\left(1-\frac{2}{3}\right)=\frac{1}{3}\cdot$

Clearly, E_1 and E_2 are independent events.

$\therefore \quad P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \left(\frac{1}{4} \times \frac{1}{3}\right) = \frac{1}{12}.$

Required probability $= P(E_1 \text{ or } E_2) = P(E_1 \cup E_2)$

$$= P(E_1) + P(E_2) - P(E_1 \cap E_2) = \left(\frac{1}{4} + \frac{1}{3} - \frac{1}{12}\right) = \frac{6}{12} = \frac{1}{2}.$$

13. We have $P(A) = \frac{1}{6}, P(B) = \frac{1}{4}.$

$\therefore \quad P(\overline{A}) = 1 - P(A) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}$ and $P(\overline{B}) = 1 - P(B) = \left(1 - \frac{1}{4}\right) = \frac{3}{4}.$

(i) P(both are selected)

$= P(A \cap B) = P(A) \times P(B).$

(ii) P(only one of them is selected)

$= P[(A \text{ and not } B) \text{ or } (B \text{ and not } A)]$

$= P(A \cap \overline{B}) + P(B \cap \overline{A})$

$= \{P(A) \times P(\overline{B})\} + \{P(B) \times P(\overline{A})\}.$

(iii) P(none is selected) $= P(\text{not } A \text{ and not } B)$

$= P(\overline{A} \cap \overline{B}) = P(\overline{A}) \times P(\overline{B}).$

(iv) P(at least one is selected) $= 1 - P$(none is selected).

14. Let E_1 = event that A can solve the problem,
and E_2 = event that B can solve the problem.
Then, E_1 and E_2 are clearly independent events.

$\therefore \quad P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \left(\frac{2}{3} \times \frac{3}{5}\right) = \frac{2}{5}.$

(i) P(at least one of A and B can solve the problem)

$= P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \left(\frac{2}{3} + \frac{3}{5} - \frac{2}{5}\right) = \frac{13}{15}.$

(ii) P(none can solve the problem)

$= 1 - P$(at least one can solve the problem)

$= \left(1 - \frac{13}{15}\right) = \frac{2}{15}.$

15. Let A, B, C be the events of solving the problem by the 1st, 2nd and 3rd student respectively. Then,

$P(A) = \frac{1}{4}, P(B) = \frac{1}{5}$ and $P(C) = \frac{1}{6}$

$\Rightarrow \quad P(\overline{A}) = \left(1 - \frac{1}{4}\right) = \frac{3}{4}; P(\overline{B}) = \left(1 - \frac{1}{5}\right) = \frac{4}{5}$ and $P(\overline{C}) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}.$

$\therefore \quad P$(none solves the problem) $= P[(\text{not } A) \text{ and } (\text{not } B) \text{ and } (\text{not } C)]$

$= P(\overline{A} \text{ and } \overline{B} \text{ and } \overline{C}) = P(\overline{A}) \times P(\overline{B}) \times P(\overline{C}).$

16. Given $P(A) = \frac{1}{3}, P(B) = \frac{1}{4}$ and $P(C) = \frac{1}{6}$

$\Rightarrow \quad P(\overline{A}) = \left(1 - \frac{1}{3}\right) = \frac{2}{3}, P(\overline{B}) = \left(1 - \frac{1}{4}\right) = \frac{3}{4}$ and $P(\overline{C}) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}.$

$\therefore$ required probability

$= P(\text{exactly one of them solves the problem})$

$= P[(A \cap \overline{B} \cap \overline{C}) \text{ or } (\overline{A} \cap B \cap \overline{C}) \text{ or } (\overline{A} \cap \overline{B} \cap C)]$

$= P(A \cap \overline{B} \cap \overline{C}) + P(\overline{A} \cap B \cap \overline{C}) + P(\overline{A} \cap \overline{B} \cap C)$

$= \{P(A) \times P(\overline{B}) \times P(\overline{C})\} + \{P(\overline{A}) \times P(B) \times P(\overline{C})\} + \{P(\overline{A}) \times P(\overline{B}) \times P(C)\}.$

17. $P(\text{A hits}) = \frac{4}{5}$, $P(\text{B hits}) = \frac{3}{4}$ and $P(\text{C hits}) = \frac{2}{3}$.

(i) $P(\text{A hits and B hits and C hits}) = P(\text{A}) \times P(\text{B}) \times P(\text{C})$.

(ii) $P(\text{B hits and C hits and A does not hit})$

$= P(\text{B}) \times P(\text{C}) \times P(\overline{\text{A}})$.

18. (i) $P(\text{getting all } A \text{ grades})$

$= P(A_1 A_2 A_3) = P(A_1) \times P(A_2) \times P(A_3)$.

(ii) $P(\text{getting no } A \text{ grade})$

$= P(\overline{A}_1 \overline{A}_2 \overline{A}_3) = P(\overline{A}_1) \times P(\overline{A}_2) \times P(\overline{A}_3)$.

(iii) $P(\text{getting exactly two } A \text{ grades})$

$= P[(\overline{A}_1 A_2 A_3) \text{ or } (A_1 \overline{A}_2 A_3) \text{ or } (A_1 A_2 \overline{A}_3)]$

$= P(\overline{A}_1 A_2 A_3) + P(A_1 \overline{A}_2 A_3) + P(A_1 A_2 \overline{A}_3)$

$= \{P(\overline{A}_1) \times P(A_2) \times P(A_3)\} + \{P(A_1) \times P(\overline{A}_2) \times P(A_3)\}$

$+ \{P(A_1) \times P(A_2) \times P(\overline{A}_3)\}.$

19. $P(X \text{ is defective}) = \frac{8}{100} = \frac{2}{25}$.

$P(Y \text{ is defective}) = \frac{5}{100} = \frac{1}{20}$.

$P(X \text{ is not defective}) = \left(1 - \frac{2}{25}\right) = \frac{23}{25}$.

$P(Y \text{ is not defective}) = \left(1 - \frac{1}{20}\right) = \frac{19}{20}$.

Required probability $= P(\text{assembled part is not defective})$

$= P(X \text{ is not defective and } Y \text{ is not defective})$

$= \left(\frac{23}{25} \times \frac{19}{20}\right) = \frac{437}{500}$.

20. Let E_1 = event of availability of the first engine.

And, E_2 = event of availability of the second engine.

Then, $P(E_1) = P(E_2) = 0.95$ and $P(\overline{E}_1) = P(\overline{E}_2) = (1 - 0.95) = 0.05$.

(i) $P(\text{neither of them is available when needed})$

$= P(\overline{E}_1 \text{ and } \overline{E}_2) = P(\overline{E}_1) \times P(\overline{E}_2)$.

(ii) $P(\text{an engine is available when needed})$

$= P[(E_1 \text{ and not } E_2) \text{ or } (E_2 \text{ and not } E_1)]$

$= P[(E_1 \text{ and } \overline{E}_2) \text{ or } (E_2 \text{ and } \overline{E}_1)]$

$= P(E_1 \cap \overline{E}_2) + P(E_2 \cap \overline{E}_1)$

$= \{P(E_1) \times P(\overline{E}_2)\} + \{P(E_2) \times P(\overline{E}_1)\}.$

21. Let E_1, E_2, E_3 be the respective events that the 1st, 2nd and 3rd components function. Then,

$$P(\bar{E}_1) = 0.14, P(\bar{E}_2) = 0.10 \text{ and } P(\bar{E}_3) = 0.05$$

$\Rightarrow$ $P(E_1) = (1 - 0.14) = 0.86, P(E_2) = (1 - 0.10) = 0.90$ and $P(E_3) = (1 - 0.05) = 0.95$

$\Rightarrow$ $P(\text{machine fails}) = 1 - P(\text{machine functions})$

$$= 1 - P[(E_1 \text{ and } E_2 \text{ and } E_3)]$$

$$= 1 - [P(E_1) \times P(E_2) \times P(E_3)].$$

22. Let E_1, E_2, E_3, E_4 be the respective events that the plane is hit in the 1st, 2nd, 3rd and 4th shot. Then,

$$P(E_1) = 0.4, P(E_2) = 0.3, P(E_3) = 0.2 \text{ and } P(E_4) = 0.1$$

$\therefore$ $P(\bar{E}_1) = (1 - 0.4) = 0.6, P(\bar{E}_2) = (1 - 0.3) = 0.7,$

$$P(\bar{E}_3) = (1 - 0.2) = 0.8, P(\bar{E}_4) = (1 - 0.1) = 0.9.$$

$\therefore$ $P(\text{at least one shot hits the plane}) = 1 - P(\bar{E}_1 \text{ and } \bar{E}_2 \text{ and } \bar{E}_3 \text{ and } \bar{E}_4)$

$$= 1 - \{P(\bar{E}_1) \times P(\bar{E}_2) \times P(\bar{E}_3) \times P(\bar{E}_4)\}.$$

23. $P(\text{current flows from A to B})$

$$= P(S_1 \text{ is closed and } S_2 \text{ is closed})$$

$$= P(S_1 \text{ and } S_2) = P(S_1) \times P(S_2).$$

24. $P(\text{the current flows}) = P(S_1 \text{ or } S_2)$

$$= P(S_1 \cup S_2) = P(S_1) + P(S_2) - P(S_1 \cap S_2)$$

$$= P(S_1) + P(S_2) - P(S_1) \times P(S_2).$$

25. $S = \{H1, H2, H3, H4, H5, H6, TT, TH\} \rightarrow n(S) = 8.$

(i) $P(\text{two tails}) = \frac{1}{8}\cdot$

(ii) $P(\text{head and the number 6}) = \frac{1}{8}\cdot$

(iii) $P(\text{head and an even number}) = \frac{3}{8}\cdot$

30. BAYES'S THEOREM AND ITS APPLICATIONS

Theorem of Total Probability

THEOREM *Let $E_1, E_2, \ldots, E_n$ be mutually exclusive and exhaustive events associated with a random experiment and let E be an event that occurs with some E_i. Then, prove that*

$$P(E) = \sum_{i=1}^{n} P(E/E_i) \cdot P(E_i).$$

PROOF Let S be the sample space. Then,

$$S = E_1 \cup E_2 \cup \ldots \cup E_n \text{ and } E_i \cap E_j = \phi \text{ for } i \neq j.$$

$$\therefore \quad E = E \cap S = E \cap (E_1 \cup E_2 \cup \ldots \cup E_n)$$

$$= (E \cap E_1) \cup (E \cap E_2) \cup \ldots \cup (E \cap E_n)$$

$$\Rightarrow \quad P(E) = P\{(E \cap E_1) \cup (E \cap E_2) \cup \ldots \cup (E \cap E_n)\}$$

$$= P(E \cap E_1) + P(E \cap E_2) + \ldots + P(E \cap E_n)$$

$$\{\because (E \cap E_1), (E \cap E_2), \ldots, (E \cap E_n) \text{ are pairwise disjoint}\}$$

$$= P(E/E_1) \cdot P(E_1) + P(E/E_2) \cdot P(E_2) + \ldots + P(E/E_n) \cdot P(E_n)$$

[by multiplication theorem]

$$= \sum_{i=1}^{n} P(E/E_i) \cdot P(E_i).$$

EXAMPLE *There are three urns containing 3 white and 2 black balls; 2 white and 3 black balls; 1 black and 4 white balls respectively. There is equal probability of each urn being chosen. One ball is drawn from an urn chosen at random. What is the probability that a white ball is drawn?*

SOLUTION Let E_1, E_2 and E_3 be the events of choosing the first, second and third urn respectively. Then,

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}\cdot$$

Let E be the event that a white ball is drawn. Then,

$$P(E/E_1) = \frac{3}{5},\ P(E/E_2) = \frac{2}{5} \text{ and } P(E/E_3) = \frac{4}{5}\cdot$$

By the theorem of total probability, we have

$$P(E) = P(E/E_1) \cdot P(E_1) + P(E/E_2) \cdot P(E_2) + P(E/E_3) \cdot P(E_3)$$

$$= \left(\frac{3}{5} \times \frac{1}{3} + \frac{2}{5} \times \frac{1}{3} + \frac{4}{5} \times \frac{1}{3}\right) = \left(\frac{1}{5} + \frac{2}{15} + \frac{4}{15}\right) = \frac{9}{15} = \frac{3}{5}\cdot$$

BAYES'S THEOREM *Let $E_1, E_2, \ldots, E_n$ be mutually exclusive and exhaustive events, associated with a random experiment, and let E be any event that occurs with some E_i. Then,*

$$P(E_i/E) = \frac{P(E/E_i) \cdot P(E_i)}{\sum_{i=1}^{n} P(E/E_i) \cdot P(E_i)}; \; i = 1, 2, 3, \ldots, n.$$

PROOF By the theorem of total probability, we have

$$P(E) = \sum_{i=1}^{n} P(E/E_i) \cdot P(E_i) \qquad \ldots \text{(i)}$$

$$\therefore \quad P(E_i/E) = \frac{P(E \cap E_i)}{P(E)} \qquad \text{[by multiplication theorem]}$$

$$= \frac{P(E/E_i) \cdot P(E_i)}{P(E)} \qquad \left[\because \; P(E/E_i) = \frac{P(E \cap E_i)}{P(E_i)}\right]$$

$$= \frac{P(E/E_i) \cdot P(E_i)}{\sum_{i=1}^{n} P(E/E_i) \cdot P(E_i)} \text{ [using (i)].}$$

Hence, $P(E_i/E) = \dfrac{P(E/E_i) \cdot P(E_i)}{\sum_{i=1}^{n} P(E/E_i) \cdot P(E_i)}$.

SOLVED EXAMPLES

EXAMPLE 1 *A factory has three machines, X, Y and Z, producing* 1000, 2000 *and* 3000 *bolts per day respectively. The machine X produces* 1% *defective bolts, Y produces* 1.5% *defective bolts and Z produces* 2% *defective bolts. At the end of the day, a bolt is drawn at random and it is found to be defective. What is the probability that this defective bolt has been produced by the machine X?* **[CBSE 2002]**

SOLUTION Total number of bolts produced in a day

$= (1000 + 2000 + 3000) = 6000.$

Let E_1, E_2 and E_3 be the events of drawing a bolt produced by machines X, Y and Z respectively. Then,

$$P(E_1) = \frac{1000}{6000} = \frac{1}{6}; \; P(E_2) = \frac{2000}{6000} = \frac{1}{3} \text{ and } P(E_3) = \frac{3000}{6000} = \frac{1}{2}.$$

Let E be the event of drawing a defective bolt. Then,

$P(E/E_1)$ = probability of drawing a defective bolt, given that it is produced by the machine X

$= \dfrac{1}{100}$.

$P(E/E_2)$ = probability of drawing a defective bolt, given that it is produced by the machine Y

$$= \frac{1.5}{100} = \frac{15}{1000} = \frac{3}{200}.$$

$P(E/E_3) =$ probability of drawing a defective bolt, given that it is produced by the machine Z

$$= \frac{2}{100} = \frac{1}{50}.$$

Required probability

$= P(E_1/E)$

$=$ probability that the bolt drawn is produced by X, given that it is defective

$$= \frac{P(E_1) \cdot P(E/E_1)}{P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2) + P(E_3) \cdot P(E/E_3)}$$

$$= \frac{\left(\frac{1}{6} \times \frac{1}{100}\right)}{\left(\frac{1}{6} \times \frac{1}{100}\right) + \left(\frac{1}{3} \times \frac{3}{200}\right) + \left(\frac{1}{2} \times \frac{1}{50}\right)}$$

$$= \left(\frac{1}{600} \times \frac{600}{10}\right) = \frac{1}{10} = 0.1.$$

Hence, the required probability is 0.1.

EXAMPLE 2 *In a bolt factory, three machines, A, B, C, manufacture* 25%, 35% *and* 40% *of the total production respectively. Of their respective outputs,* 5%, 4% *and* 2% *are defective. A bolt is drawn at random from the total product and it is found to be defective. Find the probability that it was manufactured by the machine C.* **[CBSE 2006, '10]**

SOLUTION Let E_1, E_2 and E_3 be the events of drawing a bolt produced by machine A, B and C respectively. Then,

$$P(E_1) = \frac{25}{100} = \frac{1}{4},\quad P(E_2) = \frac{35}{100} = \frac{7}{20},\text{ and } P(E_3) = \frac{40}{100} = \frac{2}{5}.$$

Let E be the event of drawing a defective bolt. Then,

$P(E/E_1) =$ probability of drawing a defective bolt, given that it is produced by the machine A

$$= \frac{5}{100} = \frac{1}{20}.$$

$P(E/E_2) =$ probability of drawing a defective bolt, given that it is produced by the machine B

$$= \frac{4}{100} = \frac{1}{25}.$$

$P(E/E_3) =$ probability of drawing a defective bolt, given that it is produced by the machine C

$$= \frac{2}{100} = \frac{1}{50}.$$

Probability that the bolt drawn is manufactured by C, given that it is defective

$$= P(E_3/E)$$

$$= \frac{P(E/E_3) \cdot P(E_3)}{P(E/E_1) \cdot P(E_1) + P(E/E_2) \cdot P(E_2) + P(E/E_3) \cdot P(E_3)}$$

[by Bayes's theorem]

$$= \frac{\left(\frac{1}{50} \times \frac{2}{5}\right)}{\left(\frac{1}{20} \times \frac{1}{4}\right) + \left(\frac{1}{25} \times \frac{7}{20}\right) + \left(\frac{1}{50} \times \frac{2}{5}\right)} = \left(\frac{1}{125} \times \frac{2000}{69}\right) = \frac{16}{69}.$$

Hence, the required probability is $\frac{16}{69}$.

EXAMPLE 3 *A company has two plants to manufacture bicycles. The first plant manufactures* 60% *of the bicycles and the second plant,* 40%. *Also,* 80% *of the bicycles are rated of standard quality at the first plant and* 90% *of standard quality at the second plant. A bicycle is picked up at random and found to be of standard quality. Find the probability that it comes from the second plant.* **[CBSE 2003]**

SOLUTION Let E_1 and E_2 be the events of choosing a bicycle from the first plant and the second plant respectively. Then,

$$P(E_1) = \frac{60}{100} = \frac{3}{5}, \text{ and } P(E_2) = \frac{40}{100} = \frac{2}{5}.$$

Let E be the event of choosing a bicycle of standard quality. Then,

$P(E/E_1)$ = probability of choosing a bicycle of standard quality, given that it is produced by the first plant

$$= \frac{80}{100} = \frac{4}{5}.$$

$P(E/E_2)$ = probability of choosing a bicycle of standard quality, given that it is produced by the second plant

$$= \frac{90}{100} = \frac{9}{10}.$$

The required probability

$P(E_2/E)$ = probability of choosing a bicycle from the second plant, given that it is of standard quality

$$= \frac{P(E_2) \cdot P(E/E_2)}{P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2)} \text{ [by Bayes's theorem]}$$

$$= \frac{\left(\frac{2}{5} \times \frac{9}{10}\right)}{\left(\frac{3}{5} \times \frac{4}{5}\right) + \left(\frac{2}{5} \times \frac{9}{10}\right)} = \frac{3}{7}.$$

EXAMPLE 4 *An insurance company insured* 2000 *scooter drivers,* 4000 *car drivers and* 6000 *truck drivers. The probability of an accident involving a scooter, a car and a truck is* $\frac{1}{100}, \frac{3}{100}$ *and* $\frac{3}{20}$ *respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?* **[CBSE 2000, '02, '08]**

SOLUTION Total number of persons insured = (2000 + 4000 + 6000) = 12000.

Let E_1, E_2 and E_3 be the events of choosing a scooter driver, a car driver and a truck driver respectively. Then,

$$P(E_1) = \frac{2000}{12000} = \frac{1}{6}, P(E_2) = \frac{4000}{12000} = \frac{1}{3} \text{ and } P(E_3) = \frac{6000}{12000} = \frac{1}{2}.$$

Let E be the event of an insured person meeting with an accident. Then,

$P(E/E_1)$ = probability that an insured person meets with an accident, given that he is a scooter driver

$$= \frac{1}{100}.$$

Similarly, $P(E/E_2) = \frac{3}{100}$ and $P(E/E_3) = \frac{3}{20}$.

Required probability

= $P(E_1/E)$ [by Bayes's theorem]

= probability of choosing a scooter driver, given that he meets with an accident

$$= \frac{P(E/E_1) \cdot P(E_1)}{P(E/E_1) \cdot P(E_1) + P(E/E_2) \cdot P(E_2) + P(E/E_3) \cdot P(E_3)}$$

$$= \frac{\left(\frac{1}{100} \times \frac{1}{6}\right)}{\left(\frac{1}{100} \times \frac{1}{6}\right) + \left(\frac{3}{100} \times \frac{1}{3}\right) + \left(\frac{3}{20} \times \frac{1}{2}\right)} = \frac{1}{52}.$$

Hence, the required probability is $\frac{1}{52}$.

EXAMPLE 5 *A doctor is to visit a patient. From past experience, it is known that the probabilities that he will come by train, bus, scooter or by car are respectively* $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ *and* $\frac{2}{5}$ *. The probabilities that he will be late are* $\frac{1}{4}$, $\frac{1}{3}$ *and* $\frac{1}{12}$*, if he comes by train, bus and scooter respectively; but if he comes by car, he will not be late. When he arrives, he is late. What is the probability that he has come by train?* **[CBSE 2008C]**

SOLUTION Let E_1, E_2, E_3 and E_4 be the events that the doctor comes by train, bus, scooter and car respectively. Then,

$$P(E_1) = \frac{3}{10},\quad P(E_2) = \frac{1}{5},\quad P(E_3) = \frac{1}{10} \text{ and } P(E_4) = \frac{2}{5}.$$

Let E be the event that the doctor is late. Then,

$P(E/E_1)$ = probability that the doctor is late, given that he comes by train $= \frac{1}{4}$.

$P(E/E_2)$ = probability that the doctor is late, given that he comes by bus $= \frac{1}{3}$.

$P(E/E_3)$ = probability that the doctor is late, given that he comes by scooter $= \frac{1}{12}$.

$P(E/E_4)$ = probability that the doctor is late, given that he comes by car $= 0$.

Probability that he comes by train, given that he is late

$$= P(E_1/E)$$

$$= \frac{P(E_1)\cdot P(E/E_1)}{P(E_1)\cdot P(E/E_1) + P(E_2)\cdot P(E/E_2) + P(E_3)\cdot P(E/E_3) + P(E_4)\cdot P(E/E_4)}$$

[by Bayes's theorem]

$$= \frac{\left(\frac{3}{10}\times\frac{1}{4}\right)}{\left(\frac{3}{10}\times\frac{1}{4}\right) + \left(\frac{1}{5}\times\frac{1}{3}\right) + \left(\frac{1}{10}\times\frac{1}{12}\right) + \left(\frac{2}{5}\times 0\right)}$$

$$= \left(\frac{3}{40}\times\frac{120}{18}\right) = \frac{1}{2}.$$

Hence, the required probability is $\frac{1}{2}$.

EXAMPLE 6 *A man is known to speak the truth* 3 *out of* 4 *times. He throws a die and reports that it is a six. Find the probability that it is actually a six.*

[CBSE 2005, '11]

SOLUTION In a throw of a die, let

E_1 = event of getting a six,

E_2 = event of not getting a six, and

E = event that the man reports that it is a six.

Then, $P(E_1) = \frac{1}{6}$, and $P(E_2) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}\cdot$

$P(E/E_1)$ = probability that the man reports that six occurs, when six has actually occurred

= probability that the man speaks the truth

$= \frac{3}{4}\cdot$

$P(E/E_2)$ = probability that the man reports that six occurs, when six has not actually occurred

= probability that the man does not speak the truth

$= \left(1 - \frac{3}{4}\right) = \frac{1}{4}\cdot$

Probability of getting a six, given that the man reports it to be six

$= P(E_1/E)$

$$= \frac{P(E/E_1)\cdot P(E_1)}{P(E/E_1)\cdot P(E_1) + P(E/E_2)\cdot P(E_2)} \quad \text{[by Bayes's theorem]}$$

$$= \frac{\left(\frac{3}{4}\times\frac{1}{6}\right)}{\left(\frac{3}{4}\times\frac{1}{6}\right) + \left(\frac{1}{4}\times\frac{5}{6}\right)} = \left(\frac{1}{8}\times 3\right) = \frac{3}{8}\cdot$$

Hence, the required probability is $\frac{3}{8}\cdot$

EXAMPLE 7 *In an examination, an examinee either guesses or copies or knows the answer to a multiple-choice question with four choices. The probability that he makes a guess is* (1/3) *and the probability that he copies the answer is* (1/6)*. The probability that his answer is correct, given that he copied it, is* (1/8)*. The probability that his answer is correct, given that he guessed it, is* (1/4)*. Find the probability that he knew the answer to the question, given that he correctly answered it.*

SOLUTION Let E_1 = event that the examinee guesses the answer,

E_2 = event that he copies the answer,

E_3 = event that he knows the answer, and

E = event that he answers correctly.

Then, $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{6}$, and $P(E_3) = 1 - \left(\frac{1}{3} + \frac{1}{6}\right) = \frac{1}{2}$

[$\because E_1, E_2, E_3$ are mutually exclusive and exhaustive].

$\therefore$ $P(E/E_1)$ = probability that he answers correctly, given that he guesses

$= \frac{1}{4}$.

$P(E/E_2)$ = probability that he answers correctly, given that he copies

$= \frac{1}{8}$.

$P(E/E_3)$ = probability that he answers correctly, given that he knew the answer

$= 1$.

Required probability

$= P(E_3/E)$

$$= \frac{P(E/E_3) \cdot P(E_3)}{P(E/E_1) \cdot P(E_1) + P(E/E_2) \cdot P(E_2) + P(E/E_3) \cdot P(E_3)}$$

[by Bayes's theorem]

$$= \frac{\left(1 \times \frac{1}{2}\right)}{\left(\frac{1}{4} \times \frac{1}{3}\right) + \left(\frac{1}{8} \times \frac{1}{6}\right) + \left(1 \times \frac{1}{2}\right)} = \frac{24}{29}.$$

Hence, the required probability is $\frac{24}{29}$.

EXAMPLE 8 *By examining the chest X-ray, the probability that a person is diagnosed with* TB *when he is actually suffering from it, is* 0.99. *The probability that the doctor incorrectly diagnoses a person to be having* TB, *on the basis of X-ray reports, is* 0.001. *In a certain city,* 1 *in* 1000 *persons suffers from* TB. *A person is selected at random and is diagnosed to have* TB. *What is the chance that he actually has* TB?

SOLUTION Let E = event that the doctor diagnoses TB,

E_1 = event that the person selected is suffering from TB, and

E_2 = event that the person selected is not suffering from TB.

Then, $P(E_1) = \frac{1}{1000}$ and $P(E_2) = \left(1 - \frac{1}{1000}\right) = \frac{999}{1000}$.

$P(E/E_1)$ = probability that TB is diagnosed, when the person actually has TB

$= \frac{99}{100}$.

$P(E/E_2)$ = probability that TB is diagnosed, when the person has no TB

$$= \frac{1}{1000}.$$

Using Bayes's theorem, we have

$P(E_1/E)$ = probability of a person actually having TB, if it is known that he is diagnosed to have TB

$$= \frac{P(E/E_1) \cdot P(E_1)}{P(E/E_1) \cdot P(E_1) + P(E/E_2) \cdot P(E_2)}$$

$$= \frac{\left(\frac{99}{100} \times \frac{1}{1000}\right)}{\left(\frac{99}{100} \times \frac{1}{1000}\right) + \left(\frac{1}{1000} \times \frac{999}{1000}\right)} = \frac{110}{221}.$$

Hence, the required probability is $\frac{110}{221}$.

EXAMPLE 9 *Bag A contains 2 white and 3 red balls, and bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from bag B.* **[CBSE 2004C]**

SOLUTION Let E_1 = event of choosing bag A,

E_2 = event of choosing bag B, and

E = event of drawing a red ball.

Then, $P(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{2}$.

Also, $P(E/E_1)$ = event of drawing a red ball from bag $A = \frac{3}{5}$, and

$P(E/E_2)$ = event of drawing a red ball from bag $B = \frac{5}{9}$.

Probability of drawing a ball from B, it being given that it is red

$= P(E_2/E)$

$$= \frac{P(E/E_2) \cdot P(E_2)}{P(E/E_1) \cdot P(E_1) + P(E/E_2) \cdot P(E_2)} \quad \text{[by Bayes's theorem]}$$

$$= \frac{\left(\frac{5}{9} \times \frac{1}{2}\right)}{\left(\frac{3}{5} \times \frac{1}{2}\right) + \left(\frac{5}{9} \times \frac{1}{2}\right)} = \frac{25}{52}.$$

Hence, the required probability is $\frac{25}{52}$.

EXAMPLE 10 *There are 5 bags, each containing 5 white balls and 3 black balls. Also, there are 6 bags, each containing 2 white balls and 4 black balls. A white ball is drawn at random. Find the probability that this white ball is from a bag of the first group.*

SOLUTION Let E_1 = event of selecting a bag from the first group,
E_2 = event of selecting a bag from the second group, and
E = event of drawing a white ball.

Then, $P(E_1) = \frac{5}{11}$ and $P(E_2) = \frac{6}{11}$.

$P(E/E_1)$ = probability of getting a white ball, given that it is from a bag of the first group
$= \frac{5}{8}$.

$P(E/E_2)$ = probability of getting a white ball, given that it is from a bag of the second group
$= \frac{2}{6} = \frac{1}{3}$.

Probability of getting the ball from a bag of the first group, given that it is white

$$= P(E_1/E)$$

$$= \frac{P(E/E_1)\cdot P(E_1)}{P(E/E_1)\cdot P(E_1) + P(E/E_2)\cdot P(E_2)} \quad \text{[by Bayes's theorem]}$$

$$= \frac{\left(\frac{5}{8}\times\frac{5}{11}\right)}{\left(\frac{5}{8}\times\frac{5}{11}\right)+\left(\frac{1}{3}\times\frac{6}{11}\right)} = \frac{75}{123}.$$

EXAMPLE 11 *Urn A contains 1 white, 2 black and 3 red balls; urn B contains 2 white, 1 black and 1 red ball; and urn C contains 4 white, 5 black and 3 red balls. One urn is chosen at random and two balls are drawn. These happen to be one white and one red. What is the probability that they come from urn A?* **[CBSE 2009]**

SOLUTION Let E_1, E_2, E_3 be the events that the balls are drawn from urn A, urn B and urn C respectively, and let E be the event that the balls drawn are one white and one red. Then, $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$.

$P(E/E_1)$ = probability that the balls drawn are one white and one red, given that the balls are from urn A

$$= \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{3}{15} = \frac{1}{5}.$$

$P(E/E_2)$ = probability that the balls drawn are one white and one red, given that the balls are from urn B

$$= \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} = \frac{2}{6} = \frac{1}{3}.$$

$P(E/E_3)$ = probability that the balls drawn are one white and one red, given that the balls are from urn C

$$= \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{12}{66} = \frac{2}{11}.$$

Probability that the balls drawn are from urn A, it being given that the balls drawn are one white and one red

$= P(E_1/E)$

$$= \frac{P(E/E_1)\cdot P(E_1)}{P(E/E_1)\cdot P(E_1) + P(E/E_2)\cdot P(E_2) + P(E/E_3)\cdot P(E_3)}$$

[by Bayes's theorem]

$$= \frac{\left(\frac{1}{5}\times\frac{1}{3}\right)}{\left(\frac{1}{5}\times\frac{1}{3}\right)+\left(\frac{1}{3}\times\frac{1}{3}\right)+\left(\frac{2}{11}\times\frac{1}{3}\right)}$$

$$= \left(\frac{1}{15}\times\frac{495}{118}\right) = \frac{33}{118}.$$

Hence, the required probability is $\frac{33}{118}$.

EXAMPLE 12 *A card from a pack of* 52 *cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both spades. Find the probability of the lost card being a spade.* **[CBSE 2002]**

SOLUTION Let E_1, E_2, E_3 and E_4 be the events of losing a card of spades, clubs, hearts and diamonds respectively.

Then, $P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{13}{52} = \frac{1}{4}$.

Let E be the event of drawing 2 spades from the remaining 51 cards. Then,

$P(E/E_1)$ = probability of drawing 2 spades, given that a card of spades is missing

$$= \frac{{}^{12}C_2}{{}^{51}C_2} = \frac{(12\times 11)}{2!}\times\frac{2!}{(51\times 50)} = \frac{22}{425}.$$

$P(E/E_2)$ = probability of drawing 2 spades, given that a card of clubs is missing

$$= \frac{{}^{13}C_2}{{}^{51}C_2} = \frac{(13\times 12)}{2!}\times\frac{2!}{(51\times 50)} = \frac{26}{425}.$$

$P(E/E_3)$ = probability of drawing 2 spades, given that a card of hearts is missing

$$= \frac{^{13}C_2}{^{51}C_2} = \frac{26}{425}.$$

$P(E/E_4)$ = probability of drawing 2 spades, given that a card of diamonds is missing

$$= \frac{^{13}C_2}{^{51}C_2} = \frac{26}{425}.$$

$\therefore$ $P(E_1/E)$ = probability of the lost card being a spade, given that 2 spades are drawn from the remaining 51 cards

$$= \frac{P(E_1)\cdot P(E/E_1)}{P(E_1)\cdot P(E/E_1) + P(E_2)\cdot P(E/E_2) + P(E_3)\cdot P(E/E_3) + P(E_4)\cdot P(E/E_4)}$$

$$= \frac{\left(\frac{1}{4}\times\frac{22}{425}\right)}{\left(\frac{1}{4}\times\frac{22}{425}\right)+\left(\frac{1}{4}\times\frac{26}{425}\right)+\left(\frac{1}{4}\times\frac{26}{425}\right)+\left(\frac{1}{4}\times\frac{26}{425}\right)}$$

$$= \frac{22}{100} = 0.22.$$

Hence, the required probability is 0.22.

EXERCISE 30

Long-Answer Questions

1. In a bulb factory, three machines, A, B, C, manufacture 60%, 25% and 15% of the total production respectively. Of their respective outputs, 1%, 2% and 1% are defective. A bulb is drawn at random from the total product and it is found to be defective. Find the probability that it was manufactured by machine C.

2. A company manufactures scooters at two plants, A and B. Plant A produces 80% and plant B produces 20% of the total product. 85% of the scooters produced at plant A and 65% of the scooters produced at plant B are of standard quality. A scooter produced by the company is selected at random and it is found to be of standard quality. What is the probability that it was manufactured at plant A?

3. In a certain college, 4% of boys and 1% of girls are taller than 1.75 metres. Furthermore, 60% of the students are girls. If a student is selected at random and is taller than 1.75 metres, what is the probability that the selected student is a girl?

4. In a class, 5% of the boys and 10% of the girls have an IQ of more than 150. In this class, 60% of the students are boys. If a student is selected at random and found to have an IQ of more than 150, find the probability that the student is a boy.

5. Suppose 5% of men and 0.25% of women have grey hair. A grey-haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females. **[CBSE 2011]**

6. Two groups are competing for the positions on the board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and when the second group wins, the corresponding probability is 0.3. Find the probability that the new product introduced was by the second group.

7. A bag A contains 1 white and 6 red balls. Another bag contains 4 white and 3 red balls. One of the bags is selected at random and a ball is drawn from it, which is found to be white. Find the probability that the ball drawn is from the bag A. **[CBSE 2005]**

8. There are two bags I and II. Bag I contains 3 white and 4 black balls, and bag II contains 5 white and 6 black balls. One ball is drawn at random from one of the bags and is found to be white. Find the probability that it was drawn from bag I. **[CBSE 2005C]**

9. A box contains 2 gold and 3 silver coins. Another box contains 3 gold and 3 silver coins. A box is chosen at random, and a coin is drawn from it. If the selected coin is a gold coin, find the probability that it was drawn from the second box. **[CBSE 2004C]**

10. Three urns A, B and C contain 6 red and 4 white; 2 red and 6 white; and 1 red and 5 white balls respectively. An urn is chosen at random and a ball is drawn. If the ball drawn is found to be red, find the probability that the ball was drawn from the urn A. **[CBSE 2004]**

11. Three urns contain 2 white and 3 black balls; 3 white and 2 black balls, and 4 white and 1 black ball respectively. One ball is drawn from an urn chosen at random and it was found to be white. Find the probability that it was drawn from the first urn.

12. There are three boxes, the first one containing 1 white, 2 red and 3 black balls; the second one containing 2 white, 3 red and 1 black ball and the third one containing 3 white, 1 red and 2 black balls. A box is chosen at random and from it two balls are drawn at random. One ball is red and the other, white. What is the probability that they come from the second box?

13. Urn A contains 7 white and 3 black balls; urn B contains 4 white and 6 black balls; urn C contains 2 white and 8 black balls. One of these urns is chosen at random with probabilities 0.2, 0.6 and 0.2 respectively. From the chosen urn, two balls are drawn at random without replacement. Both the balls happen to be white. Find the probability that the balls drawn are from the urn C.

14. There are 3 bags, each containing 5 white and 3 black balls. Also, there are 2 bags, each containing 2 white and 4 black balls. A white ball is drawn at random. Find the probability that this ball is from a bag of the first group.

15. There are four boxes, A, B, C and D, containing marbles. A contains 1 red, 6 white and 3 black marbles; B contains 6 red, 2 white and 2 black marbles; C contains 8 red, 1 white and 1 black marbles; and D contains 6 white and 4 black marbles. One of the boxes is selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from the box A?

16. A car manufacturing factory has two plants X and Y. Plant X manufactures 70% of the cars and plant Y manufactures 30%. At plant X, 80% of the cars are rated of standard quality and at plant Y, 90% are rated of standard quality. A car is picked up at random and is found to be of standard quality. Find the probability that it has come from plant X. **[CBSE 2005, '06C]**

17. An insurance company insured 2000 scooters and 3000 motorcycles. The probability of an accident involving a scooter is 0.01 and that of a motorcycle is 0.02. An insured vehicle met with an accident. Find the probability that the accidented vehicle was a motorcycle. **[CBSE 2005]**

18. In a bulb factory, machines A, B and C manufacture 60%, 30% and 10% bulbs respectively. Out of these bulbs 1%, 2% and 3% of the bulbs produced respectively by A, B and C are found to be defective. A bulb is picked up at random from the total production and found to be defective. Find the probability that this bulb was produced by the machine A. **[CBSE 2008]**

ANSWERS (EXERCISE 30)

1. $\frac{2}{25}$ **2.** $\frac{68}{81}$ **3.** $\frac{3}{11}$ **4.** $\frac{3}{7}$ **5.** $\frac{2}{3}$ **6.** $\frac{2}{9}$ **7.** $\frac{1}{5}$ **8.** $\frac{33}{68}$

9. $\frac{5}{9}$ **10.** $\frac{36}{61}$ **11.** $\frac{2}{9}$ **12.** $\frac{6}{11}$ **13.** $\frac{1}{40}$ **14.** $\frac{45}{61}$ **15.** $\frac{1}{15}$ **16.** $\frac{56}{83}$

17. $\frac{3}{4}$ **18.** $\frac{2}{5}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 30)

2. Let E_1 = event that the selected scooter is produced at plant A, and
E_2 = event that the selected scooter is produced at plant B.

Then, $P(E_1) = \frac{80}{100} = \frac{4}{5}$ and $P(E_2) = \frac{20}{100} = \frac{1}{5}$.

Let E be the event of choosing a scooter which is of standard quality.

Then, $P(E/E_1) = \frac{85}{100} = \frac{17}{20}$, and $P(E/E_2) = \frac{65}{100} = \frac{13}{20}$.

Probability that the selected scooter was produced at plant A, given that it is of standard quality

$$= P(E_1/E)$$
$$= \frac{P(E/E_1) \cdot P(E_1)}{P(E/E_1) \cdot P(E_1) + P(E/E_2) \cdot P(E_2)}.$$

3. Let E_1 and E_2 be the events of selecting a boy and a girl respectively.

Then, $P(E_1)=\frac{40}{100}=\frac{2}{5}$, and $P(E_2)=\frac{60}{100}=\frac{3}{5}$.

Let E = event that the student selected is taller than 1.75 m.

Then, $P(E/E_1)=\frac{4}{100}=\frac{1}{25}$ and $P(E/E_2)=\frac{1}{100}$.

Probability that the selected student is a girl, given that she is taller than 1.75 m

$$= P(E_2/E)$$
$$= \frac{P(E/E_2)\cdot P(E_2)}{P(E/E_1)\cdot P(E_1)+P(E/E_2)\cdot P(E_2)}.$$

4. Let E_1 and E_2 be the events of selecting a boy and a girl respectively. Then,

$$P(E_1)=\frac{60}{100}=\frac{3}{5}, \text{ and } P(E_2)=\frac{40}{100}=\frac{2}{5}.$$

Let E be the event of selecting a student having an IQ of more than 150.

Then, $P(E/E_1)=\frac{5}{100}=\frac{1}{20}$, and $P(E/E_2)=\frac{10}{100}=\frac{1}{10}$.

Required probability $= P(E_1/E)$

$$= \frac{P(E/E_1)\cdot P(E_1)}{P(E/E_1)\cdot P(E_1)+P(E/E_2)\cdot P(E_2)}.$$

5. Let there be 1000 males and 1000 females.

Let E_1 and E_2 be the events of choosing a male and a female respectively. Then,

$$P(E_1)=\frac{1000}{2000}=\frac{1}{2}, \text{ and } P(E_2)=\frac{1000}{2000}=\frac{1}{2}.$$

Let E be the event of choosing a grey haired person. Then,

$$P(E/E_1)=\frac{50}{1000}=\frac{1}{20}, \text{ and } P(E/E_2)=\frac{25}{1000}=\frac{1}{40}.$$

Probability of selecting a male person, given that the person selected is a grey haired

$$= P(E_1/E)$$
$$= \frac{P(E/E_1)\cdot P(E_1)}{P(E/E_1)\cdot P(E_1)+P(E/E_2)\cdot P(E_2)}.$$

6. Let E_1 = event that the first group wins,
E_2 = event that the second group wins, and
E = event that a new product is introduced.

Then, $P(E_1)=0.6,\ P(E_2)=0.4,$
$P(E/E_1)=0.7,\ P(E/E_2)=0.3.$

Required probability $= P(E_2/E)$

$$= \frac{P(E/E_2)\cdot P(E_2)}{P(E/E_1)\cdot P(E_1)+P(E/E_2)\cdot P(E_2)}.$$

10. Let E_1, E_2 and E_3 be the events of choosing the urns A, B and C respectively, and let E be the event of drawing a red ball. Then,

$$P(E_1)=P(E_2)=P(E_3)=\frac{1}{3}.$$

$$P\left(\frac{E}{E_1}\right)=\frac{6}{10},\ P\left(\frac{E}{E_2}\right)=\frac{2}{8},\text{ and } P\left(\frac{E}{E_3}\right)=\frac{1}{6}.$$

Required probability $= P\left(\frac{E_1}{E}\right)$

$$= \frac{P(E_1) \times P\left(\frac{E}{E_1}\right)}{P(E_1) \times P\left(\frac{E}{E_1}\right) + P(E_2) \times P\left(\frac{E}{E_2}\right) + P(E_3) \times P\left(\frac{E}{E_3}\right)}$$

$$= \frac{\left(\frac{1}{3} \times \frac{6}{10}\right)}{\left(\frac{1}{3} \times \frac{6}{10}\right) + \left(\frac{1}{3} \times \frac{2}{8}\right) + \left(\frac{1}{3} \times \frac{1}{6}\right)} = \frac{36}{61}.$$

13. $P(A) = 0.2,\ P(B) = 0.6$ and $P(C) = 0.2$.

Let E be the event that 2 white balls are drawn. Then,

$$P(E/A) = \frac{{}^7C_2}{{}^{10}C_2};\ P(E/B) = \frac{{}^4C_2}{{}^{10}C_2};\ P(E/C) = \frac{{}^2C_2}{{}^{10}C_2}.$$

$\therefore$ required probability

$$= P(C/E) = \frac{P(E/C) \cdot P(C)}{P(E/A) \cdot P(A) + P(E/B) \cdot P(B) + P(E/C) \cdot P(C)}.$$

14. Let E_1 = event of selecting a bag from the first group, and
E_2 = event of selecting a bag from the second group.

Then, $P(E_1) = \frac{3}{5}$ and $P(E_2) = \frac{2}{5}$.

Let E = event that the ball drawn is white. Then,

$$P(E/E_1) = \frac{5}{8},\ P(E/E_2) = \frac{2}{6} = \frac{1}{3}.$$

$$\therefore\quad P(E_1/E) = \frac{P(E/E_1) \cdot P(E_1)}{P(E/E_1) \cdot P(E_1) + P(E/E_2) \cdot P(E_2)}.$$

15. Let E_1, E_2, E_3, E_4 be the events of selecting boxes A, B, C, D respectively. Then,

$$P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}.$$

Let E = event that the marble drawn is red. Then,

$$P(E/E_1) = \frac{1}{10},\ P(E/E_2) = \frac{6}{10} = \frac{3}{5},\ P(E/E_3) = \frac{8}{10} = \frac{4}{5},\ P(E/E_4) = 0.$$

$$\therefore\quad P(E_1/E) = \frac{P(E/E_1) \cdot P(E_1)}{P(E/E_1) \cdot P(E_1) + P(E/E_2) \cdot P(E_2) + P(E/E_3) \cdot P(E_3) + P(E/E_4) \cdot P(E_4)}$$

16. Let E_1 and E_2 be the events that the car is manufactured by plant X and Y respectively. Let E be the event that the car is of standard quality. Then,

$$P(E_1) = \frac{70}{100} = \frac{7}{10},\ P(E_2) = \frac{30}{100} = \frac{3}{10};$$

$$P(E/E_1) = \frac{80}{100} = \frac{4}{5},\ P(E/E_2) = \frac{90}{100} = \frac{9}{10}.$$

$$\therefore \quad P(E_1/E) = \frac{P(E_1) \times P(E/E_1)}{P(E_1) \times P(E/E_1) + P(E_2) \times P(E/E_2)}$$

$$= \frac{\left(\frac{7}{10} \times \frac{4}{5}\right)}{\left(\frac{7}{10} \times \frac{4}{5}\right) + \left(\frac{3}{10} \times \frac{9}{10}\right)} = \frac{56}{83}.$$

17. Let E_1 and E_2 be the events that an insured vehicle is a scooter and a motorcycle respectively.

Let E be the event that the insured vehicle meets an accident.

$$P(E_1) = \frac{2000}{(2000 + 3000)} = \frac{2}{5}, \quad P(E_2) = \frac{3000}{5000} = \frac{3}{5},$$

$P(E/E_1) = 0.01$ and $P(E/E_2) = 0.02$.

$$\therefore \quad P(E_2/E) = \frac{P(E_2) \cdot P(E/E_2)}{P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2)}$$

$$= \frac{\left(\frac{3}{5} \times 0.02\right)}{\left(\frac{2}{5} \times 0.01\right) + \left(\frac{3}{5} \times 0.02\right)} = \frac{3}{4}.$$

31. PROBABILITY DISTRIBUTION

RANDOM VARIABLE *Let S be the sample space associated with a given random experiment. A real-valued function X which assigns a unique real number X(w) to each $w \in S$, is called a random variable.*

A random variable which can assume only a finite number of values is called a *discrete random variable*.

Example Suppose that a coin is tossed twice.

Then, sample space $S = \{TT, HT, TH, HH\}$.

Consider a real-valued function X on S, defined by

$X : S \to R : X(w)$ = number of heads in w, for all $w \in S$.

Then, X is a random variable such that

$X(TT) = 0$, $X(HT) = 1$, $X(TH) = 1$ and $X(HH) = 2$.

Clearly, range $(X) = \{0, 1, 2\}$.

Probability Distribution of a Random Variable

A description giving the values of a random variable along with the corresponding probabilities is called the probability distribution of the random variable.

If a random variable X takes the values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$ then the probability distribution of X is given by

X	x_1	x_2	x_3	...	...	x_n
$P(X)$	p_1	p_2	p_3	...	...	p_n

REMARK The above probability distribution of X is defined only when

(i) each $p_i \geq 0$ (ii) $\sum_{i=1}^{n} p_i = 1$

Mean and Variance of Random Variables

Let a random variable X assume values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ respectively such that each $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$. Then *mean* of X, denoted by μ [or expected value of X, denoted by $E(X)$], is defined as

$$\mu = E(X) = \sum_{i=1}^{n} x_i\, p_i.$$

And, the *variance*, denoted by σ^2, is defined as

$$\sigma^2 = (\Sigma x_i^2 p_i - \mu^2).$$

Standard deviation, σ, is given by

$$\sigma = \sqrt{\text{variance}}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the mean, variance and standard deviation of the number of tails in two tosses of a coin.*

SOLUTION In two tosses of a coin, the sample space is given by

$$S = \{HH, HT, TH, TT\}.$$

$\therefore \quad n(S) = 4.$

So, every single outcome has a probability $\frac{1}{4}\cdot$

Let X = number of tails in two tosses.

In two tosses, we may have no tail, 1 tail or 2 tails.

So, the possible values of X are 0, 1, 2.

$P(X = 0) = P(\text{getting no tail}) = P(HH) = \frac{1}{4}\cdot$

$P(X = 1) = P(\text{getting 1 tail}) = P(HT \text{ or } TH) = \frac{2}{4} = \frac{1}{2}\cdot$

$P(X = 2) = P(\text{getting 2 tails}) = P(TT) = \frac{1}{4}\cdot$

Hence, the probability distribution of X is given by

$X = x_i$	0	1	2
p_i	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$\therefore \quad \text{mean, } \mu = \Sigma x_i p_i = \left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{1}{2}\right) + \left(2 \times \frac{1}{4}\right) = 1.$

Variance, $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$

$$= \left[\left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{1}{2}\right) + \left(4 \times \frac{1}{4}\right)\right] - 1^2$$

$$= \frac{1}{2}\cdot$$

Standard deviation, $\sigma = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.414}{2} = 0.707.$

EXAMPLE 2 *Find the mean, variance and standard deviation of the number of heads when three coins are tossed.*

SOLUTION Here, $S = \{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH\}$.

$\therefore \quad n(S) = 8.$

So, every single outcome has a probability $\frac{1}{8}\cdot$

Let X = number of heads in tossing three coins.

The number of heads may be 0, 1, 2, or 3.

So, the possible values of X are 0, 1, 2, 3.

$P(X = 0) = P(\text{getting no head}) = P(TTT) = \frac{1}{8}\cdot$

$P(X = 1) = P(\text{getting 1 head}) = P(TTH \text{ or } THT \text{ or } HTT) = \frac{3}{8}\cdot$

$P(X = 2) = P(\text{getting 2 heads}) = P(THH, HTH, HHT) = \frac{3}{8}\cdot$

$P(X = 3) = P(\text{getting 3 heads}) = P(HHH) = \frac{1}{8}\cdot$

Thus, we have the following probability distribution:

$X = x_i$	0	1	2	3
p_i	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\therefore \quad \text{mean, } \mu = \Sigma x_i\, p_i = \left(0 \times \frac{1}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right) = \frac{3}{2}\cdot$$

$$\text{Variance, } \sigma^2 = \Sigma x_i^2 p_i - \mu^2$$

$$= \left[\left(0 \times \frac{1}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(4 \times \frac{3}{8}\right) + \left(9 \times \frac{1}{8}\right) - \frac{9}{4}\right] = \frac{3}{4}\cdot$$

$$\text{Standard deviation, } \sigma = \frac{\sqrt{3}}{2}\cdot$$

EXAMPLE 3 *A die is tossed once. If the random variable X is defined as*

$$X = \begin{cases} 1, & \textit{if the die results in an even number} \\ 0, & \textit{if the die results in an odd number} \end{cases}$$

then find the mean and variance of X.

SOLUTION In tossing a die once, the sample space is given by

$$S = \{1, 2, 3, 4, 5, 6\}.$$

$$\therefore \quad P(\text{getting an even number}) = \frac{3}{6} = \frac{1}{2},$$

$$P(\text{getting an odd number}) = \frac{3}{6} = \frac{1}{2}\cdot$$

As given, X takes the value 0 or 1.

$$P(X=0) = P(\text{getting an odd number}) = \frac{1}{2}\cdot$$

$$P(X=1) = P(\text{getting an even number}) = \frac{1}{2}\cdot$$

Thus, the probability distribution of X is given by

$X = x_i$	0	1
p_i	$\frac{1}{2}$	$\frac{1}{2}$

$\therefore$ mean, $\mu = \Sigma x_i\, p_i = \left(0 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{2}\right) = \frac{1}{2}\cdot$

Variance, $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$

$$= \left(0 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{4}\cdot$$

EXAMPLE 4 *Find the mean, variance and standard deviation of the number of sixes in two tosses of a die.*

SOLUTION In a single toss, we have

probability of getting a six $= \frac{1}{6}$, and

probability of getting a non-six $= \left(1 - \frac{1}{6}\right) = \frac{5}{6}\cdot$

Let X denote the number of sixes in two tosses.

Then, clearly X can assume the value 0, 1, or 2.

$P(X=0) = P$[(non-six in the 1st draw) and (non-six in the 2nd draw)]

$= P(\text{non-six in the 1st draw}) \times P(\text{non-six in the 2nd draw})$

$= \left(\frac{5}{6} \times \frac{5}{6}\right) = \frac{25}{36}\cdot$

$P(X=1) = P$[(six in the 1st draw and non-six in the 2nd draw) or(non-six in the 1st draw and six in the 2nd draw)]

$= P(\text{six in the 1st draw and non-six in the 2nd draw})$
$+ P(\text{non-six in the 1st draw and six in the 2nd draw})$

$= \left(\frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6}\right) = \left(\frac{5}{36} + \frac{5}{36}\right) = \frac{10}{36} = \frac{5}{18}\cdot$

$P(X=2) = P$[six in the 1st draw and six in the 2nd draw]

$= P(\text{six in the 1st draw}) \times P(\text{six in the 2nd draw})$

$$= \left(\frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{36}.$$

Hence, the probability distribution is given by

$X = x_i$	0	1	2
p_i	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

$\therefore$ mean, $\mu = \Sigma x_i \, p_i = \left(0 \times \frac{25}{36}\right) + \left(1 \times \frac{5}{18}\right) + \left(2 \times \frac{1}{36}\right) = \frac{6}{18} = \frac{1}{3}.$

Variance, $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$

$$= \left[\left(0 \times \frac{25}{36}\right) + \left(1 \times \frac{5}{18}\right) + \left(4 \times \frac{1}{36}\right) - \frac{1}{9}\right] = \frac{5}{18}.$$

Standard deviation, $\sigma = \sqrt{\frac{5}{18}} = \frac{1}{3} \cdot \sqrt{\frac{5}{2}}.$

EXAMPLE 5 *Two cards are drawn successively with replacement from a well-shuffled pack of* 52 *cards. Find the mean and variance of the number of kings.*

[CBSE 2005C]

SOLUTION Let X be the random variable. Then,

X = number of kings obtained in two draws.

Clearly, X can assume the value 0, 1 or 2.

P(drawing a king) $= \frac{4}{52} = \frac{1}{13}.$

P(not drawing a king) $= \left(1 - \frac{1}{13}\right) = \frac{12}{13}.$

$P(X = 0) = P$(not a king in the 1st draw and not a king in the 2nd draw)

$$= \left(\frac{12}{13} \times \frac{12}{13}\right) = \frac{144}{169}.$$

$P(X = 1) = P$(a king in the 1st draw and not a king in the 2nd draw) or P(not a king in the 1st draw and a king in the 2nd draw)

$$= \left(\frac{1}{13} \times \frac{12}{13} + \frac{12}{13} \times \frac{1}{13}\right) = \frac{24}{169}.$$

$P(X = 2) = P$(a king in the 1st draw and a king in the 2nd draw)

$$= \left(\frac{1}{13} \times \frac{1}{13}\right) = \frac{1}{169}.$$

Hence, the probability distribution is given by

$X = x_i$	0	1	2
p_i	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

$$\therefore \quad \text{mean, } \mu = \Sigma x_i\, p_i = \left(0 \times \frac{144}{169}\right) + \left(1 \times \frac{24}{169}\right) + \left(2 \times \frac{1}{169}\right) = \frac{2}{13}.$$

$$\text{Variance, } \sigma^2 = \Sigma x_i^2 p_i - \mu^2$$

$$= \left[\left(0 \times \frac{144}{169}\right) + \left(1 \times \frac{24}{169}\right) + \left(4 \times \frac{1}{169}\right) - \frac{4}{169}\right] = \frac{24}{169}.$$

EXAMPLE 6 *Two cards are drawn simultaneously (or successively without replacement) from a well-shuffled pack of* 52 *cards. Find the mean and variance of the number of aces.* **[CBSE 2001, '12]**

SOLUTION Let X be the random variable.

Then, X denotes the number of aces in a draw of 2 cards.

$\therefore$ X can assume the value 0, 1 or 2.

Number of ways of drawing 2 cards out of $52 = C(52, 2)$.

$P(X = 0) = P$(both non-aces, i.e., 2 non-aces out of 48)

$$= \frac{^{48}C_2}{^{52}C_2} = \left(\frac{48 \times 47}{2 \times 1} \times \frac{2}{52 \times 51}\right) = \frac{188}{221}.$$

$P(X = 1) = P$[(one ace out of 4) and (one non-ace out of 48)]

$$= \frac{^{4}C_1 \times {}^{48}C_1}{^{52}C_2} = \left(\frac{4 \times 48}{52 \times 51} \times 2\right) = \frac{32}{221}.$$

$$P(X = 2) = P(\text{both aces}) = \frac{^{4}C_2}{^{52}C_2} = \left(\frac{4 \times 3}{2 \times 1} \times \frac{2 \times 1}{52 \times 51}\right) = \frac{1}{221}.$$

Thus, we have the following probability distribution:

$X = x_i$	0	1	2
p_i	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

$$\therefore \quad \text{mean, } \mu = \Sigma x_i\, p_i = \left(0 \times \frac{188}{221}\right) + \left(1 \times \frac{32}{221}\right) + \left(2 \times \frac{1}{221}\right) = \frac{2}{13}.$$

$$\text{Variance, } \sigma^2 = \Sigma x_i^2 p_i - \mu^2$$

$$= \left(0 \times \frac{188}{221}\right) + \left(1 \times \frac{32}{221}\right) + \left(4 \times \frac{1}{221}\right) - \frac{4}{169}$$

$$= \left(\frac{36}{221} - \frac{4}{169}\right) = \frac{400}{2873}.$$

EXAMPLE 7 *Three defective bulbs are mixed with 7 good ones. Let X be the number of defective bulbs when 3 bulbs are drawn at random. Find the mean and variance of X.*

SOLUTION Let X denote the random variable showing the number of defective bulbs.

Then, X can take the value 0, 1, 2 or 3.

$\therefore$ $P(X=0) = P$(none of the bulbs is defective)

$= P$(all the 3 bulbs are good ones)

$$= \frac{{}^7C_3}{{}^{10}C_3} = \left(\frac{7\times6\times5}{3\times2\times1}\times\frac{3\times2\times1}{10\times9\times8}\right) = \frac{7}{24}.$$

$P(X=1) = P$(1 defective and 2 non-defective bulbs)

$$= \frac{{}^3C_1\times{}^7C_2}{{}^{10}C_3} = \left(3\times\frac{7\times6}{2\times1}\times\frac{3\times2\times1}{10\times9\times8}\right) = \frac{21}{40}.$$

$P(X=2) = P$(2 defective and 1 good one)

$$= \frac{{}^3C_2\times{}^7C_1}{{}^{10}C_3} = \left(\frac{3\times2}{2\times1}\times7\times\frac{3\times2\times1}{10\times9\times8}\right) = \frac{7}{40}.$$

$P(X=3) = P$(3 defective bulbs)

$$= \frac{{}^3C_3}{{}^{10}C_3} = \left(1\times\frac{3\times2\times1}{10\times9\times8}\right) = \frac{1}{120}.$$

Thus, the probability distribution is given by

$X=x_i$	0	1	2	3
p_i	$\frac{7}{24}$	$\frac{21}{40}$	$\frac{7}{40}$	$\frac{1}{120}$

$$\therefore \text{ mean, } \mu = \Sigma x_i p_i = \left(0\times\frac{7}{24}\right)+\left(1\times\frac{21}{40}\right)+\left(2\times\frac{7}{40}\right)+\left(3\times\frac{1}{120}\right) = \frac{9}{10}.$$

Variance, $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$

$$= \left(0\times\frac{7}{24}\right)+\left(1\times\frac{21}{40}\right)+\left(4\times\frac{7}{40}\right)+\left(9\times\frac{1}{120}\right)-\frac{81}{100}$$

$$= \left(\frac{13}{10}-\frac{81}{100}\right) = \frac{49}{100}.$$

EXAMPLE 8 *An urn contains 4 white and 3 red balls. Let X be the number of red balls in a random draw of 3 balls. Find the mean and variance of X.*

SOLUTION When 3 balls are drawn at random, there may be no red ball, 1 red ball, 2 red balls or 3 red balls.

Let X denote the random variable showing the number of red balls in a draw of 3 balls.

Then, X can take the value 0, 1, 2 or 3.

$P(X=0) = P(\text{getting no red ball})$
$= P(\text{getting 3 white balls})$
$= \frac{{}^4C_3}{{}^7C_3} = \left(\frac{4\times3\times2}{3\times2\times1} \times \frac{3\times2\times1}{7\times6\times5}\right) = \frac{4}{35}.$

$P(X=1) = P(\text{getting 1 red and 2 white balls})$
$= \frac{{}^3C_1 \times {}^4C_2}{{}^7C_3} = \left(\frac{3\times4\times3}{2} \times \frac{3\times2\times1}{7\times6\times5}\right) = \frac{18}{35}.$

$P(X=2) = P(\text{getting 2 red and 1 white ball})$
$= \frac{{}^3C_2 \times {}^4C_1}{{}^7C_3} = \left(\frac{3\times2}{2\times1} \times 4 \times \frac{3\times2\times1}{7\times6\times5}\right) = \frac{12}{35}.$

$P(X=3) = P(\text{getting 3 red balls}) = \frac{{}^3C_3}{{}^7C_3} = \frac{1\times3\times2\times1}{7\times6\times5} = \frac{1}{35}.$

Thus, the probability distribution of X is given below.

$X = x_i$	0	1	2	3
p_i	$\frac{4}{35}$	$\frac{18}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

$\therefore$ mean, $\mu = \Sigma x_i p_i = \left(0\times\frac{4}{35}\right) + \left(1\times\frac{18}{35}\right) + \left(2\times\frac{12}{35}\right) + \left(3\times\frac{1}{35}\right) = \frac{9}{7}.$

Variance, $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$

$$= \left[\left(0\times\frac{4}{35}\right) + \left(1\times\frac{18}{35}\right) + \left(4\times\frac{12}{35}\right) + \left(9\times\frac{1}{35}\right) - \frac{81}{49}\right]$$

$$= \left(\frac{15}{7} - \frac{81}{49}\right) = \frac{24}{49}.$$

EXAMPLE 9 *In a game, 3 coins are tossed. A person is paid ₹ 5 if he gets all heads or all tails; and he is supposed to pay ₹ 3 if he gets one head or two heads. What can he expect to win on an average per game?*

SOLUTION In tossing 3 coins, the sample space is given by

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$

$\therefore \quad n(S) = 8.$

$P(\text{getting all heads or all tails}) = \frac{2}{8} = \frac{1}{4}.$

$P(\text{getting one head or 2 heads}) = \frac{6}{8} = \frac{3}{4}.$

Let X = number of rupees the person gets.

Then, possible values of X are 5 and –3.

$P(X=5) = \frac{1}{4}$ and $P(X=-3) = \frac{3}{4}.$

Thus, we have

$X = x_i$	5	–3
p_i	$\frac{1}{4}$	$\frac{3}{4}$

$\therefore$ the required expectations = mean, $\mu = \Sigma x_i \, p_i$

$$= \left(5 \times \frac{1}{4}\right) + (-3) \times \frac{3}{4} = -1,$$

i.e., he loses ₹ 1 per toss.

EXERCISE 31

1. Find the mean (μ), variance (σ^2) and standard deviation (σ) for each of the following probability distributions:

(i)

x	0	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

[CBSE 2007]

(ii)

x_i	1	2	3	4
p_i	0.4	0.3	0.2	0.1

(iii)

x_i	–3	–1	0	2
p_i	0.2	0.4	0.3	0.1

(iv)

x_i	–2	–1	0	1	2
p_i	0.1	0.2	0.4	0.2	0.1

2. Find the mean and variance of the number of heads when two coins are tossed simultaneously.

3. Find the mean and variance of the number of tails when three coins are tossed simultaneously.

4. A die is tossed twice. ‘Getting an odd number on a toss’ is considered a success. Find the probability distribution of number of successes. Also, find the mean and variance of the number of successes.

5. A die is tossed twice. ‘Getting a number greater than 4’ is considered a success. Find the probability distribution of number of successes. Also, find the mean and variance of the number of successes.

6. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability distribution of number of successes. Also, find the mean and variance of number of successes. **[CBSE 2008]**

7. A coin is tossed 4 times. Let X denote the number of heads. Find the probability distribution of X. Also, find the mean and variance of X. **[CBSE 2005]**

8. Let X denote the number of times 'a total of 9' appears in two throws of a pair of dice. Find the probability distribution of X. Also, find the mean, variance and standard deviation of X.

9. There are 5 cards, numbered 1 to 5, one number on each card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two cards drawn. Find the mean and variance of X.

10. Two cards are drawn from a well-shuffled pack of 52 cards. Find the probability distribution of number of kings. Also, compute the variance for the number of kings. **[CBSE 2007]**

11. A box contains 16 bulbs, out of which 4 bulbs are defective. Three bulbs are drawn at random from the box. Let X be the number of defective bulbs drawn. Find the mean and variance of X.

12. 20% of the bulbs produced by a machine are defective. Find the probability distribution of the number of defective bulbs in a sample of 4 bulbs chosen at random. **[CBSE 2004C]**

13. Four bad eggs are mixed with 10 good ones. Three eggs are drawn one by one without replacement. Let X be the number of bad eggs drawn. Find the mean and variance of X.

14. Four rotten oranges are accidentally mixed with 16 good ones. Three oranges are drawn at random from the mixed lot. Let X be the number of rotten oranges drawn. Find the mean and variance of X.

15. Three balls are drawn simultaneously from a bag containing 5 white and 4 red balls. Let X be the number of red balls drawn. Find the mean and variance of X.

16. Two cards are drawn without replacement from a well-shuffled deck of 52 cards. Let X be the number of face cards drawn. Find the mean and variance of X.

17. Two cards are drawn one by one with replacement from a well-shuffled deck of 52 cards. Find the mean and variance of the number of aces.

18. Three cards are drawn successively with replacement from a well-shuffled deck of 52 cards. A random variable X denotes the number of hearts in the three cards drawn. Find the mean and variance of X.

19. Five defective bulbs are accidently mixed with 20 good ones. It is not possible to just look at a bulb and tell whether or not it is defective. Find the probability distribution from this lot. **[CBSE 2007]**

ANSWERS (EXERCISE 31)

1. (i) Mean = 1.2, variance = 0.56, SD =0.74

(ii) Mean = 2, variance = 1, SD = 1

(iii) Mean = – 0.8, variance = 2.6, SD = 1.612

(iv) Mean = 0, variance = 1.2, SD = 1.095

2. Mean = 1, variance = 0.5

3. Mean =1.5 variance = 0.75

4.

$X = x_i$	0	1	2
p_i	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Mean = 1, variance = 0.5

5.

$X = x_i$	0	1	2
p_i	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

Mean = $\frac{2}{3}$, variance = $\frac{4}{9}$

6.

$X = x_i$	0	1	2	3	4
p_i	$\frac{625}{1296}$	$\frac{125}{324}$	$\frac{25}{216}$	$\frac{5}{324}$	$\frac{1}{1296}$

Mean = $\frac{2}{3}$, variance = $\frac{5}{9}$

7.

$X = x_i$	0	1	2	3	4
p_i	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

Mean = 2, variance = 1

8.

$X = x_i$	0	1	2
p_i	$\frac{64}{81}$	$\frac{16}{81}$	$\frac{1}{81}$

Mean = $\frac{2}{9}$, variance = $\frac{16}{81}$, SD = $\frac{4}{9}$

9.

$X = x_i$	3	4	5	6	7	8	9
p_i	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

Mean = 6, variance = 3

10.

$X = x_i$	0	1	2
p_i	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

Variance $= \frac{400}{2873}$

11. Mean $= \frac{3}{4}$, variance $= \frac{39}{80}$

13. Mean $= \frac{6}{7}$, variance $= \frac{30}{49}$

14. Mean $= \frac{3}{5}$, variance $= \frac{68}{125}$

15. Mean $= \frac{4}{3}$, variance $= \frac{5}{9}$

16. Mean $= \frac{20}{13}$, variance $= \frac{1000}{2873}$

17. Mean $= \frac{2}{13}$, variance $= \frac{24}{169}$

18. Mean $= \frac{3}{4}$, variance $= \frac{9}{16}$

19.

X	0	1	2	3	4
$P(X)$	$\frac{969}{2530}$	$\frac{114}{253}$	$\frac{38}{253}$	$\frac{4}{253}$	$\frac{1}{2530}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 31)

2. $S = \{HH, HT, TH, TT\}$.
Let X be the number of heads. Then, $X = 0, 1$ or 2.

$P(X = 0) = P(\text{getting no head}) = \frac{1}{4}$.

$P(X = 1) = P(\text{getting 1 head}) = \frac{2}{4} = \frac{1}{2}$.

$P(X = 2) = P(\text{getting 2 heads}) = \frac{1}{4}$.

$X = x_i$	0	1	2
p_i	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

3.

$X = x_i$	0	1	2	3
p_i	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

4. In a single toss,

$P(\text{success}) = P(\text{getting an odd number}) = \frac{3}{6} = \frac{1}{2}$, and

$P(\text{non-success}) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$.

Let X be the number of successes. Then, $X = 0, 1$ or 2.

$P(X = 0) = P[(\text{non-success in the 1st toss and non-success in the 2nd})]$

$$= \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{4}.$$

$P(X = 1) = P$[(success in the 1st toss and non-success in the 2nd) or (non-success in 1st toss and success in the 2nd)]

$$= \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{2}.$$

$P(X = 2) = P$[(success in the 1st toss and success in the 2nd)]

$$= \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{4}.$$

$X = x_i$	0	1	2
p_i	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

5. In a single toss, $P(\text{success}) = \frac{2}{6} = \frac{1}{3}$ and $P(\text{non-success}) = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$.

$P(X = 0) = P$(non-success in the 1st draw and non-success in the second)

$$= \left(\frac{2}{3} \times \frac{2}{3}\right) = \frac{4}{9}.$$

$P(X = 1) = P$(success in the 1st toss and non-success in the 2nd) or (non-success in the 1st toss and success in the 2nd)]

$$= \left(\frac{1}{3} \times \frac{2}{3}\right) + \left(\frac{2}{3} \times \frac{1}{3}\right) = \frac{4}{9}.$$

$P(X = 2) = P$[(success in the 1st toss and success in the 2nd)]

$$= \left(\frac{1}{3} \times \frac{1}{3}\right) = \frac{1}{9}.$$

$X = x_i$	0	1	2
p_i	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

6. In a single throw, $P(\text{doublet}) = \frac{6}{36} = \frac{1}{6}$, and $P(\text{non-doublet}) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}$.

Let X be the number of doublets. Then, $X = 0, 1, 2$ or 3.

$P(X = 0) = P$(non-doublet in each case)

$$= P(\overline{D}_1\ \overline{D}_2\ \overline{D}_3\ \overline{D}_4) = \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}\right) = \frac{625}{1296}.$$

$P(X = 1) = P$(one doublet)

$$= P(D_1\ \overline{D}_2\ \overline{D}_3\ \overline{D}_4) \text{ or } P(\overline{D}_1\ D_2\ \overline{D}_3\ \overline{D}_4)$$

$$\text{or } P(\overline{D}_1\ \overline{D}_2\ D_3\ \overline{D}_4) \text{ or } P(\overline{D}_1\ \overline{D}_2\ \overline{D}_3\ D_4)$$

$$= \left(\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)$$

$$= \left(4 \times \frac{125}{1296}\right) = \frac{125}{324}.$$

$P(X=2) = P(\text{two doublets})$

$$= P(D_1 \; D_2 \; \overline{D}_3 \; \overline{D}_4) \text{ or } P(D_1 \; \overline{D}_2 \; D_3 \; \overline{D}_4) \text{ or } P(D_1 \; \overline{D}_2 \; \overline{D}_3 \; D_4)$$

$$\text{or } P(\overline{D}_1 \; D_2 \; D_3 \; \overline{D}_4) \text{ or } P(\overline{D}_1 \; D_2 \; \overline{D}_3 \; D_4) \text{ or } P(\overline{D}_1 \; \overline{D}_2 \; D_3 \; D_4)$$

$$= \left(\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + \left(\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)$$

$$+ \left(\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}\right)$$

$$= \left(6 \times \frac{25}{1296}\right) = \frac{25}{216}.$$

$P(X=3) = P(\text{three doublets})$

$$= P(D_1 \; D_2 \; D_3 \; \overline{D}_4) \text{ or } P(D_1 \; D_2 \; \overline{D}_3 \; D_4)$$

$$\text{or } P(D_1 \; \overline{D}_2 \; D_3 \; D_4) \text{ or } P(\overline{D}_1 \; D_2 \; D_3 \; D_4)$$

$$= \left(\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\right)$$

$$= \left(4 \times \frac{5}{1296}\right) = \frac{5}{324}.$$

$P(X=4) = P(\text{four doublets}) = P(D_1 \; D_2 \; D_3 \; D_4)$

$$= \left(\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{1296}.$$

Thus, we have

$X = x_i$	0	1	2	3	4
p_i	$\frac{625}{1296}$	$\frac{125}{324}$	$\frac{25}{216}$	$\frac{5}{324}$	$\frac{1}{1296}$

Now, find μ and σ^2.

7. In a single throw of a coin, $P(H) = \frac{1}{2}$ and $P(\overline{H}) = P(T) = \frac{1}{2}$.

Let X show the number of heads. Then, $X = 0, 1, 2, 3$ or 4.

$$P(X=0) = P(\text{no head}) = P(\overline{H}_1 \; \overline{H}_2 \; \overline{H}_3 \; \overline{H}_4) = \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{16}.$$

$$P(X=1) = P(\text{one head}) = P(H_1 \; \overline{H}_2 \; \overline{H}_3 \; \overline{H}_4) \text{ or } (\overline{H}_1 \; H_2 \; \overline{H}_3 \; \overline{H}_4)$$

$$\text{or } P(\overline{H}_1 \; \overline{H}_2 \; H_3 \; \overline{H}_4) \text{ or } P(\overline{H}_1 \; \overline{H}_2 \; \overline{H}_3 \; H_4)$$

$$= 4\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \left(4 \times \frac{1}{16}\right) = \frac{1}{4}.$$

$$P(X=2) = P(\text{two heads}) = P(H_1 \; H_2 \; \overline{H}_3 \; \overline{H}_4) \text{ or } P(H_1 \; \overline{H}_2 \; H_3 \; \overline{H}_4)$$

$$\text{or } P(H_1 \; \overline{H}_2 \; \overline{H}_3 \; H_4) \text{ or } P(\overline{H}_1 \; H_2 \; H_3 \; \overline{H}_4)$$

$$\text{or } P(\overline{H}_1 \; H_2 \; \overline{H}_3 \; H_4) \text{ or } P(\overline{H}_1 \; \overline{H}_2 \; H_3 \; H_4)$$

$$= \left(6 \times \frac{1}{16}\right) = \frac{3}{8}.$$

$P(X = 3) = P(\text{three heads}) = P(H_1\ H_2\ H_3\ \overline{H}_4)$ or $(H_1\ H_2\ \overline{H}_3\ H_4)$

or $P(H_1\ \overline{H}_2\ H_3\ H_4)$ or $P(\overline{H}_1\ H_2\ H_3\ H_4)$

$$= 4 \times \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \left(4 \times \frac{1}{16}\right) = \frac{1}{4}.$$

$$P(X = 4) = P(\text{four heads}) = P(H_1\ H_2\ H_3\ H_4) = \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{16}.$$

Thus, we have

$X = x_i$	0	1	2	3	4
p_i	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

8. Let $E = \{(3, 6), (6, 3), (4, 5), (5, 4)\}$. So, $n(E) = 4$.

$\therefore \quad P(E) = \frac{4}{36} = \frac{1}{9}$, and $P(\overline{E}) = \left(1 - \frac{1}{9}\right) = \frac{8}{9}.$

Let X be the number of times 'a total of 9' appears in 2 throws.

Then, $X = 0, 1$ or 2.

$$P(X = 0) = P(\overline{E}_1\ \overline{E}_2) = \left(\frac{8}{9} \times \frac{8}{9}\right) = \frac{64}{81}.$$

$$P(X = 1) = P[(E_1\ \overline{E}_2) \text{ or } (\overline{E}_1\ E_2)] = P(E_1\ \overline{E}_2) + P(\overline{E}_1\ E_2)$$

$$= \left(\frac{1}{9} \times \frac{8}{9}\right) + \left(\frac{8}{9} \times \frac{1}{9}\right) = \frac{16}{81}.$$

$$P(X = 2) = P(E_1\ E_2) = P(E_1) \times P(E_2) = \left(\frac{1}{9} \times \frac{1}{9}\right) = \frac{1}{81}.$$

$X = x_i$	0	1	2
p_i	$\frac{64}{81}$	$\frac{16}{81}$	$\frac{1}{81}$

9. $S = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4)\}$

Then $X = 3, 4, 5, 6, 7, 8$ and 9.

$$P(X = 3) = \frac{2}{20} = \frac{1}{10};\ P(X = 4) = \frac{2}{20} = \frac{1}{10};\ P(X = 5) = \frac{4}{20} = \frac{1}{5};$$

$$P(X = 6) = \frac{4}{20} = \frac{1}{5};\ P(X = 7) = \frac{4}{20} = \frac{1}{5};\ P(X = 8) = \frac{2}{20} = \frac{1}{10};$$

$$P(X = 9) = \frac{2}{20} = \frac{1}{10}.$$

Thus, we have

$X = x_i$	3	4	5	6	7	8	9
p_i	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

10. Let X be the number of kings. Then, $X = 0, 1$ or 2.

$P(X = 0) = P(\text{none is a king}) = P(\text{both are non-kings})$

$$= \frac{{}^{48}C_2}{{}^{52}C_2} = \left(\frac{48 \times 47}{2} \times \frac{2}{52 \times 51}\right) = \frac{188}{221}.$$

$P(X = 1) = P(\text{one king and one non-king})$

$$= \frac{{}^{4}C_1 \times {}^{48}C_1}{{}^{52}C_2} = \left(4 \times 48 \times \frac{2}{52 \times 51}\right) = \frac{32}{221}.$$

$P(X = 2) = P(\text{both are kings})$

$$= \frac{{}^{4}C_2}{{}^{52}C_2} = \left(\frac{4 \times 3}{2 \times 1} \times \frac{2 \times 1}{52 \times 51}\right) = \frac{1}{221}.$$

Thus, we have

$X = x_i$	0	1	2
p_i	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

11. Three bulbs drawn one by one without replacement is the same as drawing 3 bulbs simultaneously.

Let X = number of defective bulbs in a lot of 3 bulbs drawn.

Then, $X = 0, 1, 2$ or 3.

$P(X = 0) = P(\text{none of the bulbs is defective})$

$$= \frac{{}^{12}C_3}{{}^{16}C_3} = \left(\frac{12 \times 11 \times 10}{3 \times 2 \times 1} \times \frac{3 \times 2 \times 1}{16 \times 15 \times 14}\right) = \frac{11}{28}.$$

$P(X = 1) = P(\text{1 defective bulb and 2 nondefective bulbs})$

$$= \frac{{}^{4}C_1 \times {}^{12}C_2}{{}^{16}C_3} = \left(\frac{4 \times 12 \times 11}{2 \times 1} \times \frac{3 \times 2 \times 1}{16 \times 15 \times 14}\right) = \frac{33}{70}.$$

$P(X = 2) = P(\text{2 defective bulbs and 1 nondefective bulb})$

$$= \frac{({}^{4}C_2 \times {}^{12}C_1)}{{}^{16}C_3} = \left(\frac{4 \times 3}{2 \times 1} \times 12 \times \frac{3 \times 2 \times 1}{16 \times 15 \times 14}\right) = \frac{9}{70}.$$

$P(X = 3) = P(\text{3 defective bulbs})$

$$= \frac{{}^{4}C_3}{{}^{16}C_3} = \left(\frac{4 \times 3 \times 2}{3 \times 2 \times 1} \times \frac{3 \times 2 \times 1}{16 \times 15 \times 14}\right) = \frac{1}{140}.$$

$X = x_i$	0	1	2	3
p_i	$\frac{11}{28}$	$\frac{33}{70}$	$\frac{9}{70}$	$\frac{1}{140}$

12. Let there be 100 bulbs in all and let X be the number of defective bulbs. Then,

$$P(X=0) = P(\text{none is defective}) = \frac{{}^{80}C_4}{{}^{100}C_4}.$$

$$P(X=1) = P(1 \text{ defective and 3 nondefective})$$
$$= \frac{({}^{20}C_1 \times {}^{80}C_3)}{{}^{100}C_4}.$$

$$P(X=2) = \frac{({}^{20}C_2 \times {}^{80}C_2)}{{}^{100}C_4}; \quad P(X=3) = \frac{({}^{20}C_3 \times {}^{80}C_1)}{{}^{100}C_4}; \quad P(X=4) = \frac{{}^{20}C_4}{{}^{100}C_4}.$$

15. $P(X=0) = \frac{{}^{5}C_3}{{}^{9}C_3} = \frac{5}{42}$, $P(X=1) = \frac{({}^{4}C_1 \times {}^{5}C_2)}{{}^{9}C_3} = \frac{10}{21}$,

$$P(X=2) = \frac{({}^{4}C_2 \times {}^{5}C_1)}{{}^{9}C_3} = \frac{5}{14}, \quad P(X=3) = \frac{{}^{4}C_3}{{}^{9}C_3} = \frac{1}{21}.$$

16. There are 12 face cards (4 kings, 4 queens and 4 jacks).

Clearly, $X = 0$ or 1 or 2.

$$P(X=0) = P(\text{no face card})$$
$$= P(\text{drawing 2 cards out of 40 non-face cards})$$
$$= \frac{{}^{40}C_2}{{}^{52}C_2} = \left(\frac{40 \times 39}{2 \times 1} \times \frac{2 \times 1}{52 \times 51}\right) = \frac{10}{17}.$$

$$P(X=1) = P(1 \text{ face card and 1 non-face card})$$
$$= \frac{({}^{12}C_1 \times {}^{40}C_1)}{{}^{52}C_2} = \left(12 \times 40 \times \frac{2 \times 1}{52 \times 51}\right) = \frac{80}{221}.$$

$$P(X=2) = P(2 \text{ face cards})$$
$$= \frac{{}^{40}C_2}{{}^{52}C_2} = \left(\frac{40 \times 39}{2 \times 1} \times \frac{2 \times 1}{52 \times 51}\right) = \frac{10}{17}.$$

Thus, we have

$X = x_i$	0	1	2
p_i	$\frac{10}{17}$	$\frac{80}{221}$	$\frac{10}{17}$

Now, find the mean and variance.

17. See Example 5.

18. There are 13 hearts and 39 other cards.

Let E = event of drawing a heart.

Then, $P(E) = \frac{13}{52} = \frac{1}{4}$ and $P(\bar{E}) = \left(1 - \frac{1}{4}\right) = \frac{3}{4}$.

Let X = number of hearts in a draw.

Then, $X = 0, 1, 2$ or 3.

$$P(X=0) = P(\overline{E}\,\overline{E}\,\overline{E}) = P(\overline{E}) \times P(\overline{E}) \times P(\overline{E}) = \left(\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}\right) = \frac{27}{64}.$$

$$P(X=1) = P[(E\,\overline{E}\,\overline{E}) \text{ or } (\overline{E}\,E\,\overline{E}) \text{ or } (\overline{E}\,\overline{E}\,E)]$$

$$= P(E\,\overline{E}\,\overline{E}) + P(\overline{E}\,E\,\overline{E}) + P(\overline{E}\,\overline{E}\,E)$$

$$= \left(\frac{1}{4} \times \frac{3}{4} \times \frac{3}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{3}{4} \times \frac{3}{4} \times \frac{1}{4}\right) = \frac{27}{64}.$$

$$P(X=2) = P[(E\,E\,\overline{E}) \text{ or } (E\,\overline{E}\,E) \text{ or } (\overline{E}\,E\,E)]$$

$$= P(E\,E\,\overline{E}) + P(E\,\overline{E}\,E) + P(\overline{E}\,E\,E)$$

$$= \left(\frac{1}{4} \times \frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{1}{4} \times \frac{3}{4} \times \frac{1}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4} \times \frac{1}{4}\right) = \frac{9}{64}.$$

$$P(X=3) = P(E\,E\,E) = P(E) \times P(E) \times P(E) = \left(\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}\right) = \frac{1}{64}.$$

$X = x_i$	0	1	2	3
p_i	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

Find the mean and variance.

32. BINOMIAL DISTRIBUTION

SUCCESS AND FAILURE IN AN EXPERIMENT There are certain kinds of experiments which have two possible outcomes. One of these two outcomes is called a *success*, while the other is called a *failure*.

For example, in tossing a coin, we get either a head or a tail. If getting head is taken as a success then getting a tail is a failure.

BERNOULLI'S THEOREM *Let there be n independent trials in an experiment and let the random variable X denote the number of successes in these trials. Let the probability of getting a success in a single trial be p and that of getting a failure be q so that* $p+q=1$. *Then,*

$$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)}.$$

PROOF Let us denote a success by S and a failure by F.
Number of ways of getting r successes in n trials $= {}^nC_r$.

$$\therefore \quad P(X=r) = {}^nC_r \cdot \left\{ \underbrace{SSS \ldots S}_{r \text{ times}} \text{ and } \underbrace{FFF \ldots F}_{(n-r) \text{ times}} \right\}$$

$$= {}^nC_r \cdot \{P(S) \cdot P(S) \ldots r \text{ times}\} \times \{P(F) \cdot P(F) \ldots (n-r) \text{ times}\}$$

$$= {}^nC_r \cdot (p \cdot p \cdot p \ldots r \text{ times}) \times [q \cdot q \cdot q \ldots (n-r) \text{ times}]$$

$$= {}^nC_r \cdot p^r \cdot q^{(n-r)}.$$

Hence, $P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)}$.

REMARK We have

$$P(X=0) = q^n;\ P(X=1) = npq^{(n-1)};\ P(X=2) = {}^nC_2 \cdot p^2 \cdot q^{(n-2)}, \text{ etc.}$$

The probability distribution of X may be expressed as

$$\begin{pmatrix} X: & 0 & 1 & \ldots & r \\ P(X): & q^n & npq^{(n-1)} & \ldots & C_r \cdot p^r \cdot q^{(n-r)} \end{pmatrix}.$$

This distribution is called a *binomial distribution*.

Conditions for the Applicability of a Binomial Distribution

(i) The experiment is performed for a finite and fixed number of trials.

(ii) Each trial must give either a success or a failure.

(iii) The probability of a success in each trial is the same.

SOLVED EXAMPLES

EXAMPLE 1 *A coin is tossed 4 times. If X is the number of heads observed, find the probability distribution of X.*

SOLUTION When a coin is tossed, we have $S = \{H, T\}$.

$$P(\text{getting a head}) = \frac{1}{2}\text{, and}$$

$$P(\text{not getting a head}) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}.$$

Let X be the random variable denoting the number of heads.

In 4 trials, we may get 0 or 1 or 2 or 3 or 4 heads.

So, X may assume the values 0, 1, 2, 3, 4.

$$P(X = 0) = {}^4C_0 \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^{(4-0)} = \frac{1}{16}.$$

$$P(X = 1) = {}^4C_1 \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^{(4-1)} = \frac{1}{4}.$$

$$P(X = 2) = {}^4C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{(4-2)} = \frac{3}{8}.$$

$$P(X = 3) = {}^4C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{(4-3)} = \frac{1}{4}.$$

$$P(X = 4) = {}^4C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^{(4-4)} = \frac{1}{16}.$$

Hence, the required probability distribution is given by

$$\begin{pmatrix} X: & 0 & 1 & 2 & 3 & 4 \\ P(X): & \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} \end{pmatrix}.$$

EXAMPLE 2 *Find the probability distribution of the number of sixes in three tosses of a die.*

SOLUTION When a die is tossed, we have $S = \{1, 2, 3, 4, 5, 6\}$.

$$\therefore \quad P(\text{getting a six}) = \frac{1}{6} \text{ and } P(\text{not getting a six}) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}.$$

Let X be the random variable denoting the number of sixes.

In 3 trials, the number of sixes may be 0 or 1 or 2 or 3.

So, X may assume the values 0, 1, 2, 3.

$$P(X = 0) = {}^3C_0 \cdot \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^{(3-0)} = \frac{125}{216}.$$

$$P(X = 1) = {}^3C_1 \cdot \left(\frac{1}{6}\right)^1 \cdot \left(\frac{5}{6}\right)^{(3-1)} = \frac{25}{72}.$$

$$P(X=2) = {}^3C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^{(3-2)} = \frac{5}{72}.$$

$$P(X=3) = {}^3C_3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^{(3-3)} = \frac{1}{216}.$$

The required probability distribution of X is given below:

$$\begin{pmatrix} X: & 0 & 1 & 2 & 3 \\ P(X): & \frac{125}{216} & \frac{25}{72} & \frac{5}{72} & \frac{1}{216} \end{pmatrix}.$$

EXAMPLE 3 *Find the probability distribution of the number of doublets in four throws of a pair of dice.*

SOLUTION When a pair of dice is thrown, there are 36 possible outcomes.

$\therefore \ n(S) = 36.$

All possible doublets are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6).

$P(\text{getting a doublet}) = \frac{6}{36} = \frac{1}{6}$, and

$P(\text{not getting a doublet}) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}.$

Let X denote the number of doublets.

In 4 throws, we can have 0 or 1 or 2 or 3 or 4 doublets.

$$P(X=0) = {}^4C_0 \cdot \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^4 = \frac{625}{1296}.$$

$$P(X=1) = {}^4C_1 \cdot \left(\frac{1}{6}\right)^1 \cdot \left(\frac{5}{6}\right)^3 = \frac{125}{324}.$$

$$P(X=2) = {}^4C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^2 = \frac{25}{216}.$$

$$P(X=3) = {}^4C_3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^1 = \frac{5}{324}.$$

$$P(X=4) = {}^4C_4 \cdot \left(\frac{1}{6}\right)^4 \cdot \left(\frac{5}{6}\right)^0 = \frac{1}{1296}.$$

The required probability distribution is given below:

$$\begin{pmatrix} X: & 0 & 1 & 2 & 3 & 4 \\ P(X): & \frac{625}{1296} & \frac{125}{324} & \frac{25}{216} & \frac{5}{324} & \frac{1}{1296} \end{pmatrix}.$$

EXAMPLE 4 *An unbiased coin is tossed 6 times. Find, using binomial distribution, the probability of getting at least 5 heads.* **[CBSE 2000]**

SOLUTION In a single throw of a coin, we have $S = \{H, T\}$.

$P(\text{getting a head}) = \frac{1}{2}$, and $P(\text{not getting a head}) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}.$

$\therefore \quad p = \frac{1}{2}$ and $q = \frac{1}{2}$.

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^6C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{(6-r)} = {}^6C_r \cdot \left(\frac{1}{2}\right)^6$$

$\therefore \quad P$(getting at least 5 heads)

$$= P(X \geq 5)$$
$$= P(X = 5) + P(X = 6)$$
$$= {}^6C_5 \cdot \left(\frac{1}{2}\right)^6 + {}^6C_6 \cdot \left(\frac{1}{2}\right)^6 = \left(\frac{3}{32} + \frac{1}{64}\right) = \frac{7}{64}.$$

Hence, the required probability is $\frac{7}{64}$.

EXAMPLE 5 *An unbiased coin is tossed 8 times. Find, by using binomial distribution, the probability of getting at least 3 heads.* **[CBSE 2000]**

SOLUTION In a single throw of a coin, we have $S = \{H, T\}$.

P(getting a head) $= \frac{1}{2}$, and P(not getting a head) $= \left(1 - \frac{1}{2}\right) = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}$ and $q = \frac{1}{2}$.

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^8C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{(8-r)} = {}^8C_r \left(\frac{1}{2}\right)^8$$

$\therefore \quad P$(getting at least 3 heads)

$$= P(X \geq 3)$$
$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1 - \left[{}^8C_0 \cdot \left(\frac{1}{2}\right)^8 + {}^8C_1 \cdot \left(\frac{1}{2}\right)^8 + {}^8C_2 \cdot \left(\frac{1}{2}\right)^8\right]$$
$$= 1 - \frac{1}{256} \cdot (1 + 8 + 28) = \left(1 - \frac{37}{256}\right) = \frac{219}{256}.$$

Hence, the required probability is $\frac{219}{256}$.

EXAMPLE 6 *Six coins are tossed simultaneously. Find the probability of getting*

(i) 3 heads (ii) no head (iii) at least one head

(iv) not more than 3 heads. **[CBSE 2003]**

SOLUTION The experiment may be taken as throwing a single coin 6 times.

In a single throw of a coin, we have $S = \{H, T\}$.

P(getting a head) $= \frac{1}{2}$, and P(not getting a head) $= \left(1 - \frac{1}{2}\right) = \frac{1}{2}$.

Let X be the random variable showing the number of heads.

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^6C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{6-r} = {}^6C_r \cdot \left(\frac{1}{2}\right)^6$$

(i) $P(\text{getting 3 heads}) = P(X = 3) = {}^6C_3 \cdot \left(\frac{1}{2}\right)^6 = \frac{5}{16}\cdot$

(ii) $P(\text{getting no head}) = P(X = 0) = {}^6C_0 \cdot \left(\frac{1}{2}\right)^6 = \frac{1}{64}\cdot$

(iii) $P(\text{getting at least 1 head})$

$$= 1 - P(X = 0) = 1 - \left[{}^6C_0 \cdot \left(\frac{1}{2}\right)^6\right]$$

$$= \left(1 - \frac{1}{64}\right) = \frac{63}{64}\cdot$$

(iv) $P(\text{getting not more than 3 heads})$

$= P(\text{no head or 1 head or 2 heads or 3 heads})$

$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$$= {}^6C_0 \cdot \left(\frac{1}{2}\right)^6 + {}^6C_1 \cdot \left(\frac{1}{2}\right)^6 + {}^6C_2 \cdot \left(\frac{1}{2}\right)^6 + {}^6C_3 \cdot \left(\frac{1}{2}\right)^6$$

$$= \left(\frac{1}{2}\right)^6 \cdot (1 + 6 + 15 + 20) = \left(\frac{1}{64} \times 42\right) = \frac{21}{32}\cdot$$

EXAMPLE 7 *A die is thrown* 5 *times. If getting an odd number is a success, find the probability of getting at least* 4 *successes.*

SOLUTION When a die is thrown, we have $S = \{1, 2, 3, 4, 5, 6\}$.

$\therefore$ $P(\text{getting an odd number}) = \frac{3}{6} = \frac{1}{2}\cdot$

$\therefore$ $P(\text{a success}) = \frac{1}{2}$, and $P(\text{not a success}) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}\cdot$

Let X be the random variable showing the number of successes.

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^5C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{(5-r)} = {}^5C_r \cdot \left(\frac{1}{2}\right)^5.$$

$P(\text{at least 4 successes}) = P(\text{4 successes or 5 successes})$

$= P(X = 4) + P(X = 5)$

$$= {}^5C_4 \cdot \left(\frac{1}{2}\right)^5 + {}^5C_5 \cdot \left(\frac{1}{2}\right)^5 = \left(\frac{5}{32} + \frac{1}{32}\right) = \frac{6}{32} = \frac{3}{16}\cdot$$

EXAMPLE 8 *In* 4 *throws with a pair of dice, what is the probability of throwing doublets at least twice?*

SOLUTION In a single throw of a pair of dice, the number of all possible outcomes is 36.

All doublets are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6).

$$P(\text{getting a doublet}) = \frac{6}{36} = \frac{1}{6}, \text{ and}$$

$$P(\text{not getting a doublet}) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}.$$

Let X be the random variable denoting the number of doublets.

Then, $P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^4C_r \cdot \left(\frac{1}{6}\right)^r \cdot \left(\frac{5}{6}\right)^{(4-r)}$.

P(at least 2 doublets)

$= P(X = 2) + P(X = 3) + P(X = 4)$

$$= {}^4C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^{(4-2)} + {}^4C_3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^{(4-3)} + {}^4C_4 \cdot \left(\frac{1}{6}\right)^4 \cdot \left(\frac{5}{6}\right)^{(4-4)}$$

$$= 6 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^2 + 4 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^1 + \left(\frac{1}{6}\right)^4 \cdot \left(\frac{5}{6}\right)^0$$

$$= \left(\frac{25}{216} + \frac{5}{324} + \frac{1}{1296}\right) = \frac{171}{1296}.$$

EXAMPLE 9 *The bulbs produced in a factory are supposed to contain* 5% *defective bulbs. What is the probability that a sample of* 10 *bulbs will contain not more than* 2 *defective bulbs?*

SOLUTION $P(\text{getting a defective bulb}) = \frac{5}{100} = \frac{1}{20}$, and

$$P(\text{getting a nondefective bulb}) = \left(1 - \frac{1}{20}\right) = \frac{19}{20}.$$

Then, $p = \frac{1}{20}$ and $q = \frac{19}{20}$.

Let X denote the number of defective bulbs.

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^{10}C_r \cdot \left(\frac{1}{20}\right)^r \cdot \left(\frac{19}{20}\right)^{(10-r)}$$

P(getting not more than 2 defective bulbs)

$= P(X = 0 \text{ or } X = 1 \text{ or } X = 2)$

$= P(X = 0) + P(X = 1) + P(X = 2)$

$$= {}^{10}C_0 \cdot \left(\frac{1}{20}\right)^0 \cdot \left(\frac{19}{20}\right)^{10} + {}^{10}C_1 \cdot \left(\frac{1}{20}\right)^1 \cdot \left(\frac{19}{20}\right)^9 + {}^{10}C_2 \cdot \left(\frac{1}{20}\right)^2 \cdot \left(\frac{19}{20}\right)^8$$

$$= \left(\frac{19}{20}\right)^{10} + \frac{1}{2} \cdot \left(\frac{19}{20}\right)^9 + \frac{9}{80} \cdot \left(\frac{19}{20}\right)^8 = \left(\frac{19}{20}\right)^8 \cdot \left(\frac{149}{100}\right).$$

Let $A = \left(\frac{19}{20}\right)^8 \cdot \left(\frac{149}{100}\right)$. Then,

$\log A = 8(\log 19 - \log 20) + \log 149 - \log 100$

$= 8(1.2788 - 1.3010) + 2.1732 - 2$

$= -0.0044 = \bar{1}.9956.$

$\therefore \quad A = \text{antilog}\,(\bar{1}.9956) = 0.99.$

Hence, the required probability $= \dfrac{99}{100}$.

EXAMPLE 10 *If on an average, out of* 10 *ships, one gets drowned then what is the probability that out of* 5 *ships at least* 4 *reach the shore safely?*

SOLUTION Probability of a ship to reach the shore safely $= \dfrac{9}{10}$.

Probability that a ship gets drowned $= \left(1 - \dfrac{9}{10}\right) = \dfrac{1}{10}$.

Let X be the random variable showing the number of ships reaching the shore safely.

$\therefore \quad P(\text{at least 4 reaching safely})$

$= P(\text{4 reaching safely or 5 reaching safely})$

$= P(\text{4 reaching safely}) + P(\text{5 reaching safely})$

$= P(X = 4) + P(X = 5)$

$$= {}^5C_4 \cdot \left(\frac{9}{10}\right)^4 \cdot \left(\frac{1}{10}\right)^{(5-4)} + {}^5C_5 \cdot \left(\frac{9}{10}\right)^5 \cdot \left(\frac{1}{10}\right)^0$$

$$= \frac{1}{2} \cdot \left(\frac{9}{10}\right)^4 + \left(\frac{9}{10}\right)^5 = \left(\frac{9}{10}\right)^4 \left(\frac{1}{2} + \frac{9}{10}\right) = \frac{7}{5} \cdot \left(\frac{9}{10}\right)^4.$$

Let $A = \dfrac{7}{5} \cdot \left(\dfrac{9}{10}\right)^4 = \dfrac{7 \times (9)^4}{5 \times (10)^4}$

$\Rightarrow \quad \log A = \log 7 + 4 \times \log 9 - \log 5 - 4 \times \log 10$

$= (0.8451 + 4 \times 0.9542 - 0.6990 - 4)$

$= -0.0371 = \bar{1}.9629$

$\Rightarrow \quad A = \text{antilog}\ (\bar{1}.9629) = 0.9181.$

Hence, the required probability is 0.9181.

Mean and Variance of a Binomial Distribution

MEAN *If a random variable X assumes the values* $x_1, x_2, \ldots, x_n$ *with probabilities* $p_1, p_2, \ldots, p_n$ *respectively then the mean of X is defined by*

$$\mu = \sum_{i=1}^{n} x_i\, p_i.$$

To Find the Mean of a Binomial Distribution

For the binomial distribution

$$P(X=r) = P(r) = {}^nC_r \cdot p^r \cdot q^{(n-r)}, \text{ where } r = 0, 1, 2, \ldots, n$$

the *mean*, μ, is given by

$$\mu = \sum_{r=0}^{n} r \cdot p(r) = \sum_{r=0}^{n} r \cdot {}^nC_r \cdot p^r \cdot q^{(n-r)}$$

$$= {}^nC_1 \cdot p \cdot q^{n-1} + 2 \cdot {}^nC_2 \cdot p^2 \cdot q^{(n-2)} + \ldots + n \cdot {}^nC_n \cdot p^n q^0$$

$$= 1 \cdot np \cdot q^{n-1} + n(n-1) \cdot p^2 \cdot q^{(n-2)} + \ldots + np^n$$

$$= np \cdot [{}^{(n-1)}C_0 \cdot p^0 \cdot q^{(n-1)} + {}^{(n-1)}C_1 \cdot p^1 \cdot q^{(n-2)} + \ldots + {}^{(n-1)}C_{(n-1)} \cdot p^{n-1} \cdot q^0]$$

$$= (np) \cdot (q+p)^{n-1} = (np) \qquad [\because \; q + p = 1].$$

Hence, the *mean* is given by $\mu = np$.

The *variance* $= \sigma^2$ is given by

$$\sigma^2 = \sum_{r=0}^{n} r^2 \cdot p(r) - (\text{mean})^2$$

$$= \sum_{r=0}^{n} r^2 \cdot {}^nC_r \cdot p^r q^{(n-r)} - (np)^2 \qquad [\because \; \text{mean} = np]$$

$$= \sum_{r=0}^{n} \{r + r(r-1)\} \cdot {}^nC_r \, p^r q^{(n-r)} - (np)^2$$

$$= \sum_{r=0}^{n} r \cdot {}^nC_r \, p^r q^{(n-r)} + \sum_{r=0}^{n} r(r-1) \cdot {}^nC_r \cdot p^r \cdot q^{(n-r)} - (np)^2$$

$$= np + \sum_{r=2}^{n} r(r-1) \cdot \frac{n(n-1)}{r(r-1)} \cdot {}^{(n-2)}C_{(r-2)} \cdot p^2 \cdot p^{(r-2)} \cdot q^{(n-r)} - (np)^2$$

$$\left[\because \; \sum_{r=0}^{n} r \cdot {}^nC_r \cdot p^r \cdot q^{(n-r)} = \text{mean} = np\right]$$

$$= np + n(n-1) \cdot p^2 \cdot \left(\sum_{r=0}^{n} {}^{(n-2)}C_{(r-2)} \cdot p^{(r-2)} \cdot q^{(n-r)}\right) - n^2p^2$$

$$= np + n(n-1)\, p^2 (q+p)^{(n-2)} - n^2 p^2$$

$$= np + n(n-1)\, p^2 - n^2 p^2 \qquad [\because \; q + p = 1]$$

$$= np - np^2 = np(1-p) = npq.$$

Hence, *variance* $= npq$.

$\therefore$ standard deviation $= \sqrt{npq}$.

Recurrence Relation for a Binomial Distribution

We have

$$P(r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} \text{ and } P(r+1) = {}^nC_{(r+1)} \cdot p^{(r+1)} \cdot q^{n-r-1}.$$

$$\therefore \quad \frac{P(r+1)}{P(r)} = \frac{{}^nC_{(r+1)} \cdot p^{(r+1)} \cdot q^{n-r-1}}{{}^nC_r \cdot p^r \cdot q^{(n-r)}}$$

$$= \frac{(n!)}{(r+1)! \cdot (n-r-1)!} \cdot \frac{(r)! \cdot (n-r)!}{(n)!} \cdot \frac{p}{q} = \frac{(n-r)}{(r+1)} \cdot \frac{p}{q}$$

$$\therefore \quad P(r+1) = \frac{(n-r)}{(r+1)} \cdot \frac{p}{q} \cdot P(r) \cdot$$

EXAMPLE 11 *If X follows a binomial distribution with mean* 3 *and variance* (3/2), *find (i)* $P(X \geq 1)$ *(ii)* $P(X \leq 5)$. **[CBSE 2001C]**

SOLUTION We know that mean $= np$ and variance $= npq$.

$$\therefore \quad np = 3 \text{ and } npq = \frac{3}{2} \Rightarrow 3q = \frac{3}{2} \Rightarrow q = \frac{1}{2} \cdot$$

$$\therefore \quad p = (1-q) = \left(1 - \frac{1}{2}\right) = \frac{1}{2} \cdot$$

Now, $np = 3$ and $p = \frac{1}{2} \Rightarrow n \times \frac{1}{2} = 3 \Rightarrow n = 6$.

So, the binomial distribution is given by

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^6C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{(6-r)} = {}^6C_r \left(\frac{1}{2}\right)^6.$$

(i) $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^6C_0 \cdot \left(\frac{1}{2}\right)^6 = \left(1 - \frac{1}{64}\right) = \frac{63}{64} \cdot$$

(ii) $P(X \leq 5) = 1 - P(X = 6)$

$$= 1 - {}^6C_6 \left(\frac{1}{2}\right)^6 = \left(1 - \frac{1}{64}\right) = \frac{63}{64} \cdot$$

EXAMPLE 12 *If X follows a binomial distribution with mean* 4 *and variance* 2, *find* $P(X \geq 5)$. **[CBSE 2001C]**

SOLUTION We know that mean $= np$ and variance $= npq$.

$$\therefore \quad np = 4 \text{ and } npq = 2.$$

Now, $np = 4$ and $npq = 2 \Rightarrow 4q = 2 \Rightarrow q = \frac{1}{2} \cdot$

$$\therefore \quad p = (1-q) = \left(1 - \frac{1}{2}\right) = \frac{1}{2} \cdot$$

Now, $np = 4$ and $p = \frac{1}{2} \Rightarrow \frac{1}{2}n = 4 \Rightarrow n = 8$.

So, the binomial distribution is given by

$$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^8C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{(8-r)} = {}^8C_r \cdot \left(\frac{1}{2}\right)^8.$$

$\therefore \quad P(X \geq 5) = P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$$= {}^8C_5 \cdot \left(\frac{1}{2}\right)^8 + {}^8C_6 \cdot \left(\frac{1}{2}\right)^8 + {}^8C_7 \cdot \left(\frac{1}{2}\right)^8 + {}^8C_8 \cdot \left(\frac{1}{2}\right)^8$$

$$= [{}^8C_3 + {}^8C_2 + {}^8C_1 + 1]\left(\frac{1}{2}\right)^8$$

$$= (56 + 28 + 8 + 1) \cdot \frac{1}{256} = \frac{93}{256}.$$

EXAMPLE 13 *Find the binomial distribution for which the mean and variance are* 12 *and* 3 *respectively.* **[CBSE 2004C]**

SOLUTION Let X be a binomial variate for which mean = 12 and variance = 3.

Then, $np = 12$ and $npq = 3 \Leftrightarrow 12 \times q = 3 \Leftrightarrow q = \frac{1}{4}.$

$\therefore \quad p = (1-q) = \left(1 - \frac{1}{4}\right) = \frac{3}{4}.$

And, $np = 12 \Leftrightarrow n = \frac{12}{p} = \left(12 \times \frac{4}{3}\right) = 16.$

Thus, $n = 16$, $p = \frac{3}{4}$ and $q = \frac{1}{4}.$

Hence, the binomial distribution is given by

$$P(X=r) = {}^{16}C_r \cdot \left(\frac{3}{4}\right)^r \cdot \left(\frac{1}{4}\right)^{(16-r)}, \text{ where } r = 0, 1, 2, 3, \ldots, 15.$$

EXAMPLE 14 *If the sum of the mean and variance of a binomial distribution for* 5 *trials is* 1.8, *find the distribution.* **[CBSE 2004]**

SOLUTION We know that

mean = np and variance = npq.

It is being given that $n = 5$ and mean + variance = 1.8.

$\therefore \quad np + npq = 1.8$, where $n = 5$

$\Leftrightarrow \quad 5p + 5pq = 1.8$

$\Leftrightarrow \quad p + p(1-p) = 0.36 \quad [\because \quad q = (1-p)]$

$\Leftrightarrow \quad p^2 - 2p + 0.36 = 0$

$\Leftrightarrow \quad 100p^2 - 200p + 36 = 0$

$\Leftrightarrow \quad 25p^2 - 50p + 9 = 0$

$\Leftrightarrow \quad 25p^2 - 45p - 5p + 9 = 0$

$$\Leftrightarrow \quad 5p(5p-9)-(5p-9)=0$$
$$\Leftrightarrow \quad (5p-9)(5p-1)=0$$
$$\Leftrightarrow \quad p=\frac{1}{5}=0.2 \quad [\because \quad p \text{ cannot exceed } 1].$$

Thus, $n=5$, $p=0.2$, and $q=(1-p)=(1-0.2)=0.8$.

Let X denote the binomial variate. Then, the required distribution is

$$P(X=r)={}^nC_r \cdot p^r \cdot q^{(n-r)}={}^5C_r \cdot (0.2)^r \cdot (0.8)^{(5-r)},$$

where $r=0, 1, 2, 3, 4, 5$.

EXAMPLE 15 *The sum and the product of the mean and variance of a binomial distribution are 24 and 128 respectively. Find the distribution.*

SOLUTION We have

$$np+npq=24 \text{ and } np \times npq=128$$
$$\Leftrightarrow \quad (np)(1+q)=24 \text{ and } n^2p^2 \times q=128$$
$$\Leftrightarrow \quad n^2p^2=\frac{576}{(1+q)^2} \text{ and } n^2p^2 \times q=128$$
$$\Leftrightarrow \quad \frac{576}{(1+q)^2}=\frac{128}{q} \quad \Leftrightarrow \quad 2(1+q^2+2q)=9q$$
$$\Leftrightarrow \quad 2q^2-5q+2=0 \quad \Leftrightarrow \quad (2q-1)(q-2)=0$$
$$\Leftrightarrow \quad q=\frac{1}{2} \quad [\because \quad q \neq 2].$$
$$\therefore \quad p=(1-q)=\left(1-\frac{1}{2}\right)=\frac{1}{2}.$$

Now, $np(1+q)=24 \quad \Leftrightarrow \quad n \times \frac{1}{2}\left(1+\frac{1}{2}\right)=24 \quad \Leftrightarrow \quad n=32.$

Hence, the required probability distribution is given by

$$P(X=r)={}^{32}C_r \cdot \left(\frac{1}{2}\right)^{32}.$$

EXAMPLE 16 *In a binomial distribution, prove that*

mean > variance.

SOLUTION Let X be a binomial variate with parameters n and p. Then,

mean $=np$ and variance $=npq$.

$$\therefore \quad (\text{mean})-(\text{variance})=(np-npq)=np(1-q)=np^2>0$$

$[\because \quad (1-q)=p \text{ and } np^2>0 \text{ as } n \in N]$

$$\Rightarrow \quad [(\text{mean})-(\text{variance})]>0$$
$$\Rightarrow \quad \text{mean}>\text{variance}.$$

Hence, mean > variance.

EXAMPLE 17 *A die is tossed thrice. Getting an even number is considered a success. What is the variance of the binomial distribution?*

SOLUTION Here, $n = 3$.

Let p = probability of getting an even number in a single throw

$$\Rightarrow \quad p = \frac{3}{6} = \frac{1}{2}.$$

$$\Rightarrow \quad q = (1 - p) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}.$$

$$\therefore \quad \text{variance} = npq = \left(3 \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{3}{4}.$$

EXAMPLE 18 *A die is rolled 20 times. Getting a number greater than 4 is a success. Find the mean and variance of the number of successes.*

SOLUTION In a single throw of a die, we have

$$p = \text{probability of getting a number greater than } 4 = \frac{2}{6} = \frac{1}{3}.$$

$$\therefore \quad q = (1 - p) = \left(1 - \frac{1}{3}\right) = \frac{2}{3}.$$

Also, $n = 20$ (given).

$$\therefore \quad \text{mean} = np = \left(20 \times \frac{1}{3}\right) = 6.67, \text{ and}$$

$$\text{variance} = npq = \left(20 \times \frac{1}{3} \times \frac{2}{3}\right) = 4.44.$$

EXAMPLE 19 *A die is tossed* 180 *times. Find the expected number* (μ) *of times the face with the number* 5 *will appear. Also, find the standard deviation* (σ), *and variance* (σ^2).

SOLUTION In a single throw of a die, $S = \{1, 2, 3, 4, 5, 6\}$.

$$p = (\text{probability of getting the number } 5) = \frac{1}{6}.$$

$$\therefore \quad q = (1 - p) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}.$$

Thus, $n = 180$, $p = \frac{1}{6}$ and $q = \frac{5}{6}$.

$$\therefore \quad \mu = np = \left(180 \times \frac{1}{6}\right) = 30.$$

$$\text{Variance} = \sigma^2 = npq = \left(180 \times \frac{1}{6} \times \frac{5}{6}\right) = 25.$$

$$\text{Standard deviation} = \sigma = \sqrt{25} = 5.$$

EXERCISE 32

1. A coin is tossed 6 times. Find the probability of getting at least 3 heads.
2. A coin is tossed 5 times. What is the probability that a head appears an even number of times?
3. 7 coins are tossed simultaneously. What is the probability that a tail appears an odd number of times?
4. A coin is tossed 6 times. Find the probability of getting
   (i) exactly 4 heads (ii) at least 1 head (iii) at most 4 heads.
5. 10 coins are tossed simultaneously. Find the probability of getting
   (i) exactly 3 heads (ii) not more than 4 heads
   (iii) at least 4 heads.
6. A die is thrown 6 times. If 'getting an even number' is a success, find the probability of getting
   (i) exactly 5 successes (ii) at least 5 successes
   (iii) at most 5 successes.
7. A die is thrown 4 times. 'Getting a 1 or a 6' is considered a success. Find the probability of getting
   (i) exactly 3 successes (ii) at least 2 successes
   (iii) at most 2 successes.
8. Find the probability of a 4 turning up at least once in two tosses of a fair die.
9. A pair of dice is thrown 4 times. If 'getting a doublet' is considered a success, find the probability of getting 2 successes.
10. A pair of dice is thrown 7 times. If 'getting a total of 7' is considered a success, find the probability of getting
    (i) no success (ii) exactly 6 successes
    (iii) at least 6 successes (iv) at most 6 successes.
11. There are 6% defective items in a large bulk of items. Find the probability that a sample of 8 items will include not more than one defective item.
12. In a box containing 60 bulbs, 6 are defective. What is the probability that out of a sample of 5 bulbs (i) none is defective, (ii) exactly 2 are defective?
13. The probability that a bulb produced by a factory will fuse after 6 months of use is 0.05. Find the probability that out of 5 such bulbs
    (i) none will fuse after 6 months of use
    (ii) at least one will fuse after 6 months of use
    (iii) not more than one will fuse after 6 months of use.
14. In the items produced by a factory, there are 10% defective items. A sample of 6 items is randomly chosen. Find the probability that this sample contains (i) exactly 2 defective items, (ii) not more than 2 defective items, (iii) at least 3 defective items.

15. Assume that on an average one telephone number out of 15, called between 3 p.m. and 4 p.m. on weekdays, will be busy. What is the probability that if six randomly selected telephone numbers are called, at least 3 of them will be busy?

16. Three cars participate in a race. The probability that any one of them has an accident is 0.1. Find the probability that all the cars reach the finishing line without any accident.

17. Past records show that 80% of the operations performed by a certain doctor were successful. If he performs 4 operations in a day, what is the probability that at least 3 operations will be successful?

18. The probability of a man hitting a target is (1/4). If he fires 7 times, what is the probability of his hitting the target at least twice?

19. In a hurdles race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is (5/6). What is the probability that he will knock down fewer than 2 hurdles?

20. A man can hit a bird, once in 3 shots. On this assumption he fires 3 shots. What is the chance that at least one bird is hit?

21. If the probability that a man aged 60 will live to be 70 is 0.65, what is the probability that out of 10 men, now 60, at least 8 will live to be 70?

22. A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn one by one with replacement, what is the probability that (i) none is white, (ii) all are white, (iii) at least one is white?

23. A policeman fires 6 bullets at a burglar. The probability that the burglar will be hit by a bullet is 0.6. What is the probability that the burglar is still unhurt?

24. A die is tossed thrice. A success is 1 or 6 on a toss. Find the mean and variance of successes.

25. A die is thrown 100 times. Getting an even number is considered a success. Find the mean and variance of successes.

26. Determine the binomial distribution whose mean is 9 and variance is 6. **[CBSE 2006]**

27. Find the binomial distribution whose mean is 5 and variance is 2.5.

28. The mean and variance of a binomial distribution are 4 and (4/3) respectively. Find $P(X \geq 1)$. **[CBSE 2004]**

29. For a binomial distribution, the mean is 6 and the standard deviation is $\sqrt{2}$. Find the probability of getting 5 successes.

30. In a binomial distribution, the sum and the product of the mean and the variance are (25/3) and (50/3) respectively. Find the distribution.

31. Obtain the binomial distribution whose mean is 10 and standard deviation is $2\sqrt{2}$.

32. Bring out the fallacy, if any, in the following statement:
'The mean of a binomial distribution is 6 and its variance is 9'.

ANSWERS (EXERCISE 32)

1. $\frac{21}{32}$ **2.** $\frac{1}{2}$ **3.** $\frac{1}{2}$

4. (i) $\frac{15}{64}$ (ii) $\frac{63}{64}$ (iii) $\frac{57}{64}$ **5.** (i) $\frac{15}{128}$ (ii) $\frac{193}{512}$ (iii) $\frac{53}{64}$

6. (i) $\frac{3}{32}$ (ii) $\frac{7}{64}$ (iii) $\frac{63}{64}$ **7.** (i) $\frac{8}{81}$ (ii) $\frac{11}{27}$ (iii) $\frac{8}{9}$

8. $\frac{11}{36}$ **9.** $\frac{25}{216}$ **10.** (i) $\left(\frac{5}{6}\right)^7$ (ii) $\frac{35}{6^7}$ (iii) $\frac{1}{6^5}$ (iv) $\left(1-\frac{1}{6^7}\right)$

11. $\left(\frac{47}{50}\right)^7 \times \left(\frac{71}{50}\right)$ **12.** (i) $\left(\frac{9}{10}\right)^5$ (ii) $\frac{729}{10000}$

13. (i) $\left(\frac{19}{20}\right)^5$ (ii) $1-\left(\frac{19}{20}\right)^5$ (iii) $\left(\frac{19}{20}\right)^4 \times \left(\frac{6}{5}\right)$

14. (i) $\frac{3}{20} \times \left(\frac{9}{10}\right)^4$ (ii) $\frac{3}{2} \times \left(\frac{19}{20}\right)^4$ (iii) $1-\left[\frac{3}{2} \times \left(\frac{19}{20}\right)^4\right]$

15. $1-\left(\frac{14}{15}\right)^4 \cdot \left(\frac{59}{45}\right)$ **16.** $\frac{729}{1000}$ **17.** $\frac{512}{625}$

18. $\frac{4547}{8192}$ **19.** $\frac{5^{10}}{(2 \times 6^9)}$ **20.** $\frac{19}{27}$

21. 0.2615 **22.** (i) $\frac{81}{256}$ (ii) $\frac{1}{256}$ (iii) $\frac{175}{256}$

23. 0.004096 **24.** $\mu = 1,\ \sigma^2 = \frac{2}{3}$ **25.** $\mu = 50,\ \sigma^2 = 25$

26. ${}^{27}C_r \cdot \left(\frac{1}{3}\right)^r \cdot \left(\frac{2}{3}\right)^{(27-r)}$, where $r = 0, 1, 2, 3, \ldots, 27$

27. ${}^{10}C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{(10-r)}$, $0 \le r \le 10$ **28.** $\frac{728}{729}$

29. ${}^{9}C_5 \cdot \left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^4$ **30.** ${}^{15}C_r \cdot \left(\frac{1}{3}\right)^r \cdot \left(\frac{2}{3}\right)^{(15-r)}$

31. ${}^{50}C_r \cdot \left(\frac{1}{5}\right)^r \cdot \left(\frac{4}{5}\right)^{(50-r)}$, $0 \le r \le 50$

32. The probability of getting a failure (i.e., q) cannot be greater than 1.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 32)

1. In a single toss, $P(H)=\frac{1}{2}$ and $P(\text{not } H)=\frac{1}{2}\cdot$

$\therefore \quad p=\frac{1}{2},\ q=\frac{1}{2}$ and $n=6$.

Let X show the number of heads. Then,

$$P(X=r)={}^{n}C_{r}\cdot p^{r}\cdot q^{(n-r)}={}^{6}C_{r}\left(\frac{1}{2}\right)^{6}.$$

Required probability $=P(X=3)+P(X=4)+P(X=5)+P(X=6)$.

2. Here, $p=\frac{1}{2},\ q=\frac{1}{2}$ and $n=5$.

Let X show the number of heads. Then,

$$P(X=r)={}^{n}C_{r}\cdot p^{r}\cdot q^{(n-r)}={}^{5}C_{r}\left(\frac{1}{2}\right)^{5}.$$

Required probability $=P(X=0)+P(X=2)+P(X=4)$.

3. 7 coins being tossed simultaneously is the same as one coin being tossed 7 times.

$\therefore \quad p=\frac{1}{2},\ q=\frac{1}{2}$ and $n=7$.

Let X show the number of tails. Then,

$$P(X=r)={}^{n}C_{r}\cdot p^{r}\cdot q^{n-r}={}^{7}C_{r}\cdot\left(\frac{1}{2}\right)^{7}\cdot$$

Required probability $=P(X=1)+P(X=3)+P(X=5)+P(X=7)$.

4. $P(\text{a head})=\frac{1}{2}$ and $P(\text{not a head})=\frac{1}{2}\cdot$

$\therefore \quad p=\frac{1}{2},\ q=\frac{1}{2}$ and $n=6$.

$$\therefore \quad P(X=r)={}^{n}C_{r}\cdot p^{r}\cdot q^{(n-r)}={}^{6}C_{r}\cdot\left(\frac{1}{2}\right)^{6}\cdot$$

(i) $P(\text{exactly 4 heads})={}^{6}C_{4}\cdot\left(\frac{1}{2}\right)^{6}.$

(ii) $P(\text{at least 1 head})=1-P(\text{no head})$

$$=1-P(0\text{ head})=\left[1-{}^{6}C_{0}\cdot\left(\frac{1}{2}\right)^{6}\right]\cdot$$

(iii) $P(\text{at the most 4 heads})=P(4\text{ or less than 4 heads})$

$$=1-P[5\text{ heads or 6 heads}]$$

$$=1-\left[({}^{6}C_{5}+{}^{6}C_{6})\left(\frac{1}{2}\right)^{6}\right]\cdot$$

5. 10 coins being tossed simultaneously is the same as one coin being tossed 10 times.

$$P(X=r)={}^{n}C_{r}\cdot p^{r}\cdot q^{(n-r)}={}^{10}C_{r}\cdot\left(\frac{1}{2}\right)^{10}.$$

(i) $P(\text{exactly 3 heads})={}^{10}C_{3}\cdot\left(\frac{1}{2}\right)^{10}.$

(ii) P(not more than 4 heads)

$= P(X \le 4)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4)\left(\frac{1}{2}\right)^{10}.$

(iii) P(at least 4 heads)

$= P$(4 heads or 5 heads or … or 10 heads)

$= 1 - P$(0 head or 1 head or 2 heads or 3 heads)

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$= 1 - ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3)\left(\frac{1}{2}\right)^{10}.$

6. $p = \frac{3}{6} = \frac{1}{2}, q = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$ and $n = 6.$

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{n-r} = {}^6C_5 \left(\frac{1}{2}\right)^6.$

(i) $P(\text{exactly 5 successes}) = {}^6C_5 \left(\frac{1}{2}\right)^6.$

(ii) $P(\text{at least 5 successes}) = P[(5 \text{ successes}) \text{ or } (6 \text{ successes})]$

$= ({}^6C_5 + {}^6C_6)\left(\frac{1}{2}\right)^6.$

(iii) $P(\text{at most 5 successes}) = P[0 \text{ or } 1 \text{ or } 2 \text{ or } 3 \text{ or } 5 \text{ successes}]$

$= 1 - P(6 \text{ successes}) = \left[1 - {}^6C_6 \cdot \left(\frac{1}{2}\right)^6\right].$

7. $p = \frac{2}{6} = \frac{1}{3}, q = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$ and $n = 4.$

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^4C_r \cdot \left(\frac{1}{3}\right)^r \cdot \left(\frac{2}{3}\right)^{(4-r)}.$

(i) $P(\text{exactly 3 successes}) = {}^4C_3 \cdot \left(\frac{1}{3}\right)^3 \cdot \left(\frac{2}{3}\right)^1.$

(ii) $P(\text{at least 2 successes}) = [P(X=2) \text{ or } P(X=3) \text{ or } P(X=4)]$

$= 1 - [P(X=0) + P(X=1)].$

(iii) $P(\text{at most 2 successes}) = P[(X=0) \text{ or } (X=1) \text{ or } (X=2)].$

8. $p = \frac{1}{6},\ q = \left(1 - \frac{1}{6}\right) = \frac{5}{6}$ and $n = 2.$

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^2C_r \cdot \left(\frac{1}{6}\right)^r \cdot \left(\frac{5}{6}\right)^{(2-r)}.$

$P[\text{at least one } 4] = P(X=1 \text{ or } X=2) = P(X=1) + P(X=2).$

9. $p = \frac{6}{36} = \frac{1}{6}, q = \left(1 - \frac{1}{6}\right) = \frac{5}{6}$ and $n = 4.$

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^4C_r \cdot \left(\frac{1}{6}\right)^r \cdot \left(\frac{5}{6}\right)^{(4-r)}.$

$\therefore \quad P(X=2) = {}^4C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^2.$

10. Let $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$.

$\therefore \quad p = \frac{6}{36} = \frac{1}{6},\ q = \left(1 - \frac{1}{6}\right) = \frac{5}{6},\ n = 7.$

$$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^7C_r \cdot \left(\frac{1}{6}\right)^r \cdot \left(\frac{5}{6}\right)^{(7-r)}$$

(i) $P(X=0) = {}^7C_0 \cdot \left(\frac{5}{6}\right)^7.$

(ii) $P(X=6) = {}^7C_6 \cdot \left(\frac{1}{6}\right)^6 \cdot \left(\frac{5}{6}\right).$

(iii) $P(\text{at least 6 successes}) = P(\text{6 successes or 7 successes})$

$$= P(X=6) + P(X=7)$$

$$= {}^7C_6 \cdot \left(\frac{1}{6}\right)^6 \cdot \left(\frac{5}{6}\right) + {}^7C_7 \cdot \left(\frac{1}{6}\right)^7.$$

(iv) $P(\text{at most 6 successes}) = P(X \le 6)$

$$= [1 - P(X=7)] = 1 - {}^7C_7 \cdot \left(\frac{1}{6}\right)^7.$$

11. $p = \frac{6}{100} = \frac{3}{50},\ q = \left(1 - \frac{3}{50}\right) = \frac{47}{50}$ and $n = 8$.

$$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^8C_r \cdot \left(\frac{3}{50}\right)^r \cdot \left(\frac{47}{50}\right)^{(8-r)}$$

Required probability $= P(\text{0 defective or 1 defective})$

$$= P(X=0) + P(X=1)$$

$$= {}^8C_0 \cdot \left(\frac{47}{50}\right)^8 + {}^8C_1 \cdot \left(\frac{3}{50}\right)\left(\frac{47}{50}\right)^7.$$

12. $p = \frac{6}{60} = \frac{1}{10}, q = \left(1 - \frac{1}{10}\right) = \frac{9}{10}$ and $n = 5$.

$$P(X=r) = {}^nC_r \cdot p^r \cdot q^{n-r} = {}^5C_r \cdot \left(\frac{1}{10}\right)^r \cdot \left(\frac{9}{10}\right)^{(5-r)}$$

(i) $P(\text{none is defective}) = P(X=0) = {}^5C_0 \cdot \left(\frac{9}{10}\right)^5.$

(ii) $P(\text{exactly 2 are defective}) = P(X=2) = {}^5C_2 \cdot \left(\frac{1}{10}\right)^2 \cdot \left(\frac{9}{10}\right)^3.$

13. $p = \frac{1}{20},\ q = \left(1 - \frac{1}{20}\right) = \frac{19}{20}$ and $n = 5$.

$$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^5C_r \cdot \left(\frac{1}{20}\right)^r \cdot \left(\frac{19}{20}\right)^{(5-r)}$$

(i) $P(X=0) = {}^5C_0 \cdot \left(\frac{19}{20}\right)^5.$

(ii) $P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0)$

$$= 1 - {}^5C_0 \cdot \left(\frac{19}{20}\right)^5.$$

(iii) $P(X \leq 1) = P(X = 0) + P(X = 1)$

$$= {}^5C_0 \cdot \left(\frac{19}{20}\right)^5 + {}^5C_1 \cdot \left(\frac{1}{20}\right)\left(\frac{19}{20}\right)^4.$$

14. $p = \dfrac{10}{100} = \dfrac{1}{10}$, $q = \left(1 - \dfrac{1}{10}\right) = \dfrac{9}{10}$ and $n = 6$.

$$\therefore \quad P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^6C_r \cdot \left(\frac{1}{10}\right)^r \cdot \left(\frac{9}{10}\right)^{(6-r)}.$$

(i) $P(X = 2) = {}^6C_2 \cdot \left(\dfrac{1}{10}\right)^2 \cdot \left(\dfrac{9}{10}\right)^4.$

(ii) $P(X \leq 2) = {}^6C_0 \cdot \left(\dfrac{9}{10}\right)^6 + {}^6C_1 \cdot \dfrac{1}{10} \times \left(\dfrac{9}{10}\right)^5 + {}^6C_2 \cdot \dfrac{1}{100} \times \left(\dfrac{9}{10}\right)^4.$

(iii) $P(X \geq 3) = 1 - P(X < 3) = 1 - P(X \leq 2).$

15. $p = \dfrac{1}{15}$, $q = \left(1 - \dfrac{1}{15}\right) = \dfrac{14}{15}$ and $n = 6$.

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^6C_r \cdot \left(\frac{1}{15}\right)^r \cdot \left(\frac{14}{15}\right)^{(6-r)}.$$

$P(X \geq 3) = 1 - P(X < 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$

$$= 1 - \left[{}^6C_0 \cdot \left(\frac{14}{15}\right)^6 + {}^6C_1 \cdot \left(\frac{1}{15}\right) \cdot \left(\frac{14}{15}\right)^5 + {}^6C_2 \cdot \left(\frac{1}{15}\right)^2 \cdot \left(\frac{14}{15}\right)^4\right]$$

$$= 1 - \left(\frac{14}{15}\right)^4 \left(\frac{59}{45}\right).$$

16. $p = \dfrac{1}{10}$, $q = \left(1 - \dfrac{1}{10}\right) = \dfrac{9}{10}$ and $n = 3$.

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^3C_r \cdot \left(\frac{1}{10}\right)^r \cdot \left(\frac{9}{10}\right)^{(3-r)}.$$

$$P(X = 0) = {}^3C_0 \cdot \left(\frac{9}{10}\right)^3.$$

17. $p = \dfrac{80}{100} = \dfrac{4}{5}$, $q = \left(1 - \dfrac{4}{5}\right) = \dfrac{1}{5}$ and $n = 4$.

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^4C_r \cdot \left(\frac{4}{5}\right)^r \cdot \left(\frac{1}{5}\right)^{(4-r)}.$$

$P(X \geq 3) = P(X = 3) + P(X = 4)$

$$= \left[{}^4C_3 \cdot \left(\frac{4}{5}\right)^3 \cdot \left(\frac{1}{5}\right) + {}^4C_4 \cdot \left(\frac{4}{5}\right)^4\right].$$

18. $p=\frac{1}{4}$, $q=\left(1-\frac{1}{4}\right)=\frac{3}{4}$ and $n=7$.

$$P(X=r)={}^nC_r\cdot p^r\cdot q^{(n-r)}={}^7C_r\cdot\left(\frac{1}{4}\right)^r\cdot\left(\frac{3}{4}\right)^{(n-r)}$$

$$\begin{aligned}P(X\geq 2)&=1-P(X<2)\\&=1-P[(X=0)\text{ or }(X=1)]\\&=1-[P(X=0)+P(X=1)].\end{aligned}$$

19. $p=P(\text{knocking down 1 hurdle})=\left(1-\frac{5}{6}\right)=\frac{1}{6}$, $q=\frac{5}{6}$ and $n=10$.

$$P(X=r)={}^nC_r\cdot p^r\cdot q^{(n-r)}={}^{10}C_r\cdot\left(\frac{1}{6}\right)^r\cdot\left(\frac{5}{6}\right)^{(10-r)}.$$

$$\begin{aligned}P(X<2)&=P(X=0)+P(X=1)\\&=\left[{}^{10}C_0\cdot\left(\frac{5}{6}\right)^{10}+{}^{10}C_1\cdot\left(\frac{1}{6}\right)^1\cdot\left(\frac{5}{9}\right)^9\right].\end{aligned}$$

20. $p=\frac{1}{3}$, $q=\left(1-\frac{1}{3}\right)=\frac{2}{3}$ and $n=3$.

$$P(X=r)={}^nC_r\cdot p^r\cdot q^{(n-r)}={}^3C_r\cdot\left(\frac{1}{3}\right)^r\cdot\left(\frac{2}{3}\right)^{(3-r)}.$$

$$\begin{aligned}P(X\geq 1)&=P(X=1)+P(X=2)+P(X=3)\\&=\left[{}^3C_1\times\frac{1}{3}\times\left(\frac{2}{3}\right)^2\right]+\left[{}^3C_2\times\left(\frac{1}{3}\right)^2\times\frac{2}{3}\right]+\left[{}^3C_3\times\left(\frac{1}{3}\right)^3\right]\\&=\left(\frac{4}{9}+\frac{2}{9}+\frac{1}{9}\right)=\frac{7}{9}.\end{aligned}$$

21. $$\begin{aligned}P(X\geq 8)&=P(X=8)+P(X=9)+P(X=10)\\&=\left[{}^{10}C_8\cdot\left(\frac{13}{20}\right)^8\cdot\left(\frac{7}{20}\right)^2+{}^{10}C_9\cdot\left(\frac{13}{20}\right)^9\cdot\left(\frac{7}{20}\right)+{}^{10}C_{10}\cdot\left(\frac{13}{20}\right)^{10}\right]\\&=\left[{}^{10}C_2\cdot\left(\frac{13}{20}\right)^8\cdot\frac{49}{400}+{}^{10}C_1\cdot\left(\frac{13}{20}\right)^9\cdot\frac{7}{20}+\left(\frac{13}{20}\right)^{10}\right].\end{aligned}$$

$$\therefore\quad P=\left(\frac{13}{20}\right)^8\cdot\left[\frac{441}{80}+\frac{91}{40}+\frac{169}{400}\right]=\left[\left(\frac{13}{20}\right)^8\times\frac{821}{100}\right].$$

$$\begin{aligned}\Rightarrow\quad \log P&=[8(\log 13-\log 20)+\log 821-\log 100]\\&=[-1.4968+2.9143-2]=-0.5825\\&=\bar{1}.4175\end{aligned}$$

$$\Rightarrow\quad P=\text{antilog}\,(\bar{1}.4175).$$

22. $P(\text{white})=\frac{5}{20}=\frac{1}{4}$, $P(\text{nonwhite})=\frac{3}{4}$.

$\therefore\quad p=\frac{1}{4}$, $q=\frac{3}{4}$ and $n=4$.

$$P(X=r)={}^nC_r\cdot p^r\cdot q^{(n-r)}={}^4C_r\cdot\left(\frac{1}{4}\right)^r\cdot\left(\frac{3}{4}\right)^{(4-r)}.$$

(i) $P(X=0) = {}^4C_0 \cdot \left(\frac{3}{4}\right)^4.$

(ii) $P(X=4) = {}^4C_4 \cdot \left(\frac{1}{4}\right)^4.$

(iii) $P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0).$

23. $p = \frac{6}{10} = \frac{3}{5}, q = \left(1 - \frac{3}{5}\right) = \frac{2}{5}$ and $n = 6.$

$$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^6C_r \cdot \left(\frac{3}{5}\right)^r \cdot \left(\frac{2}{5}\right)^{(6-r)}.$$

$\therefore \quad P(X=0) = {}^6C_0 \cdot \left(\frac{2}{5}\right)^6.$

24. $p = \frac{2}{6} = \frac{1}{3},\ q = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$ and $n = 3.$

$\therefore \quad \mu = np$ and $\sigma^2 = npq.$

28. $\left(np = 4 \text{ and } npq = \frac{4}{3}\right) \Rightarrow q = \frac{1}{3}.$

$\therefore \quad p = (1-q) = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$ and $n = \frac{4}{p} = \left(4 \times \frac{3}{2}\right) = 6.$

$\therefore \quad P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)}$

$$= {}^6C_r \cdot \left(\frac{2}{3}\right)^r \cdot \left(\frac{1}{3}\right)^{(6-r)}.$$

$P(X \geq 1) = 1 - P(X=0)$

$$= 1 - {}^6C_0 \cdot \left(\frac{2}{3}\right)^0 \cdot \left(\frac{1}{3}\right)^6$$

$$= \left(1 - \frac{1}{3^6}\right) = \frac{728}{729}.$$

29. $np = 6$ and $npq = 2 \Rightarrow q = \frac{1}{3}$ and $p = \frac{2}{3}.$

$$\left(n \times \frac{2}{3}\right) = 6 \Rightarrow n = \left(6 \times \frac{3}{2}\right) = 9.$$

$\therefore \quad P(5 \text{ successes}) = {}^9C_5 \cdot \left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^4.$

32. $np = 6$ and $npq = 9 \Rightarrow q = \frac{npq}{np} = \frac{9}{6} = \frac{3}{2}.$

But, q cannot be greater than 1.

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. If A and B are mutually exclusive events such that $P(A) = 0.4$, $P(B) = x$ and $P(A \cup B) = 0.5$, then $x = ?$

   (a) 0.2 (b) 0.1 (c) $\frac{4}{5}$ (d) none of these

2. If A and B are independent events such that $P(A) = 0.4$, $P(B) = x$ and $P(A \cup B) = 0.5$, then $x = ?$

   (a) $\frac{4}{5}$ (b) 0.1 (c) $\frac{1}{6}$ (d) none of these

3. If $P(A) = 0.8$, $P(B) = 0.5$ and $P(B/A) = 0.4$, then $P(A/B) = ?$

   (a) 0.32 (b) 0.64 (c) 0.16 (d) 0.25

4. If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, then $P(A/B) = ?$

   (a) $\frac{5}{6}$ (b) $\frac{5}{7}$ (c) $\frac{6}{7}$ (d) $\frac{4}{5}$

5. If A and B are events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(A' \cup B') = \frac{1}{4}$, then A and B are

   (a) independent (b) mutually exclusive

   (c) both 'a' and 'b' (d) none of these

6. It is given that the probability that A can solve a given problem is $\frac{3}{5}$ and the probability that B can solve the same problem is $\frac{2}{3}$. The probability that at least one of A and B can solve a problem is

   (a) $\frac{2}{5}$ (b) $\frac{1}{15}$ (c) $\frac{13}{15}$ (d) $\frac{2}{15}$

7. The probabilities of A, B and C of solving a problem are $\frac{1}{6}$, $\frac{1}{5}$ and $\frac{1}{3}$ respectively. What is the probability that the problem is solved?

   (a) $\frac{4}{9}$ (b) $\frac{5}{9}$ (c) $\frac{1}{3}$ (d) none of these

8. A can hit a target 4 times in 5 shots, B can hit 3 times in 4 shots, and C can hit 2 times in 3 shots. The probability that B and C hit and A does not hit is

   (a) $\frac{1}{10}$ (b) $\frac{2}{5}$

   (c) $\frac{7}{12}$ (d) none of these

9. A machine operates only when all of its three components function. The probabilities of the failures of the first, second and third component are 0.2, 0.3 and 0.5 respectively. What is the probability that the machine will fail?

(a) 0.70 (b) 0.72 (c) 0.07 (d) none of these

10. A die is rolled. If the outcome is an odd number, what is the probability that it is prime?

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{5}{12}$ (d) none of these

11. If A and B are events such that $P(A) = 0.3$, $P(B) = 0.2$ and $P(A \cap B) = 0.1$, then $P(\overline{A} \cap B) = ?$

(a) 0.2 (b) 0.1 (c) 0.4 (d) 0.5

12. If $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{5}$, then $P(\overline{B}/\overline{A}) = ?$

(a) $\frac{11}{15}$ (b) $\frac{11}{45}$ (c) $\frac{23}{60}$ (d) $\frac{37}{45}$

13. If A and B are events such that $P(A) = 0.4$, $P(B) = 0.8$ and $P(B/A) = 0.6$, then $P(A/B) = ?$

(a) 0.2 (b) 0.3 (c) 0.4 (d) 0.5

14. If A and B are independent events, then $P(\overline{A}/\overline{B}) = ?$

(a) $1 - P(A)$ (b) $1 - P(B)$ (c) $1 - P(A/\overline{B})$ (d) $-P(\overline{A}/B)$

15. If A and B are two events such that $P(A \cup B) = \frac{5}{6}$, $P(A \cap B) = \frac{1}{3}$ and $P(\overline{B}) = \frac{1}{2}$, then the events A and B are

(a) independent (b) dependent
(c) mutually exclusive (d) none of these

16. A die is thrown twice and the sum of the numbers appearing is observed to be 7. What is the conditional probability that the number 2 has appeared at least once?

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{2}{7}$ (d) $\frac{3}{5}$

17. Two numbers are selected at random from integers 1 through 9. If the sum is even, what is the probability that both numbers are odd?

(a) $\frac{1}{6}$ (b) $\frac{2}{3}$ (c) $\frac{4}{9}$ (d) $\frac{5}{8}$

18. In a class, 60% of the students read mathematics, 25% biology and 15% both mathematics and biology. One student is selected at random. What is the probability that he reads mathematics, if it is known that he reads biology?

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{3}{8}$ (d) $\frac{5}{8}$

19. A couple has 2 children. What is the probability that both are boys, if it is known that one of them is a boy?

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) $\frac{1}{4}$

20. An unbiased die is tossed twice. What is the probability of getting a 4, 5 or 6 on the first toss and a 1, 2, 3 or 4 on the second toss?

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) $\frac{5}{6}$

21. A fair coin is tossed 6 times. What is the probability of getting at least 3 heads?

(a) $\frac{11}{16}$ (b) $\frac{21}{32}$ (c) $\frac{1}{18}$ (d) $\frac{3}{64}$

22. A coin is tossed 5 times. What is the probability that tail appears an odd number of times?

(a) $\frac{3}{5}$ (b) $\frac{2}{15}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

23. A coin is tossed 5 times. What is the probability that head appears an even number of times?

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{4}{15}$ (d) $\frac{1}{2}$

24. 8 coins are tossed simultaneously. The probability of getting 6 heads is

(a) $\frac{7}{64}$ (b) $\frac{57}{64}$ (c) $\frac{37}{256}$ (d) $\frac{249}{256}$

25. A die is thrown 5 times. If getting an odd number is a success, then what is the probability of getting at least 4 successes?

(a) $\frac{4}{5}$ (b) $\frac{7}{16}$ (c) $\frac{3}{16}$ (d) $\frac{3}{20}$

26. In 4 throws of a pair of dice, what is the probability of throwing doublets at least twice?

(a) $\frac{7}{36}$ (b) $\frac{17}{144}$ (c) $\frac{19}{144}$ (d) none of these

27. A pair of dice is thrown 7 times. If getting a total of 7 is considered a success, what is the probability of getting at most 6 successes?

(a) $\left(\frac{5}{6}\right)^7$ (b) $\left(\frac{1}{6}\right)^7$ (c) $\left(1-\frac{1}{6^7}\right)$ (d) none of these

28. The probability that a man can hit a target is $\frac{3}{4}\cdot$ He tries five times. What is the probability that he will hit the target at least 3 times?

(a) $\frac{459}{512}$ (b) $\frac{291}{364}$ (c) $\frac{371}{464}$ (d) none of these

29. The probability of the safe arrival of one ship out of 5 is $\frac{1}{5}\cdot$ What is the probability of the safe arrival of at least 3 ships?

(a) $\frac{1}{31}$ (b) $\frac{3}{52}$ (c) $\frac{181}{3125}$ (d) $\frac{184}{3125}$

30. The probability that an event E occurs in one trial is 0.4. Three independent trials of the experiment are performed. What is the probability that E occurs at least once?

(a) 0.784 (b) 0.936 (c) 0.964 (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (b)	**2.** (c)	**3.** (b)	**4.** (d)	**5.** (d)	**6.** (c)	**7.** (b)	**8** (a)	**9.** (b)	**10.** (a)
11. (b)	**12.** (d)	**13.** (b)	**14.** (a)	**15.** (a)	**16.** (b)	**17.** (d)	**18.** (b)	**19.** (a)	**20.** (a)
21. (b)	**22.** (c)	**23.** (d)	**24.** (c)	**25.** (c)	**26.** (c)	**27.** (c)	**28.** (a)	**29.** (c)	**30.** (a)

HINTS TO THE GIVEN OBJECTIVE QUESTIONS

1. We have $A \cap B = \phi \Rightarrow P(A \cap B) = 0.$

$\therefore \quad P(A \cup B) = P(A) + P(B) \Rightarrow 0.5 = 0.4 + x \Rightarrow x = 0.1.$

2. We have $P(A \cap B) = P(A) \times P(B) = 0.4 \times x.$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad 0.5 = 0.4 + x - 0.4x \Rightarrow 0.6x = 0.1 \Rightarrow x = \frac{1}{6}\cdot$

3. $P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}\cdot$

$\Rightarrow \quad P(A \cap B) = P(B/A)\cdot P(A) = (0.4 \times 0.8) = 0.32.$

$\Rightarrow \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.32}{0.5} = 0.64.$

4. $P(A \cup B) = P(A) + P(B) - P(A \cap B).$

$\Rightarrow \quad P(A \cap B) = \left(\frac{6}{11} + \frac{5}{11} - \frac{7}{11}\right) = \frac{4}{11}\cdot$

$\Rightarrow \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{(4/11)}{(5/11)} = \frac{4}{5} = 0.8.$

5. $P\{(A \cap B)'\} = \frac{1}{4} \Rightarrow P(A \cap B) = \left(1 - \frac{1}{4}\right) = \frac{3}{4}\cdot$

Since $P(A \cap B) \neq 0$, so A and B are not mutually exclusive.

$$P(A) \times P(B) = \left(\frac{1}{2} \times \frac{7}{12}\right) = \frac{7}{24} \neq P(A \cap B).$$

$\Rightarrow$ A and B are not independent.

6. Let E_1 be the event that A can solve the problem and E_2 be the event that B can solve the problem. Then, E_1 and E_2 are independent.

$\therefore \quad P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \left(\frac{3}{5} \times \frac{2}{3}\right) = \frac{2}{5}.$

$\therefore \quad P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \left(\frac{3}{5} + \frac{2}{3} - \frac{2}{5}\right) = \frac{13}{15}.$

7. $P(A') = \left(1 - \frac{1}{6}\right) = \frac{5}{6}, P(B') = \left(1 - \frac{1}{5}\right) = \frac{4}{5}$ and $P(C') = \left(1 - \frac{1}{3}\right) = \frac{2}{3}.$

P(that the problem is not solved) $= P(A'$ and B' and $C')$

$= P(A') \times P(B') \times P(C') = \left(\frac{5}{6} \times \frac{4}{5} \times \frac{2}{3}\right) = \frac{4}{9}.$

P(that the problem is solved) $= \left(1 - \frac{4}{9}\right) = \frac{5}{9}.$

8. Let E_1, E_2 and E_3 be the events that A, B and C can hit respectively.

Then, $P(E_1) = \frac{4}{5}, P(E_2) = \frac{3}{4}$ and $P(E_3) = \frac{2}{3}.$

Required probability $= P(E_2$ and E_3 but not $E_1) = P(E_2) \times P(E_3) \times P(E_1')$

$= \left\{\frac{3}{4} \times \frac{2}{3} \times \left(1 - \frac{4}{5}\right)\right\} = \left(\frac{1}{2} \times \frac{1}{5}\right) = \frac{1}{10}.$

9. Let E_1, E_2 and E_3 be the events that the 1st, 2nd and 3rd component function. Then,

$P(E_1) = (1 - 0.2) = 0.8, P(E_2) = (1 - 0.3) = 0.7$ and $P(E_3) = (1 - 0.5) = 0.5.$

P(machine fails) $= 1 - P$(machine functions)

$= 1 - P(E_1, E_2$ and $E_3) = 1 - \{P(E_1) \times P(E_2) \times P(E_3)\}$

$= 1 - (0.8 \times 0.7 \times 0.5) = (1 - 0.280) = 0.72.$

10. Here $S = \{1, 2, 3, 4, 5, 6\}.$

Let $A = \{1, 3, 5\}, B = \{2, 3, 5\}.$ Then, $A \cap B = \{3, 5\}.$

$\therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}, P(B) = \frac{3}{6} = \frac{1}{2}$ and $P(A \cap B) = \frac{2}{6} = \frac{1}{3}.$

$\therefore \quad P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{(1/3)}{(1/2)} = \frac{2}{3}.$

11. $B = (\overline{A} \cup A) \cap B = (\overline{A} \cap B) \cup (A \cap B)$

$\Rightarrow \quad P(B) = P(\overline{A} \cap B) + P(A \cap B)$

$\Rightarrow \quad P(\overline{A} \cap B) = P(B) - P(A \cap B) = (0.2 - 0.1) = 0.1.$

12. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \left(\frac{1}{4} + \frac{1}{3} - \frac{1}{5}\right) = \frac{23}{60}.$

$\therefore \quad P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = \left(1 - \frac{23}{60}\right) = \frac{37}{60}.$

$P(\overline{B}/\overline{A}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{A})} = \left(\frac{37}{60} \times \frac{4}{3}\right) = \frac{37}{45}.$

13. $P(B/A) = \dfrac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = 0.6 \times 0.4 = 0.24.$

$\therefore \quad P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.24}{0.8} = \dfrac{24}{80} = \dfrac{3}{10} = 0.3.$

14. $P(\overline{A}/\overline{B}) = \dfrac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \dfrac{P(\overline{A}) \cdot P(\overline{B})}{P(\overline{B})} = P(\overline{A}) = 1 - P(A).$

15. $P(B) = 1 - P(\overline{B}) = \left(1 - \dfrac{1}{2}\right) = \dfrac{1}{2}.$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow P(A) = \dfrac{2}{3}$

$P(A) \cdot P(B) = \left(\dfrac{2}{3} \times \dfrac{1}{2}\right) = \dfrac{1}{3} = P(A \cap B).$

16. Let $A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

and $B = \{(1,2),(2,1),(2,2),(3,2),(2,3),(4,2),(2,4),(5,2),(2,5),(6,2),(2,6)\}$

$\therefore \quad A \cap B = \{(2,5),(5,2)\}$

$P(B/A) = \dfrac{n(A \cap B)}{n(A)} = \dfrac{2}{6} = \dfrac{1}{3}.$

17. Out of given 9 numbers, 4 are even and 5 odd.

Let A = event of choosing odd number and B be the event of getting the sum even.

$n(B) = ({}^4C_2 + {}^5C_2) = 16$ and $n(A \cap B) = {}^5C_2 = 10.$

$\therefore \quad P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{10}{16} = \dfrac{5}{8}.$

18. Let A = event of reading mathematics and B of reading biology.

$P(A) = \dfrac{60}{100} = \dfrac{3}{5}, P(B) = \dfrac{25}{100} = \dfrac{1}{4}$ and $P(A \cap B) = \dfrac{15}{100} = \dfrac{3}{20}.$

$\therefore \quad P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \left(\dfrac{3}{20} \times \dfrac{4}{1}\right) = \dfrac{3}{5}.$

19. Let $S = \{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}.$

Let A = event that both are boys and B = event that one of the two is a boy.

Then, $A = \{B_1B_2\}$, $B = \{B_1B_2, B_1G_2, G_1B_2\}$ and $A \cap B = \{B_1B_2\}.$

$\therefore \quad P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{1}{3}.$

20. Here $S = \{1,2,3,4,5,6\}$. Let $A = \{4,5,6\}$ and $B = \{1,2,3,4\}.$

$\therefore \quad P(A) = \dfrac{3}{6} = \dfrac{1}{2}$ and $P(B) = \dfrac{4}{6} = \dfrac{2}{3}.$

Clearly, A and B are independent events.

$\therefore \quad P(A \cap B) = P(A) \times P(B) = \left(\dfrac{1}{2} \times \dfrac{2}{3}\right) = \dfrac{1}{3}.$

21. In a single throw, we have $P(H) = \dfrac{1}{2}$ and $P(\text{not } H) = \dfrac{1}{2}.$

$\therefore \quad p = \dfrac{1}{2}, q = \dfrac{1}{2}$ and $n = 6.$

Required probability = P(3 heads or 4 heads or 5 heads or 6 heads)

$$= {}^6C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^3 + {}^6C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2 + {}^6C_5 \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right) + {}^6C_6 \cdot \left(\frac{1}{2}\right)^6$$

$$= \left(20 \times \frac{1}{64} + 15 \times \frac{1}{64} + 6 \times \frac{1}{64} + \frac{1}{64}\right) = \frac{42}{64} = \frac{21}{32}.$$

22. In a single throw, we have $P(T) = \frac{1}{2}$ and $P(\text{not } T) = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}, q = \frac{1}{2}$ and $n = 5$.

Required probability = $P(X = 1)$ or $P(X = 3)$ or $P(X = 5)$

$$= P(X = 1) + P(X = 3) + P(X = 5)$$

$$= {}^5C_1 \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^4 + {}^5C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2 + {}^5C_5 \cdot \left(\frac{1}{2}\right)^5$$

$$= \left(\frac{5}{32} + \frac{10}{32} + \frac{1}{32}\right) = \frac{16}{32} = \frac{1}{2}.$$

23. In a single throw, we have $P(H) = \frac{1}{2}$ and $P(\text{not } H) = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}, q = \frac{1}{2}$ and $n = 5$.

Required probability = $P(X = 0)$ or $P(X = 2)$ or $P(X = 4)$

$$= P(X = 0) + P(X = 2) + P(X = 4)$$

$$= {}^5C_0 \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^5 + {}^5C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^3 + {}^5C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^1$$

$$= \left(\frac{1}{32} + \frac{10}{32} + \frac{5}{32}\right) = \frac{16}{32} = \frac{1}{2}.$$

24. In a single throw, we have $P(H) = \frac{1}{2}$ and $P(\text{not } H) = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}, q = \frac{1}{2}$ and $n = 8$.

Required probability = P(6 heads or 7 heads or 8 heads)

$$= P(\text{6 heads}) + P(\text{7 heads}) + P(\text{8 heads})$$

$$= {}^8C_6 \cdot \left(\frac{1}{2}\right)^6 \cdot \left(\frac{1}{2}\right)^2 + {}^8C_7 \cdot \left(\frac{1}{2}\right)^7 \cdot \left(\frac{1}{2}\right)^1 + {}^8C_8 \cdot \left(\frac{1}{2}\right)^8$$

$$= \left(\frac{28}{256} + \frac{8}{256} + \frac{1}{256}\right) = \frac{37}{256}.$$

25. In a single throw of a die, P(getting an odd number) $= \frac{3}{6} = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}, q = (1 - p) = \frac{1}{2}$ and $n = 5$.

Required probability = P(4 successes or 5 successes)

$$= {}^5C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^1 + {}^5C_5 \cdot \left(\frac{1}{2}\right)^5 = \left(\frac{5}{32} + \frac{1}{32}\right) = \frac{6}{32} = \frac{3}{16}.$$

26. In a single throw of a pair of dice, we have

$n(S) = (6 \times 6) = 36.$

Let $E = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$.

Then, $P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}$, $P(\text{not } E) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}\cdot$

$\therefore \quad p = \frac{1}{6}, q = \frac{5}{6}$ and $n = 4$.

Required probability $= P(2 \text{ or } 3 \text{ or } 4 \text{ successes})$

$$= {}^4C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^2 + {}^4C_3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right) + {}^4C_4 \cdot \left(\frac{1}{6}\right)^4$$

$$= \left(\frac{25}{216} + \frac{5}{324} + \frac{1}{1296}\right) = \frac{171}{1296} = \frac{19}{144}\cdot$$

27. In a single throw of a pair of dice, we have

$n(S) = (6 \times 6) = 36.$

Let $E = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$

$\therefore \quad p = \frac{6}{36} = \frac{1}{6}, q = \left(1 - \frac{1}{6}\right) = \frac{5}{6}$ and $n = 7$.

$P(7 \text{ successes}) = {}^7C_7 \cdot \left(\frac{1}{6}\right)^7 = \left(\frac{1}{6}\right)^7\cdot$

$P(\text{at most 6 successes}) = 1 - P(7 \text{ successes}) = 1 - \left(\frac{1}{6}\right)^7 = \left(1 - \frac{1}{6^7}\right)\cdot$

28. Here $p = \frac{3}{4}, q = \left(1 - \frac{3}{4}\right) = \frac{1}{4}$ and $n = 5$.

$P(3 \text{ or } 4 \text{ or } 5 \text{ successes}) = P(3 \text{ successes}) + P(4 \text{ successes}) + P(5 \text{ successes})$

$$= {}^5C_3 \cdot \left(\frac{3}{4}\right)^3 \cdot \left(\frac{1}{4}\right)^2 + {}^5C_4 \cdot \left(\frac{3}{4}\right)^4 \cdot \frac{1}{4} + {}^5C_5 \cdot \left(\frac{3}{4}\right)^5$$

$$= \left(\frac{135}{512} + \frac{405}{1024} + \frac{243}{1024}\right) = \frac{918}{1024} = \frac{459}{512}\cdot$$

29. Here $p = \frac{1}{5}, q = \left(1 - \frac{1}{5}\right) = \frac{4}{5}$ and $n = 5$.

$\therefore \quad P(3 \text{ or } 4 \text{ or } 5 \text{ successes}) = P(3 \text{ successes}) + P(4 \text{ successes}) + P(5 \text{ successes})$

$$= {}^5C_3 \cdot \left(\frac{1}{5}\right)^3 \cdot \left(\frac{4}{5}\right)^2 + {}^5C_4 \cdot \left(\frac{1}{5}\right)^4 \cdot \left(\frac{4}{5}\right)^1 + {}^5C_5 \cdot \left(\frac{1}{5}\right)^5$$

$$= \left(\frac{32}{625} + \frac{4}{625} + \frac{1}{3125}\right) = \frac{181}{3125}\cdot$$

30. $p = 0.4, q = (1 - 0.4) = 0.6$ and $n = 3$.

Required probability $= P(E \text{ occuring at least once})$

$$= {}^3C_1 \cdot (0.4)^1 \times (0.6)^2 + {}^3C_2 \cdot (0.4)^2 \times (0.6)^1 + {}^3C_3 \cdot (0.4)^3$$

$$= \left(\frac{54}{125} + \frac{36}{125} + \frac{8}{125}\right) = \frac{98}{125} = 0.784.$$

33. LINEAR PROGRAMMING

Linear Inequations in Two Variables

The inequalities of the form

$$ax + by \le c,\ ax + by < c,\ ax + by \ge c \text{ and } ax + by > c$$

are called *linear inequations* in two variables x and y.

The points (x, y) for which the inequation is true, constitute its solution set.

Graph of a Linear Inequation

Let us consider an inequation, $ax + by \le c$. For drawing its graph to obtain a solution set, we proceed as under.

STEP 1 Consider the equation $ax + by = c$, and plot the resulting line. In case of strict inequalities < or >, draw the line as dotted, otherwise mark it thick.

STEP 2 Choose a point [if possible (0, 0)], not lying on this line. Substitute its coordinates in the inequation. If the inequation is satisfied then shade the portion of the plane which contains the chosen point; otherwise shade the portion which does not contain this point.

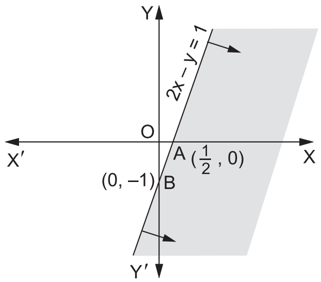
The shaded portion represents the solution set. The dotted line is not a part of the shaded region while the thick line is a part of it.

EXAMPLE *Graph the solution set of the inequation* $2x - y \ge 1$.

SOLUTION Consider the equation $2x - y = 1$.

We may write it as $\dfrac{x}{(1/2)} + \dfrac{y}{-1} = 1$.

This shows that the line $2x - y = 1$ makes intercepts of $\dfrac{1}{2}$ and -1 on the axes. Thus, the line meets the x-axis at $A\left(\dfrac{1}{2}, 0\right)$ and the y-axis at $B(0, -1)$. We plot these points and join them by a thick line.



Consider $O(0, 0)$. Clearly, (0, 0) does not satisfy the given inequation.

So, out of the portions divided by this line, the one not containing $O(0, 0)$, together with the points on the line, forms the solution set.

Simultaneous Inequations

The solution set of a system of linear inequations in two variables is the set of all points (x, y) which satisfy all the inequations in the system simultaneously.

So, we find the region of the plane common to all the portions comprising the solution sets of the given inequations. When there is no region common to all the solutions of the given inequations, we say that the solution set of the system is empty.

The linear inequations are also known as *linear constraints.*

SOLVED EXAMPLES

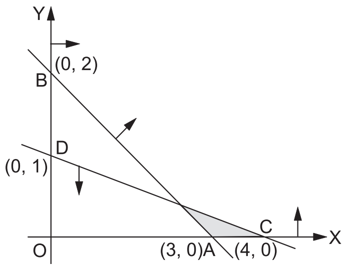
EXAMPLE 1 *Draw the graph of the solution set of the system of inequations*

$$2x + 3y \geq 6,\ x + 4y \leq 4,\ x \geq 0 \text{ and } y \geq 0.$$

SOLUTION Consider the equations

$$2x + 3y = 6,\ x + 4y = 4,\ x = 0 \text{ and } y = 0.$$

Now, $2x + 3y = 6 \Rightarrow \dfrac{x}{3} + \dfrac{y}{2} = 1.$



This line meets the axes at $A(3, 0)$ and $B(0, 2)$. Join these points and draw a thick line. Clearly, the portion not containing (0, 0) represents the solution set of the inequation $2x + 3y \geq 6$.

Again, $x + 4y = 4 \Rightarrow \dfrac{x}{4} + \dfrac{y}{1} = 1.$

This line meets the axes at $C(4, 0)$ and $D(0, 1)$. Join these points and draw a thick line. Clearly, the portion containing (0, 0) represents the solution set of the inequation $x + 4y \leq 4$.

Clearly, $x \geq 0$ is represented by the y-axis and the portion on its right-hand side.

Also, $y \geq 0$ is represented by the x-axis and the portion above the x-axis.

Hence, the shaded region represents the solution set of the given inequations.

EXAMPLE 2 *Exhibit graphically the solution set of the system of linear inequations*

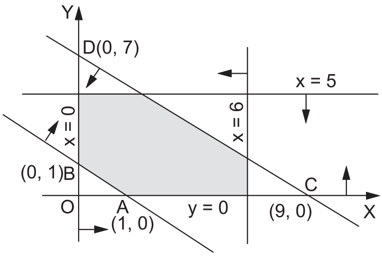
$$x + y \geq 1, 7x + 9y \leq 63, y \leq 5, x \leq 6, x \geq 0 \text{ and } y \geq 0.$$

SOLUTION $x + y = 1$ meets the axes at $A(1, 0)$ and $B(0, 1)$.

Join these points by a thick line. Clearly, the portion not containing $O(0, 0)$ is the solution set of $x + y \geq 1$.

$$7x + 9y = 63 \Rightarrow \frac{x}{9} + \frac{y}{7} = 1.$$

This line meets the axes at $C(9, 0)$ and $D(0, 7)$. Join these points by a thick line. Clearly, the portion containing (0, 0) is the solution set of $7x + 9y \leq 63$.



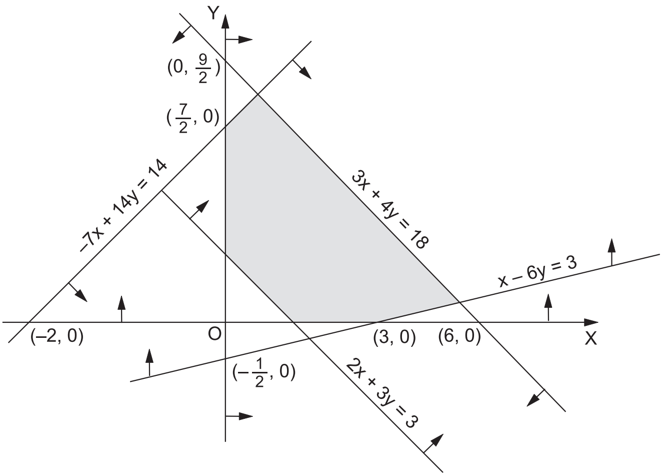
$y = 5$ is a line parallel to the x-axis at a distance 5 from the x-axis and the portion containing $O(0, 0)$ is the solution set of the inequation $y \leq 5$.

$x = 6$ is a line parallel to the y-axis at a distance 6 from the y-axis and the portion containing $(0, 0)$ is the solution set of $x \leq 6$.

Clearly, $x \geq 0$ has a solution represented by the y-axis and the portion on its right. Also, $y \geq 0$ has a solution represented by the x-axis and the portion above it.

The shaded region represents the solution set of the given system of inequations.

EXAMPLE 3 *Find the linear constraints for which the shaded area in the figure below is the solution set.*



SOLUTION Consider the line $3x + 4y = 18$.

Clearly, $O(0, 0)$ satisfies $3x + 4y \leq 18$.

Clearly, the shaded area and (0, 0) lie on the same side of the line $3x + 4y = 18$.

So, we must have $3x + 4y \leq 18$.

Consider the line $x - 6y = 3$.

Clearly, (0, 0) satisfies the inequation $x - 6y \leq 3$.

Also, the shaded area and (0, 0) lie on the same side of the line $x - 6y = 3$.

So, we must have $x - 6y \leq 3$.

Consider the line $2x + 3y = 3$.

Clearly, (0, 0) satisfies the inequation $2x + 3y \leq 3$.

But, the shaded region and the point (0, 0) lie on the opposite sides of the line $2x + 3y = 3$.

So, we must have $2x + 3y \geq 3$.

Consider the line $-7x + 14y = 14$.

Clearly, (0, 0) satisfies the inequation $-7x + 14y \leq 14$.

Also, the shaded region and the point (0, 0) lie on the same side of the line $-7x + 14y = 14$.

So, we must have $-7x + 14y \leq 14$.

The shaded region is above the x-axis and on the right-hand side of the y-axis, so we have $y \geq 0$ and $x \geq 0$.

Thus, the linear constraints for which the shaded area in the given figure is the solution set, are

$$3x + 4y \leq 18,\ x - 6y \leq 3,\ 2x + 3y \geq 3,$$
$$-7x + 14y \leq 14,\ x \geq 0 \text{ and } y \geq 0.$$

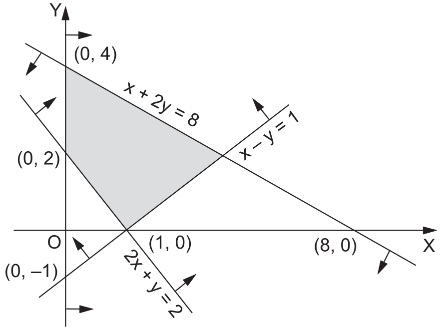
EXERCISE 33A

Graph the solution sets of the following inequations:

1. $x + y \geq 4$
2. $x - y \leq 3$
3. $x + 2y > 1$
4. $2x - 3y < 4$
5. $x \geq y - 2$
6. $y - 2 \leq 3x$

Solve each of the following systems of simultaneous inequations:

7. $2x + y > 1$ and $2x - y \geq -3$
8. $x - 2y \geq 0,\ 2x - y \leq -2$
9. $3x + 4y \geq 12,\ x \geq 0,\ y \geq 1$ and $4x + 7y \leq 28$
10. Show that the solution set of the following linear constraints is empty:
    $x - 2y \geq 0,\ 2x - y \leq -2,\ x \geq 0$ and $y \geq 0$.
11. Find the linear constraints for which the shaded area in the figure given is the solution set.



ANSWERS (EXERCISE 33A)

11. $x \geq 0, y \geq 0, 2x + y \geq 2, x - y \leq 1$ and $x + 2y \leq 8$

Linear Programming

Linear programming is the method used in decision making in business for obtaining the maximum or minimum value of a linear expression, subject to satisfying certain given linear inequations.

The linear expression is known as an *objective function* and the linear inequations are known as *linear constraints.*

LINEAR CONSTRAINTS In business or industry we want to make the best use of our limited resources like money, labour, time, materials, etc.

The limitations on the resources can often be expressed in the form of linear inequations, known as *linear constraints.*

OBJECTIVE FUNCTION *A linear function of the involved variables, which we want to maximize or minimize, subject to the given linear constraints, is known as an objective function.*

OPTIMAL VALUE OF AN OBJECTIVE FUNCTION *The maximum or minimum value of an objective function is known as its optimal value.*

FEASIBLE SOLUTION *A set of values of the variables satisfying all the constraints is known as a feasible solution of the system of inequations.*

OPTIMAL SOLUTION *A feasible solution which leads to the optimal value of an objective function is known as an optimal solution of the system of inequations.*

OPTIMIZATION TECHNIQUES *The processes of obtaining the optimal values of a system of inequations are called optimization techniques.*

A Linear Programming Problem (LPP)

A general linear programming problem consists of maximizing or minimizing an objective function, subject to certain given constraints.

Formulation of a Linear Programming Problem (LPP)

Working rules

STEP 1 Identify the unknowns in the given LPP. Denote them by x and y.

STEP 2 Formulate the objective function in terms of x and y. Be sure whether it is to be maximized or minimized.

STEP 3 Translate all the constraints in the form of linear inequations.

STEP 4 Solve these inequations simultaneously. Mark the common area by a shaded region. This is the feasible region.

STEP 5 Find the coordinates of all the vertices of the feasible region.

STEP 6 Find the value of the objective function at each vertex of the feasible region.

STEP 7 Find the values of x and y for which the objective function $Z = ax + by$ has maximum or minimum value (as the case may be).

Graphical Solution of an LPP

We shall restrict ourselves to the case of an LPP in two variables. We shall consider at least three constraints or inequations. Each inequation gives rise to a line in the plane. For a simultaneous solution of these inequations, we consider the region common to their solution sets. In each case we obtain such a region, a convex polygon, i.e., a closed region bounded by straight lines with the property that the line joining any two points of the region lies wholly in the region.

The maximum or minimum value of a linear function over a convex polygon occurs at some vertex of the polygon.

So, we look at the values of the objective function at the vertices of the set of feasible solutions. The largest of these values is the maximum value of the objective function and the smallest of these values is the minimum.

Graphical Method

It will be clear from the following solved examples.

SOLVED EXAMPLES

EXAMPLE 1 *Solve the following problem graphically:*
Minimize and maximize $z = 3x + 9y$, *subject to the constraints*

$$x + 3y \le 60,\ x + y \ge 10,\ x \le y,\ x \ge 0 \text{ and } y \ge 0.$$

SOLUTION ***Region represented by*** $\boldsymbol{x + 3y \le 60}$

Consider the equation $x + 3y = 60$.

$$x = 0 \Rightarrow 3y = 60 \Rightarrow y = 20;\ y = 0 \Rightarrow x = 60$$

Plot the points $A(0, 20)$ and $B(60, 0)$. Join AB and produce it both ways.

Putting $x = 0$ and $y = 0$, we get $0 + 3 \times 0 = 0 \leq 60$.

$\therefore$ $O(0, 0)$ lies in the region $x + 3y \leq 60$.

So, the region containing the origin is the solution set of $x + 3y \leq 60$.

Region represented by $\boldsymbol{x + y \geq 10}$

Consider the equation $x + y = 10$.

$x = 0 \Rightarrow y = 10$; $y = 0 \Rightarrow x = 10$.

Plot the points $C(0, 10)$ and $D(10, 0)$. Join CD and produce it both ways.

Now, $x = 0$, $y = 0 \Rightarrow 0 + 0 \geq 10$ is not true.

$\therefore$ $O(0, 0)$ does not lie in the region $x + y \geq 10$.

Region represented by $\boldsymbol{x \leq y}$***, i.e.,*** $x - y \leq 0$

Consider the equation $x = y$, i.e., $x - y = 0$.

Clearly, $x = 10 \Rightarrow y = 10$. And, $x = 20 \Rightarrow y = 20$.

Plot the points $E(10, 10)$ and $F(20, 20)$. Join EF and produce it both ways.

Clearly, $O(0, 0)$ satisfies $x - y \leq 0$.

$\therefore$ $O(0, 0)$ lies in the region $x - y \leq 0$.

We know that:

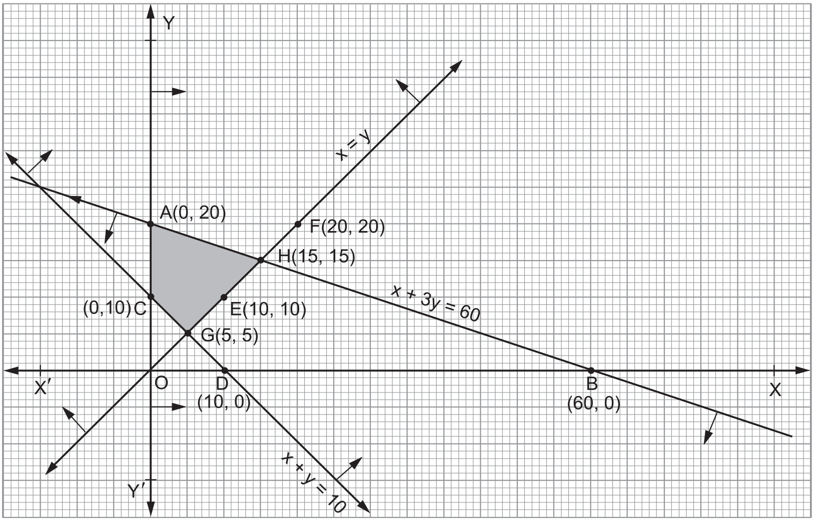
$x \geq 0$ is the y-axis and the region on its RHS.

$y \geq 0$ is the x-axis and the region above the x-axis.

On solving $x = y$ and $x + y = 10$, we get the point $G(5, 5)$.

On solving $x = y$ and $x + 3y = 60$, we get $H(15, 15)$.

Thus, the feasible region is $ACGH$, as shown in the figure.



Value of $z = 3x + 9y$:

(i) At $A(0, 20)$ it is $(3 \times 0 + 9 \times 20) = 180$.

(ii) At $C(0, 10)$ it is $(3 \times 0 + 9 \times 10) = 90$.

(iii) At $G(5, 5)$ it is $(3 \times 5 + 9 \times 5) = 60$.

(iv) At $H(15, 15)$ it is $(3 \times 15 + 9 \times 15) = 180$.

So, the minimum value of z is 60 and its maximum value is 180.

EXAMPLE 2 *A furniture dealer deals in only two items: tables and chairs. He has* ₹ 5000 *to invest and a space to store at most* 60 *pieces. A table costs him* ₹ 250 *and a chair,* ₹ 50. *He can sell a table at a profit of* ₹ 50 *and a chair at a profit of* ₹ 15. *Assuming that he can sell all the items that he buys, how should he invest his money in order that he may maximize his profit?* **[CBSE 2006C]**

SOLUTION This problem can be formulated as under.

Let x and y be the required numbers of tables and chairs respectively. Then, clearly we have

$x \geq 0$, $y \geq 0$; $x + y \leq 60$;

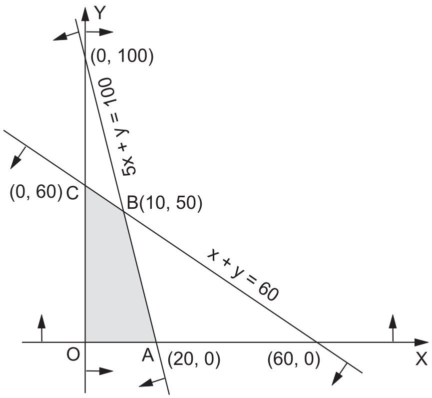
$250x + 50y \leq 5000$, i.e., $5x + y \leq 100$.

Let P be the profit function. Then, $P = 50x + 15y$.

Now, we have to maximize P.

Now, $x + y = 60 \Rightarrow \frac{x}{60} + \frac{y}{60} = 1$.

This line meets the axes at (60, 0) and (0, 60). Plot these points and join them to get the line $x + y = 60$.



Also, $5x + y = 100 \Rightarrow \frac{x}{20} + \frac{y}{100} = 1$.

This line meets the axes at (20, 0) and (0, 100). Plot these points and join them to get the line $5x + y = 100$.

Also, the line $x = 0$ is the y-axis and the line $y = 0$ is the x-axis.

These four straight lines enclose the quadrilateral $OABC$.

The coordinates of the points O, A, B, C are (0, 0), (20, 0), (10, 50) and (0, 60) respectively.

At these points, the corresponding values of $P = 50x + 15y$ are 0, 1000, 1250 and 900 respectively.

Clearly, it is maximum at $B(10, 50)$.

So, for a maximum profit, the dealer should purchase 10 tables and 50 chairs.

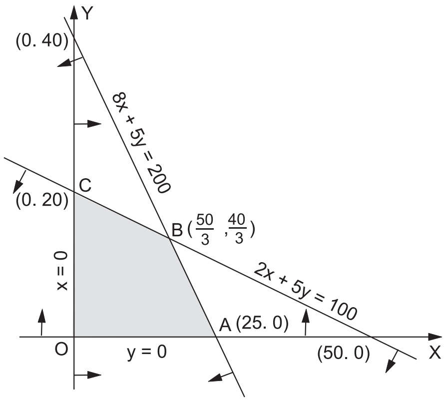
EXAMPLE 3 *If a young man rides his motorcycle at* 25 km *per hour, he has to spend* ₹ 2 *per kilometre on petrol; if he rides it at a faster speed of* 40 km *per hour, the petrol cost increases to* ₹ 5 *per kilometre. He has* ₹ 100 *to spend on petrol and wishes to find the maximum distance he can travel within one hour. Express this as a linear programming problem and then solve it.* **[CBSE 2013C]**

SOLUTION Suppose that the young man rides x km at 25 km per hour and y km at 40 km per hour. Then, we have to maximize $P = x + y$.

Clearly, $x \geq 0$, $y \geq 0$, $2x + 5y \leq 100$.

Since the available time is at most one hour, we have

$$\frac{x}{25} + \frac{y}{40} \leq 1 \text{ or } 8x + 5y \leq 200.$$



Now, we solve the system of the inequations.

$$2x + 5y = 100 \Rightarrow \frac{x}{50} + \frac{y}{20} = 1.$$

This line meets the axes at (50, 0) and (0, 20). Plot these points and join them to get the line $2x + 5y = 100$.

Also, $8x + 5y = 200 \Rightarrow \frac{x}{25} + \frac{y}{40} = 1.$

This line meets the axes at (25, 0) and (0, 40). Plot these points and obtain the line $8x + 5y = 200$.

$x = 0$ is the y-axis and $y = 0$ is the x-axis.

We find that the solution set of the above system is the shaded region $OABC$.

The coordinates of O, A, B, C are (0, 0), (25, 0), $\left(\frac{50}{3}, \frac{40}{3}\right)$ and (0, 20) respectively.

The values of $P = x + y$ at these points are 0, 25, 30 and 20 respectively.

So, $P = x + y$ is maximum when $x = \frac{50}{3}$ and $y = \frac{40}{3}\cdot$

Thus, the young man can cover the maximum distance of 30 km, if he rides $\frac{50}{3}$ km at 25 km/h and $\frac{40}{3}$ km at 40 km/h.

EXAMPLE 4 *Suppose every gram of wheat provides* 0.1 g *of proteins and* 0.25 g *of carbohydrates, and the corresponding values for rice are* 0.05 g *and* 0.5 g *respectively. Wheat costs* ₹ 5 *and rice* ₹ 20 *per kilogram. The minimum daily requirements of proteins and carbohydrates for an average man are* 50 g *and* 200 g *respectively. In what quantities should wheat and rice be mixed in the daily diet to provide the minimum daily requirements of proteins and carbohydrates at minimum cost, assuming that both wheat and rice are to be taken in the diet?*

SOLUTION Let x g of wheat and y g of rice be mixed to fulfil the requirements.

Then, we have to minimize the cost function

$$Z = \frac{5x}{1000} + \frac{20y}{1000}, \text{ i.e., } Z = \frac{x}{200} + \frac{y}{50}\cdot \qquad \text{... (i)}$$

x g of wheat and y g of rice must give at least 50 g of proteins.

So, we must have

$$0.1x + 0.05y \geq 50 \text{ or } 2x + y \geq 1000.$$

Similarly, x g of wheat and y g of rice must give at least 200 g of carbohydrates.

So, we must have

$$0.25x + 0.5y \geq 200 \text{ or } x + 2y \geq 800.$$

Thus, we have to minimize $Z = \frac{x}{200} + \frac{y}{50}$, subject to the constraints

$$x > 0, y > 0, 2x + y \geq 1000 \text{ and } x + 2y \geq 800.$$

$2x + y = 1000 \Rightarrow \frac{x}{500} + \frac{y}{1000} = 1.$

This line meets the axes at $A(500, 0)$ and $B(0, 1000)$.

Plot these points and join them to obtain the line $2x + y = 1000$.

Clearly, $(0, 0)$ does not satisfy $2x + y \geq 1000$.

Again, $x + 2y = 800 \Rightarrow \dfrac{x}{800} + \dfrac{y}{400} = 1$.

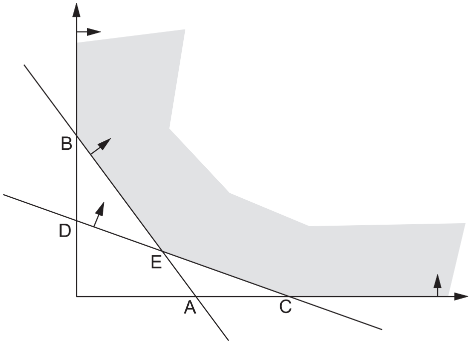
This line meets the axes at $C(800, 0)$ and $D(0, 400)$.

Plot these points and join them to obtain the line $x + 2y = 800$.

Clearly, $(0, 0)$ does not satisfy $x + 2y \geq 800$.

$x = 0$ is the y-axis and $y = 0$ is the x-axis.

We obtain the solution set of the above system, as shown by the shaded region.



Solving $2x + y = 1000$ and $x + 2y = 800$, we get the point of intersection of AB and CD, given by $E(400, 200)$.

The minimum value of $Z = \dfrac{x}{200} + \dfrac{y}{50}$ would be at some vertex of the unbounded feasible region BEC.

Clearly, at B we have $x = 0$, and at C we have $y = 0$.

Also, the value of Z at $E(400, 200) = \dfrac{400}{200} + \dfrac{200}{50} = 6$.

So, we must have 400 g of wheat and 200 g of rice.

EXAMPLE 5 *A firm manufactures two types of products, A and B, and sells them at a profit of ₹ 5 per unit of type A and ₹ 3 per unit of type B. Each product is processed on two machines, M_1 and M_2. One unit of type A requires one minute of processing time on M_1 and two minutes of processing time on M_2; whereas one unit of type B requires one minute of processing time on M_1 and one minute on M_2. Machines M_1 and M_2 are respectively available for at most 5 hours and 6 hours in a day. Find out how many units of each type of product should the firm produce a day in order to maximize the profit. Solve the problem graphically.* **[CBSE 2000]**

SOLUTION Let x units of A and y units of B be produced in order to have a maximum profit.

Then, clearly $x \geq 0$ and $y \geq 0$.

x units of A and y units of B will take $(x + y)$ minutes on M_1.

$\therefore \quad x + y \leq 300.$

x units of A and y units of B will take $(2x + y)$ minutes on M_2.

$\therefore \quad 2x + y \leq 360.$

Let Z be the profit function. Then, $Z = 5x + 3y$.

We have to maximize $Z = 5x + 3y$, subject to the constraints

$x \geq 0, \ y \geq 0, \ x + y \leq 300$ and $2x + y \leq 360$.

Now, $x + y = 300 \ \Rightarrow \ \dfrac{x}{300} + \dfrac{y}{300} = 1.$

This line meets the axes in (300, 0) and (0, 300).

Joining these points, we get the line $x + y = 300$.

Since (0, 0) satisfies the inequation $x + y \leq 300$, the region below the line $x + y = 300$ containing $O(0, 0)$ represents $x + y \leq 300$.

Again, $2x + y = 360 \ \Rightarrow \ \dfrac{x}{180} + \dfrac{y}{360} = 1.$

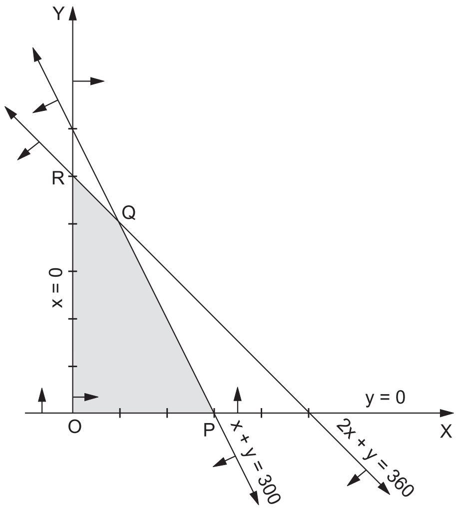
This line meets the axes in (180, 0) and (0, 360).

Joining these points, we get the line $2x + y = 360$.

Since (0, 0) satisfies $2x + y \leq 360$, the region below the line $2x + y = 360$ containing (0, 0) represents $2x + y \leq 360$.

Also $x = 0$ is the y-axis and $y = 0$ is the x-axis.

On drawing these lines and shading the feasible region, we obtain a figure, given below.



On solving $x = 0$ and $x + y = 300$, we get the point $R(0, 300)$.

On solving $x + y = 300$ and $2x + y = 360$, we get the point $Q(60, 240)$.

$\therefore$ the vertices of the feasible region are

$$O(0, 0),\ P(180, 0),\ Q(60, 240) \text{ and } R(0, 300).$$

Values of $Z = 5x + 3y$ at O, P, Q, R are 0, 900, 1020, 900 respectively.

$\therefore$ Z is maximum when $x = 60$ and $y = 240$.

EXAMPLE 6 *An aeroplane of an airline can carry a maximum of* 200 *passengers. A profit of* ₹ 400 *is made on each first-class ticket and a profit of* ₹ 300 *is made on each economy-class ticket. The airline reserves at least* 20 *seats for first class. However, at least* 4 *times as many passengers prefer to travel by economy class than by first class. Determine how many of each type of tickets must be sold in order to maximize the profit for the airline. What is the maximum profit?*

SOLUTION Let x tickets of first class and y tickets of economy class be sold to maximize the profit. Then,

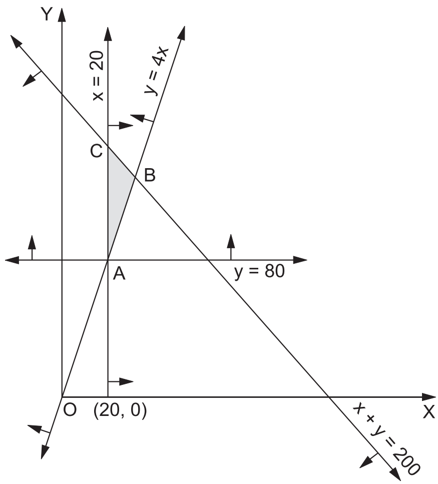
$$x \geq 20,\ y \geq 4x,\ y \geq 80 \text{ and } x + y \leq 200.$$

The profit function is given by $Z = 400x + 300y$.

Draw the graphs of the lines

$$x = 20,\ y = 4x,\ y = 80 \text{ and } x + y = 200$$

as shown below.



Graph of the inequation $x \geq 20$

Since (0, 0) does not satisfy $x \geq 20$, the line $x = 20$ together with the region to its right-hand side, not containing (0, 0), represents the region $x \geq 20$.

Graph of the inequation $y \geq 4x$

Clearly, since (20, 0) does not satisfy the inequation $y \geq 4x$, the line

$y = 4x$ together with the region to its left, not containing (20, 0), represents $y \geq 4x$.

Graph of the inequation $y \geq 80$

Clearly, the line $y = 80$ and the region above this line represents $y \geq 80$.

Graph of the inequation $x + y \leq 200$

Clearly, (0, 0) satisfies $x + y \leq 200$. So, the line $x + y = 200$ together with the region containing $O(0, 0)$ represents $x + y \leq 200$.

Thus, the shaded region in the given figure is the feasible region, whose vertices are A, B and C. A is the point of intersection of $x = 20$ and $y = 80$.

So, its coordinates are $A(20, 80)$.

On solving $y = 4x$ and $x + y = 200$, we get $B(40, 160)$.

On solving $x = 20$ and $x + y = 200$, we get $C(20, 180)$.

The values of $Z = 400x + 300y$ at $A(20, 80)$, $B(40, 160)$ and $C(20, 180)$ are respectively ₹ 32000, ₹ 64000 and ₹ 62000.

$\therefore$ Z is maximum at $x = 40$, $y = 160$.

EXAMPLE 7 *A chemical industry produces two compounds, A and B. The following table gives the units of ingredients C and D (per* kg*) of compounds A and B as well as minimum requirements of C and D, and costs per* kg *of A and B.*

	Compound (in units)		*Minimum requirement (in units)*
	A	*B*	
Ingredient C (per kg*)*	1	2	80
Ingredient D (per kg*)*	3	1	75
Cost per kg (in ₹)	4	6	

Find the quantities of A and B which would minimize the cost.

SOLUTION Let x kg of A and y kg of B be produced. Then,

$$x \geq 0,\ y \geq 0,\ x + 2y \geq 80 \text{ and } 3x + y \geq 75.$$

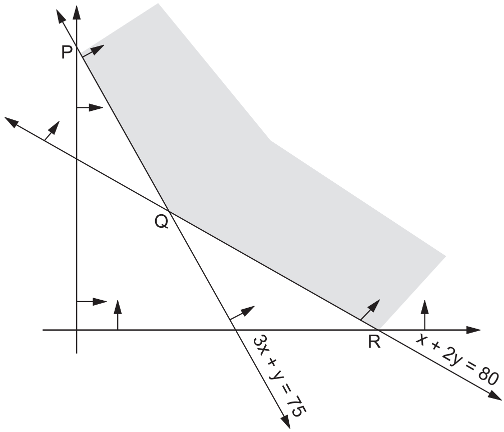
Then cost function is given by $Z = 4x + 6y$.

Thus, we have to minimize $Z = 4x + 6y$, subject to the constraints:

$$x \geq 0,\ y \geq 0,\ x + 2y \geq 80 \text{ and } 3x + y \geq 75.$$

Draw the graphs of the lines

$$x = 0,\ y = 0,\ x + 2y = 80 \text{ and } 3x + y = 75.$$



Since (0, 0) does not satisfy the inequation $x + 2y \geq 80$, the line $x + 2y = 80$ together with the region not containing (0, 0) represents $x + 2y \geq 80$.

Since (0, 0) does not satisfy the inequation $3x + y \geq 75$, the line $3x + y = 75$ together with the region not containing (0, 0) represents $3x + y \geq 75$.

Thus, the shaded region is the feasible region.

The vertices of this region are P, Q and R.

On solving $x = 0$ and $3x + y = 75$, we get the point $P(0, 75)$.

On solving $x + 2y = 80$ and $3x + y = 75$, we get the point $Q(14, 33)$.

On solving $y = 0$ and $x + 2y = 80$, we get the point $R(80, 0)$.

The values of $Z = 4x + 6y$ at the points $P(0, 75)$, $Q(14, 33)$ and $R(80, 0)$ are 450, 254 and 320 respectively.

Thus, Z is minimum at $Q(14, 33)$.

Hence, for a minimum cost, 14 kg of A and 33 kg of B must be taken.

EXAMPLE 8 *A company makes two types of belts, A and B; profits on these belts being ₹ 4 and ₹ 3 each respectively. Each belt of type A requires twice as much time as a belt of type B, and if all belts were of type B, the company could make* 1000 *belts per day. The supply of leather is sufficient for only* 800 *belts per day (both A and B combined). At the most* 400 *buckles for belts of type A and* 700 *for those of type B are available per day. How many belts of each type should the company make per day so as to maximize the profit?*

SOLUTION Let x belts of type A and y belts of type B be made.

Then, $x \geq 0$, $y \geq 0$, $x \leq 400$, $y \leq 700$ and $x + y \leq 800$.

Now, 1000 belts of type B can be made in 1 day.

$\therefore$ 500 belts of type A can be made in 1 day.

$\therefore$ time taken to make x belts of type A and y belts of type B

$$= \left(\frac{x}{500} + \frac{y}{1000}\right) \text{ days.}$$

$\therefore$ $\frac{x}{500} + \frac{y}{1000} \leq 1$, i.e., $2x + y \leq 1000.$

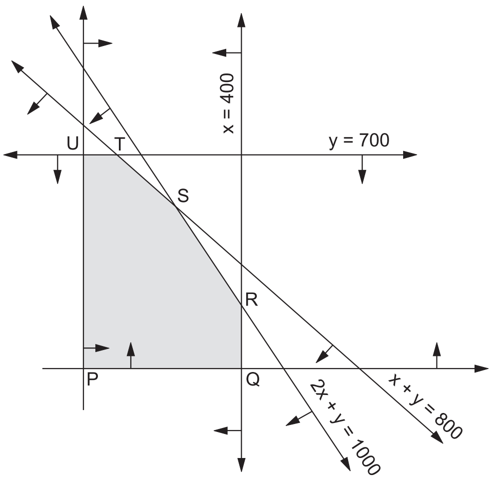
We have to maximize $Z = 4x + 3y$, subject to the constraints

$$x \geq 0,\ y \geq 0,\ x \leq 400,\ y \leq 700,\ x + y \leq 800 \text{ and } 2x + y \leq 1000.$$

We draw the graphs of the lines

$$x = 0,\ y = 0,\ x = 400,\ y = 700,\ x + y = 800,\ 2x + y = 1000$$

as shown below.



Since (0, 0) satisfies $x \leq 400$, the line $x = 400$ together with the region containing $O(0, 0)$ represents $x \leq 400$.

Since (0, 0) satisfies $y \leq 700$, the line $y = 700$ together with the region containing $O(0, 0)$ represents $y \leq 700$.

Since (0, 0) satisfies $x + y \leq 800$, the line $x + y = 800$ together with the region containing $O(0, 0)$ represents $x + y \leq 800$.

Since (0, 0) satisfies $2x + y \leq 1000$, the line $2x + y = 1000$ together with the region containing $O(0, 0)$ represents $2x + y \leq 1000$.

The y-axis and the region to its right-hand side represents $x \geq 0$.

The x-axis and the region above it represents $y \geq 0$.

Thus, the shaded region represents the feasible region, whose vertices are P, Q, R, S, T and U.

Clearly, the coordinates of P and Q are (0, 0) and (400, 0) respectively.

On solving $x = 400$ and $2x + y = 1000$, we get $R(400, 200)$.

On solving $x + y = 800$ and $2x + y = 1000$, we get $S(200, 600)$.

On solving $y = 700$ and $x + y = 800$, we get $T(100, 700)$.

On solving $x = 0$ and $y = 700$, we get $U(0, 700)$.

The values of $Z = 4x + 3y$ at the points P, Q, R, S, T and E are respectively 0, 1600, 2200, 2600, 2500 and 2100.

The maximum of these values is 2600 occurring at $S(200, 600)$.

$\therefore$ Z is maximum when $x = 200$ and $y = 600$.

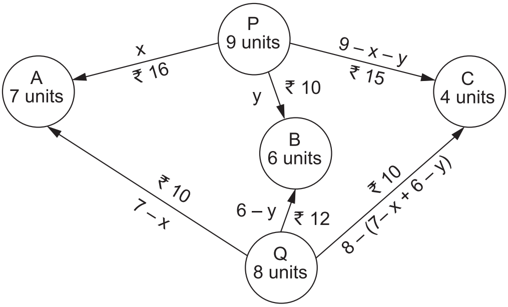
Thus, the company should make 200 belts of type A and 600 belts of type B to have a maximum profit.

EXAMPLE 9 *A company has factories located at each of the two places P and Q. From these locations, a certain commodity is delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively* 7, 6 *and* 4 *units of the commodity while the weekly production capacities of the factories at P and Q are respectively* 9 *and* 8 *units. The cost of transportation per unit is given below.*

To / From	Cost (in ₹)		
	A	*B*	*C*
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum? Formulate the above LPP *mathematically and then solve it.*

SOLUTION This problem can be explained diagramatically as follows:



Let x units and y units of the commodity be transported from the factory at P to the depots at A and B respectively. Then, $(9 - x - y)$ units will be transported from the factory at P to the depot at C.

$\therefore$ $x \geq 0$, $y \geq 0$, and $9 - x - y \geq 0 \Rightarrow x + y \leq 9$.

The weekly requirement of the depot at A is 7 units. So, $(7 - x)$ units will be transported to A from the factory at Q.

Similarly, $(6 - y)$ units will be transported to B from the factory at Q.

And, $8-(7-x+6-y)=(x+y-5)$ units will be transported to C from the factory at Q.

$\therefore \quad 7-x\geq 0,\ 6-y\geq 0$ and $x+y-5\geq 0,$
i.e., $x\leq 7,\ y\leq 6$ and $x+y\geq 5.$

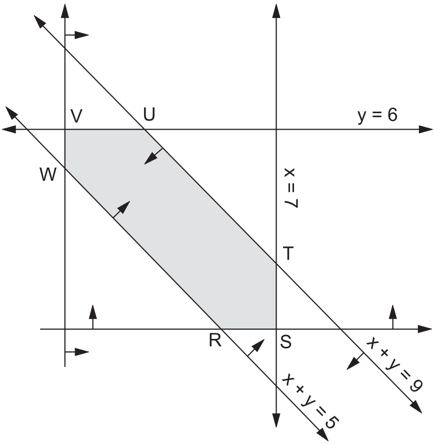
The total cost of transportation is

$$Z=16x+10(7-x)+10y+12(6-y)+15(9-x-y)+10(x+y-5)$$

$\Rightarrow\ Z=x-7y+227.$

Now, we have to find the values of x and y which minimize
$Z=x-7y+227$, subject to the constraints
$x\geq 0,\ y\geq 0,\ x+y\leq 9,\ x\leq 7,\ y\leq 6$ and $x+y\geq 5.$

We draw the graphs of the lines
$x=0,\ y=0,\ x+y=9,\ x=7,\ y=6$ and $x+y=5$
as shown below.



Since (0, 0) satisfies $x+y\leq 9$, the line $x+y=9$ together with the region containing $O(0, 0)$ represents $x+y\leq 9.$

Since (0, 0) does not satisfy $x+y\geq 5$, the line $x+y=5$ together with the region not containing $O(0, 0)$ represents $x+y\geq 5.$

Since (0, 0) satisfies $x\leq 7$, the line $x=7$ together with the region containing $O(0, 0)$ represents $x\leq 7.$

Since (0, 0) satisfies $y\leq 6$, the line $y=6$ together with the region containing $O(0, 0)$ represents $y\leq 6.$

The y-axis and the region to its right-hand side represents $x\geq 0.$

The x-axis and the region above it represents $y\geq 0.$

Thus, the shaded region represents the feasible region whose vertices are R, S, T, U, V and W.

On solving $y=0$ and $x+y=5$, we get $R(5, 0)$.

On solving $y=0$ and $x=7$, we get $S(7, 0)$.

On solving $x = 7$ and $x + y = 9$, we get $T(7, 2)$.

On solving $x + y = 9$ and $y = 6$, we get $U(3, 6)$.

On solving $x = 0$ and $y = 6$, we get $V(0, 6)$.

On solving $x = 0$ and $x + y = 5$, we get $W(0, 5)$.

The values of $Z = x - 7y + 227$ at R, S, T, U, V and W are

232, 234, 220, 188, 185 and 192 respectively.

Thus, Z is minimum at $V(0, 6)$, i.e., when $x = 0$ and $y = 6$.

$\therefore$ from the factory at P there must be a delivery of 0, 6 and 3 units to A, B, C respectively.

And, the factory at Q must deliver 7, 0 and 1 units to A, B, C respectively.

EXAMPLE 10 *A factory owner purchases two types of machines A and B for his factory. The requirements and the limitations for the machines are as follows:*

Machine	*Area occupied*	*Labour force on each machine*	*Daily output (in units)*
A	$1000\ m^2$	12 men	60
B	$1200\ m^2$	8 men	40

He has maximum area of $9000\ m^2$ *available and* 72 *skilled labourers who can operate both the machines. How many machines of each type should he buy to maximize the daily output?* **[CBSE 2008]**

SOLUTION Let x machines of type A and y machines of type B be bought and let z be the daily output.

Then, $z = 60x + 40y$. … (i)

Maximum area available = $9000\ m^2$.

$\therefore \quad 1000x + 1200y \le 9000$

$\Rightarrow \quad 5x + 6y \le 45$. … (ii)

Maximum labour available = 72 men.

$\therefore \quad 12x + 8y \le 72 \quad \Rightarrow \quad 3x + 2y \le 18$. … (iii)

Now, we have to maximize $z = 60x + 40y$, subject to the constraints

$$5x + 6y \le 45,$$
$$3x + 2y \le 18,$$
$$x \ge 0 \text{ and } y \ge 0.$$

Now, $5x + 6y = 45 \quad \Rightarrow \quad \dfrac{x}{9} + \dfrac{y}{(15/2)} = 1$.

This line meets the axes at $A(9, 0)$ and $B\left(0, \dfrac{15}{2}\right)$.

Plot these points and join them to obtain the line $5x + 6y = 45$.

Clearly, (0, 0) satisfies $5x + 6y \leq 45$.

So, the region below AB represents $5x + 6y \leq 45$.

Again, $3x + 2y = 18 \Rightarrow \frac{x}{6} + \frac{y}{9} = 1$.

This line meets the axes at $C(6, 0)$ and $D(0, 9)$.

Plot these points and join them to obtain the line $3x + 2y = 18$.

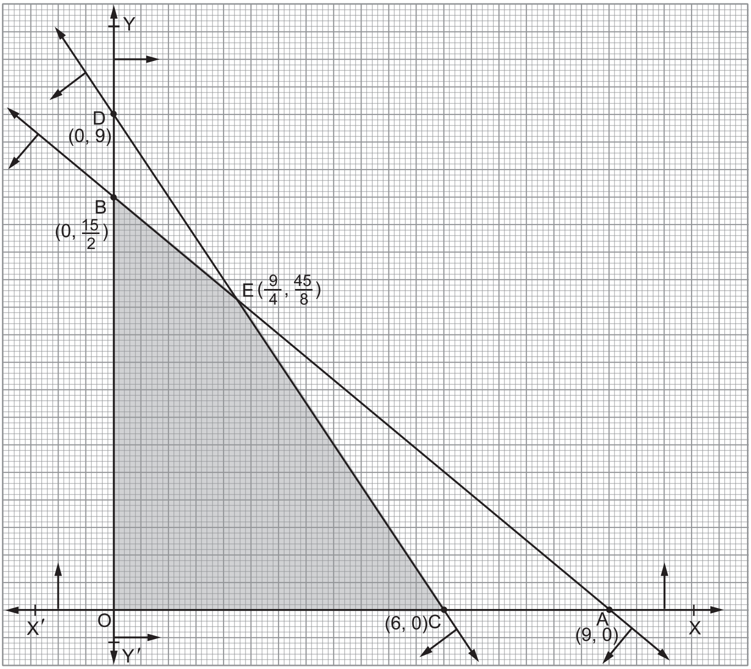
Clearly, (0, 0) satisfies $3x + 2y \leq 18$.

So, the region below CD represents $3x + 2y \leq 18$.

$x \geq 0$ is the region to the right of the y-axis.

And, $y \geq 0$ is the region above the x-axis.

On solving $5x + 6y = 45$ and $3x + 2y = 18$ simultaneously, we get $x = \frac{9}{4}$ and $y = \frac{45}{8}$.



So, the lines AB and CD intersect at $E\left(\frac{9}{4}, \frac{45}{8}\right)$.

Thus, the corner points of the feasible region are

$$O(0, 0),\ C(6, 0),\ E\left(\frac{9}{4}, \frac{45}{8}\right) \text{ and } B\left(0, \frac{15}{2}\right).$$

Value of daily output $z = 60x + 40y$:

(i) At $O(0, 0)$ it is $z = (60 \times 0 + 40 \times 0) = 0$.

(ii) At $C(6, 0)$ it is $z = (60 \times 6 + 40 \times 0) = 360$.

(iii) At $E\left(\frac{9}{4}, \frac{45}{8}\right)$ it is $z = \left(60 \times \frac{9}{4} + 40 \times \frac{45}{8}\right) = 360$.

(iv) At $B\left(0, \frac{15}{2}\right)$ it is $z = \left(60 \times 0 + 40 \times \frac{15}{2}\right) = 300$.

Thus, either (6 machines of type A and no machine of type B)
or (2 machines of type A and 6 machines of type B) be used to have maximum output.

[**NOTE** $\frac{9}{4}$ machines $\equiv$ 2 machines and $\frac{45}{8}$ machines $\equiv$ 6 machines.]

EXAMPLE 11 *A retired person has ₹ 70000 to invest and two types of bonds are available in the market for investment. First type of bond yields an annual income of 8% on the amount invested and the second type of bond yields 10% per annum. As per norms, he has to invest minimum of ₹ 10000 in the first type and not more than ₹ 30000 in the second type. How should he plan his investment, so as to get maximum return, after one year of investment?* **[CBSE 2007]**

SOLUTION Let bonds A be at 8% and bonds B be at 10%.

Suppose he plans to invest ₹ x in bonds A and ₹ y in bonds B.

Then, clearly $x + y = 70000$. ... (i)

He invests minimum of ₹ 10000 in bonds A.

$\therefore \quad x \geq 10000$. ... (ii)

Also, he invests not more than ₹ 30000 in bonds B.

$\therefore \quad y \leq 30000$ ... (iii)

Let z be the annual return on these investments. Then,

$$z = \frac{8x}{100} + \frac{10y}{100} \Rightarrow z = 0.08x + 0.1y. \quad \text{... (iv)}$$

Thus, we have to maximize z, subject to the conditions

$$\left.\begin{aligned} x + y &= 70000 \\ x &\geq 10000 \\ y &\leq 30000 \end{aligned}\right\}$$

Now, $x + y = 70000 \Rightarrow \frac{x}{70000} + \frac{y}{70000} = 1$.

This line meets the axes at $A(70000, 0)$ and $B(0, 70000)$.

Plot these points and join them to obtain the line $x + y = 70000$.

Clearly, $(0, 0)$ satisfies $x + y \leq 70000$.

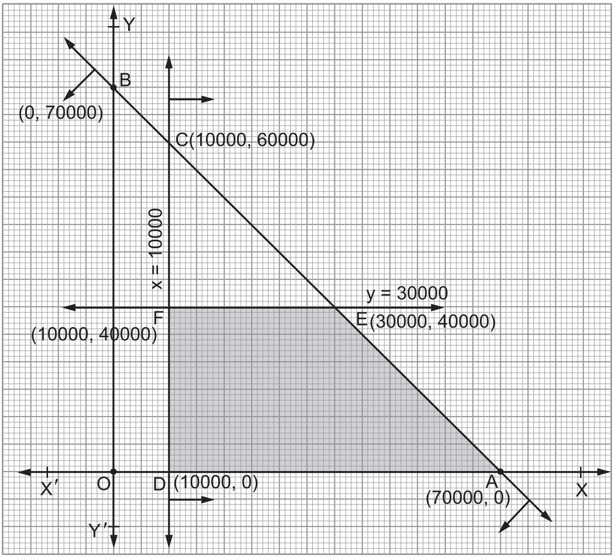
So, the region below AB represents $x + y \leq 70000$.

$x \geq 10000$ is the region parallel to the y-axis and to the right of it beyond the line $x = 10000$.

Thus, the region to the right of line CD represents $x \geq 10000$.

$y \leq 30000$ is the region parallel to the x-axis and above it, but below the line $y = 30000$.

Thus, the region below the line EF and above the x-axis represents $y \leq 30000$.



Thus, the corner points of the feasible region are

$D(10000, 0)$, $A(70000, 0)$, $E(30000, 40000)$ and $F(10000, 40000)$.

Value of annual return $z = 0.08x + 0.1y$:

(i) At $D(10000, 0)$ it is $z = (0.08 \times 10000 + 0.1 \times 0) = 800$.

(ii) At $A(70000, 0)$ it is $z = (0.08 \times 70000 + 0.1 \times 0) = 5600$.

(iii) At $E(30000, 40000)$ it is $z = (0.08 \times 30000 + 0.1 \times 40000) = 6400$.

(iv) At $F(10000, 40000)$ it is $z = (0.08 \times 10000 + 0.1 \times 40000) = 4800$.

So, in order to get a maximum annual return, he should invest ₹ 30000 in bond A and ₹ 40000 in bond B.

EXERCISE 33B

Long-Answer Questions

1. Find the maximum value of $Z = 7x + 7y$, subject to the constraints
$$x \geq 0, y \geq 0, x + y \geq 2 \text{ and } 2x + 3y \leq 6.$$

2. Maximize $Z = 4x + 9y$, subject to the constraints
$$x \geq 0, y \geq 0, x + 5y \leq 200, 2x + 3y \leq 134.$$

3. Find the maximum value of $Z = 3x + 5y$, subject to the constraints
$$-2x + y \leq 4, x + y \geq 3, x - 2y \leq 2, x \geq 0 \text{ and } y \geq 0.$$

4. Minimize $Z = 2x + 3y$, subject to the constraints
$$x \geq 0, y \geq 0, x + 2y \geq 1 \text{ and } x + 2y \leq 10.$$

5. Maximize $Z = 3x + 5y$, subject to the constraints
$$x + 2y \leq 2000, x + y \leq 1500, y \leq 600, x \geq 0 \text{ and } y \geq 0.$$

6. Find the maximum and minimum values of $Z = 2x + y$, subject to the constraints
$$x + 3y \geq 6, x - 3y \leq 3, 3x + 4y \leq 24,$$
$$-3x + 2y \leq 6, 5x + y \geq 5, x \geq 0 \text{ and } y \geq 0.$$

7. Mr Dass wants to invest ₹ 12000 in Public Provident Fund (PPF) and in National bonds. He has to invest at least ₹ 1000 in PPF and at least ₹ 2000 in bonds. If the rate of interest on PPF is 12% per annum and that on bonds is 15% per annum, how should he invest the money to earn maximum annual income? Also find the maximum annual income.

8. A small firm manufactures necklaces and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes 1 hour to make a bracelet and half an hour to make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹ 100 and that on a bracelet is ₹ 300, how many of each should be produced daily to maximize the profit? It is being given that at least one of each must be produced.

9. A man has ₹ 1500 to purchase rice and wheat. A bag of rice and a bag of wheat cost ₹ 180 and ₹ 120 respectively. He has a storage capacity of 10 bags only. He earns a profit of ₹ 11 and ₹ 8 per bag of rice and wheat respectively. How many bags of each must he buy to make maximum profit? **[CBSE 2008C, '09C]**

10. A manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machine B to produce a packet of nuts while it takes 3 hours on machine A and 1 hour on machine B to produce a packet of bolts. He earns a profit of ₹ 17.50 per packet on nuts and ₹ 7 per packet on bolts. How many packets of each should be produced each day so as to maximize his profit if he operates his machines for at the most 12 hours a day? Also find the maximum profit. **[CBSE 2009C, '12]**

11. Two tailors, A and B, earn ₹ 300 and ₹ 400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. How many days should each of them work if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost?

12. A dealer wishes to purchase a number of fans and sewing machines. He has only ₹ 5760 to invest and has space for at most 20 items. A fan costs him ₹ 360 and a sewing machine, ₹ 240. He expects to gain ₹ 22 on a fan and ₹ 18 on a sewing machine. Assuming that he can sell all the items he can buy, how should he invest the money in order to maximize the profit?
[CBSE 2009, '09C]

13. A firm manufactures two types of products, *A* and *B*, and sells them at a profit of ₹ 2 on type *A* and ₹ 2 on type *B*. Each product is processed on two machines, M_1 and M_2. Type *A* requires one minute of processing time on M_1 and two minutes on M_2. Type *B* requires one minute on M_1 and one minute on M_2. The machine M_1 is available for not more than 6 hours 40 minutes while M_2 is available for at most 10 hours a day.

    Find how many products of each type the firm should produce each day in order to get maximum profit.

14. A manufacturer produces two types of soap bars using two machines, *A* and *B*. *A* is operated for 2 minutes and *B* for 3 minutes to manufacture the first type, while it takes 3 minutes on machine *A* and 5 minutes on machine *B* to manufacture the second type. Each machine can be operated at the most for 8 hours per day. The two types of soap bars are sold at a profit of ₹ 0.25 and ₹ 0.50 each. Assuming that the manufacturer can sell all the soap bars he can manufacture, how many bars of soap of each type should be manufactured per day so as to maximize his profit?

15. A manufacturer of a line of patent medicines is preparing a production plan on medicines *A* and *B*. There are sufficient ingredients available to make 20000 bottles of *A* and 40000 bottles of *B* but there are only 45000 bottles into which either of the medicines can be put. Furthermore, it takes 3 hours to prepare enough material to fill 1000 bottles of *A* and it takes 1 hour to prepare enough material to fill 1000 bottles of *B*, and there are 66 hours available for this operation. The profit is ₹ 8 per bottle for *A* and ₹ 7 per bottle for *B*.

    How should the manufacturer schedule the production in order to maximize his profit? Also, find the maximum profit.

16. A toy company manufactures two types of dolls, *A* and *B*. Each doll of type *B* takes twice as long to produce as one of type *A*, and the company would have time to make a maximum of 2000 per day, if it produces only type *A*. The supply of plastic is sufficient to produce 1500 dolls per day (both *A* and *B* combined). Type *B* requires a fancy dress of which there are only 600 per day available. If the company makes profits of ₹ 3 and ₹ 5 per doll respectively on dolls *A* and *B*, how many of each should be produced per day in order to maximize the profit? Also, find the maximum profit.

17. A small manufacturer has employed 5 skilled men and 10 semiskilled men and makes an article in two qualities, a de luxe model and an ordinary model. The making of a de luxe model requires 2 hours' work by a skilled man and 2 hours' work by a semiskilled man. The ordinary model requires 1 hour by a skilled man and 3 hours by a semiskilled man. By union rules, no man can work more than 8 hours per day. The manufacturer gains ₹ 15 on the de luxe model and ₹ 10 on the ordinary model. How many of each type should be made in order to maximize his total daily profit? Also, find the maximum daily profit.

18. A company producing soft drinks has a contract which requires a minimum of 80 units of chemical A and 60 units of chemical B to go in each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. Supplier X has a mix of 4 units of A and 2 units of B that costs ₹ 10, and the supplier Y has a mix of 1 unit of A and 1 unit of B that costs ₹ 4. How many mixes from X and Y should the company purchase to honour the contract requirement and yet minimize the cost? **[CBSE 2012]**

19. A small firm manufactures gold rings and chains. The combined number of rings and chains manufactured per day is at most 24. It takes 1 hour to make a ring and half an hour for a chain. The maximum number of hours available per day is 16. If the profit on a ring is ₹ 300 and that on a chain is ₹ 190, how many of each should be manufactured daily so as to maximize the profit?

20. A manufacturer makes two types, A and B, of teapots. Three machines are needed for the manufacture and the time required for each teapot on the machines is given below.

Machine / Type	Time (in minutes)		
	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each teapot of type A is 75 paise and that on each teapot of type B is 50 paise, show that 15 teapots of type A and 30 of type B should be manufactured in a day to get the maximum profit.

21. A manufacturer makes two products, A and B. Product A sells at ₹ 200 each and takes $\frac{1}{2}$ hour to make. Product B sells at ₹ 300 each and takes 1 hour to make. There is a permanent order for 14 of product A and 16 of product B. A working week consists of 40 hours of production and the weekly turnover must not be less than ₹ 10000. If the profit on each of the product A is ₹ 20 and on product B, it is ₹ 30 then how many of each should be produced so that the profit is maximum? Also, find the maximum profit.

22. A man owns a field of area 1000 m^2. He wants to plant fruit trees in it. He has a sum of ₹ 1400 to purchase young trees. He has the choice of two types of trees. Type *A* requires 10 m^2 of ground per tree and costs ₹ 20 per tree, and type *B* requires 20 m^2 of ground per tree and costs ₹ 25 per tree. When full grown, a type-*A* tree produces an average of 20 kg of fruit which can be sold at a profit of ₹ 2 per kg and a type-*B* tree produces an average of 40 kg of fruit which can be sold at a profit of ₹ 1.50 per kg. How many of each type should be planted to achieve maximum profit when trees are fully grown? What is the maximum profit?

23. A publisher sells a hardcover edition of a book for ₹ 72 and a paperback edition of the same for ₹ 40. Costs to the publisher are ₹ 56 and ₹ 28 respectively in addition to weekly costs of ₹ 9600. Both types require 5 minutes of printing time although the hardcover edition requires 10 minutes of binding time and the paperback edition requires only 2 minutes. Both the printing and binding operations have 4800 minutes available each week. How many of each type of books should be produced in order to maximize the profit? Also, find the maximum profit per week.

24. A gardener has a supply of fertilizers of the type I which consists of 10% nitrogen and 6% phosphoric acid, and of the type II which consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, he finds that he needs at least 14 kg of nitrogen and 14 kg of phosphoric acid for his crop. If the type-I fertilizer costs 60 paise per kg and the type-II fertilizer costs 40 paise per kg, determine how many kilograms of each type of fertilizer should be used so that the nutrient requirements are met at a minimum cost. What is the minimum cost? **[CBSE 2008]**

25. Two godowns, *A* and *B*, have a grain storage capacity of 100 quintals and 50 quintals respectively. Their supply goes to three ration shops, *D*, *E* and *F*, whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table.

	Cost of transportation (in ₹ per quintal)	
From / To	*A*	*B*
D	6.00	4.00
E	3.00	2.00
F	2.50	3.00

How should the supplies be transported in order that the transportation cost is minimum?

26. A brick manufacturer has two depots, P and Q, with stocks of 30000 and 20000 bricks respectively. He receives orders from three buildings A, B, C, for 15000, 20000 and 15000 bricks respectively. The cost of transporting 1000 bricks to the buildings from the depots are given below.

	Cost of transportation (in ₹ per 1000 bricks)		
To / From	A	B	C
P	40	20	30
Q	20	60	40

How should the manufacturer fulfil the orders so as to keep the cost of transportation minimum?

27. A medicine company has factories at two places, X and Y. From these places, supply is made to each of its three agencies situated at P, Q and R. The monthly requirements of the agencies are respectively 40 packets, 40 packets and 50 packets of medicines, while the production capacity of the factories at X and Y are 60 packets and 70 packets respectively. The transportation cost per packet from the factories to the agencies are given as follows.

	Transportation cost per packet (in ₹)	
From / To	X	Y
P	5	4
Q	4	2
R	3	5

How many packets from each factory should be transported to each agency so that the cost of transportation is minimum? Also, find the minimum cost.

28. An oil company has two depots, A and B, with capacities of 7000 L and 4000 L respectively. The company is to supply oil to three petrol pumps, D, E, F, whose requirements are 4500 L, 3000 L and 3500 L respectively. The distances (in km) between the depots and the petrol pumps are given in the following table:

	Distance (in km)	
From / To	*A*	*B*
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost per km is Re 1 per litre, how should the delivery be scheduled in order that the transportation cost is minimum?

29. A firm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. They need certain nutrients, named as *X*, *Y*, *Z*. The pigs are fed on two products, *A* and *B*. One unit of product *A* contains 36 units of *X*, 3 units of *Y* and 20 units of *Z*, while one unit of product *B* contains 6 units of *X*, 12 units of *Y* and 10 units of *Z*. The minimum requirements of *X*, *Y*, *Z* are 108 units, 36 units and 100 units respectively. Product *A* costs ₹ 20 per unit and product *B* costs ₹ 40 per unit. How many units of each product must be taken to minimize the cost? Also, find the minimum cost.

30. A dietician wishes to mix two types of food, *X* and *Y*, in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 10 units of vitamin C. Food *X* contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C, while food *Y* contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs ₹ 5 per kg to purchase the food *X* and ₹ 7 per kg to purchase the food *Y*. Determine the minimum cost of such a mixture. **[CBSE 2011C, '12]**

31. A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 calories. Two foods, *A* and *B*, are available at a cost of ₹ 4 and ₹ 3 per unit respectively. If one unit of *A* contains 200 units of vitamins, 1 unit of minerals and 40 calories, and 1 unit of *B* contains 100 units of vitamins, 2 units of minerals and 40 calories, find what combination of foods should be used to have the least cost. **[CBSE 2008]**

32. A housewife wishes to mix together two kinds of food, *X* and *Y*, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. **[CBSE 2009C]**

    The vitamin contents of 1 kg of each food are given below.

	Vitamin A	Vitamin B	Vitamin C
Food *X*	1	2	3
Food *Y*	2	2	1

If 1 kg of food *X* costs ₹ 6 and 1 kg of food *Y* costs ₹ 10, find the minimum cost of the mixture which will produce the diet.

33. A firm manufactures two types of products, A and B, and sells them at a profit of ₹ 5 per unit of type A and ₹ 3 per unit of type B. Each product is processed on two machines, M_1 and M_2. One unit of type A requires one minute of processing time on M_1 and two minutes of processing time on M_2; whereas one unit of type B requires one minute of processing time on M_1 and one minute on M_2. Machines M_1 and M_2 are respectively available for at most 5 hours and 6 hours in a day. Find out how many units of each type of product the firm should produce a day in order to maximize the profit. Solve the problem graphically. **[CBSE 2000]**

34. A small firm manufactures items A and B. The total number of items that it can manufacture in a day is at the most 24. Item A takes one hour to make while item B takes only half an hour. The maximum time available per day is 16 hours. If the profit on one unit of item A be ₹ 300 and that on one unit of item B be ₹ 160, how many of each type of item should be produced to maximize the profit? Solve the problem graphically. **[CBSE 2000, '04]**

35. A manufacturer produces two types of steel trunks. He has two machines, A and B. The first type of trunk requires 3 hours on machine A and 3 hours on machine B. The second type requires 3 hours on machine A and 2 hours on machine B. Machines A and B can work at most for 18 hours and 15 hours per day respectively. He earns a profit of ₹ 30 and ₹ 25 per trunk of the first type and second type respectively. How many trunks of each type must he make each day to make the maximum profit? **[CBSE 2005, '12]**

36. A company manufactures two types of toys, A and B. Type A requires 5 minutes each for cutting and 10 minutes each for assembling. Type B requires 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours available for cutting and 4 hours available for assembling in a day. The profit is ₹ 50 each on type A and ₹ 60 each on type B. How many toys of each type should the company manufacture in a day to maximize the profit? **[CBSE 2001]**

37. Kellogg is a new cereal formed of a mixture of bran and rice, that contains at least 88 grams of protein and at least 36 milligrams of iron. Knowing that bran contains 80 grams of protein and 40 milligrams of iron per kilogram, and that rice contains 100 grams of protein and 30 milligrams of iron per kilogram, find the minimum cost of producing this new cereal if bran costs ₹ 5 per kilogram and rice costs ₹ 4 per kilogram. **[CBSE 2002]**

38. A dealer wishes to purchase a number of fans and sewing machines. He has only ₹ 5760 to invest and has space for at most 20 items. A fan costs him ₹ 360 and a sewing machine ₹ 240. He expects to sell a fan at a profit of ₹ 22 and a sewing machine at a profit of ₹ 18. Assuming that he can sell all the items that he buys, how should he invest his money to maximize the profit? Solve graphically and find the maximum profit. **[CBSE 2002, '03, '06]**

39. Anil wants to invest at the most ₹ 12000 in bonds A and B. According to rules, he has to invest at least ₹ 2000 in Bond A and at least ₹ 4000 in Bond B. If the rate of interest of Bond A is 8% per annum and on Bond B, it is 10% per annum, how should he invest his money for maximum interest? Solve the problem graphically. **[CBSE 2004C]**

40. Maximize $z = 60x + 15y$, subject to the constraints
$$x + y \leq 50, \quad 3x + y \leq 90, \quad x, y \geq 0.$$ **[CBSE 2005]**

41. A company manufactures two types of toys A and B. Type A requires 5 minutes each for cutting and 10 minutes each for assembling. Type B requires 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours available for cutting and 4 hours available for assembling in a day. He earns a profit of ₹ 50 each on type A and ₹ 60 each on type B. How many toys of each type should the company manufacture in a day to maximize the profit? **[CBSE 2007]**

42. One kind of cake requires 200 g of flour and 25 g of fat, another kind of cake requires 100 g of flour and 50 g of fat. Find the maximum number of cakes which can be made from 5 kg of flour and 1 kg of fat, assuming that there is no shortage of the other ingredients used in making the cakes. Make it an LPP and solve it graphically. **[CBSE 2014C]**

43. A manufacturing company makes two types of teaching aids A and B of Mathematics for class XII. Each type of A requires 9 labour hours of fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ₹ 80 on each piece of type A and ₹ 120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit? Make it as an LPP and solve graphically. What is the maximum profit per week? **[CBSE 2014]**

ANSWERS (EXERCISE 33B)

1. Maximum $Z = 21$ at $x = 3, y = 0$

2. Maximum $Z = 382$ when $x = 10$ and $y = 38$

3. Minimum $Z = 9\frac{2}{3}$ at $\left(\frac{8}{3}, \frac{1}{3}\right)$ **4.** Minimum $Z = \frac{3}{2}$ at $\left(0, \frac{1}{2}\right)$

5. Maximum $Z = 5500$ at $x = 1000$ and $y = 500$

6. Maximum $Z = 14\frac{1}{3}$ at $\left(\frac{84}{13}, \frac{15}{13}\right)$ and minimum $Z = 3\frac{1}{14}$ at $\left(\frac{9}{14}, \frac{25}{14}\right)$

7. Rs 1000 in PPF and ₹ 11000 in bonds; ₹ 1770

8. $x = 16$ and $y = 8$ **9.** 5 bags of each

10. Three packets each of nuts and bolts; maximum profit = ₹ 73.50

11. 5 days and 3 days

12. 24 sewing machines only **13.** 200, 200

14. For maximum profit, 96 soap bars of 2nd type must be manufactured.

15. 10500 bottles of A and 34500 bottles of B, and maximum profit is ₹ 325500

16. 1000 of type A and 500 of type B, maximum profit = ₹ 5500

17. 10 de luxe, 20 ordinary; maximum daily gain = ₹ 400

18. 10 mixes from X and 40 mixes from Y

19. Number of rings = 8, number of chains = 16

21. $A = 48$, $B = 16$; maximum profit = ₹ 1440

22. Type $A = 20$, type $B = 40$; maximum profit = ₹ 3200

23. 360 and 600; ₹ 2880

24. 100 kg of type I and 80 kg of type II; minimum cost = ₹ 92

25. 10 q, 50 q and 40 q from A to D, E, F respectively, and 50 q, 0 q and 0 q from B to D, E, F respectively

26. From P: 0, 20000, 10000 bricks to A, B, C respectively
From Q: 15000, 0, 5000 bricks to A, B, C respectively

27. From X: 10 pkt, 0 pkt and 50 pkt to P, Q, R respectively
From Y: 30 pkt, 40 pkt and 0 pkt to P, Q, R respectively
Minimum cost = ₹ 400

28. From A: 500 L, 3000 L, 3500 L to D, E, F respectively
From B: 4000 L, 0 L, 0 L to D, E, F respectively

29. 2 units of A, 4 units of B; minimum cost = ₹ 160

30. $x = 2$, $y = 4$; ₹ 38 **31.** 5 units of A and 30 units of B **32.** ₹ 52

33. Z is maximum at $P(60, 240)$ and its maximum value is ₹ 1020.

34. Z is maximum at (8, 16) and its maximum value is ₹ 4960.

35. For getting a maximum profit of ₹ 165, 3 trunks of each type should be manufactured.

36. For getting a maximum profit of ₹ 1500, 12 toys of type A and 15 toys of type B should be manufactured.

37. The minimum cost of producing this cereal is ₹ 4.60 per kg.

38. The profit is maximum at $E(8, 12)$ and it is ₹ 392.

39. The interest is maximum at $E(2000, 10000)$ and it is ₹ 1160.

40. The maximum value of z is at $C(30, 0)$.

41. 12 toys of type A and 15 toys of type B to earn maximum profit ₹ 1500

42. Max. no. of cakes: 1st kind—20, 2nd kind—10

43. For max. profit, 12 pieces of type A and 8 pieces of type B; max. profit = ₹ 1680

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 33B)

7. Maximize $Z = \frac{12x}{100} + \frac{15y}{100}$ subject to $x \geq 1000, y \geq 2000, x + y \leq 12000$.

8. Maximize $P = 100x + 300y$ subject to $x > 0, y > 0, x + y \leq 24, \frac{1}{2}x + y \leq 16$.

10. Maximize $P = 17.5x + 7y$ subject to $x \geq 0, y \geq 0, x + 3y \leq 12$ and $3x + y \leq 12$.

11. Let A and B work for x and y days respectively.
Minimize $Z = 300x + 400y$ subject to $x \geq 0, y \geq 0, 6x + 10y \geq 60$ and $4x + 4y \geq 32$.

12. Maximize $P = 22x + 18y$ subject to
$x \geq 0, y \geq 0, x + y \leq 20, 360x + 240y \leq 5760$.

13. Maximize $Z = 2x + 2y$ subject to $x \geq 0, y \geq 0, x + y \leq 400, 2x + y \leq 600$.

14. Maximize $P = \frac{x}{4} + \frac{y}{2}$ subject to $x \geq 0, y \geq 0, 2x + 3y \leq 480$ and $3x + 5y \leq 480$.

15. Maximize $Z = 8x + 7y$ subject to $3x + y \leq 66000, x + y \leq 45000, x \leq 20000, y \leq 40000$, $x \geq 0$ and $y \geq 0$.

16. Maximize $Z = 3x + 5y$ subject to $x + 2y \leq 2000, x + y \leq 1500, y \leq 600, x \geq 0$ and $y \geq 0$.

17. Suppose x de luxe models and y ordinary models of articles be made each day. Then,
maximize $Z = 15x + 10y$, subject to the constraints
$x \geq 0, y \geq 0, 2x + y \leq 40$ and $2x + 3y \leq 80$.

18. Let the company purchase x mixes from X and y mixes from Y. Then,
minimize $Z = 10x + 4y$, subject to the constraints
$x \geq 0, y \geq 0, 4x + y \geq 80$ and $2x + y \geq 60$.

20. Let x teapots of type A and y teapots of type B be manufactured. Then,
$x \geq 0, y \geq 0, 12x + 6y \leq 6 \times 60, 18x + 0y \leq 6 \times 60, 6x + 9y \leq 6 \times 60$
$\Rightarrow x \geq 0, y \geq 0, 2x + y \leq 60, x \leq 20, 2x + 3y \leq 120$.
Profit function is $Z = \frac{75}{100}x + \frac{50}{100}y \Rightarrow Z = \frac{3}{4}x + \frac{1}{2}y$.
Maximize Z, subject to the constraints
$x \geq 0, y \geq 0, x \leq 20, 2x + y \leq 60, 2x + 3y \leq 120$.
Then $x = 15$ and $y = 30$ will give the maximum value of Z.

21. Let the number of articles produced per week be x of A and y of B.
Then, $\frac{1}{2}x + y \leq 40, 200x + 300y \geq 10000, x \geq 14, y \geq 16$.
Profit function is $Z = 20x + 30y$.
Maximize $Z = 20x + 30y$, subject to the constraints
$x + 2y \leq 80, 2x + 3y \geq 1000, x \geq 14, y \geq 16$.

22. Let x plants of type A and y plants of type B be planted.
Then, $x \geq 0, y \geq 0, 20x + 25y \leq 1400, 10x + 20y \leq 1000$.
Profit function is $Z = 2 \times 20x + \frac{3}{2} \times 40y$, i.e., $Z = 40x + 60y$.

23. Let x copies of the hardcover edition and y copies of the paperback edition be prepared. Then,

$$C = 9600 + 56x + 28y \text{ and } S = 72x + 40y.$$

Profit function is $Z = (S - C)$, i.e., $Z = 16x + 12y - 9600$.

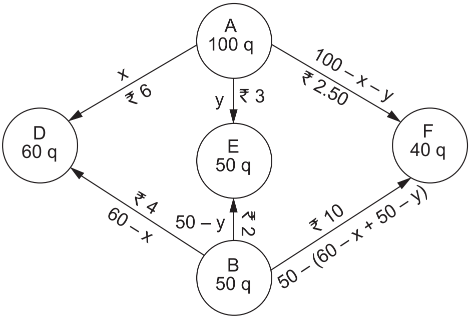
Maximize Z subject to $x \geq 0, y \geq 0, (5x + 10x) + (5y + 2y) \leq 4800$.

24. Let x kg of type I and y kg of type II be used. Then,

$$x \geq 0, y \geq 0, \frac{10}{100}x + \frac{5}{100}y \geq 14, \frac{6}{100}x + \frac{10}{100}y \geq 14.$$

Cost function is $Z = \frac{60}{100}x + \frac{40}{100}y$, i.e., $Z = \frac{3}{5}x + \frac{2}{5}y$.

25. The entire problem can be explained diagramatically as shown below.



Let x q and y q of grains be transported from A to shops D and E respectively. Then, $[100 - (x + y)]$ q will be transported from A to F.

$\therefore \quad x \geq 0, y \geq 0, 100 - (x + y) \geq 0 \;\Rightarrow\; x + y \leq 100.$

D, E, F require 60 q, 50 q and 40 q respectively.

From B, there will be a supply of $(60 - x)$q to D; $(50 - y)$q to E and $[50 - (60 - x + 50 - y)]$ q to F.

$\therefore \quad (60 - x) \geq 0, (50 - y) \geq 0, (x + y - 60) \geq 0$

$\Rightarrow \quad x \leq 60, y \leq 50$ and $x + y \geq 60$.

$\therefore$ cost of transportation is given by

$$Z = 6x + 4(60 - x) + 3y + 2(50 - y) + \frac{5}{2} + (100 - x - y) + 3(x + y - 60)$$

$$\Rightarrow \quad Z = \frac{5}{2}x + \frac{3}{2}y + 410.$$

28. Let x and y litres be transported from A to D and E.

Then, $(7000 - x - y)$ litres will be transported to F from A.

From B, $(4500 - x)$ litres, $(3000 - y)$ litres, $[4000 - (4500 - x + 3000 - y)]$ litres will be transported to D, E and F respectively.

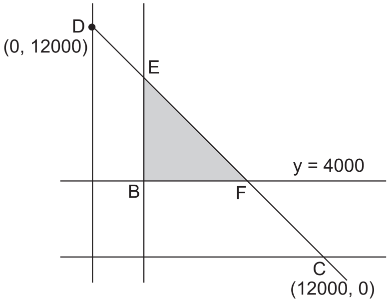
Cost of transportation is

$$Z = 7x + 6y + 3(7000 - x - y) + 3(45000 - x) + 4(3000 - y) + 2(x + y - 3500).$$

Minimize $Z = 3x + y + 39500$, subject to

$$x \geq 0, y \geq 0, x + y \leq 7000, x \leq 4500, y \leq 3000, x + y \geq 3500 \text{ and } x = 500.$$

29. Let x units of A and y units of B be taken. Then,
minimize $C = 20x + 40y$, subject to
$$x \geq 0, y \geq 0, 36x + 6y \geq 108, 3x + 12y \geq 36, 20x + 10y \geq 100.$$
30. Let x kg of the food X and y kg of the food Y be taken. Then,
$$x \geq 0, y \geq 0, 2x + y \geq 8 \text{ and } x + 2y \geq 10.$$
The cost function is $Z = 5x + 7y$.
31. Let x units of the food A and y units of the food B be taken. Then,
$$x \geq 0, y \geq 0, 200x + 100y \geq 4000, x + 2y \geq 50, 40x + 40y \geq 1400.$$
The cost function is $Z = 4x + 3y$.
32. Let x kg of the food X and y kg of the food Y be taken. Then,
$$x \geq 0, y \geq 0, x + 2y \geq 10, 2x + 2y \geq 12 \text{ and } 3x + y \geq 8.$$
The cost function is $Z = 6x + 10y$.
33. Let x units of the type A and y units of the type B be produced.
Then, $x_1 + x_2 \leq 300; 2x_1 + x_2 \leq 360; x_1 \geq 0$ and $x_2 \geq 0$.
Maximize $Z = 5x + 3y$.
34. Let x items of A and y items of B be produced. Then,
$$x + y \leq 24, 1 \cdot x + \frac{1}{2} \cdot y \leq 16; x \geq 0 \text{ and } y \geq 0.$$
Maximize $P = 300x + 160y$.
35. Let x trunks of the first type and y trunks of the second type be manufactured. Then,
$$3x + 3y \leq 18;\ 3x + 2y \leq 15;\ x \geq 0 \text{ and } y \geq 0.$$
Maximize $Z = 30x + 25y$.
36. Let x and y be the number of toys of type A and type B respectively. Then,
$$5x + 8y \leq 180,\ 10x + 8y \leq 240 \text{ and } x \geq 0,\ y \geq 0.$$
Maximize $Z = 50x + 60y$.
37. Let the cereal contain x kg of bran and y kg of rice.
Maximize $Z = 5x + 4y$, subject to the conditions
$$\left(x \times \frac{80}{1000}\right) + \left(y \times \frac{100}{1000}\right) \geq \frac{88}{1000};$$
$$\left(x \times \frac{40}{1000}\right) + \left(y \times \frac{30}{1000}\right) \geq \frac{36}{1000};$$
$$x \geq 0 \text{ and } y \geq 0.$$
38. Let the number of fans and sewing machines bought be x and y respectively.
Maximize $Z = 22x + 18y$, subject to
$$x + y \leq 20,\ 360x + 240y \leq 5760 \text{ and } x \geq 0, y \geq 0.$$
39. Maximize $Z = \frac{8x}{100} + \frac{10y}{100} = \left(\frac{2x}{25} + \frac{y}{10}\right)$ subject to $x \geq 2000, y \geq 4000$ and $x + y \leq 12000$.
Draw the graphs of $x = 2000,\ y = 4000$ and $\frac{x}{12000} + \frac{y}{12000} = 1$.
Shade $x \geq 2000,\ y \geq 4000$ and $\frac{x}{12000} + \frac{y}{12000} \leq 1$ to give $\triangle BEF$.



Values of $\left(\frac{2x}{25}+\frac{y}{10}\right)$ at B, E and F are respectively 560, 1160 and 1040.

So, it is maximum at $E(2000, 10000)$.

41. Maximize $z = 50x + 60y$, subject to the constraints

$5x + 8y \le 180,$

$5x + 4y \le 120,$

$x \ge 0$ and $y \ge 0$.

The corner points of the feasible region are $O(0, 0)$, $C(24, 0)$, $D(0, 22.5)$ and $E(12, 15)$.

z is maximum at $E(12, 15)$.

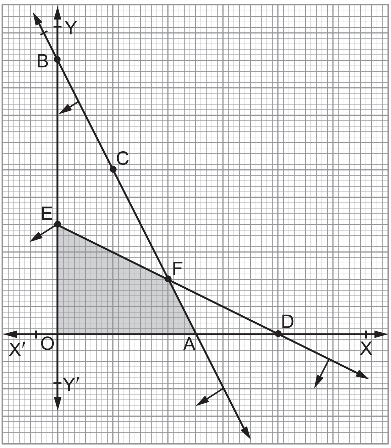
42. Let the number of cakes of the first and second type be x and y respectively.

Then, we have to maximise $z = x + y$ subject to the constraints

$200x + 100y \le 5000 \Rightarrow 2x + y \le 50$ ... (i)

$25x + 50y \le 1000 \Rightarrow x + 2y \le 40$ ... (ii)

and $x \ge 0$, $y \ge 0$.



First we draw the line $2x + y = 50$.

For this, we plot the points $A(25, 0)$, $B(0, 50)$ and $C(10, 30)$.

Draw the line ACB.

The region below this line represents $2x + y \leq 50$.

Now, we draw the line $x + 2y = 40$.

For this, we plot the points $D(40, 0)$, $E(0, 20)$ and $F(20, 10)$.

Draw the line *DFE*.

The region below this line represetns $x + 2y \leq 40$.

The feasible region contains the points $A(25, 0)$, $F(20, 10)$ and $E(0, 20)$.

Value of z at $A(25, 0) = 25 + 0 = 25$.

Value of z at $F(20, 10) = 20 + 10 = 30$.

Value of z at $E(0, 20) = 0 + 20 = 20$.

For maximum value of z, we have $x = 20$ and $y = 10$.

Maximum number of cakes required for first and second type are 20 and 10 respectively.

43. Let the number of pieces of type A and type B manufactured per week be x and y respectively.

Then, we have to maximize $P = 80x + 120y$ subject to the constraints

$$9x + 12y \leq 180 \Rightarrow 3x + 4y \leq 60, \quad \text{... (i)}$$

$$x + 3y \leq 30 \quad \text{... (ii)}$$

and $x \geq 0,\ y \geq 0$.

We leave it to the reader to draw the graphs.

For maximum profit, we shall have 12 pieces of type A and 6 pieces of type B; and the maximum profit is ₹ 1680.

Sample Question Paper

MATHEMATICS
CLASS 12

Time : 3 hrs. Max. Marks: 100

General Instructions:

1. All questions are compulsory.
2. There are 29 questions in all.
3. Section A contains 4 questions of 1 mark each.
   Section B contains 8 questions of 2 marks each.
   Section C contains 11 questions of 4 marks each.
   Section D contains 6 questions of 6 marks each.

SECTION A

Question numbers* 1 *to* 4 *carry* 1 *mark each.

1. If A is a square matrix of order 3 and $|3A| = k|A|$ then find the value of k.
2. If $*$ is a binary operation on the set R of all real numbers defined by $a * b = a + b - 3$ then find the identity element for $*$.
3. Give an example to show that the relation $R = \{(a, b) : a \le b^2\}$ on the set R of all real numbers is not reflexive.
4. If $\vec{a}$ and $\vec{b}$ are two nonzero vectors such that $|\vec{a} \times \vec{b}| = \vec{a} \cdot \vec{b}$ then find the angle between $\vec{a}$ and $\vec{b}$.

SECTION B

Question numbers* 5 *to* 12 *carry* 2 *marks each.

5. Prove that each diagonal element of a skew-symmetric matrix is zero.
6. If $y = \tan^{-1}\left(\dfrac{5x}{1-6x^2}\right)$, where $\dfrac{-1}{\sqrt{6}} < x < \dfrac{1}{\sqrt{6}}$ then prove that
$$\frac{dy}{dx} = \left(\frac{2}{(1+4x^2} + \frac{3}{(1+9x^2)}\right).$$
7. Evaluate: $\displaystyle\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx.$
8. If $|\vec{a} + \vec{b}| = 60$, $|\vec{a} - \vec{b}| = 40$ and $|\vec{a}| = 22$ then find $|\vec{b}|$.

9. If $P(A) = \frac{2}{5}, P(B) = \frac{1}{3}, P(A \cap B) = \frac{1}{5}$ then find $P(\overline{A} \mid \overline{B})$.

10. If x changes from 5 to 5.01 then find the approximate change in $\log_e x$.

11. Obtain the differential equation of the family of circles passing through the points $A(a, 0)$ and $B(-a, 0)$.

12. Simplify: $\cot^{-1} \frac{1}{\sqrt{x^2 - 1}}$ for $x < -1$.

SECTION C

Question numbers* 13 *to* 23 *carry* 4 *marks each.

13. If $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$ then using A^{-1} solve the following system of equations: $3x + 4y = 5,\ x - y = -3.$

14. Discuss the differentiability of the function

$$f(x) = \begin{cases} 2x - 1, \text{when } x < \frac{1}{2} \\ 3 - 6x, \text{when } x \geq \frac{1}{2} \end{cases}$$

at $x = \frac{1}{2}\cdot$

Or, For what value of k is the following function continuous at $x = \frac{-\pi}{6}$?

$$f(x) = \begin{cases} \dfrac{\sqrt{3}\sin x + \cos x}{\left(x + \frac{\pi}{6}\right)}, & \text{when } x \neq \frac{-\pi}{6} \\ k, & \text{when } x = \frac{-\pi}{6}\cdot \end{cases}$$

15. If $x = a \sin pt$ and $y = b \cos pt$ then show that

$$(a^2 - x^2)y \cdot \frac{d^2y}{dx^2} + b^2 = 0.$$

16. Find the equation of normal to the curve $2y = x^2$ which passes through the point (2, 1).

Or, Separate the interval $\left[0, \frac{\pi}{2}\right]$ into subintervals in which the function $f(x) = \sin^4 x + \cos^4 x$ is strictly increasing or strictly decreasing.

17. Evaluate: $\int \frac{\sin x}{(\cos^2 x + 1)(\cos^2 x + 4)} dx.$

18. A magazine seller has 500 subscribers and collects annual subscription charges of ₹ 300 per subscriber. He proposes to increase the annual

subscription charges and it is believed that for every increase of ₹ 1, one subscriber will discontinue. What increase will bring maximum income to him? Make appropriate assumptions in order to apply derivatives to reach the solution.

Write one important role of magazines in our lives.

19. Solve the differential equation $(1+e^{x/y})dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0.$

Or, Find the general solution of the differential equation

$$(1+\tan y)(dx - dy) + 2xdy = 0.$$

20. Prove that: $\vec{a}\cdot\{(\vec{b}+\vec{c})\times(\vec{a}+2\vec{b}+3\vec{c})\} = [\vec{a}\ \vec{b}\ \vec{c}]$.

21. If the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-a}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ are the skew lines then find the value of a.

22. A bag contains 4 green and 6 white balls. Two balls are drawn one by one without replacement. If the second ball drawn is white, what is the probability that the first ball drawn is also white?

23. Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of diamond cards drawn. Also, find the mean and variance of the distribution.

SECTION D

Question numbers* 24 *to* 29 *carry* 6 *marks each.

24. Let Q be the set of all rational numbers and let $*$ be a binary operation on $Q\times Q$ defined by $(a, b) * (c, d) = (ac, b+ad)$.

Determine whether $*$ is commutative and associative. Find the identity element for $*$ and invertible elements of $Q\times Q$.

Or, Let $f:[0,\infty)\to R$ be a function defined by $f(x) = 9x^2+6x-5$. Prove that f is not invertible. Modify only the codomain of f to make f invertible and then find its inverse.

25. Using the properties of determinants, prove that

$$\begin{vmatrix} \frac{(a+b)^2}{c} & c & c \\ a & \frac{(b+c)^2}{a} & a \\ b & b & \frac{(c+a)^2}{b} \end{vmatrix} = 2(a+b+c)^3.$$

Or, If $p \neq 0, q \neq 0$ and $\begin{vmatrix} p & q & p\alpha + q \\ q & r & q\alpha + r \\ p\alpha + q & q\alpha + r & 0 \end{vmatrix} = 0$ then using the properties of determinants, prove that at least one of the following statements is true:

(a) p, q, r are in GP (b) α is a root of the equation $px^2 + 2qx + r = 0$.

26. Using integration, find the area of the region bounded by the curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$.

27. Evaluate: $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$.

Or, Evaluate: $\int_0^4 (x + e^{2x}) dx$ as the limit of a sum.

28. Find the equation of the plane through the point (4, –3, 2) and perpendicular to the line of intersection of the planes $x - y + 2z - 3 = 0$ and $2x - y - 3z = 0$. Find the point of intersection of the line $\vec{r} = \hat{i} + 2\hat{j} - \hat{k} + \lambda(\hat{i} + 3\hat{j} - 9\hat{k})$ and the plane obtained above.

29. In a mid-day meal programme, an NGO wants to provide vitamin-rich diet to the students of an MCD school. The dietician of the NGO wishes to mix two types of food in such a way that vitamin contents of the mixture contains at least 8 units of vitamin A and 10 units of vitamin C. Food 1 contains 2 untis per kg of vitamin A and 1 unit per kg of vitamin C. Food 2 contains 1 unit per kg of vitamin A and 2 units per kg of vitamin C. It costs ₹ 50 per kg to purchase Food 1 and ₹ 70 per kg to purchase Food 2. Formulate the problems as LPP and solve it graphically for the minimum cost of such a mixture?

ANSWERS

SECTION A

1. $k = 27$ **2.** $e = 3$ **3.** $\left(\frac{1}{2}, \frac{1}{2}\right) \notin R$ **4.** $45°$

SECTION B

7. $\frac{e^x}{(1+x)^2} + C$ **8.** $|\vec{b}| = 46$ **9.** $\frac{7}{10}$ **10.** 0.002

11. $\frac{dy}{dx} = \frac{2xy}{(x^2 - y^2 - a^2)}$ **12.** $\pi - \sec^{-1} x$, as $0 < \pi - \theta < \frac{\pi}{2}$

SECTION C

13. $x = -1, y = 2$

14. Not differentiable

Or, $k = 2$

16. $x + 2^{2/3}y = 2 + 2^{2/3}$

Or, Strictly decreasing in $\left[0, \frac{\pi}{4}\right]$, strictly increasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

17. $-\frac{1}{3}\tan^{-1}(\cos x) + \frac{1}{6}\tan^{-1}\left(\frac{\cos x}{2}\right) + C$

18. Increase subscription charges by ₹ 100 to have maximum income. Through reading magazines, our mind and point of view are consolidated and enriched.

19. $\left(e^{x/y} + \frac{x}{y}\right)y = C$

Or, $x(\cos y + \sin y)e^y = e^y \sin y + C$

21. $\lambda = -1, \mu = -1$ **22.** $\frac{5}{9}$ **23.** Mean $= \frac{1}{2}$, Variance $= \frac{3}{8}$

SECTION D

24. $*$ is not commutative, $*$ is associative, (1, 0) is the identity, inverse of (a, b) is $\left(\frac{1}{a}, \frac{-b}{a}\right)$

Or, $f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$

26. $\frac{-1}{2} + \frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right)$

27. $\frac{\pi^2}{16}$

Or, $8 + \frac{1}{2}(e^8 - 1)$

28. $-\hat{j} + 8\hat{k}$ **29.** Minimum cost = ₹ 380

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3360	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4028	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8335	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9043	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1393	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8402	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20