MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. The 3 digit numbers which are divisible either 3 or 4 but not divisible by 48

   (1) 414 (2) 420

   (3) 429 (4) 432

Answer (4)

Sol. No's divisible by 3 = 300

No's divisible by 4 = 225

No's divisible by 12 = 75

No's divisible by 48 = 18

Total no's = 300 + 225 – 75 – 18

= 432

2. The letters of the word GHOTU is arranged alphabetically as in a dictionary. The rank of the word TOUGH is

   (1) 84 (2) 79

   (3) 74 (4) 89

Answer (4)

Sol. GHOTU

G _ _ _ _	24
H _ _ _ _	24
O _ _ _ _	24
T G _ _ _	6
T H _ _ _	6
T O G _ _	2
T O H _ _	2
T O U G H	1
=	89

3. $\int_{1/2}^{2} \frac{\tan^{-1}x}{x}dx$ equals

   (1) $\frac{\pi}{2}\ln 2$ (2) $\frac{\pi}{4}\ln 2$

   (3) $\pi \ln 2$ (4) ln 2

Answer (1)

Sol. $I = \int_{1/2}^{2} \frac{\tan^{-1}x}{x}dx$

Put $x = \frac{1}{t}$

$$I = \int_{1/2}^{2} \frac{\tan^{-1}\left(\frac{1}{t}\right)}{t}dt$$

$$\therefore \quad 2I = \int_{1/2}^{2} \frac{\pi}{2}\frac{dt}{t}$$

$$\Rightarrow \quad I = \frac{\pi}{2}\ln 2$$

4. Shortest distance between lines:

   $\frac{x-1}{2} = \frac{2y-2}{3} = \frac{z-3}{1}$ and $\frac{x-2}{3} = \frac{y-1}{2} = \frac{z+2}{4}$ is

   (1) $\frac{13}{\sqrt{165}}$ (2) $\frac{15}{\sqrt{165}}$

   (3) $\frac{18}{\sqrt{165}}$ (4) $\frac{19}{\sqrt{165}}$

Answer (1)

Sol. Lines are

$$\frac{x-1}{2} = \frac{y-1}{\frac{3}{2}} = \frac{z-3}{1} \;\&\; \frac{x-2}{3} = \frac{y-1}{2} = \frac{z+2}{4}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & \frac{3}{2} & 1 \\ 3 & 2 & 4 \end{vmatrix} = \hat{i}(4) - \hat{j}(5) + \hat{k}\left(-\frac{1}{2}\right)$$

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (\hat{i} - 5\hat{k})\left(4\hat{i} - 5\hat{j} - \frac{\hat{k}}{2}\right)$$

$$= 4 + \frac{5}{2} = \frac{13}{2}$$

$$\therefore \quad \text{Shortest distance} = \frac{\frac{13}{2}}{\sqrt{16 + 25 + \frac{1}{4}}} = \frac{3}{\sqrt{165}}$$

5. $R = \{(a, b) : 2a + 3b$ is divisible by 5 and $a, b \in N\}$ is

(1) Transitive but not symmetric

(2) Equivalence relation

(3) Symmetric but not transitive

(4) Not equivalence

Answer (2)

Sol. Let $(a, b) \in R$

$f(a, b) = 2a + 3b$

For Reflexive

$f(a, a) = 2a + 3a = 5a$ *i.e*, divisible by 5

$\Rightarrow (a, a) \in R$

For symmetric

$f(b, a) = 2b + 3a = \underbrace{5a+5b}_{\text{divisible by 5}} - \underbrace{(2a+3b)}_{\text{divisible by 5}}$

$f(b, a)$ is divisible by 5

$\Rightarrow (b, a) \in R$

For transitive

$f(a, b) = 2a + 3b$ is divisible by 5

$f(b, c) = 2b + 3c$ is divisible by 5

$\Rightarrow 2a + 5b + 3c$ is divisible by 5

So, $2a + 3c$ is divisible by 5

$\Rightarrow (a, c) \in R$

6. $(\sim A) \vee B$ is equivalent to

(1) $A \to B$

(2) $A \leftrightarrow B$

(3) $\sim A \wedge B$

(4) $B \to A$

Answer (1)

Sol. Making truth table

A	B	$\sim B$	$(\sim A) \vee B$	$A \to B$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The truth table clearly shows.

$(\sim A) \vee B \equiv A \to B$

7. The value of $\int_{\frac{1}{2}}^{2}\left(\frac{t^4+1}{t^6+1}\right)dt =$

(1) $\tan^{-1}2+\tan^{-1}8+\frac{2\pi}{3}$

(2) $2\tan^{-1}2+\frac{2}{3}\tan^{-1}8-\frac{2\pi}{3}$

(3) $2\tan^{-1}2+\frac{2}{3}\tan^{-1}8+\frac{2\pi}{3}$

(4) $2\tan^{-1}2-\frac{2}{3}\tan^{-1}(8)+\frac{2\pi}{3}$

Answer (2)

Sol. $= \int_{\frac{1}{2}}^{2}\frac{t^4+1}{t^6+1}dt$

$= \int_{\frac{1}{2}}^{2}\frac{(t^4+1)(t^2+1)}{(t^6+1)(t^2+1)}dt = \int_{\frac{1}{2}}^{2}\frac{t^6+1+t^2(t^2+1)}{(t^6+1)(t^2+1)}dt$

$= \int_{\frac{1}{2}}^{2}\frac{dt}{t^2+1}+\frac{1}{3}\int_{\frac{1}{2}}^{2}\frac{3t^2dt}{t^6+1}$

$= \tan^{-1}t\Big|_{\frac{1}{2}}^{2}+\frac{1}{3}\tan^{-1}(t^3)\Big|_{\frac{1}{2}}^{2}$

$= \left(\tan^{-1}2-\tan^{-1}\left(\frac{1}{2}\right)\right)+\frac{1}{3}\left(\tan^{-1}(8)-\tan^{-1}\left(\frac{1}{8}\right)\right)$

$= \left(\tan^{-1}2-\cot^{-1}2\right)+\frac{1}{3}\left(\tan^{-1}8-\cot^{-1}8\right)$

$= \tan^{-1}2-\left(\frac{\pi}{2}-\tan^{-1}2\right)+\frac{1}{3}\left(\tan^{-1}8-\left(\frac{\pi}{2}-\tan^{-1}8\right)\right)$

$= 2\left(\tan^{-1}2\right)+\frac{2}{3}\tan^{-1}8-\frac{\pi}{2}-\frac{\pi}{6}$

$= 2\tan^{-1}2+\frac{2}{3}\tan^{-1}8-\frac{2\pi}{3}$

8. Area of region

$|\cos x-\sin x| \le y \le \sin x$ for $x \in (0, \pi/2)$ is

(1) $\left(-1+2\sqrt{2}\right)$ sq. unit

(2) $\left(1-\frac{1}{\sqrt{2}}\right)$ sq. unit

(3) $\left(\sqrt{5}+1-2\sqrt{2}\right)$ sq. unit

(4) $\left(\sqrt{5}-\sqrt{2}\right)$ sq. unit

Answer (3)

Sol. $A = \int_{\theta}^{\pi/2}(\sin x - |\cos x - \sin x|)dx$

Where $\theta = \tan^{-}\frac{1}{2}$

$= \int_{\theta}^{\pi/4}(\sin x - \cos x + \sin x)dx +$

$\int_{\pi/4}^{\pi/2}(\sin x + \cos x - \sin x)dx$

$= -2\cos x - \sin x|_{\theta}^{\pi/4} + \left(1 - \frac{1}{\sqrt{2}}\right)$

$= -\left(\sqrt{2} + \frac{1}{\sqrt{2}} - 2\cos\theta - \sin\theta\right) + 1 - \frac{1}{\sqrt{2}}$

$= -\sqrt{2} - \frac{1}{\sqrt{2}} + (2\cos\theta + \sin\theta) + 1 - \frac{1}{\sqrt{2}}$

$= 1 - 2\sqrt{2} + 2\cdot\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}}$

$= \sqrt{5} + 1 - 2\sqrt{2}$

9. For solution of differential equation $x\ln x\frac{dy}{dx} + y = x^2\ln x, y(2) = 2$ then $y(e)$ is equal to

(1) $1 + \frac{e^2}{4}$ (2) $1 - \frac{e^2}{4}$

(3) $\frac{e^2}{2}$ (4) $1 + \frac{e^2}{2}$

Answer (1)

Sol. $x\ln x\frac{dy}{dx} + y = x^2\ln x$

$\frac{dy}{dx} + \frac{y}{x\ln x} = x$

I.F $= e^{\int\frac{1}{x\ln x}dx} = e^{\ln|\ln x|} = |\ln x|$

Solution of equation is

$y(\text{IF}) = \int x\cdot|\ln x|\,dx$

$y|\ln x| = |\ln x|\frac{x^2}{2} - \frac{x^2}{4} + c$

$\boxed{x = 2}$, 2 |ln2| = |ln2|· 2 – 1 + c

$\Rightarrow \boxed{c = 1}$

For $x = e$

$y = \frac{e^2}{2} - \frac{e^2}{4} + 1$

$\Rightarrow y(e) = 1 + \frac{e^2}{4}$

10. Let $f(x) = x^2 + 2x + 5$ and α, β be roots of $f\left(\frac{1}{t}\right) = 0$, then $\alpha + \beta =$

(1) $\frac{-2}{5}$ (2) –2

(3) $\frac{5}{2}$ (4) $\frac{-5}{2}$

Answer (1)

Sol. $f(x) = x^2 + 2x + 5$

$f\left(\frac{1}{t}\right) = 0 \Rightarrow \frac{1}{t^2} + \frac{2}{t} + 5 = 0$

$\Rightarrow 5t^2 + 2t + 1 = 0$ $[t \neq 0]$

This equation has roots α and β.

$\therefore \alpha + \beta = \frac{-2}{5}$

11. If the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+3}{1}$ and $\frac{x-11}{4} = \frac{y-9}{2} = \frac{z-4}{3}$ intersects at point p, then perpendicular distance of p from plane $3x + 2y + 6z = 10$ is

(1) $\frac{2}{7}$ (2) $\frac{3}{7}$

(3) $\frac{4}{7}$ (4) $\frac{5}{7}$

Answer (2)

Sol. Given

$L_1 \equiv \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+3}{1} = \lambda$

and $L_2 \equiv \frac{x-11}{4} = \frac{y-9}{2} = \frac{z-4}{3} = \mu$

Finding intersection, we get $\lambda = 1$, $\mu = -2$

$\therefore p \equiv (3, 5, -2)$

Distance from given plane $= \left|\frac{9 + 10 - 12 - 10}{\sqrt{9 + 4 + 36}}\right| = \frac{3}{7}$

12. If $\cos^2 2x - \sin^4 x - 2\cos^2 x = \lambda$, has a solution $\forall x \in R$, then the range of λ is

(1) $\left[-\frac{1}{2}, 1\right]$ (2) $\left[-\frac{4}{3}, 0\right]$

(3) (0, 2) (4) None of these

Answer (2)

Sol. $\cos^2 2x - \sin^4 x - 2\cos^2 x = \lambda$

$$\Rightarrow \left(\cos^2 x - \sin^2 x\right)^2 - \sin^4 x - 2\cos^2 x = \lambda$$

$$\Rightarrow 3\cos^4 x - 4\cos^2 x = \lambda$$

$$\Rightarrow 3\left(\left(\cos^2 x - \frac{2}{3}\right)^2 - \frac{4}{9}\right) = \lambda$$

$$\lambda_{\min} = -\frac{4}{3}$$

$$\lambda_{\max} = 0$$

$$\therefore \quad \lambda \in \left[-\frac{4}{3}, 0\right]$$

13. $\vec{a} = 9\hat{i} + 2\hat{k}, \vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 7\hat{i} - 3\hat{j} + 2\hat{k}$ are three given vectors. Let there be a $\vec{r}$ such that $(\vec{r} \times \vec{b}) + (\vec{b} \times \vec{c}) = \vec{0}$ and $\bar{r} \cdot \bar{a} = 0$ then $\bar{r} \cdot \bar{c}$ is

(1) $\frac{280}{11}$ (2) 28

(3) $\frac{279}{13}$ (4) $\frac{290}{11}$

Answer (1)

Sol. $\vec{r} \times \vec{b} - \vec{c} \times \vec{b} = \vec{0}$

$$(\vec{r} - \vec{c}) \times \vec{b} = 0$$

$$\Rightarrow \vec{r} = \vec{c} + \lambda\vec{b}$$

$$\vec{r} \cdot \vec{a} = 0$$

$$\vec{c} \cdot \vec{a} + \lambda\vec{b} \cdot \vec{a} = 0$$

$$67 + \lambda\,(11) = 0$$

$$\boxed{\lambda = -\frac{67}{11}}$$

$$\vec{r} \cdot \vec{c} = (\vec{c} + \lambda\vec{b}) \cdot \vec{c}$$

$$= |\vec{c}|^2 + \lambda\,\vec{b} \cdot \vec{c}$$

$$= 62 - \frac{67}{11}(7 - 3 + 2)$$

$$= 62 - \frac{67}{11}(6)$$

$$= \frac{682 - 402}{11} = \frac{280}{11}$$

14. For observation set x data obtained is

$x_i = \{11, 12, 13, \ldots 41\}$

For another observation set y data obtained is

$y_i = \{61, 62, 63, \ldots 91\}$

Then value of $\bar{x} + \bar{y} + \sigma^2$ where $\bar{x}, \bar{y}$ are means of respective data set while 6^2 is variance of combined data is

(1) 801 (2) 754

(3) 807 (4) 774

Answer (3)

Sol. $\bar{x} = \dfrac{\frac{31}{2}(11+41)}{31} = \dfrac{1}{2} \times 52 = 26$

$$\bar{y} = \frac{\frac{31}{2}(61+91)}{31} = \frac{1}{2} \times 152 = 76$$

$$\sigma^2 = \frac{\Sigma x_i^2 + \Sigma y_i^2}{62} - \left(\frac{\Sigma x_i + \Sigma y_i}{62}\right)^2$$

$$= \frac{(11^2 + 12^2 + \ldots + 41^2) + (61^2 + 62^2 + \ldots + 91^2)}{62} - (51)^2$$

$$= \frac{\left(\frac{41 \cdot 42 \cdot 83}{6} - \frac{10 \cdot 11 \cdot 21}{6}\right) + \left(\frac{91 \cdot 92 \cdot 183}{6} - \frac{60 \cdot 61 \cdot 121}{6}\right)}{62} - (51)^2$$

$$= \frac{(41 \cdot 7 \cdot 83 - 11 \cdot 35) + (91 \cdot 46 \cdot 61 - 10 \cdot 61 \cdot 121)}{62} - (51)^2$$

$$= \frac{23436 + 181536}{62} - (51)^2$$

$$= 3306 - 2601 = 705$$

$$\bar{x} + \bar{y} + \sigma^2$$

$$= 26 + 76 + 705$$

$$= 807$$

15. If curve represented by $y = \dfrac{(x-a)}{(x-3)(x-2)}$ passes through (1, –3) then equation of normal at (1, –3) to the curve is given by

(1) $2x + 3y = -7$ (2) $3x - 2y = 9$

(3) $3x - 4y = 21$ (4) $x - 4y = 13$

Answer (4)

Sol. If passes through (1, –3)

$$-3 = \frac{1-a}{(-2)(-1)}$$

$$\boxed{a = 7}$$

$$f'(x) = \frac{(x-3)(x-2)-(x-7)(2x-5)}{((x-3)(x-2))^2}$$

$$f'(1) = \frac{2-18}{(2)^2} = -4$$

Slope of normal $= \frac{1}{4}$

Equation of normal

$$y+3 = \frac{1}{4}(x-1)$$

or $4y+12 = x-1$

or $x-4y = 13$

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

21. Find the number of four-digit numbers N such that $GCD(N, 54) = 2$

Answer (3000)

Sol. N should be divisible by 2 but not by 3.

N = (Number of numbers divisible by 2) – (Number of numbers divisible by 6)

$$= \frac{9000}{2} - \frac{9000}{6} = 3000$$

22. If $f(1)+2f(2)+3f(3)+......+nf(n) = n(n+1)f(n)$ and $f(1)=1$, then $\frac{1}{f(2022)}+\frac{1}{f(2028)} =$

Answer (4050)

Sol. $f(1)+2f(2)+......+nf(n) = n(n+1)f(n)$ ...(i)

$f(1)+2f(2)+......+nf(n)+(n+1)f(n+1)$

$= (n+1)(n+2)f(n+1)$ ...(ii)

Using (i) in (ii),

$$n(n+1)f(n)+(n+1)f(n+1) = (n+1)(n+2)f(n+1)$$

$$\Rightarrow \quad f(n+1) = \frac{n}{n+1}f(n)$$

$$f(1) = 1$$

$$\Rightarrow \quad f(2) = \frac{1}{2},\ f(3) = \frac{1}{3},\ .....,f(n) = \frac{1}{n}$$

$$\frac{1}{f(2022)}+\frac{1}{f(2028)} = 2022+2028 = 4050$$

23. A line $x + y = 3$ cuts the circle having center (2, 3) and radius 4 at two points A and B. Tangents drawn at A and B intersect at (α, β). Find the value of $4\alpha - 7\beta$.

Answer (11)

Sol. The given line is the polar of (α, β) w.r.t. given circle

Circle : $x^2 + y^2 - 4x - 6y - 3 = 0$

Chord of contact :

$\alpha x + \beta y - 2(x + \alpha) - 3(y + \beta) - 3 = 0$

$(\alpha - 2)x + (\beta - 3)y - (2\alpha + 3\beta + 3) = 0$

But equation of chord of contact is

$x + y - 3 = 0$

Comparing the coefficients,

$$\frac{\alpha-2}{1} = \frac{\beta-3}{1} = \frac{-(2\alpha+3\beta+3)}{-3}$$

$\Rightarrow \quad \alpha = -6, \beta = -5$

$\Rightarrow \quad 4\alpha - 7\beta = 11$

24. Consider a sequence a_1, a_2 ……, a_n given by $a_n = a_{n-1} + 2^{n-1}, a_1 = 1$ and another sequence given by $b_n = b_{n-1} + a_{n-1}$, $b_1 = 1$. Also

$P = \sum_{n=0}^{10} \frac{b_n}{2^{n-1}}$ and $Q = \sum_{n=0}^{10} \frac{n}{2^{n-1}}$, then $2^7(P - 2Q)$ is

Answer (07.50)

Sol.

$a_2 - a_1 = 2^1$	$b_2 - b_1 = a_1$
$a_3 - a_2 = 2^2$	$b_3 - b_2 = a_2$
$\vdots$	$\vdots$
$a_n - a_{n-1} = 2^{n-1}$	$b_n - b_{n-1} = a_{n-1}$
$a_n = 2^n - 1$	$b_n = 2^n - n$

$$P - 2Q = \sum_{n=1}^{10} \frac{2^n - n}{2^n} - \frac{2n}{2^{n-1}}$$

$$= \sum_{n=1}^{10} \left(1 - \frac{5n}{2^n}\right) = 10 - 5\Sigma \frac{n}{2^n}$$

Let $S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n}$

$$\frac{S}{2} = \frac{1}{2^2} + \frac{2}{2^3} + \ldots + \frac{n-1}{2n} + \frac{n}{2^{n+1}}$$

$$\frac{S}{2} = \left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}\right) - \frac{n}{2^{n+1}}$$

$$\Rightarrow \frac{S}{2} = \frac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^n - 1\right)}{-\frac{1}{2}} - \frac{n}{2^{n+1}}$$

$$\Rightarrow S = 2\left(1 - \left(\frac{1}{2}\right)^n - \frac{n}{2^{n+1}}\right)$$

$$= 2\left(1 - \left(\frac{1}{2}\right)^{10} - \frac{10}{2^{11}}\right) = 2\left(1 - \frac{12}{2^{11}}\right)$$

$$P - 2Q = 10 - 5 \times 2\left(1 - \frac{12}{2^{11}}\right)$$

$$= 10 - 10 + \frac{120}{2^{11}} = \frac{60}{2^{10}}$$

$$\Rightarrow 2^7(P - 2Q) = \frac{60}{8}$$

$$= 07.50$$

25.

26.

27.

28.

29.

30.

❑ ❑ ❑