JEE-Main-24-06-2022-Shift-1 (Memory Based)

MATHEMATICS

Question: If the sum of the squares of the reciprocal of the roots α and β of the equation $3x^2+\lambda x+1=0$ is 15, then $6\left(\alpha^3+\beta^3\right)^2$ is equal to

Options:
(a) 18
(b) 24
(c) 36
(d) 96

Answer: (b)

Solution:

Given, $3x^2+\lambda x-1=0$

$$\alpha+\beta=-\frac{\lambda}{3}$$

$$\alpha\beta=\frac{-1}{3}$$

Also, $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=1$

$$\therefore\ \frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}=15$$

$$\frac{\frac{\lambda^2}{9}+\frac{2}{3}}{\frac{1}{9}}=15$$

$$\lambda^2+6=15$$

$$\lambda^2=9$$

$$\lambda=\pm 3$$

Now, $\left(\alpha^3+\beta^3\right)^2=6\left[(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)\right]^2$

$$=6\left[(-1)^3-3\left(\frac{-1}{3}\right)(-1)\right]^2$$

$$=6[-1-1]^2$$

$$=24$$

Question: Find the remainder when 3^{2022} is divided by 5:

Options:

(a) 1

(b) 2

(c) 4

(d) 0

Answer: (c)

Solution:

Given, 3^{2022}

$\left(3^2\right)^{1011}$

$(9)^{1011}$

$(10-1)^{1011}$

${}^{1011}C_0 10^{1011} - {}^{1011}C_1 10^{1010} + \ldots + {}^{1011}C_{1010} 10^1 - {}^{1011}C_{1011}$

$\therefore\ 10(\text{Integer})-1$

or $10(\text{Integer})-1-4+4$

$\Rightarrow 10(\text{Integer})-5+4$

$\Rightarrow 5(2\text{ Integer}-1)+4$

$\therefore$ Remainder when it is divided by 5 will be '4'.

Question: The Boolean expression $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to:

Answer: ()

Solution:

p	q	$p \to q$	$q \to \sim p$	$(p \to q) \wedge (q \to \sim p)$
T	T	T	F	F
T	F	F	T	F
F	T	T	T	T
F	F	T	T	T

Question: $\left(\tan^{-1} x\right)^3 + \left(\cot^{-1} x\right)^3 = k\pi^3$, find the range of k :

Options:

(a) $\left(\frac{1}{32}, \frac{9}{8}\right)$

(b) $\left[\frac{1}{32}, \frac{7}{8}\right)$

(c)

(d)

Answer: (b)

Solution:

$\left(\tan^{-1}x\right)^3+\left(\cot^{-1}x\right)^3=k\pi^3$

Now, $\left(\tan^{-1}x+\cot^{-1}x\right)^3-3\tan^{-1}x\cot^{-1}x\left(\tan^{-1}x+\cot^{-1}x\right)$

$\frac{\pi^3}{8}-3\tan^{-1}x\left(\frac{\pi}{2}-\tan^{-1}x\right)\left(\frac{\pi}{2}\right)$

Let $\tan^{-1}x=t$

$\Rightarrow\frac{\pi^3}{8}-\frac{3\pi^2}{4}t+\frac{3\pi}{2}t^2$

$\Rightarrow\frac{3\pi}{2}\left[t^2-\frac{\pi}{2}t\right]+\frac{\pi^3}{8}$

$\Rightarrow\frac{3\pi}{2}\left[t^2-2\frac{\pi}{4}t+\frac{\pi^2}{16}-\frac{\pi^2}{16}\right]+\frac{\pi^3}{8}$

$\Rightarrow\frac{3\pi}{2}\left(\tan^{-1}x-\frac{\pi}{4}\right)^2-\frac{3\pi^3}{32}+\frac{\pi^3}{8}$

Now, minimum value when $\tan^{-1}x=\frac{\pi}{4}$

$\Rightarrow\frac{3\pi}{2}(0)+\frac{\pi^3}{32}$

$=\frac{\pi^3}{32}$

Maximum when $\tan^{-1}x=-\frac{\pi}{2}$

$\frac{3\pi}{2}\left(\frac{9\pi^2}{16}\right)+\frac{\pi^3}{32}$

$\Rightarrow\frac{28\pi^3}{32}$

$\Rightarrow 7\frac{\pi^3}{8}$

So, $k\in\left[\frac{1}{32},\frac{7}{8}\right]$

Question: If $f(\theta)=\sin\theta+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\sin\theta+t\cos\theta\right)\cdot f(t)\,d\theta$, then $\left|\int_0^{\frac{\pi}{2}}f(\theta)\,d\theta\right|$ is

Options:

(a) $1+\pi t\, f(t)$

(b) $1-\pi t f(t)$

(c)

(d)

Answer: (a)

Solution:

Given, $f(\theta)=\sin\theta+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin\theta+t\cos\theta)\cdot f(t)d\theta$

$$=\sin\theta+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\theta\cdot f(t)d\theta+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}t\cos\theta f(t)d\theta$$

$$=\sin\theta+f(t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\theta\, d\theta+t\, f(t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\, d\theta$$

$$\sin\theta+0+t\, f(t)[\sin\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$f(\theta)=\sin\theta+2t\, f(t)$$

Now, $\left|\int_0^{\frac{\pi}{2}} f(\theta)d\theta\right|=\left|\int_0^{\frac{\pi}{2}}(\sin\theta+2t\, f(t))d\theta\right|$

$$=\left|\int_0^{\frac{\pi}{2}}\sin\theta d\theta+2t\, f(t)\int_0^{\frac{\pi}{2}}1\cdot d\theta\right|$$

$$=\left|[-\cos\theta]_0^{\frac{\pi}{2}}+2t\, f(t)\frac{\pi}{2}\right|$$

$$=\left|1+t\, f(t)(x)\right|$$

$$=1+\pi t\, f(t)$$

Question: $\langle a_i\rangle$ sequence is an A.P. with common difference 1 and $\sum_{i=1}^{n}a_i=192$, $\sum_{i=1}^{\frac{n}{2}}a_{2i}=120$, then find the value of n, where n is an even integer.

Options:

(a) 48

(b) 96

(c) 18

(d) 36

Answer: (b)

Solution:

Given, $\sum_{i=1}^{n} a_i = 192 \Rightarrow a_1 + a_2 + ... + a_n = 192$

$\Rightarrow \frac{n}{2}(a_1 + a_n) = 192$

$a_1 + a_n = \frac{384}{n} \qquad ...(1)$

Also, $\sum_{i=1}^{\frac{n}{2}} a_{x_i} = 120$

$\Rightarrow a_2 + a_4 + a_6 + ... + a_n = 120$

$\Rightarrow \frac{\frac{n}{2}}{2}[a_2 + a_n] = 120$

$\Rightarrow \frac{n}{4}[a_1 + 1 + a_n] = 120$

$a_1 + a_n + 1 = \frac{480}{n} \qquad ...(2)$

Now, $\frac{384}{n} + 1 = \frac{480}{n}$

$\Rightarrow \frac{96}{n} = 1$

$n = 96$

Question: Find domain: $\frac{\cos^{-1}\left(\frac{x^2 - 5x + 6}{x^2 - 9}\right)}{\ln\left(x^2 - 3x + 2\right)}$

Answer: $\left[-\frac{1}{2}, 1\right) \cup (2, \infty) - \{3\}$

Solution:

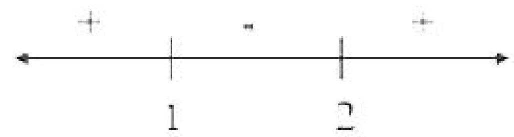
Given, $\frac{\cos^{-1}\left(\frac{x^2 - 5x + 6}{x^2 - 9}\right)}{\ln\left(x^2 - 3x + 2\right)}$

For domain

$x^2 - 3x + 2 > 0$

$x^2 - 2x - x + 2 > 0$

$(x - 2)(x - 1) > 0$



$x \in (-\infty, 1) \cup (2, \infty)$

&

$-1 \le \dfrac{x^2 - 5x + 6}{x^2 - 9} \le 1$

$-1 \le \dfrac{(x-2)(x-3)}{(x-3)(x+3)} \le 1$

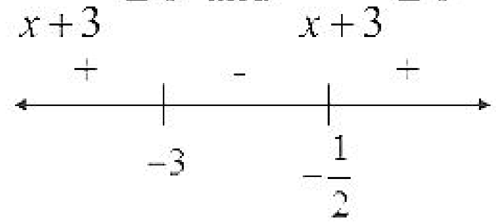
$x \ne 3$, then

$-1 \le \dfrac{x-2}{x+3} \le 1$

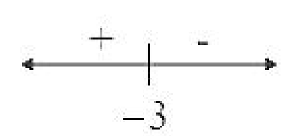
$\dfrac{x-2}{x+3} \ge -1$ and $\dfrac{x-2}{x+3} \le 1$

$\dfrac{x-2+x+3}{x+3} \ge 0$ and $\dfrac{x-2-x-3}{x+3} \le 0$

$\dfrac{2x+1}{x+3} \ge 0$ and $\dfrac{-5}{x+3} \le 0$



&



So, $x \in \left[-\dfrac{1}{2}, \infty\right)$

Combining both we get

$\left[-\dfrac{1}{2}, 1\right) \cup (2, \infty) - \{3\}$

Question: If $A = \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, where $a \in N$ from 1 to 50 and $\sum_{a=1}^{50} |adj\ A| = 100k$, then the value of k is:

Options:

(a) $\dfrac{1723}{2}$

(b) $\dfrac{1717}{2}$

(c) $\dfrac{1719}{4}$

(d) $\dfrac{1821}{4}$

Answer: (d)

Solution:

Given, $\sum_{a=1}^{50} |adj\ A| = 100k$

Now, $|adj\ A| = |A|^{n-1} = |A|^2$

Here, $|A| = a+1$

$\therefore |A|^2 = (a+1)^2$

$\sum |A|^2 = \sum_{a=1}^{50} (a+1)^2 = 2^2 + 3^2 + 4^2 + ... + 51^2$

$= 1^2 + 2^2 + ... + 51^2 - 1^2$

$= \frac{(51)(52)(103)}{6} - 1$

$= \frac{1821}{4}$

Question: A tangent $ax - \mu y = 2$ to hyperbola $\frac{a^4 x^2}{\lambda^2} - \frac{b^2 y^2}{1} = 4$, then the value of $\left(\frac{\lambda}{a}\right)^2 - \left(\frac{\mu}{b}\right)^2$ is:

Options:

(a) 0

(b) 1

(c) 2

(d) 3

Answer: (b)

Solution:

Given, $\frac{a^4 x^2}{\lambda^2} - \frac{b^2 y^2}{1} = 4$

$\Rightarrow \frac{x^2}{4\left(\frac{\lambda^2}{a^4}\right)} - \frac{y^2}{4\left(\frac{1}{b^2}\right)} = 1$

$\Rightarrow \frac{x^2}{\left(\frac{2\lambda}{a^2}\right)^2} - \frac{y^2}{\left(\frac{2}{b}\right)^2} = 1$

Now, tangent to above hyperbola is

$y = mx \pm \sqrt{a^2 m^2 - b^2}$

Here, $ax - \mu y = 2$

$\Rightarrow \mu y = ax - 2$

$$y=\frac{a}{\mu}x-\frac{2}{\mu}$$

$$\Rightarrow -\frac{2}{\mu}=\sqrt{\frac{4\lambda^2}{a^4}\cdot\frac{a^2}{\mu^2}-\frac{4}{b^2}}$$

$$\Rightarrow \frac{4}{\mu^2}=\frac{4\lambda^2}{a^2\mu^2}-\frac{4}{b^2}$$

$$\Rightarrow \left(\frac{\lambda}{a}\right)^2-\left(\frac{\mu}{b}\right)^2=1$$

Question: A tangent at (x_1, y_1) to the curve $y=x^3+2x^2+4$ and passes through origin, then (x_1, y_1) is:

Options:

(a) $(0,4)$

(b) $(-1,5)$

(c) $(1,7)$

(d) $(2,20)$

Answer: (c)

Solution:

Given, $y=x^3+2x^2+4$

$\therefore \frac{dy}{dx}=3x^2+4x$

At $(x_1, y_1)\frac{dy}{dx}=3x_1^2+4x_1$

Now, slope of tangent is also

$m=\frac{y_1-0}{x_1-0}=\frac{y_1}{x_1}$

$\Rightarrow \frac{y_1}{x_1}=3x_1^2+4x_1$

$y_1=3x_1^3+4x_1^2$

$\Rightarrow x_1^3+2x_1^2+4=3x_1^3+4x_1^2$

$2x_1^2+2x_1^2-4=0$

$x_1^3+x_1^2-2=0$

Here, $x_1=1,$ satisfying equation

$\therefore x_1=1,\ y_1=7$

$(1,7)$

Question: A circle of equation $x^2+y^2+ax+by+c=0$ passes through (0, 6) and touches $y=x^2$ at (2, 4). Find $a+c$.

Answer: 16.00

Solution:

Given, $x^2+y^2+ax+by+c=0$ passes through (0, 6)

Then, $0+36+0+6b+c=0$

$6b+c=-36 \quad ...(1)$

Touches $y=x^2$ at $(2,4)$

Thus, passes through $(2,4)$ and also tangent at $(2,4)$ is having same slope.

$\therefore\ 4+16+2a+4b+c=0$

$2a+4b+c=-20 \quad ....(2)$

Now, for slope

$$\frac{dy}{dx}=2x$$

$$\frac{dy}{dx}_{(2,4)}=4$$

For circle

$$2x+2y\frac{dy}{dx}+a+b\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-a-2a}{2y+b}$$

$$\frac{dy}{dx}_{(2,4)}=\frac{-a-4}{8+b}$$

$$\therefore\ 4=\frac{-a-4}{8+b}$$

$4b+32=-a-4$

$a+4b=-36 \quad ....(3)$

Putting in (2)

$a+a+4b+c=-20$

$a+(-36)+c=-20$

$a+c=16$

Question: $S=\left\{\theta:\theta\in[-\pi,\pi]-\left\{\pm\frac{\pi}{2}\right\} \& \sin\theta\tan\theta+\tan\theta=\sin 2\theta\right\}$. Let $T=\sum\cos 2\theta$ where $\theta\in S$, then $T+n(S)=?$

Answer: 9.00

Solution:

Here, $\sin\theta\tan\theta+\tan\theta-\sin\theta=0$

$\Rightarrow \sin\theta\left[\tan\theta+\sec\theta-2\cos\theta\right]=0$

$\tan\theta\left(2\sin^2\theta+\sin\theta-1\right)=0$

$\tan\theta=0,\ \sin\theta=-1,\ \sin\theta=\frac{-1}{2}$

$\theta=0,\pi,-\pi,\ \theta=-\frac{\pi}{2},\ \theta=\frac{\pi}{6},\frac{5\pi}{6}$

But $\theta\neq-\frac{\pi}{2}$

$\therefore n(S)=5$

Now $T=\sum\cos 2\theta,\ \theta\in S$

$\Rightarrow T=\cos 2(-\pi)+\cos(2\pi)+\cos(0)+\cos 2\left(\frac{\pi}{6}\right)+\cos 2\left(\frac{5\pi}{6}\right)$

$=1+1+1+\frac{1}{2}+\frac{1}{2}$

$=4$

$\therefore T+n(S)=4+5=9$

Question: Image of $A\left(\frac{3}{\sqrt{a}},\sqrt{a}\right)$ in y-axis is B & image of B is x-axis is C. Point $D(3\cos\theta, a\sin\theta)$ lies in 4th quadrant. If maximum area of $\Delta ACD=12$ sq. units, then find a.

Answer: 8.00

Solution:

Image of $A\left(\frac{3}{\sqrt{a}},\sqrt{a}\right)$ in y-axis will be $\left(-\frac{3}{\sqrt{a}},\sqrt{a}\right)B$,

Also, image of B in x-axis will be $\left(-\frac{3}{\sqrt{a}},-\sqrt{a}\right)C$

Given, $D(3\cos\theta, a\sin\theta)$ and IVth quadrant

$$\text{Area of } \Delta ACD=\frac{1}{2}\begin{vmatrix}\frac{3}{\sqrt{a}} & \sqrt{a} & 1\\ -\frac{3}{\sqrt{a}} & -\sqrt{a} & 1\\ 3\cos\theta & a\sin\theta & 1\end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 2 \\ -\frac{3}{\sqrt{a}} & -\sqrt{a} & 1 \\ 3\cos\theta & a\sin\theta & 1 \end{vmatrix} \left(R_1 \to R_1 + R_2\right)$$

$$12 = \frac{1}{2}\left|2\left(-3\sqrt{a}\sin\theta + 3\sqrt{a}\cos\theta\right)\right|$$

$$12 = 3\sqrt{a}\left(\cos\theta - \sin\theta\right)$$

Now, maximum value of $\cos\theta - \sin\theta = \sqrt{2}$

$$\therefore 12 = 3\sqrt{a}\left(\sqrt{2}\right)$$

$$\sqrt{a} = \frac{12}{3\sqrt{2}}$$

$$a = \frac{16}{2} = 8$$