JEE-Main-25-06-2022-Shift-1 (Memory Based)

MATHEMATICS

Question: For $\hat{a},\hat{b}$, which is correct:

Options:

(a) $\left|\hat{a}+\hat{b}\right|=\left|\hat{a}-\hat{b}\right|\tan\frac{\theta}{2}$

(b) $\left|\hat{a}-\hat{b}\right|=\left|\hat{a}+\hat{b}\right|\tan\frac{\theta}{2}$

(c) $\left|\hat{a}+\hat{b}\right|=\left|\hat{a}-\hat{b}\right|\cos\frac{\theta}{2}$

(d) $\left|\hat{a}-\hat{b}\right|=\left|\hat{a}+b\right|\cos\frac{\theta}{2}$

Answer: (b)

Solution:

Given, $\hat{a}.\hat{b}$ are unit vectors

$\therefore \left|\hat{a}-\hat{b}\right|^2=\left|\hat{a}\right|^2+\left|\hat{b}\right|^2-2\,\hat{a}\cdot\hat{b}=1+1-2\cos\theta$

$\left|\hat{a}+\hat{b}\right|^2=\left|\hat{a}\right|^2+\left|\hat{b}\right|^2+2\hat{a}\cdot\hat{b}=1+1+2\cos\theta$

$\Rightarrow \left|\hat{a}-\hat{b}\right|^2=2\left(1-\cos\theta\right)=4\sin^2\frac{\theta}{2}$

and $\left|\hat{a}+\hat{b}\right|^2=2\left(1+\cos\theta\right)=4\cos^2\frac{\theta}{2}$

$\therefore \frac{\left|\hat{a}-\hat{b}\right|^2}{\left|\hat{a}+\hat{b}\right|^2}=\tan^2\frac{\theta}{2}$

$\Rightarrow \left|\hat{a}-\hat{b}\right|=\left|\hat{a}+\hat{b}\right|\tan\frac{\theta}{2}$

Question: $(x+1)\frac{dy}{dx}-y=e^{3x}\left(1+x\right)^2$. $y(0)=\frac{1}{3}$ then $x=\frac{-3}{y}$ at $x=-4$ is point of

Options:

(a) minimum

(b) maximum

Answer: (a)

Solution:

Given, $\frac{dy}{dx}-\frac{y}{x+1}=e^{3x}\left(1+x\right)$

$$\text{I.F.} = e^{\int -\frac{dx}{(x+1)}} = e^{\ln(x+1)^{-1}}$$

$$\text{I.F.} = \frac{1}{x+1}$$

$$\Rightarrow \frac{y}{x+1} = \int e^{3x} dx$$

$$\frac{y}{x+1} = \frac{e^{3x}}{3} + c$$

$$y = \frac{e^{3x}(x+1)}{3} + c(x+1)$$

$$y(0) = \frac{1}{3}$$

$$\Rightarrow \frac{1}{3} = \frac{1}{3} + c$$

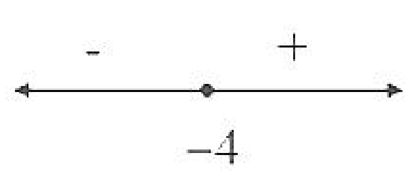
$$\Rightarrow c = 0$$

$$\therefore\ y = \frac{e^{3x}}{3}(x+1)$$

$$\text{Now, } \frac{dy}{dx} = \frac{1}{3}\left[\frac{e^{3x}}{3}(x+1) + e^{3x}\right]$$

$$= \frac{e^{3x}}{9}(x+1+3)$$

$$= \frac{e^{3x}}{9}(x+4)$$



At $x = -4$, its will make minimum

Question: $f(x) = x^3 + x - 5$ and $f(g(x)) = x$ and $g'(63) =$

Answer: $\frac{1}{49}$

Solution:

Given, $f(g(x)) = x$

$$g(x) = f^{-1}(x)$$

$$\Rightarrow \frac{d}{dx} g(x) = \frac{d}{dx} f^{-1}(x)$$

$$\Rightarrow \frac{d}{dx} f^{-1}(x)_{\text{at } x=63} = \frac{1}{\frac{d}{dx} f(x)_{\text{at } x=4}}$$

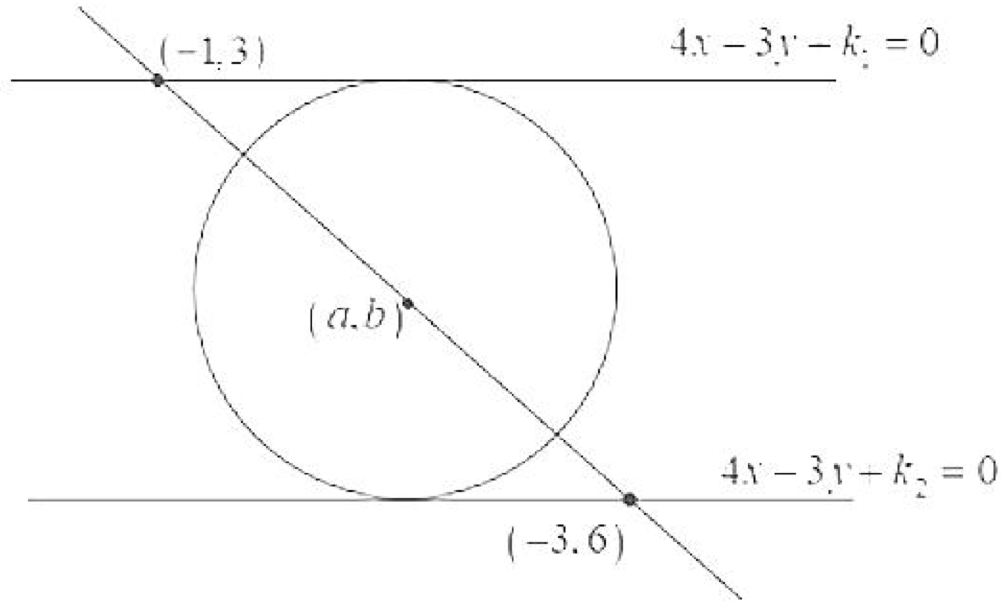
$$= \frac{1}{(3x^2+1)_{\text{at } x=4}}$$

$= \dfrac{1}{49}$

Question: Circle is touched by lines $4x-3y+k_1=0$ and $4x-3y+k_2=0$. A line passing through centre touches lines respectively at $(-1,3)$ and $(-3,6)$. Equation of circles is

Answer: $(x+2)^2+\left(y-\dfrac{9}{2}\right)^2=\left(\dfrac{17}{10}\right)^2$

Solution:
Given,



$\therefore (-1,3)$ satisfy $4x-3y+k_1=0$

$\therefore -4-9+k_1=0$

$k_1=13$

$(-3,6)$ satisfy $4x-3y+k_2=0$

$\therefore -12-18+k_2=0$

$k_2=30$

Now, diameter of circle is distance between lines

$\therefore d=\left|\dfrac{k_2-k_1}{\sqrt{4^2+3^2}}\right|=\left|\dfrac{17}{5}\right|$

$\Rightarrow r=\dfrac{17}{10}$

Centre of circle $\left(\dfrac{-1-3}{2},\dfrac{3+6}{2}\right)=\left(-2,\dfrac{9}{2}\right)$

$\therefore$ Radius of circle

$(x+2)^2+\left(y-\dfrac{9}{2}\right)^2=\left(\dfrac{17}{10}\right)^2$

Question: $\frac{1}{2\times3^{10}}+\frac{1}{2^2\times3^9}+\frac{1}{2^3\times3^8}+...+\frac{1}{2^{10}\times3}=\frac{k}{2^{10}\times3^{10}}$. Find remainder when k is divided by 6.

Answer: 5.00

Solution:

Given, $\frac{1}{2\times3^{10}}+\frac{1}{2^2\times3^9}+\frac{1}{2^3\times3^8}+...+\frac{1}{2^{10}\times3}=\frac{k}{2^{10}\times3^{10}}$

It is G.P. series, with common ratio $\frac{3}{2}$

$$S_{10}=\frac{1}{2\times3^{10}}\left[\frac{\left(\frac{3}{2}\right)^{10}-1}{\frac{3}{2}-1}\right]$$

$$=\frac{1}{2\times3^{10}}\left[\frac{3^{10}-2^{10}}{2^{10}\left(\frac{1}{2}\right)}\right]$$

$$=\frac{1}{2^{10}\times3^{10}}\left(3^{10}-2^{10}\right)$$

$\therefore\ k=3^{10}-2^{10}=\left(9^5\right)-\left(4\right)^5$

$k=(9-4)\left(9^4+9^3\cdot4+9^2\cdot4^2+9\cdot4^3+4^4\right)$

$=5\left(9^4+4^4\right)+5\left(6\text{ Integer}\right)$

$=5\left(6561+256\right)+6\left(\text{Integer}\right)$

$=34085+6\text{ Integer}$

So remainder is 5 when k is divided by 6

Question: $\frac{a+b}{7}=\frac{b+c}{8}=\frac{c+a}{9}$. Find $\frac{R}{r}$.

Answer: $\frac{5}{2}$

Solution:

Given, $\frac{a+b}{7}=\frac{b+c}{8}=\frac{c+a}{9}=k$

$\therefore\ a+b=7k$

$b+c=8k$

$c+a=9k$

$2(a+b+c)=24k$

$a+b+c=12k$

$\Rightarrow c=5k,\ a=4k,\ b=3k$

$$S=\frac{a+b+c}{2}=6k$$

$$\Rightarrow \Delta=\sqrt{S(S-a)(S-b)(S-c)}=\sqrt{6k(2k)(3k)(k)}$$

$$\Delta=6k^2$$

Now, $\Delta=\frac{abc}{4R}$

$$6k^2=\frac{5k\cdot 4k\cdot 3k}{4R}$$

$$R=\frac{5}{2}k$$

Also, $r=\frac{\Delta}{S}=\frac{6k^2}{6k}=k$

$$\therefore \frac{R}{r}=\frac{\frac{5}{2}k}{k}=\frac{5}{2}$$

Question: $\int_0^{\pi}\frac{e^{\cos x}}{(1+\cos^2 x)}e^{\cos x}+e^{-\cos x}$

Answer: $\frac{\pi}{2\sqrt{2}}$

Solution:

$$I=\int_0^{\pi}\frac{e^{\cos x}}{(1+\cos^2 x)(e^{\cos x}+e^{-\cos x})}dx \qquad ....(1)$$

Applying property

$$I=\int_0^{\pi}\frac{e^{-\cos x}}{(1+\cos^2 x)(e^{-\cos x}+e^{\cos x})}dx \qquad ....(2)$$

(1) + (2)

$$2I=\int_0^{\pi}\frac{dx}{1+\cos^2 x}$$

$$I=\int_0^{\frac{\pi}{2}}\frac{dx}{1+\cos^2 x}$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sec^2 x\,dx}{2+\tan^2 x}$$

$$=\frac{1}{\sqrt{2}}\left[\tan^{-1}(\tan x)\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{\sqrt{2}}\left(\frac{\pi}{2}\right)$$

$$=\frac{\pi}{2\sqrt{2}}$$

Question: Solve: $y^2dx+\left(x^2-xy+y^2\right)dy=0$

Answer: $\ln\left(y\right)-\tan^{-1}\left(\frac{y}{x}\right)=c$

Solution:

Given, $y^2dx+\left(x^2-xy+y^2\right)dy=0$

$$\frac{dy}{dx}=\frac{-y^2}{x^2-xy+y^2}$$

Put $y=vx,\ \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{-v^2x^2}{x^2-vx^2+v^2x^2}$$

$$x\frac{dv}{dx}=\frac{-v^2-v+v^2-v^3}{1-v+v^2}$$

$$\int\frac{1-v+v^2}{v^3+v}dv=-\int\frac{dx}{x}$$

$$\int\left(\frac{v^2+1}{v\left(v^2+1\right)}-\frac{v}{v\left(v^2+1\right)}\right)dv=-\int\frac{dx}{x}$$

$$\int\left(\frac{1}{v}-\frac{1}{v^2+1}\right)dv=-\int\frac{dx}{x}$$

$$\ln v-\tan^{-1}v=-\ln x+c$$

$$\Rightarrow \ln y-\tan^{-1}\left(\frac{y}{x}\right)=c$$

Question: Number of 3 digit numbers which have sum of digits is 7

Answer: 28.00

Solution:

Let $x+y+z=7$

Here, x is in Hundred's place digit, y is ten's place and z is one's place digit

Using multinomial theorem

Coefficient of a^7 in $\left(a^1+a^2+\ldots+a^7\right)\left(a^0+a^1+\ldots+a^7\right)^2$

Coefficient of a^7 in $a\left(\frac{1-a^7}{1-a}\right)\left(\frac{1-a^8}{1-a}\right)^2$

Coefficient of a^6 in $\left(1-a^7\right)\left(1-a^8\right)^2\left(1-a\right)^{-3}$

Coefficient of a^6 in $\left(1-a^8\right)^2\left(1-a\right)^{-3}$

Coefficient of a^6 in $\left(1-a\right)^{-3} = {}^8C_6 = 28$

Question: $x^2-31x+228=0$, let $a_n=\alpha^n-\beta^n$. Then find the value of $\frac{a_{10}-31a_9}{57a_8}=$

Answer: 4.00

Solution:

Given, $x^2-31x+228=0$

$\therefore \alpha+\beta=31$

$\alpha\beta=228$

Also, $\alpha^2=31\alpha-228$ & $\alpha^2-31\alpha=228$

$\beta^2=31\alpha-228$ & $\beta^2-31\beta=228$

Now, $\frac{a_{10}-31a_9}{57a_8}=\frac{\left(\alpha^{10}-\beta^{10}\right)-31\left(\alpha^9-\beta^9\right)}{57a_8}$

$=\frac{\alpha^8\left(\alpha^2-31\alpha\right)-\beta^8\left(\beta^2-31\beta\right)}{57a_8}$

$=\frac{228\left(\alpha^8-\beta^8\right)}{57a_8}$

$=\frac{228}{57}$

$=4$