JAM 2006

MATHEMATICS TEST PAPER

NOTATIONS USED

$\mathbb{R}$: The set of all real numbers
Z : The set of all integers

IMPORTANT NOTE FOR CANDIDATES

<u>Objective Part</u>:
Attempt ALL the objective questions (Questions 1-15). Each of these questions carries *six* marks. Each incorrect answer carries *minus two*. Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 7 only.

<u>Subjective Part:</u>
Attempt ALL subjective questions (Questions 16-29). Each of these questions carries *fifteen* marks.

1. $\lim_{n\to\infty} \frac{2^{n+1}+3^{n+1}}{2^n+3^n}$ equals

(A) 3
(B) 2
(C) 1
(D) 0

2. Let $f(x)=(x-2)^{17}(x+5)^{24}$. Then

(A) f does not have a critical point at 2
(B) f has a minimum at 2
(C) f has a maximum at 2
(D) f has neither a minimum nor a maximum at 2

3. Let $f(x,y)=x^5y^2\tan^{-1}\left(\frac{y}{x}\right)$. Then $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$ equals

(A) $2f$
(B) $3f$
(C) $5f$
(D) $7f$

4. Let G be the set of all irrational numbers. The interior and the closure of G are denoted by G^0 and $\overline{G}$, respectively. Then

(A) $G^0=\phi$, $\overline{G}=G$
(B) $G^0=\mathbb{R}$, $\overline{G}=\mathbb{R}$
(C) $G^0=\phi$, $\overline{G}=\mathbb{R}$
(D) $G^0=G$, $\overline{G}=\mathbb{R}$

5. Let $f(x) = \int_{\sin x}^{\cos x} e^{-t^2}\, dt$. Then $f'(\pi/4)$ equals

(A) $\sqrt{1/e}$
(B) $-\sqrt{2/e}$
(C) $\sqrt{2/e}$
(D) $-\sqrt{1/e}$

6. Let C be the circle $x^2 + y^2 = 1$ taken in the anti-clockwise sense. Then the value of the integral

$$\int_C \left[\left(2xy^3 + y\right)dx + \left(3x^2y^2 + 2x\right)dy\right]$$

equals

(A) 1
(B) $\pi/2$
(C) π
(D) 0

7. Let r be the distance of a point $P(x, y, z)$ from the origin O. Then ∇r is a vector

(A) orthogonal to $\overrightarrow{OP}$
(B) normal to the level surface of r at P
(C) normal to the surface of revolution generated by OP about x-axis
(D) normal to the surface of revolution generated by OP about y-axis

8. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_2, 0).$$

If $N(T)$ and $R(T)$ denote the null space and the range space of T respectively, then

(A) $\dim N(T) = 2$
(B) $\dim R(T) = 2$
(C) $R(T) = N(T)$
(D) $N(T) \subset R(T)$

9. Let S be a closed surface for which $\iint_S \vec{r}.\hat{n}\, d\sigma = 1$. Then the volume enclosed by the surface is

(A) 1
(B) 1/3
(C) 2/3
(D) 3

10. If $\left(c_1 + c_2 \ln x\right)/x$ is the general solution of the differential equation

$$x^2 \frac{d^2 y}{dx^2} + kx \frac{dy}{dx} + y = 0, \qquad x > 0,$$

then k equals

(A) 3
(B) –3
(C) 2
(D) –1

11. If A and B are 3×3 real matrices such that $rank(AB) = 1$, then $rank(BA)$ **cannot** be

(A) 0
(B) 1
(C) 2
(D) 3

12. The differential equation representing the family of circles touching y-axis at the origin is

(A) linear and of first order
(B) linear and of second order
(C) nonlinear and of first order
(D) nonlinear and of second order

13. Let G be a group of order 7 and $\phi\,(x) = x^4,\ x \in G$. Then ϕ is

(A) not one – one
(B) not onto
(C) not a homomorphism
(D) one – one, onto and a homomorphism

14. Let R be the ring of all 2×2 matrices with integer entries. Which of the following subsets of R is an integral domain?

(A) $\left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} : x,\ y \in \mathbf{Z} \right\}$

(B) $\left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x,\ y \in \mathbf{Z} \right\}$

(C) $\left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in \mathbf{Z} \right\}$

(D) $\left\{ \begin{pmatrix} x & y \\ y & z \end{pmatrix} : x,\ y,\ z \in \mathbf{Z} \right\}$

15. Let $f_n(x) = n \sin^{2n+1} x \cos x$. Then the value of

$$\lim_{n\to\infty} \int_0^{\pi/2} f_n(x)dx - \int_0^{\pi/2} \left(\lim_{n\to\infty} f_n(x)\right) dx$$

is

(A) $1/2$
(B) 0
(C) $-1/2$
(D) $-\infty$

16. (a) Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!\, 3^n} \qquad (6)$$

(b) Show that

$$\ln(1+\cos x) \le \ln 2 - \frac{x^2}{4}$$

for $0 \le x \le \pi/2$. (9)

17. Find the critical points of the function

$$f(x, y) = x^3 + y^2 - 12x - 6y + 40.$$

Test each of these for maximum and minimum. (15)

18. (a) Evaluate $\iint_R xe^{y^2}\, dx\, dy$, where R is the region bounded by the lines $x = 0$, $y = 1$ and the parabola $y = x^2$. (6)

(b) Find the volume of the solid bounded above by the surface $z = 1 - x^2 - y^2$ and below by the plane $z = 0$. (9)

19. Evaluate the surface integral

$$\iint_S x\left(12y - y^4 + z^2\right) d\sigma,$$

where the surface S is represented in the form $z = y^2$, $0 \le x \le 1$, $0 \le y \le 1$. (15)

20. Using the change of variables, evaluate $\iint_R xy\, dx\, dy$, where the region R is bounded by the curves $xy = 1$, $xy = 3$, $y = 3x$ and $y = 5x$ in the first quadrant. (15)

21. (a) Let u and v be the eigenvectors of A corresponding to the eigenvalues 1 and 3 respectively. Prove that $u + v$ is not an eigenvector of A. (6)

(b) Let A and B be real matrices such that the sum of each row of A is 1 and the sum of each row of B is 2. Then show that 2 is an eigenvalue of AB. (9)

22. Suppose W_1 and W_2 are subspaces of $\mathbb{R}^4$ spanned by $\{(1,2,3,4),(2,1,1,2)\}$ and $\{(1,0,1,0),(3,0,1,0)\}$ respectively. Find a basis of $W_1 \cap W_2$. Also find a basis of $W_1 + W_2$ containing $\{(1,0,1,0),(3,0,1,0)\}$. (15)

23. Determine y_0 such that the solution of the differential equation

$$y' - y = 1 - e^{-x}, \; y(0) = y_0$$

has a finite limit as $x \to \infty$. (15)

24. Let $\phi(x, y, z) = e^x \sin y$. Evaluate the surface integral $\iint_S \frac{\partial \phi}{\partial n} d\sigma$, where S is the surface of the cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$ and $\frac{\partial \phi}{\partial n}$ is the directional derivative of ϕ in the direction of the unit outward normal to S. Verify the divergence theorem. (15)

25. Let $y = f(x)$ be a twice continuously differentiable function on $(0,\infty)$ satisfying

$$f(1) = 1 \text{ and } f'(x) = \frac{1}{2} f\left(\frac{1}{x}\right), \quad x > 0.$$

Form the second order differential equation satisfied by $y = f(x)$, and obtain its solution satisfying the given conditions. (15)

26. Let $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{Z} \right\}$ be the group under matrix addition and H be the subgroup of G consisting of matrices with even entries. Find the order of the quotient group G/H. (15)

27. Let

$$f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ \sqrt{x} & x > 1. \end{cases}$$

Show that f is uniformly continuous on $[0, \infty)$. (15)

28. Find $M_n = \max\limits_{x \ge 0} \left\{ \frac{x}{n(1 + nx^3)} \right\}$, and hence prove that the series

$$\sum_{n=1}^{\infty} \frac{x}{n\left(1 + n x^3\right)}$$

is uniformly convergent on $[0, \infty)$. (15)

29. Let R be the ring of polynomials with real coefficients under polynomial addition and polynomial multiplication. Suppose

$$I = \{ p \in R : \text{ sum of the coefficients of } p \text{ is zero} \}.$$

Prove that I is a maximal ideal of R. (15)