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$1+2+2+3+3+5+4 = 20$ m

CONTINUITY AND DIFFERENTIABILITY

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	$\sin x$	$\cos x$
k	0	$\cos x$	$-\sin x$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\tan x$	$\sec^2 x$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\cot x$	$-\operatorname{cosec}^2 x$
e^x	e^x	$\sec x$	$\sec x \tan x$
a^x	$a^x \log a$	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\log x$	$\frac{1}{x}$	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
x	1	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
		$\tan^{-1} x$	$\frac{1}{1+x^2}$
		$\cot^{-1} x$	$\frac{-1}{1+x^2}$
		$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
		$\operatorname{cosec}^{-1} x$	$\frac{-1}{x\sqrt{x^2-1}}$

$$\frac{d}{dx}(uv) = uv' + vu'$$

$$\frac{d}{dx}(uvw) = u'vw + uv'w + uvw'$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

Composite function is function inside a function.

Eg: $y = \sqrt{\sin\sqrt{\sin\sqrt{x}}}$

$y = e^{m\sin^{-1}x}$

$y = \tan(\sec(\log\sqrt{x}))$

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To differentiate composite functions, we follow chain rule, i.e,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{dv} \cdots \frac{dv}{dx}$$

Differentiate the following functions :

1] $y = \sin 2x - 3\sin x + 4\cot\sqrt{x}$

$\frac{dy}{dx} = \cos 2x \cdot 2 - 3\cos x \cdot 4(-)\frac{\operatorname{cosec}^2\sqrt{x}}{2\sqrt{x}}$

2] $y = \log(2x^2 - 3x + 5)$

$\frac{dy}{dx} = \frac{1}{2x^2 - 3x + 5} \cdot (4x - 3)$

3] $y = \cos(e^{3x})$

$\frac{dy}{dx} = -\sin(e^{3x}) \cdot e^{3x} \cdot 3$

4] $y = \tan 3x + 4\operatorname{cosec} x - 5\sec\sqrt{x}$

$\frac{dy}{dx} = \sec^2 3x \cdot 3 - 4\operatorname{cosec}x \cdot \cot x - \frac{5\sec\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}}$

5] $y = \log(\sin e^{3x})$

$\frac{dy}{dx} = \frac{1}{\sin e^{3x}} \cdot \cos e^{3x} \cdot e^{3x} \cdot 3$

** 6] $y = e^{\log \sin 2x} = \sin 2x$

$\frac{dy}{dx} = 2\cos 2x$

NOTE : 1] $e^{\log x} = x$

2] $a^{\log_a x} = x$

7] $y = e^{\log \cos 2x} = \cos 2x$

$\frac{dy}{dx} = -2\sin 2x$

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8] $y = e^{\log \sec 2x} = \sec 2x$

$y' = 2\sec 2x \tan 2x$

9] $y = e^{\log \tan 2x} = \tan 2x$

$y' = 2\sec^2 2x$

10] $y = e^{\log \cot 2x} = \cot 2x$

$y' = -2\,\mathrm{cosec}^2 2x$

11] $y = e^{\log \mathrm{cosec}\, 2x} = \mathrm{cosec}\, 2x$

$y' = -2\,\mathrm{cosec}\, 2x \cot 2x$

12] $y = e^{3\log x} = e^{\log x^3} = x^3$

$y' = 3x^2$

13] $y = e^{4\log x} = e^{\log x^4} = x^4$

$y' = 4x^3$

14] $y = e^{5\log x} = e^{\log x^5} = x^5$

$y' = 5x^4$

15] $y = e^{7\log x} = e^{\log x^7} = x^7$

$y' = 7x^6$

16] $y = e^{3\log \sin x} = e^{\log \sin^3 x} = \sin^3 x$

$y' = 3\sin^2 x \cos x$

17] $y = e^{4\log \cos x} = e^{\log \cos^4 x} = \cos^4 x$

$y' = -4\cos^3 x \cdot \sin x$

18] $y = e^{\frac{1}{2}\log \tan x}$

$= e^{\log \sqrt{\tan x}} = \sqrt{\tan x}$

$y' = \dfrac{\sec^2 x}{2\sqrt{\tan x}}$

19] $y = e^{\frac{1}{2}\log \sec x} = e^{\log \sqrt{\sec x}}$

$= \sqrt{\sec x}$

$y' = \dfrac{\sec x \tan x}{2\sqrt{\sec x}}$

20] $y = e^{\frac{1}{2}\log \mathrm{cosec}\, x}$

$= e^{\log \sqrt{\mathrm{cosec}\, x}} = \sqrt{\mathrm{cosec}\, x}$

$y' = \dfrac{-\mathrm{cosec}\, x \cot x}{2\sqrt{\mathrm{cosec}\, x}}$

21] $y = e^{\frac{1}{2}\log \cot x}$

$= e^{\log \sqrt{\cot x}} = \sqrt{\cot x}$

$y' = \dfrac{-\mathrm{cosec}^2 x}{2\sqrt{\cot x}}$

22] $y = 3^{2\log_3 x} = 3^{\log_3 x^2} = x^2$

$y' = 2x$

23] $y = 4^{3\log_4 x} = 4^{\log_4 x^3} = x^3$

$y' = 3x^2$

24] $y = 5^{4\log_5 x} = 5^{\log_5 x^4} = x^4$

$y' = 4x^3$

25] $y = 6^{7\log_6 x} = 6^{\log_6 x^7} = x^7$

$y' = 7x^6$

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EXERCISE 5·4

** 1] $y = \dfrac{e^x}{\sin x}$

$y' = \dfrac{\sin x e^x - e^x \cos x}{\sin^2 x} = \dfrac{e^x(\sin x - \cos x)}{\sin^2 x}$

2] $y = e^{\sin^{-1}x}$

$y' = \dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

3] $y = e^{x^3}$

$y' = e^{x^3} \cdot 3x^2$

4] $y = \sin(\tan^{-1} e^{-x})$

$y' = \dfrac{\cos(\tan^{-1}(e^{-x}))}{\sqrt{1+(e^{-x})^2}} \cdot e^{-x}(-1)$

$= -\dfrac{e^{-x}\cos(\tan^{-1}e^{-x})}{\sqrt{1+e^{-2x}}}$

5] $y = \log(\cos e^x)$

$y' = \dfrac{-\sin e^x}{\cos e^x} \cdot e^x$

$= -e^x(\tan e^x)$

6] $y = e^x + e^{x^2} + \cdots + e^{x^5}$

$y = e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$

$y' = e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$

7] $y = \sqrt{e^{\sqrt{x}}}$

$y' = \dfrac{e^{\sqrt{x}}}{2\sqrt{e^{\sqrt{x}}}} \cdot \dfrac{1}{2\sqrt{x}}$

8] $y = \log(\log x)$

$y' = \dfrac{1}{\log x} \cdot \dfrac{1}{x} = \dfrac{1}{x \log x}$

9] $y = \dfrac{\cos x}{\log x}$

$y' = \dfrac{-\log x \sin x - \cos x\left(\frac{1}{x}\right)}{(\log x)^2}$

10] $y = \cos(\log x + e^x)$

$y' = -\sin(\log x + e^x)\left[\frac{1}{x} + e^x\right]$

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EXERCISE 5.2

1] ~~sin~~ $y = \sin(x^2+5)$

$y' = \cos(x^2+5)\cdot 2x$

2] $y = \cos(\sin x)$

$y' = -\sin(\sin x)\cdot \cos x$

3] $y = \sin(ax+b)$

$y' = \cos(ax+b)\cdot a$

**4] a) $y = \sec(\tan\sqrt{x})$

$y' = \sec(\tan\sqrt{x})\cdot\tan(\tan\sqrt{x})\cdot\frac{\sec^2\sqrt{x}}{2\sqrt{x}}$

8] $y = \cos\sqrt{x}$

$y' = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$

b) $y = \operatorname{cosec}(\cot\sqrt{x})$

$y' = -\operatorname{cosec}(\cot\sqrt{x})\cdot\cot(\cot\sqrt{x})\cdot\frac{-\operatorname{cosec}^2\sqrt{x}}{2\sqrt{x}}$

$= \operatorname{cosec}(\cot\sqrt{x})\cdot\cot(\cot\sqrt{x})\cdot\frac{\operatorname{cosec}^2\sqrt{x}}{2\sqrt{x}}$

5] $y = \frac{\sin(ax+b)}{\cos(cx+d)}$

$y' = \frac{\cos(cx+d)\cdot\cos(ax+b)\cdot a + \sin(ax+b)\sin(cx+d)\cdot c}{\cos^2(cx+d)}$

6] $y = \cos x^3\cdot\sin^2(x^5)$

$y' = -\sin x^3\cdot 3x^2\cdot\sin^2(x^5) + \cos x^3\cdot 2\sin(x^5)\cdot\cos(x^5)\cdot 5x^4$

7] $y = 2\sqrt{\cot x^2}$

$y' = \frac{2(-)\operatorname{cosec}^2(x^2)}{2\sqrt{\cot x^2}}\cdot 2x = \frac{-2x\cdot\operatorname{cosec}^2(x^2)}{\sqrt{\cot x^2}}$

EXERCISE 5.3

NOTE: Explicit function : A function of the form $y = f(x)$ is called an explicit function (variables are seperated in LHS & RHS)

Implicit function : A function of the form $f(x,y)$ or $f(x,y,z)$ or $f(r,\theta)$ are called implicit functions where the variables are inseperable.

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To differentiate implicit function, differentiate the equation as it stands and simplify to get $\frac{dy}{dx}$

Find $\frac{dy}{dx}$ for the following :

** 1] $2x + 3y = \sin x$

$2 + 3\frac{dy}{dx} = \cos x$

$\frac{dy}{dx} = \frac{\cos x - 2}{3}$

2] $2x + 3y = \sin y$

$2 + 3\frac{dy}{dx} = \cos y \cdot \frac{dy}{dx}$

$\frac{dy}{dx}\left[3 - \cos y\right] = -2$

$\frac{dy}{dx} = \frac{-2}{3-\cos y} = \frac{2}{\cos y - 3}$

3] $ax + by^2 = \cos y$

$a + b.2y.\frac{dy}{dx} = -\sin y \cdot \frac{dy}{dx}$

$\frac{dy}{dx}(2by + \sin y) = -a$

$\frac{dy}{dx} = \frac{-a}{2by + \sin y}$

4] $xy + y^2 = \tan x + y$

$x\frac{dy}{dx} + y + 2y\cdot\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$

$\frac{dy}{dx}(x + 2y - 1) = \sec^2 x - y$

$\frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}$

5] $x^2 + xy + y^2 = 100$

$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left[x + 2y\right] = -2x - y$

$\frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$

6] $x^3 + x^2y + xy^2 + y^3 = 81$

$3x^2 + x^2\frac{dy}{dx} + y2x + x\,2y\cdot\frac{dy}{dx} + y^2 + 3y^2\cdot\frac{dy}{dx}$

$\frac{dy}{dx}\left(x^2 + 2xy + 3y^2\right) = -3x^2 - 2xy - y^2$

$\frac{dy}{dx} = \frac{-(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}$

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7] $\sin^2 y + \cos(xy) = k$

$2\sin y \cdot \cos y \cdot \frac{dy}{dx} - \sin(xy)\left[x\frac{dy}{dx} + y\right] = 0$

$\frac{dy}{dx}\left[2\sin y \cdot \cos y - x\sin(xy)\right] = y\sin(xy)$

$\frac{dy}{dx} = \frac{y\sin(xy)}{\sin 2y - x\sin(xy)}$

8] $\sin^2 x + \cos^2 y = 1$

$2\sin x \cdot \cos x - 2\cos y \sin y \cdot \frac{dy}{dx} = 0$

$\sin 2x - \sin 2y \cdot \frac{dy}{dx} = 0$

$\frac{\sin 2x}{\sin 2y} = \frac{dy}{dx}$

Differentiate the following:

*] $y = \sin^3 2x$

$y' = 3\sin^2 2x \cdot \cos 2x \cdot 2$

*] $y = \cos^3 3x$

$y' = -2\cos 3x \cdot \sin 3x \cdot 3$

*] $y = \tan^4 x$

$y' = 4\tan^3 x \cdot \sec^2 x$

*] $y = \cot^3\left(\frac{x}{2}\right)$

$y' = 3\cot^2\left(\frac{x}{2}\right)(-)\operatorname{cosec}^2\left(\frac{x}{2}\right)\cdot\left(\frac{1}{2}\right)$

*] $y = \sec^2 3x$

$y' = 2\sec 3x \cdot \sec 3x \cdot \tan 3x \cdot 3$

$= 6\sec^2 3x \cdot \tan 3x$

*] $y = \sec^2\left(\frac{a}{x^2}\right)$

$y' = 2\sec\left(\frac{a}{x^2}\right)\cdot\sec\left(\frac{a}{x^2}\right)\tan\left(\frac{a}{x^2}\right)$

$(-2ax^{-3})$

*] $y = \cos^4(\log x)$

$y' = \frac{4\cos^3(\log x)(-)\sin(\log x)}{x}$

*] $y = \cos^2\sqrt{x}$

$y' = \frac{2\cos\sqrt{x}(-)\sin\sqrt{x}}{2\sqrt{x}}$

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*] $y = \operatorname{cosec}^2 6x$

$y' = -2\operatorname{cosec} 6x \cdot \operatorname{cosec} 6x \cot 6x \cdot 6$

*] $y = \tan^3 3x$

$y' = 3\tan^2 3x \cdot \sec^2 3x \cdot 3$

*] $y = \log(\sec x + \tan x)$

$y' = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$= \dfrac{\sec x(\tan x + \sec x)}{(\sec x + \tan x)}$

$= \sec x$

*] $y = (1 - x + x^2)^3$

$y' = 3(1 - x + x^2)^2 \cdot (-1 + 2x)$

*] $y = \dfrac{1}{\sqrt[3]{2x+3}} = (2x+3)^{\frac{-1}{3}}$

$y' = -\frac{1}{3}(2x+3)^{-\frac{4}{3}} \cdot 2$

*] $y = \sqrt{1-x}$

$y' = \dfrac{(-1)}{2\sqrt{1-x}}$

*] $y = \log(\sec x)$

$y' = \dfrac{\sec x \cdot \tan x}{\sec x} = \tan x$

*] $y = \sqrt{\log x}$

$y' = \dfrac{1}{2\sqrt{\log x}\,(x)}$

*] $y = \sqrt{\sqrt{\cos x}}$

$y' = \dfrac{1}{2\sqrt{\sqrt{\cos x}}} \cdot \dfrac{-\sin x}{2\sqrt{\cos x}}$

*] $y = e^{\cos 3x}$

$y' = -e^{\cos 3x} \cdot \sin 3x \cdot 3$

*] $y = (1 - 3x^4)^{\frac{5}{2}}$

$y' = \frac{5}{3}(1 - 3x^4)^{\frac{3}{2}} \cdot (-12x^3)$

*] $y = \cos\left(\frac{1}{x^2}\right) = \cos(x^{-2})$

$y' = -\sin(x^{-2}) \cdot (-2)x^{-3}$

*] $y = \sqrt{a^2 + x^2}$

$y' = \dfrac{\cancel{2}x}{\cancel{2}\sqrt{a^2+x^2}} = \dfrac{x}{\sqrt{a^2+x^2}}$

*] $y = \sqrt{1-x^2}$

$y' = \dfrac{-(\cancel{2}x)}{\cancel{2}\sqrt{1-x^2}} = \dfrac{-x}{\sqrt{1-x^2}}$

*] $y = \operatorname{cosec}^2\left(\frac{1}{x}\right)$

$y' = -2\operatorname{cosec}\left(\frac{1}{x}\right)\left(\operatorname{cosec}\left(\frac{1}{x}\right) \cdot \cot\left(\frac{1}{x}\right)\right) \cdot \left(\frac{-1}{x^2}\right)$

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*] $y = \tan\sqrt{\cos x}$

$y' = \dfrac{\sec^2\sqrt{\cos x}}{2\sqrt{\cos x}}\,(-)\sin x$

*] $y = \tan^2(1-e^x)$

$y' = 2\tan(1-e^x)\cdot\sec^2(1-e^x)\cdot e^x$

*] $y = \cancel{e^{\sin x \cdot \cos x}}\; e^{\sin x}\cdot\cos x$

$y' = \cancel{e^{\sin x \cdot \sin x +}}\; e^{\sin x}.(-\sin x) + \cos x \cdot e^{\sin x}\cdot\cos x$

$= e^{\sin x}(\cos^2 x - \sin x)$

*] $y = \log(\sin x)\cdot e^{\sqrt{x}}$

$y' = \log(\sin x)\cdot\dfrac{e^{\sqrt{x}}}{2\sqrt{x}} + e^{\sqrt{x}}\cdot\dfrac{\cos x}{\sin x}$

*] $y = (x^3+1)\ \sec^2 x$

$y' = (x^3+1)\,2\sec x\cdot\sec x\cdot\tan x + \sec^2 x\cdot 3x^2$

$= (x^3+1)\,2\sec^2 x\cdot\tan x + \sec^2 x\cdot 3x^2$

*] $y = e^{ax}\sin bx$

$y' = e^{ax}\,b\cos x + \sin bx + a e^{ax}$

**] $y = \sin x^\circ$ (degree can't be differentiated)

$y = \sin\left(\dfrac{\pi x}{180}\right)$

$y' = \cos\left(\dfrac{\pi x}{180}\right)\cdot\left(\dfrac{\pi}{180}\right)$

*] $y = \cos x^\circ = \cos\left(\dfrac{\pi x}{180}\right)$

$y' = -\sin\left(\dfrac{\pi x}{180}\right)\cdot\left(\dfrac{\pi}{180}\right)$

*] $y = \tan x^\circ = \tan\left(\dfrac{\pi x}{180}\right)$

$y' = \sec^2\left(\dfrac{\pi x}{180}\right)\cdot\left(\dfrac{\pi}{180}\right)$

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*] $y = \log_7(\log x)$ $\left[\log_b a = \dfrac{\log a}{\log b}\right]$

$y = \dfrac{\log(\log x)}{\log 7}$

$y' = \dfrac{1}{\log 7} \cdot \dfrac{1}{\log x(x)}$

*] $y = \sin^3 4x \cdot \cos^2 3x$

$y' = \sin^3 4x \cdot 2\cos 3x \cdot (-)\sin 3x \cdot 3 + \cos^2 3x \cdot 3\sin^2 4x \cdot \cos 4x \cdot 4$

*] $y = \tan^2 2x \cdot \sec^4 3x$

$y' = \tan^2 2x \cdot 4\sec^3 3x \cdot \sec 3x \cdot \tan 3x \cdot 3 + \sec^4 3x \cdot 2\tan 2x \cdot \sec^2 2x \cdot 2$

*] $y = \dfrac{\sqrt{1+x}}{x^2}$

$y' = \dfrac{\dfrac{x^2}{2\sqrt{1+x}} - \sqrt{1+x} \cdot 2x}{x^4}$

*] $y = \dfrac{x}{\sqrt{a^2-x^2}}$

$y' = \dfrac{\sqrt{a^2-x^2} - x\dfrac{(-\cancel{2}x)}{\cancel{2}\sqrt{a^2-x^2}}}{a^2-x^2}$

$= \dfrac{a^2 - x^2 + x^2}{(a^2-x^2)\sqrt{a^2-x^2}} = \dfrac{a^2}{(a^2-x^2)^{\frac{3}{2}}}$

*] $y = \sqrt{\dfrac{1+x}{1-x}}$

$y' = \dfrac{\sqrt{1-x}}{2\sqrt{1+x}} \cdot \left[\dfrac{(1-\cancel{x}) + (1+\cancel{x})}{(1-x)^2}\right]$

$= \dfrac{\cancel{\sqrt{1-x}}}{\cancel{2}\sqrt{1+x}}\left[\dfrac{\cancel{2}}{(1-x)\cancel{\sqrt{1-x}}\sqrt{1-x}}\right] = \dfrac{1}{(1-x)\sqrt{1-x^2}}$

*] $y = \sqrt{\dfrac{1-\cos x}{1+\cos x}}$

$y' = \dfrac{\sqrt{1+\cos x}}{2\sqrt{1-\cos x}} \cdot \left[\dfrac{(1+\cos x)\sin x - (1-\cos x)\sin x}{(1+\cos x)^2}\right]$

$= \dfrac{\cancel{\sqrt{1+\cos x}}}{2\sqrt{1-\cos x}}\left[\dfrac{\sin x\,(1+\cancel{\cos x} + 1 - \cancel{\cos x})}{(1+\cos x)\cancel{\sqrt{1+\cos x}}\sqrt{1+\cos x}}\right]$

$= \dfrac{\cancel{2}\cancel{\sin x}}{\cancel{2}\sqrt{1-\cos^2 x}\,(1+\cos x)} = \dfrac{1}{1+\cos x}$

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*] $y = \sqrt{\dfrac{1-\sin x}{1+\sin x}}$

$y' = \dfrac{\sqrt{1+\sin x}}{2\sqrt{1-\sin x}}\left[\dfrac{(1+\sin x)(-\cos x) - (1-\sin x)(\cos x)}{(1+\sin x)^2}\right]$

$= \dfrac{\sqrt{1+\sin x}}{2\sqrt{1-\sin x}} \quad \dfrac{\cos x(-1-\cancel{\sin x}-1+\cancel{\sin x})}{(1+\sin x)^2}$

$= \dfrac{\cancel{\sqrt{1+\sin x}}}{\cancel{2}\sqrt{1-\sin x}} \quad \dfrac{-\cancel{2}\cos x}{(1+\sin x)\cancel{\sqrt{1+\sin x}}\ \sqrt{1+\sin x}}$

$= \dfrac{-\cancel{\cos x}}{\sqrt{1-\cancel{\sin^2 x}}\ (1+\sin x)} = \dfrac{-1}{1+\sin x}$

*] $y = \log_x x = \dfrac{\log x}{\log x} = 1$

$y' = 0$

*] $y = \log_x (\log x) = \dfrac{\log(\log x)}{\log x}$

$y' = \dfrac{\dfrac{\cancel{\log x}}{x\cdot\cancel{\log x}} - \dfrac{\log(\log x)}{x}}{(\log x)^2} = \dfrac{1-\log(\log x)}{x(\log x)^2}$

*] $y = \log\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$ Show that $y' = \dfrac{-2a^2x}{a^4-x^4}$

$y = \log\left(\dfrac{a^2-x^2}{a^2+x^2}\right)^{\frac{1}{2}} = \dfrac{1}{2}\log\left(\dfrac{a^2-x^2}{a^2+x^2}\right)$

$= \dfrac{1}{2}\left[\log(a^2-x^2) - \log(a^2+x^2)\right]$

$y' = \dfrac{1}{2}\left[\dfrac{-2x}{a^2-x^2} - \dfrac{2x}{a^2+x^2}\right]$

$= \dfrac{-\cancel{2}x}{\cancel{2}}\left[\dfrac{1}{a^2-x^2} + \dfrac{1}{a^2+x^2}\right] = -x\,\dfrac{(a^2+\cancel{x^2}+a^2-\cancel{x^2})}{(a^2-x^2)(a^2+x^2)}$

$= \dfrac{-2a^2x}{a^4-x^4}$

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*] If $y = x + \sqrt{1+x^2}$. prove that $y' = \frac{y}{\sqrt{1+x^2}}$

$y' = 1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \frac{y}{\sqrt{1+x^2}}$

*] $y = \sqrt{x+1} + \sqrt{x-1}$ prove that $4(x^2-1)\cdot\left(\frac{dy}{dx}\right)^2 = y^2$

$y = \sqrt{x+1} + \sqrt{x-1}$

$y' = \frac{1}{2\sqrt{x+1}} + \frac{1}{2\sqrt{x-1}}$

$= \frac{\sqrt{x-1}+\sqrt{x+1}}{2\sqrt{x^2-1}} = \frac{y}{2\sqrt{x^2-1}}$

$\Rightarrow 2\sqrt{x^2-1}\cdot\frac{dy}{dx} = y$

(squaring on both sides)

$4(x^2-1)\left(\frac{dy}{dx}\right)^2 = y^2$

*] If $y = \sqrt{x\log x}$. show that $2y\frac{dy}{dx} = 1 + \log x$

$y' = \frac{1}{2\sqrt{x\log x}}\cdot\left[x\cdot\frac{1}{x} + \log x\right]$

$y' = \frac{1}{2y}\left[1 + \log x\right]$

$\Rightarrow 2y\frac{dy}{dx} = 1 + \log x$

*] If $y = \left(x + \sqrt{a^2+x^2}\right)^n$. Show that $(a^2+x^2)y_1^2 = n^2y^2$

$y_1 = n\left(x+\sqrt{a^2+x^2}\right)^{n-1}\cdot\left[1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{a^2+x^2}}\right]$

$= n\frac{\left(x+\sqrt{a^2+x^2}\right)^n}{x+\sqrt{a^2+x^2}}\cdot\left[\frac{x+\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}}\right]$

$= \frac{ny}{\sqrt{a^2+x^2}}$

$\Rightarrow y_1\sqrt{a^2+x^2} = ny \quad \Rightarrow (a^2+x^2)y_1^2 = n^2y^2$

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*] If $y = \log\left[\frac{a+b\tan x}{a-b\tan x}\right]$. Show that $\frac{dy}{dx} = \frac{2ab\sec^2 x}{a^2-b^2\tan^2 x}$

$$y' = \left[\frac{\cancel{a-b\tan x}}{a+b\tan x}\right]\cdot\left[\frac{a-b\tan x\,(b\sec^2 x) - a+b\tan x\,(-b\sec^2 x)}{(a-b\tan x)^{\cancel{2}}}\right]$$

$$= \left(\frac{\cancel{a-b\tan x}}{a+b\tan x}\right)\frac{b\sec^2 x\,(a-\cancel{b\tan x}+a+\cancel{b\tan x})}{(a-b\tan x)}$$

$$= \frac{2ab\sec^2 x}{a^2-b^2\tan^2 x}$$

Types of Differentiation:

Differentiation of inverse trignometric funⁿ by substitution:

EXERCISE 5.3

9] $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ $\quad x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$y = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$$

$$y' = \frac{2}{1+x^2}$$

10] $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$ $\quad x = \tan\theta$, $\theta = \tan^{-1}x$

$$= \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta = 3\tan^{-1}x$$

$$y' = \frac{3}{1+x^2}$$

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11] $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ 0 < x < 1$ $x = \tan\theta$

$\theta = \tan^{-1}x$

$= \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cancel{\cos^{-1}}(\cancel{\cos} 2\theta)$

$= 2\theta = 2\tan^{-1}x$

$y' = \frac{2}{1+x^2}$

12] $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ 0 < x < 1$ $x = \tan\theta$

$\theta = \tan^{-1}x$

$= \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}(\cos 2\theta) = \cancel{\sin^{-1}}\left[\cancel{\sin}\left(\frac{\pi}{2} - 2\theta\right)\right]$

$= \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\tan^{-1}x$

$y' = 0 - \frac{2}{1+x^2} = \frac{-2}{1+x^2}$

13] $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right),\ -1 < x < 1$ $x = \tan\theta$

$\theta = \tan^{-1}x$

$= \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\sin 2\theta) = \cancel{\cos^{-1}}\left[\cancel{\cos}\left(\frac{\pi}{2} - 2\theta\right)\right]$

$= \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\tan^{-1}x$

$y' = 0 - \frac{2}{1+x^2} = \frac{-2}{1+x^2}$

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14] $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ $x = \sin\theta$

$\theta = \sin^{-1}x$

$= \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$= \sin^{-1}(2\sin\theta\cos\theta)$

$= \cancel{\sin^{-1}}(\cancel{\sin}\,2\theta) = 2\theta = 2\sin^{-1}x$

$y' = \frac{2}{\sqrt{1-x^2}}$

15] $y = \sec^{-1}\left[\frac{1}{2x^2-1}\right]$, $0 < x < \frac{1}{\sqrt{2}}$ $x = \cos\theta$

$\theta = \cos^{-1}x$

$= \sec^{-1}\left[\frac{1}{2\cos^2\theta - 1}\right]$

$= \sec^{-1}\left(\frac{1}{\cos 2\theta}\right) = \cancel{\sec^{-1}}(\cancel{\sec}\,2\theta) = 2\theta = 2\cos^{-1}x$

$y' = \frac{-2}{\sqrt{1-x^2}}$

Logarithmic differentiation :

→ If a function has x or y in power, then logarithm is applied and then differentiated.

→ If a function is product of many terms, then application of log simplifies the differentiation.

→ If log is not applied for functions with power as x/y, differentiation will be incomplete.

*] $y = x^x$

$\log y = x \log x$

$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x} + \log x$

$\frac{dy}{dx} = y(1+\log x)$

$\frac{dy}{dx} = x^x(1+\log x)$

*] $y = \sin x^x + x^{x^2}$

$y = u + v \longrightarrow (1)$

$u = \sin x^x$

$\log u = x \log(\sin x)$

$\frac{1}{u}\cdot\frac{du}{dx} = \frac{x\cos x}{\sin x} + \log(\sin x)$

$\frac{du}{dx} = (\sin x)^x\left[x\cot x + \log(\sin x)\right]$

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① $\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= (\sin x)^x [x \cot x + \log(\sin x)] + x^x (1 + \log x)$

EXERCISE 5.5

Eg 30] Differentiate $\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$ w.r.t x (Apply log on both side)

$\log y = \frac{1}{2}\left[\log(x-3) + \log(x^2+4) - \log(3x^2+4x+5)\right]$

$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$

$= \frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$

Eg 31) Differentiate a^x w.r.t x.

$y = a^x$

$\log y = x \log a$

$\frac{1}{y}\frac{dy}{dx} = \log a \cdot 1$

$\frac{dy}{dx} = y \cdot \log a = a^x \log a$

Eg 32] $y = x^{\sin x}$, $x > 0$ w.r.t x (Apply log on both sides)

$\log y = \sin x \cdot \log x$

$\frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + \log x \cdot \cos x$

$\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x \cdot \cos x\right]$

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Eg 33] Find $\frac{dy}{dx}$ if $y^x + x^y + x^x = a^b$

$u + v + w = a^b$ → constant

$\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0 \rightarrow (1)$

$w = x^x$

$\frac{dw}{dx} = x^x(1 + \log x)$

$u = y^x$ (Apply log on both sides)

$\log u = x \log y$

$\frac{1}{u}\frac{du}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y$

$\frac{du}{dx} = y^x\left(\frac{x}{y}\cdot\frac{dy}{dx} + \log y\right)$

$\frac{du}{dx} = xy^{x-1}\frac{dy}{dx} + y^x \log y$

$v = x^y$ (Apply log)

$\log v = y \log x$

$\frac{1}{v}\frac{dv}{dx} = \frac{y}{x} + \log x \frac{dy}{dx}$

$\frac{dv}{dx} = x^y\left(\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right)$

$\frac{dv}{dx} = x^{y-1}y + x^y \cdot \log x \cdot \frac{dy}{dx}$

(1) $\Rightarrow xy^{x-1}\frac{dy}{dx} + y^x \log y + x^{y-1}y + x^y \log x \cdot \frac{dy}{dx} + x^x(1 + \log x) = 0$

$\frac{dy}{dx}\left[xy^{x-1} + x^y \cdot \log x\right] = -\left[y^x \log y + x^{y-1}y + x^x(1 + \log x)\right]$

$\therefore \frac{dy}{dx} = -\frac{\left[y^x \log y + x^{y-1}y + x^x(1 + \log x)\right]}{\left[xy^{x-1} + x^y \cdot \log x\right]}$

1] $\cos x \cdot \cos 2x \cdot \cos 3x$

$y = \cos x \cdot \cos 2x \cdot \cos 3x$ (Apply log)

$\log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$

$\frac{1}{y}\frac{dy}{dx} = \frac{-\sin x}{\cos x} + \frac{-\sin 2x(2)}{\cos 2x} + \frac{-\sin 3x(3)}{\cos 3x}$

$\frac{dy}{dx} = -\cos x \cdot \cos 2x \cdot \cos 3x \left(\tan x + 2\tan 2x + 3\tan 3x\right)$

2] $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

$\log y = \frac{1}{2}\left[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\right]$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$

$\frac{dy}{dx} = \frac{1}{2}(y)\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$

3] $y = (\log x)^{\cos x}$

$\log y = \cancel{\log(\log x)^{\cos x}}\ \cos x \cdot \log(\log x)$

$\frac{1}{y}\frac{dy}{dx} = \left[\frac{\cos x}{x \log x} + \log(\log x)(-)\sin x\right]$

$\frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{(x)\log x} - \log(\log x)\cdot \sin x\right]$

4] $y = x^x - 2^{\sin x}$

$y = u - v \longrightarrow (1)$

$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

$u = x^x$	$v = 2^{\sin x}$
$\frac{du}{dx} = x^x(1+\log x)$	$\frac{dv}{dx} = 2^{\sin x}\log 2 \cdot \cos x$

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① $\Rightarrow \dfrac{dy}{dx} = x^{x}(1+\log x) - 2^{\sin x}\log 2 \cdot \cos x$

5] $(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$

$y = (x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$ (Apply log)

$\log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$

$\dfrac{dy}{dx} = y\left[\dfrac{2}{x+3} + \dfrac{3}{x+4} + \dfrac{4}{x+5}\right]$

7] $y = (\log x)^{x} + x^{\log x}$

$y = u + v$

$\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} \rightarrow ①$

$u = (\log x)^{x}$

$\log u = x \log(\log x)$

$\dfrac{1}{u}\dfrac{du}{dx} = \dfrac{x}{x \cdot \log x} + \log(\log x)$

$\dfrac{du}{dx} = u\left[\dfrac{1}{\log x} + \log(\log x)\right] = (\log x)^{x}\left[\dfrac{1}{\log x} + \log(\log x)\right]$

$v = x^{\log x}$

$\log v = \log x \cdot \log x = (\log x)^2$

$\dfrac{1}{v} \cdot \dfrac{dv}{dx} = \dfrac{2\log x}{x}$

$\dfrac{dv}{dx} = x^{\log x}\left(\dfrac{2\log x}{x}\right)$

① $\Rightarrow \dfrac{dy}{dx} = (\log x)^{x}\left[\dfrac{1}{\log x} + \log(\log x)\right] + x^{\log x}\left(\dfrac{2\log x}{x}\right)$

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9] $y = x^{\sin x} + \sin x^{\cos x}$

$y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \rightarrow (1)$

$u = x^{\sin x}$

$\log u = \sin x \cdot \log x$

$\frac{1}{u}\frac{du}{dx} = \frac{\sin x}{x} + \log x \cdot \cos x$

$\frac{du}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x \cdot \cos x\right]$

$v = \sin x^{\cos x}$

$\log v = \cos x \cdot \log(\sin x)$

$\frac{1}{v}\frac{dv}{dx} = \frac{\cos x}{\sin x} \cdot (\overset{\cos}{\cancel{\sin}}\, x) + \log(\sin x)(-\sin x)$

$\frac{dv}{dx} = \sin x^{\cos x}\left[\cos x \cdot \cot x - \log(\sin x) \cdot \sin x\right]$

(1) $\Rightarrow \frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x \cdot \cos x\right] + \sin x^{\cos x}\left[\cos x \cdot \cot x - \log(\sin x) \sin x\right]$

6] $y = \left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$

$y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \longrightarrow (1)$

$u = \left(x + \frac{1}{x}\right)^x$

$\log u = x \cdot \log\left(x + \frac{1}{x}\right)$

$\frac{1}{u} \cdot \frac{du}{dx} = \left(\frac{x}{x + \frac{1}{x}}\right)\left(1 - \frac{1}{x^2}\right) + \log\left(x + \frac{1}{x}\right)$

$= \left(\frac{x^2}{x^2+1}\right)\left(\frac{x^2-1}{x^2}\right) + \log\left(x + \frac{1}{x}\right)$

$\frac{du}{dx} = \left(x + \frac{1}{x}\right)^x \left[\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right]$

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$v = x^{\left(1+\frac{1}{x}\right)}$

$\log v = \left(1+\frac{1}{x}\right)\cdot \log x$

$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{\left(1+\frac{1}{x}\right)}{x} + \log x\left(0-\frac{1}{x^2}\right)$

$= \left(\frac{x+1}{x^2}\right) + \left(\frac{-1}{x^2}\right)\log x$

$= \frac{x+1-\log x}{x^2}$

$\frac{dv}{dx} = x^{\left(1+\frac{1}{x}\right)}\left[\frac{x+1-\log x}{x^2}\right]$

① $\Rightarrow \frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)}\left[\frac{x+1-\log x}{x^2}\right]$

8] $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \rightarrow$ ①

$u = (\sin x)^x$

$\log u = x\cdot\log(\sin x)$

$\frac{1}{u}\frac{du}{dx} = \frac{x(\cos x)}{\sin x} + \log(\sin x)\cdot 1$

$\frac{du}{dx} = (\sin x)^x\left[x\cot x + \log(\sin x)\right]$

$v = \sin^{-1}\sqrt{x}$

$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$

$\frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}}$

① $\Rightarrow \frac{dy}{dx} = (\sin x)^x\left[x\cot x + \log(\sin x)\right] + \frac{1}{2\sqrt{x-x^2}}$

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10] $y = x^{x \cdot \cos x} + \frac{x^2+1}{x^2-1}$

$y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \rightarrow (1)$

$u = x^{x\cos x}$

$\log u = x \cdot \cos x \cdot \log x$

$\frac{1}{u} \frac{du}{dx} = 1 \cdot \cos x \cdot \log x + x(-\sin x) \log x + \frac{x \cdot \cos x}{x}$

$\frac{du}{dx} = x^{x\cos x}(\cos x \cdot \log x - x \cdot \sin x \cdot \log x + \cos x)$

$v = \frac{x^2+1}{x^2-1}$

$\frac{dv}{dx} = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{2x[x^2-1-x^2-1]}{(x^2-1)^2}$

$\frac{dv}{dx} = \frac{2x(-2)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$

$(1) \Rightarrow \frac{dy}{dx} = x^{x\cos x}(\cos x \cdot \log x - x \cdot \sin x \cdot \log x + \cos x) - \frac{4x}{(x^2-1)^2}$

11] $y = (x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

$y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \rightarrow (1)$

$u = (x\cos x)^x$

$\log u = x \cdot \log(x \cdot \cos x)$

$\log u = x \cdot (\log x + \log(\cos x))$

$\frac{1}{u} \frac{du}{dx} = x\left[\frac{1}{x} + \frac{-\sin x}{\cos x}\right] + (\log x + \log(\cos x)) \cdot 1$

$\frac{du}{dx} = (x\cos x)^x [1 - x\tan x + \log(x \cdot \cos x)]$

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$v = (x \sin x)^{\frac{1}{x}}$

$\log v = \frac{1}{x} \log(x \sin x)$

$\log v = \frac{1}{x}\left[\log x + \log(\sin x)\right]$

$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{1}{x}\left(\frac{1}{x} + \frac{\cos x}{\sin x}\right) + \left(\log x + \log(\sin x)\left(\frac{-1}{x^2}\right)\right)$

$\frac{1}{v} \cdot \frac{dv}{dx} = \left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log(x \cdot \sin x)}{x^2}\right]$

$\frac{dv}{dx} = (x \sin x)^{\frac{1}{x}}\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log(x \cdot \sin x)}{x^2}\right]$

① $\Rightarrow \frac{dy}{dx} = (x\cos x)^x\left[1 - x\tan x + \log(x \cdot \cos x)\right] + (x \cdot \sin x)^{\frac{1}{x}}\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log(x \sin x)}{x^2}\right]$

Find $\frac{dy}{dx}$ of the functions:

12] $x^y + y^x = 1$

$u + v = 1$

$\frac{du}{dx} + \frac{dv}{dx} = 0 \rightarrow ①$

$u = x^y$ | $v = y^x$

$\log u = y \log x$ | $\log v = x \log y$

$\frac{1}{u} \cdot \frac{du}{dx} = \frac{y}{x} + \log x \frac{dy}{dx}$ | $\frac{1}{v} \frac{dv}{dx} = \frac{x}{y} y' + \log y$

$\frac{du}{dx} = x^y\left[\frac{y}{x} + \log x \, y'\right]$ | $\frac{dv}{dx} = y^x\left[\frac{x}{y} y' + \log y\right]$

$u' = yx^{y-1} + x^y \log x \, y'$ | $\frac{dv}{dx} = xy^{x-1} y' + y^x \log y$

① $\Rightarrow yx^{y-1} + x^y \log x \, y' + xy^{x-1} y' + y^x \log y = 0$

$y' = \frac{-(yx^{y-1} + y^x \log y)}{x^y \log x + xy^{x-1}}$

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13] $x^y = y^x$

$y \log x = x \log y$

$\frac{y}{x} + \log x \, y' = \frac{x}{y} y' + \log y$

$y' = \frac{\log y - \frac{y}{x}}{-\frac{x}{y} + \log x} = \frac{(x \log y - y)\, y}{(-x + y \log x)\, x}$

$= \frac{y(x \log y - y)}{x(y \log x - x)}$

14] $(\cos x)^y = (\cos y)^x$

$y \log \cos x = x \log \cos y$

$\frac{y(-\sin x)}{\cos x} + \log \cos x \cdot y' = \frac{x(-\sin y)}{\cos y} y' + \log(\cos y)$

$y'[\log(\cos x) + x \tan y] = \log(\cos y) + y \tan x$

$y' = \frac{\log(\cos y) + y \tan x}{\log(\cos x) + x \tan y}$

15] $xy = e^{x-y}$ Apply log,

$\log x + \log y = x - y \log e$ $[\because \log e = 1]$

$\log x + \log y = x - y$ (1)

$\frac{1}{x} + \frac{1}{y} \cdot y' = 1 - y'$

$y'\left[\frac{1}{y} + 1\right] = 1 - \frac{1}{x}$

$y'\left(\frac{1+y}{y}\right) = \left(\frac{x-1}{x}\right)$

$y' = \frac{(x-1)y}{(1+y)x}$

16] Find the derivation of the fun^n^ given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$

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Given : $f(x) = y = (1+x)(1+x^2)(1+x^4)(1+x^8)$

$\log y = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$

$\frac{1}{y} \cdot y' = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}$

$y' = y\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$

$y']_{x=1} = 2^4\left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right]$

$= (\cancel{16}^{8})\left(\frac{15}{\cancel{2}}\right) = 120$

Paramatric Differentiation :

→ Generally used variables are x and y. These variables can be inturn expressed in other special variables like θ, t, ϕ, ψ etc

→ A pair of equations x and y expressed in special variables form paramatric equations. Special variables θ, t, ϕ, ψ are called parameters.

→ To differentiate parametric equations, we use

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{dy/dt}{dx/dt} \dots$

→ Differentiation of one function w.r.t another function is also a type of paramtric differentiation.

Here, we call the first fun^n as y and 2^{nd} fun^n as z and hence required is

$\frac{dy}{dz} = \frac{dy/d\theta}{dz/d\theta} = \frac{dy/dx}{dz/dx} \dots$

** Mis Eg 48] Differentiate $\underset{y}{\sin^2 x}$ w.r.t $\underset{z}{e^{\cos x}}$.

$y = \sin^2 x$	$z = e^{\cos x}$
$\frac{dy}{dx} = 2\sin x \cdot \cos x$	$\frac{dz}{dx} = e^{\cos x} \cdot (-\sin x)$

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$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{2\sin x \cdot \cos x}{-e^{\cos x} \cdot \sin x} = -2\cos x \cdot e^{-\cos x}$

Eg 34] Find $\frac{dy}{dx}$ if $x = a\cos\theta$ & $y = a\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = a\cos\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{a}\cos\theta}{-\cancel{a}\sin\theta} = -\cot\theta$

Eg 35] Find $\frac{dy}{dx}$ if $x = at^2$, $y = 2at$

$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cancel{a}}{2at} = \frac{1}{t}$

**** Eg 36] Find $\frac{dy}{dx}$ if $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$

$\frac{dx}{d\theta} = a[1+\cos\theta] = a\, 2\cos^2\frac{\theta}{2}$ | $\frac{dy}{d\theta} = a(0+\sin\theta) = a\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{2a}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2a}\cos^{\cancel{2}}\frac{\theta}{2}} = \tan\frac{\theta}{2}$

Eg 37] Find y' if $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ $\left(\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}\right)$

Given : $x^{2/3} + y^{2/3} = a^{2/3} \rightarrow (1)$

$\begin{bmatrix} \text{put } x = a\cos^3\theta \\ y = a\sin^3\theta \end{bmatrix}$

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LHS $= a^{\frac{2}{3}}\cos^2\theta + a^{2/3}\sin^2\theta$

$= a^{\frac{2}{3}}(\cos^2\theta + \sin^2\theta)$

$= a^{\frac{2}{3}}(1)$

$= a^{\frac{2}{3}} =$ RHS

$x = a\cos^3\theta$ | $y = a\sin^3\theta$

$\frac{dx}{d\theta} = a\,3\cos^2\theta\cdot(-\sin\theta)$ | $\frac{dy}{d\theta} = a\,3\sin^2\theta\cdot\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{3a}\sin^{\cancel{2}}\theta\,\cancel{\cos\theta}}{-\cancel{3a}\cos^{\cancel{2}}\theta(+\sin\theta)} = \frac{-\sin\theta}{\cos\theta} \rightarrow ②$$

but $x^{\frac{1}{3}} = a^{\frac{1}{3}}\cos\theta$ IIIly $y^{\frac{1}{3}} = a^{\frac{1}{3}}\sin\theta$

$\cos\theta = \left(\frac{x}{a}\right)^{\frac{1}{3}}$ $\sin\theta = \left(\frac{y}{a}\right)^{\frac{1}{3}}$

$$② \Rightarrow \frac{dy}{dx} = \frac{-\left(\frac{y}{a}\right)^{\frac{1}{3}}}{\left(\frac{x}{a}\right)^{\frac{1}{3}}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}} = -\sqrt[3]{\frac{y}{x}}$$

EXERCISE 5.6

1] $x = 2at^2$ $y = at^4$

$\frac{dx}{dt} = 4at$ $\frac{dy}{dt} = 4at^3$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{4a}t^3}{\cancel{4a}t} = t^2$$

2] $x = a\cos\theta,\ y = b\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

3] $x = \sin t$ $y = \cos 2t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin 2t}{\cos t} = \frac{-2(2\sin t\cos\theta)}{\cos t} = -4\sin t$$

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4] $x = 4t \quad , \quad y = \frac{4}{t}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{4}{t^2} \cdot \frac{1}{4} = -\frac{1}{t^2}$

5] $x = \cos\theta - \cos 2\theta \quad , \quad y = \sin\theta - \sin 2\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta - 2\cos 2\theta}{-\sin\theta + 2\sin 2\theta}$

** 6] $x = a(\theta - \sin\theta) \quad , \quad y = a(1 + \cos\theta)$

$\frac{dx}{d\theta} = a[1 - \cos\theta] = a \cdot 2\sin^2\frac{\theta}{2}$ | $\frac{dy}{d\theta} = a(-\sin\theta) = -a\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\cancel{a}\cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cancel{2}\cancel{a}\sin^2\frac{\theta}{2}} = -\cot\frac{\theta}{2}$

** 8] $x = a\left[\cos t + \log\left(\tan\frac{t}{2}\right)\right] \quad , \quad y = a\sin t$

$\frac{dx}{dt} = a\left\{-\sin t + \frac{\sec^2\left(\frac{t}{2}\right)}{\tan\left(\frac{t}{2}\right)} \cdot \frac{1}{2}\right\}$ $\qquad \frac{dy}{dt} = a\cos t$

$= a\left\{-\sin t + \frac{\cos t/2}{2\cos^2 t/2 \cdot \sin t/2}\right\}$

$= a\left\{-\sin t + \frac{1}{\sin t}\right\} = a\left\{\frac{1 - \sin^2 t}{\sin t}\right\} = a\left\{\frac{\cos^2 t}{\sin t}\right\}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{a}\cos t}{\cancel{a}\cos^2 t}\sin t = \tan t$

** 10] $x = a[\cos\theta + \theta\sin\theta] \quad , \quad y = a[\sin\theta - \theta\cos\theta]$

$\frac{dx}{d\theta} = a\{-\cancel{\sin\theta} + \theta\cos\theta + \cancel{\sin\theta}\} = a\theta\cos\theta$ | $\frac{dy}{d\theta} = a\{\cancel{\cos\theta} + \theta\sin\theta - \cancel{\cos\theta}\} = a\theta\sin\theta$

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$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

11] $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$ show that $\frac{dy}{dx} = \frac{-y}{x}$

$x = \sqrt{a^{\sin^{-1}t}}$ Squaring on both sides,

$x^2 = a^{\sin^{-1}t}$, $y^2 = a^{\cos^{-1}t} \longrightarrow (1)$

$$2x\cdot\frac{dx}{dt} = \frac{a^{\sin^{-1}t}\log a}{\sqrt{1-t^2}} \qquad 2y\cdot\frac{dy}{dt} = \frac{-a^{\cos^{-1}t}\log a}{\sqrt{1-t^2}}$$

$$\therefore \frac{dx}{dt} = \frac{a^{\sin^{-1}t}\log a}{2x\sqrt{1-t^2}} \qquad \frac{dy}{dt} = \frac{-a^{\cos^{-1}t}\log a}{2y\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a^{\cos^{-1}t}\cancel{\log a}}{\cancel{2}y\cancel{\sqrt{1-t^2}}}\cdot\frac{\cancel{2}x\cancel{\sqrt{1-t^2}}}{a^{\sin^{-1}t}\cancel{\log a}}$$

using (1), $\frac{dy}{dx} = \frac{-y^2 x}{y x^2} = \frac{-y}{x}$

<u>Second order derivative</u>:

If $y = f(x) \rightarrow y$ is a function of 'x'

Differentiating w.r.t 'x'

$\frac{dy}{dx} = f'(x) = y_1 = y' \rightarrow 1^{st}$ order derivative.

Differentiating again w.r.t x

$\frac{d^2y}{dx^2} = f''(x) = y_2 = y'' \rightarrow 2^{nd}$ order derivative

Differentiating again w.r.t x

$\frac{d^3y}{dx^3} = f'''(x) = y_3 = y''' \rightarrow 3^{rd}$ order derivative

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EXERCISE 5.7

1] $y = x^2 + 3x + 2$
$y_1 = 2x + 3$
$y_2 = 2$

2] $y = x^{20}$
$y_1 = 20x^{19}$
$y_2 = (20)(19)x^{18}$

*] $y = x^6$
$y_1 = 6x^5$; $y_2 = 6 \cdot 5 \cdot x^4$; $y_3 = 6 \cdot 5 \cdot 4 \cdot x^3$
$y_4 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot x^2$; $y_5 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot x$; $y_5 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6$

NOTE: $y = x^n \Rightarrow y_n = n!$

3] $y = x \cdot \cos x$
$y_1 = -x\sin x + \cos x$
$y_2 = -x \cdot \cos x + (-\sin x)1 - \sin x$
$= -[2\sin x + x\cos x]$

4] $y = \log x$
$y_1 = \frac{1}{x}$
$y_2 = -\frac{1}{x^2}$

5] $y = x^3 \cdot \log x$
$y_1 = \frac{x^{3}}{x} + \log x \cdot 3x^2 = x^2[1 + 3\log x]$

$y_2 = \frac{3x^2}{x} + (1 + 3\log x) \cdot 2x$
$= x[5 + 6\log x]$

6] $y = e^x \cdot \sin 5x$
$y_1 = e^x \cdot \cos 5x \cdot 5 + \sin 5x \cdot e^x$
$y_1 = e^x(5\cos 5x + \sin 5x)$
$y_2 = e^x[-25\sin 5x + 5\cos 5x] + (5\cos 5x + \sin 5x)e^x$
$= e^x(-24\sin 5x + 10\cos 5x)$

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7] $y = e^{6x}\cdot\cos 3x$

$y_1 = e^{6x}\cdot(-3\sin 3x) + (\cos 3x\cdot e^{6x}\cdot 6)$

$y_1 = e^{6x}(-3\sin 3x + 6\cos 3x)$

$y_2 = e^{6x}(-9\cos 3x - 18\sin 3x) + (-3\sin 3x + 6\cos 3x)\cdot e^{6x}\cdot 6$

$y_2 = e^{6x}(-9\cos 3x - 18\sin 3x - 18\sin 3x + 36\cos 3x)$

$y_2 = e^{6x}(27\cos 3x - 36\sin 3x)$

$y_2 = 9e^{6x}(3\cos 3x - 4\sin 3x)$

8] $y = \tan^{-1}x$

$y_1 = \frac{1}{1+x^2}$

$y_2 = \frac{-1}{(1+x^2)^2}\cdot 2x$

9] $y = \log(\log x)$

$y_1 = \frac{1}{x\cdot\log x}$

$y_2 = \frac{-1}{(x\cdot\log x)^2}\cdot\left[\frac{x}{x} + \log x\right]$

$y_2 = \frac{-1}{(x\cdot\log x)^2}(1 + \log x)$

10] $y = \sin(\log x)$

$y_1 = \frac{\cos(\log x)}{x}$

$y_2 = \frac{x\left(\frac{-\sin(\log x)}{x}\right) - \cos(\log x)}{x^2}$

$= \frac{-[\sin(\log x) + \cos(\log x)]}{x^2}$

** 11] If $y = 5\cos x - 3\sin x$. Prove that $\frac{d^2y}{dx^2} + y = 0$

$y = 5\cos x - 3\sin x$

$\frac{dy}{dx} = -5\sin x - 3\cos x$

$\frac{d^2y}{dx^2} = -5\cos x + 3\sin x$

$= -(5\cos x - 3\sin x)$

$= -y$

$\frac{d^2y}{dx^2} + y = 0$

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Eg39] If $y = A\sin x + B\cos x$. Prove that $\frac{d^2y}{dx^2} + y = 0$

$y = A\sin x + B\cos x$

$\frac{dy}{dx} = A\cos x - B\sin x$

$\frac{d^2y}{dx^2} = -A\sin x - B\cos x$

$= -(A\sin x + B\cos x) = -y$

$\frac{d^2y}{dx^2} + y = 0$

15] If $y = 500e^{7x} + 600e^{-7x}$. Prove $\frac{d^2y}{dx^2} = 49y$

$y = 500e^{7x} + 600e^{-7x}$

$\frac{dy}{dx} = (500)(7)e^{7x} + (600)(-7)e^{-7x}$

$\frac{d^2y}{dx^2} = 500(49)e^{7x} + (600)(49)e^{-7x}$

$= 49(500e^{7x} + 600e^{-7x})$

$\frac{d^2y}{dx^2} = 49y$

14] If $y = Ae^{mx} + Be^{nx}$. Show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Eg 40] If $y = 3e^{2x} + 2e^{3x}$. Show that $\frac{d^2y}{dx^2} - (2+3)\frac{dy}{dx} + 6y = 0$

14 Ans] $y = Ae^{mx} + Be^{nx}$

$\frac{dy}{dx} = Ame^{mx} + Bne^{nx}$

$\frac{d^2y}{dx^2} = Am^2e^{mx} + Bn^2e^{nx}$

LHS $= \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

$= (Am^2e^{mx} + Bn^2e^{nx}) - (m+n)(Ame^{mx} + Bne^{nx}) +$
$mn(Ae^{mx} + Be^{nx})$

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$LHS = A\cancel{m^2}e^{mx} + B\cancel{n^2}e^{nx} - A\cancel{m^2}e^{mx} - B\cancel{mn}e^{nx} - A\cancel{mn}e^{mx} - B\cancel{n^2}e^{nx} + A\cancel{mn}e^{mx} + B\cancel{mn}e^{nx}$

$= 0$

$= RHS$

Eg 40 Ans] $y = 3e^{2x} + 2e^{3x}$

$\frac{dy}{dx} = 6e^{2x} + 6e^{3x}$

$\frac{d^2y}{dx^2} = 12e^{2x} + 18e^{3x}$

$LHS = \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y$

$= 12e^{2x} + 18e^{3x} - 5[6e^{2x} + 6e^{3x}] + 6[3e^{2x} + 2e^{3x}]$

$= 12e^{2x} + 18e^{3x} - 30e^{2x} - 30e^{3x} + 18e^{2x} + 12e^{3x}$

$= \cancel{12}e^{2x} + \cancel{18}e^{3x} - \cancel{12}e^{2x} - \cancel{18}e^{3x}$

$= 0$

$= RHS$

** 13] If $y = 3\cos(\log x) + 4\sin(\log x)$. Show that $x^2y_2 + xy_1 + y = 0$

$y = 3\cos(\log x) + 4\sin(\log x)$

$y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x}$

$xy_1 = -3\sin(\log x) + 4\cos(\log x)$

$xy_2 + y_1 \cdot 1 = \frac{-3\cos(\log x)}{x} - \frac{4\sin(\log x)}{x}$

$x^2y_2 + xy_1 = -[3\cos(\log x) + 4\sin(\log x)]$

$x^2y_2 + xy_1 = -y$

$x^2y_2 + xy_1 + y = 0$

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** 17] If $y=(\tan^{-1}x)^2$ show that $(x^2+1)^2 y_2 + 2x(x^2+1)y_1 = 2$

$y=(\tan^{-1}x)^2$

$y_1 = \dfrac{2\tan^{-1}x}{1+x^2}$

$(1+x^2)y_1 = 2\tan^{-1}x$

$(1+x^2)y_2 + y_1(2x) = \dfrac{2}{1+x^2}$

$(1+x^2)\left[(1+x^2)y_2 + y_1(2x)\right] = 2$

$(1+x^2)^2 y_2 + 2x(x^2+1)y_1 = 2$

** Eg 41] If $y=\sin^{-1}x$. Show that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0$

$y=\sin^{-1}x$

$y_1 = \dfrac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\cdot y_1 = 1$ squaring on both sides,

$(1-x^2)y_1^2 = 1$

$(1-x^2)\cdot 2y_1\cdot y_2 + y_1^2(-2x) = 0$

Divide by '$2y_1$'

$(1-x^2)\cdot y_2 - xy_1 = 0$

$(1-x^2)\dfrac{dy^2}{dx^2} - x\dfrac{dy}{dx} = 0$

12] If $y=\cos^{-1}x$. find $\dfrac{d^2y}{dx^2}$ in terms of y alone.

$y=\cos^{-1}x$

~~$y=\cos$~~

$\cos y = x$

$-\sin y\dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = -\dfrac{1}{\sin y} = -\operatorname{cosec} y$

$\dfrac{d^2y}{dx^2} = (-)(-)\operatorname{cosec} y\cdot\cot y\cdot\dfrac{dy}{dx}$

$= \operatorname{cosec} y\cdot\cot y(-\operatorname{cosec} y)$

$= -\operatorname{cosec}^2 y\cdot\cot y$

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16] If $e^y(x+1)=1$. Show That (a) $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$ (b) $\frac{dy}{dx}=-e^y$

(a) $e^y(x+1)=1 \rightarrow ①$

$e^y+(x+1)\cdot e^y\cdot\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-\cancel{e^y}}{\cancel{e^y}(x+1)}=\frac{-1}{x+1}$

$\frac{d^2y}{dx^2}=(-)(-)\frac{1}{(x+1)^2}=\left(\frac{-1}{x+1}\right)^2$

$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$

(b) $\frac{dy}{dx}=\frac{-e^y}{e^y(x+1)}=\frac{-e^y}{(1)}=-e^y$ [using ①]

Mis Ex 16] If $\cos y = x\cos(a+y)$, $\cos a \neq \pm 1$, Prove $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$

Given : $\cos y = x\cdot\cos(a+y)$

$\Rightarrow x=\frac{\cos y}{\cos(a+y)}$, differentiating w.r.t x

$\frac{dx}{dy}=\frac{\cos(a+y)(-\sin y)+\cos y\cdot\sin(a+y)}{\cos^2(a+y)}$

$\frac{dx}{dy}=\frac{\sin(a+\cancel{y}-\cancel{y})}{\cos^2(a+y)}=\frac{\sin a}{\cos^2(a+y)}$

$\therefore \frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$

Mis Ex 17] If $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$ find $\frac{d^2y}{dx^2}$

$\frac{dx}{dt}=a[-\cancel{\sin t}+t\cos t+\cancel{\sin t}]$

$=at\cos t \rightarrow ①$

$\frac{dy}{dt}=a[\cancel{\cos t}+t\sin t-\cancel{\cos t}]$

$=at\sin t$

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$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx}$$

$$= \frac{\sec^2 t}{at\cos t} \quad [\text{using } ①]$$

$$\frac{d^2y}{dx^2} = \frac{\sec^3 t}{at}$$

Mis Ex 23] If $y = e^{a\cos^{-1}x}$; $-1 \le x \le 1$. Show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2 y = 0$

$$y = e^{a\cos^{-1}x}$$

$$\frac{dy}{dx} = e^{a\cos^{-1}x} \cdot \frac{(-a)}{\sqrt{1-x^2}} = \frac{-ay}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,\frac{dy}{dx} = -ay \quad (\text{squaring on both sides})$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = +a^2y^2$$

$$(1-x^2)\cdot 2y_1 \cdot y_2 + y_1^2(-2x) = a^2 \cdot 2y \cdot y_1$$

Divide by $2y_1$

$$(1-x^2)\cdot y_2 + y_1(\cancel{-2x}\,-x) = a^2 y$$

$$(1-x^2)y_2 - y_1 x = a^2 y$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2 y = 0$$

Mis Ex 1] $y = (3x^2 - 9x + 5)^9$

$$y_1 = 9(3x^2-9x+5)^8 \cdot (6x-9)$$

Mis Ex 2] $y = \sin^3 x + \cos^6 x$

$$y_1 = 3\sin^2 x \cdot \cos x - 6\cos^5 x \cdot \sin x$$

$$= 3\sin x\cos x(\sin x - 2\cos^4 x)$$

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Mis Ex 3] $y = (5x)^{3\cos 2x}$

power has 'x', hence apply log.

$\log y = 3\cos 2x \cdot \log 5x$

$\frac{1}{y}\frac{dy}{dx} = \frac{3\cos 2x}{\cancel{5}x}\cdot\cancel{5} + \log 5x \cdot \cancel{3\sin}\ 3(-2\sin 2x)$

$\frac{dy}{dx} = y\left[\frac{3\cos 2x}{x} - 6\sin 2x \cdot \log 5x\right]$

Mis Ex 4] $y = \sin^{-1}(x\sqrt{x}) = \sin^{-1}(x^{3/2})$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^3}}\cdot\frac{3}{2}\cdot x^{\frac{1}{2}}$

Mis Ex 5] $y = \frac{\cos^{-1}\left(\frac{x}{2}\right)}{\sqrt{2x+7}}$, $-2 < x < 2$

$\frac{dy}{dx} = \frac{(\sqrt{2x+7})\cdot\frac{-1}{\sqrt{1-(\frac{x}{2})^2}}\cdot\frac{1}{2} - \cos^{-1}\left(\frac{x}{2}\right)\cdot\frac{\cancel{2}}{\cancel{2}\sqrt{2x+7}}}{(\sqrt{2x+7})^2}$

Mis Ex 6] $y = \cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right)$, $0 < x < \frac{\pi}{2}$

$y = \cot^{-1}\left[\frac{\cos\frac{x}{2} + \cancel{\sin\frac{x}{2}} + \cos\frac{x}{2} - \cancel{\sin\frac{x}{2}}}{\cancel{\cos\frac{x}{2}} + \sin\frac{x}{2} - \cancel{\cos\frac{x}{2}} + \sin\frac{x}{2}}\right]$

$= \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right] = \cancel{\cot}^{-1}\left(\cancel{\cot}\ \frac{x}{2}\right) = \frac{x}{2}$

$\frac{dy}{dx} = \frac{1}{2}$

Mis Ex 7] $y = \log x^{\log x}$ power has x, apply log on both sides,

$\log y = \log x \cdot \log(\log x)$

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$\frac{1}{y}\frac{dy}{dx} = \log x \cdot \frac{1}{x \cdot \log x} + \frac{\log(\log x)}{x}$

$\frac{dy}{dx} = \frac{\log x^{\log x}}{x}\left[1 + \log(\log x)\right]$

Mis Ex 8] $y = \cos(a\cos x + b\sin x)$ for some constant a, b.

$\frac{dy}{dx} = -\sin(a\cos x + b\sin x)\cdot(-a\sin x + b\cos x)$

Mis Ex 9] $y = (\sin x - \cos x)^{\sin x - \cos x}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$

$\log y = \sin x - \cos x \cdot \log(\sin x - \cos x)$

$\frac{1}{y}\frac{dy}{dx} = \frac{\sin x - \cos x}{\sin x - \cos x}(\cos x + \sin x) + \log(\sin x - \cos x)(\cos x + \sin x)$

$\frac{dy}{dx} = y(\cos x + \sin x)\left[1 + \log(\sin x - \cos x)\right]$

Mis Ex 10] $y = x^x + x^a + a^x + a^a$ for some fixed $a > 0, x > 0$

$\frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x\log a + 0$;

$u = x^x$ (Apply log)

$\log u = x\log x$

$\frac{1}{u}\frac{du}{dx} = \frac{x}{x} + \log x$

$\frac{du}{dx} = x^x(1 + \log x)$

Mis Ex 11] $y = x^{x^2-3} + (x-3)^{x^2}$

$y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \to ①$

$u = x^{x^2-3}$

$\log u = x^2 - 3 \cdot \log x$

$\frac{1}{u}\frac{du}{dx} = \frac{x^2-3}{x} + \log x(2x)$

$v = (x-3)^{x^2}$

$\log v = x^2\log(x-3)$

$\frac{1}{v}\frac{dv}{dx} = \frac{x^2}{x-3} + \log(x-3)(2x)$

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① ⟹ $\frac{dy}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + \log x (2x)\right] + (x-3)^{x^2}\left[\frac{x^2}{x-3} + \log(x-3)(2x)\right]$

Mis Ex12] Find $\frac{dy}{dx}$ if $y = 12(1-\cos t)$, $x = 10(t - \sin t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

$\frac{dy}{dt} = 12(0+\sin t) + (1-\cos t)\cdot 0$ | $\frac{dx}{dt} = 10(1-\cos t) + (t-\sin t)\cdot 0$

$\frac{dy}{dt} = 12\sin t$ | $\frac{dx}{dt} = 10(1-\cos t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12\sin t}{10(1-\cos t)} = \frac{6\sin t}{5(2\sin^2\frac{t}{2})} \quad \frac{6(2)\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{5(2)\sin^2\frac{t}{2}}$$

$$= \frac{6}{5}\cot t/2$$

MisEx13] $\frac{dy}{dx}$ if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$, $0 < x < 1$

put $x = \sin\theta$, $\theta = \sin^{-1}x$

$$y = \sin^{-1}(\sin\theta) + \sin^{-1}\left(\sqrt{1-\sin^2\theta}\right)$$
$$= \theta + \sin^{-1}(\cos\theta)$$
$$= \theta + \sin^{-1}\left[\sin\left(\frac{\pi}{2}-\theta\right)\right]$$
$$= \cancel{\theta} + \frac{\pi}{2} - \cancel{\theta}$$
$$= \frac{\pi}{2}$$
$$\frac{dy}{dx} = 0$$

NOTE: $\sin^{-1}\sqrt{1-x^2} = \cos^{-1}x$

$\cos^{-1}\sqrt{1-x^2} = \sin^{-1}x$

$\therefore\ y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2} = \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Mis Ex14] If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ for $\underbrace{-1 < x < 1}_{\text{or } x \neq y}$, prove $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$
$$x\sqrt{1+y} = -y\sqrt{1+x}$$

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squaring on both sides,

$x^2(1+y) = +y^2(1+x)$

$x^2+x^2y = y^2+xy^2$

$(x^2-y^2)+(x^2y-xy^2) = 0$

$(x-y)(x+y) + xy(x-y) = 0$

$\cancel{(x-y)[(x+y)+(xyy)] = 0}$

$(x-y)(x+y+xy) = 0$ but $x \neq y$

$x+y+xy = 0$

$y(1-x) = -x$

$y = \frac{-x}{1+x}$

$\frac{dy}{dx} = \frac{(1+x)(-1)+x}{(1+x)^2} = \frac{-1-\cancel{x}+\cancel{x}}{(1+x)^2} = \frac{-1}{(1+x)^2}$

MisEx 15] If $(x-a)^2+(y-b)^2 = c^2$, for some $c>0$. prove
$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ is a constant independent of a & b.

Sol] Gn: $(x-a)^2+(y-b)^2 = c^2$

$2(x-a)+2(y-b)y' = 0$

$y' = \frac{-2(x-a)}{2(y-b)} = -\frac{(x-a)}{(y-b)}$

$y'' = \frac{(y-b)(-1)+(x-a)y'}{(y-b)^2} = \frac{-(y-b)+(x-a)\left[-\frac{(x-a)}{(y-b)}\right]}{(y-b)^2}$

$= \frac{-(y-b)^2-(x-a)^2}{(y-b)^3} = \frac{-[(y-b)^2+(x-a)^2]}{(y-b)^3}$

Gn: $\frac{[1+(y')^2]^{\frac{3}{2}}}{y''} = \frac{\left(1+\left[-\frac{(x-a)}{(y-b)}\right]^2\right)^{\frac{3}{2}}}{\frac{-[(y-b)^2+(x-a)^2]}{(y-b)^3}}$

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$$= \left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}} \cdot \frac{(y-b)^3}{-[(y-b)^2 + (x-a)^2]}$$

$$= \frac{(c^2)^{\frac{3}{2}}}{((y-b)^2)^{\frac{3}{2}}} \cdot \frac{(y-b)^3}{-c^2} = \frac{c^3}{(y-b)^3} \cdot \frac{(y-b)^3}{-c^2}$$

$$= \frac{c^3}{-c^2} = -c$$

$\therefore \dfrac{[1+(y')^2]^{\frac{3}{2}}}{y''}$ is a constant independent of a and b

EXERCISE 5.6

7] $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$. Find $\dfrac{dy}{dx}$

$$\frac{dx}{dt} = \frac{\sqrt{\cos 2t} \cdot 3\sin^2 t \cdot \cos t + \sin^3 t \cdot \frac{\cancel{2} \sin 2t}{\cancel{2}\sqrt{\cos 2t}}}{(\sqrt{\cos 2t})^2}$$

$$= \frac{\cos 2t \cdot 3\sin^2 t \cdot \cos t + \sin^3 t \cdot \sin 2t}{(\sqrt{\cos 2t})^3}$$

$$\frac{dy}{dt} = \frac{-\sqrt{\cos 2t} \cdot 3\cos^2 t \cdot \sin t + \cos^3 t \cdot \frac{\cancel{2} \sin 2t}{\cancel{2}\sqrt{\cos 2t}}}{(\sqrt{\cos 2t})^2}$$

$$= \frac{-\cos 2t \cdot 3\cos^2 t \cdot \sin t + \cos^3 t \cdot \sin 2t}{(\sqrt{\cos 2t})^3}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

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$$\frac{dy}{dx} = \frac{\dfrac{-\cos 2t \cdot 3\cos^2 t \cdot \sin t + \cos^3 t \cdot \sin 2t}{(\sqrt{\cos 2t})^3}}{\dfrac{\cos 2t \cdot 3\sin^2 t \cdot \cos t + \sin^3 t \cdot \sin 2t}{(\sqrt{\cos 2t})^3}}$$

$$= \frac{-\cos 2t \cdot 3\cos^2 t \sin t + \cos^3 t \sin 2t}{\cos 2t \cdot 3\sin^2 t \cdot \cos t + \sin^3 t \cdot \sin 2t}$$

$$= \frac{-(2\cos^2 t - 1)\, 3\cos^2 t \sin t + \cos^3 t \cdot 2\sin t \cos t}{(1-2\sin^2 t)\, 3\sin^2 t \cdot \cos t + \sin^3 t \cdot 2\sin t \cdot \cos t}$$

$$= \frac{-6\cos^4 t \sin t + 3\cos^2 t \sin t + 2\cos^4 t \sin t}{3\sin^2 t \cos t - 6\sin^4 t \cos t + 2\sin^4 t \cdot \cos t}$$

$$= \frac{3\cos^2 t \sin t - 4\cos^4 t \sin t}{3\sin^2 t \cos t - 4\sin^4 t \cos t}$$

$$= \frac{\cos t \cdot \sin t\,(3\cos t - 4\cos^3 t)}{\sin t \cdot \cos t\,(3\sin t - 4\sin^3 t)}$$

$$= \frac{3\cos t - 4\cos^3 t}{3\sin t - 4\sin^3 t}$$

$$= \frac{-(4\cos^3 t - 3\cos t)}{3\sin t - 4\sin^3 t} = -\frac{\cos 3t}{\sin 3t}$$

$$= -\cot 3t$$

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Continuity :

A function $f(x)$ is said to be continuous if $\boxed{\lim_{x\to a} f(x) = f(a)}$

i.e, LHL = $f(a)$ = RHL

EXERCISE 5.1

17] Find the relation b/w a & b so that the funⁿ f defined by

$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$

Gn : function is continuous at $x = 3$

$\Rightarrow \lim_{x\to 3^-} f(x) = f(3) = \lim_{x\to 3^+} f(x)$

$\lim_{x\to 3^-} (ax+1) = \lim_{x\to 3^+} (bx+3)$

$3a + 1 = 3b + 3$

$3a = 3b + 2$

$a = \frac{3b+2}{3}$

26] $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & \text{if } x \ne \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$. k = ?

Gn : funⁿ is continuous at $x = \frac{\pi}{2}$

$\Rightarrow \lim_{x\to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$

$\lim_{x\to \frac{\pi}{2}} \left[\frac{k\cos x}{\pi - 2x}\right] = 3$

$\lim_{\theta\to 0} \left[\frac{k\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}{\theta}\right] = 3$

let $\pi - 2x = \theta$

substituting x

$\pi - \theta = 2x$

$x = \frac{\pi}{2} - \frac{\theta}{2}$

as $x \to \frac{\pi}{2}$,

$\pi - 2\left(\frac{\pi}{2}\right) = \theta$

$\theta = 0$

$\theta \to 0$

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$$\lim_{\frac{\theta}{2}\to 0}\left[\frac{k\sin\frac{\theta}{2}}{2\cdot\frac{\theta}{2}}\right]=3$$

$$\frac{k}{2}\cdot 1 = 3$$

$$\therefore k=6$$

27] $f(x)=\begin{cases}kx^2, & \text{if } x\le 2\\ 3, & \text{if } x>2\end{cases}$ if continuas at $x=2$, $k=?$

Gn function is continuous at $x=2$

$$\Rightarrow \lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x)$$

$$\lim_{x\to 2^-}(kx^2) = \lim_{x\to 2^+}(3)$$

$$4k = 3$$

$$k = \frac{3}{4}$$

28] $f(x)=\begin{cases}kx+1, & \text{if } x\le \pi\\ \cos x, & \text{if } x>\pi\end{cases}$ is continuous at $x=\pi$, $k=?$

Gn $f(x)$ is continuous at $x=\pi$

$$\Rightarrow \lim_{x\to\pi^-} f(x) = f(\pi) = \lim_{x\to\pi^+} f(x)$$

$$\lim_{x\to\pi^-}(kx+1) = \lim_{x\to\pi^+}(\cos x)$$

$$\pi k+1 = +\cos\pi = -1$$

$$\pi k = -2$$

$$k = \frac{-2}{\pi}$$

** 29] $f(x)=\begin{cases}kx+1, & \text{if } x\le 5\\ 3x-5, & \text{if } x>5\end{cases}$ at $x=5$, $k=?$

$$\Rightarrow \lim_{x\to 5^-} f(x) = f(5) = \lim_{x\to 5^+} f(x)$$

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$$\lim_{x \to 5^+} (kx+1) = \lim_{x \to 5^+} (3x-5)$$

$$5k+1 = 15-5$$

$$5k = 9$$

$$k = \frac{9}{5}$$

30] Find a, b such that the funⁿ defined by $f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$ is a continuous funⁿ.

Gn funⁿ is continuous

at $x = 2$

$$\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x)$$

$$5 = \lim_{x \to 2^+} (ax+b)$$

$$5 = 2a + b \rightarrow ①$$

at $x = 10$

$$\lim_{x \to 10^-} f(x) = f(10) = \lim_{x \to 10^+} f(x)$$

$$\lim_{x \to 10^-} (ax+b) = 21$$

$$10a + b = 21 \rightarrow ②$$

② − ① ⇒

$$\begin{array}{r} 10a + b = 21 \\ -\ \ 2a + b = 5 \\ \hline 8a = 16 \end{array}$$

$$\boxed{a = 2}$$

① ⇒ $4 + b = 5$

$$\boxed{b = 1}$$

18] For what value of λ is the funⁿ defined by $f(x) = \begin{cases} \lambda(x^2-2x) & \text{if } x \le 0 \\ 4x+1 & \text{if } x > 0 \end{cases}$ is continuous at $x = 0$. What about continuity at $x = 1$.

Gn funⁿ is continuous at $x = 0$

$$\lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^+} f(x)$$

$$\lim_{x \to 0^-} [\lambda(x^2-2x)] = \lim_{x \to 0^+} (4x+1)$$

$$\lambda(0) = 1 \quad \Rightarrow \text{'}\lambda\text{' vanishes for any valu of } x$$

$\cancel{\lambda = 1}$

∴ For no value of λ $f(x)$ is continuous at $x = 0$.

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At $x=1$: $\lim_{x\to 1} f(x) = \lim_{x\to 1} (4x+1) = 5$

at $x=1$, $f(x)$ is a constant at any value of x it is continuous.

1] Prove that the funⁿ $f(x) = 5x-3$ is continuous i) at $x=0$
ii) at $x=-3$
and iii) at $x=5$

i] at $x=0$,

$f(0) = 5(0)-3 = -3$

$\lim_{x\to 0} f(x) = \lim_{x\to 0}(5x-3) = 5(0)-3 = -3$

$\because \lim_{x\to 0} f(x) = f(0) \Rightarrow$ funⁿ is contⁿ at $x=0$

ii] at $x=-3$

$f(-3) = 5(-3)-3 = -18$

$\lim_{x\to -3} f(x) = \lim_{x\to -3}(5x-3) = 5(-3)-3 = -18$

$\because \lim_{x\to -3} f(x) = f(-3) \Rightarrow$ funⁿ is contⁿ at $x=-3$

iii] at $x=5$

$f(5) = 5(5)-3 = 22$

$\lim_{x\to 5} f(x) = \lim_{x\to 5}(5x-3) = 5(5)-3 = 22$

$\because \lim_{x\to 5} f(x) = f(5) \Rightarrow$ funⁿ is contⁿ at $x=5$

2] Examine the conᵗʸ of the funⁿ $f(x) = 2x^2-1$ at $x=3$.

at $x=3$; consider, $\lim_{x\to 3} f(x) = f(3)$

$\lim_{x\to 3}(2x^2-1) = 2(3)^2-1$

$2(3^2)-1 = 2(3)^2-1$

$17 = 17$

$\therefore$ funⁿ is contⁿ at $x=3$

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5] Is the fun^n f defined by $f(x) = \begin{cases} x, & \text{if } \le 1 \\ 5, & \text{if } > 1 \end{cases}$ continuous at $x = 0$? at $x = 1$? at $x = 2$?

At $x = 1$, $\lim_{x \to 1^-} f(x) = f(1) = \lim_{x \to 1^+} f(x)$

$$\lim_{x \to 1^-} (x) = 5$$

but $1 \neq 5$

$\therefore$ Gn fun^n is not $cont^n$ at $x = 1$ and hence it is $cont^n$ at all other points.

At $x = 0$, $\lim_{x \to 0} f(x) = \lim_{x \to 0} (x) = 0$

$\Rightarrow f(x)$ is $cont^n$ at $x = 0$

At $x = 2$, $\lim_{x \to 2} f(x) = \lim_{x \to 2} (5) = 5$

$\Rightarrow f(x)$ is $cont^n$ at $x = 2$

Find all points of discontinuity of f, where f is defined by

6] $f(x) = \begin{cases} 2x + 3, & \text{if } x \le 2 \\ 2x - 3, & \text{if } x > 2 \end{cases}$

At $x = 2$, $\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x)$

$$\lim_{x \to 2^-} (2x+3) = \lim_{x \to 2^+} (2x-3)$$

but $4 + 3 \neq 4 - 3$

$7 \neq 1$

LHL is not equal to RHL

$\therefore$ The given fun^n is not $cont^n$ at $x = 2$.

Point of discontinuity is $x = 2$.

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7] $f(x) = \begin{cases} |x|+3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x+2, & \text{if } x \ge 3 \end{cases}$

(a) At $x = -3$

$$\lim_{x \to -3^-} f(x) = f(-3) = \lim_{x \to -3^+} f(x)$$

$$\lim_{x \to -3^-} (|x|+3) = \lim_{x \to -3^+} (-2x)$$

$$3+3 = -2(-3)$$

$$6 = 6$$

$\therefore$ The given funⁿ is contⁿ at $x = -3$

(b) At $x = 3$:

$$\lim_{x \to 3^-} f(x) = f(3) = \lim_{x \to 3^+} f(x)$$

$$\lim_{x \to 3^-} (-2x) = \lim_{x \to 3^+} (6x+2)$$

but $-6 \ne 20$

$\therefore$ The given funⁿ is not continuous at $x = 3$

$\therefore$ The point is discontinuous at $x = 3$

8] $f(x) = \begin{cases} \frac{|x|}{x} ; & x \ne 0 \\ 0 ; & x = 0 \end{cases}$

LHL $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$

$f(0) = 0$

RHL $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{x} = 1$

LHL $\ne$ RHL, $f(x)$ is not contⁿ

$\therefore$ pt. of discontity is $x = 0$.

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9] $f(x) = \begin{cases} \frac{x}{|x|}, & x < 0 \\ -1, & x \geq 0 \end{cases}$

LHL : $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} \frac{x}{-x} = -1$

$f(0) = -1$

RHL : $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (-1) = -1$

$\because$ LHL = RHL, $f(x)$ is continuous.

$\therefore$ no point of discontinuity.

10] $f(x) = \begin{cases} x+1, & x \geq 1 \\ x^2+1, & x < 1 \end{cases}$

LHL : $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2+1) = (1^2+1) = 2$

$f(1) = 1+1 = 2$

RHL : $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x+1) = 1+1 = 2$

$\because$ LHL = RHL, $f(x)$ is continuous.

$\therefore$ no point of discontinuity.

11] $f(x) = \begin{cases} x^3-3, & x \leq 2 \\ x^2+1, & x > 2 \end{cases}$

$\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x)$

$\lim_{x \to 2^-} (x^3-3) = \lim_{x \to 2^+} (x^2+1)$

$8-3 = 4+1$

$5 = 5$

$\therefore f(x)$ is contn at $x = 2$

There is no pt. of discontinuity.

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12] $f(x) = \begin{cases} x^{10}-1, & x \le 1 \\ x^2, & x > 1 \end{cases}$

$LHL = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^{10}-1) = (1^{10}-1) = 0$

$f(1) = 1^{10}-1 = 0$

$RHL = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2) = 1^2 = 1$

$LHL \ne RHL$ $\therefore$ $f(x)$ is not contⁿ at $x = 1$.

$\therefore$ pt. of discontinuity is $x = 1$

13] If the funⁿ defined by $f(x) = \begin{cases} x+5, & x \le 1 \\ x-5, & x > 1 \end{cases}$ is a contⁿ fun

$\lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^-} (x+5) \ne \lim_{x\to 1^+} (x-5)$

$1+5 \ne 1-5$

$6 \ne -4$

$\therefore$ It is not continuous.

14] Discuss the contnity of funⁿ f where f is defined by

$f(x) = \begin{cases} 3, & 0 \le x \le 1 \\ 4, & 1 < x < 3 \\ 5, & 3 \le x \le 10 \end{cases}$

A $x = 1$: $\lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^-} (3) = (3) = \lim_{x\to 1^+} (4)$

$3 \ne 4$

$\therefore$ $f(x)$ is discontinuous at $x = 1$

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At $x=3$: $\lim_{x\to 3^-} f(x) = f(3) = \lim_{x\to 3^+} f(x)$

$\lim_{x\to 3^-}(4) = \lim_{x\to 3^+}(5)$

$4 \neq 5$

$\therefore f(x)$ is discontinuous at $x=3$

15] $f(x) = \begin{cases} 2x, & x<0 \\ 0, & 0\le x\le 1 \\ 4x, & x>1 \end{cases}$

At $x=0$: $\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$

$\lim_{x\to 0^-}(2x) = \lim_{x\to 0^+}(0)$

$2(0) = 0$

$0 = 0$

$\therefore f(x)$ is contⁿ at $x=0$

At $x=1$: $\lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^-}(0) = \lim_{x\to 1^+}(4x)$

$0 \neq 4(1)$

$0 \neq 4$

$\therefore f(x)$ is not contⁿ at $x=1$.

25] Examine the contity of f where f is defined by

$f(x) = \begin{cases} \sin x - \cos x, & x\neq 0 \\ -1, & x=0 \end{cases}$

$\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$

$\lim_{x\to 0^-} \cancel{\cos}(\sin x - \cos x) = -1 = \lim_{x\to 0^+}(\sin x - \cos x)$

$(0-1) = -1 = (0-1)$

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$-1 = -1$

$\because$ LHL = RHL, $f(x)$ is continuous.

16] $f(x) = \begin{cases} -2, & x \le -1 \\ 2x, & -1 \le x \le 1 \\ 2, & x > 1 \end{cases}$

At $x = -1$: $\lim_{x \to -1^-} f(x) = f(-1) = \lim_{x \to -1^+} f(x)$

$$\lim_{x \to -1^-} (-2) = \lim_{x \to 1^+} (2x)$$

$$-2 = -2$$

$\therefore f(x)$ is contⁿ at $x = -1$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = f(1) = \lim_{x \to 1^+} f(x)$

$$\lim_{x \to 1^-} (2x) = \lim_{x \to 1^+} (2)$$

$$2(1) = 2$$

$$2 = 2$$

$\therefore f(x)$ is contⁿ at $x = 1$

Algebra of Continuous functions:

Theorem 1: Suppose f and g are two real funⁿs continuous at a real no. c, then

(a) $f + g$ is contⁿ at $x = c$

(b) $f - g$ is contⁿ at $x = c$

(c) $f \cdot g$ is contⁿ at $x = c$

(d) $\frac{f}{g}$ is contⁿ at $x = c$; $g(x) \neq 0$

Theorem 2: Suppose f and g are real valued funⁿs such that $(f o g)$ is defined at c. If g and f are both contⁿ at c, then their composition $(f o g)$ is also contⁿ at c.

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Eg 19] Show that the funⁿ $f(x) = \sin(x^2)$ is a contⁿ funⁿ

Let $f(x) = \sin(x^2)$,

$g(x) = \sin x$, $h(x) = x^2$

$g(x)$ being trignometric is contⁿ funⁿ

$h(x)$ being polynomial is a contⁿ funⁿ

Their composition : $g \circ h(x) = g[h(x)] = g(x^2) = \sin(x^2) = f(x)$

$\because$ g and h are contⁿ funⁿs, their composition $g \circ h(x) = f(x)$ is also contⁿ.

31] Show that funⁿ $f(x) = \cos(x^2)$ is a contⁿ funⁿ.

Let $f(x) = \cos(x^2)$,

$g(x) = \cos x$, $h(x) = x^2$

$g(x)$ being trignometric is contⁿ funⁿ

$h(x)$ being polynomial is a contⁿ funⁿ

Their composition : $g \circ h(x) = g[h(x)] = g(x^2) = \cos(x^2) = f(x)$

$\because$ g and h are contⁿ funⁿs, their composition $g \circ h(x) = f(x)$ is also contⁿ.

32] Show that $f(x) = |\cos x|$ is a contⁿ funⁿ.

Gn: $f(x) = |\cos x|$

let $g(x) = |x|$, $h(x) = \cos x$

their composition $g \circ h(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)$

$g(x)$ being modulus funⁿ is contⁿ funⁿ

$h(x)$ being trignometric funⁿ is contⁿ funⁿ.

$\therefore$ g and h are contⁿ funⁿs, their composition $g \circ h(x) = f(x)$ is also contⁿ.

33] Examine that $\sin|x|$ is a contⁿ funⁿ

$f(x) = \sin|x|$

let $g(x) = \sin x$, $h(x) = |x|$

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Their composition : $goh(x) = g[h(x)] = g(|x|) = \sin|x| = f$

$g(x)$ being trigonometric funⁿ is contⁿ

$h(x)$ being modulus funⁿ is contⁿ

$\because$ g and h are contⁿ funⁿ, their composition $goh(x) = f(x)$ is also contⁿ.

34] Find all pts of discontinuity of f, $f(x) = |x| - |x+1|$

Sol] Let $g(x) = |x|$

$h(x) = x+1$

$goh(x) = g(h(x)) = g(x+1) = |x+1|$

$|x|$ and $x+1$ both being continuous functions on R,

$goh(x)$ is also continuous function on $\mathbb{R}$.

Also $|x|$ is continuous function.

$\therefore$ Their difference, $|x| - |x+1|$ is also continuous on $\mathbb{R}$

$\Rightarrow f(x)$ is continuous on R

$\Rightarrow$ f has no point of discontinuity

Eg 20] Show that $f(x) = |1-x+|x||$ where x is any real no. is a contⁿ funⁿ.

Gn: $f(x) = |1-x+|x||$

let, $g(x) = |x|$, $h(x) = 1-x+|x|$

Their composition: $goh(x) = g[h(x)] = g(1-x+|x|) = |1-x+|x|| = f$

$\therefore$ $g(x)$ being modulus funⁿ is contⁿ

$h(x)$ being sum of a polynomial and a modulus funⁿ is contⁿ.

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$\therefore$ Their composition $go\,h(x) = f(x)$ is also cont^n.

Mean Value Theorems:

Theorem 6: Rolle's Theorem, (RT)

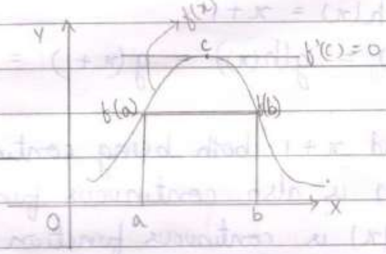
Let f be a fun^n $f: [a,b] \to R$ be cont (i) on $[a,b]$

(ii) differentiable on (a,b)

(iii) $f(a) = f(b)$ where a and b are some real no.s;

then there exists atleast one pt c such that

$f'(c) = 0$

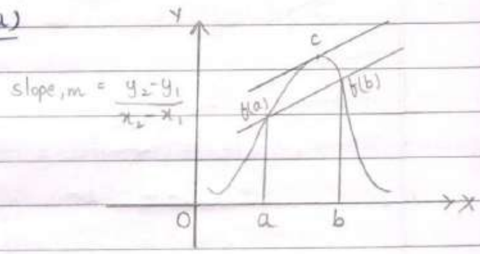


(LMVT)

Theorem 7: Let a fun^n $f: [a,b] \to R$ be

i) be cont^n $[a,b]$ ii) differentiable in (a,b), then there exist atleast one pt. c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



EXERCISE 5.8

1] Verify rolle's theorem for the fun^n $f(x) = x^2 + 2x - 8$; $x \in [-4, 2]$

Gn: $f(x) = x^2 + 2x - 8$

(1) $f(x)$ being polynomial is cont^n in $[-4, 2]$

(2) $f'(x) = 2x + 2 \Rightarrow f(x)$ is differentiable in $(-4, 2)$

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(3) $a = -4$, $f(a) = f(-4) = 16 - 8 - 8 = 0$

$b = 2$, $f(b) = f(2) = 4 + 4 - 8 = 0$

$f(a) = f(b)$

$\because$ all 3 conditions of RT are satisfied, there exists atleast one pt $\because$ c such that

$f'(c) = 0$

$2c + 2 = 0$

$2c = -2$

$c = -1 \in (-4, 2)$

Hence, RT verified

2] Examine if rolles theorem is applicable, if so verify

i] $f(x) = x^2 - 1$ for $x \in [1, 2]$

Gn : $f(x) = x^2 - 1$

(1) $f(x)$ being polynomial is contn in $[1, 2]$

(2) $f'(x) = 2x \Rightarrow f(x)$ is differentiable in $(1, 2)$

(3) $a = 1$, $f(a) = f(1) = 1^2 - 1 = 0$

$b = 2$, $f(b) = f(2) = 4 - 1 = 3$

$\because f(a) \neq f(b)$

$\therefore$ RT is not applicable

ii] $f(x) = [x]$; $x \in [5, 9]$

Gn: $f(x) = [x]$ is not a contn funn, here it is discontinous at the integral pt.s 5, 6, 7, 8, 9

Hence, RT is not applicable.

** iii] $f(x) = [x]$, $x \in [-2, 2]$

Gn : $f(x) = [x]$ is not a contn funn, here it is discontinous at the integral pt.s $-2, -1, 0, 1, 2$

Hence, RT is not applicable.

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Eg 42] Verify RT for $y = x^2+2$, $a=-2$, $b=2$

Gn: $f(x) = x^2+2 \quad x \in [-2,2]$

(1) $f(x)$ being polynomial is contn in $[-2,2]$

(2) $f'(x) = 2x \Rightarrow f(x)$ is differentiable in $(-2,2)$

(3) $a=-2$, $f(a) = f(-2) = (-2)^2+2 = 4+2 = 6$

$b=2$, $f(b) = f(2) = 2^2+2 = 4+2 = 6$

$\because f(a) = f(b)$

$\because$ all 3 conditions of RT are satisfied, there exists atleast one pt c such that

$f'(c) = 0$

$2c = 0$

$c = 0, \in (-2,2)$

Hence, RT verified.

Eg 43] Verify MVT for $f(x) = x^2$, $[2,4]$

Gn: $f(x) = x^2$

(1) $f(x)$ being polynomial is contn in $[2,4]$

(2) $f'(x) = 2x \Rightarrow f(x)$ is differentiable in $(2,4)$

$\because$ 2 conditions of MVT are satisfied, there exists atleast one pt c such that

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$f(a) = f(2) = 4$

$f(b) = f(4) = 16$

$b-a = 4-2 = 2$

$2c = \frac{16-4}{2}$

$2c = \frac{12}{2} = 6$

$c = 3 \in (2,4)$

Hence, MVT verified.

4] Verify MVT if $f(x) = x^2-4x-3$ $x \in [a,b]$ where $a=1, b=4$.

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Gn : $f(x) = x^2 - 4x - 3 \quad x \in [1, 4]$

(1) $f(x)$ being polynomial is contn in $[1, 4]$

(2) $f'(x) = 2x - 4 \Rightarrow f(x)$ is differentiable in $(1, 4)$

$\because$ 2 conditions of MVT are satisfied, there exists atleast one pt. c such that

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$f(a) = f(1) = 1 - 4 - 3 = -6$

$f(b) = f(4) = 16 - 16 - 3 = -3$

$b - a = 4 - 1 = 3$

$2c - 4 = \dfrac{-3 - (-6)}{3}$

$2c - 4 = 1$

$2c = 5$

$\frac{5}{2} = c = 2.5 \in (1, 4)$

$\therefore$ MVT verified.

**5] Verify MVT if $f(x) = x^3 - 5x^2 - 3x$, $x \in [a, b]$ where $a = 1$, $b = 3$. find all $c \in (1, 3)$ for all $f'(c) = 0$.

Gn : $f(x) = x^3 - 5x^2 - 3x$

(a) (1) $f(x)$ being polynomial is contn in $[1, 3]$.

(2) $f'(x) = 3x^2 - 10x - 3 \Rightarrow f(x)$ is differentiable in $(1, 3)$

$\because$ 2 conditions of MVT are satisfied, there exists atleast one pt c such that

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$f(a) = f(1) = 1 - 5 - 3 = -7$

$f(b) = f(3) = 27 - 45 - 9 = -27$

$b - a = 3 - 1 = 2$

$3c^2 - 10c - 3 = \dfrac{-27 + 7}{2}$

~~$3c^2 - 10c = -10 + 3$~~

~~$3c^2 - 10c - 13 = 0$~~

$3c^2 - 10c - 3 = -10$

$3c^2 - 10c + 7 = 0$

$3c^2 - 3c - 7c + 7 = 0$

$3c(c - 1) - 7(c - 1) = 0$

$(3c - 7), (c - 1)$

$c = \frac{7}{3} = 2.33 \in (1, 3)$

$c = 1 \notin (1, 3)$

$\therefore c = 2.33 \in (1, 3)$

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(b) $f'(c) = 0$

$3c^2 - 10c - 3 = 0$

$c = \frac{10 \pm \sqrt{100+36}}{6}$

$c = 3.6, -0.27$

$c_1 = 3.6 \notin (1,3)$

$c_2 = -0.27 \notin (1,3)$

$\therefore$ there exists no value of $c \in (1,3)$ when $f'(c) = 0$

(b) Find k if $f(x) = \begin{cases} \frac{1-\cos 2x}{1-\cos x} & ; x \neq 0 \\ k & ; x = 0 \end{cases}$ is continuous @ $x=0$

Gn: funⁿ is contⁿ at $x=0$

$\Rightarrow \lim_{x\to 0} f(x) = f(0)$

$\lim_{x\to 0} \frac{1-\cos 2x}{1-\cos x} = k$

$\lim_{x\to 0} \frac{2\sin^2 x}{2\sin^2\frac{x}{2}} = k$

$\lim_{x\to 0} \left[\frac{\left(\frac{\sin^2 x}{x^2}\right)\cdot x^2}{\left(\frac{\sin^2(\frac{x}{2})}{\frac{x^2}{4}}\right)\cdot\frac{x^2}{4}}\right] = k$

$\frac{1}{1}\cdot 4 = k$

$\boxed{k=4}$

10] (a) PT $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; \text{ if } f(2a-x) = f(x) \\ 0 & ; \text{ if } f(2a-x) = -f(x) \end{cases}$

and hence evaluate $\int_0^{2\pi} \cos^5 x\,dx$

In LHS : In 2ⁿᵈ integral : $x = 2a-t$, $dx = -dt$ | $x=a, t=a$; $x=2a, t=0$

$I = \int_0^a f(x)\,dx + \int_a^0 f(2a-t)(-)\,dt$

$= \int_0^a f(x)\,dx - \int_a^0 f(2a-t)\,dt = \int_0^a f(x)\,dx + \int_0^a f(2a-t)\,dt$

$\therefore \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

case (i): $f(2a-x) = f(x)$

$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$

case (ii): $f(2a-x) = -f(x)$

$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0$

$f(2\pi-x) = \cos^5(2\pi-x) = \cos^5 x = f(x)$

$\Rightarrow I = 2\int_0^{\pi} \cos^5 x\,dx$

(b) PT $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

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LHS $= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$ $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$

$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$