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2005 – MA

Test Paper Code : MA

Time : 3 Hours **Max. Marks : 300**

INSTRUCTIONS

1. The question-cum-answer book has **44** pages and has **32** questions. Please ensure that the copy of the question-cum-answer book you have received contains all the questions.
2. Write your **Roll Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on page No. **7**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used :
   (a) For each objective question, you will be awarded **6 (six)** marks if you have written only the correct answer.
   (b) In case you have not written any answer for a question you will be awarded **0 (zero)** mark for that question.
   (c) In all other cases, you will be awarded **–2 (minus two)** marks for the question.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing later only will be evaluated.
7. All answers must be written in blue/ black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Logarithmic Tables / Calculator of any kind / cellular phone / pager / electronic gadgets are not allowed.**
11. The question-cum-answer book must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this book.
12. Refer to special instruction/useful data on the reverse.

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READ THE INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

ROLL NUMBER					

Name :

Test Centre :

Do not write your Roll Number or Name anywhere else in this question-cum-answer book.

I have read all the instructions and shall abide by them.

....

Signature of the Candidate

I have verified the information filled by the Candidate above.

....

Signature of the Invigilator

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<u>IMPORTANT NOTE FOR CANDIDATES</u>

Objective Part :

Attempt ALL the objective questions (Questions 1 – 15). Each of these questions carries <u>six</u> marks. Write the answers to the objective questions ONLY in the *Answer Table for Objective Questions* provided on page 7.

Subjective Part :

Attempt ALL questions in the *Core Section* (Questions 16 – 22). Questions 16 – 21 carry *<u>twenty one</u>* marks each and Question 22 carries *<u>twelve</u>* marks. There are Five Optional Sections (A, B, C, D and E). Each Optional Section has two questions, each of the questions carries *<u>eighteen</u>* marks. Attempt both questions from *<u>any two</u>* Optional Sections. Thus in the Subjective Part, attempt a total of 11 questions.

1. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers such that $b_n = a_{2n}$ and $c_n = a_{2n+1}$. Then $\{a_n\}$ is convergent

   (A) implies $\{b_n\}$ is convergent but $\{c_n\}$ need not be convergent

   (B) implies $\{c_n\}$ is convergent but $\{b_n\}$ neet not be convergent

   (C) implies both $\{b_n\}$ and $\{c_n\}$ are convergent

   (D) if both $\{b_n\}$ and $\{c_n\}$ are convergent

2. An integrating factor of $x\dfrac{dy}{dx} + (3x+1)y = xe^{-2x}$ is

   (A) $x\,e^{3x}$

   (B) $3x\,e^{x}$

   (C) $x\,e^{x}$

   (D) x^3e^{x}

3. The general solution of $x^2 \frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 9y = 0$ is

(A) $(c_1 + c_2 x)e^{3x}$

(B) $(c_1 + c_2 \ln x)x^3$

(C) $(c_1 + c_2 x)x^3$

(D) $(c_1 + c_2 \ln x)e^{x^3}$

(Here c_1 and c_2 are arbitrary constants.)

4. Let $\vec{r} = x\,\hat{\mathrm{i}} + y\,\hat{\mathrm{j}} + z\,\hat{\mathrm{k}}$. If $\phi(x,y,z)$ is a solution of the Laplace equation then the vector field $\left(\vec{\nabla}\phi + \vec{r}\right)$ is

(A) neither solenoidal nor irrotational

(B) solenoidal but not irrotational

(C) both solenoidal and irrotational

(D) irrotational but not solenoidal

5. Let $\vec{F} = x\,\hat{\mathrm{i}} + 2y\,\hat{\mathrm{j}} + 3z\,\hat{\mathrm{k}}$, S be the surface of the sphere $x^2 + y^2 + z^2 = 1$ and $\hat{\mathrm{n}}$ be the inward unit normal vector to S. Then $\iint_S \vec{F}\cdot\hat{\mathrm{n}}\,dS$ is equal to

(A) 4π

(B) -4π

(C) 8π

(D) -8π

6. Let A be a 3×3 matrix with eigenvalues $1, -1$ and 3. Then

(A) $A^2 + A$ is non-singular

(B) $A^2 - A$ is non-singular

(C) $A^2 + 3A$ is non-singular

(D) $A^2 - 3A$ is non-singular

7. Let $T : \boldsymbol{R}^3 \longrightarrow \boldsymbol{R}^3$ be a linear transformation and I be the identity transformation of $\boldsymbol{R}^3$. If there is a scalar c and a non-zero vector $x \in \boldsymbol{R}^3$ such that $T(x) = cx$, then $\text{rank}(T - cI)$

(A) cannot be 0

(B) cannot be 1

(C) cannot be 2

(D) cannot be 3

8. In the group $\{1, 2, \ldots, 16\}$ under the operation of multiplication modulo 17, the order of the element 3 is

(A) 4

(B) 8

(C) 12

(D) 16

9. A ring R has maximal ideals

(A) if R is infinite

(B) if R is finite

(C) if R is finite with at least 2 elements

(D) only if R is finite

10. The integral $\int_0^1 \left[\int_0^{1-z} \left(\int_0^2 dx\right) dy\right] dz$ is equal to

(A) $\int_0^1 \left[\int_0^{1-y} \left(\int_0^2 dx\right) dz\right] dy$

(B) $\int_0^1 \left[\int_0^{1-y} \left(\int_0^1 dx\right) dz\right] dy$

(C) $\int_0^2 \left[\int_0^2 \left(\int_0^{1-z} dx\right) dz\right] dy$

(D) $\int_0^2 \left[\int_0^2 \left(\int_0^{1-y} dx\right) dz\right] dy$

11. Let $f: \boldsymbol{R} \longrightarrow \boldsymbol{R}$ be continuous and $g, h: \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}$ be differentiable. Let $F(u,v) = \int_v^u f(t)dt$, where $u = g(x,y)$ and $v = h(x,y)$. Then $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} =$

(A) $f(g(x,y))\left[\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\right] - f(h(x,y))\left[\frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\right]$

(B) $f(h(x,y))\left[\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\right] - f(g(x,y))\left[\frac{\partial h}{\partial x} - \frac{\partial h}{\partial y}\right]$

(C) $f(h(x,y))\left[\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\right] + f(g(x,y))\left[\frac{\partial h}{\partial x} - \frac{\partial h}{\partial y}\right]$

(D) $f(g(x,y))\left[\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}\right] + f(g(x,y))\left[\frac{\partial h}{\partial x} - \frac{\partial h}{\partial y}\right]$

12. Let $y = f(x)$ be a smooth curve such that $0 < f(x) < K$ for all $x \in [a,b]$. Let

L = length of the curve between $x = a$ and $x = b$

A = area bounded by the curve, x-axis, and the lines $x = a$ and $x = b$

S = area of the surface generated by revolving the curve about x-axis between $x = a$ and $x = b$

Then

(A) $2\pi KL < S < 2\pi A$

(B) $S \leq 2\pi A < 2\pi KL$

(C) $2\pi A \leq S < 2\pi KL$

(D) $2\pi A < 2\pi KL < S$

13. Let $f : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ be defined by $f(t) = t^2$ and let U be any non-empty open subset of $\boldsymbol{R}$. Then

(A) $f(U)$ is open

(B) $f^{-1}(U)$ is open

(C) $f(U)$ is closed

(D) $f^{-1}(U)$ is closed

14. Let $f:(-1,1) \longrightarrow \boldsymbol{R}$ be such that $f^{(n)}(x)$ exists and $\left|f^{(n)}(x)\right| \leq 1$ for every $n \geq 1$ and for every $x \in (-1,1)$. Then f has a convergent power series expansion in a neighbourhood of

(A) every $x \in (-1,1)$

(B) every $x \in \left(\frac{-1}{2}, 0\right)$ only

(C) no $x \in (-1,1)$

(D) every $x \in \left(0, \frac{1}{2}\right)$ only

15. Let $a > 1$ and $f, g, h : [-a, a] \longrightarrow \boldsymbol{R}$ be twice differentiable functions such that for some c with $0 < c < 1 < a$,

$f(x) = 0$ only for $x = -a, 0, a$;

$f'(x) = 0 = g(x)$ only $x = -1, 0, 1$;

$g'(x) = 0 = h(x)$ only for $x = -c, c$.

The possible relations between f, g, h are

(A) $f = g'$ and $h = f'$

(B) $f' = g$ and $g' = h$

(C) $f = -g'$ and $h' = g$

(D) $f = -g'$ and $h' = f$

A

CORE SECTION

16. (a) Solve the initial value problem

$$\frac{d^2y}{dx^2} - y = x\left(\sin x + e^x\right), \qquad y(0) = y'(0) = 1 \qquad (12)$$

(b) Solve the differential equation

$$(2y\sin x + 3y^4\sin x \cos x)dx - (4y^3\cos^2 x + \cos x)dy = 0 \qquad (9)$$

A

17. Let G be a finite abelian group of order n with identity e. If for all $a \in G$, $a^3 = e$, then, by induction on n, show that $n = 3^k$ for some nonnegative integer k. (21)

A

18. (a) Let $f:[a,b] \longrightarrow \boldsymbol{R}$ be a differentiable function. Show that there exist points $c_1, c_2 \in (a,b)$ such that

$$2f(c_1)f'(c_1) = f'(c_2)\,[f(a)+f(b)]$$ (9)

(b) Let

$$f(x,y) = \begin{cases} (x^2+y^2)\left[\ln(x^2+y^2)+1\right] & \text{for } (x,y) \neq (0,0) \\ \alpha & \text{for } (x,y) = (0,0) \end{cases}$$

Find a suitable value for α such that f is continuous. For this value of α, is f differentiable at $(0,0)$? Justify your claim. (12)

A

19. (a) Let S be the surface $x^2 + y^2 + z^2 = 1,\ z \geq 0$. Use Stoke's theorem to evaluate

$$\int_C [(2x - y)\,dx - y\,dy - z\,dz]$$

where C is the circle $x^2 + y^2 = 1,\ z = 0$, oriented anticlockwise. (12)

(b) Show that the vector field $\vec{F} = \left(2xy - y^4 + 3\right)\hat{\imath} + \left(x^2 - 4xy^3\right)\hat{\jmath}$ is conservative. Find its potential and also the work done in moving a particle from $(1,0)$ to $(2,1)$ along some curve. (9)

A

20. Let $T : \boldsymbol{R}^3 \longrightarrow \boldsymbol{R}^3$ be defined by $T(x, y, z) = (y + z, z, 0)$. Show that T is a linear transformation. If $v \in \boldsymbol{R}^3$ is such that $T^2(v) \neq 0$, then show that $B = \{v, T(v), T^2(v)\}$ forms a basis of $\boldsymbol{R}^3$. Compute the matrix of T with respect to B. Also find a $v \in \boldsymbol{R}^3$ such that $T^2(v) \neq 0$. (21)

A

21. (a) For each $n \in \mathbf{N}$, define $f_n : [-1, 1] \longrightarrow \mathbf{R}$ by

$$f_n(x) = \begin{cases} 4n^2 x & \text{for } x \in \left[0, \frac{1}{2n}\right) \\ -4n^2\left(x - \frac{1}{n}\right) & \text{for } x \in \left[\frac{1}{2n}, \frac{1}{n}\right) \\ 0 & \text{for } x \in \left[\frac{1}{n}, 1\right] \end{cases}$$

Compute $\int_0^1 f_n(x)\,dx$ for each n. Analyse pointwise and uniform convergence of the sequence of functions $\{f_n\}$. (12)

(b) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a continuous function with $|f(x) - f(y)| \geq |x - y|$ for every $x, y \in \mathbf{R}$. Is f one-one? Show that there cannot exist three points $a, b, c \in \mathbf{R}$ with $a < b < c$ such that $f(a) < f(c) < f(b)$. (9)

A

22. Find the volume of the cylinder with base as the disk of unit radius in the xy-plane centred at (1, 1, 0) and the top being the surface $z = \left[(x-1)^2 + (y-1)^2\right]^{3/2}$. (12)

A

OPTIONAL SECTION : A

23. (a) Bag A contains 3 white and 4 red balls, and bag B contains 6 white and 3 red balls. A biased coin, twice as likely to come up heads as tails, is tossed once. If it shows head, a ball is drawn from bag A, otherwise, from bag B. Given that a white ball was drawn, what is the probability that the coin came up tail? (9)

(b) Let the random variables X and Y have the joint probability density function $f(x,y)$ given by

$$f(x,y)=\begin{cases} y^2e^{-y(x+1)} & x \geq 0,\ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Are the random variables X and Y independent? Justify your answer. (9)

A

24. (a) Let $X_1, X_2, \ldots, X_n$ be independently identically distributed random variables (rv's) with common probability density function (pdf) $f_X(x,\theta) = \frac{1}{\theta} e^{-x/\theta}$; $x > 0$, $\theta > 0$. Obtain the moment generating function (mgf) of $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. Also find the mgf of the rv $Y = 2n\overline{X}/\theta$. (9)

(b) Let $X_1, X_2, \ldots, X_9$ be an independent random sample from $N(2,4)$ and Y_1, Y_2, Y_3, Y_4 be an independent random sample from $N(1,1)$. Find $P(\overline{X} > \overline{Y})$, where $\overline{X}$ and $\overline{Y}$ are sample means.

[Given $P(Z > 1.2) = 0.1151$, where $Z \sim N(0,1)$] (9)

OPTIONAL SECTION : B

25. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution having pdf

$$f(x; x_0, \alpha) = \begin{cases} \dfrac{\alpha x_0^{\alpha}}{x^{\alpha+1}} & \text{for } x > x_0 \\ 0 & \text{otherwise} \end{cases}$$

where $x_0 > 0$, $\alpha > 0$. Find the maximum likelihood estimator of α if x_0 is known. (9)

(b) Let $X_1, X_2, \ldots, X_5$ be a random sample from the standard normal population. Determine the constant c such that the random variable

$$Y = \frac{c(X_1 + X_2)}{\sqrt{X_3^2 + X_4^2 + X_5^2}}$$

will have a t-distribution. (9)

A

26. (a) A random sample of size $n = 1$ is drawn from pdf $f_X(x,\theta) = \frac{1}{\theta} e^{-x/\theta}$; $x > 0$, $\theta > 0$. It is decided to test $H_0 : \theta = 5$ against $H_1 : \theta = 7$ based on the criterion: reject H_0 if the observed value is greater than 10. Obtain the probabilities of type I and type II errors. (9)

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population $N(\mu, \sigma^2)$. Find the best test for testing $H_0 : \mu = 0, \sigma^2 = 1$ against $H_1 : \mu = 1, \sigma^2 = 4$. (9)

A

OPTIONAL SECTION : C

27. (a) Let $f, g : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ be such that for $x, y \in \boldsymbol{R}$,

$$\phi(x+iy) = e^{x}[f(y)+ig(y)]$$

is an analytic function. Find a differential equation of order 2 satisfied by f. (9)

(b) Compute $\int_{|z+1|=2} (2z+1)e^{\left(\sqrt{2}\, +\, 1/z\right)}\, dz.$ (9)

A

28. (a) Let $f(z)$ be analytic in the whole complex plane such that for all $r > 0$,

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right| d\theta \le \sqrt{r}$$

Find $\dfrac{f^{(n)}(0)}{n!}$ for all $n \ge 0$. (9)

(b) Find all values of $\alpha \in \boldsymbol{C}$ such that $f(z) = (z + \overline{z})^2 + 2\alpha |z|^2 + \alpha(\overline{z})^2$ is analytic at some point z having non-zero real part. (9)

OPTIONAL SECTION : D

29. A hemispherical bowl of radius 12 cm is fixed such that its rim is horizontal. A light rod of length 20 cm with weights w and W attached to its two ends is placed inside the bowl. In equilibrium, the weight w is just touching the rim of the bowl. Find the ratio $w : W$. (18)

A

30. A uniform ladder of length $2a$ and mass m lies in a vertical plane with one end against a smooth wall, the other end being supported on a horizontal floor. The ladder is released from rest when inclined at an angle α to the horizontal. Find the inclination of the ladder to the horizontal when it ceases to touch the wall. (18)

A

OPTIONAL SECTION : E

31. (a) Estimate the error in evaluating the integral $\int_0^8 (1+x^2)e^{-x}dx$ by Simpson's $\frac{1}{3}$rd rule with spacing $h = 0.25$. (9)

(b) Using Newton-Raphson method, compute the point of intersection of the curves $y = x^3$ and $y = 8x + 4$ near the point $x = 3$, correct up to 2 decimal places.

[Round-off the first iteration up to 2 decimal places for further computation] (9)

A

32. The polynomial $p_3(x) = x^3 + x^2 - 2$ interpolates the function $f(x)$ at the points $x = -1, 0, 1$ and 2. If the data $f(3) = -14$ is added, find the new interpolating polynomial by using Newton's forward difference formula. Also find $f(2.5)$ by using Newton's backward difference formula with pivot value 3. Justify whether the value obtained will be the same if pivot value 2 is taken. (18)