27/3/17

(1 + 2 + 5 + 4 = 12 m)

DETERMINANTS

Determinant of a 3rd order matrix :

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

along R_1 :

$\Delta A = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

Minor of $b_1 = \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$

Cofactor of $b_1 = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$

ΔA = along $C_3 = a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

Eg: $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

$\Delta_1 = 1(45-48) - 2(36-42) + 3(32-35)$
$= -3 + 12 - 9 = 0$

$\Delta_2 = 3(32-35) - 6(8-14) + 9(5-8)$
$= -9 + 36 - 27 = 0$

EXERCISE 4.1

1] $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = -2 + 20 = 18$

2] (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$

(ii) $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix} = (x^2-x+1)(x+1) - (x+1)(x-1)$

$= x^3 - x^2 + x + x^2 - x + 1 - x^2 + 1$

$= x^3 - x^2 + 2$

5] (i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

$\Delta = 3(0-5) + 1(0+3) - 2(0-0)$

$= -15 + 3 - 0$

$= -12$

(ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$ column : $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\Delta = 3(1+6) + 4(1+4) + 5(3-2)$

$= 21 + 20 + 5$

$= 46$

$\Delta_1 = \begin{vmatrix} 1 & -4 & 5 \\ 2 & 1 & -2 \\ 3 & 3 & 1 \end{vmatrix}$

$\Delta = 1(1+6) + 4(2+6) + 5(6-3)$

$= 7 + 32 + 15$

$= 54$

$\Delta_2 = \begin{vmatrix} 3 & 1 & 5 \\ 1 & 2 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 3(2+6) - 1(1+4) + 5(3-4)$

$= 24 - 5 - 5$

$= 14$

$\Delta_3 = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{vmatrix} = 3(3-6) + 4(3-4) + 1(3-2)$

$= -9 - 4 + 1$

$= -12$

(ii) $\Delta = \begin{vmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 0 & -4 \end{vmatrix} \quad \begin{array}{c} d \\ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{array}$

$\Delta = 0(8-0) - 1(-4-3) + 2(0+2)$
$= 0+7+4$
$= 11$

$\Delta_1 = \begin{vmatrix} -1 & 1 & 2 \\ 0 & -2 & 3 \\ 1 & 0 & -4 \end{vmatrix} = -1(8-0) - 1(0-3) + 2(0+2)$
$= -8+3+4$
$= -1$

$\Delta_2 = \begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & -4 \end{vmatrix} = 0(0-3) + 1(-4-3) + 2(1-0)$
$= 0-7+2$
$= -5$

$\Delta_3 = \begin{vmatrix} 0 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0(-2-0) - 1(1-0) - 1(0+2)$
$= 0-1-2$
$= -3$

(iii) $\Delta = \begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} = 0(0-6) - 1(0-6) + 2(-3+0)$
$= 0+6-6$
$= 0$

$\Delta = \begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 2(0-5) + 1(0+3) - 2(0-6)$
$= -10+3+12$
$= 5$

** 3] If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ Show that $|2A| = 4|A|$

LHS: $|2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 8-32 = -24$

RHS: $4|A| = 4(2-8) = 4(-6) = 24$

LHS = RHS

4] If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ show that $|3A| = 27|A|$

LHS : $|3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix} = 3(36-0) - 0(0-0) + 3(0-0)$

$= 108$

RHS : $27|A| = 27[1(4-0) - 0(0-0) + 1(0-0)]$

$= 27(4)$

$= 108$

LHS = RHS.

$\boxed{|kA|_{n\times n} = k^n|A|}$

Eg: $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ (i) find $|2A|$

$|2A| = 2^2|A| = 4|A| = 4(2-8) = -24$

(ii) find $|3A| = 3^2|A| = 9(-6) = -54$

Eg: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$ find $|4A|$

$|4A| = 4^3|A| = 64[1(4)] = 256$

Eg: If A is a matrix of order 4 & $|A| = -3$. Find $|2A|$.

Sol: Given : order = 4, $|A| = -3$

$|2A| = 2^4|A| = 16(-3) = -48$

Eg: If A is a matrix of order 3 & $|A| = 4$, Find $|5A|$.

Sol: $|5A| = 5^3(4) = 125(4) = 500$

Solving equations by matrix method:

1] matrix of cofactors: is the matrix which gives the cofactors of every element of any square matrix A.

2] Minor of a_{ij}: minor of an element a_{ij} is obtained by taking the determinant after deleting i^{th} row and j^{th} column and is denoted as M_{ij}.

3] Cofactor of a_{ij}: cofactor of an element a_{ij} is defined as $(-1)^{i+j} \times M_{ij}$ and is denoted as A_{ij}.

$\therefore \; A_{ij} = (-1)^{i+j} M_{ij}$

Sign Convention in a matrix:

Every element in a matrix has a sign given by $(-1)^{i+j}$ if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{bmatrix}$$

$$\therefore \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Adjoint of a matrix A = $\boxed{\text{adj}A = \text{transpose of matrix of cofactors}}$.

i.e, $\boxed{\text{adj}A = C_A'}$

Eg: $A = \begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$

$$C_A = \begin{bmatrix} \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} & -\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} & \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} \\ -\begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -20 & 46 & 30 \\ 4 & -19 & -13 \\ -12 & 22 & 18 \end{bmatrix}$$

$|A| = -40 - 3(46) + 150 = -28$

$$\text{adj } A = C_A' = \begin{bmatrix} -20 & 4 & -12 \\ 46 & -19 & 22 \\ 30 & -13 & 18 \end{bmatrix}$$

EXERCISE 4.5

Find adj of the following:

1] $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad \text{Adj } A = C_A' = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

2] $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

$$C_A = \begin{bmatrix} \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix}$$

$|A| = 3 + 12 + 12 = 27$; $\text{adj } A = C_A' = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$

Verify $A\cdot(\text{adj}\,A) = (\text{adj}\,A)\cdot A = |A|\,I$

3] $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ $\quad |A| = -12+12 = 0$

$\text{adj}\,A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$

$LHS = A(\text{adj}\,A) = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$RHS = |A|\,I = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$LHS = RHS$

4] $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

$C_A = \begin{bmatrix} \begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix}$

$|A| = 0 + 11 + 0 = 11$; $\text{adj}\,A = C_A' = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$

$LHS = A(\text{adj}\,A) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$

$= 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 11\,I = |A|\,I = RHS$

Find inverse of following:

5] $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ $A^{-1} = \frac{1}{|A|}(\text{adj}A) = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

6] $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ $A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

12] $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$

LHS: $AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{|AB|}\text{adj}AB = \frac{1}{-2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \rightarrow (1)$

$A^{-1} = \frac{1}{|A|}\text{adj}A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|}\text{adj}B = \frac{1}{-2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{-2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \rightarrow (2)$

From (1) & (2) $(AB)^{-1} = B^{-1}A^{-1}$

13] If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ Show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$5A = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$

$7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$\therefore A^2 - 5A + 7I$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = RHS$

Hence,

$A^2 - 5A + 7I = 0$ post multiply by A^{-1}

$A^2A^{-1} - 5AA^{-1} + 7IA^{-1} = 0A^{-1}$

$A(AA^{-1}) - 5(AA^{-1}) + 7A^{-1} = 0$

$A - 5I + 7A^{-1} = 0$

$7A^{-1} = 5I - A$

$A^{-1} = \frac{1}{7}(5I - A)$

$= \frac{1}{7}\left[\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\right]$

$= \frac{1}{7}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$

Eg 26] $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ Show that $A^2 - 4A + I = 0$, Find A^{-1}

$A^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

$4A = \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix}$

$\therefore A^2 - 4A + I$

$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = RHS$

Hence,

$A^2 - 4A + I = 0$ post multiply by A^{-1}

$A^2A^{-1} - 4AA^{-1} + IA^{-1} = 0A^{-1}$

$A(AA^{-1}) - 4(AA^{-1}) + A^{-1} = 0$

$A - 4I + A^{-1} = 0$

$A^{-1} = 4I - A$

$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

15] $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$. Show that $A^3 - 6A^2 + 5A + 11I = O$. Hence, find A^{-1}.

Sol] $A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$

$A^3 = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$

$6A^2 = 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} = \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$

$5A = 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix}$

$11I = 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$

$\therefore A^3 - 6A^2 + 5A + 11I = O$

$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$

Hence, $A^3 - 6A^2 + 5A + 11I = O$; post multiply by A^{-1}

$A^3A^{-1} - 6A^2A^{-1} + 5AA^{-1} + 11IA^{-1} = OA^{-1}$

$A^2 - 6A + 5I + 11A^{-1} = O$

$11A^{-1} = 6A - 5I - A^2$

$A^{-1} = \frac{1}{11}\left[6A - 5I - A^2\right]$

$$A^{-1} = \frac{1}{11}\left[\begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\right]$$

$$A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

16] $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Verify that $A^3 - 6A^2 + 9A - 4I = 0$ and hence find A^{-1}

Sol] $$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$6A^2 = 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix}$$

$$9A = 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix}$$

$$4I = 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

LHS = $\therefore A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 = \text{RHS}$$

Hence, $A^3 - 6A^2 + 9A - 4I = O$ post multiply by A^{-1}

$$A^3A^{-1} - 6A^2A^{-1} + 9AA^{-1} - 4IA^{-1} = OA^{-1}$$

$$A^2 - 6A + 9I - 4A^{-1} = O$$

$$4A^{-1} = A^2 - 6A + 9I$$

$$A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9I\right]$$

$$A^{-1} = \frac{1}{4}\left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right]$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Find inverses of the following:

7] $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

$$C_A = \begin{bmatrix} \begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}$$

$|A| = 10 + 0 + 0 = 10$; $\text{adj}A = C_A' = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{adj}A) = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

8] $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

$$C_A = \begin{bmatrix} \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ 5 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$|A| = -3 + 0 + 0 = -3 \; ; \; \text{adj } A = C_A' = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj} A = \frac{1}{-3} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

9] $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

$$C_A = \begin{bmatrix} \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 4 & 0 \\ -7 & 1 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ -7 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -7 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -7 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & -4 & 1 \\ 5 & 23 & -11 \\ 3 & +12 & -6 \end{bmatrix}$$

$$|A| = -2 - 4 + 3 = 5 \; ; \; \text{adj } A = C_A' = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & +12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} (\text{adj} A) = \frac{1}{-3} \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & +12 \\ 1 & -11 & -6 \end{bmatrix}$$

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10] $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

$$C_A = \begin{bmatrix} \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ +2 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$$

$$|A| = 2 + 9 - 12 = -1 \quad ; \quad \text{adj} A = C_A' = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj} A = -1 \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

11] $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

$$C_A = \begin{bmatrix} \begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} & -\begin{vmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{vmatrix} & \begin{vmatrix} 0 & \cos\alpha \\ 0 & \sin\alpha \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ \sin\alpha & -\cos\alpha \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -\cos\alpha \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ \cos\alpha & \sin\alpha \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$|A| = -1 + 0 + 0 = -1 \quad ; \quad \text{adj} A = C_A' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj} A) = -1 \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

EXERCISE 4.6

Solve system of linear equations using matrix method

7] $5x+2y=4 \qquad 7x+3y=5$

$AX = B \rightarrow ①$

$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$① \Rightarrow X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12-10 \\ -28+25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$x = 2 \ ; \ y = -3$

8] $2x - y = -2 \qquad 3x + 4y = 3$

$AX = B \rightarrow ①$

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$① \Rightarrow X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} -\frac{5}{11} \\ \frac{12}{11} \end{bmatrix}$$

$x = -\frac{5}{11}, \ y = \frac{12}{11}$

9] $4x - 3y = 3$, $3x - 5y = 7$

$AX = B \rightarrow (1)$

$$\begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj} A = \frac{1}{-11}\begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}$$

$(1) \Rightarrow X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{11}\begin{bmatrix} 6 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-6}{11} \\ \frac{-19}{11} \end{bmatrix}$$

$x = \frac{-6}{11}$, $y = \frac{-19}{11}$

10] $5x + 2y = 3$, $3x + 2y = 5$

$AX = B \longrightarrow (1)$

$$\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj} A) = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$(1) \Rightarrow X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6 - 10 \\ -9 + 25 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$x = -1$, $y = 4$

Eg 28] $3x - 2y + 3z = 8$

$2x + y - z = 1$

$4x - 3y + 2z = 4$

Sol] $AX = B \longrightarrow (1)$

$$\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$C_A = \begin{bmatrix} \begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} \\ -\begin{vmatrix} -2 & 3 \\ -3 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 3 \\ 4 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} \\ \begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}$$

$|A| = -3 + 16 - 30 = -17$; $\operatorname{adj} A = C_A' = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{-17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$(1) \Rightarrow X = A^{-1}B = \frac{-1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{17} \begin{bmatrix} -8-5-4 \\ -64-6+36 \\ -80+1+28 \end{bmatrix} = \frac{-1}{17} \begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$x = 1, \; y = 2, \; z = 3$

Eg 29] Let the no.s be x, y, z

$$x + y + z = 6$$
$$y + 3z = 11$$
$$x + z = 2y \quad \text{(or)} \quad x - 2y + z = 0$$

$$A \quad X \quad = B \rightarrow ①$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$C_A = \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 7 & 3 & -1 \\ -3 & 0 & 3 \\ 2 & -3 & 1 \end{bmatrix}$$

$$|A| = 7 + 3 - 1 = 9 \; ; \; adj A = C_A' = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj A = \frac{1}{9} \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

$$① \Rightarrow X = A^{-1} B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 42 - 33 + 0 \\ 18 + 0 + 0 \\ -6 + 33 + 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x = 1, \; y = 2, \; z = 3$$

16] The cost of 4kg onion, 3kg wheat and 2kg rice is Rs 60. The cost of 2kg onion, 4kg wheat and 6kg rice is Rs 90. The cost of 6kg onion, 2kg wheat and 3kg rice is Rs 70. Find cost of each item per kg by matrix method.

Sol] Let onion, wheat, rice be x, y, z respectively.

According to data,
$$4x + 3y + 2z = 60$$
$$2x + 4y + 6z = 90$$
$$6x + 2y + 3z = 70$$

$$AX = B \rightarrow ①$$

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$C_A = \begin{bmatrix} \begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 6 \\ 6 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 6 & 2 \end{vmatrix} \\ -\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 6 & 2 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 4 & 6 \end{vmatrix} & -\begin{vmatrix} 4 & 2 \\ 2 & 6 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ 2 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}$$

$$|A| = 0 + 90 - 40 = 50$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$① \Rightarrow X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0 - 450 + 700 \\ 1800 - 0 - 1400 \\ -1200 + 900 + 700 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

Cost of 1kg of onion = ₹ 5
Cost of 1kg of wheat = ₹ 8
Cost of 1kg of rice = ₹ 8

15] $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations:

$2x - 3y + 5z = 11,\ 3x + 2y - 4z = -5,\ x + y - 2z = -3$

Sol] $C_A = \begin{bmatrix} \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}$

$|A| = 0 - 6 + 5 = -1$; $\operatorname{adj} A = C_A' = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(\operatorname{adj} A) = -1 \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$

~~Givenen~~ Given equations: $2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

$AX = B \rightarrow ①$

$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

$① \Rightarrow X = A^{-1}B$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$x = 1,\ y = 2,\ z = 3$

11] $2x + y + z = 1$

$x - 2y - z = \frac{3}{2}$

$3y - 5z = 9$

Sol] $AX = B \longrightarrow (1)$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$$C_A = \begin{bmatrix} \begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & -5 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{bmatrix}$$

$$|A| = 26 + 5 + 3 = 34 \; ; \; \text{adj} A = C_A' = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj} A = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$(1) \Rightarrow X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{-3}{2} \end{bmatrix}$$

$x = 1, \; y = \frac{1}{2}, \; z = \frac{-3}{2}$

12] $x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$

Sol] $AX = B \longrightarrow (1)$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$C_A = \begin{bmatrix} \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

$$|A| = 4 + 5 + 1 = 10 \; ; \; \text{adj} A = C_A' = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj} A) = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$(1) \Rightarrow X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16 + 0 + 4 \\ -20 + 0 + 10 \\ 4 + 0 + 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$x = 2, \; y = -1, \; z = 1$

13] $2x + 3y + 3z = 5$

$x - 2y + z = -4$

$3x - y - 2z = 3$

Sol] $AX = B \longrightarrow (1)$

$$\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$C_A = \begin{bmatrix} \begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 3 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}$$

$$|A| = 10 + 15 + 15 = 40 \; ; \; \text{adj}\,A = C_A' = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$(1) \Rightarrow X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$x = 1, \; y = 2, \; z = -1$

14] $x - y + 2z = 7$
$3x + 4y - 5z = -5$
$2x - y + 3z = 12$

Sol] $AX = B \rightarrow (1)$

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$C_A = \begin{bmatrix} \begin{vmatrix} 4 & -5 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & -5 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 4 & -5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}$$

$$|A| = 7 + 19 - 22 = 4 \; ; \; adj A = C_A' = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj A) = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$(1) \Rightarrow X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49 - 5 - 36 \\ -133 + 5 + 132 \\ -77 + 5 + 84 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$x = 2, \; y = 1, \; z = 3$

13] Solve by matrix method:

$2x + 3y + 3z = 5$

$x - 2y + z = -4$

$3x - y - 2z = 3$

$AX = B \longrightarrow (1)$

$$\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$C_A = \begin{array}{cc|cc|cc} -2 & 1 & 1 & 1 & 1 & -2 \\ -1 & -2 & 3 & -2 & 3 & -1 \\ 3 & 3 & 2 & 3 & 2 & 3 \\ -1 & -2 & 3 & -2 & 3 & -1 \\ 3 & 3 & 2 & 3 & 2 & 3 \\ -2 & 1 & 1 & 1 & 1 & -2 \end{array} = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}$$

$|A| = 10 + 15 + 15 = 40$; $\operatorname{adj} A = C_A' = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$

$$A^{-1} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$x = 1,\ y = 2,\ z = -1$

EXERCISE 4·1

Evaluate the determinants:

1] $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = -2+20 = 18$

2] i] $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$

ii] $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix} = (x+1)(x^2-x+1)-(x^2-1)$

$= x^3+1-x^2+1$

$= x^3-x^2+2$

3] If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ show that $|2A| = 4|A|$

$|2A| = 2^2|A| = 4(2-8) = -24$

4] If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, show that $|3A| = 27|A|$

$|3A| = 3^3|A| = 27[1(4-0)+0+1(0-0)]$

$= 27(4) = 108$

5] Evaluate the determinants:

i] $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 3(0-5)+1(3)-2(0)$

$= -15+3$

$= -12$

ii] $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 3(1+6)+4(1+4)+5(3-2)$

$= 21+20+5$

$= 46$

iii] $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} = 0(0+9) - 1(0-6) + 2(-3+0)$
$= 0 + 6 - 6$
$= 0$

iv] $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 2(0-5) + 1(0+3) - 2(0-6)$
$= -10 + 3 + 12$
$= 5$

6] If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$
$= 1(-9+12) - 1(-18+15) - 2(8-5)$
$= 3 + 3 - 6$
$= 0$

7] Find values of x if:

i] $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$2 - 20 = 2x^2 - 24$
$-18 + 24 = 2x^2$
$6 = 2x^2$
$x = \pm\sqrt{3}$

ii] $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

$10 - 12 = 5x - 6x$
$-2 = -x$
$x = 2$

8] If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$ $x = ?$

(A) 6 (B) ± 6 (C) 6 (D) 0

$x^2 - 36 = 36 - 36$
$x^2 = 36$
$x = \pm 6$

Area of a triangle :

*] • — • — • If 3 points are collinear
area of $\Delta = 0$

*] Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

*] Area of $\Delta = \pm K$ if unknown value exists in vertices

*] Equation of a line if 2 pts are given $A(x_1, y_1)$, $B(x_2, y_2)$.
Step 1 : $P(x, y)$ be any point on the line AB.
Step 2 : Now pts A, B, P are collinear
$\therefore$ area of $\Delta = 0$

EXERCISE 4.3

1] Find the area of Δ with vertices given,

(i) $\underset{A}{\underset{x_1\,y_1}{(1,0)}}, \underset{B}{\underset{x_2\,y_2}{(6,0)}}, \underset{C}{\underset{x_3\,y_3}{(4,3)}}$

Ar of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \frac{1}{2}\left[1(0-3) - 0 + 1(18-0)\right]$

$= \frac{1}{2}(-3+18) = \frac{15}{2}$ square unit

ii] $\overset{A}{(2,7)}, \overset{B}{(1,1)}, \overset{C}{(10,8)}$

Ar of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix} = \frac{1}{2}\left[2(1-8) - 7(1-10) + 1(8-10)\right]$

$= \frac{1}{2}\left[-14 + 63 - 2\right] = \frac{47}{2}$ sq. unit

iii] $\overset{A}{(-2,-3)}, \overset{B}{(3,2)}, \overset{C}{(-1,-8)}$

Are of $\Delta ABC = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix} = \frac{1}{2}\left[-2(2+8) + 3(3+1) + 1(-24+2)\right]$

$= \frac{1}{2}(-20 + 12 - 22)$

$= \frac{1}{2}(-30)$

$= |-15| = 15$ sq. units

$\begin{array}{r} -42 \\ +12 \\ \hline 30 \end{array}$

2] Show that the pts $A(a, b+c)$, $B(b, c+a)$, $C(c, a+b)$ are collinear

Sol] Consider area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$C_2 \to C_2 + C_1$

$= \frac{1}{2}\begin{vmatrix} a & a+b+c & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix}$

$= \frac{1}{2}(a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}$

$\Delta = 0 \; (\because C_2 \sim C_3)$

$\Rightarrow A, B, C$ are collinear.

Consistency of system of equation :

If A is a non singular matrix, system of equations is consistent and has a unique solution.
ie, $|A| \neq 0$, unique solution and consistent.

If $|A| = 0$,
(a) $(\text{adj} A)B \neq 0$ then solution does not exist, system is inconsistent.
(b) $(\text{adj} A)B = 0$ then system may be consistent with infinite solutions or may be inconsistent with no solution.

EXERCISE 4.6

$x + 2y = 2$; $2x + 3y = 3$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$|A| = 3 - 4 = -1 \neq 0$

$\Rightarrow$ system is consistent and has unique solution.

$2x - y = 5$; $x + y = 4$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$|A| = 2 + 2 = 4 \neq 0$

$\Rightarrow$ system is consistent and has unique solution.

3) $x + 3y = 5$; $2x + 6y = 8$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$|A| = 6 - 6 = 0$

$$(\mathrm{adj}\,A)\,B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq 0$$

$\Rightarrow$ system is inconsistent and has no solution.

4) $x + y + z = 1$; $2x + 3y + 2z = 2$; $ax + ay + 2az = 4$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$|A| = 1(6a - 2a) - 1(4a - 2a) + 1(2a - 3a)$

$= 4a - 2a - a = a \neq 0$

$\Rightarrow$ system is consistent and has unique solution.

5) $3x - y - 2z = 2$; $2y - z = -1$; $3x - 5y = 3$

$$\begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$|A| = 3(0 - 5) + 1(0 + 3) - 2(0 - 6)$

$= -15 + 3 + 12 = 0$

$$C_A = \begin{bmatrix} \begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix} \\ -\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} \\ \begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -5 & & \\ 10 & & \\ 5 & & \end{bmatrix}$$

$$(\text{adj } A) = C_A' = \begin{bmatrix} -5 & 10 & 5 \\ & & \\ & & \end{bmatrix}$$

$$\therefore (\text{adj } A)\, B = \underset{3\times 3}{\begin{bmatrix} -5 & 10 & 5 \\ & & \\ & & \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}} = \begin{bmatrix} -5 \\ \\ \end{bmatrix} \neq O$$

$\Rightarrow$ system is inconsistent and has no solution.

6] $5x - y + 4z = 5$; $2x + 3y + 5z = 2$; $5x - 2y + 6z = -1$

$$\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$|A| = 5(18+10) + 1(12-25) + 4(-4-15)$$
$$= 140 - 13 - 76$$
$$= 51 \neq 0$$

$\Rightarrow$ system is consistent and has unique solution.

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DETERMINANTS

EXERCISE 4.3

3] Find the value of k if the area of triangle is 4 sq. units

i] $\overset{A}{(k,0)}, \overset{B}{(4,0)}, \overset{C}{(0,2)}$

Given : $\Delta ABC = 4$

$$\frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 4$$

$k(-2) + 1(8) = \pm 8$

$-2k + 8 = \pm 8$

$8 \mp 8 = 2k$

$2k = 0$ | $2k = 8+8 = 16$

$k = 0$ | $k = 8$

ii] $\overset{A}{(-2,0)}, \overset{B}{(0,4)}, \overset{C}{(0,k)}$

Given: $\Delta ABC = 4$

$$\frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} = 4$$

$-2(4-k) - 0 + 1(0-0) = \pm 8$

$-8 + 2k = \pm 8$

$2k = \pm 8 + 8$

$2k = 8+8$ | $2k = -8+8$

$k = 8$ | $k = 0$

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5] If area of Δ^{le} is 35 sq units, with vertices $\overset{A}{(2,-6)}, \overset{B}{(5,4)}, \overset{C}{(k,4)}$ then k =

(A) 12 (B) −2 (C) −12, −2 (D) 12, −2

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} = 35$$

$$2(4-4) + 6(5-k) + 1(20-4k) = \pm 70$$

$$0 + 30 - 6k + 20 - 4k = \pm 70$$

$$50 - 10k = \pm 70$$

$$10k = 50 \pm 70$$

$10k = 50 + 70$, $k = 12$ | $10k = 50 - 70$, $k = -2$

4] i] Find eq. of the line joining (1,2), (3,6) using determinants.

Let $A = (1,2)$, $B = (3,6)$ $P = (x,y)$ on $\overline{AB}$

$\because$ APB are collinear, $\Delta PAB = 0$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0$$

$$x(2-6) - y(1-3) + 1(6-6) = 0$$

$$2x - 6x - y + 3y = 0$$

$$-4x + 2y = 0$$

$$4x - 2y = 0$$

$$\boxed{2x - y = 0}$$

ii] Find the eq. of line joining (3,1), (9,3) using determinants

Let $A(3,1)$, $B(9,3)$, $P(x,y)$ be any pt. on $\overline{AB}$

$\because$ APB are collinear, $\Delta APB = 0$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0$$

$x(1-3) - y(3-9) + 1(9-9) = 0$

$x - 3x - 3y + 9y = 0$

$-2x + 6y = 0$

$2x - 6y = 0$

$\boxed{x - 3y = 0}$

EXERCISE 4.4

Write minors (M_{ij}) and co-factors (A_{ij}) of the elements of following determinants:

1] i] $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

M_{11} = minor of $2 = 3$; $A_{11} = (-1)^{1+1}\, 3 = 3$

$M_{12} = 0$; $A_{12} = -0 = 0$

$M_{21} = -4$; $A_{21} = -(-4) = 4$

$M_{22} = 2$; $A_{22} = +2 = 2$

ii] $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

$M_{11} = d$; $A_{11} = d$

$M_{12} = b$; $A_{12} = -b$

$M_{21} = c$; $A_{12} = -c$

$M_{22} = a$; $A_{22} = a$

2] i] $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$; $A_{11} = +1 = (-1)^{1+1} M_{11}$

$M_{12} = 0$; $A_{12} = -0 = 0$

$M_{13} = 0$; $A_{13} = 0$

$M_{21} = 0$; $A_{21} = 0$

$M_{22} = 1$; $A_{22} = +1$

$M_{23} = 0$; $A_{23} = 0$

$M_{31} = 0$; $A_{31} = 0$

$M_{32} = 0$; $A_{32} = 0$

$M_{33} = 1$; $A_{33} = 1$

$C_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ii] $A = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

$M_{11} = 11 \ ; \ A_{11} = 11$ $\quad M_{31} = -20 \ ; \ A_{31} = -20$
$M_{12} = 6 \ ; \ A_{12} = -6$ $\quad M_{32} = -13 \ ; \ A_{32} = +13$
$M_{13} = 3 \ ; \ A_{13} = 3$ $\quad M_{33} = 5 \ ; \ A_{33} = 5$
$M_{21} = -4 \ ; \ A_{21} = +4$
$M_{22} = 2 \ ; \ A_{22} = 2$
$M_{23} = 1 \ ; \ A_{23} = -1$

Properties under expansion of determinants:

Property i] Elements of any row or column multiplied by their own cofactors and added gives the value of the determinant.

3] Eg: Using cofactors of elements of 2nd row, evaluate Δ:

$$\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

extra along R_1: $5\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 8\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}$

$\Delta = 5(-2) - 3(5) + 8(4) = -10 - 15 + 32 = \boxed{7}$

along R_2: $\Delta = -2\begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} + 0 - 1\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$

$\Delta = -2(-7) - 1(7) = 14 - 7 = \boxed{7}$

extra along C_3: $8\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 5 & 3 \\ 2 & 0 \end{vmatrix}$

$\Delta = 32 - 7 - 18 = \boxed{7}$

Property ii] If elements of any row or column are multiplied by cofactors of another row or column, and added, their sum is always equal to zero.

Eg: Along R_1 and R_3: $\Delta = 5\begin{vmatrix} 3 & 8 \\ 0 & 1 \end{vmatrix} - 3\begin{vmatrix} 5 & 8 \\ 2 & 1 \end{vmatrix} + 8\begin{vmatrix} 5 & 3 \\ 2 & 0 \end{vmatrix}$

$= 15 + 33 - 48$

$= 0$

Eg: Along C_1 and C_2 : $= -5\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 5 & 8 \\ 2 & 1 \end{vmatrix}$

$= -25 + 14 + 11$

$= 0$

Theorem: If A is any square matrix of order n, then
$A(\text{adj} A) = (\text{adj} A)A = |A|I$ $\quad [\text{Adj} A = C_A']$

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$; $C_A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$\text{Adj} A = C_A' = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$\text{LHS} = A \cdot \text{adj} A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I$

$= \text{RHS}$

Theorem: If A and B are non-singular matrices of same order, then AB and BA are also non singular matrices of same order.

** Theorem: Determinant of product of matrices is equal to the product of their respective determinant.

$\boxed{|AB| = |A|\,|B|}$

Eg: If $A = \begin{bmatrix} 2 & -1 \\ 4 & -5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ -7 & 2 \end{bmatrix}$; find $|AB|$.

$|AB| = |A||B| = (-6)(23) = -138$

** NOTE : ~~Determinant of adj A~~

$|\text{adj } A| = |A|^{n-1}$ where n is order of matrix.

Eg: If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$; Find : $|\text{adj } A|$.

$|A| = 1(16-9) - 3(4-3) + 3(3-4)$

$= 7 - 3 - 3 = 1$

$|\text{adj } A| = |A|^{n-1}$

$= |1|^{3-1} = (1)^2 = 1$

Eg: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ $|A| = 1(6-3) - 1(3+6) + 1(-1-4)$

$= 3 - 9 - 5 = -11$

$|\text{adj } A| = |A|^{n-1}$

$= (-11)^{3-1} = (-11)^2 = 121$

Inverse of a Matrix :

By theorem, we know $A \cdot \text{adj } A = |A| I = \text{adj } A \cdot A$ '$\div |A|$'

$A \cdot \left(\frac{1}{|A|} \text{adj } A\right) = \left(\frac{1}{|A|} \text{adj } A\right) A = I$

$\Rightarrow A^{-1} = \frac{1}{|A|} \text{adj } A \quad (|A| \neq 0)$

ie, A is non singular matrix.

Eg 29) The sum of 3 no. is 6. If we multiply 3rd no by 3 and add 2nd no. to it, we get 11. By adding 1st and 3rd no.s we get double the 2nd no. Find the no. using matrix method.

Sol] According to data, the eq. are :

$x + y + z = 6$

$3z + y = 11 \Rightarrow y + 3z = 11$

$x + z = 2y \Rightarrow x - 2y + z = 0$

$A \quad X = B \longrightarrow ①$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$C_A = \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 7 & 3 & -1 \\ -3 & 0 & 3 \\ 2 & -3 & 1 \end{bmatrix}$$

$$|A| = 7 + 3 - 1 = 9 \;; \; \text{adj} A = C_A' = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj} A = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

$① \Rightarrow X = A^{-1}B$

$$= \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 42 - 33 + 0 \\ 18 + 0 + 0 \\ -6 + 33 + 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow x = 1, \; y = 2, \; z = 3$$

4] Using cofactors of elements of 3rd column, evaluate

$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

Along C_3 : $\Delta = yz \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} - zx \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} + xy \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix}$

$= yz(z-y) - zx(z-x) + xy(y-x)$

$= yz^2 - y^2z - z^2x + zx^2 + xy^2 - x^2y$

5] If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is cofactor of a_{ij}, then value of Δ is given by,

A] $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$

B] $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31} = 0$

C] $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} = 0$

D] $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} = \Delta$

Properties of Determinants :

1] The value of Δ remains unchanged if its rows and columns are interchanged (Transpose)

Proof: $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$

Rows $\longleftrightarrow$ Columns

$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1$

$= \Delta$

$\therefore \Delta' = \Delta$

2] If any 2 rows (or columns) are interchanged in a Δ, then the sign of the Δ changes.

Eg: $\Delta = \begin{vmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & -5 \end{vmatrix} = 2(0-2) - 3(5-0) + 1(-1-0)$
$= -4 - 15 - 1$
$= -20$

$R_1 \leftrightarrow R_2$

$\Delta' = \begin{vmatrix} -1 & 0 & 2 \\ 2 & 3 & 1 \\ 0 & 1 & -5 \end{vmatrix} = -1(-15-1) - 0 + 2(2-0)$
$= +16 + 4$
$= 20$

Proof: $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$
$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$

$R_1 \leftrightarrow R_2$

$\Delta' = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = b_1(a_2c_3 - a_3c_2) - b_2(a_1c_3 - a_3c_1) + b_3(a_1c_2 - a_2c_1)$
$= a_2b_1c_3 - a_3b_1c_2 - a_1b_2c_3 + a_3b_2c_1 + a_1b_3c_2 - a_2b_3c_1$
$= -(a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1)$

$\therefore \Delta' = -\Delta$

3] If any 2 rows (or 2 columns) of a Δ are identical, then value of Δ is zero.

Eg: $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 5 & 3 & 7 \end{vmatrix} = 1(14-12) - 2(7-20) + 4(3-10)$
$= 2 + 26 - 28$
$= 0$

Proof: Let $R_1 \sim R_2$

$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(a_2c_3 - a_3c_2) - a_2(a_1c_3 - a_3c_1) + a_3(a_1c_2 - a_2c_1)$
$= \cancel{a_1a_2c_3} - \cancel{a_1a_3c_2} - \cancel{a_1a_2c_3} + \cancel{a_2a_3c_1} + \cancel{a_1a_3c_2} - \cancel{a_2a_3c_1}$
$= 0$

** 4] If Each element of a row (or a column) of a Δ is multiplied by a constant 'k', then its values gets multiplied by k

Eg: $\Delta = \begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix} = 12 - 4 = 8$

let $R_1 \to 2R_1$

$\Delta' = \begin{vmatrix} 4 & 8 \\ 1 & 6 \end{vmatrix} = 24 - 8 = 16 = 2(8)$

$\Delta' = 2\Delta$

Proof: $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$
$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$

Let $R_1 \to kR_1$

$\Delta' = \begin{vmatrix} ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = ka_1(b_2c_3 - b_3c_2) - ka_2(b_1c_3 - b_3c_1) + ka_3(b_1c_2 - b_2c_1)$
$= k[a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1]$
$= k\Delta$

$\therefore \Delta' = k\Delta$

** 5] If some or all elements of a row (or column) of a Δ are expressed as sum of 2 (or more) terms, then the Δ can be expressed as sum of 2 (or more) Δ's.

Eg 10] Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

$$\text{LHS} = \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$$

$\because R_1 \sim R_2$

$$= 0 + 2\begin{vmatrix} a & b & c \\ x & y & z \\ x & y & z \end{vmatrix}$$

$\because R_2 \sim R_3$

$= 0 + 2(0)$

$= 0 = \text{RHS}$

Proof: Let elements of R_1 be added with $\lambda_1, \lambda_2, \lambda_3$ respectively.

$$\Delta' = \begin{vmatrix} a_1+\lambda_1 & a_2+\lambda_2 & a_3+\lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (a_1+\lambda_1)(b_2c_3-b_3c_2)-(a_2+\lambda_2)(b_1c_3-b_3c_1)+(a_3+\lambda_3)(b_1c_2-b_2c_1)$$

$$= [a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)]+$$
$$[\lambda_1(b_2c_3-b_3c_2)-\lambda_2(b_1b_3-b_3c_1)+\lambda_3(b_1c_2-b_2c_1)]$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Hence, proved.

6] If, to each element of any row (or column) of a Δ, the equimultiples of corresponding elements of other row or column are added, then value of Δ remains the same. i.e, $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Proof:
$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$R_1 \to R_1 + kR_3$

$$\Delta' = \begin{vmatrix} a_1+kc_1 & a_2+kc_2 & a_3+kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} kc_1 & kc_2 & kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\Delta' = \Delta + k\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\Delta' = \Delta + 0 \quad (\because R_1 \sim R_3)$

$\Rightarrow \Delta' = \Delta$

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7] Value of the triangular Δ is product of the PD elements.

Proof : $\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = a_1(b_2c_3 - 0) - a_2(0-0) + a_3(0-0)$
$= a_1b_2c_3 - 0 - 0 - 0 - 0$
$= a_1b_2c_3$

8] If all elements of any one row (or column) are equal to zero, value of the Δ is zero.

Proof: $\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(0-0) - a_2(0-0) + a_3(0-0)$
$= 0 - 0 + 0$
$= 0$

EXERCISE 4.2

1] Prove that, $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

$C_1 \rightarrow C_1 + C_2$

LHS $= \begin{vmatrix} x+a & a & x+a \\ y+b & b & y+b \\ z+c & c & z+c \end{vmatrix} = 0 \quad (\because C_1 \sim C_3)$

2] Prove that, $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

$C_1 \rightarrow C_1 + C_2 + C_3$

$\begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0 \quad (\because C_1 = 0)$

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**3] Prove that $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

LHS $= \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$

$C_2 \longrightarrow 9C_2 + C_1$

$= \begin{vmatrix} 2 & 65 & 65 \\ 3 & 75 & 75 \\ 5 & 86 & 86 \end{vmatrix} = 0 \quad (\because C_2 \sim C_3)$

4] Prove that $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

LHS $= \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

$C_3 \longrightarrow C_3 + C_2$

$= \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix} = ab+bc+ca \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$

$= (ab+bc+ca)(0) = 0 \quad (\because C_1 \sim C_3)$

5] Prove that $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

LHS $= \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$

$R_1 \longrightarrow R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= 2\begin{vmatrix} a & p & x \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix} = (-)(-)2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

6] $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$: Prove

3^{rd} ordered skew symmetric matrix

$\therefore \Delta = 0$

7] Prove that $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$

LHS $= \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$ Take out a, b, c common from R_1, R_2, R_3 respectively.

$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$ Taking out a, b, c again common from columns

$= a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$= a^2b^2c^2\left[-1(1-1) - 1(-1-1) + 1(1+1)\right]$

$= a^2b^2c^2\left[0 + 2 + 2\right]$

$= 4a^2b^2c^2$

Type I 8] (a) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(b) $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(c) $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(a) LHS $= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix}$

$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$

$R_3 \to R_3 - R_2$

$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$

$= -(a-b)(c-a)(c-b)$

$= (a-b)(b-c)(c-a)$

$=$ RHS

(b) LHS = $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$ $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$= \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix}$ $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$= \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & (b-a)(b^2+ab+a^2) \\ 0 & c-a & (c-a)(c^2+ac+a^2) \end{vmatrix}$

$= (b-a)(c-a) \begin{vmatrix} 1 & a & a^3 \\ 0 & 1 & a^2+ab+b^2 \\ 0 & 1 & c^2+ac+a^2 \end{vmatrix}$

$R_3 \to R_3 - R_2$: $= -(a-b)(c-a) \begin{vmatrix} 1 & a & a^3 \\ 0 & 1 & a^2+ab+b^2 \\ 0 & 0 & \underbrace{(c-b)(c+b)+a(c-b)}_{(c-b)[a+b+c]} \end{vmatrix}$

$= -(a-b)(c-a)(c-b) \begin{vmatrix} 1 & a & a^3 \\ 0 & 1 & a^2+ab+b^2 \\ 0 & 0 & a+b+c \end{vmatrix}$

$= (a-b)(b-c)(c-a)(a+b+c)$

(c) LHS = $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix}$

$= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & (b-a)(b+a) & (b-a)(b^2+ab+a^2) \\ 0 & (c-a)(c+a) & (c-a)(c^2+ac+a^2) \end{vmatrix}$

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$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & b^2+ab+a^2 \\ 0 & c+a & c^2+ab+a^2 \end{vmatrix}$$

$R_3 \to R_3 - R_2$

$$= -(a-b)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & b^2+ab+a^2 \\ 0 & c-b & (c-b)(c+b)+a(c-b) \end{vmatrix}$$

$$= -(a-b)(c-a)(c-b)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & a^2+ab+b^2 \\ 0 & 1 & a+b+c \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)\left[1(\cancel{ab}+\cancel{b^2}+bc+\cancel{a^2}+ab+ac-\cancel{b^2}-\cancel{ab}-\cancel{a^2}) + 0 + 0\right]$$

$$= (a-b)(b-c)(c-a)(ab+bc+ca)$$

9)
$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

$$\text{LHS} = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} \quad R_1 \to R_1 - R_2 ;\ R_2 \to R_2 - R_3$$

$$= \begin{vmatrix} x-y & x^2-y^2 & -z(x-y) \\ y-z & y^2-z^2 & -x(y-z) \\ z & z^2 & xy \end{vmatrix} = \begin{vmatrix} x-y & (x-y)(x+y) & -z(x-y) \\ y-z & (y-z)(y+z) & -x(y-z) \\ z & z^2 & xy \end{vmatrix}$$

$$= (x-y)(y-z)\begin{vmatrix} 1 & x+y & -z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix} \quad R_1 \to R_1 - R_2$$

$$= (x-y)(y-z)\begin{vmatrix} 0 & (x-z) & (x-z) \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

$$= (x-y)(y-z)(x-z)\begin{vmatrix} 0 & 1 & 1 \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

$$= (x-y)(y-z)(-)(z-x)\left[-1(xy+xz) + 1(z^2 - zy - z^2)\right]$$

$$= (x-y)(y-z)(z-x)(-)(-xy - xz - zy)$$

$$= (x-y)(y-z)(z-x)(xy + xz + zy)$$

Type II

10] i] $(R_1+R_2+R_3)$ or $(C_1+C_2+C_3)$

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

LHS = $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$

$$= (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix}$$

$$= (5x+4)(4-x)^2 = RHS$$

ii] $$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = (3y+k)(k^2)$$

LHS = $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix} = (3y+k)\begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$= (3y+k)\begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix}$$

$$= (3y+k)(k^2) = RHS$$

11] i] $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

LHS $=$ $R_1 \to R_1 + R_2 + R_3$

$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$= C_2 \to C_2 - C_1 \; ; \; C_3 \to C_3 - C_1$

$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$

$= (a+b+c)(-b-c-a)(-c-a-b) = (a+b+c)(-)(-)(a+b+c)(a+b+c)$

$= (a+b+c)(a+b+c)^2$

$= (a+b+c)^3$

ii] $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

LHS $\cdot$ $C_1 \to C_1 + C_2 + C_3$

$= \begin{vmatrix} 2x+2y+2z & x & y \\ 2y+2z+2x & y+z+2x & y \\ 2z+2x+2y & x & z+x+2y \end{vmatrix}$

$= \cancel{x+y+z}\; 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$

$R_2 \to R_2 - R_1 \;;\; R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & z+x+y \end{vmatrix} 2(x+y+z)$$

$$= 2(x+y+z)(x+y+z)^2$$
$$= 2(x+y+z)^3$$

***12]

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

$$\text{LHS} = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} \quad R_1 \to R_1 - xR_2 \;;\; R_2 \to R_2 - xR_3$$

$$= \begin{vmatrix} 1-x^3 & 0 & 0 \\ 0 & 1-x^3 & 0 \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

***13] Prove that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

LHS : $C_1 \to C_1 - bC_3 \;;\; C_2 \to C_2 + aC_3$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

$2b - (b - a^2b - b^3)$
$= b + a^2b + b^3$
$= b(1+a^2+b^2)$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$-2a + a - a^3 - ab^2$
$= -a - a^3 - ab^2$
$= -a(1+a^2+b^2)$

$$= (1+a^2+b^2)^2 (1-a^2-b^2+2a^2+2b^2)$$
$$= (1+a^2+b^2)^2 (1+a^2+b^2)$$
$$= (1+a^2+b^2)^3$$

**14]
$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

LHS : Multiply and divide R_1, R_2, R_3 by a, b, c respectively.

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ ac^2 & bc^2 & c(c^2+1) \end{vmatrix}$$

Take out a, b, c common from C_1, C_2, C_3 respectively.

$$= \frac{ab\cancel{c}}{abc}\begin{vmatrix} 1+a^2 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

$$= 1+a^2+b^2+c^2 \begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

$C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$

$$= 1+a^2+b^2+c^2 \begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix}$$

$= (1+a^2+b^2+c^2)(1)$

$= 1+a^2+b^2+c^2$

$=$ RHS.

Eg12] Without expanding prove that $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$

$R_1 \to R_1 + R_2$

LHS : $\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

$= x+y+z \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

$= 0 \quad [\because R_1 \sim R_3]$

Eg11] $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

$R_2 \to R_2 - 2R_1 \; ; \; R_3 \to R_3 - 3R_1$

LHS : $\begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$ $R_3 \to R_3 - 3R_2$

$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$

$= a^3 = \text{RHS}$

Eg13] Evaluate : $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$R_2 \to R_2 - R_1 \; ; \; R_3 \to R_3 - R_1$

LHS : $\Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & ca-bc \\ 0 & c-a & ab-bc \end{vmatrix}$

$= \begin{vmatrix} 1 & a & bc \\ 0 & b-a & -c(b-a) \\ 0 & c-a & -b(c-a) \end{vmatrix}$

$= (b-a)(c-a) \begin{vmatrix} 1 & a & b \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix}$

$R_3 \to R_3 - R_2$

$= \begin{vmatrix} 1 & a & b \\ 0 & 1 & -c \\ 0 & 0 & -b+c \end{vmatrix} (b-a)(c-a)$

$= (b-a)(c-a)(c-b)$

$= (-)(a-b)(-)(b-c)(c-a)$

$= (a-b)(b-c)(c-a)$

Eg14] $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

LHS : $R_1 \to R_1 - R_2 - R_3$

$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 2 \begin{vmatrix} 0 & -c & -b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

$R_2 \to R_2 + R_1$; $R_3 \to R_3 + R_1$

$= 2 \begin{vmatrix} 0 & -c & -b \\ b & a & 0 \\ c & 0 & a \end{vmatrix}$

$= 2[c(ab) - b(-ac)]$

$= 2(abc + abc)$

$= 2(2abc) = 4abc$

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Eg 15] If x, y, z are all different and $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

Then show that $1 + xyz = 0$

$$\Delta \Rightarrow \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$$

$C_1 \leftrightarrow C_3 \; ; \; C_2 \leftrightarrow C_3$

$$\Rightarrow (-)(-) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} (1 + xyz) = 0$$

$R_2 \to R_2 - R_1 \; ; \; R_3 \to R_3 - R_1$

$$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} (1 + xyz) = 0$$

$$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & (y-x)(y+x) \\ 0 & z-x & (z-x)(z+x) \end{vmatrix} (1 + xyz) = 0$$

Eg 16] Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc + bc + ac + ab$

$$LHS = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

Take out a, b, c common from R_1, R_2, R_3 respectively

$$= abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$

$$= (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(1)$$

$$= (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

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$$\Rightarrow (y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}(1+xyz)=0$$

$\Rightarrow \cancel{(-)(x-y)(z-x)}$ $R_3 \to R_3 - R_2$

$$\Rightarrow (-)(x-y)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z-y \end{vmatrix}(1+xyz)=0$$

$\Rightarrow (-)(x-y)(z-x)(z-y)(1+xyz)=0$

$\Rightarrow (-)(-)(x-y)(y-z)(z-x)(1+xyz)=0$

$\Rightarrow (x-y)(y-z)(z-x)(1+xyz)=0$

Given x, y, z are all different

$\Rightarrow$ only $(1+xyz)=0$