

JEE-Main-26-07-2022-Shift-2 (Memory Based)

MATHEMATICS

Question: The interval in which abscissa of point P on $y = x^2$ lies such that its distance from $(x-1)^2 + (y+1)^2 = 1$ is minimum is:

Options:

- (a) $0 < x < \frac{1}{4}$
- (b) $\frac{1}{4} < x < \frac{1}{2}$
- (c) $\frac{1}{2} < x < \frac{3}{4}$
- (d) $\frac{3}{4} < x < 1$

Answer: (b)

Solution:

Let $P(x, x^2)$

Distance of P from given circle:

$$l = \sqrt{(x-1)^2 + (x^2+1)^2} - 1$$

For least value of l , we need to minimize:

$$f(x) = (x-1)^2 + (x^2+1)^2$$

$$f'(x) = 2(x-1) + 4x(x^2+1)$$

$$= 2[2x^3 + 3x - 1] = 0$$

$$\therefore f'\left(\frac{1}{4}\right) \text{ is -ve and } f'\left(\frac{1}{2}\right) \text{ is +ve}$$

$$\text{So, } f'(x) = 0 \text{ for some } x \in \left(\frac{1}{4}, \frac{1}{2}\right)$$

Question: If $z = x + iy$, $|z| - 2 = 0$ and $|z - i| - |z + 5i| = 0$, then which of the following is TRUE:

Options:

- (a) $x^2 + 2y + 4 = 0$
- (b) $x^2 - 2y + 4 = 0$
- (c) $x + y = 0$
- (d) $x^2 - y + 4 = 0$

Answer: (a)

Solution:

As $z = x + iy$

$$x^2 + y^2 = 4 \quad \dots(1)$$

$$\text{And } y = -2 \quad \dots(2)$$

So, $x = 0$

Hence, only $x^2 + 2y + 4 = 0$ is true.

Question: $x \sim B(n, p)$, mean = 4, variance = $\frac{4}{3}$, find $P(x \leq 2)$.

Answer: $\frac{73}{729}$

Solution:

Given, $np = 4$

$$npq = \frac{4}{3}$$

$$\therefore q = \frac{1}{3}$$

$$q = \frac{2}{3}$$

Thus, $n = 6$

Now, $P(x \leq 2) = P(x = 0) + P(x = 1) + P(x = 2)$

$$\begin{aligned} &= {}^6C_0 p^0 q^6 + {}^6C_1 p^1 q^5 + {}^6C_2 p^2 q^4 \\ &= \left(\frac{1}{3}\right)^6 + 6\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^5 + 15\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^4 \end{aligned}$$

Question: Find area between $y = |x^2 - 1|$ & $y = 1$.

Answer: $\frac{4}{3}(\sqrt{2} - 1)$

Solution:

$$\int_0^1 -\sqrt{1-y} + \sqrt{1+y}$$

$$\frac{2}{3}(1-y)^{\frac{3}{2}} + \frac{2}{3}(1+y)^{\frac{3}{2}} \Big|_0^1$$

$$\frac{2}{3} \times 2^{\frac{3}{2}} - \left(\frac{4}{3}\right)$$

$$\frac{4\sqrt{2}}{3} - \frac{4}{3} = \frac{4}{3}(\sqrt{2} - 1)$$

Question: How 4 digit numbers lying between 1000 & 3000 can be made which are divisible by 4, using digits 1, 2, 3, 4, 5, 6 with no repetition.

Answer: 30.00

Solution:

We will solve the Question in two cases.

Case I: When first digit is 1.

Then last two digits can be 24, 32, 36, 52, 56 and 64.

Number of such numbers = $6 \times 3 = 18$

Case II: When first digit is 2

Then last two digits can be 16, 36, 56 or 64

Number of such numbers = $4 \times 3 = 12$

Total numbers of numbers = $18 + 12 = 30$

Question: $\int_0^{20\pi} (|\sin x| + |\cos x|)^2 dx$

Answer: $20\pi + 40$

Solution:

$$\begin{aligned} & \int_0^{20\pi} (|\sin x| + |\cos x|)^2 dx \\ & \Rightarrow \int_0^{20\pi} ((\sin^2 x + \cos^2 x) + |\sin 2x|) dx \\ & \Rightarrow \int_0^{20\pi} 1 dx + \int_0^{20\pi} |\sin 2x| dx \\ & \Rightarrow 20\pi + 40 \int_0^{\frac{\pi}{2}} \sin 2x dx \\ & \Rightarrow 20\pi + 40 \left[\frac{-\cos 2x}{2} \right]_0^{\frac{\pi}{2}} \\ & \Rightarrow 20\pi + 20(1+1) \\ & \Rightarrow 20\pi + 40 \end{aligned}$$

Question: Find equation of common tangent to $y = x^2$ & $y = -(x-2)^2$.

Answer: ()

Solution:

Given, $y = x^2$ & $y = -(x-2)^2$

Tangent for $y = x^2$

$$y = mx - \frac{1}{4}m^2$$

Tangent for $y = -(x-2)^2$

$$y = m(x-2) + \frac{1}{4}m^2$$

$$y = mx - 2m + \frac{1}{4}m^2$$

$$-\frac{1}{4}m^2 = -2m + \frac{1}{4}m^2 \text{ (For common tangents)}$$

$$2m = \frac{1}{2}m^2$$

$$m^2 - 4m = 0$$

$$m = 0, m = 4$$

Thus, equation of tangents are $y = 0$ or $y = 4x - 4$

Question: If $\sin^{-1}\left(\frac{x}{\alpha}\right) = \cos^{-1}\left(\frac{x}{\beta}\right)$ then find value of $\sin\left(\frac{2\pi}{\alpha + \beta}\right)$.

Answer: 0

Solution:

$$\text{Given, } \sin^{-1}\left(\frac{x}{\alpha}\right) = \cos^{-1}\left(\frac{x}{\beta}\right) = k$$

$$\Rightarrow \alpha = \frac{\sin^{-1} x}{k}, \beta = \frac{\cos^{-1} x}{k}$$

$$\therefore \sin\left(\frac{2\pi\alpha}{\alpha + \beta}\right) = \sin\left(\frac{2\pi \frac{\sin^{-1} x}{k}}{\frac{\sin^{-1} x + \cos^{-1} x}{k}}\right)$$

$$\Rightarrow \sin\left(\frac{2\pi(\sin^{-1} x)}{\frac{\pi}{2}}\right)$$

$$\Rightarrow \sin(4\sin^{-1} x)$$

$$\Rightarrow \sin\left(2\left(\sin^{-1}\left(2x\sqrt{1-x^2}\right)\right)\right)$$

$$\Rightarrow 2\left(2x\sqrt{1-x^2}\right)\sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}$$

$$\Rightarrow 4x\sqrt{1-x^2}\sqrt{1-4x^2(1-x^2)}$$

$$\Rightarrow 4x\sqrt{1-x^2}(2x^2-1)$$

Question: $\ln 2 \times \frac{d}{dx} \left(\frac{\log \operatorname{cosec} x}{\log \cos x} \right) \Big|_{\frac{\pi}{4}}$

Answer: 4.00

Solution:

$$\ln 2 \times \frac{d}{dx} \left(\frac{\log \operatorname{cosec} x}{\log \cos x} \right)$$

$$\ln 2 \times \frac{d}{dx} \left(-\frac{\log \sin x}{\log \cos x} \right)$$

$$\ln 2 \left(\frac{(\log \cos x) \left(-\frac{\cos x}{\sin x} \right) + \log \sin x \left(-\frac{\sin x}{\cos x} \right)}{(\log \cos x)^2} \right) \text{ at } x = \frac{\pi}{4}$$

$$\ln 2 \left(\frac{-2 \log \frac{1}{\sqrt{2}}}{\left(\log \frac{1}{\sqrt{2}} \right)^2} \right)$$

$$\frac{-2 \ln 2}{\log 2^{\frac{1}{2}}} = \frac{-2}{-\frac{1}{2}} = 4$$

Question: If $\beta = \lim_{x \rightarrow 0} \frac{\alpha x - (e^{3x} - 1)}{\alpha x (e^{3x} - 1)}$ then $\alpha + \beta = ?$

Answer: $\frac{5}{2}$

Solution:

Given, $\beta = \lim_{x \rightarrow 0} \frac{\alpha x - (e^{3x} - 1)}{\alpha x (e^{3x} - 1)}$

$$\beta = \lim_{x \rightarrow 0} \frac{\alpha x - \left(1 + 3x + \frac{9x^2}{2} - 1 \right)}{\alpha x \left(1 + 3x + \frac{9x^2}{2} - 1 \right)}$$

$$\beta = \lim_{x \rightarrow 0} \frac{x(\alpha - 3) - \frac{9}{2}x^2}{\alpha x(3x)}$$

$$\beta = \frac{1}{3\alpha} \lim_{x \rightarrow 0} \frac{x(\alpha - 3) - \frac{9}{2}x^2}{x^2}$$

$$\therefore \alpha = 3, \beta = \frac{1}{3 \times 3} \times \left(-\frac{9}{2}\right) = \frac{-1}{2}$$

$$\therefore \alpha + \beta = 3 - \frac{1}{2} = \frac{5}{2}$$

Question: Find minimum value of sum of squares of roots of $x^2 + (3-a)x = 2a-1$

Answer: 6.00

Solution:

Let α, β be the roots of the equation

$$x^2 + (3-a)x + 1 - 2a = 0$$

Then, $\alpha + \beta = a - 3, \alpha\beta = 1 - 2a$

$$\therefore \alpha^2 + \beta^2 = (a-3)^2 - 2(1-2a)$$

$$= a^2 - 2a + 7$$

$$= (a-1)^2 + 6$$

\therefore Minimum value of $\alpha^2 + \beta^2 = 6$

Question: If $\sum_{k=1}^{10} \frac{k}{(k^4 + k^2 + 1)} = \frac{m}{n}$, such that m and n are coprime, then $m+n$ is equal to ____

Answer: $\frac{55}{111}$

Solution:

$$\sum_{k=1}^{10} \frac{k}{(k^4 + k^2 + 1)} = \sum_{k=1}^{10} \frac{k}{(k^2 + k + 1)(k^2 - k + 1)}$$

$$= \sum_{k=1}^{10} \frac{1}{2} \left(\frac{1}{k^2 - k + 1} - \frac{1}{k^2 + k + 1} \right)$$

$$= \frac{1}{2} \left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \dots + \left(\frac{1}{91} - \frac{1}{111}\right) \right]$$

$$= \frac{1}{2} \left(1 - \frac{1}{111}\right)$$

$$\sum_{k=1}^{10} \frac{k}{k^4 + k^2 + 1} = \frac{55}{111}$$

$\therefore m = 55, n = 111$

$\therefore m + n = 166$

Question: If $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $B = \begin{bmatrix} 9^2 & 10^2 & 11^2 \\ 12^2 & -13^2 & 14^2 \\ 15^2 & 16^2 & -17^2 \end{bmatrix}$, then $A'BA$ is equal to:

Answer: 665.00

Solution:

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, B = \begin{bmatrix} 9^2 & 10^2 & 11^2 \\ 12^2 & -13^2 & 14^2 \\ 15^2 & 16^2 & -17^2 \end{bmatrix}$$

$$A' = [1 \quad 1 \quad 1]$$

$$A'B = [9^2 + 12^2 + 15^2 \quad 10^2 - 13^2 + 16^2 \quad 11^2 + 14^2 - 17^2]$$

$$A'BA = [9^2 + 12^2 + 15^2 \quad 10^2 - 13^2 + 16^2 \quad 11^2 + 14^2 - 17^2] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A'BA = [9^2 + 12^2 + 15^2 + 10^2 - 13^2 + 16^2 + 11^2 + 14^2 - 17^2] \\ = [665]$$

Question: If $ax^2 + by^2 + 2gx + 2fy + c = 0$ is a circle whose diametric end points are given by $x^2 - 4x - 9 = 0$ & $y^2 + 2x - 4 = 0$ then find $a + b - c$.

Answer: 15.00

Solution:

Diametric points be (x_1, y_1) & (x_2, y_2) and equation of circle will be

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\Rightarrow x^2 - x(x_1 + x_2) + x_1x_2 + y^2 - y(y_1 + y_2) + y_1y_2 = 0$$

$$\Rightarrow x^2 - x(4) + (-9) + y^2 - y(-2) - 4 = 0$$

$$\Rightarrow x^2 + y^2 - 4x + 2y - 13 = 0$$

Composing with $ax^2 + by^2 + 2gx + 2fy + c = 0$

$$a = 1, b = 1, c = -13$$

$$a + b - c = 1 + 1 + 13 = 15$$

Question: Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{3, 4, 6, 7, 9\}$ and $C = A \cup B$, then number of elements in cartesian product of $C \times B$ is _____.

Answer: 40.00

Solution:

$$A = \{1, 2, 3, 4, 5, 6\}, B = \{3, 4, 6, 7, 9\}$$

$$\therefore C = A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9\}$$

$$\therefore n(C \times B) = 8 \times 5 = 40$$

Question: $2 \sin^2 \theta - \cos 2\theta = 0$, $2 \cos^2 \theta + 3 \sin \theta = 0$. If sum of all solutions of θ in $[0, 2\pi]$ is $k\pi$, then find k .

Answer: 3.00

Solution:

$$\text{Given, } 2 \sin^2 \theta - \cos 2\theta = 0$$

$$2 \sin^2 \theta - 1 + 2 \sin^2 \theta = 0$$

$$4 \sin^2 \theta = 1$$

$$\sin^2 \theta = \frac{1}{4}$$

$$\sin \theta = \pm \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$2 \cos^2 \theta + 3 \sin \theta = 0$$

$$2 - 2 \sin^2 \theta + 3 \sin \theta = 0$$

$$2 \sin^2 \theta - 3 \sin \theta - 2 = 0$$

$$2 \sin^2 \theta - 4 \sin \theta + \sin \theta - 2 = 0$$

$$(\sin \theta - 2)(2 \sin \theta + 1) = 0$$

$$\sin \theta = -\frac{1}{2}$$

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

\therefore Sum of all value of common θ

$$\frac{7\pi}{6} + \frac{11\pi}{6} = \frac{18\pi}{6} = 3\pi$$

$$\therefore k = 3$$