

JAM 2006

MATHEMATICAL STATISTICS TEST PAPER

Special Instructions / Useful Data

1. For an event A , $P(A)$ denotes the probability of the event A .
2. The complement of an event is denoted by putting a superscript “ c ” on the event, e.g. A^c denotes the complement of the event A .
3. For a random variable X , $E(X)$ denotes the expectation of X and $V(X)$ denotes its variance.
4. $N(\mu, \sigma^2)$ denotes a normal distribution with mean μ and variance σ^2 .
5. Standard normal random variable is a random variable having a normal distribution with mean 0 and variance 1.
6. $P(Z > 1.96) = 0.025$, $P(Z > 1.65) = 0.050$, $P(Z > 0.675) = 0.250$ and $P(Z > 2.33) = 0.010$, where Z is a standard normal random variable.
7. $P(\chi_2^2 \geq 9.21) = 0.01$, $P(\chi_2^2 \geq 0.02) = 0.99$, $P(\chi_3^2 \geq 11.34) = 0.01$, $P(\chi_4^2 \geq 9.49) = 0.05$,
 $P(\chi_4^2 \geq 0.71) = 0.95$, $P(\chi_5^2 \geq 11.07) = 0.05$ and $P(\chi_5^2 \geq 1.15) = 0.95$, where $P(\chi_n^2 \geq c) = \alpha$, where χ_n^2 has a Chi-square distribution with n degrees of freedom.
8. $n!$ denotes the factorial of n .
9. The determinant of a square matrix A is denoted by $|A|$.
10. \mathbb{R} : The set of all real numbers.
11. \mathbb{R}^n : n -dimensional Euclidean space.
12. y' and y'' denote the first and second derivatives respectively of the function $y(x)$ with respect to x .

NOTE: This Question-cum-Answer book contains **THREE** sections, the **Compulsory Section A**, and the **Optional Sections B and C**.

- Attempt **ALL** questions in the compulsory section A. It has 15 objective type questions of *six* marks each and also *nine* subjective type questions of *fifteen* marks each.
- Optional Sections B and C have *five* subjective type questions of *fifteen* marks each.
- Candidates seeking admission to either of the two programmes, M.Sc. in Applied Statistics & Informatics at IIT Bombay and M.Sc. in Statistics & Informatics at IIT Kharagpur, are required to attempt **ONLY** Section B (Mathematics) from the Optional Sections.
- Candidates seeking admission to the programme, M.Sc. in Statistics at IIT Kanpur, are required to attempt **ONLY** Section C (Statistics) from the Optional Sections.

You must therefore attempt either Optional Section B or Optional Section C depending upon the programme(s) you are seeking admission to, and accordingly tick one of the boxes given below.

Optional Section Attempted	B	
	C	

- *The negative marks for the Objective type questions will be carried over to the total marks.*
- Write the answers to the objective questions in the Answer Table for Objective Questions provided on page MS 11/63 only.

Compulsory Section A

1. If $a_n > 0$ for $n \geq 1$ and $\lim_{n \rightarrow \infty} (a_n)^{1/n} = L < 1$, then which of the following series is not convergent?
- (A) $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$
- (B) $\sum_{n=1}^{\infty} a_n^2$
- (C) $\sum_{n=1}^{\infty} \sqrt{a_n}$
- (D) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{a_n}}$
2. Let E and F be two mutually disjoint events. Further, let E and F be independent of G . If $p = P(E) + P(F)$ and $q = P(G)$, then $P(E \cup F \cup G)$ is
- (A) $1 - pq$
- (B) $q + p^2$
- (C) $p + q^2$
- (D) $p + q - pq$
3. Let X be a continuous random variable with the probability density function symmetric about 0. If $V(X) < \infty$, then which of the following statements is true?
- (A) $E(|X|) = E(X)$
- (B) $V(|X|) = V(X)$
- (C) $V(|X|) < V(X)$
- (D) $V(|X|) > V(X)$
4. Let
- $$f(x) = x|x| + |x-1|, \quad -\infty < x < \infty.$$
- Which of the following statements is true?
- (A) f is not differentiable at $x=0$ and $x=1$.
- (B) f is differentiable at $x=0$ but not differentiable at $x=1$.
- (C) f is not differentiable at $x=0$ but differentiable at $x=1$.
- (D) f is differentiable at $x=0$ and $x=1$.
5. Let $A\tilde{x} = \tilde{b}$ be a non-homogeneous system of linear equations. The augmented matrix $[A : \tilde{b}]$ is given by

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ -1 & 2 & 3 & -1 & 0 \\ 0 & 3 & 1 & 0 & -1 \end{array} \right].$$

Which of the following statements is true?

- (A) Rank of A is 3.
- (B) The system has no solution.
- (C) The system has unique solution.
- (D) The system has infinite number of solutions.

6. An archer makes 10 independent attempts at a target and his probability of hitting the target at each attempt is $\frac{5}{6}$. Then the conditional probability that his last two attempts are successful given that he has a total of 7 successful attempts is

- (A) $\frac{1}{5^5}$
- (B) $\frac{7}{15}$
- (C) $\frac{25}{36}$
- (D) $\frac{8!}{3! 5!} \left(\frac{5}{6}\right)^7 \left(\frac{1}{6}\right)^3$

7. Let

$$f(x) = (x-1)(x-2)(x-3)(x-4)(x-5), \quad -\infty < x < \infty.$$

The number of distinct real roots of the equation $\frac{d}{dx} f(x) = 0$ is exactly

- (A) 2
- (B) 3
- (C) 4
- (D) 5

8. Let

$$f(x) = \frac{k|x|}{(1+|x|)^4}, \quad -\infty < x < \infty.$$

Then the value of k for which $f(x)$ is a probability density function is

- (A) $\frac{1}{6}$
- (B) $\frac{1}{2}$
- (C) 3
- (D) 6

9. If $M_X(t) = e^{3t+8t^2}$ is the moment generating function of a random variable X , then $P(-4.84 < X \leq 9.60)$ is

- (A) equal to 0.700
- (B) equal to 0.925
- (C) equal to 0.975
- (D) greater than 0.999

10. Let X be a binomial random variable with parameters n and p , where n is a positive integer and $0 \leq p \leq 1$. If $\alpha = P(|X - np| \geq \sqrt{n})$, then which of the following statements holds true for all n and p ?

- (A) $0 \leq \alpha \leq \frac{1}{4}$
- (B) $\frac{1}{4} < \alpha \leq \frac{1}{2}$
- (C) $\frac{1}{2} < \alpha < \frac{3}{4}$
- (D) $\frac{3}{4} \leq \alpha \leq 1$

11. Let X_1, X_2, \dots, X_n be a random sample from a Bernoulli distribution with parameter p ; $0 \leq p \leq 1$. The

bias of the estimator $\frac{\sqrt{n} + 2 \sum_{i=1}^n X_i}{2(n + \sqrt{n})}$ for estimating p is equal to

- (A) $\frac{1}{\sqrt{n} + 1} \left(p - \frac{1}{2} \right)$
- (B) $\frac{1}{n + \sqrt{n}} \left(\frac{1}{2} - p \right)$
- (C) $\frac{1}{\sqrt{n} + 1} \left(\frac{1}{2} + \frac{p}{\sqrt{n}} \right) - p$
- (D) $\frac{1}{\sqrt{n} + 1} \left(\frac{1}{2} - p \right)$

12. Let the joint probability density function of X and Y be

$$f(x, y) = \begin{cases} e^{-x}, & \text{if } 0 \leq y \leq x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Then $E(X)$ is

- (A) 0.5
- (B) 1
- (C) 2
- (D) 6

13. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(t) = \begin{cases} \frac{\tan t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Then the value of $\lim_{x \rightarrow 0} \frac{1}{x^2} \int_{x^2}^{x^3} f(t) dt$

- (A) is equal to -1
- (B) is equal to 0
- (C) is equal to 1
- (D) does not exist

14. Let X and Y have the joint probability mass function;

$$P(X = x, Y = y) = \frac{1}{2^{y+2}(y+1)} \left(\frac{2y+1}{2y+2} \right)^x, \quad x, y = 0, 1, 2, \dots$$

Then the marginal distribution of Y is

- (A) Poisson with parameter $\lambda = \frac{1}{4}$
- (B) Poisson with parameter $\lambda = \frac{1}{2}$
- (C) Geometric with parameter $p = \frac{1}{4}$
- (D) Geometric with parameter $p = \frac{1}{2}$

15. Let X_1, X_2 and X_3 be a random sample from a $N(3, 12)$ distribution. If $\bar{X} = \frac{1}{3} \sum_{i=1}^3 X_i$ and

$S^2 = \frac{1}{2} \sum_{i=1}^3 (X_i - \bar{X})^2$ denote the sample mean and the sample variance respectively, then

$P(1.65 < \bar{X} \leq 4.35, 0.12 < S^2 \leq 55.26)$ is

- (A) 0.49
- (B) 0.50
- (C) 0.98
- (D) none of the above

16. (a) Let X_1, X_2, \dots, X_n be a random sample from an exponential distribution with the probability density function;

$$f(x; \theta) = \begin{cases} \theta e^{-\theta x}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$. Obtain the maximum likelihood estimator of $P(X > 10)$. **9 Marks**

(b) Let X_1, X_2, \dots, X_n be a random sample from a discrete distribution with the probability mass function given by

$$P(X = 0) = \frac{1-\theta}{2}; \quad P(X = 1) = \frac{1}{2}; \quad P(X = 2) = \frac{\theta}{2}, \quad 0 \leq \theta \leq 1.$$

Find the method of moments estimator for θ . **6 Marks**

17. (a) Let A be a non-singular matrix of order n ($n > 1$), with $|A| = k$. If $adj(A)$ denotes the adjoint of the matrix A , find the value of $|adj(A)|$. **6 Marks**

(b) Determine the values of a, b and c so that $(1, 0, -1)$ and $(0, 1, -1)$ are eigenvectors of the matrix,

$$\begin{bmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 3 & b & c \end{bmatrix}.$$

9 Marks

18. (a) Using Lagrange's mean value theorem, prove that

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2},$$

where $0 < \tan^{-1} a < \tan^{-1} b < \frac{\pi}{2}$.

6 Marks

(b) Find the area of the region in the first quadrant that is bounded by $y = \sqrt{x}$, $y = x - 2$ and the x -axis.

9 Marks

19. Let X and Y have the joint probability density function;

$$f(x, y) = \begin{cases} c x y e^{-(x^2 + 2y^2)}, & \text{if } x > 0, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the constant c and $P(X^2 > Y^2)$.

20. Let PQ be a line segment of length β and midpoint R . A point S is chosen at random on PQ . Let X , the distance from S to P , be a random variable having the uniform distribution on the interval $(0, \beta)$. Find the probability that PS , QS and PR form the sides of a triangle.

21. Let X_1, X_2, \dots, X_n be a random sample from a $N(\mu, 1)$ distribution. For testing $H_0: \mu = 10$ against

$H_1: \mu = 11$, the most powerful critical region is $\bar{X} \geq k$, where $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$. Find k in terms of n such

that the size of this test is 0.05.

Further determine the minimum sample size n so that the power of this test is at least 0.95.

22. Consider the sequence $\{s_n\}$, $n \geq 1$, of positive real numbers satisfying the recurrence relation

$$s_{n-1} + s_n = 2 s_{n+1} \quad \text{for all } n \geq 2.$$

(a) Show that $|s_{n+1} - s_n| = \frac{1}{2^{n-1}} |s_2 - s_1|$ for all $n \geq 1$.

(b) Prove that $\{s_n\}$ is a convergent sequence.

23. The cumulative distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{5} (1 + x^3), & \text{if } 0 \leq x < 1, \\ \frac{1}{5} [3 + (x-1)^2], & \text{if } 1 \leq x < 2, \\ 1, & \text{if } x \geq 2. \end{cases}$$

Find $P(0 < X < 2)$, $P(0 \leq X \leq 1)$ and $P\left(\frac{1}{2} \leq X \leq \frac{3}{2}\right)$.

24. Let A and B be two events with $P(A|B) = 0.3$ and $P(A|B^c) = 0.4$. Find $P(B|A)$ and $P(B^c|A^c)$ in terms of $P(B)$. If $\frac{1}{4} \leq P(B|A) \leq \frac{1}{3}$ and $\frac{1}{4} \leq P(B^c|A^c) \leq \frac{9}{16}$, then determine the value of $P(B)$.

Optional Section B

25. Solve the initial value problem

$$y' - y + y^2 (x^2 + 2x + 1) = 0, \quad y(0) = 1.$$

26. Let $y_1(x)$ and $y_2(x)$ be the linearly independent solutions of

$$x y'' + 2 y' + x e^x y = 0.$$

If $W(x) = y_1(x) y_2'(x) - y_2(x) y_1'(x)$ with $W(1) = 2$, find $W(5)$.

27. (a) Evaluate $\int_0^1 \int_y^1 x^2 e^{xy} dx dy$.

9 Marks

(b) Evaluate $\iiint_W z dx dy dz$, where W is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = 1$ and the cylinder $x^2 + y^2 = 1$ with $x \geq 0$, $y \geq 0$.

6 Marks

28. A linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is given by

$$T(x, y, z) = (3x + 11y + 5z, x + 8y + 3z).$$

Determine the matrix representation of this transformation relative to the ordered bases $\{(1, 0, 1), (0, 1, 1), (1, 0, 0)\}$, $\{(1, 1), (1, 0)\}$. Also find the dimension of the null space of this transformation.

29. (a) Let $f(x, y) = \begin{cases} \frac{x^2 + y^2}{x + y}, & \text{if } x + y \neq 0, \\ 0, & \text{if } x + y = 0. \end{cases}$

Determine if f is continuous at the point $(0, 0)$.

6 Marks

(b) Find the minimum distance from the point $(1, 2, 0)$ to the cone $z^2 = x^2 + y^2$.

9 Marks

Optional Section C

30. Let X_1, X_2, \dots, X_n be a random sample from an exponential distribution with the probability density function;

$$f(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$. Derive the Cramér-Rao lower bound for the variance of any unbiased estimator of θ .

Hence, prove that $T = \frac{1}{n} \sum_{i=1}^n X_i$ is the uniformly minimum variance unbiased estimator of θ .

31. Let X_1, X_2, \dots be a sequence of independently and identically distributed random variables with the probability density function;

$$f(x) = \begin{cases} \frac{1}{2} x^2 e^{-x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\lim_{n \rightarrow \infty} P(X_1 + \dots + X_n \geq 3(n - \sqrt{n})) \geq \frac{1}{2}$.

32. Let X_1, X_2, \dots, X_n be a random sample from a $N(\mu, \sigma^2)$ distribution, where both μ and σ^2 are unknown. Find the value of b that minimizes the mean squared error of the estimator

$$T_b = \frac{b}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \text{ for estimating } \sigma^2, \text{ where } \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i.$$

33. Let X_1, X_2, \dots, X_5 be a random sample from a $N(2, \sigma^2)$ distribution, where σ^2 is unknown. Derive the most powerful test of size $\alpha = 0.05$ for testing $H_0: \sigma^2 = 4$ against $H_1: \sigma^2 = 1$.

34. Let X_1, X_2, \dots, X_n be a random sample from a continuous distribution with the probability density function;

$$f(x; \lambda) = \begin{cases} \frac{2x}{\lambda} e^{-\frac{x^2}{\lambda}}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda > 0$. Find the maximum likelihood estimator of λ and show that it is sufficient and an unbiased estimator of λ .