- $\mathbb{C}$ denotes the set of complex numbers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{Q}$ denotes the set of rational numbers.
- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{N}$ denotes the set of positive integers.

Q 1. Let $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ be a continuous function and $\{x_n\}$ a sequence of real numbers converging to x. Define

$$\begin{aligned} g_n(y) &= f(x_n, y),\ 0 \leq y \leq 1, \\ g(y) &= f(x, y),\ 0 \leq y \leq 1. \end{aligned}$$

Show that g_n converges to g uniformly on $[0,1]$.

Q 2. Let f be a real valued continuous function on $[-1,1]$ such that $f(x) = f(-x)$ for all $x \in [-1,1]$. Show that for every $\epsilon > 0$ there is a polynomial $p(x)$ with rational coefficients such that for every $x \in [-1,1]$,

$$|f(x) - p(x^2)| < \epsilon.$$

Q 3. Show that every bijection $f : \mathbb{R} \to [0,\infty)$ has infinitely many points of discontinuity.

Q 4. (a) Let f be an entire function such that $\lim_{|z|\to\infty} |f(z)| = \infty$. Prove that f is a polynomial.

(b) Let f be an entire function which is not a polynomial. Then prove that the image of the set $\{z \in \mathbb{C} : |z| > 1\}$ under f is dense in $\mathbb{C}$.

Q 5. Let $P(z)$ be a monic polynomial with complex coefficients with all roots distinct and in $\{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$.

(a) Prove that the sum of all the residues of $\dfrac{P'}{P}$ is the degree of the polynomial P.

(b) Prove that P' has no real root.

Q 6. Let A be a Lebesgue measurable subset of $\mathbb{R}$ and $\lambda(A) = 1$, where λ is the Lebesgue measure on $\mathbb{R}$. Prove that there exists a Lebesgue measurable subset B of A such that $\lambda(B) = 1/2$.

Q 7. Let $p \geq 1$ and f be a Lebesgue measurable function on $\mathbb{R}$ such that $\int_{\mathbb{R}} |f(x)|^p dx < \infty$. Show that,

$$\int_{\mathbb{R}} |f(x)|^p dx = \int_0^\infty p t^{p-1} \lambda(\{x : |f(x)| > t\}) dt,$$

where λ denotes the Lebesgue measure.

Q 8. Let $\Omega \subset \mathbb{R}^n$ be an open set and $K \subset \Omega$ compact. Prove that there exists an $r > 0$ such that the set

$$\{y \in \mathbb{R}^n : \|y - x\| \leq r \text{ for some } x \in K\}$$

is a compact subset of Ω.

Q 9. Let $\sim$ be an equivalence relation on a topological space X such that each equivalence class is connected and the quotient space $X/\sim$ is connected. Show that X is connected.

Q 10. Let C be a curve in $\mathbb{R}^2$ passing through $(3, 5)$ and $L(x, y)$ denote the segment of the tangent line to C at (x, y) lying in the first quadrant. Assuming that each point (x, y) of C in the first quadrant is the midpoint of $L(x, y)$, find the curve.