JRF Mathematics Examination
RM I

Solve any six questions.

1. Suppose that X is a Hausdorff space and $\alpha : [0,1] \to X$ is a continuous function. If α is one-one, then prove that the image of α is homeomorphic to $[0,1]$. Give an example where α is not one-one but the image of α is homeomorphic to $[0,1]$.

2. Prove that the following collection of subsets defines a topology on the set of natural numbers $\mathbb{N}$:

$$\emptyset, \mathbb{N}, U_n = \{1, \ldots, n\}, \quad n \in \mathbb{N}.$$

   Is $\mathbb{N}$ compact in this topology? What are the continuous functions from this space to the space $\mathbb{R}$ of real numbers with standard topology?

3. Let $\{a_n\}$ be a sequence of non-zero real numbers. Show that it has a subsequence $\{a_{n_k}\}$ such that $\lim \frac{a_{n_{k+1}}}{a_{n_k}}$ exists and belongs to $\{0, 1, \infty\}$.

4. Let D denote the open ball of unit radius about origin in the complex plane $\mathbb{C}$. Let f be a continuous complex-valued function on its closure $\overline{D}$ which is analytic on D. If $f(e^{it}) = 0$ for $0 < t < \dfrac{\pi}{2}$, show that $f(z) = 0$ for all z.

5. Let U be an open connected set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ be a continuous map such that $z \mapsto f(z)^n$ is analytic for some positive integer n. Prove that f is analytic.

6. Let C be a closed convex set in $\mathbb{R}^2$. For any $x \in C$, define

$$C_x = \{y \in \mathbb{R}^2 \mid x + ty \in C, \quad \forall\, t \geq 0\}.$$

Prove that for any two points $x, x' \in C$, we have $C_x = C_{x'}$.

7. Let $U = B(c, R)$ denote the open ball of radius R around $c \in \mathbb{C}$. Let f be an analytic function on U such that there exists $M > 0$ with $|f'(z)| \leq M$ for all $z \in U$. Prove that

$$|f(z_1) - f(z_2)| \leq M|z_1 - z_2| \quad \forall z_1, z_2 \in U.$$

8. Let a, b be two real numbers. If the function $x(t)$ is a solution of

$$\frac{d^2x}{dt^2} - 2a\frac{dx}{dt} + bx = 0$$

with $x(0) = x(1) = 0$, then show that $x(n) = 0$ for all $n \in \mathbb{N}$.
(Hint: Show that there exists a constant c such that $x(t+1) = cx(t)$ for all t.)

9. Let μ be a measure on the Borel σ-field of $\mathbb{R}$ such that $\mu(\mathbb{R}) = 1$. Recall that the support of μ is the largest closed set C such that for all open sets U with $U \cap C \neq \emptyset$, we have $\mu(U) > 0$. Assume that every continuous real-valued function on $\mathbb{R}$ is integrable with respect to μ. Prove that the support of μ is compact.

10. Find all pairs $(x_0, y_0) \in \mathbb{R}^2$ such that the following initial value problem has a unique solution in a neighbourhood of (x_0, y_0):

$$y' = y^{\frac{1}{3}} + x; \; y(x_0) = y_0.$$

11. Let (X, d) be a complete metric space. Let $f : X \to X$ be a function such that for all distinct $x, y \in X$,

$$d(f^k(x), f^k(y)) < c \cdot d(x, y),$$

for some real number $c < 1$ and an integer $k > 1$. Show that f has a unique fixed point.
(Here f^r denotes the r-fold composition of f with itself.)