JEE-Main-27-07-2022-Shift-2 (Memory Based)

MATHEMATICS

Question: Let $A=\begin{bmatrix} 4 & -2 \\ \alpha & \beta \end{bmatrix}$. If $A^2+\gamma A+18I=0$, then $\det(A)$ equals:

Options:
(a) -18
(b) 18
(c) -50
(d) 50

Answer: (b)

Solution:

Characteristic equation of matrix:

$$\begin{bmatrix} 4-\lambda & -2 \\ \alpha & \beta-\lambda \end{bmatrix}=0$$

$$\Rightarrow 4\beta+\lambda^2-(\beta+4)\lambda+2\alpha=0$$

$$\therefore A^2-(\beta+4)A+2\alpha I=0$$

$$\Rightarrow \gamma=0-\beta+4 \ \& \ 2\alpha+4\beta=18$$

$$\det(A)=4\beta+2\alpha=18$$

Question: The area of region enclosed by $y\le 4x^2, x^2\le 9y, y\le 4$ is equal to:

Options:

(a) $\frac{40}{3}$

(b) $\frac{56}{3}$

(c) $\frac{112}{3}$

(d) $\frac{80}{3}$

Answer: (d)

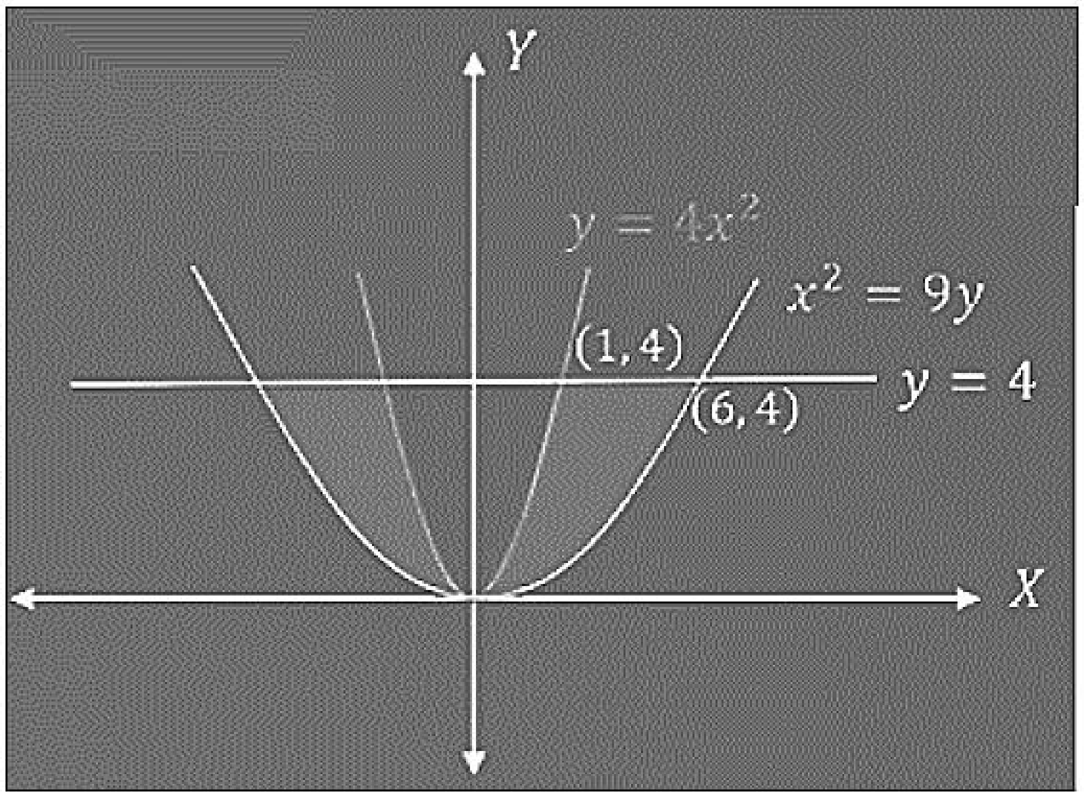
Solution:

Required Area $=2\int_0^4\left(3\sqrt{y}-\frac{\sqrt{y}}{2}\right)dy$

$$=2\cdot\frac{5}{2}\int_0^4\sqrt{y}dy$$

$$=5\left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^4$$

$= \frac{10}{3}(4)^{\frac{3}{2}} = \frac{80}{3}$



Question: If the length of the latus rectum of a parabola whose focus is (a, a) and tangent at its vertex is $x + y = a$, is 16. Then $|a|$ is equal to:

Options:

(a) $2\sqrt{3}$

(b) $2\sqrt{2}$

(c) $4\sqrt{2}$

(d) 4

Answer: (c)

Solution:

Length of perpendicular from focus to tangent at vertex:

$l = \left|\frac{a}{\sqrt{2}}\right|$

So length of latus rectum will be, $4l = 16$

$\Rightarrow 2\sqrt{2}|a| = 16$

$\Rightarrow |a| = 4\sqrt{2}$

Question: Let $f(x) = \frac{\left(729p(1+x)^{\frac{1}{7}}\right) - 3}{\left(729(1+qx)^{\frac{1}{3}}\right) - 9}$, and $f(x)$ is continuous at $x = 0$, then:

Options:

(a) $21qf(0) - p = 0$

(b) $21q^2 f(0) - p^3 = 0$

(c) $21p^2 f(0) - q^3 = 0$

(d) $p^2 f(0) - 7q^2 = 0$

Answer: (a)

Solution:

$\lim_{x\to 0} f(x)$ exists if numerator of $f(x)$ is zero at $x=0$.

Clearly, $p = 3$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{3\left[(x+1)^{\frac{1}{7}} - 1\right]}{9\left[(1+qx)^{\frac{1}{3}} - 1\right]}$$

$$= \frac{1}{3}\left(\frac{\frac{1}{7}}{\frac{q}{3}}\right) = \frac{1}{7q} = f(0)$$

So, $21qf(0) = 3 = p$

$\Rightarrow 21qf(0) - p = 0$

Question: Let $f(x) = \min\{[x],[x-1],[x-2],...,[x-10]\}$ where $[\ ]$ denotes greatest integer function. Then $\int_0^{10} \left(f(x) + |f(x)| + f^2(x)\right) dx$ is equal to:

Options:

(a) 55

(b) 385

(c) 5050

(d) 270

Answer: (b)

Solution:

Clearly $f(x) = [x-10]$

Here $f(x) \le 0\ \forall\ x \in (0,10)$

So, $\int_0^{10} \left(f(x) + |f(x)|\right) dx = 0$

Now, $\int_0^{10} f^2(x)\, dx = \int_0^{10} \left([x]-10\right)^2 dx$

$$= \int_0^1 100\, dx + \int_1^2 81\, dx + \int_2^3 64\, dx + ... + \int_9^{10} 1\, dx$$

$= \left(1^2 + 2^2 + 3^2 + ... + 10^2\right)$

$= \dfrac{10 \times 11 \times 21}{6}$

$= 385$

Question: The value of $\int_0^2 \left(\left|2x^3 - 3x\right| + \left[x - \frac{1}{2}\right] \right) dx$, where $[.]$ is greatest integer function is:

Options:

(a) $\dfrac{7}{6}$

(b) $\dfrac{19}{12}$

(c) $\dfrac{17}{4}$

(d) $\dfrac{3}{2}$

Answer: (c)

Solution:

Given, $\int_0^2 \left(\left|2x^3 - 3x\right| + \left[x - \frac{1}{2}\right] \right) dx$

$\int_0^2 \left|2x^3 - 3x\right| dx + \int_0^2 \left[x - \frac{1}{2}\right] dx$

$= \int_0^{\sqrt{\frac{3}{2}}} \left(3x - 2x^3\right) dx + \int_{\sqrt{\frac{3}{2}}}^2 \left(2x^3 - 3x\right) dx + \int_0^{\frac{1}{2}} \left[x - \frac{1}{2}\right] dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \left[x - \frac{1}{2}\right] dx + \int_{\frac{3}{2}}^2 \left[x - \frac{1}{2}\right] dx$

$= \left[\dfrac{3x^2 - x^4}{2}\right]_0^{\sqrt{\frac{3}{2}}} + \left[\dfrac{x^4 - 3x^2}{2}\right]_{\sqrt{\frac{3}{2}}}^2 + \left(-\dfrac{1}{2}\right) + 0 + \left(\dfrac{1}{2}\right)$

$= \dfrac{9}{8} + 2 + \dfrac{9}{8}$

$= \dfrac{17}{4}$

Question: If the line of intersection of the planes $ax + by = 3$ and $ax + by + cz = 0$ makes an angle $30°$ with the plane $y - z + 2 = 0$, then the direction cosines of line are:

Options:

(a) $\dfrac{1}{\sqrt{2}}, 0, \dfrac{1}{2}$

(b) $\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}, 0$

(c) $\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0$

(d) $\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0$

Answer: (b)

Solution:

Direction ratios of line of intersection $(b, -a, 0)$

As angle between this line and $y - z + 2 = 0$ is $30°$

$\therefore \sin\theta = \left|\frac{a}{\sqrt{a^2 + b^2} \cdot \sqrt{2}}\right|$

$\Rightarrow a^2 = b^2$

$\therefore$ Possible combination is $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0$

Question: If $A = \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \alpha+\gamma & \alpha+\beta \end{bmatrix}$ and $\frac{\left|adj\left(adj\left(adj\left(adj\left(A\right)\right)\right)\right)\right|}{(\alpha-\beta)^{16}(\beta-\gamma)^{16}(\gamma-\alpha)^{16}} = 2^{32} \cdot 3^{16}$, where α, β, γ are distinct natural number, then number of triplets of (α, β, γ) is _____.

Answer: 55.00

Solution:

$$A = \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \alpha+\gamma & \alpha+\beta \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\Rightarrow |A| = (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$= (\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

$\because \left|adj\left(adj\left(adj\left(adj\left(A\right)\right)\right)\right)\right| = |A|^{(2)^4} = |A|^{16}$

$|A| = (\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

Clearly $(\alpha+\beta+\gamma)^{16} = 2^{32} \cdot 3^{16}$

$\Rightarrow (\alpha+\beta+\gamma) = 12$

Number of positive integral solutions $= {}^{11}C_2 = 55$

Question: $\frac{\left(2^3-1^3\right)}{\left(1\times 7\right)}+\frac{\left\{\left(4^3-3^3\right)+\left(2^3-1^3\right)\right\}}{\left(2\times 11\right)}+\frac{\left\{\left(6^3-5^3\right)+\left(4^3-3^3\right)+\left(2^3-1^3\right)\right\}}{\left(3\times 15\right)}+\ldots$ upto 15 terms

Answer: 120.00

Solution:

$\frac{2^3-1^3}{1\times 7}+\frac{4^3-3^3+2^3-1^3}{2\times 11}+\ldots.$

$=1+2+3+\ldots.$

$=\left(\frac{15\times 16}{2}\right)$

$=120$

Question: Domain of $f\left(x\right)=\sin^{-1}\left[2x^2-3\right]+\log_2\left(\log_{\frac{1}{2}}\left(x^2-5x+5\right)\right)$

Answer: $1,\frac{5-\sqrt{5}}{2}$

Solution:

$-1\le\left[2x^2-3\right]\le 1$

$-1\le\left(2x^2-3\right)<2$

$2\le 2x^2<5$

$1\le x^2<\frac{5}{2}\qquad \ldots.(1)$

$\log_{\frac{1}{2}}\left(x^2-5x+5\right)>0$

$0<x^2-5x+5<1$

$\Rightarrow x^2-5x+5=0$ and $x^2-5x+4<0$

$x\in\left(-\infty,\frac{5-\sqrt{5}}{2}\right)\cup\left(\frac{5+\sqrt{5}}{2},\infty\right)\qquad \ldots.(2)$

and $x\in\left(1,4\right)$

Taking intersection of (1) and (2)

$x\in\left(1,\frac{5-\sqrt{5}}{2}\right)$

Question: Let n^{th} term of any sequence is given by $T_n=\frac{-1^3+2^3-3^3+4^3+\ldots+\left(2n\right)^3}{n\left(4n+3\right)}$, then $\sum_{n-1}^{15}T_n$ is equal to ____.

Answer: 120.00

Solution:

$$T_n = \frac{2\left[2^3 + 4^3 + \ldots + \left(2n^3\right)\right] - \left[1^3 + 2^3 + 3^3 + \ldots + \left(2n\right)^3\right]}{n\left(4n+3\right)}$$

$$T_n = \frac{16\left(\frac{n\left(n+1\right)}{2}\right)^2 - \left(\frac{2n\left(2n+1\right)}{2}\right)^2}{n\left(4n+2\right)}$$

$$= \frac{n^2\left(4n+3\right)}{n\left(4n+3\right)} = n$$

$$\therefore \sum_{n=1}^{15} T_n = \frac{15\times16}{2} = 120$$