B

2007 – MA

Test Paper Code : MA

Time : 3 Hours Maximum Marks : 300

INSTRUCTIONS

1. The question-cum-answer book has **40** pages and has **29** questions. Please ensure that the copy of the question-cum-answer book you have received contains all the questions.
2. Write your **Roll Number, Name** and **the Name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on Page No. **7**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used :
   (a) For each objective question, you will be awarded **6 (six)** marks if you have written only the correct answer.
   (b) In case you have not written any answer for a question, you will be awarded **0 (zero)** mark for that question.
   (c) In all other cases, you will be awarded **–2 (minus two)** marks for the question.
   (d) Negative marks for objective part will be carried over to total marks.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing later only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Logarithmic Tables / Calculator of any kind / cellular phone / pager / electronic gadgets are not allowed.**
11. The question-cum-answer book must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this book.
12. Refer to notations used on the reverse.

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READ THE INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

ROLL NUMBER						

Name :

Test Centre :

Do not write your Roll Number or Name anywhere else in this question-cum-answer book.

I have read all the instructions and shall abide by them.

....

Signature of the Candidate

I have verified the information filled by the Candidate above.

....

Signature of the Invigilator

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NOTATIONS USED

R : The set of all real numbers

Z : The set of all integers

N : The set of all natural numbers 1, 2, 3, ...

$i = \sqrt{-1}$

DO NOT WRITE ON THIS PAGE

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IMPORTANT NOTE FOR CANDIDATES

- **Attempt ALL the 29 questions.**
- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-29 (subjective questions) carry *fifteen* marks each.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 7 only.**

1. Let $A(t)$ denote the area bounded by the curve $y=e^{-|x|}$, the x-axis and the straight lines $x=-t$ and $x=t$. Then $\lim\limits_{t\to\infty} A(t)$ is equal to

(A) 2
(B) 1
(C) 1/2
(D) 0

2. If k is a constant such that $xy+k=e^{(x-1)^2/2}$ satisfies the differential equation $x\dfrac{dy}{dx}=(x^2-x-1)y+(x-1)$, then k is equal to

(A) 1
(B) 0
(C) –1
(D) –2

3. Which of the following functions is uniformly continuous on the domain as stated?

(A) $f(x)=x^2, \quad x\in \mathbf{R}$
(B) $f(x)=\dfrac{1}{x}, \quad x\in[1,\infty)$
(C) $f(x)=\tan x, \quad x\in(-\pi/2,\pi/2)$
(D) $f(x)=[x], \quad x\in[0,1]$
([x] is the greatest integer less than or equal to x)

Space for rough work

4. Let R be the ring of polynomials over $\mathbf{Z}_2$ and let I be the ideal of R generated by the polynomial x^3+x+1. Then the number of elements in the quotient ring R/I is

(A) 2
(B) 4
(C) 8
(D) 16

5. Which of the following sets is a basis for the subspace

$$W = \left\{ \begin{bmatrix} x & y \\ 0 & t \end{bmatrix} : x + 2y + t = 0,\ y + t = 0 \right\}$$

of the vector space of all real 2×2 matrices?

(A) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

(B) $\left\{ \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\}$

(C) $\left\{ \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \right\}$

(D) $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\}$

6. Let G be an Abelian group of order 10. Let $S = \{g \in G \ : \ g^{-1} = g\}$. Then the number of non-identity elements in S is

(A) 5
(B) 2
(C) 1
(D) 0

Space for rough work

7. Let (a_n) be an increasing sequence of positive real numbers such that the series $\sum_{k=1}^{\infty} a_k$ is divergent. Let $s_n = \sum_{k=1}^{n} a_k$ for $n = 1,2,\ldots$ and $t_n = \sum_{k=2}^{n} \frac{a_k}{s_{k-1}s_k}$ for $n = 2,3,\ldots$. Then $\lim_{n\to\infty} t_n$ is equal to

(A) $1/a_1$

(B) 0

(C) $1/(a_1 + a_2)$

(D) $a_1 + a_2$

8. For every function $f : [0,1] \to \mathbf{R}$ which is twice differentiable and satisfies $f'(x) \geq 1$ for all $x \in [0,1]$, we must have

(A) $f''(x) \geq 0$ for all $x \in [0,1]$

(B) $f(x) \geq x$ for all $x \in [0,1]$

(C) $f(x_2) - x_2 \leq f(x_1) - x_1$ for all $x_1, x_2 \in [0,1]$ with $x_2 \geq x_1$

(D) $f(x_2) - x_2 \geq f(x_1) - x_1$ for all $x_1, x_2 \in [0,1]$ with $x_2 \geq x_1$

9. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Which of the following statements holds regarding the continuity and the existence of partial derivatives of f at $(0,0)$?

(A) Both partial derivatives of f exist at $(0,0)$ and f is continuous at $(0,0)$

(B) Both partial derivatives of f exist at $(0,0)$ and f is NOT continuous at $(0,0)$

(C) One partial derivative of f does NOT exist at $(0,0)$ and f is continuous at $(0,0)$

(D) One partial derivative of f does NOT exist at $(0,0)$ and f is NOT continuous at $(0,0)$

Space for rough work

10. Suppose (c_n) is a sequence of real numbers such that $\lim_{n\to\infty} |c_n|^{1/n}$ exists and is non-zero. If the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is equal to r, then the radius of convergence of the power series $\sum_{n=1}^{\infty} n^2 c_n x^n$ is

(A) less than r
(B) greater than r
(C) equal to r
(D) equal to 0

11. The rank of the matrix $\begin{bmatrix} 1 & 4 & 8 \\ 2 & 10 & 22 \\ 0 & 4 & 12 \end{bmatrix}$ is

(A) 3
(B) 2
(C) 1
(D) 0

12. Let $f: \mathbf{R} \to \mathbf{R}$ be a continuous function. If $\int_0^x f(2t)\,dt = \frac{x}{\pi}\sin(\pi x)$ for all $x \in \mathbf{R}$, then $f(2)$ is equal to

(A) –1
(B) 0
(C) 1
(D) 2

Space for rough work

13. Let $\vec{u} = (a e^x \sin y - 4x)\hat{i} + (2y + e^x \cos y)\hat{j} + az\hat{k}$, where a is a constant. If the line integral $\oint_C \vec{u} \cdot d\vec{r}$ over every closed curve C is zero, then a is equal to

(A) -2

(B) -1

(C) 0

(D) 1

14. One of the integrating factors of the differential equation $(y^2 - 3xy)dx + (x^2 - xy)dy = 0$ is

(A) $1/(x^2 y^2)$

(B) $1/(x^2 y)$

(C) $1/(x y^2)$

(D) $1/(x y)$

15. Let C denote the boundary of the semi-circular disk $D = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1,\ y \geq 0\}$ and let $\varphi(x,y) = x^2 + y$ for $(x,y) \in D$. If $\hat{n}$ is the outward unit normal to C, then the integral $\oint_C (\vec{\nabla}\varphi) \cdot \hat{n}\, ds$, evaluated counter-clockwise over C, is equal to

(A) 0

(B) $\pi - 2$

(C) π

(D) $\pi + 2$

Space for rough work

Answer Table for Objective Questions

Write the Code of your chosen answer only in the 'Answer' column against each Question No. Do not write anything else on this page:

Question No.	Answer	Do not write in this column
01		
02		
03		
04		
05		
06		
07		
08		
09		
10		
11		
12		
13		
14		
15		

FOR EVALUATION ONLY

No. of Correct Answers		Marks	(+)
No. of Incorrect Answers		Marks	(−)
Total Marks in Question Nos. 1-15			( )

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16. (a) Let $M = \begin{bmatrix} 1+i & 2i & i+3 \\ 0 & 1-i & 3i \\ 0 & 0 & i \end{bmatrix}$. Determine the eigenvalues of the matrix $B = M^2 - 2M + I$. (9)

(b) Let N be a square matrix of order 2. If the determinant of N is equal to 9 and the sum of the diagonal entries of N is equal to 10, then determine the eigenvalues of N. (6)

17. (a) Using the method of variation of parameters, solve the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2,$$

given that x and $\frac{1}{x}$ are two solutions of the corresponding homogeneous equation.

(9)

(b) Find the real number α such that the differential equation

$$\frac{d^2y}{dx^2} + 2(\alpha - 1)(\alpha - 3)\frac{dy}{dx} + (\alpha - 2)y = 0$$

has a solution $y(x) = a\cos(\beta x) + b\sin(\beta x)$ for some non-zero real numbers a, b, β. (6)

18. (a) Let a, b, c be non-zero real numbers such that $(a-b)^2 = 4ac$. Solve the differential equation $a\left(x+\sqrt{2}\right)^2 \dfrac{d^2y}{dx^2} + b\left(x+\sqrt{2}\right)\dfrac{dy}{dx} + cy = 0$. (9)

(b) Solve the differential equation

$dx + (e^{y\sin y} - x)(y\cos y + \sin y)dy = 0$. (6)

19. Let $f(x, y)=x(x-2y^2)$ for $(x, y)\in \mathbf{R}^2$. Show that f has a local minimum at $(0, 0)$ on every straight line through $(0, 0)$. Is $(0, 0)$ a critical point of f? Find the discriminant of f at $(0, 0)$. Does f have a local minimum at $(0, 0)$? Justify your answers. (15)

20. (a) Find the finite volume enclosed by the paraboloids $z=2-x^2-y^2$ and $z=x^2+y^2$. (9)

(b) Let $f:[0,3]\to\mathbf{R}$ be a continuous function with $\int_0^3 f(x)dx=3$. Evaluate

$$\int_0^3 [x f(x)+\int_0^x f(t)dt]\,dx.$$ (6)

21. (a) Let S be the surface $\{(x,y,z)\in\mathbf{R}^3 : x^2+y^2+2z=2, z\geq 0\}$, and let $\hat{n}$ be the outward unit normal to S. If $\vec{F}=y\,\hat{\mathrm{i}}+xz\,\hat{\mathrm{j}}+(x^2+y^2)\,\hat{\mathrm{k}}$, then evaluate the integral $\iint_S \vec{F}\cdot\hat{n}\,dS$. (9)

(b) Let $\vec{r}=x\,\hat{\mathrm{i}}+y\,\hat{\mathrm{j}}+z\,\hat{\mathrm{k}}$ and $r=|\vec{r}|$. If a scalar field φ and a vector field $\vec{u}$ satisfy $\vec{\nabla}\varphi=\vec{\nabla}\times\vec{u}+f(r)\,\vec{r}$, where f is an arbitrary differentiable function, then show that $\nabla^2\varphi=r\,f'(r)+3f(r)$. (6)

22. (a) Let D be the region bounded by the concentric spheres $S_1 : x^2 + y^2 + z^2 = a^2$ and $S_2 : x^2 + y^2 + z^2 = b^2$, where $a < b$. Let $\hat{n}$ be the unit normal to S_1 directed away from the origin. If $\nabla^2 \varphi = 0$ in D and $\varphi = 0$ on S_2, then show that

$$\iiint_D |\vec{\nabla}\varphi|^2 \, dV + \iint_{S_1} \varphi \, (\vec{\nabla}\varphi) \cdot \hat{n} \, dS = 0 .$$ (9)

(b) Let C be the curve in $\mathbf{R}^3$ given by $x^2 + y^2 = a^2$, $z = 0$ traced counter-clockwise, and let $\vec{F} = x^2 y^3 \, \hat{\mathrm{i}} + \hat{\mathrm{j}} + z \, \hat{\mathrm{k}}$. Using Stokes' theorem, evaluate $\oint_C \vec{F} \cdot d\vec{r}$. (6)

23. Let V be the subspace of $\mathbf{R}^4$ spanned by the vectors (1,0,1,2), (2,1,3,4) and (3,1,4,6). Let $T: V \to \mathbf{R}^2$ be a linear transformation given by $T(x,y,z,t) = (x-y, z-t)$ for all $(x,y,z,t) \in V$. Find a basis for the null space of T and also a basis for the range space of T. (15)

24. (a) Compute the double integral $\iint\limits_D (x+2y)\,dx\,dy$, where D is the region in the xy-plane bounded by the straight lines $y=x+3$, $y=x-3$, $y=-2x+4$ and $y=-2x-2$. (9)

(b) Evaluate $\int_0^{\pi/2}\left[\int_{\pi/2}^{\pi}\frac{\sin x}{x}dx\right]dy+\int_{\pi/2}^{\pi}\left[\int_y^{\pi}\frac{\sin x}{x}dx\right]dy$. (6)

25. (a) Does the series $\sum_{k=1}^{\infty} \frac{(-1)^k k + x^k}{k^2}$ converge uniformly for $x \in [-1, 1]$? Justify. (9)

(b) Suppose (f_n) is a sequence of real-valued functions defined on $\mathbf{R}$ and f is a real-valued function defined on $\mathbf{R}$ such that $|f_n(x) - f(x)| \le |a_n|$ for all $n \in \mathbf{N}$ and $a_n \to 0$ as $n \to \infty$. Must the sequence (f_n) be uniformly convergent on $\mathbf{R}$? Justify. (6)

26. (a) Suppose f is a real-valued thrice differentiable function defined on $\mathbf{R}$ such that $f'''(x)>0$ for all $x\in\mathbf{R}$. Using Taylor's formula, show that

$f(x_2)-f(x_1)>(x_2-x_1)f'\left(\frac{x_1+x_2}{2}\right)$ for all x_1 and x_2 in $\mathbf{R}$ with $x_2>x_1$. (9)

(b) Let (a_n) and (b_n) be sequences of real numbers such that $a_n \le a_{n+1} \le b_{n+1} \le b_n$ for all $n \in \mathbf{N}$. Must there exist a real number x such that $a_n \le x \le b_n$ for all $n \in \mathbf{N}$? Justify your answer. (6)

27. Let G be the group of all 2×2 matrices with real entries with respect to matrix multiplication. Let G_1 be the smallest subgroup of G containing $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and G_2 be the smallest subgroup of G containing $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Determine all elements of G_1 and find their orders. Determine all elements of G_2 and find their orders. Does there exist a one-to-one homomorphism from G_1 onto G_2? Justify. (15)

28. (a) Let p be a prime number and let $\mathbf{Z}$ be the ring of integers. If an ideal J of $\mathbf{Z}$ contains the set $p\mathbf{Z}$ properly, then show that $J = \mathbf{Z}$. (Here $p\mathbf{Z} = \{ px : x \in \mathbf{Z} \}$.) (9)

(b) Consider the ring $R = \{ a + ib : a, b \in \mathbf{Z} \}$ with usual addition and multiplication. Find all invertible elements of R. (6)

29. (a) Suppose E is a non-empty subset of $\mathbf{R}$ which is bounded above, and let $\alpha = \sup E$. If E is closed, then show that $\alpha \in E$. If E is open, then show that $\alpha \notin E$. (9)

(b) Find all limit points of the set $E = \left\{ n + \frac{1}{2m} : n, m \in \mathbf{N} \right\}$. (6)

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2007 – MA Objective Part (Q. Nos. 1 – 15)	
Total Marks	Signature

Subjective Part					
Q. No.	Marks	Signature	Q. No.	Marks	Signature
16			23		
17			24		
18			25		
19			26		
20			27		
21			28		
22			29		
Total Marks in Subjective Part					

Total (Objective Part) :	
Total (Subjective Part) :	
Grand Total :	
Total Marks (in words) :	
Signature of Examiner(s) :	
Signature of Head Examiner(s) :	
Signature of Scrutinizer :	
Signature of Chief Scrutinizer :	
Signature of Coordinating Head Examiner :	