Exercise 7.1

Question 1:

sin 2*x*

Answer

The anti derivative of sin 2*x* is a function of *x* whose derivative is sin 2*x*.It is known that,

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x$$

$$\Rightarrow \sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$

$$\therefore \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

Therefore, the anti derivative of $\sin 2x \text{ is } -\frac{1}{2}\cos 2x$

Question 2:

Cos 3*x*

Answer

The anti derivative of cos 3*x* is a function of *x* whose derivative is cos 3*x*.

It is known that,

$$\frac{d}{dx}(\sin 3x) = 3\cos 3x$$

$$\Rightarrow \cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$$

$$\therefore \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Therefore, the anti derivative of $\cos 3x \text{ is } \frac{1}{3}\sin 3x$.

Question 3:

e^{2x}

Answer

The anti derivative of e^{2x} is the function of *x* whose derivative is e^{2x}.

It is known that,

$$\frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$$

$$\Rightarrow e^{2x} = \frac{1}{2}\frac{d}{dx}\left(e^{2x}\right)$$

$$\therefore e^{2x} = \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Therefore, the anti derivative of e^{2x} is $\frac{1}{2}e^{2x}$.

Question 4:

$(ax+b)^2$

Answer

The anti derivative of $(ax+b)^2$ is the function of *x* whose derivative is $(ax+b)^2$.

It is known that,

$$\frac{d}{dx}(ax+b)^3 = 3a(ax+b)^2$$

$$\Rightarrow (ax+b)^2 = \frac{1}{3a}\frac{d}{dx}(ax+b)^3$$

$$\therefore (ax+b)^2 = \frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

Therefore, the anti derivative of $(ax+b)^2$ is $\frac{1}{3a}(ax+b)^3$.

Question 5:

$\sin 2x - 4e^{3x}$

Answer

The anti derivative of $\left(\sin 2x - 4e^{3x}\right)$ is the function of *x* whose derivative is $\left(\sin 2x - 4e^{3x}\right)$.

It is known that,

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x-\frac{4}{3}e^{3x}\right)=\sin 2x-4e^{3x}$$

Therefore, the anti derivative of $\left(\sin 2x-4e^{3x}\right)$ is $\left(-\frac{1}{2}\cos 2x-\frac{4}{3}e^{3x}\right)$.

Question 6:

$$\int\left(4e^{3x}+1\right)dx$$

Answer

$$\int\left(4e^{3x}+1\right)dx$$

$$=4\int e^{3x}dx+\int 1dx$$

$$=4\left(\frac{e^{3x}}{3}\right)+x+C$$

$$=\frac{4}{3}e^{3x}+x+C$$

Question 7:

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

Answer

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

$$=\int\left(x^2-1\right)dx$$

$$=\int x^2dx-\int 1dx$$

$$=\frac{x^3}{3}-x+C$$

Question 8:

$$\int\left(ax^2+bx+c\right)dx$$

Answer

$$\int\left(ax^2+bx+c\right)dx$$

$$=a\int x^2dx+b\int xdx+c\int 1.dx$$

$$=a\left(\frac{x^3}{3}\right)+b\left(\frac{x^2}{2}\right)+cx+C$$

$$=\frac{ax^3}{3}+\frac{bx^2}{2}+cx+C$$

Question 9:

$$\int\left(2x^2+e^x\right)dx$$

Answer

$$\int\left(2x^2+e^x\right)dx$$

$$=2\int x^2dx+\int e^xdx$$

$$=2\left(\frac{x^3}{3}\right)+e^x+C$$

$$=\frac{2}{3}x^3+e^x+C$$

Question 10:

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx$$

Answer

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx$$

$$=\int\left(x+\frac{1}{x}-2\right)dx$$

$$=\int xdx+\int\frac{1}{x}dx-2\int 1.dx$$

$$=\frac{x^2}{2}+\log|x|-2x+C$$

Question 11:

$$\int \frac{x^3+5x^2-4}{x^2}dx$$

Answer

$$\int \frac{x^3+5x^2-4}{x^2}dx$$

$$= \int \left(x+5-4x^{-2}\right)dx$$

$$= \int x dx + 5\int 1.dx - 4\int x^{-2}dx$$

$$= \frac{x^2}{2}+5x-4\left(\frac{x^{-1}}{-1}\right)+C$$

$$= \frac{x^2}{2}+5x+\frac{4}{x}+C$$

Question 12:

$$\int \frac{x^3+3x+4}{\sqrt{x}}dx$$

Answer

$$\int \frac{x^3+3x+4}{\sqrt{x}}dx$$

$$= \int \left(x^{\frac{5}{2}}+3x^{\frac{1}{2}}+4x^{-\frac{1}{2}}\right)dx$$

$$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}}+\frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}}+\frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}}+C$$

$$= \frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8x^{\frac{1}{2}}+C$$

$$= \frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8\sqrt{x}+C$$

Question 13:

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$$

Answer

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$$

On dividing, we obtain

$$= \int \left(x^2 + 1\right) dx$$

$$= \int x^2 dx + \int 1 dx$$

$$= \frac{x^3}{3} + x + C$$

Question 14:

$$\int (1 - x)\sqrt{x}\, dx$$

Answer

$$\int (1 - x)\sqrt{x}\, dx$$

$$= \int \left(\sqrt{x} - x^{\frac{3}{2}}\right) dx$$

$$= \int x^{\frac{1}{2}} dx - \int x^{\frac{3}{2}} dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= \frac{2}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} + C$$

Question 15:

$$\int \sqrt{x}\left(3x^2 + 2x + 3\right) dx$$

Answer

$$\int \sqrt{x}\left(3x^2+2x+3\right)dx$$

$$=\int\left(3x^{\frac{5}{2}}+2x^{\frac{3}{2}}+3x^{\frac{1}{2}}\right)dx$$

$$=3\int x^{\frac{5}{2}}dx+2\int x^{\frac{3}{2}}dx+3\int x^{\frac{1}{2}}dx$$

$$=3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right)+2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)+3\frac{\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}}+C$$

$$=\frac{6}{7}x^{\frac{7}{2}}+\frac{4}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}+C$$

Question 16:

$$\int\left(2x-3\cos x+e^x\right)dx$$

Answer

$$\int\left(2x-3\cos x+e^x\right)dx$$

$$=2\int x dx-3\int\cos x dx+\int e^x dx$$

$$=\frac{2x^2}{2}-3(\sin x)+e^x+C$$

$$=x^2-3\sin x+e^x+C$$

Question 17:

$$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$$

Answer

$$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$$

$$= 2\int x^2 dx - 3\int \sin x dx + 5\int x^{\frac{1}{2}} dx$$

$$= \frac{2x^3}{3} - 3(-\cos x) + 5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{\frac{3}{2}} + C$$

Question 18:

$$\int \sec x(\sec x + \tan x)dx$$

Answer

$$\int \sec x(\sec x + \tan x)dx$$

$$= \int(\sec^2 x + \sec x \tan x)dx$$

$$= \int \sec^2 x dx + \int \sec x \tan x dx$$

$$= \tan x + \sec x + C$$

Question 19:

$$\int \frac{\sec^2 x}{\cos ec^2 x} dx$$

Answer

$$\int \frac{\sec^2 x}{\cos ec^2 x} dx$$

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} dx$$

$$= \int \frac{\sin^2 x}{\cos^2 x} dx$$

$$= \int \tan^2 x dx$$

$$= \int \left(\sec^2 x - 1\right) dx$$

$$= \int \sec^2 x dx - \int 1 dx$$

$$= \tan x - x + C$$

Question 20:

$$\int \frac{2 - 3\sin x}{\cos^2 x} dx$$

Answer

$$\int \frac{2 - 3\sin x}{\cos^2 x} dx$$

$$= \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x} \right) dx$$

$$= \int 2\sec^2 x dx - 3\int \tan x \sec x dx$$

$$= 2\tan x - 3\sec x + C$$

Question 21:

The anti derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ equals

(A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ **(B)** $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ **(D)** $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

Answer

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$$

$$= \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

Hence, the correct Answer is C.

Question 22:

If $\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$, then $f(x)$ is

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$ **(B)** $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$ **(D)** $x^3 + \frac{1}{x^4} - \frac{129}{8}$

Answer

It is given that,

$$\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}$$

∴Anti derivative of $4x^3 - \frac{3}{x^4} = f(x)$

$$\therefore f(x) = \int 4x^3 - \frac{3}{x^4}\,dx$$

$$f(x) = 4\int x^3 dx - 3\int (x^{-4})dx$$

$$f(x) = 4\left(\frac{x^4}{4}\right) - 3\left(\frac{x^{-3}}{-3}\right) + C$$

$$\therefore f(x) = x^4 + \frac{1}{x^3} + C$$

Also,

$$f(2) = 0$$

$$\therefore f(2) = (2)^4 + \frac{1}{(2)^3} + C = 0$$

$$\Rightarrow 16 + \frac{1}{8} + C = 0$$

$$\Rightarrow C = -\left(16 + \frac{1}{8}\right)$$

$$\Rightarrow C = \frac{-129}{8}$$

$$\therefore f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$$

Hence, the correct Answer is A.

Exercise 7.2

Question 1:

$$\frac{2x}{1+x^2}$$

Answer

Let $1+x^2 = t$

$\therefore 2x\, dx = dt$

$$\Rightarrow \int \frac{2x}{1+x^2}\,dx = \int \frac{1}{t}\,dt$$

$$= \log|t| + C$$

$$= \log\left|1+x^2\right| + C$$

$$= \log\left(1+x^2\right) + C$$

Question 2:

$$\frac{(\log x)^2}{x}$$

Answer

Let $\log|x| = t$

$$\therefore \frac{1}{x}\,dx = dt$$

$$\Rightarrow \int \frac{(\log|x|)^2}{x} dx = \int t^2 dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{(\log|x|)^3}{3} + C$$

Question 3:

$$\frac{1}{x + x \log x}$$

Answer

$$\frac{1}{x + x\log x} = \frac{1}{x(1+\log x)}$$

Let 1 + log x = t

$$\therefore \frac{1}{x} dx = dt$$

$$\Rightarrow \int \frac{1}{x(1+\log x)} dx = \int \frac{1}{t} dt$$

$$= \log|t| + C$$

$$= \log|1 + \log x| + C$$

Question 4:

sin x · sin (cos x)

Answer

$\sin x \cdot \sin(\cos x)$

Let $\cos x = t$

$\therefore -\sin x\, dx = dt$

$$\Rightarrow \int \sin x \cdot \sin(\cos x)\, dx = -\int \sin t\, dt$$
$$= -[-\cos t] + C$$
$$= \cos t + C$$
$$= \cos(\cos x) + C$$

Question 5:

$\sin(ax+b)\cos(ax+b)$

Answer

$$\sin(ax+b)\cos(ax+b) = \frac{2\sin(ax+b)\cos(ax+b)}{2} = \frac{\sin 2(ax+b)}{2}$$

Let $2(ax+b) = t$

$\therefore 2a\,dx = dt$

$$\Rightarrow \int \frac{\sin 2(ax+b)}{2}dx = \frac{1}{2}\int \frac{\sin t\ dt}{2a}$$

$$= \frac{1}{4a}\left[-\cos t\right] + C$$

$$= \frac{-1}{4a}\cos 2(ax+b) + C$$

Question 6:

$\sqrt{ax+b}$

Answer

Let $ax + b = t$

$\Rightarrow adx = dt$

$$\therefore dx = \frac{1}{a}dt$$

$$\Rightarrow \int (ax+b)^{\frac{1}{2}}\,dx = \frac{1}{a}\int t^{\frac{1}{2}}dt$$

$$= \frac{1}{a}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3a}(ax+b)^{\frac{3}{2}} + C$$

Question 7:

$x\sqrt{x+2}$

Answer

Let $(x+2) = t$

$\therefore dx = dt$

$$\Rightarrow \int x\sqrt{x+2}\,dx = \int (t-2)\sqrt{t}\,dt$$
$$= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right) dt$$
$$= \int t^{\frac{3}{2}} dt - 2\int t^{\frac{1}{2}} dt$$
$$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + C$$
$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$$

Question 8:

$x\sqrt{1+2x^2}$

Answer

Let $1 + 2x^2 = t$

$\therefore 4x\,dx = dt$

$$\Rightarrow \int x\sqrt{1+2x^2}\,dx = \int \frac{\sqrt{t}\,dt}{4}$$

$$= \frac{1}{4}\int t^{\frac{1}{2}}\,dt$$

$$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{6}\left(1+2x^2\right)^{\frac{3}{2}} + C$$

Question 9:

$$(4x+2)\sqrt{x^2+x+1}$$

Answer

Let $x^2+x+1=t$

$\therefore (2x + 1)dx = dt$

$$\int (4x+2)\sqrt{x^2+x+1}\,dx$$

$$= \int 2\sqrt{t}\,dt$$

$$= 2\int \sqrt{t}\,dt$$

$$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}} + C$$

Question 10:

$$\frac{1}{x-\sqrt{x}}$$

Answer

$$\frac{1}{x-\sqrt{x}}=\frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}$$

Let $\left(\sqrt{x}-1\right)=t$

$\therefore \frac{1}{2\sqrt{x}}dx=dt$

$$\Rightarrow \int\frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}dx=\int\frac{2}{t}dt$$

$$=2\log|t|+\text{C}$$

$$=2\log\left|\sqrt{x}-1\right|+\text{C}$$

Question 11:

$$\frac{x}{\sqrt{x+4}},x>0$$

Answer

Let $x+4=t$

$\therefore dx = dt$

$$\int \frac{x}{\sqrt{x+4}}dx = \int \frac{(t-4)}{\sqrt{t}}dt$$

$$= \int\left(\sqrt{t} - \frac{4}{\sqrt{t}}\right)dt$$

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t \cdot t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t^{\frac{1}{2}}(t-12) + C$$

$$= \frac{2}{3}(x+4)^{\frac{1}{2}}(x+4-12) + C$$

$$= \frac{2}{3}\sqrt{x+4}(x-8) + C$$

Question 12:

$$\left(x^3-1\right)^{\frac{1}{3}} x^5$$

Answer

Let $x^3 - 1 = t$

$\therefore 3x^2 dx = dt$

$$\Rightarrow \int\left(x^3-1\right)^{\frac{1}{3}} x^5 dx = \int\left(x^3-1\right)^{\frac{1}{3}} x^3 \cdot x^2 dx$$

$$= \int t^{\frac{1}{3}}(t+1)\frac{dt}{3}$$

$$= \frac{1}{3}\int\left(t^{\frac{4}{3}} + t^{\frac{1}{3}}\right)dt$$

$$= \frac{1}{3}\left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}} + \frac{t^{\frac{4}{3}}}{\frac{4}{3}}\right] + C$$

$$= \frac{1}{3}\left[\frac{3}{7}t^{\frac{7}{3}} + \frac{3}{4}t^{\frac{4}{3}}\right] + C$$

$$= \frac{1}{7}\left(x^3-1\right)^{\frac{7}{3}} + \frac{1}{4}\left(x^3-1\right)^{\frac{4}{3}} + C$$

Question 13:

$$\frac{x^2}{\left(2+3x^3\right)^3}$$

Answer

Let $2+3x^3 = t$

$\therefore 9x^2\, dx = dt$

$$\Rightarrow \int \frac{x^2}{\left(2+3x^3\right)^3} dx = \frac{1}{9}\int \frac{dt}{(t)^3}$$

$$= \frac{1}{9}\left[\frac{t^{-2}}{-2}\right] + C$$

$$= \frac{-1}{18}\left(\frac{1}{t^2}\right) + C$$

$$= \frac{-1}{18\left(2+3x^3\right)^2} + C$$

Question 14:

$\frac{1}{x(\log x)^m}$, $x > 0$

Answer

Let $\log x = t$

$\therefore \frac{1}{x} dx = dt$

$$\Rightarrow \int \frac{1}{x(\log x)^m} dx = \int \frac{dt}{(t)^m}$$

$$= \left(\frac{t^{-m+1}}{1-m}\right) + C$$

$$= \frac{(\log x)^{1-m}}{(1-m)} + C$$

Question 15:

$\frac{x}{9-4x^2}$

Answer

Let $9-4x^2=t$

∴ $-8x\,dx = dt$

$$\Rightarrow \int \frac{x}{9-4x^2}dx = \frac{-1}{8}\int \frac{1}{t}dt$$
$$= \frac{-1}{8}\log|t|+C$$
$$= \frac{-1}{8}\log\left|9-4x^2\right|+C$$

Question 16:

e^{2x+3}

Answer

Let $2x+3=t$

∴ $2dx = dt$

$$\Rightarrow \int e^{2x+3}dx = \frac{1}{2}\int e^t\,dt$$
$$= \frac{1}{2}\left(e^t\right)+C$$
$$= \frac{1}{2}e^{(2x+3)}+C$$

Question 17:

$$\frac{x}{e^{x^2}}$$

Answer

Let $x^2 = t$

$\therefore 2xdx = dt$

$$\Rightarrow \int \frac{x}{e^{x^2}} dx = \frac{1}{2} \int \frac{1}{e^t} dt$$

$$= \frac{1}{2} \int e^{-t} dt$$

$$= \frac{1}{2} \left(\frac{e^{-t}}{-1} \right) + C$$

$$= -\frac{1}{2} e^{-x^2} + C$$

$$= \frac{-1}{2e^{x^2}} + C$$

Question 18:

$$\frac{e^{\tan^{-1} x}}{1+x^2}$$

Answer

Let $\tan^{-1} x = t$

$$\therefore \frac{1}{1+x^2} dx = dt$$

$$\Rightarrow \int \frac{e^{\tan^{-1}x}}{1+x^2}dx = \int e^t dt$$

$$= e^t + C$$

$$= e^{\tan^{-1}x} + C$$

Question 19:

$$\frac{e^{2x}-1}{e^{2x}+1}$$

Answer

$$\frac{e^{2x}-1}{e^{2x}+1}$$

Dividing numerator and denominator by e^x, we obtain

$$\frac{\frac{\left(e^{2x}-1\right)}{e^x}}{\frac{\left(e^{2x}+1\right)}{e^x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Let $e^x + e^{-x} = t$

$\therefore \left(e^x - e^{-x}\right)dx = dt$

$$\Rightarrow \int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$$

$$= \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log\left|e^x + e^{-x}\right| + C$$

Question 20:

$$\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$$

Answer

Let $e^{2x}+e^{-2x}=t$

$\therefore \left(2e^{2x}-2e^{-2x}\right)dx=dt$

$\Rightarrow 2\left(e^{2x}-e^{-2x}\right)dx=dt$

$$\Rightarrow \int\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)dx=\int\frac{dt}{2t}$$

$$=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+C$$

$$=\frac{1}{2}\log\left|e^{2x}+e^{-2x}\right|+C$$

Question 21:

$\tan^2(2x-3)$

Answer

$\tan^2(2x-3)=\sec^2(2x-3)-1$

Let $2x-3=t$

$\therefore 2dx=dt$

$$\Rightarrow \int \tan^2(2x-3)\,dx = \int\left[\left(\sec^2(2x-3)\right)-1\right]dx$$

$$= \frac{1}{2}\int\left(\sec^2 t\right)dt - \int 1dx$$

$$= \frac{1}{2}\int \sec^2 t\,dt - \int 1dx$$

$$= \frac{1}{2}\tan t - x + C$$

$$= \frac{1}{2}\tan(2x-3) - x + C$$

Question 22:

$\sec^2(7-4x)$

Answer

Let 7 − 4x = t

∴ −4dx = dt

$$\therefore \int \sec^2(7-4x)\,dx = \frac{-1}{4}\int \sec^2 t\,dt$$

$$= \frac{-1}{4}(\tan t) + C$$

$$= \frac{-1}{4}\tan(7-4x) + C$$

Question 23:

$\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

Answer

Let $\sin^{-1}x = t$

$\therefore \dfrac{1}{\sqrt{1-x^2}}dx = dt$

$\Rightarrow \displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx = \int t\,dt$

$= \dfrac{t^2}{2} + C$

$= \dfrac{\left(\sin^{-1} x\right)^2}{2} + C$

Question 24:

$\dfrac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$

Answer

$\dfrac{2\cos x - 3\sin x}{6\cos x + 4\sin x} = \dfrac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)}$

Let $3\cos x + 2\sin x = t$

$\therefore (-3\sin x + 2\cos x)dx = dt$

$$\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}dx = \int \frac{dt}{2t}$$

$$= \frac{1}{2}\int \frac{1}{t}dt$$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log|2\sin x + 3\cos x| + C$$

Question 25:

$$\frac{1}{\cos^2 x(1-\tan x)^2}$$

Answer

$$\frac{1}{\cos^2 x(1-\tan x)^2} = \frac{\sec^2 x}{(1-\tan x)^2}$$

Let $(1-\tan x) = t$

$\therefore -\sec^2 x dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{(1-\tan x)^2} dx = \int \frac{-dt}{t^2}$$

$$= -\int t^{-2} dt$$

$$= +\frac{1}{t} + C$$

$$= \frac{1}{(1-\tan x)} + C$$

Question 26:

$$\frac{\cos\sqrt{x}}{\sqrt{x}}$$

Answer

Let $\sqrt{x} = t$

$\therefore \frac{1}{2\sqrt{x}} dx = dt$

$$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int \cos t\, dt$$
$$= 2\sin t + C$$
$$= 2\sin\sqrt{x} + C$$

Question 27:

$\sqrt{\sin 2x}\cos 2x$

Answer

Let sin 2x = t

$\therefore$ $2\cos 2x\, dx = dt$

$$\Rightarrow \int \sqrt{\sin 2x}\ \cos 2x\, dx = \frac{1}{2}\int \sqrt{t}\, dt$$
$$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{1}{3}t^{\frac{3}{2}} + C$$
$$= \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$$

Question 28:

$\dfrac{\cos x}{\sqrt{1+\sin x}}$

Answer

Let $1+\sin x = t$

$\therefore \cos x\, dx = dt$

$$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}} dx = \int \frac{dt}{\sqrt{t}}$$

$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2\sqrt{t} + C$$

$$= 2\sqrt{1+\sin x} + C$$

Question 29:

cot x log sin x

Answer

Let log sin x = t

$$\Rightarrow \frac{1}{\sin x} \cdot \cos x\ dx = dt$$

$$\therefore \cot x\ dx = dt$$

$$\Rightarrow \int \cot x \log \sin x\ dx = \int t\, dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{1}{2}(\log \sin x)^2 + C$$

Question 30:

$$\frac{\sin x}{1+\cos x}$$

Answer

Let 1 + cos x = t

$\therefore -\sin x\, dx = dt$

$$\Rightarrow \int \frac{\sin x}{1+\cos x}\, dx = \int -\frac{dt}{t}$$
$$= -\log|t| + C$$
$$= -\log|1+\cos x| + C$$

Question 31:

$$\frac{\sin x}{(1+\cos x)^2}$$

Answer

Let $1 + \cos x = t$

$\therefore -\sin x\, dx = dt$

$$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2}\, dx = \int -\frac{dt}{t^2}$$
$$= -\int t^{-2}\, dt$$
$$= \frac{1}{t} + C$$
$$= \frac{1}{1+\cos x} + C$$

Question 32:

$$\frac{1}{1+\cot x}$$

Answer

$$\text{Let } I=\int\frac{1}{1+\cot x}dx$$
$$=\int\frac{1}{1+\frac{\cos x}{\sin x}}dx$$
$$=\int\frac{\sin x}{\sin x+\cos x}dx$$
$$=\frac{1}{2}\int\frac{2\sin x}{\sin x+\cos x}dx$$
$$=\frac{1}{2}\int\frac{(\sin x+\cos x)+(\sin x-\cos x)}{(\sin x+\cos x)}dx$$
$$=\frac{1}{2}\int 1\,dx+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$$
$$=\frac{1}{2}(x)+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$$

Let $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\, dx = dt$

$$\therefore I=\frac{x}{2}+\frac{1}{2}\int\frac{-(dt)}{t}$$
$$=\frac{x}{2}-\frac{1}{2}\log|t|+C$$
$$=\frac{x}{2}-\frac{1}{2}\log|\sin x+\cos x|+C$$

Question 33:

$$\frac{1}{1-\tan x}$$

Answer

$$\text{Let } I = \int \frac{1}{1-\tan x} dx$$
$$= \int \frac{1}{1-\frac{\sin x}{\cos x}} dx$$
$$= \int \frac{\cos x}{\cos x - \sin x} dx$$
$$= \frac{1}{2} \int \frac{2\cos x}{\cos x - \sin x} dx$$
$$= \frac{1}{2} \int \frac{(\cos x - \sin x) + (\cos x + \sin x)}{(\cos x - \sin x)} dx$$
$$= \frac{1}{2} \int 1\, dx + \frac{1}{2} \int \frac{\cos x + \sin x}{\cos x - \sin x} dx$$
$$= \frac{x}{2} + \frac{1}{2} \int \frac{\cos x + \sin x}{\cos x - \sin x} dx$$

Put $\cos x - \sin x = t \Rightarrow (-\sin x - \cos x)\, dx = dt$

$$\therefore I = \frac{x}{2} + \frac{1}{2} \int \frac{-(dt)}{t}$$
$$= \frac{x}{2} - \frac{1}{2} \log|t| + C$$
$$= \frac{x}{2} - \frac{1}{2} \log|\cos x - \sin x| + C$$

Question 34:

$$\frac{\sqrt{\tan x}}{\sin x \cos x}$$

Answer

Let $I = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

$= \int \frac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x} dx$

$= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x} dx$

$= \int \frac{\sec^2 x \, dx}{\sqrt{\tan x}}$

Let $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$\therefore I = \int \frac{dt}{\sqrt{t}}$

$= 2\sqrt{t} + C$

$= 2\sqrt{\tan x} + C$

Question 35:

$\frac{(1+\log x)^2}{x}$

Answer

Let 1 + log x = t

$\therefore \frac{1}{x} dx = dt$

$\Rightarrow \int \frac{(1+\log x)^2}{x} dx = \int t^2 dt$

$= \frac{t^3}{3} + C$

$= \frac{(1+\log x)^3}{3} + C$

Question 36:

$$\frac{(x+1)(x+\log x)^2}{x}$$

Answer

$$\frac{(x+1)(x+\log x)^2}{x}=\left(\frac{x+1}{x}\right)(x+\log x)^2=\left(1+\frac{1}{x}\right)(x+\log x)^2$$

Let $(x+\log x)=t$

$\therefore \left(1+\frac{1}{x}\right)dx=dt$

$$\Rightarrow \int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx=\int t^2\,dt$$

$$=\frac{t^3}{3}+C$$

$$=\frac{1}{3}(x+\log x)^3+C$$

Question 37:

$$\frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$$

Answer

Let $x^4=t$

$\therefore 4x^3\,dx=dt$

$$\Rightarrow \int \frac{x^3 \sin\left(\tan^{-1} x^4\right)}{1+x^8} dx = \frac{1}{4} \int \frac{\sin\left(\tan^{-1} t\right)}{1+t^2} dt \qquad ...(1)$$

Let $\tan^{-1} t = u$

$$\therefore \frac{1}{1+t^2} dt = du$$

From (1), we obtain

$$\int \frac{x^3 \sin\left(\tan^{-1} x^4\right) dx}{1+x^8} = \frac{1}{4} \int \sin u \, du$$

$$= \frac{1}{4}\left(-\cos u\right) + C$$

$$= \frac{-1}{4} \cos\left(\tan^{-1} t\right) + C$$

$$= \frac{-1}{4} \cos\left(\tan^{-1} x^4\right) + C$$

Question 38:

$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$ equals

(A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$

(C) $\left(10^x - x^{10}\right)^{-1} + C$ (D) $\log\left(10^x + x^{10}\right) + C$

Answer

Let $x^{10} + 10^x = t$

$$\therefore \left(10x^9 + 10^x \log_e 10\right) dx = dt$$

$$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx = \int \frac{dt}{t}$$

$$= \log t + C$$

$$= \log\left(10^x + x^{10}\right) + C$$

Hence, the correct Answer is D.

Question 39:

$\int \frac{dx}{\sin^2 x \cos^2 x}$ equals

A. $\tan x + \cot x + C$

B. $\tan x - \cot x + C$

C. $\tan x \cot x + C$

D. $\tan x - \cot 2x + C$

Answer

$$\text{Let } I = \int \frac{dx}{\sin^2 x \cos^2 x}$$

$$= \int \frac{1}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx$$

$$= \tan x - \cot x + C$$

Hence, the correct Answer is B.

Exercise 7.3

Question 1:

$\sin^2(2x+5)$

Answer

$$\sin^2(2x+5) = \frac{1-\cos 2(2x+5)}{2} = \frac{1-\cos(4x+10)}{2}$$

$$\Rightarrow \int \sin^2(2x+5)\,dx = \int \frac{1-\cos(4x+10)}{2}dx$$

$$= \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos(4x+10)\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right) + C$$

$$= \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + C$$

Question 2:

$\sin 3x \cos 4x$

Answer

It is known that, $\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

$$\therefore \int \sin 3x \cos 4x\,dx = \frac{1}{2}\int \{\sin(3x+4x) + \sin(3x-4x)\}\,dx$$

$$= \frac{1}{2}\int \{\sin 7x + \sin(-x)\}\,dx$$

$$= \frac{1}{2}\int \{\sin 7x - \sin x\}\,dx$$

$$= \frac{1}{2}\int \sin 7x\,dx - \frac{1}{2}\int \sin x\,dx$$

$$= \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) - \frac{1}{2}(-\cos x) + C$$

$$= \frac{-\cos 7x}{14} + \frac{\cos x}{2} + C$$

Question 3:

cos 2*x* cos 4*x* cos 6*x*

Answer

It is known that, $\cos A\cos B=\frac{1}{2}\{\cos(A+B)+\cos(A-B)\}$

$$\therefore\int\cos 2x(\cos 4x\cos 6x)\,dx=\int\cos 2x\left[\frac{1}{2}\{\cos(4x+6x)+\cos(4x-6x)\}\right]dx$$

$$=\frac{1}{2}\int\{\cos 2x\cos 10x+\cos 2x\cos(-2x)\}\,dx$$

$$=\frac{1}{2}\int\{\cos 2x\cos 10x+\cos^2 2x\}\,dx$$

$$=\frac{1}{2}\int\left[\left\{\frac{1}{2}\cos(2x+10x)+\cos(2x-10x)\right\}+\left(\frac{1+\cos 4x}{2}\right)\right]dx$$

$$=\frac{1}{4}\int(\cos 12x+\cos 8x+1+\cos 4x)\,dx$$

$$=\frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+x+\frac{\sin 4x}{4}\right]+\mathrm{C}$$

Question 4:

$\sin^3(2x+1)$

Answer

Let $I=\int\sin^3(2x+1)$

$$\Rightarrow\int\sin^3(2x+1)\,dx=\int\sin^2(2x+1)\cdot\sin(2x+1)\,dx$$

$$=\int\left(1-\cos^2(2x+1)\right)\sin(2x+1)\,dx$$

$$\text{Let}\cos(2x+1)=t$$

$$\Rightarrow -2\sin(2x+1)\,dx=dt$$

$$\Rightarrow\sin(2x+1)\,dx=\frac{-dt}{2}$$

$$\Rightarrow I = \frac{-1}{2}\int\left(1-t^2\right)dt$$

$$= \frac{-1}{2}\left\{t-\frac{t^3}{3}\right\}$$

$$= \frac{-1}{2}\left\{\cos(2x+1)-\frac{\cos^3(2x+1)}{3}\right\}$$

$$= \frac{-\cos(2x+1)}{2}+\frac{\cos^3(2x+1)}{6}+C$$

Question 5:

$\sin^3 x \cos^3 x$

Answer

$$\text{Let } I = \int \sin^3 x \cos^3 x \cdot dx$$

$$= \int \cos^3 x \cdot \sin^2 x \cdot \sin x \cdot dx$$

$$= \int \cos^3 x\left(1-\cos^2 x\right)\sin x \cdot dx$$

$$\text{Let } \cos x = t$$

$$\Rightarrow -\sin x \cdot dx = dt$$

$$\Rightarrow I = -\int t^3\left(1-t^2\right)dt$$

$$= -\int\left(t^3-t^5\right)dt$$

$$= -\left\{\frac{t^4}{4}-\frac{t^6}{6}\right\}+C$$

$$= -\left\{\frac{\cos^4 x}{4}-\frac{\cos^6 x}{6}\right\}+C$$

$$= \frac{\cos^6 x}{6}-\frac{\cos^4 x}{4}+C$$

Question 6:

$\sin x \sin 2x \sin 3x$

Answer

It is known that, $\sin A \sin B = \frac{1}{2}\left\{\cos(A-B) - \cos(A+B)\right\}$

$$\therefore \int \sin x \sin 2x \sin 3x \, dx = \int \left[\sin x \cdot \frac{1}{2}\left\{\cos(2x-3x) - \cos(2x+3x)\right\}\right] dx$$

$$= \frac{1}{2}\int \left(\sin x \cos(-x) - \sin x \cos 5x\right) dx$$

$$= \frac{1}{2}\int \left(\sin x \cos x - \sin x \cos 5x\right) dx$$

$$= \frac{1}{2}\int \frac{\sin 2x}{2}\, dx - \frac{1}{2}\int \sin x \cos 5x \, dx$$

$$= \frac{1}{4}\left[\frac{-\cos 2x}{2}\right] - \frac{1}{2}\int \left\{\frac{1}{2}\sin(x+5x) + \sin(x-5x)\right\} dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\int \left(\sin 6x + \sin(-4x)\right) dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{4}\right] + C$$

$$= \frac{-\cos 2x}{8} - \frac{1}{8}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{2}\right] + C$$

$$= \frac{1}{8}\left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x\right] + C$$

Question 7:

sin 4x sin 8x

Answer

It is known that, $\sin A \sin B = \frac{1}{2}\cos(A-B) - \cos(A+B)$

$$\therefore \int \sin 4x \sin 8x \; dx = \int \left\{ \frac{1}{2} \cos(4x - 8x) - \cos(4x + 8x) \right\} dx$$

$$= \frac{1}{2} \int \left(\cos(-4x) - \cos 12x \right) dx$$

$$= \frac{1}{2} \int \left(\cos 4x - \cos 12x \right) dx$$

$$= \frac{1}{2} \left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12} \right]$$

Question 8:

$$\frac{1 - \cos x}{1 + \cos x}$$

Answer

$$\frac{1 - \cos x}{1 + \cos x} = \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} \qquad \left[2\sin^2 \frac{x}{2} = 1 - \cos x \text{ and } 2\cos^2 \frac{x}{2} = 1 + \cos x \right]$$

$$= \tan^2 \frac{x}{2}$$

$$= \left(\sec^2 \frac{x}{2} - 1 \right)$$

$$\therefore \int \frac{1 - \cos x}{1 + \cos x} dx = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx$$

$$= \left[\frac{\tan \frac{x}{2}}{\frac{1}{2}} - x \right] + C$$

$$= 2 \tan \frac{x}{2} - x + C$$

Question 9:

$$\frac{\cos x}{1 + \cos x}$$

Answer

$$\frac{\cos x}{1+\cos x}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \left[\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2} \text{ and } \cos x=2\cos^2\frac{x}{2}-1\right]$$

$$=\frac{1}{2}\left[1-\tan^2\frac{x}{2}\right]$$

$$\therefore \int\frac{\cos x}{1+\cos x}dx=\frac{1}{2}\int\left(1-\tan^2\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\int\left(1-\sec^2\frac{x}{2}+1\right)dx$$

$$=\frac{1}{2}\int\left(2-\sec^2\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\left[2x-\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]+C$$

$$=x-\tan\frac{x}{2}+C$$

Question 10:

$\sin^4 x$

Answer

$$\sin^4 x = \sin^2 x \sin^2 x$$
$$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$$
$$= \frac{1}{4}(1-\cos 2x)^2$$
$$= \frac{1}{4}\left[1+\cos^2 2x - 2\cos 2x\right]$$
$$= \frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right)-2\cos 2x\right]$$
$$= \frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$$
$$= \frac{1}{4}\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$$

$$\therefore \int \sin^4 x \, dx = \frac{1}{4}\int\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right] dx$$
$$= \frac{1}{4}\left[\frac{3}{2}x+\frac{1}{2}\left(\frac{\sin 4x}{4}\right)-\frac{2\sin 2x}{2}\right]+C$$
$$= \frac{1}{8}\left[3x+\frac{\sin 4x}{4}-2\sin 2x\right]+C$$
$$= \frac{3x}{8}-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x + C$$

Question 11:

$\cos^4 2x$

Answer

$$\cos^4 2x = \left(\cos^2 2x\right)^2$$
$$= \left(\frac{1+\cos 4x}{2}\right)^2$$
$$= \frac{1}{4}\left[1+\cos^2 4x + 2\cos 4x\right]$$
$$= \frac{1}{4}\left[1+\left(\frac{1+\cos 8x}{2}\right)+2\cos 4x\right]$$
$$= \frac{1}{4}\left[1+\frac{1}{2}+\frac{\cos 8x}{2}+2\cos 4x\right]$$
$$= \frac{1}{4}\left[\frac{3}{2}+\frac{\cos 8x}{2}+2\cos 4x\right]$$
$$\therefore \int \cos^4 2x\,dx = \int\left(\frac{3}{8}+\frac{\cos 8x}{8}+\frac{\cos 4x}{2}\right)dx$$
$$= \frac{3}{8}x + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + C$$

Question 12:

$$\frac{\sin^2 x}{1+\cos x}$$

Answer

$$\frac{\sin^2 x}{1+\cos x} = \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}} \quad \left[\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2};\ \cos x = 2\cos^2\frac{x}{2}-1\right]$$
$$= \frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$
$$= 2\sin^2\frac{x}{2}$$
$$= 1-\cos x$$
$$\therefore \int \frac{\sin^2 x}{1+\cos x}dx = \int(1-\cos x)dx$$
$$= x - \sin x + C$$

Question 13:

$$\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos \alpha}$$

Answer

$$\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos \alpha}=\frac{-2\sin\frac{2x+2\alpha}{2}\sin\frac{2x-2\alpha}{2}}{-2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}} \qquad \left[\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$=\frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=\frac{\left[2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)\right]\left[2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$=2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right]$$

$$=2\left[\cos(x)+\cos\alpha\right]$$

$$=2\cos x+2\cos\alpha$$

$$\therefore \int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos \alpha}dx=\int 2\cos x+2\cos\alpha$$

$$=2\left[\sin x+x\cos\alpha\right]+C$$

Question 14:

$$\frac{\cos x-\sin x}{1+\sin 2x}$$

Answer

$$\frac{\cos x-\sin x}{1+\sin 2x}=\frac{\cos x-\sin x}{\left(\sin^2 x+\cos^2 x\right)+2\sin x\cos x}$$

$$\left[\sin^2 x+\cos^2 x=1;\ \ \sin 2x=2\sin x\cos x\right]$$

$$=\frac{\cos x-\sin x}{\left(\sin x+\cos x\right)^2}$$

Let $\sin x+\cos x=t$

$\therefore \left(\cos x-\sin x\right)dx=dt$

$$\Rightarrow \int\frac{\cos x-\sin x}{1+\sin 2x}dx=\int\frac{\cos x-\sin x}{\left(\sin x+\cos x\right)^2}dx$$

$$=\int\frac{dt}{t^2}$$

$$=\int t^{-2}dt$$

$$=-t^{-1}+\mathrm{C}$$

$$=-\frac{1}{t}+\mathrm{C}$$

$$=\frac{-1}{\sin x+\cos x}+\mathrm{C}$$

Question 15:

$\tan^3 2x\sec 2x$

Answer

$$\tan^3 2x \sec 2x = \tan^2 2x \tan 2x \sec 2x$$
$$= \left(\sec^2 2x - 1\right) \tan 2x \sec 2x$$
$$= \sec^2 2x \cdot \tan 2x \sec 2x - \tan 2x \sec 2x$$
$$\therefore \int \tan^3 2x \sec 2x \, dx = \int \sec^2 2x \tan 2x \sec 2x \, dx - \int \tan 2x \sec 2x \, dx$$
$$= \int \sec^2 2x \tan 2x \sec 2x \, dx - \frac{\sec 2x}{2} + C$$

Let $\sec 2x = t$

$\therefore 2 \sec 2x \tan 2x \, dx = dt$

$$\therefore \int \tan^3 2x \sec 2x \, dx = \frac{1}{2} \int t^2 dt - \frac{\sec 2x}{2} + C$$
$$= \frac{t^3}{6} - \frac{\sec 2x}{2} + C$$
$$= \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + C$$

Question 16:

$\tan^4 x$

Answer

$$\tan^4 x$$
$$= \tan^2 x \cdot \tan^2 x$$
$$= \left(\sec^2 x - 1\right) \tan^2 x$$
$$= \sec^2 x \tan^2 x - \tan^2 x$$
$$= \sec^2 x \tan^2 x - \left(\sec^2 x - 1\right)$$
$$= \sec^2 x \tan^2 x - \sec^2 x + 1$$

$$\therefore \int \tan^4 x \, dx = \int \sec^2 x \tan^2 x \, dx - \int \sec^2 x \, dx + \int 1 \cdot dx$$
$$= \int \sec^2 x \tan^2 x \, dx - \tan x + x + C \quad \ldots(1)$$

Consider $\int \sec^2 x \tan^2 x \, dx$

Let $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$\Rightarrow \int \sec^2 x \tan^2 x dx = \int t^2 dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

From equation (1), we obtain

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

Question 17:

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$$

Answer

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}$$

$$= \tan x \sec x + \cot x \operatorname{cosec} x$$

$$\therefore \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\, dx = \int (\tan x \sec x + \cot x \operatorname{cosec} x)\, dx$$

$$= \sec x - \operatorname{cosec} x + C$$

Question 18:

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$

Answer

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos 2x + (1 - \cos 2x)}{\cos^2 x} \qquad [\cos 2x = 1 - 2\sin^2 x]$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\therefore \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\, dx = \int \sec^2 x\, dx = \tan x + C$$

Question 19:

$$\frac{1}{\sin x \cos^3 x}$$

Answer

$$\frac{1}{\sin x\cos^3 x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x}$$

$$= \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x}$$

$$= \tan x \sec^2 x + \frac{\frac{1\cos^2 x}{\sin x \cos x}}{\cos^2 x}$$

$$= \tan x \sec^2 x + \frac{\sec^2 x}{\tan x}$$

$$\therefore \int \frac{1}{\sin x \cos^3 x} dx = \int \tan x \sec^2 x \; dx + \int \frac{\sec^2 x}{\tan x} \; dx$$

Let $\tan x = t \Rightarrow \sec^2 x \; dx = dt$

$$\Rightarrow \int \frac{1}{\sin x \cos^3 x} dx = \int t dt + \int \frac{1}{t} dt$$

$$= \frac{t^2}{2} + \log|t| + C$$

$$= \frac{1}{2}\tan^2 x + \log|\tan x| + C$$

Question 20:

$$\frac{\cos 2x}{(\cos x + \sin x)^2}$$

Answer

$$\frac{\cos 2x}{(\cos x+\sin x)^2}=\frac{\cos 2x}{\cos^2 x+\sin^2 x+2\sin x\cos x}=\frac{\cos 2x}{1+\sin 2x}$$

$$\therefore \int\frac{\cos 2x}{(\cos x+\sin x)^2}\,dx=\int\frac{\cos 2x}{(1+\sin 2x)}\,dx$$

Let $1+\sin 2x=t$

$\Rightarrow 2\cos 2x\,dx=dt$

$$\therefore \int\frac{\cos 2x}{(\cos x+\sin x)^2}\,dx=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+C$$

$$=\frac{1}{2}\log|1+\sin 2x|+C$$

$$=\frac{1}{2}\log\left|(\sin x+\cos x)^2\right|+C$$

$$=\log|\sin x+\cos x|+C$$

Question 21:

$\sin^{-1}(\cos x)$

Answer

$\sin^{-1}(\cos x)$

Let $\cos x=t$

Then, $\sin x=\sqrt{1-t^2}$

$$\Rightarrow (-\sin x)\,dx = dt$$

$$dx = \frac{-dt}{\sin x}$$

$$dx = \frac{-dt}{\sqrt{1-t^2}}$$

$$\therefore \int \sin^{-1}(\cos x)\,dx = \int \sin^{-1} t\left(\frac{-dt}{\sqrt{1-t^2}}\right)$$

$$= -\int \frac{\sin^{-1} t}{\sqrt{1-t^2}}\,dt$$

Let $\sin^{-1} t = u$

$$\Rightarrow \frac{1}{\sqrt{1-t^2}}\,dt = du$$

$$\therefore \int \sin^{-1}(\cos x)\,dx = \int 4du$$

$$= -\frac{u^2}{2} + C$$

$$= \frac{-(\sin^{1} t)^2}{2} + C$$

$$= \frac{-\left[\sin^{-1}(\cos x)\right]^2}{2} + C \qquad \ldots(1)$$

It is known that,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\therefore \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \left(\frac{\pi}{2} - x\right)$$

Substituting in equation (1), we obtain

$$\int \sin^{-1}(\cos x)\,dx = \frac{-\left[\frac{\pi}{2}-x\right]^2}{2}+C$$

$$= -\frac{1}{2}\left(\frac{\pi^2}{2}+x^2-\pi x\right)+C$$

$$= -\frac{\pi^2}{8}-\frac{x^2}{2}+\frac{1}{2}\pi x+C$$

$$= \frac{\pi x}{2}-\frac{x^2}{2}+\left(C-\frac{\pi^2}{8}\right)$$

$$= \frac{\pi x}{2}-\frac{x^2}{2}+C_1$$

Question 22:

$$\frac{1}{\cos(x-a)\cos(x-b)}$$

Answer

$$\frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x-b)-(x-a)\right]}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\frac{\left[\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)\right]}{\cos(x-a)\cos(x-b)}$$

$$= \frac{1}{\sin(a-b)}\left[\tan(x-b)-\tan(x-a)\right]$$

$$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}dx = \frac{1}{\sin(a-b)}\int\left[\tan(x-b)-\tan(x-a)\right]dx$$

$$= \frac{1}{\sin(a-b)}\left[-\log\left|\cos(x-b)\right|+\log\left|\cos(x-a)\right|\right]$$

$$= \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$$

Question 23:

$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx$ is equal to

A. tan *x* + cot *x* + C

B. tan *x* + cosec *x* + C

C. − tan *x* + cot *x* + C

D. tan *x* + sec *x* + C

Answer

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx = \int\left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x}\right) dx$$

$$= \int\left(\sec^2 x - \operatorname{cosec}^2 x\right) dx$$

$$= \tan x + \cot x + C$$

Hence, the correct Answer is A.

Question 24:

$\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$ equals

A. − cot (ex^x) + C

B. tan (xe^x) + C

C. tan (e^x) + C

D. cot (e^x) + C

Answer

$$\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$$

Let $ex^x = t$

$$\Rightarrow \left(e^x \cdot x + e^x \cdot 1\right) dx = dt$$

$$e^x\left(x+1\right) dx = dt$$

$$\therefore \int \frac{e^x\left(1+x\right)}{\cos^2\left(e^x x\right)} dx = \int \frac{dt}{\cos^2 t}$$

$$= \int \sec^2 t \; dt$$

$$= \tan t + C$$

$$= \tan\left(e^x \cdot x\right) + C$$

Hence, the correct Answer is B.

Exercise 7.4

Question 1:

$$\frac{3x^2}{x^6+1}$$

Answer

Let $x^3 = t$

$\therefore 3x^2\, dx = dt$

$$\Rightarrow \int \frac{3x^2}{x^6+1} dx = \int \frac{dt}{t^2+1}$$
$$= \tan^{1} t + C$$
$$= \tan^{-1}\left(x^3\right) + C$$

Question 2:

$$\frac{1}{\sqrt{1+4x^2}}$$

Answer

Let $2x = t$

$\therefore 2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$

$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right]+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right|+C$$

Question 3:

$$\frac{1}{\sqrt{(2-x)^2+1}}$$

Answer

Let $2 - x = t$

$\Rightarrow -dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$

$$= -\log\left|t+\sqrt{t^2+1}\right|+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right|+C$$

$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right|+C$$

Question 4:

$$\frac{1}{\sqrt{9-25x^2}}$$

Answer

Let $5x = t$

$\therefore 5dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}\,dx = \frac{1}{5}\int \frac{1}{9-t^2}\,dt$$
$$= \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}\,dt$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

Question 5:

$$\frac{3x}{1+2x^4}$$

Answer

Let $\sqrt{2}x^2 = t$

$\therefore\ 2\sqrt{2}x\ dx = dt$

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$
$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1} t\right] + C$$
$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

Question 6:

$$\frac{x^2}{1-x^6}$$

Answer

Let $x^3 = t$

$\therefore 3x^2\, dx = dt$

$$\Rightarrow \int \frac{x^2}{1-x^6} dx = \frac{1}{3} \int \frac{dt}{1-t^2}$$
$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$
$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

Question 7:

$$\frac{x-1}{\sqrt{x^2-1}}$$

Answer

$$\int \frac{x-1}{\sqrt{x^2-1}}\, dx = \int \frac{x}{\sqrt{x^2-1}}\, dx - \int \frac{1}{\sqrt{x^2-1}}\, dx \qquad \ldots(1)$$

For $\int \frac{x}{\sqrt{x^2-1}} dx$, let $x^2-1=t \Rightarrow 2x\, dx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}} dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$
$$= \frac{1}{2}\int t^{-\frac{1}{2}} dt$$
$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$
$$= \sqrt{t}$$
$$= \sqrt{x^2-1}$$

From (1), we obtain

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad \left[\int \frac{1}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$

$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

Question 8:

$$\frac{x^2}{\sqrt{x^6+a^6}}$$

Answer

Let $x^3 = t$

$\Rightarrow 3x^2\ dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$

$$= \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right| + C$$

$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C$$

Question 9:

$$\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$$

Answer

Let $\tan x = t$

$\therefore \sec^2 x\ dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

$$= \log\left|t + \sqrt{t^2 + 4}\right| + C$$

$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$$

Question 10:

$$\frac{1}{\sqrt{x^2 + 2x + 2}}$$

Answer

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(x+1)^2 + (1)^2}} dx$$

Let $x + 1 = t$

$\therefore \ dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{t^2 + 1}} dt$$

$$= \log\left|t + \sqrt{t^2 + 1}\right| + C$$

$$= \log\left|(x+1) + \sqrt{(x+1)^2 + 1}\right| + C$$

$$= \log\left|(x+1) + \sqrt{x^2 + 2x + 2}\right| + C$$

Question 11:

$$\frac{1}{\sqrt{9x^2 + 6x + 5}}$$

Answer

$$\int \frac{1}{9x^2+6x+5}dx = \int \frac{1}{(3x+1)^2+(2)^2}dx$$

Let $(3x+1) = t$

$\therefore\ 3dx = dt$

$$\Rightarrow \int \frac{1}{(3x+1)^2+(2)^2}dx = \frac{1}{3}\int \frac{1}{t^2+2^2}dt$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]+C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

Question 12:

$$\frac{1}{\sqrt{7-6x-x^2}}$$

Answer

$7-6x-x^2$ can be written as $7-(x^2+6x+9-9)$.

Therefore,

$7-(x^2+6x+9-9)$

$=16-(x^2+6x+9)$

$=16-(x+3)^2$

$=(4)^2-(x+3)^2$

$$\therefore\ \int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$$

Let $x+3 = t$

$\Rightarrow\ dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx = \int \frac{1}{\sqrt{(4)^2-(t)^2}}dt$$

$$= \sin^{-1}\left(\frac{t}{4}\right)+C$$

$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C$$

Question 13:

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

Answer

$(x-1)(x-2)$ can be written as x^2-3x+2.

Therefore,

$$x^2-3x+2$$

$$=x^2-3x+\frac{9}{4}-\frac{9}{4}+2$$

$$=\left(x-\frac{3}{2}\right)^2-\frac{1}{4}$$

$$=\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\therefore \int\frac{1}{\sqrt{(x-1)(x-2)}}dx=\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

Let $x-\frac{3}{2}=t$

$\therefore\ dx=dt$

$$\Rightarrow\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right|+C$$

$$=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$$

Question 14:

$$\frac{1}{\sqrt{8+3x-x^2}}$$

Answer

$8+3x-x^2$ can be written as $8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$.

Therefore,

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}}dx=\int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$$

Let $x-\frac{3}{2}=t$

$\therefore\ dx=dt$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx=\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

Question 15:

$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

Answer

$(x-a)(x-b)$ can be written as $x^2-(a+b)x+ab$.

Therefore,

$x^2-(a+b)x+ab$

$$=x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$$

$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$$

$$\Rightarrow\int\frac{1}{\sqrt{(x-a)(x-b)}}dx=\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx$$

Let $x-\left(\frac{a+b}{2}\right)=t$

$\therefore\ dx=dt$

$$\Rightarrow\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right|+C$$

$$=\log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$$

Question 16:

$$\frac{4x+1}{\sqrt{2x^2+x-3}}$$

Answer

Let $4x+1=A\frac{d}{dx}(2x^2+x-3)+B$

$\Rightarrow 4x+1=A(4x+1)+B$

$\Rightarrow 4x+1=4Ax+A+B$

Equating the coefficients of *x* and constant term on both sides, we obtain

$4A = 4 \Rightarrow A = 1$

$A + B = 1 \Rightarrow B = 0$

Let $2x^2 + x - 3 = t$

$\therefore (4x + 1)\, dx = dt$

$$\Rightarrow \int \frac{4x+1}{\sqrt{2x^2+x-3}}dx = \int \frac{1}{\sqrt{t}}dt$$
$$= 2\sqrt{t} + C$$
$$= 2\sqrt{2x^2+x-3} + C$$

Question 17:

$$\frac{x+2}{\sqrt{x^2-1}}$$

Answer

$$\text{Let } x+2 = A\frac{d}{dx}\left(x^2-1\right)+B \qquad \ldots(1)$$
$$\Rightarrow x+2 = A(2x)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$
$$B = 2$$

From (1), we obtain

$$(x+2)=\frac{1}{2}(2x)+2$$

$$\text{Then, }\int\frac{x+2}{\sqrt{x^2-1}}dx=\int\frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$$

$$=\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx+\int\frac{2}{\sqrt{x^2-1}}dx \qquad ...(2)$$

$$\text{In }\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx,\text{ let }x^2-1=t\Rightarrow 2xdx=dt$$

$$\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx=\frac{1}{2}\int\frac{dt}{\sqrt{t}}$$

$$=\frac{1}{2}\left[2\sqrt{t}\right]$$

$$=\sqrt{t}$$

$$=\sqrt{x^2-1}$$

$$\text{Then, }\int\frac{2}{\sqrt{x^2-1}}dx=2\int\frac{1}{\sqrt{x^2-1}}dx=2\log\left|x+\sqrt{x^2-1}\right|$$

From equation (2), we obtain

$$\int\frac{x+2}{\sqrt{x^2-1}}dx=\sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+C$$

Question 18:

$$\frac{5x-2}{1+2x+3x^2}$$

Answer

$$\text{Let }5x-2=A\frac{d}{dx}\left(1+2x+3x^2\right)+B$$

$$\Rightarrow 5x-2=A(2+6x)+B$$

Equating the coefficient of x and constant term on both sides, we obtain

$5 = 6A \Rightarrow A = \frac{5}{6}$

$2A + B = -2 \Rightarrow B = -\frac{11}{3}$

$\therefore 5x - 2 = \frac{5}{6}(2+6x) + \left(-\frac{11}{3}\right)$

$$\Rightarrow \int \frac{5x-2}{1+2x+3x^2}dx = \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2}dx$$

$$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}dx$$

Let $I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$ and $I_2 = \int \frac{1}{1+2x+3x^2}dx$

$$\therefore \int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}I_1 - \frac{11}{3}I_2 \quad ...(1)$$

$I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$

Let $1+2x+3x^2 = t$

$\Rightarrow (2+6x)dx = dt$

$\therefore I_1 = \int \frac{dt}{t}$

$I_1 = \log|t|$

$I_1 = \log|1+2x+3x^2| \quad ...(2)$

$I_2 = \int \frac{1}{1+2x+3x^2}dx$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$.

Therefore,

$$1+3\left(x^2+\frac{2}{3}x\right)$$

$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$$

$$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2=\frac{1}{3}\int\frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

$$=\frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$=\frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad ...(3)$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right] - \frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]+C$$

$$= \frac{5}{6}\log\left|1+2x+3x^2\right| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

Question 19:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}$$

Answer

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

$$\text{Let } 6x+7 = A\frac{d}{dx}\left(x^2-9x+20\right)+B$$

$$\Rightarrow 6x+7 = A(2x-9)+B$$

Equating the coefficients of *x* and constant term, we obtain

$2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

$\therefore 6x + 7 = 3\ (2x - 9) + 34$

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int \frac{1}{\sqrt{x^2-9x+20}}dx$$

Let $I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$$\therefore \int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2 \qquad ...(1)$$

Then,

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

Let $x^2 - 9x + 20 = t$

$\Rightarrow (2x-9)dx = dt$

$\Rightarrow I_1 = \frac{dt}{\sqrt{t}}$

$I_1 = 2\sqrt{t}$

$$I_1 = 2\sqrt{x^2-9x+20} \qquad ...(2)$$

and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$x^2-9x+20$ can be written as $x^2-9x+20+\frac{81}{4}-\frac{81}{4}$.

Therefore,

$$x^2-9x+20+\frac{81}{4}-\frac{81}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\frac{1}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2=\int\frac{1}{\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$I_2=\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right| \quad ...(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}}dx=3\left[2\sqrt{x^2-9x+20}\right]+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

$$=6\sqrt{x^2-9x+20}+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

Question 20:

$$\frac{x+2}{\sqrt{4x-x^2}}$$

Answer

$$\text{Let } x+2=A\frac{d}{dx}\left(4x-x^2\right)+B$$

$$\Rightarrow x+2=A(4-2x)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$-2A = 1 \Rightarrow A = -\frac{1}{2}$

$4A + B = 2 \Rightarrow B = 4$

$\Rightarrow (x+2) = -\frac{1}{2}(4-2x) + 4$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$$

Let $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$ and $I_2 \int \frac{1}{\sqrt{4x-x^2}}dx$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 + 4I_2 \quad ...(1)$$

Then, $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$

Let $4x - x^2 = t$

$\Rightarrow (4-2x)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \quad ...(2)$$

$I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$

$\Rightarrow 4x - x^2 = -(-4x + x^2)$

$= (-4x + x^2 + 4 - 4)$

$= 4 - (x-2)^2$

$= (2)^2 - (x-2)^2$

$$\therefore I_2 = \int \frac{1}{\sqrt{(2)^2 - (x-2)^2}}dx = \sin^{-1}\left(\frac{x-2}{2}\right) \quad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right)+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

$$= -\sqrt{4x-x^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

Question 21:

$$\frac{x+2}{\sqrt{x^2+2x+3}}$$

Answer

$$\int \frac{(x+2)}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \quad ...(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2)\, dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \qquad \ldots(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2 + \left(\sqrt{2}\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}dx = \log\left|(x+1)+\sqrt{x^2+2x+3}\right| \qquad \ldots(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right] + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

$$= \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

Question 22:

$$\frac{x+3}{x^2-2x-5}$$

Answer

$$\text{Let } (x+3) = A\frac{d}{dx}\left(x^2-2x-5\right) + B$$

$$(x+3) = A(2x-2) + B$$

Equating the coefficients of *x* and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$-2A + B = 3 \Rightarrow B = 4$$

$$\therefore (x+3) = \frac{1}{2}(2x-2) + 4$$

$$\Rightarrow \int \frac{x+3}{x^2-2x-5}dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

Let $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$ and $I_2 = \int \frac{1}{x^2-2x-5}dx$

$$\therefore \int \frac{x+3}{\left(x^2-2x-5\right)}dx = \frac{1}{2}I_1 + 4I_2 \quad ...(1)$$

Then, $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$

Let $x^2-2x-5=t$

$\Rightarrow (2x-2)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log\left|x^2-2x-5\right| \quad ...(2)$$

$$I_2 = \int \frac{1}{x^2-2x-5}dx$$

$$= \int \frac{1}{\left(x^2-2x+1\right)-6}dx$$

$$= \int \frac{1}{(x-1)^2+\left(\sqrt{6}\right)^2}dx$$

$$= \frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) \quad ...(3)$$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

$$= \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

Question 23:

$$\frac{5x+3}{\sqrt{x^2+4x+10}}$$

Answer

Let $5x+3 = A\frac{d}{dx}\left(x^2+4x+10\right)+B$

$\Rightarrow 5x+3 = A\left(2x+4\right)+B$

Equating the coefficients of *x* and constant term, we obtain

$2A = 5 \Rightarrow A = \frac{5}{2}$

$4A+B = 3 \Rightarrow B = -7$

$\therefore 5x+3 = \frac{5}{2}\left(2x+4\right)-7$

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}\left(2x+4\right)-7}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Let $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}I_1 - 7I_2 \quad \text{...(1)}$$

Then, $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$

Let $x^2+4x+10 = t$

$\therefore \left(2x+4\right)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \quad \text{...(2)}$$

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \int \frac{1}{\sqrt{\left(x^2+4x+4\right)+6}}dx$$

$$= \int \frac{1}{\left(x+2\right)^2+\left(\sqrt{6}\right)^2}dx$$

$$= \log\left|\left(x+2\right)\sqrt{x^2+4x+10}\right| \quad \text{...(3)}$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

Question 24:

$\int \frac{dx}{x^2+2x+2}$ equals

A. $x \tan^{-1}(x+1) + C$

B. $\tan^{-1}(x+1) + C$

C. $(x+1)\tan^{-1} x + C$

D. $\tan^{-1} x + C$

Answer

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$

$$= \int \frac{1}{(x+1)^2+(1)^2}dx$$

$$= \left[\tan^{-1}(x+1)\right] + C$$

Hence, the correct Answer is B.

Question 25:

$\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

A. $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$

B. $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right) + C$

C. $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$

D. $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Answer

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

$$=\int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

$$=\int \frac{1}{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)}dx$$

$$=\int \frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

$$=\frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}dx$$

$$=\frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C \qquad \left(\int \frac{dy}{\sqrt{a^2-y^2}}=\sin^{-1}\frac{y}{a}+C\right)$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Hence, the correct Answer is B.

Exercise 7.5

Question 1:

$$\frac{x}{(x+1)(x+2)}$$

Answer

Let $\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

$$\Rightarrow x = A(x+2) + B(x+1)$$

Equating the coefficients of *x* and constant term, we obtain

A + *B* = 1

2*A* + *B* = 0

On solving, we obtain

A = −1 and *B* = 2

$$\therefore \frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$$

$$\Rightarrow \int \frac{x}{(x+1)(x+2)} dx = \int \frac{-1}{(x+1)} + \frac{2}{(x+2)} dx$$

$$= -\log|x+1| + 2\log|x+2| + C$$

$$= \log(x+2)^2 - \log|x+1| + C$$

$$= \log\frac{(x+2)^2}{(x+1)} + C$$

Question 2:

$$\frac{1}{x^2-9}$$

Answer

Let $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$

$$1 = A(x-3) + B(x+3)$$

Equating the coefficients of x and constant term, we obtain

$A + B = 0$

$-3A + 3B = 1$

On solving, we obtain

$$A = -\frac{1}{6} \text{ and } B = \frac{1}{6}$$

$$\therefore \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)} + \frac{1}{6(x-3)}$$

$$\Rightarrow \int \frac{1}{(x^2-9)} dx = \int \left(\frac{-1}{6(x+3)} + \frac{1}{6(x-3)} \right) dx$$

$$= -\frac{1}{6} \log|x+3| + \frac{1}{6} \log|x-3| + C$$

$$= \frac{1}{6} \log \left| \frac{(x-3)}{(x+3)} \right| + C$$

Question 3:

$$\frac{3x-1}{(x-1)(x-2)(x-3)}$$

Answer

Let $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \ldots(1)$$

Substituting x = 1, 2, and 3 respectively in equation (1), we obtain

$A = 1$, $B = -5$, and $C = 4$

$$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}$$

$$\Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)} dx = \int \left\{ \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)} \right\} dx$$

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$$

Question 4:

$$\frac{x}{(x-1)(x-2)(x-3)}$$

Answer

Let $\frac{x}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$

$$x=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \qquad \ldots(1)$$

Substituting x = 1, 2, and 3 respectively in equation (1), we obtain

$$A=\frac{1}{2},\ B=-2,\ \text{and } C=\frac{3}{2}$$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}$$

$$\Rightarrow \int\frac{x}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}\right\}dx$$

$$=\frac{1}{2}\log|x-1|-2\log|x-2|+\frac{3}{2}\log|x-3|+\text{C}$$

Question 5:

$$\frac{2x}{x^2+3x+2}$$

Answer

Let $\frac{2x}{x^2+3x+2}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

$$2x=A(x+2)+B(x+1) \qquad \ldots(1)$$

Substituting x = −1 and −2 in equation (1), we obtain

A = −2 and B = 4

$$\therefore \frac{2x}{(x+1)(x+2)} = \frac{-2}{(x+1)} + \frac{4}{(x+2)}$$

$$\Rightarrow \int \frac{2x}{(x+1)(x+2)} dx = \int \left\{ \frac{4}{(x+2)} - \frac{2}{(x+1)} \right\} dx$$

$$= 4\log|x+2| - 2\log|x+1| + C$$

Question 6:

$$\frac{1-x^2}{x(1-2x)}$$

Answer

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1 - x^2)$ by $x(1 - 2x)$, we obtain

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left(\frac{2-x}{x(1-2x)}\right)$$

Let $\frac{2-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{(1-2x)}$

$$\Rightarrow (2-x) = A(1-2x) + Bx \qquad \text{...(1)}$$

Substituting $x = 0$ and $\frac{1}{2}$ in equation (1), we obtain

$A = 2$ and $B = 3$

$$\therefore \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{1-2x}$$

Substituting in equation (1), we obtain

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left\{\frac{2}{x} + \frac{3}{(1-2x)}\right\}$$

$$\Rightarrow \int \frac{1-x^2}{x(1-2x)}dx = \int\left\{\frac{1}{2} + \frac{1}{2}\left(\frac{2}{x} + \frac{3}{1-2x}\right)\right\}dx$$

$$= \frac{x}{2} + \log|x| + \frac{3}{2(-2)}\log|1-2x| + C$$

$$= \frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| + C$$

Question 7:

$$\frac{x}{(x^2+1)(x-1)}$$

Answer

Let $\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)}$

$$x = (Ax+B)(x-1) + C(x^2+1)$$

$$x = Ax^2 - Ax + Bx - B + Cx^2 + C$$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + C = 0$

$-A + B = 1$

$-B + C = 0$

On solving these equations, we obtain

$$A = -\frac{1}{2}, B = \frac{1}{2}, \text{ and } C = \frac{1}{2}$$

From equation (1), we obtain

$$\therefore \frac{x}{(x^2+1)(x-1)} = \frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1} + \frac{\frac{1}{2}}{(x-1)}$$

$$\Rightarrow \int \frac{x}{(x^2+1)(x-1)} = -\frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$$

$$= -\frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

Consider $\int \frac{2x}{x^2+1}dx$, let $(x^2+1)=t \Rightarrow 2x\,dx = dt$

$$\Rightarrow \int \frac{2x}{x^2+1}dx = \int \frac{dt}{t} = \log|t| = \log|x^2+1|$$

$$\therefore \int \frac{x}{(x^2+1)(x-1)} = -\frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$$

Question 8:

$$\frac{x}{(x-1)^2(x+2)}$$

Answer

Let $\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Substituting $x = 1$, we obtain

$$B = \frac{1}{3}$$

Equating the coefficients of x^2 and constant term, we obtain

$A + C = 0$

$-2A + 2B + C = 0$

On solving, we obtain

$A=\frac{2}{9}$ and $C=\frac{-2}{9}$

$$\therefore \frac{x}{(x-1)^2(x+2)}=\frac{2}{9(x-1)}+\frac{1}{3(x-1)^2}-\frac{2}{9(x+2)}$$

$$\Rightarrow \int\frac{x}{(x-1)^2(x+2)}dx=\frac{2}{9}\int\frac{1}{(x-1)}dx+\frac{1}{3}\int\frac{1}{(x-1)^2}dx-\frac{2}{9}\int\frac{1}{(x+2)}dx$$

$$=\frac{2}{9}\log|x-1|+\frac{1}{3}\left(\frac{-1}{x-1}\right)-\frac{2}{9}\log|x+2|+C$$

$$=\frac{2}{9}\log\left|\frac{x-1}{x+2}\right|-\frac{1}{3(x-1)}+C$$

Question 9:

$$\frac{3x+5}{x^3-x^2-x+1}$$

Answer

$$\frac{3x+5}{x^3-x^2-x+1}=\frac{3x+5}{(x-1)^2(x+1)}$$

Let $\frac{3x+5}{(x-1)^2(x+1)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}$

$$3x+5=A(x-1)(x+1)+B(x+1)+C(x-1)^2$$

$$3x+5=A(x^2-1)+B(x+1)+C(x^2+1-2x) \quad ...(1)$$

Substituting $x = 1$ in equation (1), we obtain

$B = 4$

Equating the coefficients of x^2 and x, we obtain

$A + C = 0$

$B - 2C = 3$

On solving, we obtain

$A=-\frac{1}{2}$ and $C=\frac{1}{2}$

$$\therefore \frac{3x+5}{(x-1)^2(x+1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$$

$$\Rightarrow \int \frac{3x+5}{(x-1)^2(x+1)} dx = -\frac{1}{2}\int \frac{1}{x-1} dx + 4\int \frac{1}{(x-1)^2} dx + \frac{1}{2}\int \frac{1}{(x+1)} dx$$

$$= -\frac{1}{2}\log|x-1| + 4\left(\frac{-1}{x-1}\right) + \frac{1}{2}\log|x+1| + C$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{(x-1)} + C$$

Question 10:

$$\frac{2x-3}{(x^2-1)(2x+3)}$$

Answer

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)}$$

Let $\frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(2x+3)}$

$$\Rightarrow (2x-3) = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x+1)(x-1)$$

$$\Rightarrow (2x-3) = A(2x^2+x-3) + B(2x^2+5x+3) + C(x^2-1)$$

$$\Rightarrow (2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$$

Equating the coefficients of x^2 and x, we obtain

$$B = -\frac{1}{10},\ A = \frac{5}{2},\ \text{and } C = -\frac{24}{5}$$

$$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$$

$$\Rightarrow \int \frac{2x-3}{(x^2-1)(2x+3)} dx = \frac{5}{2}\int \frac{1}{(x+1)} dx - \frac{1}{10}\int \frac{1}{x-1} dx - \frac{24}{5}\int \frac{1}{(2x+3)} dx$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5\times 2}\log|2x+3|$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C$$

Question 11:

$$\frac{5x}{(x+1)(x^2-4)}$$

Answer

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

Let $\frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \quad ...(1)$$

Substituting $x = -1, -2$, and 2 respectively in equation (1), we obtain

$$A = \frac{5}{3},\ B = -\frac{5}{2}, \text{ and } C = \frac{5}{6}$$

$$\therefore \frac{5x}{(x+1)(x+2)(x-2)} = \frac{5}{3(x+1)} - \frac{5}{2(x+2)} + \frac{5}{6(x-2)}$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2-4)} dx = \frac{5}{3}\int \frac{1}{(x+1)} dx - \frac{5}{2}\int \frac{1}{(x+2)} dx + \frac{5}{6}\int \frac{1}{(x-2)} dx$$

$$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$$

Question 12:

$$\frac{x^3+x+1}{x^2-1}$$

Answer

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing ($x^3 + x + 1$) by $x^2 - 1$, we obtain

$$\frac{x^3+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1}$$

Let $\dfrac{2x+1}{x^2-1} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-1)}$

$$2x+1 = A(x-1)+B(x+1) \qquad ...(1)$$

Substituting x = 1 and −1 in equation (1), we obtain

$$A = \frac{1}{2} \text{ and } B = \frac{3}{2}$$

$$\therefore \frac{x^3+x+1}{x^2-1} = x + \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$$

$$\Rightarrow \int \frac{x^3+x+1}{x^2-1}dx = \int x\,dx + \frac{1}{2}\int \frac{1}{(x+1)}dx + \frac{3}{2}\int \frac{1}{(x-1)}dx$$

$$= \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$$

Question 13:

$$\frac{2}{(1-x)(1+x^2)}$$

Answer

Let $\dfrac{2}{(1-x)(1+x^2)} = \dfrac{A}{(1-x)} + \dfrac{Bx+C}{(1+x^2)}$

$$2 = A(1+x^2)+(Bx+C)(1-x)$$

$$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$$

Equating the coefficient of x^2, x, and constant term, we obtain

$A - B = 0$

$B - C = 0$

$A + C = 2$

On solving these equations, we obtain

$A = 1$, $B = 1$, and $C = 1$

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)} dx = \int \frac{1}{1-x} dx + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\log|x-1| + \frac{1}{2} \log|1+x^2| + \tan^{-1} x + C$$

Question 14:

$$\frac{3x-1}{(x+2)^2}$$

Answer

$$\text{Let } \frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

$$\Rightarrow 3x-1 = A(x+2) + B$$

Equating the coefficient of x and constant term, we obtain

$A = 3$

$2A + B = -1 \Rightarrow B = -7$

$$\therefore \frac{3x-1}{(x+2)^2}=\frac{3}{(x+2)}-\frac{7}{(x+2)^2}$$

$$\Rightarrow \int\frac{3x-1}{(x+2)^2}dx=3\int\frac{1}{(x+2)}dx-7\int\frac{x}{(x+2)^2}dx$$

$$=3\log|x+2|-7\left(\frac{-1}{(x+2)}\right)+C$$

$$=3\log|x+2|+\frac{7}{(x+2)}+C$$

Question 15:

$$\frac{1}{x^4-1}$$

Answer

$$\frac{1}{(x^4-1)}=\frac{1}{(x^2-1)(x^2+1)}=\frac{1}{(x+1)(x-1)(1+x^2)}$$

$$\text{Let } \frac{1}{(x+1)(x-1)(1+x^2)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{Cx+D}{(x^2+1)}$$

$$1=A(x-1)(x^2+1)+B(x+1)(x^2+1)+(Cx+D)(x^2-1)$$

$$1=A(x^3+x-x^2-1)+B(x^3+x+x^2+1)+Cx^3+Dx^2-Cx-D$$

$$1=(A+B+C)x^3+(-A+B+D)x^2+(A+B-C)x+(-A+B-D)$$

Equating the coefficient of x^3, x^2, x, and constant term, we obtain

$$A+B+C=0$$
$$-A+B+D=0$$
$$A+B-C=0$$
$$-A+B-D=1$$

On solving these equations, we obtain

$$A=-\frac{1}{4}, B=\frac{1}{4}, C=0, \text{ and } D=-\frac{1}{2}$$

$$\therefore \frac{1}{x^4-1}=\frac{-1}{4(x+1)}+\frac{1}{4(x-1)}-\frac{1}{2(x^2+1)}$$

$$\Rightarrow \int\frac{1}{x^4-1}dx=-\frac{1}{4}\log|x-1|+\frac{1}{4}\log|x-1|-\frac{1}{2}\tan^{-1}x+C$$

$$=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$$

Question 16:

$\frac{1}{x(x^n+1)}$ [Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

Answer

$$\frac{1}{x(x^n+1)}$$

Multiplying numerator and denominator by x^{n-1}, we obtain

$$\frac{1}{x(x^n+1)}=\frac{x^{n-1}}{x^{n-1}x(x^n+1)}=\frac{x^{n-1}}{x^n(x^n+1)}$$

$$\text{Let } x^n=t \Rightarrow x^{n-1}dx=dt$$

$$\therefore \int\frac{1}{x(x^n+1)}dx=\int\frac{x^{n-1}}{x^n(x^n+1)}dx=\frac{1}{n}\int\frac{1}{t(t+1)}dt$$

$$\text{Let } \frac{1}{t(t+1)}=\frac{A}{t}+\frac{B}{(t+1)}$$

$$1=A(1+t)+Bt \qquad \ldots(1)$$

Substituting t = 0, −1 in equation (1), we obtain

A = 1 and B = −1

$$\therefore \frac{1}{t(t+1)}=\frac{1}{t}-\frac{1}{(1+t)}$$

$$\Rightarrow \int \frac{1}{x(x^n+1)}dx = \frac{1}{n}\int\left\{\frac{1}{t}-\frac{1}{(t+1)}\right\}dx$$

$$= \frac{1}{n}\left[\log|t| - \log|t+1|\right] + C$$

$$= -\frac{1}{n}\left[\log|x^n| - \log|x^n+1|\right] + C$$

$$= \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C$$

Question 17:

$\frac{\cos x}{(1-\sin x)(2-\sin x)}$ [Hint: Put sin x = t]

Answer

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

Let $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \frac{dt}{(1-t)(2-t)}$$

Let $\frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)} + \frac{B}{(2-t)}$

$$1 = A(2-t) + B(1-t) \quad \ldots(1)$$

Substituting t = 2 and then t = 1 in equation (1), we obtain

A = 1 and B = −1

$$\therefore \frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)} - \frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \left\{ \frac{1}{1-t} - \frac{1}{(2-t)} \right\} dt$$

$$= -\log|1-t| + \log|2-t| + C$$

$$= \log\left|\frac{2-t}{1-t}\right| + C$$

$$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$$

Question 18:

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$$

Answer

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \frac{(4x^2+10)}{(x^2+3)(x^2+4)}$$

$$\text{Let } \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2+4)}$$

$$4x^2+10 = (Ax+B)(x^2+4) + (Cx+D)(x^2+3)$$

$$4x^2+10 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 3Cx + Dx^2 + 3D$$

$$4x^2+10 = (A+C)x^3 + (B+D)x^2 + (4A+3C)x + (4B+3D)$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

$A + C = 0$

$B + D = 4$

$4A + 3C = 0$

$4B + 3D = 10$

On solving these equations, we obtain

$A = 0$, $B = -2$, $C = 0$, and $D = 6$

$$\therefore \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \left(\frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}\right)$$

$$\Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx = \int \left\{1 + \frac{2}{(x^2+3)} - \frac{6}{(x^2+4)}\right\} dx$$

$$= \int \left\{1 + \frac{2}{x^2+(\sqrt{3})^2} - \frac{6}{x^2+2^2}\right\}$$

$$= x + 2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right) - 6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right) + C$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + C$$

Question 19:

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Answer

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Let $x^2 = t \Rightarrow 2x\ dx = dt$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+3)} dx = \int \frac{dt}{(t+1)(t+3)} \qquad ...(1)$$

Let $\frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)} + \frac{B}{(t+3)}$

$$1 = A(t+3) + B(t+1) \qquad ...(1)$$

Substituting $t = -3$ and $t = -1$ in equation (1), we obtain

$A=\frac{1}{2}$ and $B=-\frac{1}{2}$

$$\therefore \frac{1}{(t+1)(t+3)}=\frac{1}{2(t+1)}-\frac{1}{2(t+3)}$$

$$\Rightarrow \int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx=\int\left\{\frac{1}{2(t+1)}-\frac{1}{2(t+3)}\right\}dt$$

$$=\frac{1}{2}\log|(t+1)|-\frac{1}{2}\log|t+3|+C$$

$$=\frac{1}{2}\log\left|\frac{t+1}{t+3}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+C$$

Question 20:

$$\frac{1}{x\left(x^4-1\right)}$$

Answer

$$\frac{1}{x\left(x^4-1\right)}$$

Multiplying numerator and denominator by x^3, we obtain

$$\frac{1}{x\left(x^4-1\right)}=\frac{x^3}{x^4\left(x^4-1\right)}$$

$$\therefore \int\frac{1}{x\left(x^4-1\right)}dx=\int\frac{x^3}{x^4\left(x^4-1\right)}dx$$

Let $x^4=t \Rightarrow 4x^3dx=dt$

$$\therefore \int\frac{1}{x\left(x^4-1\right)}dx=\frac{1}{4}\int\frac{dt}{t(t-1)}$$

Let $\frac{1}{t(t-1)}=\frac{A}{t}+\frac{B}{(t-1)}$

$1=A(t-1)+Bt \qquad \ldots(1)$

Substituting t = 0 and 1 in (1), we obtain

A = −1 and B = 1

$$\Rightarrow \frac{1}{t(t+1)}=\frac{-1}{t}+\frac{1}{t-1}$$

$$\Rightarrow \int\frac{1}{x\left(x^4-1\right)}dx=\frac{1}{4}\int\left\{\frac{-1}{t}+\frac{1}{t-1}\right\}dt$$

$$=\frac{1}{4}\left[-\log|t|+\log|t-1|\right]+C$$

$$=\frac{1}{4}\log\left|\frac{t-1}{t}\right|+\mathrm{C}$$

$$=\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right|+\mathrm{C}$$

Question 21:

$\frac{1}{\left(e^x-1\right)}$ [Hint: Put e^x = t]

Answer

$$\frac{1}{\left(e^x-1\right)}$$

Let $e^x = t \Rightarrow e^x\, dx = dt$

$$\Rightarrow \int\frac{1}{e^x-1}dx=\int\frac{1}{t-1}\times\frac{dt}{t}=\int\frac{1}{t(t-1)}dt$$

Let $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$

$1 = A(t-1) + Bt \qquad ...(1)$

Substituting t = 1 and t = 0 in equation (1), we obtain

A = −1 and B = 1

$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$

$$\Rightarrow \int \frac{1}{t(t-1)} dt = \log\left|\frac{t-1}{t}\right| + C$$

$$= \log\left|\frac{e^x - 1}{e^x}\right| + C$$

Question 22:

$\int \frac{x dx}{(x-1)(x-2)}$ equals

A. $\log\left|\frac{(x-1)^2}{x-2}\right| + C$

B. $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

C. $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + C$

D. $\log\left|(x-1)(x-2)\right| + C$

Answer

Let $\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$x = A(x-2) + B(x-1) \qquad ...(1)$

Substituting x = 1 and 2 in (1), we obtain

A = −1 and B = 2

$$\therefore \frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{2}{(x-2)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)} dx = \int \left\{ \frac{-1}{(x-1)} + \frac{2}{(x-2)} \right\} dx$$

$$= -\log|x-1| + 2\log|x-2| + C$$

$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

Hence, the correct Answer is B.

Question 23:

$\int \frac{dx}{x(x^2+1)}$ equals

A. $\log|x| - \frac{1}{2}\log(x^2+1) + C$

B. $\log|x| + \frac{1}{2}\log(x^2+1) + C$

C. $-\log|x| + \frac{1}{2}\log(x^2+1) + C$

D. $\frac{1}{2}\log|x| + \log(x^2+1) + C$

Answer

Let $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$1 = A(x^2+1) + (Bx+C)x$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + B = 0$

$C = 0$

$A = 1$

On solving these equations, we obtain

$A = 1$, $B = -1$, and $C = 0$

$$\therefore \frac{1}{x\left(x^2+1\right)}=\frac{1}{x}+\frac{-x}{x^2+1}$$

$$\Rightarrow \int \frac{1}{x\left(x^2+1\right)}dx=\int\left\{\frac{1}{x}-\frac{x}{x^2+1}\right\}dx$$

$$=\log|x|-\frac{1}{2}\log\left|x^2+1\right|+C$$

Hence, the correct Answer is A.

Exercise 7.6

Question 1:

$x \sin x$

Answer

Let $I = \int x \sin x \, dx$

Taking x as first function and sin x as second function and integrating by parts, we obtain

$$I = x\int \sin x \, dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sin x \, dx\right\}dx$$

$$= x(-\cos x) - \int 1\cdot(-\cos x)\,dx$$

$$= -x\cos x + \sin x + C$$

Question 2:

$x \sin 3x$

Answer

Let $I = \int x \sin 3x \, dx$

Taking x as first function and sin $3x$ as second function and integrating by parts, we obtain

$$I = x\int \sin 3x \, dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sin 3x \, dx\right\}$$

$$= x\left(\frac{-\cos 3x}{3}\right) - \int 1\cdot\left(\frac{-\cos 3x}{3}\right)dx$$

$$= \frac{-x\cos 3x}{3} + \frac{1}{3}\int \cos 3x \, dx$$

$$= \frac{-x\cos 3x}{3} + \frac{1}{9}\sin 3x + C$$

Question 3:

$x^2 e^x$

Answer

Let $I = \int x^2 e^x dx$

Taking x^2 as first function and e^x as second function and integrating by parts, we obtain

$$I = x^2 \int e^x dx - \int \left\{ \left(\frac{d}{dx} x^2 \right) \int e^x dx \right\} dx$$

$$= x^2 e^x - \int 2x \cdot e^x dx$$

$$= x^2 e^x - 2 \int x \cdot e^x dx$$

Again integrating by parts, we obtain

$$= x^2 e^x - 2 \left[x \cdot \int e^x dx - \int \left\{ \left(\frac{d}{dx} x \right) \cdot \int e^x dx \right\} dx \right]$$

$$= x^2 e^x - 2 \left[x e^x - \int e^x dx \right]$$

$$= x^2 e^x - 2 \left[x e^x - e^x \right]$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

$$= e^x \left(x^2 - 2x + 2 \right) + C$$

Question 4:

x log*x*

Answer

Let $I = \int x \log x dx$

Taking log x as first function and x as second function and integrating by parts, we obtain

$$I = \log x \int x \, dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x \, dx \right\} dx$$

$$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$$

$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$$

Question 5:

x log 2*x*

Answer

Let $I = \int x \log 2x dx$

Taking log 2*x* as first function and *x* as second function and integrating by parts, we obtain

$$I = \log 2x \int x\ dx - \int \left\{ \left(\frac{d}{dx} 2\log x \right) \int x\ dx \right\} dx$$

$$= \log 2x \cdot \frac{x^2}{2} - \int \frac{2}{2x} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx$$

$$= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C$$

Question 6:

x^2 log *x*

Answer

Let $I = \int x^2 \log x\ dx$

Taking log *x* as first function and x^2 as second function and integrating by parts, we obtain

$$I = \log x \int x^2\ dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2 dx \right\} dx$$

$$= \log x \left(\frac{x^3}{3} \right) - \int \frac{1}{x} \cdot \frac{x^3}{3} dx$$

$$= \frac{x^3 \log x}{3} - \int \frac{x^2}{3} dx$$

$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$$

Question 7:

$x\sin^{-1}x$

Answer

Let $I = \int x\sin^{-1}x\ dx$

Taking $\sin^{-1}x$ as first function and *x* as second function and integrating by parts, we obtain

$$I = \sin^{-1}x\int x\ dx - \int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int x\ dx\right\}dx$$

$$= \sin^{-1}x\left(\frac{x^2}{2}\right) - \int\frac{1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}dx$$

$$= \frac{x^2\sin^{-1}x}{2} + \frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2\sin^{-1}x}{2} + \frac{1}{2}\int\left\{\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$= \frac{x^2\sin^{-1}x}{2} + \frac{1}{2}\int\left\{\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$= \frac{x^2\sin^{-1}x}{2} + \frac{1}{2}\left\{\int\sqrt{1-x^2}dx - \int\frac{1}{\sqrt{1-x^2}}dx\right\}$$

$$= \frac{x^2\sin^{-1}x}{2} + \frac{1}{2}\left\{\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x - \sin^{-1}x\right\} + C$$

$$= \frac{x^2\sin^{-1}x}{2} + \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1}x - \frac{1}{2}\sin^{-1}x + C$$

$$= \frac{1}{4}\left(2x^2-1\right)\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} + C$$

Question 8:

$x\tan^{-1}x$

Answer

Let $I = \int x\tan^{-1}x\ dx$

Taking $\tan^{-1} x$ as first function and *x* as second function and integrating by parts, we obtain

$$I = \tan^{-1} x \int x\,dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int x\,dx \right\} dx$$

$$= \tan^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2}\,dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \left(x - \tan^{-1} x \right) + \text{C}$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + \text{C}$$

Question 9:

$x \cos^{-1} x$

Answer

Let $I = \int x \cos^{-1} x dx$

Taking $\cos^{-1} x$ as first function and *x* as second function and integrating by parts, we obtain

$$I=\cos^{-1}x\int x\,dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int x\,dx\right\}dx$$

$$=\cos^{-1}x\frac{x^2}{2}-\int\frac{-1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\left\{\sqrt{1-x^2}+\left(\frac{-1}{\sqrt{1-x^2}}\right)\right\}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\sqrt{1-x^2}dx-\frac{1}{2}\int\left(\frac{-1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}I_1-\frac{1}{2}\cos^{-1}x \qquad \ldots(1)$$

where, $I_1=\int\sqrt{1-x^2}dx$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{d}{dx}\sqrt{1-x^2}\int x\,dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{-2x}{2\sqrt{1-x^2}}.x\,dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\left\{\int\sqrt{1-x^2}\,dx+\int\frac{-dx}{\sqrt{1-x^2}}\right\}$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\left\{I_1+\cos^{-1}x\right\}$$

$$\Rightarrow 2I_1=x\sqrt{1-x^2}-\cos^{-1}x$$

$$\therefore I_1=\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\cos^{-1}x$$

Substituting in (1), we obtain

$$I=\frac{x\cos^{-1}x}{2}-\frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\cos^{-1}x\right)-\frac{1}{2}\cos^{-1}x$$

$$=\frac{\left(2x^2-1\right)}{4}\cos^{-1}x-\frac{x}{4}\sqrt{1-x^2}+C$$

Question 10:

$\left(\sin^{-1} x\right)^2$

Answer

Let $I = \int \left(\sin^{-1} x\right)^2 \cdot 1\, dx$

Taking $\left(\sin^{-1} x\right)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I = \left(\sin^{-1} x\right) \int 1 dx - \int \left\{ \frac{d}{dx}\left(\sin^{-1} x\right)^2 \cdot \int 1 \cdot dx \right\} dx$$

$$= \left(\sin^{-1} x\right)^2 \cdot x - \int \frac{2\sin^{-1} x}{\sqrt{1-x^2}} \cdot x\, dx$$

$$= x\left(\sin^{-1} x\right)^2 + \int \sin^{-1} x \cdot \left(\frac{-2x}{\sqrt{1-x^2}} \right) dx$$

$$= x\left(\sin^{-1} x\right)^2 + \left[\sin^{-1} x \int \frac{-2x}{\sqrt{1-x^2}} dx - \int \left\{ \left(\frac{d}{dx} \sin^{-1} x \right) \int \frac{-2x}{\sqrt{1-x^2}} dx \right\} dx \right]$$

$$= x\left(\sin^{-1} x\right)^2 + \left[\sin^{-1} x \cdot 2\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2}\, dx \right]$$

$$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2} \sin^{-1} x - \int 2 dx$$

$$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2} \sin^{-1} x - 2x + C$$

Question 11:

$\frac{x \cos^{-1} x}{\sqrt{1-x^2}}$

Answer

Let $I = \int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx$

$$I = \frac{-1}{2} \int \frac{-2x}{\sqrt{1-x^2}} \cdot \cos^{-1} x dx$$

Taking $\cos^{-1} x$ as first function and $\left(\frac{-2x}{\sqrt{1-x^2}}\right)$ as second function and integrating by parts, we obtain

$$I = \frac{-1}{2}\left[\cos^{-1} x \int \frac{-2x}{\sqrt{1-x^2}} dx - \int \left\{\left(\frac{d}{dx}\cos^{-1} x\right) \int \frac{-2x}{\sqrt{1-x^2}} dx\right\} dx\right]$$

$$= \frac{-1}{2}\left[\cos^{-1} x \cdot 2\sqrt{1-x^2} - \int \frac{-1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2}\, dx\right]$$

$$= \frac{-1}{2}\left[2\sqrt{1-x^2} \cos^{-1} x + \int 2\, dx\right]$$

$$= \frac{-1}{2}\left[2\sqrt{1-x^2} \cos^{-1} x + 2x\right] + C$$

$$= -\left[\sqrt{1-x^2} \cos^{-1} x + x\right] + C$$

Question 12:

$x \sec^2 x$

Answer

Let $I = \int x \sec^2 x dx$

Taking x as first function and $\sec^2 x$ as second function and integrating by parts, we obtain

$$I = x \int \sec^2 x\, dx - \int \left\{\left\{\frac{d}{dx} x\right\} \int \sec^2 x\, dx\right\} dx$$

$$= x \tan x - \int 1 \cdot \tan x dx$$

$$= x \tan x + \log |\cos x| + C$$

Question 13:

$\tan^{-1} x$

Answer

Let $I = \int 1 \cdot \tan^{-1} x dx$

Taking $\tan^{-1} x$ as first function and 1 as second function and integrating by parts, we obtain

$$I = \tan^{-1} x \int 1 dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int 1 \cdot dx \right\} dx$$

$$= \tan^{-1} x \cdot x - \int \frac{1}{1+x^2} \cdot x \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$= x \tan^{-1} x - \frac{1}{2} \log\left|1+x^2\right| + C$$

$$= x \tan^{-1} x - \frac{1}{2} \log\left(1+x^2\right) + C$$

Question 14:

$x(\log x)^2$

Answer

$$I = \int x (\log x)^2 \, dx$$

Taking $(\log x)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I = (\log x)^2 \int x \, dx - \int \left[\left\{ \left(\frac{d}{dx} \log x \right)^2 \right\} \int x \, dx \right] dx$$

$$= \frac{x^2}{2} (\log x)^2 - \left[\int 2 \log x \cdot \frac{1}{x} \cdot \frac{x^2}{2} dx \right]$$

$$= \frac{x^2}{2} (\log x)^2 - \int x \log x \, dx$$

Again integrating by parts, we obtain

$$I=\frac{x^2}{2}(\log x)^2-\left[\log x\int x\,dx-\int\left\{\left(\frac{d}{dx}\log x\right)\int x\,dx\right\}dx\right]$$

$$=\frac{x^2}{2}(\log x)^2-\left[\frac{x^2}{2}-\log x-\int\frac{1}{x}\cdot\frac{x^2}{2}dx\right]$$

$$=\frac{x^2}{2}(\log x)^2-\frac{x^2}{2}\log x+\frac{1}{2}\int x\,dx$$

$$=\frac{x^2}{2}(\log x)^2-\frac{x^2}{2}\log x+\frac{x^2}{4}+C$$

Question 15:

$\left(x^2+1\right)\log x$

Answer

Let $I=\int\left(x^2+1\right)\log x\,dx=\int x^2\log x\,dx+\int\log x\,dx$

Let $I = I_1 + I_2$ ... (1)

Where, $I_1=\int x^2\log x\,dx$ and $I_2=\int\log x\,dx$

$I_1=\int x^2\log x dx$

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$I_1=\log x-\int x^2dx-\int\left\{\left(\frac{d}{dx}\log x\right)\int x^2dx\right\}dx$$

$$=\log x\cdot\frac{x^3}{3}-\int\frac{1}{x}\cdot\frac{x^3}{3}dx$$

$$=\frac{x^3}{3}\log x-\frac{1}{3}\left(\int x^2dx\right)$$

$$=\frac{x^3}{3}\log x-\frac{x^3}{9}+C_1 \qquad \ldots(2)$$

$I_2=\int\log x\,dx$

Taking log x as first function and 1 as second function and integrating by parts, we obtain

$$I_2 = \log x \int 1 \cdot dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int 1 \cdot dx \right\}$$

$$= \log x \cdot x - \int \frac{1}{x} \cdot x dx$$

$$= x \log x - \int 1 dx$$

$$= x \log x - x + C_2 \qquad \ldots (3)$$

Using equations (2) and (3) in (1), we obtain

$$I = \frac{x^3}{3} \log x - \frac{x^3}{9} + C_1 + x \log x - x + C_2$$

$$= \frac{x^3}{3} \log x - \frac{x^3}{9} + x \log x - x + (C_1 + C_2)$$

$$= \left(\frac{x^3}{3} + x \right) \log x - \frac{x^3}{9} - x + C$$

Question 16:

$e^x (\sin x + \cos x)$

Answer

Let $I = \int e^x (\sin x + \cos x) dx$

Let $f(x) = \sin x$

□ $f'(x) = \cos x$

□ $I = \int e^x \{f(x) + f'(x)\} dx$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$\therefore I = e^x \sin x + C$

Question 17:

$$\frac{xe^x}{(1+x)^2}$$

Answer

Let $I=\int \frac{xe^x}{(1+x)^2}dx=\int e^x\left\{\frac{x}{(1+x)^2}\right\}dx$

$$=\int e^x\left\{\frac{1+x-1}{(1+x)^2}\right\}dx$$

$$=\int e^x\left\{\frac{1}{1+x}-\frac{1}{(1+x)^2}\right\}dx$$

Let $f(x)=\frac{1}{1+x}$ □ $f'(x)=\frac{-1}{(1+x)^2}$

$$\Rightarrow \int \frac{xe^x}{(1+x)^2}dx=\int e^x\{f(x)+f'(x)\}dx$$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+\mathrm{C}$

$$\therefore \int \frac{xe^x}{(1+x)^2}dx=\frac{e^x}{1+x}+\mathrm{C}$$

Question 18:

$$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$$

Answer

$$e^{x}\left(\frac{1+\sin x}{1+\cos x}\right)$$

$$=e^{x}\left(\frac{\sin^{2}\frac{x}{2}+\cos^{2}\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^{2}\frac{x}{2}}\right)$$

$$=\frac{e^{x}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^{2}}{2\cos^{2}\frac{x}{2}}$$

$$=\frac{1}{2}e^{x}\cdot\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^{2}$$

$$=\frac{1}{2}e^{x}\left[\tan\frac{x}{2}+1\right]^{2}$$

$$=\frac{1}{2}e^{2}\left(1+\tan\frac{x}{2}\right)^{2}$$

$$=\frac{1}{2}e^{x}\left[1+\tan^{2}\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$=\frac{1}{2}e^{x}\left[\sec^{2}\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$\frac{e^{x}(1+\sin x)dx}{(1+\cos x)}=e^{x}\left[\frac{1}{2}\sec^{2}\frac{x}{2}+\tan\frac{x}{2}\right] \quad \ldots(1)$$

Let $\tan\frac{x}{2}=f(x)$ □ $f'(x)=\frac{1}{2}\sec^{2}\frac{x}{2}$

It is known that, $\int e^{x}\{f(x)+f'(x)\}dx=e^{x}f(x)+C$

From equation (1), we obtain

$$\int\frac{e^{x}(1+\sin x)}{(1+\cos x)}dx=e^{x}\tan\frac{x}{2}+C$$

Question 19:

$$e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$$

Answer

$$\text{Let } I=\int e^x\left[\frac{1}{x}-\frac{1}{x^2}\right]dx$$

Also, let $\frac{1}{x}=f(x)$ □ $f'(x)=\frac{-1}{x^2}$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

$$\therefore I=\frac{e^x}{x}+C$$

Question 20:

$$\frac{(x-3)e^x}{(x-1)^3}$$

Answer

$$\int e^x\left\{\frac{x-3}{(x-1)^3}\right\}dx=\int e^x\left\{\frac{x-1-2}{(x-1)^3}\right\}dx$$

$$=\int e^x\left\{\frac{1}{(x-1)^2}-\frac{2}{(x-1)^3}\right\}dx$$

Let $f(x)=\frac{1}{(x-1)^2}$ □ $f'(x)=\frac{-2}{(x-1)^3}$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

$$\therefore \int e^x\left\{\frac{(x-3)}{(x-1)^2}\right\}dx=\frac{e^x}{(x-1)^2}+C$$

Question 21:

$$e^{2x}\sin x$$

Answer

Let $I = \int e^{2x} \sin x \, dx \qquad ...(1)$

Integrating by parts, we obtain

$$I = \sin x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \sin x \right) \int e^{2x} dx \right\} dx$$

$$\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2} dx$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x \, dx$$

Again integrating by parts, we obtain

$$I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[\cos x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \cos x \right) \int e^{2x} dx \right\} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \left[\cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \frac{e^{2x}}{2} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[\frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} I \qquad \text{[From (1)]}$$

$$\Rightarrow I + \frac{1}{4} I = \frac{e^{2x} \cdot \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow \frac{5}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow I = \frac{4}{5} \left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \right] + C$$

$$\Rightarrow I = \frac{e^{2x}}{5} \left[2 \sin x - \cos x \right] + C$$

Question 22:

$$\sin^{-1} \left(\frac{2x}{1+x^2} \right)$$

Answer

Let $x = \tan \theta$ □ $dx = \sec^2 \theta \, d\theta$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

□ $$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int 2\theta \cdot \sec^2\theta \, d\theta = 2\int \theta \cdot \sec^2\theta \, d\theta$$

Integrating by parts, we obtain

$$2\left[\theta \cdot \int \sec^2\theta d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\int \sec^2\theta d\theta\right\}d\theta\right]$$

$$= 2\left[\theta \cdot \tan\theta - \int \tan\theta d\theta\right]$$

$$= 2\left[\theta\tan\theta + \log\left|\cos\theta\right|\right] + C$$

$$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C$$

$$= 2x\tan^{-1}x + 2\log\left(1+x^2\right)^{-\frac{1}{2}} + C$$

$$= 2x\tan^{-1}x + 2\left[-\frac{1}{2}\log\left(1+x^2\right)\right] + C$$

$$= 2x\tan^{-1}x - \log\left(1+x^2\right) + C$$

Question 23:

$\int x^2 e^{x^3} dx$ equals

(A) $\frac{1}{3}e^{x^3} + C$ (B) $\frac{1}{3}e^{x^2} + C$

(C) $\frac{1}{2}e^{x^3} + C$ (D) $\frac{1}{3}e^{x^2} + C$

Answer

Let $I = \int x^2 e^{x^3} dx$

Also, let $x^3 = t$ □ $3x^2 dx = dt$

$$\Rightarrow I = \frac{1}{3}\int e^t dt$$

$$= \frac{1}{3}\left(e^t\right) + C$$

$$= \frac{1}{3}e^{x^3} + C$$

Hence, the correct Answer is A.

Question 24:

$\int e^x \sec x(1+\tan x)dx$ equals

(A) $e^x \cos x + C$ (B) $e^x \sec x + C$

(C) $e^x \sin x + C$ (D) $e^x \tan x + C$

Answer

$\int e^x \sec x(1+\tan x)dx$

Let $I = \int e^x \sec x(1+\tan x)dx = \int e^x\left(\sec x + \sec x \tan x\right)dx$

Also, let $\sec x = f(x)$ □ $\sec x \tan x = f'(x)$

It is known that, $\int e^x\left\{f(x) + f'(x)\right\}dx = e^x f(x) + C$

$\therefore I = e^x \sec x + C$

Hence, the correct Answer is B.

Exercise 7.7

Question 1:

$\sqrt{4-x^2}$

Answer

Let $I = \int\sqrt{4-x^2}\,dx = \int\sqrt{(2)^2-(x)^2}\,dx$

It is known that, $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}\,\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\text{C}$

$$\therefore I = \frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}+\text{C}$$

$$= \frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+\text{C}$$

Question 2:

$\sqrt{1-4x^2}$

Answer

Let $I = \int\sqrt{1-4x^2}\,dx = \int\sqrt{(1)^2-(2x)^2}\,dx$

Let $2x = t \Rightarrow 2\,dx = dt$

$$\therefore I = \frac{1}{2}\int\sqrt{(1)^2-(t)^2}\,dt$$

It is known that, $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\text{C}$

$$\Rightarrow I = \frac{1}{2}\left[\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]+\text{C}$$

$$= \frac{t}{4}\sqrt{1-t^2}+\frac{1}{4}\sin^{-1}t+\text{C}$$

$$= \frac{2x}{4}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+\text{C}$$

$$= \frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+\text{C}$$

Question 3:

$\sqrt{x^2+4x+6}$

Answer

$$\text{Let } I=\int\sqrt{x^2+4x+6}\;dx$$
$$=\int\sqrt{x^2+4x+4+2}\;dx$$
$$=\int\sqrt{\left(x^2+4x+4\right)+2}\;dx$$
$$=\int\sqrt{(x+2)^2+\left(\sqrt{2}\right)^2}\;dx$$

It is known that, $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\frac{2}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$$
$$=\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$$

Question 4:

$\sqrt{x^2+4x+1}$

Answer

$$\text{Let } I=\int\sqrt{x^2+4x+1}\,dx$$
$$=\int\sqrt{\left(x^2+4x+4\right)-3}\,dx$$
$$=\int\sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}\;dx$$

It is known that, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right|+C$$

Question 5:

$\sqrt{1-4x-x^2}$

Answer

$$\text{Let } I=\int\sqrt{1-4x-x^2}\,dx$$
$$=\int\sqrt{1-\left(x^2+4x+4-4\right)}\,dx$$
$$=\int\sqrt{1+4-(x+2)^2}\,dx$$
$$=\int\sqrt{\left(\sqrt{5}\right)^2-(x+2)^2}\,dx$$

It is known that, $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\text{C}$

$$\therefore I=\frac{(x+2)}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+\text{C}$$

Question 6:

$\sqrt{x^2+4x-5}$

Answer

$$\text{Let } I=\int\sqrt{x^2+4x-5}\,dx$$
$$=\int\sqrt{\left(x^2+4x+4\right)-9}\,dx$$
$$=\int\sqrt{(x+2)^2-(3)^2}\,dx$$

It is known that, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+\text{C}$

$$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+\text{C}$$

Question 7:

$\sqrt{1+3x-x^2}$

Answer

Let $I = \int \sqrt{1+3x-x^2}\,dx$

$$= \int \sqrt{1-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}\,dx$$

$$= \int \sqrt{\left(1+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2}\,dx$$

$$= \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\,dx$$

It is known that, $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$$\therefore I = \frac{x-\frac{3}{2}}{2}\sqrt{1+3x-x^2}+\frac{13}{4\times 2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right)+C$$

$$= \frac{2x-3}{4}\sqrt{1+3x-x^2}+\frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right)+C$$

Question 8:

$\sqrt{x^2+3x}$

Answer

Let $I = \int \sqrt{x^2+3x}\,dx$

$$= \int \sqrt{x^2+3x+\frac{9}{4}-\frac{9}{4}}\,dx$$

$$= \int \sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}\,dx$$

It is known that, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore I=\frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{x^2+3x}-\frac{\frac{9}{4}}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$$

$$=\frac{(2x+3)}{4}\sqrt{x^2+3x}-\frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$$

Question 9:

$\sqrt{1+\frac{x^2}{9}}$

Answer

Let $I=\int\sqrt{1+\frac{x^2}{9}}dx=\frac{1}{3}\int\sqrt{9+x^2}\,dx=\frac{1}{3}\int\sqrt{(3)^2+x^2}\,dx$

It is known that, $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$$\therefore I=\frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+9}+\frac{9}{2}\log\left|x+\sqrt{x^2+9}\right|\right]+C$$

$$=\frac{x}{6}\sqrt{x^2+9}+\frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+C$$

Question 10:

$\int\sqrt{1+x^2}\,dx$ is equal to

A. $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

B. $\frac{2}{3}\left(1+x^2\right)^{\frac{2}{3}}+C$

C. $\frac{2}{3}x\left(1+x^2\right)^{\frac{3}{2}}+C$

D. $\frac{x^2}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right|+C$

Answer

It is known that, $\int\sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$\therefore \int\sqrt{1+x^2}\,dx=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

Hence, the correct Answer is A.

Question 11:

$\int\sqrt{x^2-8x+7}\,dx$ is equal to

A. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

B. $\frac{1}{2}(x+4)\sqrt{x^2-8x+7}+9\log\left|x+4+\sqrt{x^2-8x+7}\right|+C$

C. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

D. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

Answer

$$\begin{aligned}\text{Let } I &= \int\sqrt{x^2-8x+7}\,dx\\ &= \int\sqrt{\left(x^2-8x+16\right)-9}\,dx\\ &= \int\sqrt{(x-4)^2-(3)^2}\,dx\end{aligned}$$

It is known that, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\therefore I=\frac{(x-4)}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\sqrt{x^2-8x+7}\right|+C$

Hence, the correct Answer is D.

Exercise 7.8

Question 1:

$$\int_a^b x\,dx$$

Answer

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f\left(a+(n-1)h\right)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=a,\ b=b,$ and $f(x)=x$

$$\therefore \int_a^b x\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[a+(a+h)\ldots(a+2h)\ldots a+(n-1)h\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{a+a+a+\ldots+a}_{n\text{ times}}\right)+\left(h+2h+3h+\ldots+(n-1)h\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left(1+2+3+\ldots+(n-1)\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+\frac{n(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{n}{n}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)(b-a)}{2n}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{\left(1-\frac{1}{n}\right)(b-a)}{2}\right]$$

$$=(b-a)\left[a+\frac{(b-a)}{2}\right]$$

$$=(b-a)\left[\frac{2a+b-a}{2}\right]$$

$$=\frac{(b-a)(b+a)}{2}$$

$$=\frac{1}{2}\left(b^2-a^2\right)$$

Question 2:

$$\int_0^5 (x+1)\,dx$$

Answer

Let $I = \int_0^5 (x+1)\,dx$

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)\ldots f(a+(n-1)h)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=0$, $b=5$, and $f(x)=(x+1)$

$$\Rightarrow h=\frac{5-0}{n}=\frac{5}{n}$$

$$\therefore \int_0^5 (x+1)\,dx = (5-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right)+\ldots+f\left((n-1)\frac{5}{n}\right)\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{5}{n}+1\right)+\ldots\left\{1+\left(\frac{5(n-1)}{n}\right)\right\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[(\underbrace{1+1+1\ldots1}_{n\text{ times}})+\left[\frac{5}{n}+2\cdot\frac{5}{n}+3\cdot\frac{5}{n}+\ldots(n-1)\frac{5}{n}\right]\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\{1+2+3\ldots(n-1)\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\cdot\frac{(n-1)n}{2}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5(n-1)}{2}\right]$$

$$=5\lim_{n\to\infty}\left[1+\frac{5}{2}\left(1-\frac{1}{n}\right)\right]$$

$$=5\left[1+\frac{5}{2}\right]$$

$$=5\left[\frac{7}{2}\right]$$

$$=\frac{35}{2}$$

Question 3:

$$\int_2^3 x^2 dx$$

Answer

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f(a+2h)\ldots f\{a+(n-1)h\}\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=2$, $b=3$, and $f(x)=x^2$

$$\Rightarrow h=\frac{3-2}{n}=\frac{1}{n}$$

$$\therefore \int_2^3 x^2 dx = (3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+f\left(2+\frac{2}{n}\right)\ldots f\left\{2+(n-1)\frac{1}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[(2)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+\ldots\left(2+\frac{(n-1)}{n}\right)^2\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[2^2+\left\{2^2+\left(\frac{1}{n}\right)^2+2\cdot2\cdot\frac{1}{n}\right\}+\ldots+\left\{(2)^2+\frac{(n-1)^2}{n^2}+2\cdot2\cdot\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{2^2+\ldots+2^2}_{n\text{ times}}\right)+\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+\left(\frac{n-1}{n}\right)^2\right\}+2\cdot2\cdot\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\ldots+\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{1^2+2^2+3^2\ldots+(n-1)^2\right\}+\frac{4}{n}\left\{1+2+\ldots+(n-1)\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}+\frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{4n-4}{2}\right]$$

$$=\lim_{n\to\infty}\left[4+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+2-\frac{2}{n}\right]$$

$$=4+\frac{2}{6}+2$$

$$=\frac{19}{3}$$

Question 4:

$$\int_1^4 \left(x^2 - x\right) dx$$

Answer

$$\text{Let } I = \int_1^4 \left(x^2 - x\right) dx$$
$$= \int_1^4 x^2\, dx - \int_1^4 x\, dx$$

$$\text{Let } I = I_1 - I_2, \text{ where } I_1 = \int_1^4 x^2 dx \text{ and } I_2 = \int_1^4 x\, dx \qquad \ldots(1)$$

It is known that,

$$\int_a^b f(x) dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a) + f(a+h) + f\left(a+(n-1)h\right)\right], \text{ where } h = \frac{b-a}{n}$$

$$\text{For } I_1 = \int_1^4 x^2 dx,$$

$$a = 1, b = 4, \text{ and } f(x) = x^2$$

$$\therefore h = \frac{4-1}{n} = \frac{3}{n}$$

$$I_1 = \int_1^4 x^2 dx = (4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1) + f(1+h) + \ldots + f\left(1+(n-1)h\right)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1^2 + \left(1+\frac{3}{n}\right)^2 + \left(1+2\cdot\frac{3}{n}\right)^2 + \ldots\left(1+\frac{(n-1)3}{n}\right)^2\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1^2 + \left\{1^2 + \left(\frac{3}{n}\right)^2 + 2\cdot\frac{3}{n}\right\} + \ldots + \left\{1^2 + \left(\frac{(n-1)3}{n}\right)^2 + \frac{2\cdot(n-1)\cdot 3}{n}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1^2 + \ldots + 1^2}_{n \text{ times}}\right) + \left(\frac{3}{n}\right)^2\left\{1^2 + 2^2 + \ldots + (n-1)^2\right\} + 2\cdot\frac{3}{n}\left\{1 + 2 + \ldots + (n-1)\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9}{n^2}\left\{\frac{(n-1)(n)(2n-1)}{6}\right\}+\frac{6}{n}\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{6n-6}{2}\right]$$

$$= 3\lim_{n\to\infty}\left[1+\frac{9}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+3-\frac{3}{n}\right]$$

$$= 3[1+3+3]$$

$$= 3[7]$$

$$I_1 = 21 \qquad \ldots(2)$$

For $I_2 = \int_1^4 x\,dx$,

$a = 1, b = 4,$ and $f(x) = x$

$$\Rightarrow h = \frac{4-1}{n} = \frac{3}{n}$$

$$\therefore I_2 = (4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+\ldots f(a+(n-1)h)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+(1+h)+\ldots+(1+(n-1)h)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{3}{n}\right)+\ldots+\left\{1+(n-1)\frac{3}{n}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+\ldots+1}_{n\text{ times}}\right)+\frac{3}{n}\left(1+2+\ldots+(n-1)\right)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{3}{n}\left\{\frac{(n-1)n}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{3}{2}\left(1-\frac{1}{n}\right)\right]$$

$$= 3\left[1+\frac{3}{2}\right]$$

$$= 3\left[\frac{5}{2}\right]$$

$$I_2 = \frac{15}{2} \qquad \ldots(3)$$

From equations (2) and (3), we obtain

$$I = I_1 + I_2 = 21 - \frac{15}{2} = \frac{27}{2}$$

Question 5:

$$\int_{-1}^{1} e^x dx$$

Answer

$$\text{Let } I = \int_{-1}^{1} e^x dx \qquad ...(1)$$

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)...f(a+(n-1)h)\right], \text{ where } h = \frac{b-a}{n}$$

Here, $a = -1$, $b = 1$, and $f(x) = e^x$

$$\therefore h = \frac{1+1}{n} = \frac{2}{n}$$

$$\therefore I=(1+1)\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2\cdot\frac{2}{n}\right)+...+f\left(-1+\frac{(n-1)2}{n}\right)\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2\cdot\frac{2}{n}\right)}+...e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$$

$$=2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right]$$

$$=e^{-1}\times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$$

$$=\frac{e^{-1}\times 2\left(e^2-1\right)}{\lim\limits_{\frac{2}{n}\to 0}\left(\dfrac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right)\times 2}$$

$$=e^{-1}\left[\frac{2\left(e^2-1\right)}{2}\right] \qquad \left[\lim_{h\to 0}\left(\frac{e^h-1}{h}\right)=1\right]$$

$$=\frac{e^2-1}{e}$$

$$=\left(e-\frac{1}{e}\right)$$

Question 6:

$$\int_0^4\left(x+e^{2x}\right)dx$$

Answer

It is known that,

$$\int_a^b f(x)\,dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f(a+(n-1)h)\right],\text{ where } h=\frac{b-a}{n}$$

Here, $a=0$, $b=4$, and $f(x)=x+e^{2x}$

$$\therefore h=\frac{4-0}{n}=\frac{4}{n}$$

$$\Rightarrow \int_0^4 \left(x+e^{2x}\right)dx = (4-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(h)+f(2h)+\ldots+f\left((n-1)h\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\left(0+e^0\right)+\left(h+e^{2h}\right)+\left(2h+e^{2\cdot 2h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[1+\left(h+e^{2h}\right)+\left(2h+e^{4h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\left\{h+2h+3h+\ldots+(n-1)h\right\}+\left(1+e^{2h}+e^{4h}+\ldots+e^{2(n-1)h}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[h\left\{1+2+\ldots(n-1)\right\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(h(n-1)n\right)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}\cdot\frac{(n-1)n}{2}+\left(\frac{e^8-1}{e^{\frac{8}{n}}-1}\right)\right]$$

$$= 4(2)+4\lim_{n\to\infty}\frac{\left(e^8-1\right)}{\left(\frac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$= 8+\frac{4\cdot\left(e^8-1\right)}{8} \qquad \left(\lim_{x\to 0}\frac{e^x-1}{x}=1\right)$$

$$= 8+\frac{e^8-1}{2}$$

$$= \frac{15+e^8}{2}$$

Exercise 7.9

Question 1:

$$\int_{-1}^{1}(x+1)dx$$

Answer

$$\text{Let } I=\int_{-1}^{1}(x+1)dx$$

$$\int(x+1)\,dx=\frac{x^2}{2}+x=\text{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I=\text{F}(1)-\text{F}(-1)$$
$$=\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)$$
$$=\frac{1}{2}+1-\frac{1}{2}+1$$
$$=2$$

Question 2:

$$\int_{2}^{3}\frac{1}{x}dx$$

Answer

$$\text{Let } I=\int_{2}^{3}\frac{1}{x}\,dx$$

$$\int\frac{1}{x}\,dx=\log|x|=\text{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I=\text{F}(3)-\text{F}(2)$$
$$=\log|3|-\log|2|=\log\frac{3}{2}$$

Question 3:

$$\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$$

Answer

$$\text{Let } I = \int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$$

$$\int \left(4x^3 - 5x^2 + 6x + 9\right) dx = 4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right) + 9(x)$$

$$= x^4 - \frac{5x^3}{3} + 3x^2 + 9x = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(2) - \mathrm{F}(1)$$

$$I = \left\{2^4 - \frac{5\cdot(2)^3}{3} + 3(2)^2 + 9(2)\right\} - \left\{(1)^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1)\right\}$$

$$= \left(16 - \frac{40}{3} + 12 + 18\right) - \left(1 - \frac{5}{3} + 3 + 9\right)$$

$$= 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9$$

$$= 33 - \frac{35}{3}$$

$$= \frac{99 - 35}{3}$$

$$= \frac{64}{3}$$

Question 4:

$$\int_0^{\frac{\pi}{4}} \sin 2x dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \sin 2x \, dx$$

$$\int \sin 2x \, dx = \left(\frac{-\cos 2x}{2}\right) = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= -\frac{1\pi}{2}\left[\cos 2\left(\frac{}{4}\right) - \cos 0\right]$$

$$= -\frac{1\pi}{2}\left[\cos\left(\frac{}{2}\right) - \cos 0\right]$$

$$= -\frac{1}{2}[0-1]$$

$$= \frac{1}{2}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \cos 2x \, dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \cos 2x \, dx$$

$$\int \cos 2x \, dx = \left(\frac{\sin 2x}{2}\right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{2}\right) - F(0)$$

$$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right]$$

$$= \frac{1}{2}[\sin\pi - \sin 0]$$

$$= \frac{1}{2}[0-0] = 0$$

Question 6:

$$\int_4^5 e^x \, dx$$

Answer

Let $I = \int_4^5 e^x dx$

$\int e^x dx = e^x = \text{F}(x)$

By second fundamental theorem of calculus, we obtain

$I = \text{F}(5) - \text{F}(4)$

$= e^5 - e^4$

$= e^4(e-1)$

Question 7:

$\int_0^{\frac{\pi}{4}} \tan x \, dx$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \tan x \, dx$

$\int \tan x \, dx = -\log|\cos x| = \text{F}(x)$

By second fundamental theorem of calculus, we obtain

$I = \text{F}\left(\frac{\pi}{4}\right) - \text{F}(0)$

$= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0|$

$= -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|$

$= -\log(2)^{-\frac{1}{2}}$

$= \frac{1}{2}\log 2$

Question 8:

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

Answer

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

$\int \operatorname{cosec} x \; dx = \log\left|\operatorname{cosec} x - \cot x\right| = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}\left(\frac{\pi}{4}\right) - \mathrm{F}\left(\frac{\pi}{6}\right)$$

$$= \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$$

$$= \log\left|\sqrt{2} - 1\right| - \log\left|2 - \sqrt{3}\right|$$

$$= \log\left(\frac{\sqrt{2}-1}{2-\sqrt{3}}\right)$$

Question 9:

$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Answer

Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(1) - \mathrm{F}(0)$$

$$= \sin^{-1}(1) - \sin^{-1}(0)$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Question 10:

$\int_0^1 \frac{dx}{1+x^2}$

Answer

Let $I = \int_0^1 \frac{dx}{1+x^2}$

$\int \frac{dx}{1+x^2} = \tan^{-1} x = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$I = \mathrm{F}(1) - \mathrm{F}(0)$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4}$

Question 11:

$\int_2^3 \frac{dx}{x^2-1}$

Answer

Let $I = \int_2^3 \frac{dx}{x^2-1}$

$\int \frac{dx}{x^2-1} = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$I = \mathrm{F}(3) - \mathrm{F}(2)$

$= \frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right| - \log\left|\frac{2-1}{2+1}\right|\right]$

$= \frac{1}{2}\left[\log\left|\frac{2}{4}\right| - \log\left|\frac{1}{3}\right|\right]$

$= \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right]$

$= \frac{1}{2}\left[\log\frac{3}{2}\right]$

Question 12:

$$\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx$

$$\int \cos^2 x\,dx = \int\left(\frac{1+\cos 2x}{2}\right)dx = \frac{x}{2}+\frac{\sin 2x}{4} = \frac{1}{2}\left(x+\frac{\sin 2x}{2}\right) = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \left[\mathrm{F}\left(\frac{\pi}{2}\right)-\mathrm{F}(0)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}-\frac{\sin\pi}{2}\right)-\left(0+\frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+0-0-0\right]$$

$$= \frac{\pi}{4}$$

Question 13:

$$\int_2^3 \frac{x\,dx}{x^2+1}$$

Answer

Let $I = \int_2^3 \frac{x}{x^2+1}dx$

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int\frac{2x}{x^2+1}dx = \frac{1}{2}\log\left(1+x^2\right) = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(3)-\mathrm{F}(2)$$

$$= \frac{1}{2}\left[\log\left(1+(3)^2\right)-\log\left(1+(2)^2\right)\right]$$

$$= \frac{1}{2}\left[\log(10)-\log(5)\right]$$

$$= \frac{1}{2}\log\left(\frac{10}{5}\right) = \frac{1}{2}\log 2$$

Question 14:

$$\int_0^1 \frac{2x+3}{5x^2+1}dx$$

Answer

$$\text{Let } I = \int_0^1 \frac{2x+3}{5x^2+1}dx$$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{5(2x+3)}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x+15}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)}dx$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{5}\cdot\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}}$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}x\right)$$

$$= \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(1) - \mathrm{F}(0)$$

$$= \left\{\frac{1}{5}\log(5+1) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)\right\} - \left\{\frac{1}{5}\log(1) + \frac{3}{\sqrt{5}}\tan^{-1}(0)\right\}$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

Question 15:

$$\int_0^1 xe^{x^2}dx$$

Answer

Let $I = \int_0^1 xe^{x^2}dx$

Put $x^2 = t \Rightarrow 2x\ dx = dt$

As $x \to 0, t \to 0$ and as $x \to 1, t \to 1,$

$\therefore I = \frac{1}{2}\int_0^1 e^t dt$

$\frac{1}{2}\int e^t dt = \frac{1}{2}e^t = \mathrm{F}(t)$

By second fundamental theorem of calculus, we obtain

$I = \mathrm{F}(1) - \mathrm{F}(0)$

$= \frac{1}{2}e - \frac{1}{2}e^0$

$= \frac{1}{2}(e-1)$

Question 16:

$\int_0^1 \frac{5x^2}{x^2+4x+3}$

Answer

Let $I = \int_1^2 \frac{5x^2}{x^2+4x+3}dx$

Dividing $5x^2$ by x^2+4x+3, we obtain

$I = \int_1^2 \left\{5 - \frac{20x+15}{x^2+4x+3}\right\}dx$

$= \int_1^2 5dx - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$

$= [5x]_1^2 - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$

$I = 5 - I_1$, where $I = \int_1^2 \frac{20x+15}{x^2+4x+3}dx$ ...(1)

Consider $I_1 = \int_1^2 \frac{20x+15}{x^2+4x+8} dx$

Let $20x+15 = A\frac{d}{dx}(x^2+4x+3)+B$

$= 2Ax+(4A+B)$

Equating the coefficients of x and constant term, we obtain

A = 10 and B = −25

$\Rightarrow I_1 = 10\int_1^2 \frac{2x+4}{x^2+4x+3}dx - 25\int_1^2 \frac{dx}{x^2+4x+3}$

Let $x^2+4x+3=t$

$\Rightarrow (2x+4)dx = dt$

$\Rightarrow I_1 = 10\int\frac{dt}{t} - 25\int\frac{dx}{(x+2)^2-1^2}$

$= 10\log t - 25\left[\frac{1}{2}\log\left(\frac{x+2-1}{x+2+1}\right)\right]$

$= \left[10\log\left(x^2+4x+3\right)\right]_1^2 - 25\left[\frac{1}{2}\log\left(\frac{x+1}{x+3}\right)\right]_1^2$

$= [10\log 15 - 10\log 8] - 25\left[\frac{1}{2}\log\frac{3}{5} - \frac{1}{2}\log\frac{2}{4}\right]$

$= \left[10\log(5\times3) - 10\log(4\times2)\right] - \frac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$

$= [10\log 5 + 10\log 3 - 10\log 4 - 10\log 2] - \frac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$

$= \left[10+\frac{25}{2}\right]\log 5 + \left[-10-\frac{25}{2}\right]\log 4 + \left[10-\frac{25}{2}\right]\log 3 + \left[-10+\frac{25}{2}\right]\log 2$

$= \frac{45}{2}\log 5 - \frac{45}{2}\log 4 - \frac{5}{2}\log 3 + \frac{5}{2}\log 2$

$= \frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}$

Substituting the value of I_1 in (1), we obtain

$$I = 5 - \left[\frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}\right]$$

$$= 5 - \frac{5}{2}\left[9\log\frac{5}{4} - \log\frac{3}{2}\right]$$

Question 17:

$$\int_0^{\frac{\pi}{4}}\left(2\sec^2 x + x^3 + 2\right)dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}}\left(2\sec^2 x + x^3 + 2\right)dx$$

$$\int\left(2\sec^2 x + x^3 + 2\right)dx = 2\tan x + \frac{x^4}{4} + 2x = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}\left(\frac{\pi}{4}\right) - \mathrm{F}(0)$$

$$= \left\{\left(2\tan\frac{\pi}{4} + \frac{1}{4}\left(\frac{\pi}{4}\right)^4 + 2\left(\frac{\pi}{4}\right)\right) - \left(2\tan 0 + 0 + 0\right)\right\}$$

$$= 2\tan\frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$$

$$= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

Question 18:

$$\int_0^{\pi}\left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)dx$$

Answer

Let $I = \int_0^{\pi}\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)dx$

$= -\int_0^{\pi}\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)dx$

$= -\int_0^{\pi}\cos x\ dx$

$\int \cos x\ dx = \sin x = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$I = \mathrm{F}(\pi)-\mathrm{F}(0)$

$= \sin\pi - \sin 0$

$= 0$

Question 19:

$\int_0^2 \frac{6x+3}{x^2+4}dx$

Answer

Let $I = \int_0^2 \frac{6x+3}{x^2+4}dx$

$\int \frac{6x+3}{x^2+4}dx = 3\int\frac{2x+1}{x^2+4}dx$

$= 3\int\frac{2x}{x^2+4}dx + 3\int\frac{1}{x^2+4}dx$

$= 3\log\left(x^2+4\right)+\frac{3}{2}\tan^{-1}\frac{x}{2} = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(2) - \mathrm{F}(0)$$

$$= \left\{3\log\left(2^2+4\right)+\frac{3}{2}\tan^{-1}\left(\frac{2}{2}\right)\right\}-\left\{3\log(0+4)+\frac{3}{2}\tan^{-1}\left(\frac{0}{2}\right)\right\}$$

$$= 3\log 8+\frac{3}{2}\tan^{-1}1-3\log 4-\frac{3}{2}\tan^{-1}0$$

$$= 3\log 8+\frac{3}{2}\left(\frac{\pi}{4}\right)-3\log 4-0$$

$$= 3\log\left(\frac{8}{4}\right)+\frac{3\pi}{8}$$

$$= 3\log 2+\frac{3\pi}{8}$$

Question 20:

$$\int_0^1\left(xe^x+\sin\frac{\pi x}{4}\right)dx$$

Answer

$$\text{Let } I = \int_0^1\left(xe^x+\sin\frac{\pi x}{4}\right)dx$$

$$\int\left(xe^x+\sin\frac{\pi x}{4}\right)dx = x\int e^x dx-\int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx+\left\{\frac{-\cos\frac{\pi x}{4}}{\frac{\pi}{4}}\right\}$$

$$= xe^x-\int e^x dx-\frac{4\pi}{\pi}\cos\frac{x}{4}$$

$$= xe^x-e^x-\frac{4\pi}{\pi}\cos\frac{x}{4}$$

$$= \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$

$$= \left(1.e^1 - e^1 - \frac{4}{\pi}\cos\frac{\pi}{4}\right) - \left(0.e^0 - e^0 - \frac{4}{\pi}\cos 0\right)$$

$$= e - e - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}}\right) + 1 + \frac{4}{\pi}$$

$$= 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

Question 21:

$\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ equals

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{12}$

Answer

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = F(\sqrt{3}) - F(1)$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Hence, the correct Answer is D.

Question 22:

$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$ equals

A. $\frac{\pi}{6}$

B. $\frac{\pi}{12}$

C. $\frac{\pi}{24}$

D. $\frac{\pi}{4}$

Answer

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2+(3x)^2}$$

Put $3x = t \Rightarrow 3dx = dt$

$$\begin{aligned}\therefore \int \frac{dx}{(2)^2+(3x)^2} &= \frac{1}{3}\int \frac{dt}{(2)^2+t^2} \\ &= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right] \\ &= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) \\ &= \mathrm{F}(x)\end{aligned}$$

By second fundamental theorem of calculus, we obtain

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}\cdot\frac{2}{3}\right) - \frac{1}{6}\tan^{-1} 0$$

$$= \frac{1}{6}\tan^{-1} 1 - 0$$

$$= \frac{1}{6} \times \frac{\pi}{4}$$

$$= \frac{\pi}{24}$$

Hence, the correct Answer is C.

Exercise 7.10

Question 1:

$$\int_0^1 \frac{x}{x^2+1}dx$$

Answer

$$\int_0^1 \frac{x}{x^2+1}dx$$

$$\text{Let } x^2+1=t \Rightarrow 2x\,dx=dt$$

When x = 0, t = 1 and when x = 1, t = 2

$$\therefore \int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$= \frac{1}{2}\left[\log|t|\right]_1^2$$

$$= \frac{1}{2}\left[\log 2 - \log 1\right]$$

$$= \frac{1}{2}\log 2$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\,d\phi$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\,d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\cos\phi\,d\phi$$

Also, let $\sin\phi = t \Rightarrow \cos\phi\,d\phi = dt$

When $\phi = 0$, $t = 0$ and when $\phi = \frac{\pi}{2}$, $t = 1$

$$\therefore I = \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$$

$$= \int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt$$

$$= \int_0^1 \left[t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}}\right]dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - \frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{154+42-132}{231}$$

$$= \frac{64}{231}$$

Question 3:

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Answer

Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Also, let $x = \tan\theta$ □ $dx = \sec^2\theta \, d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta\cdot\sec^2\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \theta\cdot\sec^2\theta\, d\theta$$

Takingθas first function and $\sec^2\theta$ as second function and integrating by parts, we obtain

$$I = 2\left[\theta\int\sec^2\theta\, d\theta - \int\left\{\left(\frac{d}{dx}\theta\right)\int\sec^2\theta\, d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta - \int\tan\theta\, d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$$

$$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$

$$= \frac{\pi}{2} - \log 2$$

Question 4:

$\int_0^2 x\sqrt{x+2}\ \left(\text{Put } x+2 = t^2\right)$

Answer

$\int_0^2 x\sqrt{x+2}dx$

Let $x + 2 = t^2$ □ $dx = 2tdt$

When $x = 0$, $t = \sqrt{2}$ and when $x = 2$, $t = 2$

$$\therefore \int_0^2 x\sqrt{x+2}\,dx = \int_{\sqrt{2}}^2 \left(t^2-2\right)\sqrt{t^2}\,2t\,dt$$

$$= 2\int_{\sqrt{2}}^2 \left(t^2-2\right)t^2\,dt$$

$$= 2\int_{\sqrt{2}}^2 \left(t^4-2t^2\right)dt$$

$$= 2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^2$$

$$= 2\left[\frac{32}{5}-\frac{16}{3}-\frac{4\sqrt{2}}{5}+\frac{4\sqrt{2}}{3}\right]$$

$$= 2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right]$$

$$= 2\left[\frac{16+8\sqrt{2}}{15}\right]$$

$$= \frac{16\left(2+\sqrt{2}\right)}{15}$$

$$= \frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$$

Answer

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$$

Let cos x = t ☐ −sin$x$ dx = dt

When x = 0, t = 1 and when $x=\frac{\pi}{2}, t=0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = -\int_1^0 \frac{dt}{1+t^2}$$

$$= -\left[\tan^{-1} t\right]_1^0$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= -\left[-\frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$

Question 6:

$$\int_0^2 \frac{dx}{x+4-x^2}$$

Answer

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-\left(x^2-x-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

Let $x-\frac{1}{2}=t$ ☐ $dx = dt$

When $x = 0$, $t = -\frac{1}{2}$ and when $x = 2$, $t = \frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2 - t^2}$$

$$= \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log \frac{\frac{\sqrt{17}}{2} + t}{\frac{\sqrt{17}}{2} - t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\frac{\sqrt{17}}{2} + \frac{3}{2}}{\frac{\sqrt{17}}{2} - \frac{3}{2}} - \frac{\log \frac{\sqrt{17}}{2} - \frac{1}{2}}{\log \frac{\sqrt{17}}{2} + \frac{1}{2}}\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\sqrt{17}+3}{\sqrt{17}-3} - \log \frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$$

$$= \frac{1}{\sqrt{17}} \log \frac{\sqrt{17}+3}{\sqrt{17}-3} \times \frac{\sqrt{17}+1}{\sqrt{17}-1}$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{\left(5+\sqrt{17}\right)\left(5+\sqrt{17}\right)}{25-17}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{25+17+10\sqrt{17}}{8}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{42+10\sqrt{17}}{8}\right)$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{21+5\sqrt{17}}{4}\right)$$

Question 7:

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}$$

Answer

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}=\int_{-1}^{1}\frac{dx}{\left(x^2+2x+1\right)+4}=\int_{-1}^{1}\frac{dx}{\left(x+1\right)^2+\left(2\right)^2}$$

Let $x + 1 = t$ □ $dx = dt$

When $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

$$\therefore \int_{-1}^{1}\frac{dx}{\left(x+1\right)^2+\left(2\right)^2}=\int_{0}^{2}\frac{dt}{t^2+2^2}$$

$$=\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$=\frac{1}{2}\tan^{-1}1-\frac{1}{2}\tan^{-1}0$$

$$=\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{\pi}{8}$$

Question 8:

$$\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Answer

$$\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Let $2x = t$ □ $2dx = dt$

When $x = 1$, $t = 2$ and when $x = 2$, $t = 4$

$$\therefore \int_1^2 \left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx = \frac{1}{2}\int_2^4\left(\frac{2}{t}-\frac{2}{t^2}\right)e^t dt$$

$$= \int_2^4\left(\frac{1}{t}-\frac{1}{t^2}\right)e^t dt$$

Let $\frac{1}{t} = f(t)$

Then, $f'(t) = -\frac{1}{t^2}$

$$\Rightarrow \int_2^4\left(\frac{1}{t}-\frac{1}{t^2}\right)e^t dt = \int_2^4 e^t\left[f(t)+f'(t)\right]dt$$

$$=\left[e^t f(t)\right]_2^4$$

$$=\left[e^t\cdot\frac{2}{t}\right]_2^4$$

$$=\left[\frac{e^t}{t}\right]_2^4$$

$$=\frac{e^4}{4}-\frac{e^2}{2}$$

$$=\frac{e^2\left(e^2-2\right)}{4}$$

Question 9:

The value of the integral $\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$ is

A. 6

B. 0

C. 3

D. 4

Answer

Let $I=\int_{\frac{1}{3}}^{1}\frac{\left(x-x^{3}\right)^{\frac{1}{3}}}{x^{4}}dx$

Also, let $x=\sin\theta \Rightarrow dx=\cos\theta\, d\theta$

When $x=\frac{1}{3}$, $\theta=\sin^{-1}\left(\frac{1}{3}\right)$ and when $x=1$, $\theta=\frac{\pi}{2}$

$$\Rightarrow I=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta-\sin^{3}\theta\right)^{\frac{1}{3}}}{\sin^{4}\theta}\cos\theta\, d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(1-\sin^{2}\theta\right)^{\frac{1}{3}}}{\sin^{4}\theta}\cos\theta\, d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^{4}\theta}\cos\theta\, d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^{2}\theta\sin^{2}\theta}\cos\theta\, d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\cos\theta\right)^{\frac{5}{3}}}{\left(\sin\theta\right)^{\frac{5}{3}}}\operatorname{cosec}^{2}\theta\, d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\left(\cot\theta\right)^{\frac{5}{3}}\operatorname{cosec}^{2}\theta\, d\theta$$

Let $\cot\theta = t$ □ $-\text{cosec}2\theta\ d\theta = dt$

When $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, $t = 2\sqrt{2}$ and when $\theta = \frac{\pi}{2}$, $t = 0$

$$\therefore I = -\int_{2\sqrt{2}}^{0} (t)^{\frac{5}{3}} dt$$

$$= -\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$= -\frac{3}{8}\left[(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$= -\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]$$

$$= \frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$

$$= \frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$= \frac{3}{8}[16]$$

$$= 3 \times 2$$

$$= 6$$

Hence, the correct Answer is A.

Question 10:

If $f(x) = \int_0^x t \sin t \, dt$, then $f'(x)$ is

A. $\cos x + x \sin x$

B. $x \sin x$

C. $x \cos x$

D. $\sin x + x \cos x$

Answer

$$f(x) = \int_0^x t \sin t dt$$

Integrating by parts, we obtain

$$f(x) = t\int_0^x \sin t\,dt - \int_0^x \left\{\left(\frac{d}{dt}t\right)\int \sin t\,dt\right\}dt$$

$$= \left[t(-\cos t)\right]_0^x - \int_0^x (-\cos t)\,dt$$

$$= \left[-t\cos t + \sin t\right]_0^x$$

$$= -x\cos x + \sin x$$

$$\Rightarrow f'(x) = -\left[\{x(-\sin x)\} + \cos x\right] + \cos x$$

$$= x\sin x - \cos x + \cos x$$

$$= x\sin x$$

Hence, the correct Answer is B.

Exercise 7.11

Question 1:

$$\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$$

Answer

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x\, dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right) dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Answer

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos}}{\sqrt{\cos}+\sqrt{\sin x}}dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\cdot dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 3:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}$$

Answer

$$\text{Let } I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx \qquad \ldots(1)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1\,dx$$

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

Question 4:

$$\int_0^{\frac{\pi}{2}}\frac{\cos^5 x dx}{\sin^5 x+\cos^5 x}$$

Answer

$$\text{Let } I=\int_0^{\frac{\pi}{2}}\frac{\cos^5 x}{\sin^5 x+\cos^5 x}dx \qquad \ldots(1)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin^5 x}{\sin^5 x+\cos^5 x}dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^5 x+\cos^5 x}{\sin^5 x+\cos^5 x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1\cdot dx$$

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

Question 5:

$$\int_{-5}^{5}|x+2|\,dx$$

Answer

$$\text{Let } I=\int_{-5}^{5}|x+2|\,dx$$

It can be seen that $(x+2)\le 0$ on $[-5,-2]$ and $(x+2)\ge 0$ on $[-2,5]$.

$$\therefore I = \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5}(x+2)\,dx \qquad \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)\right)$$

$$I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2)\right]$$

$$= -\left[2 - 4 - \frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10 - 2 + 4\right]$$

$$= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4$$

$$= 29$$

Question 6:

$$\int_2^8 |x-5|\,dx$$

Answer

$$\text{Let } I = \int_2^8 |x-5|\,dx$$

It can be seen that (x − 5) ≤ 0 on [2, 5] and (x − 5) ≥ 0 on [5, 8].

$$I = \int_2^5 -(x-5)\,dx + \int_5^8 (x-5)\,dx \qquad \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)\right)$$

$$= -\left[\frac{x^2}{2} - 5x\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$$

$$= -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[32 - 40 - \frac{25}{2} + 25\right]$$

$$= 9$$

Question 7:

$$\int_0^1 x(1-x)^n\,dx$$

Answer

Let $I = \int_0^1 x(1-x)^n \, dx$

$\therefore I = \int_0^1 (1-x)(1-(1-x))^n \, dx$

$= \int_0^1 (1-x)(x)^n \, dx$

$= \int_0^1 \left(x^n - x^{n+1}\right) dx$

$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$ $\quad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$= \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$

$= \frac{(n+2)-(n+1)}{(n+1)(n+2)}$

$= \frac{1}{(n+1)(n+2)}$

Question 8:

$\int_0^{\frac{\pi}{4}} \log\,(1+\tan x)\,dx$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \log\left(1+\tan x\right) dx$ ...(1)

$\therefore I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right] dx$ $\left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right\} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left\{1+\frac{1-\tan x}{1+\tan}\right\} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\frac{2}{\left(1+\tan x\right)} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2\, dx - \int_0^{\frac{\pi}{4}} \log\left(1+\tan x\right) dx$$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2dx - I$ [From (1)]

$$\Rightarrow 2I = \left[x\log 2\right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow 2I = \frac{\pi}{4}\log 2$$

$$\Rightarrow I = \frac{\pi}{8}\log 2$$

Question 9:

$\int_0^2 x\sqrt{2-x}\,dx$

Answer

Let $I=\int_0^2 x\sqrt{2-x}\,dx$

$I=\int_0^2 (2-x)\sqrt{x}\,dx \qquad \left(\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$

$=\int_0^2 \left\{2x^{\frac{1}{2}}-x^{\frac{3}{2}}\right\}dx$

$=\left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)-\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$

$=\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}\right]_0^2$

$=\frac{4}{3}(2)^{\frac{3}{2}}-\frac{2}{5}(2)^{\frac{5}{2}}$

$=\frac{4\times2\sqrt{2}}{3}-\frac{2}{5}\times4\sqrt{2}$

$=\frac{8\sqrt{2}}{3}-\frac{8\sqrt{2}}{5}$

$=\frac{40\sqrt{2}-24\sqrt{2}}{15}$

$=\frac{16\sqrt{2}}{15}$

Question 10:

$\int_0^{\frac{\pi}{2}}(2\log\sin x-\log\sin 2x)\,dx$

Answer

Let $I=\int_0^{\frac{\pi}{2}}(2\log\sin x-\log\sin 2x)\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{2\log\sin x-\log(2\sin x\cos x)\}\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{2\log\sin x-\log\sin x-\log\cos x-\log 2\}\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{\log\sin x-\log\cos x-\log 2\}\,dx$ ...(1)

It is known that, $\left(\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{\log\cos x-\log\sin x-\log 2\}\,dx$ ...(2)

Adding (1) and (2), we obtain

$2I=\int_0^{\frac{\pi}{2}}(-\log 2-\log 2)\,dx$

$\Rightarrow 2I=-2\log 2\int_0^{\frac{\pi}{2}}1\,dx$

$\Rightarrow I=-\log 2\left[\frac{\pi}{2}\right]$

$\Rightarrow I=\frac{\pi}{2}(-\log 2)$

$\Rightarrow I=\frac{\pi}{2}\left[\log\frac{1}{2}\right]$

$\Rightarrow I=\frac{\pi}{2}\log\frac{1}{2}$

Question 11:

$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\,dx$

Answer

Let $I=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\,dx$

As $\sin^2(-x)=(\sin(-x))^2=(-\sin x)^2=\sin^2 x$, therefore, $\sin^2 x$ is an even function.

It is known that if *f*(*x*) is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

$$I = 2\int_{0}^{\frac{\pi}{2}} \sin^2 x\,dx$$

$$= 2\int_{0}^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\,dx$$

$$= \int_{0}^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

$$= \left[x - \frac{\sin 2x}{2}\right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

Question 12:

$$\int_{0}^{\pi} \frac{x\,dx}{1+\sin x}$$

Answer

$$\text{Let } I = \int_{0}^{\pi} \frac{x\,dx}{1+\sin x} \qquad \ldots(1)$$

$$\Rightarrow I = \int_{0}^{\pi} \frac{(\pi - x)}{1+\sin(\pi - x)}\,dx \qquad \left(\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_{0}^{\pi} \frac{(\pi - x)}{1+\sin x}\,dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \frac{\pi}{1+\sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \{\sec^2 x - \tan x \sec x\} dx$$

$$\Rightarrow 2I = \pi [\tan x - \sec x]_0^{\pi}$$

$$\Rightarrow 2I = \pi [2]$$

$$\Rightarrow I = \pi$$

Question 13:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$$

Answer

$$\text{Let } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx \qquad \ldots(1)$$

As $\sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$, therefore, $\sin^2 x$ is an odd function.

It is known that, if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x) dx = 0$

$$\therefore I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx = 0$$

Question 14:

$$\int_0^{2\pi} \cos^5 x dx$$

Answer

$$\text{Let } I = \int_0^{2\pi} \cos^5 x dx \qquad \ldots(1)$$

$$\cos^5(2\pi - x) = \cos^5 x$$

It is known that,

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)$$
$$= 0 \text{ if } f(2a-x) = -f(x)$$

$$\therefore I = 2\int_0^{\pi} \cos^5 x\,dx$$

$$\Rightarrow I = 2(0) = 0 \qquad \left[\cos^5(\pi - x) = -\cos^5 x\right]$$

Question 15:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x}\,dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x}\,dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x \cos x}\,dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{0}{1+\sin x \cos x}\,dx$$

$$\Rightarrow I = 0$$

Question 16:

$$\int_0^{\pi} \log(1+\cos x)\,dx$$

Answer

$$\text{Let } I = \int_0^{\pi} \log(1+\cos x)\,dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\pi} \log\left(1+\cos(\pi - x)\right)dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\pi} \log(1-\cos x)\,dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \left\{\log(1+\cos x) + \log(1-\cos x)\right\} dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log\left(1-\cos^2 x\right) dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log \sin^2 x \, dx$$

$$\Rightarrow 2I = 2\int_0^{\pi} \log \sin x \, dx$$

$$\Rightarrow I = \int_0^{\pi} \log \sin x \, dx \qquad \text{...(3)}$$

sin (π − x) = sin x

$$\therefore I = 2\int_0^{\frac{\pi}{2}} \log \sin x \, dx \qquad \text{...(4)}$$

$$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = 2\int_0^{\frac{\pi}{2}} \log \cos x \, dx \qquad \text{...(5)}$$

Adding (4) and (5), we obtain

$$2I = 2\int_0^{\frac{\pi}{2}} \left(\log \sin x + \log \cos x\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\log \sin x + \log \cos x + \log 2 - \log 2\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\log 2 \sin x \cos x - \log 2\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx$$

Let 2x = t □ 2dx = dt

When x = 0, t = 0 and when $x = \frac{\pi}{2}, \pi =$

$$\therefore I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2} \log 2$$

$$\Rightarrow I = \frac{1}{2} I - \frac{\pi}{2} \log 2$$

$$\Rightarrow \frac{I}{2} = -\frac{\pi}{2} \log 2$$

$$\Rightarrow I = -\pi \log 2$$

Question 17:

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$$

Answer

$$\text{Let } I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx \qquad ...(1)$$

It is known that, $\left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$$

$$\Rightarrow 2I = \int_0^a 1 \cdot dx$$

$$\Rightarrow 2I = [x]_0^a$$

$$\Rightarrow 2I = a$$

$$\Rightarrow I = \frac{a}{2}$$

Question 18:

$$\int_0^4 |x-1|\, dx$$

Answer

$$I = \int_0^4 |x-1|\, dx$$

It can be seen that, $(x - 1) \le 0$ when $0 \le x \le 1$ and $(x - 1) \ge 0$ when $1 \le x \le 4$

$$I=\int_0^1|x-1|\,dx+\int_1^4|x-1|\,dx \qquad \left(\int_a^b f(x)=\int_a^c f(x)+\int_c^b f(x)\right)$$

$$=\int_0^1-(x-1)\,dx+\int_1^4(x-1)\,dx$$

$$=\left[x-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^4$$

$$=1-\frac{1}{2}+\frac{(4)^2}{2}-4-\frac{1}{2}+1$$

$$=1-\frac{1}{2}+8-4-\frac{1}{2}+1$$

$$=5$$

Question 19:

Show that $\int_0^a f(x)g(x)\,dx=2\int_0^a f(x)\,dx,$ if f and g are defined as $f(x)=f(a-x)$ and $g(x)+g(a-x)=4$

Answer

$$\text{Let } I=\int_0^a f(x)g(x)\,dx \qquad \ldots(1)$$

$$\Rightarrow I=\int_0^a f(a-x)g(a-x)\,dx \qquad \left(\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I=\int_0^a f(x)g(a-x)\,dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^a\{f(x)g(x)+f(x)g(a-x)\}\,dx$$

$$\Rightarrow 2I=\int_0^a f(x)\{g(x)+g(a-x)\}\,dx$$

$$\Rightarrow 2I=\int_0^a f(x)\times 4\,dx \qquad [g(x)+g(a-x)=4]$$

$$\Rightarrow I=2\int_0^a f(x)\,dx$$

Question 20:

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3+x\cos x+\tan^5 x+1\right)dx$ is

A. 0

B. 2

C. π

D. 1

Answer

$$\text{Let } I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3+x\cos x+\tan^5 x+1\right)dx$$

$$\Rightarrow I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x^3dx+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\tan^5 x dx+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}1\cdot dx$$

It is known that if *f*(*x*) is an even function, then $\int_{-a}^{a}f(x)dx=2\int_{0}^{a}f(x)dx$ and

if *f*(*x*) is an odd function, then $\int_{-a}^{a}f(x)dx=0$

$$I=0+0+0+2\int_{0}^{\frac{\pi}{2}}1\cdot dx$$

$$=2[x]_0^{\frac{\pi}{2}}$$

$$=\frac{2\pi}{2}$$

$$=\pi$$

Hence, the correct Answer is C.

Question 21:

The value of $\int_{0}^{\frac{\pi}{2}}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

A. 2

B. $\frac{3}{4}$

C. 0

D. -2

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left[\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right]dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left\{\log\left(\frac{4+3\sin x}{4+3\cos x}\right)+\log\left(\frac{4+3\cos x}{4+3\sin x}\right)\right\}dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\times\frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log 1\,dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0\,dx$$

$$\Rightarrow I = 0$$

Hence, the correct Answer is C.

Miscellaneous Solutions

Question 1:

$$\frac{1}{x-x^3}$$

Answer

$$\frac{1}{x-x^3}=\frac{1}{x\left(1-x^2\right)}=\frac{1}{x(1-x)(1+x)}$$

$$\text{Let } \frac{1}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{(1-x)}+\frac{C}{1+x} \qquad \ldots(1)$$

$$\Rightarrow 1=A\left(1-x^2\right)+Bx(1+x)+Cx(1-x)$$

$$\Rightarrow 1=A-Ax^2+Bx+Bx^2+Cx-Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain

$-A+B-C=0$

$B+C=0$

$A=1$

On solving these equations, we obtain

$$A=1, B=\frac{1}{2}, \text{ and } C=-\frac{1}{2}$$

From equation (1), we obtain

$$\frac{1}{x(1-x)(1+x)}=\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}$$

$$\Rightarrow \int\frac{1}{x(1-x)(1+x)}dx=\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{1-x}dx-\frac{1}{2}\int\frac{1}{1+x}dx$$

$$=\log|x|-\frac{1}{2}\log|(1-x)|-\frac{1}{2}\log|(1+x)|$$

$$=\log|x|-\log\left|(1-x)^{\frac{1}{2}}\right|-\log\left|(1+x)^{\frac{1}{2}}\right|$$

$$=\log\left|\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right|+C$$

$$=\log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right|+C$$

Question 2:

$$\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}$$

Answer

$$\frac{1}{\sqrt{x+a}+\sqrt{x+b}}=\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$=\frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}$$

$$=\frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$$

$$\Rightarrow\int\frac{1}{\sqrt{x+a}-\sqrt{x+b}}dx=\frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$=\frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$=\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+C$$

Question 3:

$\frac{1}{x\sqrt{ax-x^2}}$ [Hint: Put $x=\frac{a}{t}$]

Answer

$$\frac{1}{x\sqrt{ax-x^2}}$$

Let $x=\frac{a}{t} \Rightarrow dx=-\frac{a}{t^2}dt$

$$\Rightarrow \int \frac{1}{x\sqrt{ax-x^2}}dx = \int \frac{1}{\frac{a}{t}\sqrt{a\cdot\frac{a}{t}-\left(\frac{a}{t}\right)^2}}\left(-\frac{a}{t^2}dt\right)$$

$$= -\int \frac{1}{at}\cdot\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt$$

$$= -\frac{1}{a}\int \frac{1}{\sqrt{\frac{t^2}{t}-\frac{t^2}{t^2}}}dt$$

$$= -\frac{1}{a}\int \frac{1}{\sqrt{t-1}}dt$$

$$= -\frac{1}{a}\left[2\sqrt{t-1}\right]+C$$

$$= -\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+C$$

$$= -\frac{2}{a}\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)+C$$

$$= -\frac{2}{a}\left(\sqrt{\frac{a-x}{x}}\right)+C$$

Question 4:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Answer

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3}, we obtain

$$\frac{x^{-3}}{x^2 \cdot x^{-3}\left(x^4+1\right)^{\frac{3}{4}}}=\frac{x^{-3}\left(x^4+1\right)^{\frac{-3}{4}}}{x^2 \cdot x^{-3}}$$

$$=\frac{\left(x^4+1\right)^{\frac{-3}{4}}}{x^5 \cdot \left(x^4\right)^{-\frac{3}{4}}}$$

$$=\frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

$$=\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

Let $\frac{1}{x^4}=t \Rightarrow -\frac{4}{x^5}dx=dt \Rightarrow \frac{1}{x^5}dx=-\frac{dt}{4}$

$$\therefore \int \frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx=\int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx$$

$$=-\frac{1}{4}\int(1+t)^{-\frac{3}{4}}dt$$

$$=-\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C$$

$$=-\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}}+C$$

$$=-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$$

Question 5:

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}\left[\text{Hint: }\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}\text{ Put }x=t^6\right]$$

Answer

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

$$\text{Let } x=t^6 \Rightarrow dx=6t^5\,dt$$

$$\therefore \int\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=\int\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}dx$$

$$=\int\frac{6t^5}{t^2(1+t)}dt$$

$$=6\int\frac{t^3}{(1+t)}dt$$

On dividing, we obtain

$$\int\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=6\int\left\{\left(t^2-t+1\right)-\frac{1}{1+t}\right\}dt$$

$$=6\left[\left(\frac{t^3}{3}\right)-\left(\frac{t^2}{2}\right)+t-\log|1+t|\right]$$

$$=2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

$$=2\sqrt{x}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

Question 6:

$$\frac{5x}{(x+1)\left(x^2+9\right)}$$

Answer

Let $\frac{5x}{(x+1)(x^2+9)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+9)}$ ...(1)

$\Rightarrow 5x = A(x^2+9) + (Bx+C)(x+1)$

$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + B = 0$

$B + C = 5$

$9A + C = 0$

On solving these equations, we obtain

$A = -\frac{1}{2}, B = \frac{1}{2}, \text{ and } C = \frac{9}{2}$

From equation (1), we obtain

$$\frac{5x}{(x+1)(x^2+9)} = \frac{-1}{2(x+1)} + \frac{\frac{x}{2}+\frac{9}{2}}{(x^2+9)}$$

$$\int \frac{5x}{(x+1)(x^2+9)} dx = \int \left\{ \frac{-1}{2(x+1)} + \frac{(x+9)}{2(x^2+9)} \right\} dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{2}\int \frac{x}{x^2+9} dx + \frac{9}{2}\int \frac{1}{x^2+9} dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\int \frac{2x}{x^2+9} dx + \frac{9}{2}\int \frac{1}{x^2+9} dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{9}{2}\cdot\frac{1}{3}\tan^{-1}\frac{x}{3}$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + C$$

Question 7:

$$\frac{\sin x}{\sin(x-a)}$$

Answer

$$\frac{\sin x}{\sin(x-a)}$$

Let $x - a = t$ □ $dx = dt$

$$\int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin(t+a)}{\sin t}dt$$

$$= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t}dt$$

$$= \int (\cos a + \cot t \sin a)dt$$

$$= t\cos a + \sin a \log|\sin t| + C_1$$

$$= (x-a)\cos a + \sin a \log|\sin(x-a)| + C_1$$

$$= x\cos a + \sin a \log|\sin(x-a)| - a\cos a + C_1$$

$$= \sin a \log|\sin(x-a)| + x\cos a + C$$

Question 8:

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$$

Answer

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} = \frac{e^{4\log x}\left(e^{\log x} - 1\right)}{e^{2\log x}\left(e^{\log x} - 1\right)}$$

$$= e^{2\log x}$$

$$= e^{\log x^2}$$

$$= x^2$$

$$\therefore \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}dx = \int x^2 dx = \frac{x^3}{3} + C$$

Question 9:

$$\frac{\cos x}{\sqrt{4 - \sin^2 x}}$$

Answer

$$\frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Let sin x = t □ cos x dx = dt

$$\Rightarrow \int \frac{\cos x}{\sqrt{4-\sin^2 x}}dx = \int \frac{dt}{\sqrt{(2)^2-(t)^2}}$$
$$= \sin^{-1}\left(\frac{t}{2}\right)+C$$
$$= \sin^{-1}\left(\frac{\sin x}{2}\right)+C$$

Question 10:

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$$

Answer

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x} = \frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^4 x-\cos^4 x\right)}{\sin^2 x+\cos^2 x-\sin^2 x\cos^2 x-\sin^2 x\cos^2 x}$$
$$= \frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x+\cos^2 x\right)\left(\sin^2 x-\cos^2 x\right)}{\left(\sin^2 x-\sin^2 x\cos^2 x\right)+\left(\cos^2 x-\sin^2 x\cos^2 x\right)}$$
$$= \frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)}{\sin^2 x\left(1-\cos^2 x\right)+\cos^2 x\left(1-\sin^2 x\right)}$$
$$= \frac{-\left(\sin^4 x+\cos^4 x\right)\left(\cos^2 x-\sin^2 x\right)}{\left(\sin^4 x+\cos^4 x\right)}$$
$$= -\cos 2x$$

$$\therefore \int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx = \int -\cos 2x\,dx = -\frac{\sin 2x}{2}+C$$

Question 11:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Answer

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by $\sin(a-b)$, we obtain

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x+a)-(x+b)\right]}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)\cdot\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\tan(x+a)-\tan(x+b)\right]$$

$$\int\frac{1}{\cos(x+a)\cos(x+b)}dx=\frac{1}{\sin(a-b)}\int\left[\tan(x+a)-\tan(x+b)\right]dx$$

$$=\frac{1}{\sin(a-b)}\left[-\log\left|\cos(x+a)\right|+\log\left|\cos(x+b)\right|\right]+C$$

$$=\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$$

Question 12:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Answer

$$\frac{x^3}{\sqrt{1-x^8}}$$

Let $x^4 = t$ □ $4x^3\,dx = dt$

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}}dx = \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{4}\sin^{-1} t + C$$

$$= \frac{1}{4}\sin^{-1}\left(x^4\right) + C$$

Question 13:

$$\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$$

Answer

$$\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$$

Let $e^x = t$ □ $e^x\, dx = dt$

$$\Rightarrow \int \frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}dx = \int \frac{dt}{(t+1)(t+2)}$$

$$= \int\left[\frac{1}{(t+1)} - \frac{1}{(t+2)}\right]dt$$

$$= \log|t+1| - \log|t+2| + C$$

$$= \log\left|\frac{t+1}{t+2}\right| + C$$

$$= \log\left|\frac{1+e^x}{2+e^x}\right| + C$$

Question 14:

$$\frac{1}{\left(x^2+1\right)\left(x^2+4\right)}$$

Answer

$$\therefore \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$$

$$\Rightarrow 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

$A + C = 0$

$B + D = 0$

$4A + C = 0$

$4B + D = 1$

On solving these equations, we obtain

$$A = 0,\ B = \frac{1}{3},\ C = 0, \text{and } D = -\frac{1}{3}$$

From equation (1), we obtain

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)}$$

$$\int \frac{1}{(x^2+1)(x^2+4)}dx = \frac{1}{3}\int \frac{1}{x^2+1}dx - \frac{1}{3}\int \frac{1}{x^2+4}dx$$

$$= \frac{1}{3}\tan^{-1}x - \frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$$

Question 15:

$\cos^3 x e^{\log \sin x}$

Answer

$\cos^3 x e^{\log \sin x}$ = $\cos^3 x \times \sin x$

Let $\cos x = t$ ☐ $-\sin x\, dx = dt$

$$\Rightarrow \int \cos^3 x\, e^{\log \sin x} dx = \int \cos^3 x \sin x dx$$
$$= -\int t \cdot dt$$
$$= -\frac{t^4}{4} + C$$
$$= -\frac{\cos^4 x}{4} + C$$

Question 16:

$e^{3\log x}\left(x^4+1\right)^{-1}$

Answer

$$e^{3\log x}\left(x^4+1\right)^{-1} = e^{\log x^3}\left(x^4+1\right)^{-1} = \frac{x^3}{\left(x^4+1\right)}$$

Let $x^4+1=t \;\Rightarrow\; 4x^3\,dx = dt$

$$\Rightarrow \int e^{3\log x}\left(x^4+1\right)^{-1}dx = \int \frac{x^3}{\left(x^4+1\right)}dx$$
$$= \frac{1}{4}\int \frac{dt}{t}$$
$$= \frac{1}{4}\log|t| + C$$
$$= \frac{1}{4}\log\left|x^4+1\right| + C$$
$$= \frac{1}{4}\log\left(x^4+1\right) + C$$

Question 17:

$f'(ax+b)\left[f(ax+b)\right]^n$

Answer

$f'(ax+b)\left[f(ax+b)\right]^n$

Let $f(ax+b)=t \Rightarrow af'(ax+b)dx=dt$

$$\Rightarrow \int f'(ax+b)\left[f(ax+b)\right]^n dx=\frac{1}{a}\int t^n dt$$

$$=\frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right]$$

$$=\frac{1}{a(n+1)}\left(f(ax+b)\right)^{n+1}+C$$

Question 18:

$$\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}$$

Answer

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \frac{1}{\sqrt{\sin^3 x(\sin x \cos\alpha + \cos x \sin\alpha)}}$$

$$= \frac{1}{\sqrt{\sin^4 x \cos\alpha + \sin^3 x \cos x \sin\alpha}}$$

$$= \frac{1}{\sin^2 x\sqrt{\cos\alpha + \cot x \sin\alpha}}$$

$$= \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}}$$

Let $\cos\alpha + \cot x \sin\alpha = t \Rightarrow -\operatorname{cosec}^2 x \sin\alpha \, dx = dt$

$$\therefore \int \frac{1}{\sin^3 x \sin(x+\alpha)} dx = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}} dx$$

$$= \frac{-1}{\sin\alpha} \int \frac{dt}{\sqrt{t}}$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{t}\right] + C$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{\cos\alpha + \cot x \sin\alpha}\right] + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \frac{\cos x \sin\alpha}{\sin x}} + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\frac{\sin x \cos\alpha + \cos x \sin\alpha}{\sin x}} + C$$

$$= -\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$$

Question 19:

$$\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}, x \in [0,1]$$

Answer

Let $I=\int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx$

It is known that, $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\frac{\pi}{2}$

$$\Rightarrow I=\int\frac{\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\cos^{-1}\sqrt{x}}{\frac{\pi}{2}}dx$$

$$=\frac{2}{\pi}\int\left(\frac{\pi}{2}-2\cos^{-1}\sqrt{x}\right)dx$$

$$=\frac{2}{\pi}\cdot\frac{\pi}{2}\int 1\cdot dx-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}\,dx$$

$$=x-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}\,dx \qquad ...(1)$$

Let $I_1=\int\cos^{-1}\sqrt{x}\,dx$

Also, let $\sqrt{x}=t \;\Rightarrow\; dx=2t\,dt$

$$\Rightarrow I_1=2\int\cos^{-1}t\cdot t\,dt$$

$$=2\left[\cos^{-1}t\cdot\frac{t^2}{2}-\int\frac{-1}{\sqrt{1-t^2}}\cdot\frac{t^2}{2}dt\right]$$

$$=t^2\cos^{-1}t+\int\frac{t^2}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\frac{1-t^2-1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\sqrt{1-t^2}\,dt+\int\frac{1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\frac{t}{2}\sqrt{1-t^2}-\frac{1}{2}\sin^{-1}t+\sin^{-1}t$$

$$=t^2\cos^{-1}t-\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t$$

From equation (1), we obtain

$$I = x - \frac{4}{\pi}\left[t^2 \cos t - \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t\right]$$

$$= x - \frac{4}{\pi}\left[x\cos^{-1}\sqrt{x} - \frac{\sqrt{x}}{2}\sqrt{1-x} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - \frac{4}{\pi}\left[x\left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right) - \frac{\sqrt{x-x^2}}{2} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - 2x + \frac{4x}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - \frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$= -x + \frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right] + \frac{2}{\pi}\sqrt{x-x^2} + \mathrm{C}$$

$$= \frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - x + \mathrm{C}$$

Question 20:

$$\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Answer

$$I = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$$

$$\text{Let } x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta\, d\theta$$

$$I = \int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-2\sin\theta\cos\theta)\, d\theta$$

$$= -\int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin 2\theta\, d\theta$$

$$= -\int\tan\frac{\theta}{2}\cdot 2\sin\theta\cos\theta\, d\theta$$

$$= -2\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta\, d\theta$$

$$= -4\int \sin^2 \frac{\theta}{2} \cos\theta \, d\theta$$

$$= -4\int \sin^2 \frac{\theta}{2} \cdot \left(2\cos^2 \frac{\theta}{2} - 1\right) d\theta$$

$$= -4\int \left(2\sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}\right) d\theta$$

$$= -8\int \sin^2 \frac{\theta}{2} \cdot \cos^2 \frac{\theta}{2} \, d\theta + 4\int \sin^2 \frac{\theta}{2} \, d\theta$$

$$= -2\int \sin^2 \theta \, d\theta + 4\int \sin^2 \frac{\theta}{2} \, d\theta$$

$$= -2\int \left(\frac{1-\cos 2\theta}{2}\right) d\theta + 4\int \frac{1-\cos\theta}{2} \, d\theta$$

$$= -2\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right] + 4\left[\frac{\theta}{2} - \frac{\sin\theta}{2}\right] + C$$

$$= -\theta + \frac{\sin 2\theta}{2} + 2\theta - 2\sin\theta + C$$

$$= \theta + \frac{\sin 2\theta}{2} - 2\sin\theta + C$$

$$= \theta + \frac{2\sin\theta\cos\theta}{2} - 2\sin\theta + C$$

$$= \theta + \sqrt{1-\cos^2\theta} \cdot \cos\theta - 2\sqrt{1-\cos^2\theta} + C$$

$$= \cos^{-1}\sqrt{x} + \sqrt{1-x} \cdot \sqrt{x} - 2\sqrt{1-x} + C$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} + C$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + C$$

Question 21:

$$\frac{2+\sin 2x}{1+\cos 2x} e^x$$

Answer

$$I = \int\left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x$$
$$= \int\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)e^x$$
$$= \int\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)e^x$$
$$= \int\left(\sec^2 x+\tan x\right)e^x$$

Let $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$

$$\therefore I = \int\left(f(x)+f'(x)\right]e^x dx$$
$$= e^x f(x)+C$$
$$= e^x \tan x + C$$

Question 22:

$$\frac{x^2+x+1}{(x+1)^2(x+2)}$$

Answer

$$\text{Let } \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)} \quad \ldots(1)$$

$$\Rightarrow x^2+x+1 = A(x+1)(x+2)+B(x+2)+C(x^2+2x+1)$$
$$\Rightarrow x^2+x+1 = A(x^2+3x+2)+B(x+2)+C(x^2+2x+1)$$
$$\Rightarrow x^2+x+1 = (A+C)x^2+(3A+B+2C)x+(2A+2B+C)$$

Equating the coefficients of x^2, x,and constant term, we obtain

$A + C = 1$

$3A + B + 2C = 1$

$2A + 2B + C = 1$

On solving these equations, we obtain

$A = -2$, $B = 1$, and $C = 3$

From equation (1), we obtain

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{(x+1)}+\frac{3}{(x+2)}+\frac{1}{(x+1)^2}$$

$$\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx=-2\int\frac{1}{x+1}dx+3\int\frac{1}{(x+2)}dx+\int\frac{1}{(x+1)^2}dx$$

$$=-2\log|x+1|+3\log|x+2|-\frac{1}{(x+1)}+C$$

Question 23:

$\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Answer

$$I=\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$$

Let $x=\cos\theta \Rightarrow dx=-\sin\theta d\theta$

$$I=\int\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-\sin\theta d\theta)$$

$$=-\int\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin\theta d\theta$$

$$=-\int\tan^{-1}\tan\frac{\theta}{2}\cdot\sin\theta d\theta$$

$$=-\frac{1}{2}\int\theta\cdot\sin\theta d\theta$$

$$=-\frac{1}{2}\left[\theta\cdot(-\cos\theta)-\int 1\cdot(-\cos\theta)d\theta\right]$$

$$=-\frac{1}{2}[-\theta\cos\theta+\sin\theta]$$

$$=+\frac{1}{2}\theta\cos\theta-\frac{1}{2}\sin\theta$$

$$=\frac{1}{2}\cos^{-1}x\cdot x-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{x}{2}\cos^{-1}x-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{1}{2}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)+C$$

Question 24:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}$$

Answer

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right)-\log x^2\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$\text{Let } 1+\frac{1}{x^2}=t \Rightarrow \frac{-2}{x^3}dx=dt$$

$$\therefore I=\int\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$$

$$=-\frac{1}{2}\int\sqrt{t}\log t\,dt$$

$$=-\frac{1}{2}\int t^{\frac{1}{2}}\cdot\log t\,dt$$

Integrating by parts, we obtain

$$I=-\frac{1}{2}\left[\log t\cdot\int t^{\frac{1}{2}}dt-\left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$=-\frac{1}{2}\left[\log t\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}-\int\frac{1}{t}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{2}{3}\int t^{\frac{1}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{4}{9}t^{\frac{3}{2}}\right]$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\log t+\frac{2}{9}t^{\frac{3}{2}}$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\left[\log t-\frac{2}{3}\right]$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+C$$

Question 25:

$$\int_{\frac{\pi}{2}}^{\pi}e^{x}\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Answer

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x} \right) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}} \right) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{\operatorname{cosec}^2\frac{x}{2}}{2} - \cot\frac{x}{2} \right) dx$$

$$\text{Let } f(x) = -\cot\frac{x}{2}$$

$$\Rightarrow f'(x) = -\left(-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} \right) = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(f(x) + f'(x) \right] dx$$

$$= \left[e^x \cdot f(x) dx \right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^x \cdot \cot\frac{x}{2} \right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^{\pi} \times \cot\frac{\pi}{2} - e^{\frac{\pi}{2}} \times \cot\frac{\pi}{4} \right]$$

$$= -\left[e^{\pi} \times 0 - e^{\frac{\pi}{2}} \times 1 \right]$$

$$= e^{\frac{\pi}{2}}$$

Question 26:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\frac{(\sin x \cos x)}{\cos^4 x}}{\frac{(\cos^4 x + \sin^4 x)}{\cos^4 x}} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$$

Let $\tan^2 x = t \Rightarrow 2 \tan x \sec^2 x \, dx = dt$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{4}, t = 1$

$$\therefore I = \frac{1}{2} \int_0^1 \frac{dt}{1+t^2}$$

$$= \frac{1}{2}\left[\tan^{-1} t\right]_0^1$$

$$= \frac{1}{2}\left[\tan^{-1} 1 - \tan^{-1} 0\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4}\right]$$

$$= \frac{\pi}{8}$$

Question 27:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x \, dx}{\cos^2 x + 4 \sin^2 x}$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\left(1-\cos^2 x\right)} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 - 4\cos^2 x} dx$

$\Rightarrow I = \frac{-1}{3}\int_0^{\frac{\pi}{2}} \frac{4-3\cos^2 x - 4}{4-3\cos^2 x} dx$

$\Rightarrow I = \frac{-1}{3}\int_0^{\frac{\pi}{2}} \frac{4-3\cos^2 x}{4-3\cos^2 x} dx + \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4}{4-3\cos^2 x} dx$

$\Rightarrow I = \frac{-1}{3}\int_0^{\frac{\pi}{2}} 1\, dx + \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x - 3} dx$

$\Rightarrow I = \frac{-1}{3}[x]_0^{\frac{\pi}{2}} + \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\left(1+\tan^2 x\right)-3} dx$

$\Rightarrow I = -\frac{\pi}{6} + \frac{2}{3}\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x} dx$ ...(1)

Consider, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x} dx$

Let $2\tan x = t \Rightarrow 2\sec^2 x\, dx = dt$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \infty$

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x} dx = \int_0^{\infty} \frac{dt}{1+t^2}$

$= \left[\tan^{-1} t\right]_0^{\infty}$

$= \left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$

$= \frac{\pi}{2}$

Therefore, from (1), we obtain

$I = -\frac{\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2}\right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

Question 28:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}} dx$$

Answer

$$\text{Let } I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}} dx$$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x+\cos x)}{\sqrt{-(-\sin 2x)}} dx$$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{-(-1+1-2\sin x\cos x)}} dx$$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x+\cos x)}{\sqrt{1-(\sin^2 x+\cos^2 x-2\sin x\cos x)}} dx$$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x+\cos x)dx}{\sqrt{1-(\sin x-\cos x)^2}}$$

Let $(\sin x-\cos x)=t \Rightarrow (\sin x+\cos x)dx=dt$

When $x=\frac{\pi}{6}, t=\left(\frac{1-\sqrt{3}}{2}\right)$ and when $x=\frac{\pi}{3}, t=\left(\frac{\sqrt{3}-1}{2}\right)$

$$I=\int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I=\int_{-\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

As $\frac{1}{\sqrt{1-(-t)^2}}=\frac{1}{\sqrt{1-t^2}}$, therefore, $\frac{1}{\sqrt{1-t^2}}$ is an even function.

It is known that if $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx=2\int_{0}^{a} f(x)dx$

$$\Rightarrow I = 2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$= \left[2\sin^{-1} t\right]_0^{\frac{\sqrt{3}-1}{2}}$$

$$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

Question 29:

$$\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

Answer

$$\text{Let } I = \int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

$$I = \int_0^1 \frac{1}{\left(\sqrt{1+x}-\sqrt{x}\right)} \times \frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{\left(\sqrt{1+x}+\sqrt{x}\right)} dx$$

$$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx$$

$$= \int_0^1 \sqrt{1+x}\, dx + \int_0^1 \sqrt{x}\, dx$$

$$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}\left[(2)^{\frac{3}{2}}-1\right] + \frac{2}{3}[1]$$

$$= \frac{2}{3}(2)^{\frac{3}{2}}$$

$$= \frac{2\cdot 2\sqrt{2}}{3}$$

$$= \frac{4\sqrt{2}}{3}$$

Question 30:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$$

Answer

$$\text{Let } I=\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$$

Also, let $\sin x-\cos x=t \Rightarrow (\cos x+\sin x)dx=dt$

When $x=0,\ t=-1$ and when $x=\frac{\pi}{4},\ t=0$

$\Rightarrow (\sin x-\cos x)^2=t^2$

$\Rightarrow \sin^2 x+\cos^2 x-2\sin x\cos x=t^2$

$\Rightarrow 1-\sin 2x=t^2$

$\Rightarrow \sin 2x=1-t^2$

$$\therefore I=\int_{-1}^{0}\frac{dt}{9+16\left(1-t^2\right)}$$

$$=\int_{-1}^{0}\frac{dt}{9+16-16t^2}$$

$$=\int_{-1}^{0}\frac{dt}{25-16t^2}=\int_{-1}^{0}\frac{dt}{(5)^2-(4t)^2}$$

$$=\frac{1}{4}\left[\frac{1}{2(5)}\log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}$$

$$=\frac{1}{40}\left[\log(1)-\log\left|\frac{1}{9}\right|\right]$$

$$=\frac{1}{40}\log 9$$

Question 31:

$$\int_0^{\frac{\pi}{2}} \sin 2x\tan^{-1}(\sin x)\,dx$$

Answer

Let $I=\int_0^{\frac{\pi}{2}}\sin 2x\tan^{-1}(\sin x)\,dx=\int_0^{\frac{\pi}{2}}2\sin x\cos x\tan^{-1}(\sin x)\,dx$

Also, let $\sin x=t \Rightarrow \cos x\,dx=dt$

When $x=0, t=0$ and when $x=\frac{\pi}{2}, t=1$

$\Rightarrow I=2\int_0^1 t\tan^{-1}(t)\,dt \qquad ...(1)$

Consider $\int t\cdot\tan^{-1}t\,dt=\tan^{-1}t\cdot\int t\,dt-\int\left\{\frac{d}{dt}\left(\tan^{-1}t\right)\int t\,dt\right\}dt$

$$=\tan^{-1}t\cdot\frac{t^2}{2}-\int\frac{1}{1+t^2}\cdot\frac{t^2}{2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}\int\frac{t^2+1-1}{1+t^2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}\int 1\,dt+\frac{1}{2}\int\frac{1}{1+t^2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}\cdot t+\frac{1}{2}\tan^{-1}t$$

$$\Rightarrow\int_0^1 t\cdot\tan^{-1}t\,dt=\left[\frac{t^2\cdot\tan^{-1}t}{2}-\frac{t}{2}+\frac{1}{2}\tan^{-1}t\right]_0^1$$

$$=\frac{1}{2}\left[\frac{\pi}{4}-1+\frac{\pi}{4}\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{2}-1\right]=\frac{\pi}{4}-\frac{1}{2}$$

From equation (1), we obtain

$$I=2\left[\frac{\pi}{4}-\frac{1}{2}\right]=\frac{\pi}{2}-1$$

Question 32:

$$\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx$$

Answer

$$Let\ I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx \quad ...(1)$$

$$I = \int_0^{\pi} \left\{ \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} \right\} dx \quad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx \right)$$

$$\Rightarrow I = \int_0^{\pi} \left\{ \frac{-(\pi - x)\tan x}{-(\sec x + \tan x)} \right\} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx \quad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x + 1 - 1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} 1.dx - \pi \int_0^{\pi} \frac{1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi [x]_0^{\pi} - \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi^2 - \pi \int_0^{\pi} (\sec^2 x - \tan x \sec x) dx$$

$$\Rightarrow 2I = \pi^2 - \pi [\tan x - \sec x]_0^{\pi}$$

$$\Rightarrow 2I = \pi^2 - \pi [\tan \pi - \sec \pi - \tan 0 + \sec 0]$$

$$\Rightarrow 2I = \pi^2 - \pi [0 - (-1) - 0 + 1]$$

$$\Rightarrow 2I = \pi^2 - 2\pi$$

$$\Rightarrow 2I = \pi(\pi - 2)$$

$$\Rightarrow I = \frac{\pi}{2}(\pi - 2)$$

Question 33:

$$\int_1^4 \left[|x-1|+|x-2|+|x-3|\right] dx$$

Answer

Let $I = \int_1^4 \left[|x-1|+|x-2|+|x-3|\right] dx$

$\Rightarrow I = \int_1^4 |x-1|\,dx + \int_1^4 |x-2|\,dx + \int_1^4 |x-3|\,dx$

$I = I_1 + I_2 + I_3 \qquad \ldots(1)$

where, $I_1 = \int_1^4 |x-1|\,dx$, $I_2 = \int_1^4 |x-2|\,dx$, and $I_3 = \int_1^4 |x-3|\,dx$

$I_1 = \int_1^4 |x-1|\,dx$

$(x-1) \geq 0$ for $1 \leq x \leq 4$

$\therefore I_1 = \int_1^4 (x-1)\,dx$

$$\Rightarrow I_1 = \left[\frac{x^2}{x} - x\right]_1^4$$

$$\Rightarrow I_1 = \left[8 - 4 - \frac{1}{2} + 1\right] = \frac{9}{2} \qquad \ldots(2)$$

$I_2 = \int_1^4 |x-2|\,dx$

$x-2 \geq 0$ for $2 \leq x \leq 4$ and $x-2 \leq 0$ for $1 \leq x \leq 2$

$\therefore I_2 = \int_1^2 (2-x)\,dx + \int_2^4 (x-2)\,dx$

$$\Rightarrow I_2 = \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$\Rightarrow I_2 = \left[4 - 2 - 2 + \frac{1}{2}\right] + \left[8 - 8 - 2 + 4\right]$$

$$\Rightarrow I_2 = \frac{1}{2} + 2 = \frac{5}{2} \qquad \ldots(3)$$

$$I_3 = \int_1^4 |x-3|\,dx$$

$x-3 \geq 0$ for $3 \leq x \leq 4$ and $x-3 \leq 0$ for $1 \leq x \leq 3$

$$\therefore I_3 = \int_1^3 (3-x)\,dx + \int_3^4 (x-3)\,dx$$

$$\Rightarrow I_3 = \left[3x - \frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$$

$$\Rightarrow I_3 = \left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right] + \left[8 - 12 - \frac{9}{2} + 9\right]$$

$$\Rightarrow I_3 = [6-4] + \left[\frac{1}{2}\right] = \frac{5}{2} \qquad \ldots(4)$$

From equations (1), (2), (3), and (4), we obtain

$$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$$

Question 34:

$$\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$$

Answer

$$\text{Let } I = \int_1^3 \frac{dx}{x^2(x+1)}$$

$$\text{Also, let } \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + C = 0$

$A + B = 0$

$B = 1$

On solving these equations, we obtain

$A = -1$, $C = 1$, and $B = 1$

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$$

$$\Rightarrow I = \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx$$

$$= \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3$$

$$= \left[\log\left(\frac{x+1}{x} \right) - \frac{1}{x} \right]_1^3$$

$$= \log\left(\frac{4}{3} \right) - \frac{1}{3} - \log\left(\frac{2}{1} \right) + 1$$

$$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$$

$$= \log 2 - \log 3 + \frac{2}{3}$$

$$= \log\left(\frac{2}{3} \right) + \frac{2}{3}$$

Hence, the given result is proved.

Question 35:

$$\int_0^1 x e^x dx = 1$$

Answer

$$\text{Let } I = \int_0^1 x e^x \, dx$$

Integrating by parts, we obtain

$$I = x \int_0^1 e^x dx - \int_0^1 \left\{ \left(\frac{d}{dx}(x) \right) \int e^x dx \right\} dx$$

$$= \left[x e^x \right]_0^1 - \int_0^1 e^x dx$$

$$= \left[x e^x \right]_0^1 - \left[e^x \right]_0^1$$

$$= e - e + 1$$

$$= 1$$

Hence, the given result is proved.

Question 36:

$$\int_{-1}^{1} x^{17}\cos^4 x dx = 0$$

Answer

Let $I = \int_{-1}^{1} x^{17}\cos^4 x dx$

Also, let $f(x) = x^{17}\cos^4 x$

$\Rightarrow f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$

Therefore, $f(x)$ is an odd function.

It is known that if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x) dx = 0$

$\therefore I = \int_{-1}^{1} x^{17}\cos^4 x\, dx = 0$

Hence, the given result is proved.

Question 37:

$$\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \frac{2}{3}$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \sin^3 x\, dx$

$I = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \sin x\, dx$

$= \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x\, dx$

$= \int_0^{\frac{\pi}{2}} \sin x\, dx - \int_0^{\frac{\pi}{2}} \cos^2 x \cdot \sin x\, dx$

$= [-\cos x]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}}$

$= 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$

Hence, the given result is proved.

Question 38:

$$\int_0^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} 2\tan^3 x\, dx$$

$$I = 2\int_0^{\frac{\pi}{4}} \tan^2 x \tan x\, dx = 2\int_0^{\frac{\pi}{4}} \left(\sec^2 x - 1\right)\tan x\, dx$$

$$= 2\int_0^{\frac{\pi}{4}} \sec^2 x \tan x\, dx - 2\int_0^{\frac{\pi}{4}} \tan x\, dx$$

$$= 2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2\left[\log\cos x\right]_0^{\frac{\pi}{4}}$$

$$= 1 + 2\left[\log\cos\frac{\pi}{4} - \log\cos 0\right]$$

$$= 1 + 2\left[\log\frac{1}{\sqrt{2}} - \log 1\right]$$

$$= 1 - \log 2 - \log 1 = 1 - \log 2$$

Hence, the given result is proved.

Question 39:

$$\int_0^1 \sin^{-1} x\, dx = \frac{\pi}{2} - 1$$

Answer

$$\text{Let } I = \int_0^1 \sin^{-1} x\, dx$$

$$\Rightarrow I = \int_0^1 \sin^{-1} x \cdot 1 \cdot dx$$

Integrating by parts, we obtain

$$I=\left[\sin^{-1}x\cdot x\right]_0^1-\int_0^1\frac{1}{\sqrt{1-x^2}}\cdot x\,dx$$

$$=\left[x\sin^{-1}x\right]_0^1+\frac{1}{2}\int_0^1\frac{(-2x)}{\sqrt{1-x^2}}dx$$

Let $1 - x^2 = t$ □ $-2x\,dx = dt$

When $x=0,\ t=1$ and when $x=1,\ t=0$

$$I=\left[x\sin^{-1}x\right]_0^1+\frac{1}{2}\int_1^0\frac{dt}{\sqrt{t}}$$

$$=\left[x\sin^{-1}x\right]_0^1+\frac{1}{2}\left[2\sqrt{t}\right]_1^0$$

$$=\sin^{-1}(1)+\left[-\sqrt{1}\right]$$

$$=\frac{\pi}{2}-1$$

Hence, the given result is proved.

Question 40:

Evaluate $\int_0^1 e^{2-3x}dx$ as a limit of a sum.

Answer

Let $I=\int_0^1 e^{2-3x}dx$

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$$

Where, $h=\dfrac{b-a}{n}$

Here, $a=0, b=1,$ and $f(x)=e^{2-3x}$

$$\Rightarrow h=\frac{1-0}{n}=\frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x}dx=(1-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(0+h)+\ldots+f(0+(n-1)h)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2+e^{2-3h}+\ldots e^{2-3(n-1)h}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2\left\{1+e^{-3h}+e^{-6h}+e^{-9h}+\dots e^{-3(n-1)h}\right\}\right]$$

$$= \lim_{h\to\infty} \frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-3h}\right)^n}{1-\left(e^{-3h}\right)}\right\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2\left\{\frac{1-e^{-\frac{3}{n}\times n}}{1-e^{-\frac{3}{n}}}\right\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{e^2\left(1-e^{-3}\right)}{1-e^{-\frac{3}{n}}}\right]$$

$$= e^2\left(e^{-3}-1\right)\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{e^{-\frac{3}{n}}-1}\right]$$

$$= e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\left(-\frac{1}{3}\right)\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$= \frac{-e^2\left(e^{-3}-1\right)}{3}\lim_{n\to\infty}\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$= \frac{-e^2\left(e^{-3}-1\right)}{3}(1) \qquad \left[\lim_{n\to\infty}\frac{x}{e^x-1}\right]$$

$$= \frac{-e^{-1}+e^2}{3}$$

$$= \frac{1}{3}\left(e^2-\frac{1}{e}\right)$$

Question 41:

$\int \frac{dx}{e^x+e^{-x}}$ is equal to

A. $\tan^{-1}\left(e^x\right)+C$

B. $\tan^{-1}\left(e^{-x}\right)+C$

C. $\log\left(e^{x}-e^{-x}\right)+C$

D. $\log\left(e^{x}+e^{-x}\right)+C$

Answer

$$\text{Let } I=\int\frac{dx}{e^{x}+e^{-x}}dx=\int\frac{e^{x}}{e^{2x}+1}dx$$

$$\text{Also, let } e^{x}=t \Rightarrow e^{x}\,dx=dt$$

$$\therefore I=\int\frac{dt}{1+t^{2}}$$

$$=\tan^{-1}t+C$$

$$=\tan^{-1}\left(e^{x}\right)+C$$

Hence, the correct Answer is A.

Question 42:

$\int\frac{\cos 2x}{(\sin x+\cos x)^{2}}dx$ is equal to

A. $\frac{-1}{\sin x+\cos x}+C$

B. $\log\left|\sin x+\cos x\right|+C$

C. $\log\left|\sin x-\cos x\right|+C$

D. $\frac{1}{(\sin x+\cos x)^{2}}$

Answer

Let $I = \dfrac{\cos 2x}{(\cos x + \sin x)^2}$

$$I = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$$

$$= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2} dx$$

$$= \int \frac{\cos x - \sin x}{\cos + \sin x} dx$$

Let $\cos x + \sin x = t \Rightarrow (\cos x - \sin x)dx = dt$

$$\therefore I = \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log|\cos x + \sin x| + C$$

Hence, the correct Answer is B.

Question 43:

If $f(a+b-x) = f(x)$, then $\int_a^b x f(x) dx$ is equal to

A. $\dfrac{a+b}{2} \int_a^b f(b-x) dx$

B. $\dfrac{a+b}{2} \int_a^b f(b+x) dx$

C. $\dfrac{b-a}{2} \int_a^b f(x) dx$

D. $\dfrac{a+b}{2} \int_a^b f(x) dx$

Answer

Let $I = \int_a^b x f(x) dx \quad ...(1)$

$$I=\int_a^b (a+b-x) f(a+b-x)\,dx \qquad \left(\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right)$$

$$\Rightarrow I=\int_a^b (a+b-x) f(x)\,dx$$

$$\Rightarrow I=(a+b)\int_a^b f(x)\,dx \quad -I \qquad [\text{Using}(1)]$$

$$\Rightarrow I+I=(a+b)\int_a^b f(x)\,dx$$

$$\Rightarrow 2I=(a+b)\int_a^b f(x)\,dx$$

$$\Rightarrow I=\left(\frac{a+b}{2}\right)\int_a^b f(x)\,dx$$

Hence, the correct Answer is D.

Question 44:

The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

A. 1

B. 0

C. − 1

D. $\frac{\pi}{4}$

Answer

$$\text{Let } I=\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$$

$$\Rightarrow I=\int_0^1 \tan^{-1}\left(\frac{x-(1-x)}{1+x(1-x)}\right)dx$$

$$\Rightarrow I=\int_0^1 \left[\tan^{-1}x-\tan^{-1}(1-x)\right]dx \qquad \ldots(1)$$

$$\Rightarrow I=\int_0^1 \left[\tan^{-1}(1-x)-\tan^{-1}(1-1+x)\right]dx$$

$$\Rightarrow I=\int_0^1 \left[\tan^{-1}(1-x)-\tan^{-1}(x)\right]dx$$

$$\Rightarrow I=\int_0^1 \left[\tan^{-1}(1-x)-\tan^{-1}(x)\right]dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^1 \left(\tan^{-1} x + \tan^{-1}(1-x) - \tan^{-1}(1-x) - \tan^{-1} x\right)dx$$

$$\Rightarrow 2I = 0$$

$$\Rightarrow I = 0$$

Hence, the correct Answer is B.