MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}$, $x\in[-1, 1]$.

   Sum of all solutions is $\alpha-\frac{4}{\sqrt{3}}$ then α is

   (1) 1 (2) 2

   (3) –2 (4) $\sqrt{3}$

Answer (2)

Sol. $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}$

For $x < 0$,

$$2\tan^{-1}x+2\tan^{-1}x+\pi=\frac{\pi}{3}$$

$$\Rightarrow\quad 4\tan^{-1}x=-\frac{2\pi}{3}$$

$$\Rightarrow\quad x=-\frac{1}{\sqrt{3}}$$

For $x > 0$,

$$4\tan^{-1}x=\frac{\pi}{3}$$

$$\Rightarrow\quad x=\tan\frac{\pi}{12}=2-\sqrt{3}$$

$$\text{Sum }=2-\sqrt{3}-\frac{1}{\sqrt{3}}=2-\frac{4}{\sqrt{3}}$$

2. Mean of a data set is 10 and variance is 4. If one entry of data set changes from 8 to 12, then new mean becomes 10.2. Then new variance is

   (1) 3.92 (2) 3.96

   (3) 4.04 (4) 4.08

Answer (2)

Sol. Let number of observations is n

$(10.2)n = 10n - 8 + 12$

$\Rightarrow\ (10.2)n = 10n + 4$

$\Rightarrow\ \boxed{n = 20}$

For earlier observation set

$$\frac{\sum x_i^2}{20}-(10)^2=4$$

$$\sum x_i^2=(104)(20)=2080$$

After change

$$\left(\sum x_i^2\right)_{new}=2080-8^2+12^2$$

$$=2160$$

$$\text{New variance} = \frac{2160}{20}-(10.2)^2$$

$$= 108 - (10.2)^2$$

$$= 3.96$$

3. If $y = (1 + x)(x^2 + 1)(x^4 + 1)(x^8 + 1)(x^{16} + 1)$, then $y'' - y'$ is, when $x = -1$

   (1) 496 (2) 946

   (3) –496 (4) –946

Answer (3)

Sol. $y = (x + 1)(x^2 + 1)(x^4 + 1)(x^8 + 1)(x^{16} + 1)$

Multiplying and dividing by $(x - 1)$ we get

$$y=\frac{x^{32}-1}{x-1}$$

at $x = -1$, $y = 0$

$y(x - 1) = x^{32} - 1$

Diff. on both side

$y'(x - 1) + y = 32x^{31}$ …(i)

at $x = -1$

$y'(-1) = 16$

Diff. (i) on both side

$y''(x - 1) + y' + y' = 32 \times 31x^{30}$

substitute $x = -1$

$y''(-1) = -480$

$y''(-1) - y'(-1) = -480 - 16$

$= -496$

4. The logical statement $(p \wedge \sim q) \rightarrow (p \rightarrow \sim q)$ is a

(1) Tautology

(2) Fallacy

(3) Equivalent to $p \vee \sim q$

(4) Equivalent to $p \wedge \sim q$

Answer (1)

Sol. $(p \wedge \sim q) \rightarrow (p \rightarrow \sim q)$

$= (p \wedge \sim q) \rightarrow (\sim p \vee \sim q)$

$= \sim(p \wedge \sim q) \vee (\sim p \vee \sim q)$

$= \sim p \vee q \vee (\sim p \vee \sim q)$

$= \sim p \vee T = T$ (Tautology)

5. If a_r is the coefficient of x^{10-r} in expansion of $(1+x)^{10}$ then $\sum_{r=1}^{10} r^3\left(\frac{a_r}{a_{r-1}}\right)^2$ is

(1) 390 (2) 1210

(3) 485 (4) 220

Answer (2)

Sol. $a_r = {}^{10}C_{10-r}$

$$\sum_{r=1}^{10} r^3\left(\frac{{}^{10}C_{10-r}}{{}^{10}C_{11-r}}\right)^2 = \sum_{r=1}^{10} r^3\left(\frac{10!}{r!(10-r)!}\cdot\frac{(11-r)!(r-1)!}{10!}\right)^2$$

$$= \sum_{r=1}^{10} r^3\left(\frac{11-r}{r}\right)^2 = \sum_{r=1}^{10} r(11-r)^2$$

$$= \sum_{r=1}^{10} r^2(11-r)$$

$$= 11\sum_{r=1}^{10} r^2 - \sum_{r=1}^{10} r^3$$

$$= 11\left(\frac{10\cdot 11\cdot 21}{6}\right) - \left(\frac{10\cdot 11}{2}\right)^2$$

$= (11)^2\, 35 - (11)^2 \cdot 25$

$= (11)^2 \times 10 = 1210$

6. $\lim_{n\to\infty} \frac{1+2-3+4+5-6+....(3n-2)+(3n-1)-3n}{\sqrt{2n^4+3n+1}-\sqrt{n^4+n+3}}$ is equal to

(1) $\frac{3}{2}\left(\sqrt{2}+1\right)$ (2) $\frac{2}{3}\left(\sqrt{2}+1\right)$

(3) $\frac{2}{3\sqrt{2}}$ (4) $2\sqrt{2}$

Answer (1)

Sol. $$\lim_{n\to\infty} \frac{\sum_{r=1}^{n}((3r-2)+(3r-1)-3r)}{\sqrt{2n^4+3n+1}-\sqrt{n^4+n+3}}$$

$$\lim_{n\to\infty} \frac{\sum_{r=1}^{n} 3(r-1)}{\sqrt{2n^4+3n-1}-\sqrt{n^4+n+3}}$$

$$= \lim_{n\to\infty} \frac{3\frac{n(n-1)}{2}\left(\sqrt{2n^4+3n-1}+\sqrt{n^4+n+3}\right)}{\left(2n^4+3n-1\right)-\left(n^4+n+3\right)}$$

$$= \frac{3}{2}\left(\sqrt{2}+1\right)$$

7. If $|z-z_1|^2 + |z-z_2|^2 = |z_1-z_2|^2$ when $z_1 = 2+3i$ and $z_2 = 3+4i$, then locus of z is

(1) Straight line with slope $-\frac{1}{2}$

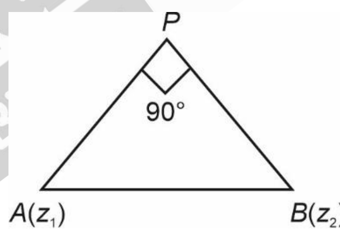
(2) Circle with radius $\frac{1}{\sqrt{2}}$

(3) Hyperbola with eccentricity $\sqrt{2}$

(4) Hyperbola with eccentricity $\frac{5}{2}$

Answer (2)

Sol.



So Locus of P is circle whose diameter is AB

$AB = \sqrt{2}$

Radius of circle $= \frac{1}{\sqrt{2}}$

8. $f(x) = \int \frac{2x}{(x^2+1)(x^2+3)}dx$ if $f(3) = \frac{1}{2}[\ln 5 - \ln 6]$, then $f(4)$ is

(1) $\frac{1}{2}[\ln 17 - \ln 19]$

(2) $\frac{1}{2}[\ln 19 - \ln 17]$

(3) $\ln 19 - \ln 17$

(4) $\ln 17 - \ln 19$

Answer (1)

Sol. $f(x)=\int \frac{2x}{(x^2+1)(x^2+3)}dx$

Let $x^2 = t$

$2xdx = dt$

$\int \frac{dt}{(t+1)(t+3)}$

$=\frac{1}{2}\int \frac{(t+3)-(t+1)}{(t+1)(t+3)}dt$

$=\frac{1}{2}[\ln|t+1|-\ln|t+3|]+\frac{C}{2}$

$=\frac{1}{2}\left[\ln|x^2+1|-\ln|x^2+3|\right]+\frac{C}{2}$

$\because\ f(3)=\frac{1}{2}[\ln 5-\ln 6]$

$\therefore\ \frac{1}{2}[\ln 5-\ln 6]=\frac{1}{2}[\ln 10-\ln 12]+\frac{C}{2}$

$\Rightarrow\ C = 0$

$\therefore\ f(x)=\frac{1}{2}\left[\ln|x^2+1|-\ln|x^2+3|\right]$

$f(4)=\frac{1}{2}[\ln 17-\ln 19]$

9. If $f(x)=\int_0^2 e^{|x-t|}\,dt$, then the minimum value of $f(x)$ is equal to

(1) $2(e-1)$ (2) $2(e+1)$

(3) $2e-1$ (4) $2e+1$

Answer (1)

Sol. For $x > 2$

$f(x)=\int_0^2 e^{x-t}dt$

$=\ e^x\left(-e^{-t}\right)\Big|_0^2$

$=\ e^x(1-e^{-2})$

For $x < 0$

$f(x)=\int_0^2 e^{t-x}dt\ =e^{-x}e^t\Big|_0^2\ =e^{-x}(e^2-1)$

For $0 \le x \le 2$

$f(x)=\int_0^x e^{x-t}\,dt+\int_x^2 e^{t-x}\,dt$

$=-e^xe^{-t}\Big|_0^x+e^{-x}e^t\Big|_x^2$

$\Rightarrow\ -e^x(e^{-x}-1)+e^{-x}(e^2-e^x)$

$\Rightarrow\ -1+e^x+e^{2-x}-1$

$=e^{2-x}+e^x-2$

$$f(x)=\begin{cases} e^x(1-e^{-2}); & x>2 \\ e^{2-x}+e^x-2; & 0\le x\le 2 \\ e^{-x}(e^2-1); & x<0 \end{cases}$$

For $x > 2$

$f(x)_{min} = e^2 - 1$

For $0 \le x \le 2$

$f'(x)=-e^{2-x}+e^x=0\ \Rightarrow e^x=e^{2-x}\Rightarrow e^{2x}=e^2\Rightarrow x=1$

$f(x)=2e-2=2(e-1)$

For $x < 0$

$f(x)_{min} = e^2 - 1$

10. If $f(x) = x^b + 3$, $g(x) = ax + c$. If $(g(fx))^{-1}=\left(\frac{x-7}{2}\right)^{\frac{1}{3}}$ then fog(ac) + gof(b) is

(1) 189 (2) 195

(3) 194 (4) 89

Answer (1)

Sol. $g(fx) = a(x^b + 3) + c$.

$(g(f(x)))^{-1}=\left[\frac{x-3a-c}{a}\right]^{\frac{1}{b}}=\left(\frac{x-7}{2}\right)^{\frac{1}{3}}$

$\Rightarrow\ a = 2$

$b = 3$

$c = 1$

$g(x) = 2x + 1$

$f(x) = x^3 + 3$

Now fog(2) + gof(3)

= 128 + 61

= 189

11. Term independent of x in expansion of $\left(2x+\frac{1}{x^7}-7x^2\right)^5$ is

(1) 1372 (2) 2744

(3) –13720 (4) 13720

Answer (3)

Sol. $\frac{1}{x^{35}}\left(2x^8+1-7x^9\right)^5=\frac{1}{x^{35}}\left(1+x^8\left(2-7x\right)\right)^5$

Term independent of x = coefficient of x^{35} in $(1 + x^8 (2 - 7x))^5$

= coefficient of x^{35} in ${}^5C_4\left(x^8\left(2-7x\right)\right)^4$

= ${}^5C_4$ coefficient of x^3 in $(2 - 7x)^4$

= ${}^5C_4 \cdot {}^4C_3\left(2^1\right)\left(-7\right)^3$

= –13720

12. The value of $A=\begin{bmatrix}1 & \log_x y & \log_x z\\ \log_y x & 2 & \log_y z\\ \log_z x & \log_z y & 3\end{bmatrix}$ then |adj (adj A^2)| is

(1) 6^4 (2) 4^8

(3) 4^5 (4) 2^8

Answer (4)

Sol. $A=\begin{bmatrix}1 & \log_x y & \log_x z\\ \log_y x & 2 & \log_y z\\ \log_z x & \log_z y & 3\end{bmatrix}$

$$|A|=\frac{1}{\log x\log y\log z}\begin{vmatrix}\log x & \log y & \log z\\ \log x & 2\log y & \log z\\ \log x & \log y & 3\log z\end{vmatrix}$$

$$|A|=\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 3\end{vmatrix}$$

$|A| = 2$

$|\text{adj (adj } A^2)| = |A|^8$

$= 2^8$

13. Sum of two positive integers is 66 and μ is the maximum value of their product

$S=\left\{x\in Z,\ x(66-x)\geq\frac{5\mu}{9}\right\},\ x\neq 0$ then probability of A when $A=\{x\in S;\ x=3k,\ x\in N\}$

(1) $\frac{1}{4}$ (2) $\frac{2}{3}$

(3) $\frac{1}{3}$ (4) $\frac{1}{2}$

Answer (3)

Sol. $\mu = 33 \times 33 = 1089$

$x(66–x) \geq 605$

$x^2 – 66x + 605 \leq 0$

$x \in [11,55]$

Favourable set of values of x for event A

= {12, 15, 18, ...54}

$P(A)=\frac{15}{45}=\frac{1}{3}$

14. Let $L_1=\frac{x-3}{1}=\frac{y-2}{2}=\frac{z-1}{3}$ and

$L_2=\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$

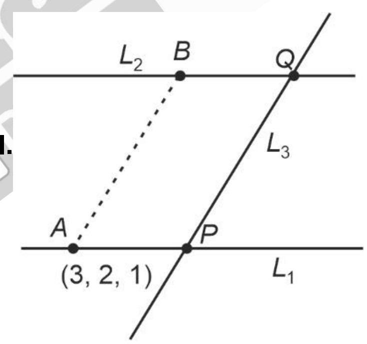
and direction ratios of line L_3 are <1, –1, 3>. P and Q are point of intersection of L_1 and L_3 and L_2 and L_3 respectively. Then distance between P and Q is

(1) $\frac{10}{3}\sqrt{6}$ (2) $\frac{8}{3}\sqrt{11}$

(3) $\frac{4}{3}\sqrt{11}$ (4) $\frac{11}{3}\sqrt{6}$

Answer (3)

Sol.



Let

$PQ = AB$

Let $A(3, 2, 1)$

Equation of line AB:

$\frac{x-3}{1}=\frac{y-2}{-1}=\frac{z-1}{3}=k \qquad (k\in R)$

$\Rightarrow\ x = k + 3,\ y = –k + 2,\ z = 3k + 1$

Let coordinates of $B(k + 3, –k + 2, 3k + 1)$

B lies on L_2

$B(\lambda + 1, 2\lambda + 2, 3\lambda + 3)$

$k + 3 = \lambda + 1 \Rightarrow \lambda – k = 2$

$2 - k = 2\lambda + 2 \Rightarrow 2\lambda + k = 0$

$\Rightarrow \; k = -2\lambda$

$\Rightarrow \; 3\lambda = 2$

$\Rightarrow \; \lambda = \frac{2}{3}$

$B\left(\frac{5}{3}, \frac{10}{3}, 5\right)$

$AB = \sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + 16}$

$= \frac{4}{3}\sqrt{11} = PQ$

15. If $\vec{a} = -\hat{i} + 2\hat{j} + \hat{k}$ is rotated by 90° about origin passing through y-axis. If new vector is $\vec{b}$ then projection of $\vec{b}$ on $\vec{c} = 5\hat{i} + 4\hat{j} + 3\hat{k}$ is equal to

(1) $\frac{6}{5}$

(2) $\frac{3}{5}$

(3) $\frac{6}{5\sqrt{3}}$

(4) $\frac{6\sqrt{3}}{5}$

Answer (1)

Sol. $\vec{b} = \lambda\vec{a} + \mu\hat{j}$

$= \left(\lambda(-\hat{i} + 2\hat{j} + \hat{k}) + \mu\hat{j}\right)$

$\vec{b} \cdot \vec{a} = 0$

$(\lambda\vec{a} + \mu\hat{j})\vec{a} = 0$

$6\lambda + 2\mu = 0$

$\mu = -3\lambda$

$\vec{b} = \lambda(\vec{a} - 3\hat{j}) = \lambda(-\hat{i} - \hat{j} + \hat{k})$

$\lambda = \pm\sqrt{2}$

Projection of $\vec{b}$ on $\vec{c}$ $= \left|\vec{b} \cdot \hat{c}\right|$

$= \left|\sqrt{2}(-\hat{i} - \hat{j} + \hat{k})\frac{(5\hat{i} + 4\hat{j} + 3\hat{k})}{5\sqrt{2}}\right|$

$= \frac{6\sqrt{2}}{5\sqrt{2}} = \frac{6}{5}$

16. Given $\frac{dy}{dx} = \frac{y}{x}\left(1 + xy^2(1 + \ln x)\right)$. If $y(1) = 3$, then the value of $\frac{y^2(3)}{9}$ is

(1) $-\frac{1}{43 + 27\ln 3}$

(2) $\frac{1}{43 + 27\ln 3}$

(3) $\frac{9}{59 - 162(1 + \ln 3)}$

(4) $\frac{1}{27 - 43\ln 3}$

Answer (3)

Sol. $\frac{dy}{dx} - \frac{y}{x} = y^3(1 + \ln x)$

$\frac{1}{y^3}\frac{dy}{dx} - \frac{1}{x}\frac{1}{y^2} = (1 + \ln x)$

$\frac{1}{y^2} = t \Rightarrow \frac{-2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$

$\therefore \; \frac{-1}{2}\frac{dt}{dx} - \frac{t}{x} = (1 + \ln x)$

$\frac{dt}{dx} + \frac{2t}{x} = -2(1 + \ln x)$

IF $e^{\int \frac{2}{x}dx} = x^2$

$\therefore \; tx^2 = \int -2(1 + \ln x)x^2 dx$

$tx^2 = -2\left[(1 + \ln x)\frac{x^3}{3} - \int \frac{x^2}{3}dx\right] + c$

$\frac{x^2}{y^2} = -2\left[\frac{x^3}{3}(1 + \ln x) - \frac{x^3}{9}\right] + c \ldots$(i)

$y(1) = 3 \Rightarrow \frac{1}{9} = -2\left(\frac{1}{3} - \frac{1}{9}\right) + c$

$\therefore \; c = \frac{5}{9}$

Now putting $x = 3, c = \frac{5}{9}$ in (1)

$\frac{9}{y^2} = -2\left(9(1 + \ln 3) - 3\right) + \frac{5}{9}$

$= \frac{59}{9} - 18(1 + \ln 3)$

$\frac{y^2}{9} = \frac{9}{59 - 162(1 + \ln 3)}$

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

21. Consider the set $S = \{1, 2, 3, 5, 7, 10, 11\}$. Number of subsets of S having sum of its elements equal to multiple of 3, is equal to.

Answer (44.00)

Sol. Out of the given numbers one is $3k$ type and 3 of $3k + 1$ type and remaining three are $3k + 2$ type.

Number of subsets with 0 elements = 1

[Considering the sum of elements of empty set equal to zero]

Number of subsets with 1 element = 1

1 of $3k$ type

Number of subsets with 2 elements

1 of $(3k + 1)$ type + 1 of $(3k + 2)$ type = 9

Number of subsets with 3 elements

1 of $3k$ type + 1 of $(3k + 1)$ type + 1 of $(3k + 2)$ type = 9

3 of $(3k + 1)$ type = 1

3 of $(3k + 2)$ type = 1

Number of subsets with 4 elements

1 of $3k$ type + 3 of $(3k + 1)$ type = 1

1 of $3k$ type + 3 of $(3k + 2)$ type = 1

2 of $(3k + 1)$ type + 2 of $(3k + 2)$ = 9

Number of subsets with 5 elements

1 of $3k$ type + 2 of $(3k + 1)$ type + 2 of $(3k + 2)$ type = 9

Number of subsets with 6 elements

3 of $3k + 1$ type + 3 of $3k + 2$ type = 1

The set itself = 1

Total = 44.

22. If $a, b \in [1, 25]$, $a, b \in N$ such that $a + b$ is multiple of 5. Find the number of ordered pair (a, b).

Answer (125)

Sol.

Type	Numbers
$5k$	5, 10, 15, 20, 25
$5k + 1$	1, 6, 11, 16, 21
$5k + 2$	2, 7, 12, 17, 22
$5k + 3$	3, 8, 13, 18, 23
$5k + 4$	4, 9, 14, 19, 24

(a, b) can be selected as

I 1 of $(5k + 1)$ and 1 of $(5k + 4) = 2 \times 25 = 50$

II 1 of $(5k + 2)$ and 1 of $(5k + 3) = 2 \times 25 = 50$

III both of the type $5k = 25$

$\therefore$ Total = 125

23. If $\log_2\left(9^{2\alpha-4}+13\right)-\log_2\left(3^{2\alpha-4}\cdot\frac{5}{2}+1\right)=2.$

Then maximum integral value of β for which equation.

$x^2-\left(\left(\sum\alpha\right)^2 x\right)-\sum(\alpha+1)^2\beta$ has real roots is

Answer (06)

Sol. $\log_2\left(9^{2\alpha-4}+13\right)-\log_2\left(3^{2x-4}\cdot\frac{5}{2}+1\right)=2$

$\therefore \dfrac{9^{2\alpha-4}+13}{3^{2\alpha-4}\cdot\frac{5}{2}+1}=4$

Let $3^{2\alpha-4} = t$

$t^2 + 13 = 10t + 4$

$t^2 - 10t + 9 = 0$

$\therefore \quad t = 9, 1$

$\Rightarrow \quad \alpha = 3, 2$

Now equation will become

$x^2 - 25x + 25\beta = 0$

has real roots

$\therefore \quad D \ge 0$

$25^2 - 4.25\,\beta \ge 0$

$\Rightarrow \quad \beta \le \dfrac{25}{4}$

Max integral value = 6

24.

25.

26.

27.

28.

29.

30.

❑ ❑ ❑