Special Instructions / Useful Data	
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}^n$	$\{(x_1,\ldots,x_n): x_i \in \mathbb{R}, i=1,\ldots,n\}$
$P(A)$	Probability of an event A
i.i.d.	Independently and identically distributed
$Bin(n, p)$	Binomial distribution with parameters n and p
$Poisson(\theta)$	Poisson distribution with mean θ
$N(\mu,\sigma^2)$	Normal distribution with mean μ and variance σ^2
$Exp(\lambda)$	The exponential distribution with probability density function $f(x\mid\lambda)=\begin{cases}\lambda e^{-\lambda x}, & x>0,\\ 0, & \text{otherwise}\end{cases}, \ \lambda>0$
t_n	Student's t distribution with n degrees of freedom
χ^2_n	Chi-square distribution with n degrees of freedom
$\chi^2_{n,\alpha}$	A constant such that $P\left(W>\chi^2_{n,\alpha}\right)=\alpha,$ where W has χ^2_n distribution
$\Phi(x)$	Cumulative distribution function of $N(0,1)$
$\phi(x)$	Probability density function of $N(0,1)$
A^C	Complement of an event A
$E(X)$	Expectation of a random variable X
$Var(X)$	Variance of a random variable X
$B(m, n)$	$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \ m>0, \ n>0$
$[x]$	The greatest integer less than or equal to real number x
f'	Derivative of function f
$\Phi(0.25)=0.5987, \Phi(0.5)=0.6915, \Phi(0.625)=0.7341, \Phi(0.71)=0.7612,$ $\Phi(1)=0.8413, \ \Phi(1.125)=0.8697, \Phi(2)=0.9772$	

SECTION – A

MULTIPLE CHOICE QUESTIONS (MCQ)

Q. 1 – Q.10 carry one mark each.

Q.1 Let

$$P=\begin{bmatrix}1 & 2 & 0 & 2\\ -1 & -2 & 1 & 1\\ 1 & 2 & -3 & -7\\ 1 & 2 & -2 & -4\end{bmatrix}.$$

Then rank of P equals

(A) 4
(B) 3
(C) 2
(D) 1

Q.2 Let α, β, γ be real numbers such that $\beta \neq 0$ and $\gamma \neq 0$. Suppose

$$P=\begin{bmatrix}\alpha & \beta\\ \gamma & 0\end{bmatrix},$$

and $P^{-1}=P$. Then

(A) $\alpha = 0$ and $\beta\gamma = 1$
(B) $\alpha \neq 0$ and $\beta\gamma = 1$
(C) $\alpha = 0$ and $\beta\gamma = 2$
(D) $\alpha = 0$ and $\beta\gamma = -1$

Q.3 Let $m > 1$. The volume of the solid generated by revolving the region between the y-axis and the curve $xy = 4, 1 \le y \le m,$ about the y-axis is 15π. The value of m is

(A) 14 (B) 15 (C) 16 (D) 17

Q.4 Consider the region S enclosed by the surface $z = y^2$ and the planes $z = 1, x = 0, x = 1, y = -1$ and $y = 1$. The volume of S is

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) 1 (D) $\frac{4}{3}$

Q.5 Let X be a discrete random variable with the moment generating function

$$M_X(t) = e^{0.5\left(e^t - 1\right)}, \; t \in \mathbb{R}.$$

Then $P\left(X \leq 1\right)$ equals

(A) $e^{-1/2}$ (B) $\frac{3}{2}e^{-1/2}$ (C) $\frac{1}{2}e^{-1/2}$ (D) $e^{-(e-1)/2}$

Q.6 Let E and F be two independent events with

$$P\left(E \mid F\right) + P\left(F \mid E\right) = 1, \; P\left(E \cap F\right) = \frac{2}{9} \text{ and } P\left(F\right) < P\left(E\right).$$

Then $P\left(E\right)$ equals

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

Q.7 Let X be a continuous random variable with the probability density function

$$f(x) = \frac{1}{(2 + x^2)^{3/2}}, \quad x \in \mathbb{R}.$$

Then $E(X^2)$

(A) equals 0
(B) equals 1
(C) equals 2
(D) does not exist

Q.8 The probability density function of a random variable X is given by

$$f(x) = \begin{cases} \alpha x^{\alpha - 1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}, \; \alpha > 0.$$

Then the distribution of the random variable $Y = \log_e X^{-2\alpha}$ is

(A) χ_2^2 (B) $\frac{1}{2}\chi_2^2$ (C) $2\chi_2^2$ (D) χ_1^2

Q.9 Let $X_1, X_2, \ldots$ be a sequence of i.i.d. $N(0,1)$ random variables. Then, as $n \to \infty$, $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ converges in probability to

(A) 0 (B) 0.5 (C) 1 (D) 2

Q.10 Consider the simple linear regression model with n random observations $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, $i = 1, \dots, n$, $(n > 2)$. β_0 and β_1 are unknown parameters, $x_1, \dots, x_n$ are observed values of the regressor variable and $\varepsilon_1, \dots, \varepsilon_n$ are error random variables with $E(\varepsilon_i) = 0$, $i = 1, \dots, n$, and for $i, j = 1, \dots, n$, $Cov(\varepsilon_i, \varepsilon_j) = \begin{cases} 0, & \text{if } i \neq j, \\ \sigma^2, & \text{if } i = j \end{cases}$. For real constants $a_1, \dots, a_n$, if $\sum_{i=1}^{n} a_i Y_i$ is an unbiased estimator of β_1, then

(A) $\sum_{i=1}^{n} a_i = 0$ and $\sum_{i=1}^{n} a_i x_i = 0$

(B) $\sum_{i=1}^{n} a_i = 0$ and $\sum_{i=1}^{n} a_i x_i = 1$

(C) $\sum_{i=1}^{n} a_i = 1$ and $\sum_{i=1}^{n} a_i x_i = 0$

(D) $\sum_{i=1}^{n} a_i = 1$ and $\sum_{i=1}^{n} a_i x_i = 1$

Q. 11 – Q. 30 carry two marks each.

Q.11 Let (X, Y) have the joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{2} y^2 e^{-x}, & \text{if } 0 < y < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Then $P(Y < 1 \mid X = 3)$ equals

(A) $\frac{1}{81}$ (B) $\frac{1}{27}$ (C) $\frac{1}{9}$ (D) $\frac{1}{3}$

Q.12 Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables having the probability density function

$$f(x) = \begin{cases} \frac{1}{B(6,4)} x^5 (1-x)^3, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y_i = \frac{X_i}{1 - X_i}$ and $U_n = \frac{1}{n} \sum_{i=1}^{n} Y_i$. If the distribution of $\frac{\sqrt{n}(U_n - 2)}{\alpha}$ converges to $N(0,1)$ as $n \to \infty$, then a possible value of α is

(A) $\sqrt{7}$ (B) $\sqrt{5}$ (C) $\sqrt{3}$ (D) 1

Q.13 Let $X_1,\ldots,X_n$ be a random sample from a population with the probability density function

$$f(x\mid\theta)=\begin{cases}4e^{-4(x-\theta)}, & x>\theta,\\ 0, & \text{otherwise}\end{cases}\quad,\quad \theta\in\mathbb{R}.$$

If $T_n=\min\{X_1,\ldots,X_n\}$, then

(A) T_n is unbiased and consistent estimator of θ
(B) T_n is biased and consistent estimator of θ
(C) T_n is unbiased but NOT consistent estimator of θ
(D) T_n is NEITHER unbiased NOR consistent estimator of θ

Q.14 Let $X_1,\ldots,X_n$ be i.i.d. random variables with the probability density function

$$f(x)=\begin{cases}e^{-x}, & x>0,\\ 0, & \text{otherwise.}\end{cases}$$

If $X_{(n)}=\max\{X_1,\ldots,X_n\}$, then $\lim_{n\to\infty}P\left(X_{(n)}-\log_e n\le 2\right)$ equals

(A) $1-e^{-2}$
(B) $e^{-e^{-0.5}}$
(C) $e^{-e^{-2}}$
(D) $e^{-e^{2}}$

Q.15 Let X and Y be two independent $N(0,1)$ random variables. Then $P\left(0<X^2+Y^2<4\right)$ equals

(A) $1-e^{-2}$ (B) $1-e^{-4}$ (C) $1-e^{-1}$ (D) e^{-2}

Q.16 Let X be a random variable with the cumulative distribution function

$$F(x)=\begin{cases}0, & x<0,\\ \dfrac{x}{8}, & 0\le x<2,\\ \dfrac{x^2}{16}, & 2\le x<4,\\ 1, & x\ge 4.\end{cases}$$

Then $E(X)$ equals

(A) $\dfrac{12}{31}$ (B) $\dfrac{13}{12}$ (C) $\dfrac{31}{21}$ (D) $\dfrac{31}{12}$

Q.17 Let $X_1, \ldots, X_n$ be a random sample from a population with the probability density function

$$f(x) = \frac{1}{2\theta} e^{-|x|/\theta}, \; x \in \mathbb{R}, \; \theta > 0.$$

For a suitable constant K, the critical region of the most powerful test for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$ is of the form

(A) $\sum_{i=1}^{n} |X_i| > K$

(B) $\sum_{i=1}^{n} |X_i| < K$

(C) $\sum_{i=1}^{n} \frac{1}{|X_i|} < K$

(D) $\sum_{i=1}^{n} \frac{1}{|X_i|} > K$

Q.18 Let $X_1, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ $(n > 4, m > 4)$ be a random sample from $N(\mu, \sigma^2)$; $\mu \in \mathbb{R}, \sigma > 0$. If $\bar{X}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\bar{X}_2 = \frac{1}{m-2}\sum_{i=n+1}^{n+m-2} X_i$, then the distribution of the random variable

$$T = \frac{X_{n+m} - X_{n+m-1}}{\sqrt{\sum_{i=1}^{n}\left(X_i - \bar{X}_1\right)^2 + \sum_{i=n+1}^{n+m-2}\left(X_i - \bar{X}_2\right)^2}}$$

is

(A) t_{n+m-2}

(B) $\sqrt{\frac{2}{n+m-1}}\, t_{n+m-1}$

(C) $\sqrt{\frac{2}{n+m-4}}\, t_{n+m-4}$

(D) t_{n+m-4}

Q.19 Let $X_1, \ldots, X_n$ $(n > 1)$ be a random sample from a $Poisson(\theta)$ population, $\theta > 0$, and $T = \sum_{i=1}^{n} X_i$. Then the uniformly minimum variance unbiased estimator of θ^2 is

(A) $\frac{T(T-1)}{n^2}$

(B) $\frac{T(T-1)}{n(n-1)}$

(C) $\frac{T(T-1)}{n(n+1)}$

(D) $\frac{T^2}{n^2}$

Q.20 Let X be a random variable whose probability mass functions $f(x \mid H_0)$ (under the null hypothesis H_0) and $f(x \mid H_1)$ (under the alternative hypothesis H_1) are given by

$X = x$	0	1	2	3
$f(x \mid H_0)$	0.4	0.3	0.2	0.1
$f(x \mid H_1)$	0.1	0.2	0.3	0.4

For testing the null hypothesis $H_0 : X \sim f(x \mid H_0)$ against the alternative hypothesis $H_1 : X \sim f(x \mid H_1)$, consider the test given by: Reject H_0 if $X > \frac{3}{2}$.

If $\alpha =$ size of the test and $\beta =$ power of the test, then

(A) $\alpha = 0.3$ and $\beta = 0.3$
(B) $\alpha = 0.3$ and $\beta = 0.7$
(C) $\alpha = 0.7$ and $\beta = 0.3$
(D) $\alpha = 0.7$ and $\beta = 0.7$

Q.21 Let $X_1, \ldots, X_n$ be a random sample from a $N(2\theta, \theta^2)$ population, $\theta > 0$. A consistent estimator for θ is

(A) $\frac{1}{n}\sum_{i=1}^{n} X_i$

(B) $\left(\frac{5}{n}\sum_{i=1}^{n} X_i^2\right)^{1/2}$

(C) $\frac{1}{5n}\sum_{i=1}^{n} X_i^2$

(D) $\left(\frac{1}{5n}\sum_{i=1}^{n} X_i^2\right)^{1/2}$

Q.22 An institute purchases laptops from either vendor V_1 or vendor V_2 with equal probability. The lifetimes (in years) of laptops from vendor V_1 have a $U(0,4)$ distribution, and the lifetimes (in years) of laptops from vendor V_2 have an $Exp(1/2)$ distribution. If a randomly selected laptop in the institute has lifetime more than two years, then the probability that it was supplied by vendor V_2 is

(A) $\frac{2}{2+e}$

(B) $\frac{1}{1+e}$

(C) $\frac{1}{1+e^{-1}}$

(D) $\frac{2}{2+e^{-1}}$

Q.23 Let $y(x)$ be the solution to the differential equation

$$x^4 \frac{dy}{dx} + 4x^3 y + \sin x = 0; \quad y(\pi) = 1, \quad x > 0.$$

Then $y\left(\frac{\pi}{2}\right)$ is

(A) $\frac{10\left(1+\pi^4\right)}{\pi^4}$

(B) $\frac{12\left(1+\pi^4\right)}{\pi^4}$

(C) $\frac{14\left(1+\pi^4\right)}{\pi^4}$

(D) $\frac{16\left(1+\pi^4\right)}{\pi^4}$

Q.24 Let $a_n = e^{-2n} \sin n$ and $b_n = e^{-n} n^2 (\sin n)^2$ for $n \geq 1$. Then

(A) $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} b_n$ does NOT converge

(B) $\sum_{n=1}^{\infty} b_n$ converges but $\sum_{n=1}^{\infty} a_n$ does NOT converge

(C) both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge

(D) NEITHER $\sum_{n=1}^{\infty} a_n$ NOR $\sum_{n=1}^{\infty} b_n$ converges

Q.25 Let

$$f(x) = \begin{cases} x \sin^2(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x (\sin x) \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then

(A) f is differentiable at 0 but g is NOT differentiable at 0

(B) g is differentiable at 0 but f is NOT differentiable at 0

(C) f and g are both differentiable at 0

(D) NEITHER f NOR g is differentiable at 0

Q.26 Let $f:[0,4]\to\mathbb{R}$ be a twice differentiable function. Further, let $f(0)=1$, $f(2)=2$ and $f(4)=3$. Then

(A) there does NOT exist any $x_1\in(0,2)$ such that $f'(x_1)=\dfrac{1}{2}$

(B) there exist $x_2\in(0,2)$ and $x_3\in(2,4)$ such that $f'(x_2)=f'(x_3)$

(C) $f''(x)>0$ for all $x\in(0,4)$

(D) $f''(x)<0$ for all $x\in(0,4)$

Q.27 Let $f(x,y)=x^2-400\,x\,y^2$ for all $(x,y)\in\mathbb{R}^2$. Then f attains its

(A) local minimum at $(0,0)$ but NOT at $(1,1)$

(B) local minimum at $(1,1)$ but NOT at $(0,0)$

(C) local minimum both at $(0,0)$ and $(1,1)$

(D) local minimum NEITHER at $(0,0)$ NOR at $(1,1)$

Q.28 Let $y(x)$ be the solution to the differential equation

$$4\,\frac{d^2y}{dx^2}+12\,\frac{dy}{dx}+9\,y=0,\ \ y(0)=1,\ \ y'(0)=-4.$$

Then $y(1)$ equals

(A) $-\dfrac{1}{2}\,e^{-3/2}$ (B) $-\dfrac{3}{2}\,e^{-3/2}$

(C) $-\dfrac{5}{2}\,e^{-3/2}$ (D) $-\dfrac{7}{2}\,e^{-3/2}$

Q.29 Let $g:[0,2]\to\mathbb{R}$ be defined by

$$g(x)=\int_0^x (x-t)e^t\,dt.$$

The area between the curve $y=g''(x)$ and the x-axis over the interval $[0,2]$ is

(A) e^2-1 (B) $2\left(e^2-1\right)$

(C) $4\left(e^2-1\right)$ (D) $8\left(e^2-1\right)$

Q.30 Let P be a 3×3 singular matrix such that $P\vec{v}=\vec{v}$ for a nonzero vector $\vec{v}$ and

$$P\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}2/5\\0\\-2/5\end{bmatrix}.$$

Then

(A) $P^3=\frac{1}{5}\left(7P^2-2P\right)$

(B) $P^3=\frac{1}{4}\left(7P^2-2P\right)$

(C) $P^3=\frac{1}{3}\left(7P^2-2P\right)$

(D) $P^3=\frac{1}{2}\left(7P^2-2P\right)$

SECTION - B

MULTIPLE SELECT QUESTIONS (MSQ)

Q. 31 – Q. 40 carry two marks each.

Q.31 For two nonzero real numbers a and $b,$ consider the system of linear equations

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b/2 \\ a/2 \end{bmatrix}.$$

Which of the following statements is (are) TRUE?

(A) If $a = b,$ the solutions of the system lie on the line $x + y = 1/2$

(B) If $a = -b,$ the solutions of the system lie on the line $y - x = 1/2$

(C) If $a \neq \pm b,$ the system has no solution

(D) If $a \neq \pm b,$ the system has a unique solution

Q.32 For $n \geq 1,$ let

$$a_n = \begin{cases} n\, 2^{-n}, & \text{if } n \text{ is odd,} \\ -3^{-n}, & \text{if } n \text{ is even.} \end{cases}$$

Which of the following statements is (are) TRUE?

(A) The sequence $\{a_n\}$ converges

(B) The sequence $\{|a_n|^{1/n}\}$ converges

(C) The series $\sum_{n=1}^{\infty} a_n$ converges

(D) The series $\sum_{n=1}^{\infty} |a_n|$ converges

Q.33 Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(x) = x\left(e^{1/x^3} - 1 + \frac{1}{x^3}\right).$$

Which of the following statements is (are) TRUE?

(A) $\lim_{x \to \infty} f(x)$ exists

(B) $\lim_{x \to \infty} x\, f(x)$ exists

(C) $\lim_{x \to \infty} x^2\, f(x)$ exists

(D) There exists $m > 0$ such that $\lim_{x \to \infty} x^m f(x)$ does NOT exist.

Q.34 For $x \in \mathbb{R}$, define $f(x) = \cos(\pi x) + [x^2]$ and $g(x) = \sin(\pi x)$. Which of the following statements is (are) TRUE?

(A) $f(x)$ is continuous at $x = 2$

(B) $g(x)$ is continuous at $x = 2$

(C) $f(x) + g(x)$ is continuous at $x = 2$

(D) $f(x)\, g(x)$ is continuous at $x = 2$

Q.35 Let E and F be two events with $0 < P(E) < 1$, $0 < P(F) < 1$ and $P(E \mid F) > P(E)$. Which of the following statements is (are) TRUE?

(A) $P(F \mid E) > P(F)$

(B) $P(E \mid F^C) > P(E)$

(C) $P(F \mid E^C) < P(F)$

(D) E and F are independent

Q.36 Let $X_1, \ldots, X_n$ $(n > 1)$ be a random sample from a $U(2\theta - 1, 2\theta + 1)$ population, $\theta \in \mathbb{R}$, and $Y_1 = \min\{X_1, \ldots, X_n\}$, $Y_n = \max\{X_1, \ldots, X_n\}$. Which of the following statistics is (are) maximum likelihood estimator (s) of θ?

(A) $\frac{1}{4}(Y_1 + Y_n)$

(B) $\frac{1}{6}(2Y_1 + Y_n + 1)$

(C) $\frac{1}{8}(Y_1 + 3Y_n - 2)$

(D) Every statistic $T(X_1, \ldots, X_n)$ satisfying $\frac{(Y_n - 1)}{2} < T(X_1, \ldots, X_n) < \frac{(Y_1 + 1)}{2}$

Q.37 Let $X_1, \ldots, X_n$ be a random sample from a $N(0, \sigma^2)$ population, $\sigma > 0$. Which of the following testing problems has (have) the region $\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \ge \chi^2_{n,\alpha}\right\}$ as the most powerful critical region of level α?

(A) $H_0 : \sigma^2 = 1$ against $H_1 : \sigma^2 = 2$

(B) $H_0 : \sigma^2 = 1$ against $H_1 : \sigma^2 = 4$

(C) $H_0 : \sigma^2 = 2$ against $H_1 : \sigma^2 = 1$

(D) $H_0 : \sigma^2 = 1$ against $H_1 : \sigma^2 = 0.5$

Q.38 Let $X_1,\ldots,X_n$ be a random sample from a $N\left(0,\, 2\theta^2\right)$ population, $\theta > 0.$ Which of the following statements is (are) TRUE?

(A) $\left(X_1,\ldots,X_n\right)$ is sufficient and complete

(B) $\left(X_1,\ldots,X_n\right)$ is sufficient but NOT complete

(C) $\sum_{i=1}^{n} X_i^2$ is sufficient and complete

(D) $\frac{1}{2n}\sum_{i=1}^{n} X_i^2$ is the uniformly minimum variance unbiased estimator for θ^2

Q.39 Let $X_1,\ldots,X_n$ be a random sample from a population with the probability density function

$$f\left(x \mid \theta\right) = \begin{cases} \theta e^{-\theta x}, & x > 0, \\ 0, & \text{otherwise} \end{cases}, \quad \theta > 0.$$

Which of the following is (are) $100\left(1-\alpha\right)\%$ confidence interval(s) for θ?

(A) $\left(\dfrac{\chi^2_{2n,1-\alpha/2}}{2\sum_{i=1}^{n} X_i}, \dfrac{\chi^2_{2n,\alpha/2}}{2\sum_{i=1}^{n} X_i}\right)$

(B) $\left(0, \dfrac{\chi^2_{2n,\alpha}}{2\sum_{i=1}^{n} X_i}\right)$

(C) $\left(\dfrac{\chi^2_{2n,1-\alpha/2}}{\sum_{i=1}^{n} X_i}, \dfrac{\chi^2_{2n,\alpha/2}}{\sum_{i=1}^{n} X_i}\right)$

(D) $\left(\dfrac{2\sum_{i=1}^{n} X_i}{\chi^2_{2n,\alpha/2}}, \dfrac{2\sum_{i=1}^{n} X_i}{\chi^2_{2n,1-\alpha/2}}\right)$

Q.40 The cumulative distribution function of a random variable X is given by

$$F\left(x\right) = \begin{cases} 0, & x < 2, \\ \frac{1}{10}\left(x^2 - \frac{7}{3}\right), & 2 \le x < 3, \\ 1, & x \ge 3. \end{cases}$$

Which of the following statements is (are) TRUE?

(A) $F\left(x\right)$ is continuous everywhere

(B) $F\left(x\right)$ increases only by jumps

(C) $P\left(X = 2\right) = \frac{1}{6}$

(D) $P\left(X = \frac{5}{2} \mid 2 \le X \le 3\right) = 0$

SECTION – C

NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q.41 Let $X_1, \ldots, X_{10}$ be a random sample from a $N(3,12)$ population. Suppose $Y_1 = \frac{1}{6}\sum_{i=1}^{6} X_i$ and $Y_2 = \frac{1}{4}\sum_{i=7}^{10} X_i$. If $\frac{(Y_1 - Y_2)^2}{\alpha}$ has a χ_1^2 distribution, then the value of α is _______________

Q.42 Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{2x}{9}, & 0 < x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

Then the upper bound of $P(|X-2| > 1)$ using Chebyshev's inequality is _______________

Q.43 Let X and Y be continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} e^{(x+y)}, & -\infty < x,\ y < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $P(X < Y) =$ ____________________

Q.44 Let X and Y be continuous random variables with the joint probability density function

$$f(x,y) = \frac{1}{2\pi} e^{-\left(x^2+y^2\right)/2}, \quad (x,\ y) \in \mathbb{R}^2.$$

Then $P(X > 0, Y < 0) =$ ________________________

Q.45 Let Y be a $Bin\left(72, \frac{1}{3}\right)$ random variable. Using normal approximation to binomial distribution, an approximate value of $P(22 \le Y \le 28)$ is _______________________

Q.46 Let X be a $Bin(2, p)$ random variable and Y be a $Bin(4, p)$ random variable, $0 < p < 1$. If $P(X \geq 1) = \frac{5}{9}$, then $P(Y \geq 1) =$ ____________

Q.47 Consider the linear transformation

$$T(x, y, z) = (2x + y + z,\ x + z,\ 3x + 2y + z).$$

The rank of T is ________________________

Q.48 The value of $\lim_{n \to \infty} n \left[e^{-n} \cos(4n) + \sin\left(\frac{1}{4n}\right) \right]$ is _________________________

Q.49 Let $f : [0, 13] \to \mathbb{R}$ be defined by $f(x) = x^{13} - e^{-x} + 5x + 6$. The minimum value of the function f on $[0,13]$ is__________________________

Q.50 Consider a differentiable function f on $[0,1]$ with the derivative $f'(x) = 2\sqrt{2x}$. The arc length of the curve $y = f(x),\ 0 \leq x \leq 1,$ is ________________________

Q. 51 – Q. 60 carry two marks each.

Q.51 Let m be a real number such that $m > 1$. If

$$\int_1^m \int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dy\, dx\, dz = e - 1,$$

then $m =$ ______________________

Q.52 Let

$$P = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}.$$

The product of the eigen values of P^{-1} is ________________________

Q.53 The value of the real number m in the following equation

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \left(x^2+y^2\right) dy\, dx = \int_{m\pi}^{\pi/2} \int_0^{\sqrt{2}} r^3\, dr\, d\theta$$

is ________________________

Q.54 Let $a_1 = 1$ and $a_n = 2 - \frac{1}{n}$ for $n \geq 2$. Then

$$\sum_{n=1}^{\infty} \left(\frac{1}{a_n^2} - \frac{1}{a_{n+1}^2} \right)$$

converges to ______________________

Q.55 Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with the probability density function

$$f(x) = \begin{cases} 4x^2 e^{-2x}, & x > 0, \\ 0, & \text{otherwise} \end{cases}$$

and let $S_n = \sum_{i=1}^{n} X_i$. Then $\lim_{n \to \infty} P\left(S_n \leq \frac{3n}{2} + \sqrt{3n} \right)$ is _____________________

Q.56 Let X and Y be continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} \dfrac{c\, x^2}{y^3}, & 0 < x < 1,\ y > 1, \\ 0, & \text{otherwise} \end{cases},$$

where c is a suitable constant. Then $E(X) =$ ________________________

Q.57 Two points are chosen at random on a line segment of length 9 cm. The probability that the distance between these two points is less than 3 cm is ______________________

Q.58 Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $P\left(\frac{1}{4} < X^2 < \frac{1}{2}\right) =$ ______________________

Q.59 If X is a $U(0,1)$ random variable, then $P\left(\min(X, 1-X) \le \frac{1}{4}\right) =$ ________________

Q.60 In a colony all families have at least one child. The probability that a randomly chosen family from this colony has exactly k children is $(0.5)^k$; $k = 1, 2, \ldots$. A child is either a male or a female with equal probability. The probability that such a family consists of at least one male child and at least one female child is _________

END OF THE QUESTION PAPER