JEE-Main-29-06-2022-Shift-2 (Memory Based)

MATHEMATICS

Question: $\lim_{x\to1}\frac{(x^2-1)\sin^2(\pi x)}{x^4-2x^3+2x-1}$ is equal to

Options:

(a) $2\pi^2$

(b) π^2

(c) $3\pi^2$

(d) $\frac{\pi^2}{2}$

Answer: (b)

Solution:

Given, $\lim_{x\to1}\frac{(x^2-1)\sin^2(\pi x)}{x^4-2x^3+2x-1}$

$=\lim_{x\to1}\frac{(x^2-1)\sin^2(\pi x)}{x^4-2x^3-x^2+x^2+2x-1}$

$=\lim_{x\to1}\frac{(x^2-1)\sin^2(\pi x)}{(x^2-1)(x^2-2x+1)}$

$=\lim_{h\to0}\frac{\sin^2(\pi(1+h))}{(1+h-1)^2}=\lim_{h\to0}\frac{\sin^2\pi h}{h^2}$

$=\lim_{h\to0}\frac{\sin^2\pi h}{h^2}$

$=\lim_{h\to0}\left(\frac{\sin\pi h}{\pi h}\right)\cdot\pi^2=\pi^2$

Question: If the line $\frac{x-2}{3}=\frac{y-2}{4}=\frac{z+6}{2}$ intersects the plane $2x+4y+3z=0$ at point P. Find the distance OP (where O is origin $(0,0,0)$).

Options:

(a) $\frac{\sqrt{8096}}{7}$

(b) $\frac{\sqrt{9053}}{14}$

(c) $\frac{\sqrt{7084}}{7}$

(d) $\frac{\sqrt{9017}}{14}$

Answer: (b)

Solution:

Let a point on line be $(3\lambda+2, 4\pi+2, 2\lambda-6)$

Now, $2(3\lambda+2)+4(4\lambda+2)+3(2\lambda-6)=0$

$28\lambda=6$

$\Rightarrow \lambda=\frac{3}{14}$

Point is: $\left(\frac{37}{14},\frac{40}{14},-\frac{78}{14}\right)$

$OP=\sqrt{\left(\frac{37}{14}\right)^2+\left(\frac{40}{14}\right)^2+\left(\frac{-78}{14}\right)^2}$

$=\frac{\sqrt{9053}}{14}$

Question: $3,6,9,....$ upto 78 terms

$5,9,13,....$ upto 59 terms.

Find the sum of common terms between them.

Options:

(a) 2223

(b) 1785

(c) 1805

(d) 2025

Answer: (a)

Solution:

$3,6,9,....$ upto 78 terms

$\Rightarrow t_{78}=3+77\times 3=234$

$5,9,13,....$ upto 59 terms

$\Rightarrow t_{59}=5+58\times 4=237$

Common difference of common terms $=\text{LCM}\{3,4\}=12$

$9,21,33,...,225$

$225=9+(n-1)12$

$\Rightarrow n=19$

$S=\frac{n}{2}[a+l]=\frac{19}{2}[9+225]=2223$

Question: If $S = 1+\frac{5}{6}+\frac{10}{6^2}+\frac{16}{6^3}+...$ then find S.

Options:

(a) $\frac{16}{216}$

(b) $\frac{301}{125}$

(c) $\frac{25}{216}$

(d) $\frac{276}{125}$

Answer: (d)

Solution:

Given,

$$S = 1+\frac{5}{6}+\frac{10}{6^2}+\frac{16}{6^3}+...$$

$$\frac{S}{6} = \quad \frac{1}{6}+\frac{5}{6^2}+\frac{10}{6^3}+....$$

$$\frac{5S}{6} = 1+\frac{4}{6}+\frac{5}{6^2}+\frac{6}{6^3}+...$$

$$\frac{5S}{6}-1 = \frac{4}{6}+\frac{5}{6^2}+\frac{6}{6^3}+...$$

$$\frac{1}{6}\left(\frac{5S}{6}-1\right) = \frac{4}{6^2}+\frac{5}{6^3}+...$$

$$\frac{5}{6}\left(\frac{5S}{6}-1\right) = \frac{4}{6}+\frac{1}{6^2}+\frac{1}{6^3}+...$$

$$\frac{5}{6}\left(\frac{5S}{6}-1\right) = \frac{4}{5}+\frac{\left(\frac{1}{36}\right)}{1-\frac{1}{6}}$$

$$\Rightarrow \frac{5S}{6}-1 = \frac{21}{25}$$

$$\Rightarrow \frac{5S}{6} = \frac{46}{25}$$

$$\Rightarrow S = \frac{46}{25}\times\frac{6}{5}$$

$$\Rightarrow S = \frac{276}{125}$$

Question: Let f be a continuous function in $[0,1]$ such that $f(x)=x+\int_0^1 (x-t)f(t)\,dt$, then which of the following points does not lie on the curve $y=f(x)$?

Options:

(a) $\left(\frac{1}{2},\frac{5}{13}\right)$

(b) $\left(\frac{1}{3},\frac{2}{13}\right)$

(c) $\left(\frac{2}{9},0\right)$

(d) $\left(\frac{1}{6},\frac{1}{13}\right)$

Answer: (d)

Solution:

$\because f(x)=x\left(1+\int_0^1 f(t)\,dt\right)-\int_0^1 t\cdot f(t)\,dt$

Let $a=1+\int_0^1 f(t)\,dt$ and $b=\int_0^1 t\cdot f(t)\,dt$

$\Rightarrow f(x)=ax-b$

$a=1+\int_0^1 f(t)\,dt$

$\Rightarrow \frac{a}{2}=1-b \quad ....(1)$

$b=\int_0^1 t\cdot(at-b)\,dt$

$\Rightarrow \frac{3b}{2}=\frac{a}{3} \quad ....(2)$

From (1) and (2), we get

$a=\frac{8}{13}$ and $b=\frac{4}{13}$

$\Rightarrow f(x)=\frac{18x-4}{13}$

Clearly $\left(\frac{1}{6},\frac{1}{13}\right)$ does not lie on the curve $y=f(x)$

Question: Find the probability that a relation $\{x,y\}\to\{x,y\}$ is symmetric as well as transitive.

Options:

(a) $\frac{1}{4}$

(b) $\frac{3}{8}$

(c) $\frac{5}{16}$

(d) $\frac{1}{8}$

Answer: (c)

Solution:

$\{x,y\}\times\{x,y\}=\{(x,x),(x,y),(y,x),(y,y)\}$

Number of possible relations $=2^4=16$

The relations which are symmetric as well as transitive are

$\phi,\{x,x\},\{y,y\},\{(x,x),(y,y)\}$ and $\{(x,x),(x,y),(y,x),(y,y)\}$

$\therefore$ Required probability $=\frac{5}{16}$

Question: The height of a pole is $20\ m$. If the angle of elevation of the tower from the top of the pole is $30°$ and the same from bottom of the tower is $60°$, then the height of the tower is

Options:

(a) $20\sqrt{3}\ m$

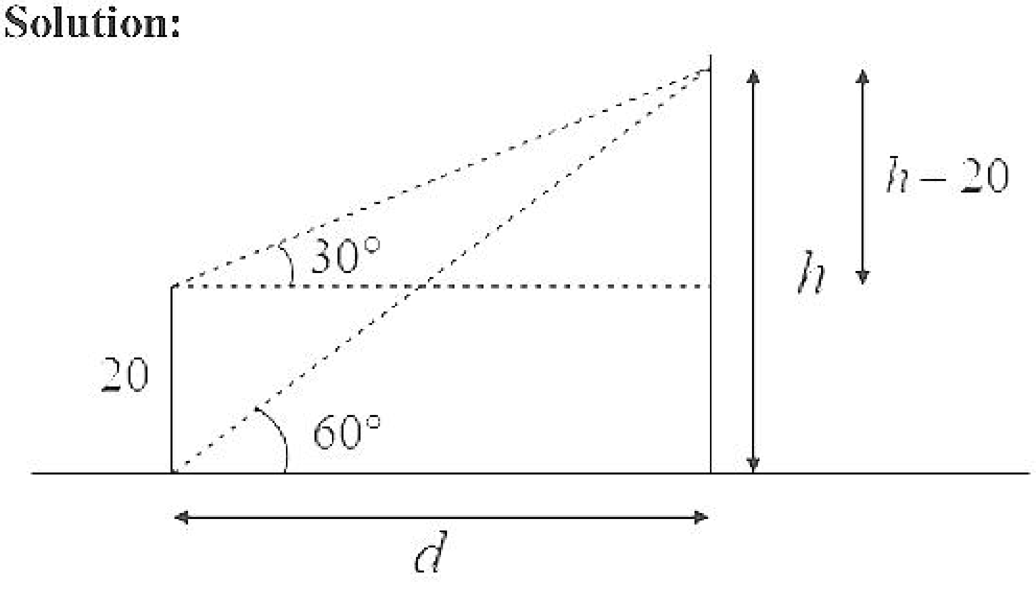
(b) $\left(20+10\sqrt{3}\right)\ m$

(c) $30\ m$

(d) $\left(10+20\sqrt{3}\right)\ m$

Answer: (c)

Solution:



$\tan 30° = \frac{h-20}{d}$

$\tan 60^\circ = \frac{h}{d}$

$\Rightarrow \frac{1}{3} = \frac{h-20}{h}$

$\Rightarrow h = 30\ m$

Question: If $\sin x = \cos^2 x,$ then the number of solutions in $x \in (0,10)$ are _____.

Answer: 4.00

Solution:

Given, $\sin x = \cos^2 x$

$\Rightarrow \sin^2 x + \sin x - 1 = 0$

$\Rightarrow \sin x = \left(\frac{-1 \pm \sqrt{5}}{2}\right)$

$\Rightarrow \sin x = \frac{\left(\sqrt{5}-1\right)}{2}$

$\Rightarrow$ Number of solutions in $(0,10) = 4$

Question: In the expansion of $\left(2x^{\frac{1}{5}} - \frac{1}{x^{\frac{1}{5}}}\right)15,$ coefficients of x^{-1} and x^{-3} are m and n respectively. If $m \cdot n^2 = {}^{15}C_r \cdot 2^r,$ then r is equal to _____.

Answer: 5.00

Solution:

$T_{r+1} = {}^{15}C_r \left(2x^{\frac{1}{5}}\right)^{15-r} \left(\frac{-1}{x^{\frac{1}{5}}}\right)^r$

For term having x^{-1}

$\frac{15-r}{5} - \frac{r}{5} = -1$

$\Rightarrow r = 10$

$\Rightarrow m = {}^{15}C_{10} \cdot 2^5 \cdot (-1)^{10}$

For term having x^{-3}

$\frac{15-2r}{5} = -3$

$\Rightarrow r = 15$

$\Rightarrow n = {}^{15}C_{15} \cdot 2^0 \cdot (-1)^{15} = -1$

$\therefore mn^2 = {}^{15}C_{10} \cdot 2^5 \cdot (-1)^2 = {}^{15}C_r \cdot 2^r$

$\Rightarrow {}^{15}C_5 \cdot 2^5 = {}^{15}C_r \cdot 2^r$

$\Rightarrow r = 5$

Question: If $A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$, then the modulus of sum of all elements of matrix B which is satisfying $B = I - {}^5C_1\, adj(A) + {}^5C_2 \left(adj(A)\right)^2 - {}^5C_3 \left(adj(A)\right)^3 + {}^5C_4 \left(adj(A)\right)^4 - {}^5C_5 \left(adj(A)\right)^5$ is ____.

Answer: 7.00

Solution:

Given, $B = I - {}^5C_1\, adj(A) + {}^5C_2 \left(adj(A)\right)^2 - {}^5C_3 \left(adj(A)\right)^3 + {}^5C_4 \left(adj(A)\right)^4 - {}^5C_5 \left(adj(A)\right)^5$

$\Rightarrow B = \left(I - adj(A)\right)^5$

$A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$

$\Rightarrow adj(A) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$B = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}^5$

$-B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^5 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

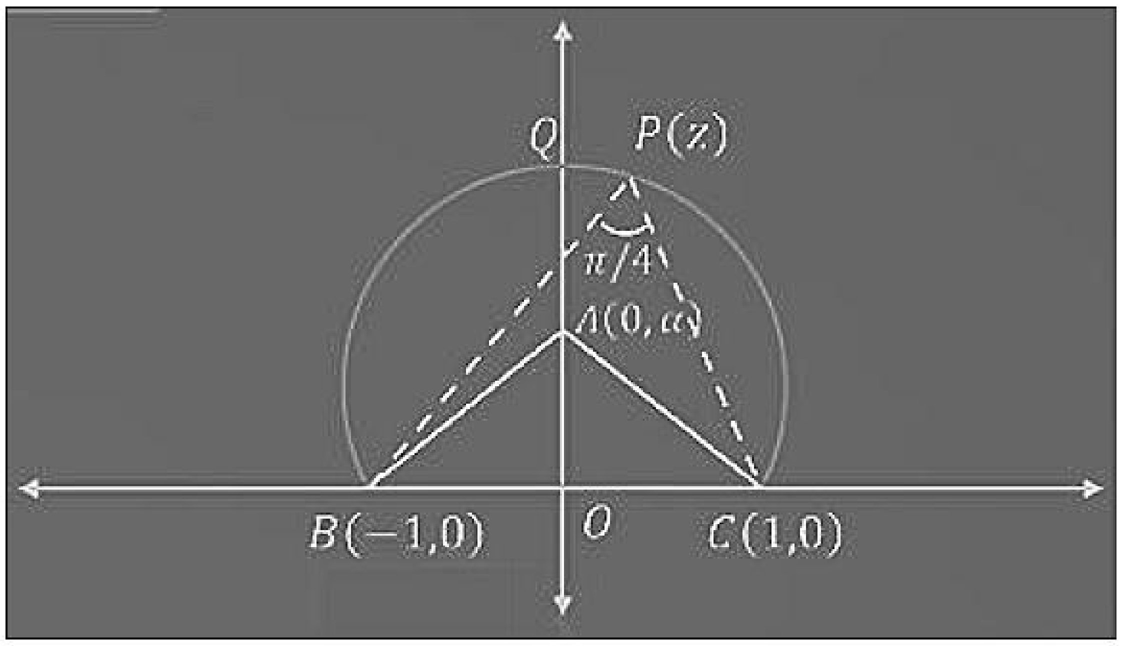
$B = \begin{bmatrix} -1 & -5 \\ 0 & -1 \end{bmatrix}$

Sum of elements $= -1 - 5 - 1 = -7$

Question: The number of complex numbers z such that $|z| = 3$ and $\arg(z-1) - \arg(z+1) = \frac{\pi}{4}$ are _____.

Answer: 0.00

Solution:



Given, $\arg(z-1)-\arg(z+1)=\frac{\pi}{4}$

$\Rightarrow \arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$

$\Rightarrow z$ is on major arc of a circle having BC as chord as shown in figure

$\angle OAC=\angle BPC=\frac{\pi}{4}$

$\Rightarrow OA=OC=1=\alpha$

Radius $=AC=\sqrt{2}$

$OQ=\alpha+\text{Radius}=1+\sqrt{2}$

$\Rightarrow Q=\left(0,1+\sqrt{2}\right)$

$|z|=3$ represents a circle of radius 3 and centre at $z=0$

Then both circles do not intersect.
Hence, no common point.

Question: Number of four digit numbers in which first three digit number is divisible by last digit i.e., fourth digit is _____.
Answer: 2545.00

Solution:

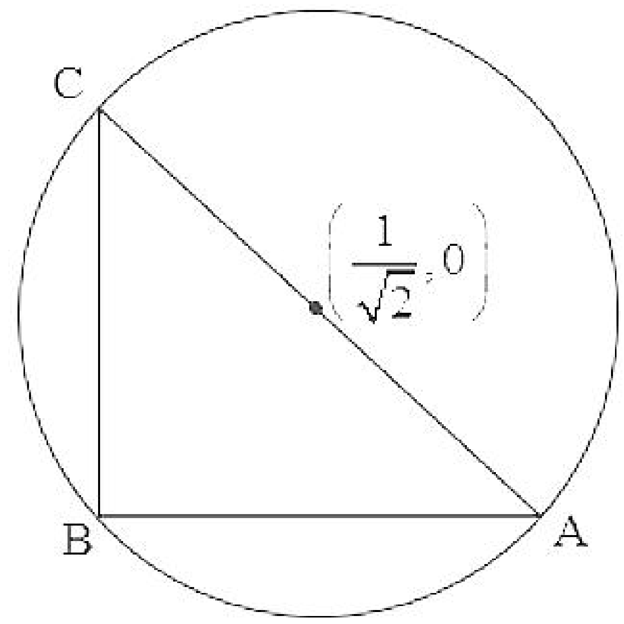
If the last digit is d then there are $\left[\frac{900}{d}\right]$ possibilities for first three digits.

Total number of 4 digit numbers $=\sum_{d=1}^{9}\left[\frac{900}{d}\right]$

$=900+450+300+225+180+150+128+112+100$
$=2545$

Question: ΔABC is inscribed in a circle $x^2-\sqrt{2}x+y^2=0$ where $\angle ABC=\frac{\pi}{2}$, then the maximum area of triangle ABC is _____.
Answer: 0.5

Solution:



Centre $\equiv \left(\frac{1}{\sqrt{2}}, 0\right)$, $r = \frac{1}{\sqrt{2}}$

$AC =$ diameter $= \sqrt{2}$

$ar(\Delta ABC) = \frac{1}{2} \times AB \times BC$

$= \frac{1}{2} \times \sqrt{2}\cos\theta \times \left(\sqrt{2}\sin\theta\right)$

$= \frac{1}{2} \times \sin 2\theta$

$\therefore ar(\Delta ABC)_{\max} = \frac{1}{2} = 0.5$

Question: $M = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}$. If $\left(I - M^2\right)N = -2I$ & $N = \sum_{k=1}^{49} M^{2k}$, then $\alpha^2 =$ ____.

Answer: 1.00

Solution:

$M = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}$

$\therefore M^2 = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}\begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix} = \begin{bmatrix} -\alpha^2 & 0 \\ 0 & -\alpha^2 \end{bmatrix} = -\alpha^2 I$

So, $N = \sum_{k=1}^{49} \left(M^2\right)^k = I\left(-\alpha^2 + \alpha^4 - \alpha^6 + \ldots 49 \text{ terms}\right) = -\frac{\alpha^2\left(1+\alpha^{98}\right)}{1+\alpha^2}$

Now, $\left(I - M^2\right)N = \left(1+\alpha^2\right)\left[-\frac{\alpha^2\left(1+\alpha^{98}\right)}{1+\alpha^2}\right]I = -2I$

$\Rightarrow \alpha^2\left(1+\alpha^{98}\right) = 2$

$\Rightarrow \alpha^2 = 1$